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Some game theoretic marketing attribution models

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Abstract
In this paper, we propose and analyse two game theoretic approaches to design attribution mechanisms for multi-channel marketing campaigns. Both approaches are based on a key performance index function that provides the benefit obtained in each of the observed paths to conversion. The first approach considers the problem as a cooperative transferable utility game, and the proposed attribution mechanisms are based on the Shapley value. The second approach models the problem as a bankruptcy problem and the proposed attribution mechanism is based on the constrained equal-losses rule. We also extend the above approaches to deal with the cases in which the position or the repetition of the channels on the paths to conversion are taken into account.

Keywords Cooperative game theory · Marketing · Multi-channel attribution · Shapley value · Bankruptcy problems · Constrained equal-losses rule

1 Introduction and literature review

The attribution of the benefits obtained from the different channels involved in a marketing campaign, is a relevant issue because it can help to optimally assign a marketing budget and, in general, a through understanding of the effects of a campaign. In fact, there is a vast amount of literature on this problem within the field of marketing that we do not intend to analyse here. The reader is referred to Jayawardane et al. (2015), Choi et al. (2020) for a review and classification of the methodologies considered in this field.

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Taking into account that an attribution problem is in essence a benefit allocation problem, we are interested in a game theoretic approach to the marketing attribution problem. Building on the models and solutions of cooperative game theory allows the decision maker to select a specific attribution rule based on a list of properties that is considered relevant to the decision maker's problem. However, the existing literature on this topic is scarce. Perhaps, the first paper on the marketing environment adopting a cooperative game approach is (Dalessandro et al., 2012), in which an attribution methodology based on a causal estimation problem that uses the concept of Shapley value (Shapley, 1953) is proposed. Since then, several papers which have not been formally published (Morales, 2016; Zhao et al., 2018) also address this problem and, in particular, they define a related cooperative game with transferable utility (TU-game) and consider its Shapley value as the attribution rule, which is precisely our first proposal. Other subsequent papers also adopt the Shapley value as an attribution methodology by means of defining TU games, whose characteristic functions are based on a probabilistic markovian approach (see, for instance, Singal, 2022). In contrast to these scarce contributions, there are other attribution problems, such as the *museum pass problem*, introduced by Ginsburgh and Zang (2003), which are closely related to this attribution problem and deserve a profuse game theoretic analysis, see for instance (Ginsburgh & Zang, 2004; Béal & Solal, 2010; Casas-Méndez et al., 2011, 2014; Estévez-Fernández et al., 2010, 2012; Bergantiños & Moreno-Ternero, 2015; Cano-Berlanga et al., 2017).

We consider two related, but basically different, models to define an attribution rule based on a game theoretic approach, which have also been considered to address the museum pass problem. The first model considers a related TU game to describe the marketing attribution problem. More specifically, the TU game we deal with is a generalisation of the museum pass game which turns out to be TU-proportional[1] to a *labeled network game*, introduced by Algaba et al. (2019c). The reader is referred to Algaba et al. (2019) for the analysis of the relationship between labeled network and museum pass games, as well as other cooperative games arising from attribute situations. On the basis of this model, we will generalise the proposed rule -which is also based on the Shapley value and is TU-proportional to the *Shapley quota allocation mechanism* for labeled network games (Algaba et al., 2019c)–to take into account those cases in which the number of times each channel appears on the observed paths to conversion is relevant, and also to obtain a measure of the relative weight each position has, when the order in which the channels appear on those paths is also relevant.

The second model follows a similar approach to that of Casas-Méndez et al. (2011, 2014); Estévez-Fernández et al. (2010, 2012); Bergantiños and Moreno-Ternero (2015), which consists on considering the attribution problem as a bankruptcy problem (O'Neill, 1982; Aumann & Maschler, 1985). However, none of the bankruptcy problems considered in these papers are applicable to our problem, with the exception of Bergantiños and Moreno-Ternero (2015). This is because all of them use the single ticket price for each of the museums involved, which makes no sense in our particular context. We specifically propose the constrained equal-losses (CEL) rule and the proportional (PROP) rule as attribution mechanisms. As stated before, on the basis of this second model, we will generalise the two proposed rules -based on CEL and PROP- to take into account those cases in which the number of times each channel appears in the observed paths to conversion is relevant, and also to obtain a measure of the relative weight each position has, when the order in which the channels appear in those paths is also relevant.

The paper is organised as follows. In Sect. 2 we formally introduce the marketing attribution problem that we are dealing with. In Sect. 3 we introduce the first model -which is

[1] See Footnote 4 in Page 8 for a rigorous definition of TU-proportionality.

based on TU games- which we will study. In addition, we analyse related games, propose several attribution rules on the grounds of different scenarios, depending on the relevance of the order or the repetitions of channels in the paths to conversion, and adapt some existing axiomatic characterisations for these rules. We also study monotonicity and decomposition properties of the mechanisms when repetition and positions are taken into account, respectively. In Sect. 4 we turn to the analysis of the second proposed model. Analogously, we analyse related bankruptcy problems, propose several attribution rules on the grounds of different scenarios, depending on the relevance of the order or the repetitions of channels in the paths to conversion, and adapt existing axiomatic characterisations for these rules. We also show the monotonicity of the proposed mechanisms when repetition is considered and we obtain some general results about their behaviour when more than one channel split their claims simultaneously. Some final conclusions are included in Sect. 5.

2 Multi-channel attribution problem

We assume that an advertising campaign exists in which an advertisement is broadcast through a set of channels. The consumers can have multiple touch-points with the campaign by watching the ad on some of those channels. Subsequently, at some point a conversion of a consumer could happen by purchasing (in a very wide sense) the advertised product, thus producing a measurable benefit. The attribution problem is then how to attribute to the different channels that were watched before the conversion, the benefit produced by that conversion. Let us formalise these ideas:

Let $N = \{1, 2, ..., n\}$ be the finite set of channels involved in the campaign. A *path to conversion*, $p = (p_{(1)}, p_{(2)}, ..., p_{(\ell_p)})$, is any finite ordered sequence of channels of N, where $p_{(j)} \in N$ is the channel in position j in path p, and ℓ_p being the length of path p. We must remark that a channel can appear more than once in a path.

Note that the cardinal of the set of *all possible* paths to conversion $\mathcal{P}(N)$ is, in principle, infinite. However, we shall consider only finite sets of paths to conversion, since in practice only a finite number of paths $P(N) \subseteq \mathcal{P}(N)$ are observed. Since the benefit generated by any path that did not occur is zero, all those paths will not belong to the support of the considered problems.

The benefit of a path is given by a Key Performance Index: $f : P(N) \longrightarrow \mathbb{R}_+$, that assigns to any observed path to conversion $p \in P(N)$ a measure $f(p) \geq 0$ of the benefit obtained by conversions of all consumers that have followed this path p. Thus, the total benefit of the campaign is $B = \sum_{p \in P(N)} f(p)$.

Hereafter, we shall assume that $f(p)$ is the sum of the benefits produced by all the consumers that have followed exactly the same path p to the conversion. We shall also assume that spontaneous conversions without having watched the advertisement in any channel, are already discounted in such a way that the benefit of the null path is zero. Formally, we define a *Multi-channel Attribution Problem* and a *Multi-channel Attribution Rule* as follows.

Definition 1 A multi-channel attribution problem (MA problem) is a 3-tuple $(N, P(N), f)$, where N is the finite set of channels, $P(N)$ is the finite set of all observed paths to conversion, and $f : P(N) \to \mathbb{R}_+$ is the KPI function.

Definition 2 A multi-channel attribution rule (MA rule) is a mapping γ that associates with each attribution problem $(N, P(N), f)$ an allocation in $\mathbb{R}^N$ indicating the amount each channel gets from the benefit $B = \sum_{p \in P(N)} f(p)$ generated by the campaign, such that:

1. $\gamma_i(N, P(N), f) \geq 0, \forall i \in N$,
2. $\sum_{i \in N} \gamma_i(N, P(N), f) = B$, where $B := \sum_{p \in P(N)} f(p)$

Regarding this problem, some classical approaches that base the attribution on the order in which the channels appear on the path to conversion are commonly used in practice (first touch, last touch, indirect last touch, time decay, among others)[2]. We propose MA rules based on the classical rules for cooperative games with transferable utility and bankruptcy problems, which allow us to get a deeper insight into the attribution mechanisms used to help the advertisers to optimally assign their marketing budget.

3 The first model: a cooperative game theoretic approach

The first model we propose to derive an MA rule is based on TU games. First, we recall some basic definitions of cooperative game theory. Then, we introduce the basic TU game on which our proposal is based and we relate it to the museum pass and labeled network games, which are closely related to it. We adopt the Shapley value as the MA rule to be used when the presence of channels in a path to conversion is the unique relevant information, which can be characterised by means of the classical axiomatisation of the Shapley value for general TU games. Next, we enhance the basic TU game approach in order to deal with more general attribution situations whereby the order in which channels appear in the conversion path, as well as the number of times each channel appears, play a relevant role. First, we consider in Sect. 3.4 the case in which only the number of times a channel appears in a conversion path is relevant, and we generalise the axiomatic characterisation of the Shapley quota allocation mechanism to derive an appropriate characterisation for the proposed rule. Finally, in Sect. 3.5 we take into account the order of appearance and we obtain a measure of the relative weight each position has in those situations.

3.1 Preliminaries. Basic concepts on cooperative games

A *cooperative game with transferable utility*, or simply a game from now on, is an ordered pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$, with $2^N = \{S \mid S \subseteq N\}$, is a *characteristic function* on N with $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S and represents the reward that coalition S can achieve by itself if all its members act together.

A game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$, for every disjoint coalitions $S \cap T \neq \emptyset$; is *monotone* if $v(S) \leq v(T)$, whenever $S \subseteq T$; and it is *convex* if $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$ for every pair $S, T \subseteq N$. Convexity can be restated as an increasing marginal contributions condition, i.e., $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$, for all $S \subseteq T \subseteq N \setminus \{i\}$, and for all $i \in N$. Thus, convexity encourages cooperation, even more than supperadditivity and monotonicity.

Given a game (N, v), and *allocation* or *payoff vector* is any $\mathbf{x} \in \mathbb{R}^N$, which gives player $i \in N$ a payoff x_i. A payoff vector is said to be *efficient* if $\sum_{i \in N} x_i = v(N)$. It is *stable* if it is efficient and $\sum_{i \in S} x_i \geq v(S)$, for every $S \subseteq N$. The set of all stable payoff vectors is called the *Core* of the game (Gillies, 1953), which will be denoted by $C(v)$. The Core of a game can be empty, however if the game is convex, it is always non-empty (Shapley, 1971).

[2] See for instance the multi-touch methodology of Google: https://adwords.googleblog.com/2016/05/move-beyond-last-click-attribution.html

A *value* φ is a map that associates with each game (N, v) a payoff vector $\varphi(N, v)$. One of the most well-known and most used[3] values is the Shapley value (Shapley, 1953) that assigns to each of the players the average of all their marginal contributions when all coalitions of the same size are equally probable, and also when all sizes are equally probable. Formally, for each $i \in N$,

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \big(v(S \cup \{i\}) - v(S)\big), \tag{1}$$

where $|S|$ is the cardinal of S. If the game is convex the Shapley value is stable (Shapley, 1971).

The Shapley value admits an alternative expression in terms of the *Harsanyi dividends* of every coalition S in (N, v), which are defined as:

$$d_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \ \forall S \subseteq N.$$

The Harsanyi dividends can be calculated recursively:

$$d_S = v(S) - \sum_{T \subsetneq S} d_T, \forall S \subseteq N. \tag{2}$$

Then, the Shapley value can be expressed from the Harsanyi dividends as follows (see Shapley, 1953):

$$\phi_i = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{d_S}{|S|} \tag{3}$$

A game (N, v) is *totally positive* when all its Harsanyi dividends are non-negative. Every totally positive game is convex, therefore its Shapley value belongs to its Core.

3.2 The sum game when only the presence of channels is relevant

In this section, we introduce the game for the multi-channel attribution problem on which we base our proposal. It was first introduced in the non-formally published contribution of Morales (2016) as the *conversion game*. In the sequel, we will refer to it as the *sum game*. We recall its definition, derive its main properties and study its relationship with museum pass and labeled network games.

We first consider the case when only the presence of channels is relevant. This approach is suitable for instance in the MA problem related to TV advertising. In this case, when you are buying TV advertising you have no control of the order in which people are exposed as you generally buy spots on multiple channels. Frequency is a possible consideration, but again it is hard to manage as when you buy more spots you reach more consumers, but at the same time you increase the frequency of exposure for those viewers that watch a lot of TV. TV ad buyers therefore, are generally more concerned about reaching consumers than about frequency, e.g. if they have exposed everyone they want to expose.

[3] The reader is referred to Roth (1988) and Algaba et al. (2019a) to delve into the study of the Shapley value, and some of its applications to specific allocation problems.

Given the MA problem $(N, P(N), f)$, when only the presence of channels is relevant, the information of a given path p that we retain is only the set of channels that appear in any position of the path (which we call its *support*):

$$S_p = \{i \in N \,/\, i \in p\} \subseteq N.$$

Thus, we can consider that we work with a set of channels $N = \{1, \ldots, n\}$ and the unique relevant information about their performance is given by the following aggregated KPI function defined over the subsets of N:

$$f(T) := \sum_{\substack{p \in P(N) \\ S_p = T}} f(p), \ \forall\, T \subseteq N, \tag{4}$$

i.e. $f(T)$ is the total benefit produced by all consumers that have seen the ad exactly in the channels in T, regardless of the order or the number of times. Now, if coalition S forms, it can be awarded the profit generated by all conversions of consumers who have seen the ad on any subset of channels of coalition S and who, in addition, have not seen it on any other channel in $N \setminus S$. That is, the worth assigned to coalition S will be given by the sum of the the KPI values of its subsets.

Definition 3 Given a set $N = \{1, 2, ..., n\}$ of players and a map $f : 2^N \to \mathbb{R}$ such that $f(S) \geq 0$ for all $S \subseteq N$, the **sum game** (N, v_Σ^f) is defined by

$$v_\Sigma^f(S) = \sum_{T \subseteq S} f(T), \ \forall\, S \subseteq N. \tag{5}$$

Remark 1 Hereafter, when the KPI function is defined over subsets of N we will refer to them as *combination* of channels, whereas *coalition* of channels is restricted to the argument of a characteristic function.

It is trivial to prove that the sum game is monotone and convex, then it is superadditive, its Core is nonempty and the Shapley value belongs to the Core. Moreover, the class of sum games coincides with the class of totally positive games.

Proposition 1 *For any given sum game (N, v_Σ^f), the Harsanyi dividends are given by the corresponding benefit function $f(\cdot)$ that defines the game. That is, $d_S = f(S)$ for all $S \subseteq N$. Therefore, the class of sum games coincides with the class of totally positive games.*

Proof We prove by induction on the size of $S \subseteq N$ that the Harsanyi dividends of a sum game verify $d_S = f(S)$. Employing the recursive formula of the Harsanyi dividends $d_S = v_\Sigma^f(S) - \sum_{T \subsetneq S} d_T$, for all $S \subseteq N$, we obtain:

$$d_\emptyset = 0; \quad d_i = f(\{i\}), \forall i \in N; \quad d_{\{i,j\}} = f(\{i, j\}), \forall \{i, j\} \subseteq N.$$

Let $S \subseteq N$ and assume that $d_T = f(T), \forall\, T \subseteq N$ such that $|T| \leq |S| - 1$. Again, by the recursive formula, it follows:

$$d_S = v_\Sigma^f(S) - \sum_{T \subsetneq S} d_T = \sum_{T \subseteq S} f(T) - \sum_{T \subsetneq S} f(T) = f(S).$$

□

The sum game associated with a multi-channel attribution problem is closely related to the museum pass game (introduced in Ginsburgh and Zang, 2003) and also to the labeled network game (Algaba et al., 2019c), which are in fact two classes of games which are also closely related between them (see Algaba et al., 2019).

In the museum pass problem $M = (N, M, K)$ there is a set N of museums participating in a pass program, a set M of customers that have bought a pass, and a map $K : M \to N$ that specifies the set of museums $K(j) \subseteq N$ visited by customer $j \in M$. In the associated *museum pass game* (N, v^M) the value of coalition S is given by the number of pass holders who only visited some or all of the museums in coalition S. Note that the sum game clearly generalises the museum pass game, in which the KPI of every conversion (which in this case is given by a pass holder) is always the same (one, or the price k of the pass). Note also, that in the museum pass problem, repetitions are not considered, since each pass allows each museum to be visited no more than one time. The order in which pass holders visit the museums is also irrelevant. However, if the museum pass game is generalised to consider the case in which visitors may pay a continuum of different prices for the pass, both classes of games will coincide when the presence of the channels is the unique relevant information. However, note that this is not a realistic assumption.

In Cano-Berlanga et al. (2017) the authors introduce the *sale channels game* to analyse the multi-channel attribution problem, which coincides with the museum pass game, since the value of a coalition of channels S equals the number of sales made by customers that have seen the ad in some of the channels of coalition S.

Within the labeled network allocation problem there is a set of agents N that control some of the arcs of a given network, a set of labelled routes $\mathcal{R}$ whose arcs may belong to different agents, and the part of one unit of flow which has effectively occurred through each of those routes. In the associated *labeled network game* (N, v^{LN}) the value of coalition S is the part of one unit of flow that has occurred through routes whose arcs belong to agents in S. Note that considering $k = \frac{1}{B}$ (where B is the global benefit of the campaign) every sum game, when only the presence of channels is relevant, is TU-proportional[4] to a *labeled network game*. In the sequel we will make use of this relationship. Again, as in the museum pass problem, the order in which the arcs are visited are usually irrelevant regarding the kind of situations considered in this framework (use of public transport systems, for instance), nor do repetitions have much sense.

3.3 The Shapley multi-channel attribution rule

Considering the definition of the multi-channel attribution rule given in Sect. 2, we can use many different MA rules based on different values of the sum game associated with each problem. However, as we have mentioned before, we adopt the Shapley value of the sum game as the proposed attribution mechanism. We base our decision on the following reasons: (i) the properties characterising the Shapley value, which are in fact very useful from a marketing point of view (primarily additivity); (ii) its simple expression in this case, which is also computationally tractable; and (iii) its widespread use in real-world applications, which has increased substantially in recent years (see Moretti and Patrone 2008; Thomson, 2019b; Sánchez-Soriano, 2019; Samek et al.,2021, among many others).

To be specific, we will refer as the *Shapley multi-channel attribution (Shapley-MA)* rule to the MA rule given by the Shapley value of the sum game (N, v_{Σ}^{f}) associated with each MA

[4] The class $\mathcal{C}$ of TU-games is said to be TU-proportional to class $\mathcal{D}$, if and only if, for every game $(N, v) \in \mathcal{C}$ there exist $k \in \mathbb{R}_+$ and $(N, w) \in \mathcal{D}$ such that $v(S) = kw(S), \forall\, S \subseteq N$.

problem $(N, P(N), f)$: $\phi(N, P(N), f) := \phi(N, v_{\Sigma}^{f})$. We must remark that this is precisely the MA rule proposed by Morales (2016) and Zhao et al. (2018). However, none of these references develop a thorough analysis of the resulting mechanism, nor generalise it to the cases in which repetitions or order have a direct impact on the contribution of each channel to the realised conversion. Moreover, taking into account that the class of sum games is TU-proportional to the class of labeled network games, it is also TU-proportional to the *Shapley quota allocation mechanism* introduced by Algaba et al. (2019c).

As well as museum pass games, sum games are also convex and, taking into account Proposition 1, their Shapley value can be expressed in terms of the KPI function $f(\cdot)$ as follows:

$$\phi_i(N, P(N), f) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{f(S)}{|S|}, \ \forall i \in N. \tag{6}$$

Although (6) is simpler than the original one, in principle, it still has the problem of involving an exponential number in n of addends. However, since in practice the number of observed combinations of channels resulting in conversions is small, the number of non-zero terms in (6) makes its calculus efficient from a computational point of view.

Zhao et al. (2018) also used (6), however they did not rely on the Harsanyi dividends of the game to deduce it.

Next, we recall the formulation of the classic axiomatic characterisation of the Shapley value based on efficiency, null player, additivity and symmetry, which is more appropriate regarding the situation at hand. See Algaba et al. (2019b) for this formulation and other alternative ones. First, we summarise the properties that characterise the Shapley value. Note that, *symmetry property* can be replaced by *equal treatment of equals* to obtain an axiomatic characterisation by a straightforward adaptation of axiomatic characterisation of the Shapley-quota allocation mechanism for Labeled Network Allocation problems given by Algaba et al. (2019c) .

Definition 4 Let γ be a MA rule. It is said to satisfy:

- *Additivity property*, if for all $i \in N$, and for all MA problems $(N, P^1(N), f^1)$ and $(N, P^2(N), f^2)$ with set of channels N, it holds:

  $$\gamma_i(N, P^{agg}(N), f^{agg}) = \gamma_i(N, P^1(N), f^1) + \gamma_i(N, P^2(N), f^2),$$

  where $P^{agg}(N) = P^1(N) \cup P^2(N)$ and $f^{agg}(p) = f^{agg}(p) = f^1(p) + f^2(p)$, where $f^j(p) := 0$ for every path $p \notin P^j(N)$, $j = 1, 2$.
- *Symmetry property*, if for every pair $i, j \in N$ of *indistinguishable* channels in terms of performance (i.e. for every conversion path $p \in P(N)$, it holds: (i) $i, j \in S_p$; (ii) $S_p \subseteq N \setminus \{i, j\}$; or (iii) for each $p = p^i$ with $S_{p^i} = S \cup \{i\}$ for some $S \subseteq N \setminus \{i, j\}$ there exists $p^j \in P(N)$ with $S_{p^j} = S \cup \{j\}$ and $f(p^i) = f(p^j)$), then $\gamma_i(N, P(N), f) = \gamma_j(N, P(N), f)$.
- *Equal treatment of equals property* if for every pair $i, j \in N$ of *equal* channels (i.e. $i \in S_p$ if, and only if, $j \in S_p$, for all $p \in P(N)$), then $\gamma_i(N, P(N), f) = \gamma_j(N, P(N), f)$.
- *Null channel property* if a channel that does not make any contribution in terms of performance (i.e. $f(p) = 0$, for every conversion path $p \in P(N)$ with $i \in S_p$), then $\gamma_i(N, P(N), f) = 0$.

Note that in this case, the additivity axiom, which is usually the most controversial one, turns out to be a key property for the marketing managers, since it allows them to aggregate campaigns and allocate the benefits of a huge campaign sequentially at different stages.

Theorem 1 *The* Shapley-MA rule *is the unique MA rule satisfying additivity, null channel and equal treatment of equals.*

Proof The proof follows the same lines than that of Algaba et al. (2019c), by induction on the number of observed conversion paths. □

Besides the characterising properties, the Shapley-MA rule satisfies other interesting properties. Next, we introduce the concept of *independent subset of channels*.

Definition 5 Let $(N, P(N), f)$ be a MA problem. Then, a subset $S^* \subseteq N$ of channels is said to be an *independent set of channels*, if for all $p \in P(N)$, $S_p \cap S^* \neq \emptyset$ if, and only if, $S_p \subseteq S^*$.

Note that the worth generated by an independent subset S^* of channels equals $\sum_{S \subseteq S^*} f(S)$, which is precisely $v^f_\Sigma(S^*)$, moreover, it can be clearly identifiable and thus should be imputed to S^*. In these situations, in which no conversion has been made by consumers exposed to combinations of channels mixing some channels of S^* with other channels that are not in S^*, channels in S^* should not receive any credit from the conversions of devices exposed to channels in $N \setminus S^*$, and the other way around.

Definition 6 Let γ be a MA rule. It is said to satisfy:

- *Stand-alone property*, if for all MA problem $(N, P(N), f)$, and for all $i \in N$, it holds:

$$\gamma_i(N, P(N), f) \geq f(\{i\}) := \sum_{\substack{p \in P(N) \\ S_p = \{i\}}} f(p)$$

- *Fair attribution property*, if for all MA problem $(N, P(N), f)$, and for all $i, j \in N$ such that:

$$f(S \cup \{i\}) := \sum_{\substack{p \in P(N) \\ S_p = S \cup \{i\}}} f(p) \geq \sum_{\substack{p \in P(N) \\ S_p = S \cup \{j\}}} f(p) =: f(S \cup \{j\}), \;\; \forall\, S \subseteq N \setminus \{i, j\},$$

  then it holds $\gamma_i(N, P(N), f) \geq \gamma_j(N, P(N), f)$.
- *Stability,* if for all MA problem $(N, P(N), f)$, and for all $S \subseteq N$ it is verified:

$$\gamma_S(N, P(N), f) := \sum_{i \in S} \gamma_i(N, P(N), f) \geq v^f_\Sigma(S) = \sum_{T \subseteq S} f(T).$$

- *No subsidizing property*, if for all MA problem $(N, P(N), f)$ with an independent subset of channels $S^* \subseteq N$, it holds:

$$\gamma_{S^*}(N, P(N), f) = \sum_{T \subseteq S^*} f(T).$$

Stand-alone property ensures that the value attributed to a channel is not less than the value it can obtain by itself. Fair attribution property states that when the combination with channel i is more profitable (or at least equally profitable) than the combination with channel j for every combination S, channel i should not be attributed worse than channel j. Stability property, which is closely related to no subsidizing, states that the global value attributed to each combination S of channels should always be greater than or equal to the value generated by all conversions from consumers exposed to any possible combination of channels in S that have not watched any other channel. No subsidizing property assures that any combination of channels should not subsidize other channels with which it does not interact.

Proposition 2 *The* Shapley-MA rule *verifies stand-alone, fair attribution, stability and no subsidizing properties.*

Proof All of them follow trivially from the properties of the Shapley value and the definition of the sum game, taking into account that the sum game is supperadditive and convex. □

3.4 Case when the number of times a channel is visited is relevant. The rShapley-like multi-channel attribution rule

In this section, we try to extend the previous analysis to the case whereby the order in which the channels are on a path to conversion is not considered relevant information, but the number of times the channels appear is. For example, paths (1, 2) and (2, 1) are considered indistinguishable in terms of information, but different from path (2, 1, 2, 2). In the latter case, we consider that the value of channel 2 should be greater than the value of channel 1 on that path to conversion. Therefore, the above approach of aggregating the values of different paths that share the same subset S of channels, must be carefully done because the information about the repetition of channels can be lost. To avoid this possible loss, we introduce the *sum game with repetitions* by repeating each original channel as many times as necessary, by properly taking into account the information about the number of times each consumer has seen the ad in each channel. As before, we rely on the Shapley value of the replicated sum game to derive the proposed MA rule, which we name as *rShapley-like multi-channel attribution rule*, and turns out to be TU-proportional to the *Doubly Proportional quota allocation mechanism* defined in Algaba et al. (2019c). We study its properties and derive an axiomatic characterisation that generalises the Shapley attribution rule axiomatisation given in Theorem 1.

In order to define the *sum game with repetitions*, let r_i the maximum number of times channel i appears in any path of the MA problem $(N, P(N), f)$, i.e.

$$r_i = \max_{p \in P_i(N)} n_i(p), \tag{7}$$

where $n_i(p)$ is the number of times channel i appears in path p and being $P_i(N) := \{p \in P(N) \mid i \in S_p\}$, for all $i = 1, \dots, n$. Then we create fictitious channels $i^1, i^2, \dots, i^{r_i}$ that substitute the original channel i. To be specific, if channel i appears $n_i(p)$ times in path p, then channel i is substituted by fictitious channels $i^1, i^2, \dots, i^{n_i(p)}$ in this path p. Let $N^r = \cup_{i=1}^n S_i^r$, where $S_i^r = \{i^1, i^2, \dots, i^{r_i}\}$, $i \in N$, be the *channel set with repetitions*, and let $P(N^r)$ denote the set of conversion paths obtained after renaming the channels. Then, the KPI function with repetitions $f^r(\cdot)$ is given by the following sum:

$$f^r(S^r) := \sum_{\substack{p \in P(N^r) \\ S^r = S_p^r}} f(p), \ \forall\, S^r \subseteq N^r, \tag{8}$$

where $S_p^r := \{i^k \in N^r \mid i^k \in p\}$, for every $p \in P(N^r)$, and being the sum over the empty set defined as 0. Now, with each MA problem $(N, P(N), f)$, we may associate a **sum game with repetitions** defined as follows.

Definition 7 Given a MA problem $(N, P(N), f)$, its associated **sum game with repetitions** $(N^r, v_\Sigma^{f^r})$ is given by:

$$v_\Sigma^{f^r}(S^r) := \sum_{T^r \subseteq S^r} f^r(T^r), \text{ for all } S^r \subseteq N^r. \tag{9}$$

Table 1 Campaign data

Path p	KPI value $f(p)$
(1)	20
(1, 2)	40
(2, 1)	10
(2, 1, 2)	30

Table 2 KPI and characteristic function with repetitions for observed combinations and coalitions

$S \subseteq N^r$	$f^r(S)$	$v_\Sigma^{f^r}(S)$
$\{1^1\}$	20	20
$\{2^1\}$	0	0
$\{2^2\}$	0	0
$\{1^1, 2^1\}$	50	70
$\{1^1, 2^2\}$	0	20
$\{2^1, 2^2\}$	0	0
$\{1^1, 2^1, 2^2\}$	30	100

Definition 8 The rShapley-like MA rule associates with each MA problem $(N, P(N), f)$, the allocation $\phi^r(N, P(N), f)$ in $\mathbb{R}^N$ given by the following sum:

$$\phi_i^r(N, P(N), f) := \phi_{i^1}(N^r, v_\Sigma^{f^r}) + \cdots + \phi_{i^{r_i}}(N^r, v_\Sigma^{f^r}), \; i \in N. \tag{10}$$

Trivially, the rule defined above is a MA rule:

- $\sum_{i=1}^n \phi_i^r(N, P(N), f) = B$.
- $\phi_i^r(N, P(N), f) \geq 0$ for all $i \in N$.

Example 1 Let us consider the MA problem $(N, P(N), f)$ with $N = \{1, 2\}$, $P(N) = \{(1), (1, 2), (2, 1), (2, 1, 2)\}$, and the following KPI values:

Then we create fictitious players $1^1, 2^1, 2^2$ and consider the sum game with repetitions $(N^r = \{1^1, 2^1, 2^2\}, v_\Sigma^{f^r})$ based on the KPI function with repetitions f^r depicted in the new Table 2.

Thus, the rShapley-like MA rule is given by:

$$\phi_1^r(N, P(N), f) = \phi_{1^1}(N^r, v_\Sigma^{f^r}) = 55,$$
$$\phi_2^r(N, P(N), f) = \phi_{2^1}(N^r, v_\Sigma^{f^r}) + \phi_{2^2}(N^r, v_\Sigma^{f^r}) = 35 + 10 = 45.$$

The Shapley-MA rule is given by $\phi_1(N, v_\Sigma^f) = 60$ and $\phi_2(N, v_\Sigma^f) = 40$. Thus, taking into account repetitions by means of the rShapley-like MA rule, increases the worth of channel 2.

Taking into account the rShapley-like MA rule definition and the expression of the Shapley-MA rule (6), we can straightforwardly deduce the following simple expression for the rShapley-like MA rule based on the KPI of the observed paths, which also shows its relation with the doubly proportional allocation mechanism for labeled network allocation problems (Algaba et al., 2019c):

$$\phi_i^r(N, P(N), f) = \sum_{\substack{S^r \subseteq N^r \\ S^r \cap S_i^r \neq \emptyset}} \frac{|S^r \cap S_i^r|}{|S^r|} f^r(S^r) = \sum_{p \in P_i(N)} \frac{n_i(p)}{\ell_p} f(p), \tag{11}$$

being $\ell_p = \sum_{i \in N} n_i(p)$ the length of path $p \in P(N)$.

Now, we introduce the property of *monotonicity with respect to channel repetition* that captures the following fact observed in Example 1: if a given channel increases its appearances while the remaining characteristics continue, its attribution will improve, or at least will not worsen, and we prove that the rShapley-like MA rule verifies it. We end up with an axiomatic characterisation of the rShapley-like MA rule, which emphasises the difference between both proposed MA rules based on the Shapley value.

Definition 9 Let $(N, P(N), f)$ and $(N, P^{+i}(N), f^{+i})$ be two MA problems with channel set N, such that they are equally apart from the existence of path $p_0 \in P(N)$ in which channel i appears one more time in $(N, P^{+i}(N), f^{+i})$ than it appears in $(N, P(N), f)$. That is:

- $P^{+i}(N) = P(N) \setminus \{p_0\} \cup \{p_0^{+i}\}$, with $n_j(p_0^{+i}) = n_j(p_0)$, for all $j \neq i$, and $n_i(p_0^{+i}) = n_i(p_0) + 1$.
- $f^{+i}(p_0^{+i}) = f(p_0)$ and $f^{+i}(p) = f(p)$, for all $p \in P(N) \setminus \{p_0\}$.

Then, a MA rule γ verifies **monotonicity with respect to channel repetition** whenever $\gamma_i(N, P(N), f) \leq \gamma_i(N, P^{+i}(N), f^{+i})$.

Proposition 3 *The rShapley-like MA rule verifies monotonicity with respect to channel repetition.*

Proof Taking into account (11), the only difference between $\phi_i^r(N, P(N), f)$ and $\phi_i^r(N, P^{+i}(N), f^{+i})$ is given by the weight corresponding to paths $p_0 \in P(N)$ and $p_0^{+i} \in P^{+i}(N)$, which are $\frac{n_i(p_0)}{\ell_{p_0}}$ and $\frac{n_i(p_0)+1}{\ell_{p_0}+1}$, respectively. Since $\ell_{p_0} \geq n_i(p_0)$, the inequality holds. □

Note that the Shapley-MA rule trivially satisfies monotonicity with respect to channel repetition, since it is in fact *indifferent* to repetitions. On the contrary, the rShapley-like MA rule is strictly monotonic whenever the enlarged path p_0 contains another channel in addition to channel i itself.

In order to obtain an axiomatic characterisation for the rShapley-like MA rule, we introduce the notion of *proportional channels* and the property of *proportional treatment of proportional channels*, which generalises the property of equal treatment of equals to properly reflect the effect of repeated exposures to the ad in the same channel over the attribution.

Definition 10 Let $(N, P(N), f)$ be a MA problem. Then, two channels $i, j \in N$ are said to be *proportional* if they are equal channels, i.e., $i \in p$ if and only if $j \in p$ for all $p \in P(N)$, and besides, there exists a positive constant $k > 0$ such that $\frac{n_i(p)}{n_j(p)} = k$, for every $p \in P_i(N) = P_j(N)$.

Definition 11 A MA rule γ verifies **proportional treatment of proportional channels** (PTP) if for all $i, j \in N$ proportional channels with $\frac{n_i(p)}{n_j(p)} = k$, for every $p \in P_i(N) = P_j(N)$, then $\frac{\gamma_i(N,P(N),f))}{\gamma_j(N,P(N),f)} = k$.

Theorem 2 *The rShapley-like MA rule is the only MA rule satisfying additivity, null channel and proportional treatment of proportional channels.*

Proof First, we prove that the properties hold for the rShapley-like MA rule. Null channel and proportional treatment of proportional channels properties follow straightforwardly from (11). In order to prove additivity, let $(N, P^1(N), f^1)$ and $(N, P^2(N), f^2)$ be two MA problems with the same set of channels N, then the channel set with repetitions for the aggregated problem $(N, P^{agg}(N), f^{agg})$ is given by $N^r_{agg} := N^r_1 \cup N^r_2$, where N^r_1, N^r_2 are the channel sets with repetitions of $(N, P^1(N), f^1)$ and $(N, P^2(N), f^2)$, with $r_i^{agg} = \max\{r_i^1, r_i^2\}$, that accounts for all possible repetitions in both campaigns. Now, in order to rely on the additivity property of the Shapley value, we extend the two sum games with repetitions $(N^r_1, v_\Sigma^{f^{1r}})$ and $(N^r_2, v_\Sigma^{f^{2r}})$ to the same player set N^r_{agg} adding the non-existing players in each game as null ones.

Clearly, if $r_i^1 < r_i^2$ ($r_i^2 < r_i^1$) then all paths p in which channel i appears $\ell > r_i^1$ ($\ell > r_i^2$) times shall have a zero KPI in the first (second) problem, and therefore the corresponding fictitious channel i^ℓ with $r_i^1 < \ell \leq r_i^2$ ($r_i^2 < \ell \leq r_i^1$) will be a *null channel* in $(N^r_1, v_\Sigma^{f^{1r}})$ $((N^r_2, v_\Sigma^{f^{2r}}))$. However, since the Shapley value verifies *Null player out* property[5] Derks and Haller (1999), it holds for all $i^\ell \in N^r_{agg}$:

$$\begin{aligned}
\phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{1r}}) &= \phi_{i^\ell}(N^r_1, v_\Sigma^{f^{1r}}), \ \forall \ell \leq r_i^1, \\
\phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{1r}}) &= 0, \ \forall r_i^1 < \ell \leq r_i^{agg}, \\
\phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{2r}}) &= \phi_{i^\ell}(N^r_2, v_\Sigma^{f^{2r}}), \ \forall \ell \leq r_i^2, \\
\phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{2r}}) &= 0, \ \forall r_i^2 < \ell \leq r_i^{agg}.
\end{aligned}$$

Thus, by definition of the rShapley-like MA rule and additivity of the Shapley value, it follows that $\phi_i^r(N, P^1(N), f^1) + \phi_i^r(N, P^2(N), f^2)$ equals:

$$\begin{aligned}
\sum_{\ell=1}^{r_i^1} \phi_{i^\ell}(N^r_1, v_\Sigma^{f^{1r}}) + \sum_{\ell=1}^{r_i^2} \phi_{i^\ell}(N^r_2, v_\Sigma^{f^{2r}}) &= \sum_{\ell=1}^{r_i^{agg}} \phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{1r}}) + \sum_{\ell=1}^{r_i^{agg}} \phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{2r}}) \\
&= \sum_{\ell=1}^{r_i^{agg}} \phi_{i^\ell}(N^r_{agg}, v_\Sigma^{f^{agg,r}}) := \phi_i^r(N, P^{agg}(N), f^{agg}),
\end{aligned}$$

for every channel $i \in N$.

Now, we are left with the question of uniqueness. It follows the same lines than the proof of uniqueness in the axiomatisation of the Shapley quota allocation mechanism given in Algaba et al. (2019c), by induction on the cardinality of $P(N)$.

Let γ be a MA rule satisfying null channel, additivity and PTP. First, let $(N, P(N), f)$ be a problem with a unique conversion path p: $|P(N)| = 1$. Then, since γ satisfies null channel and PTP, and it is efficient by definition, we obtain the following:

$$\gamma_i(N, P(N), f) = \begin{cases} \frac{n_i(p)}{\ell_p} f(p), & \text{if } i \in S_p, \\ 0, & \text{if} i \notin S_p, \end{cases}$$

[5] Removing a null player does not affect the Shapley value of the remaining players: $\phi_i(N, v) = \phi_i(N \setminus \{j\}, v_{-j})$, for all $i, j \in N$, (N, v), such that j is a null player in (N, v) and $i \neq j$.

Table 3 Campaign data

Path p	KPI value $f(p)$
(1)	30
(1, 2)	60
(2, 1)	10

for all $i \in N$, which equals the rShapley-like MA rule.

Let us suppose by induction that $\gamma = \phi^r$ for every $(N, (P(N), f)$ such that $|P(N)| \leq m-1$, $m > 1$, and let us consider $(N, P'(N), f')$ with $|P'(N)| = m$, then the equality $\gamma(N, P'(N), f') = \phi^r(N, P'(N), f')$ holds by induction hypothesis taking into account that $(N, P'(N), f')$ can be obtained as the aggregation of $(N, P'(N) \setminus \{\widetilde{p}\}, f')$ and $(N, \{\widetilde{p}\}, f')$ and both rules are additive. □

3.5 Case when the position of a channel in a path to conversion is relevant

We will now consider the case in which a channel can play distinct roles at different stages of the conversion process, i.e. a channel can have distinct impacts on consumers' decision-making at different stages of the path to conversion. In fact, as we have mentioned in Sect. 2, classical approaches that base the attribution on the order in which the channels are on the path to conversion, are commonly used in practice (first touch, last touch, indirect last touch, time decay, among others). However, in all of these cases, the relative importance that each position has is exogenously given. Our proposal is intended to obtain endogenous information about position importance. An interesting task, that deserves future analysis, is to adequately exploit the information obtained on the positions for endogenously providing an appropriate weight system to capture the relative importance of each position.

We shall first consider the case in which each channel appears once, at the most, in every path to conversion. Then again, we make use of the idea of considering fictitious channels for tracking positions. In this case, every channel i that appears in position j in some path to conversion $p \in P_i(N)$ is substituted by a new fictitious channel i^j that combines the information about the channel and its position. Let $\mathcal{IP}(i) = \{j \in \mathbb{N}_+ \mid \exists\, p \in P_i(N) \text{ s.t. } p_{(j)} = i\}$ be the set of positions channel i occupies and $p_i = |\mathcal{IP}(i)|$. Then, we create p_i fictitious channels i^j, $j \in \mathcal{IP}(i)$ that substitute the original channel i. Let $N^o = \cup_{i=1}^n S_i^o$, where $S_i^o = \{i^j \mid j \in \mathcal{IP}(i)\}$, $i \in N$, be the *set of channels positions*, and let $P(N^o)$ denote the set of conversion paths obtained after renaming the channels. Then, the KPI function $f^o(\cdot)$ and the *MA problem for channels positions* $(N^o, P(N^o), f^o)$ can be defined as in the previous Sect. 3.4.

Now, in order to assess the contribution made by each channel when it occupies different positions, we can rely on the Shapley-MA rule of the MA problem for the position of the channels. To be specific, we define the Shapley-MA of channel i in position $j \in \mathcal{IP}(i)$, as $\phi_i^j(N, P(N), f) := \phi_{ij}(N^o, P(N^o), f^o)$, for very $i \in N$, and for every MA problem $(N, P(N), f)$ with $r_i = 1$, for all $i \in N$.

Example 2 Let us consider the MA problem $(N, P(N), f)$ with $N = \{1, 2\}$, $P(N) = \{(1), (1, 2), (2, 1)\}$, and the following KPI values.

The KPI function for the channels positions $f^o(\cdot)$ and the corresponding characteristic function of the sum game $(N^o, v_\Sigma^{f^o})$, with channels positions set $N^o = \{1^1, 1^2, 2^1, 2^2\}$, are depicted in Table 4.

Table 4 KPI and characteristic functions for the channels positions set N^o

$S \subseteq N^o$	$f^o(S)$	$v_{\Sigma}^{f^o}(S)$
$\{1^1\}$	30	30
$\{2^1\}$	0	0
$\{2^2\}$	0	0
$\{1^1, 2^1\}$	0	30
$\{1^1, 2^2\}$	60	90
$\{1^2, 2^1\}$	10	10
$\{1^1, 2^1, 2^2\}$	0	90
$\{1^1, 1^2, 2^1, 2^2\}$	0	100

The Shapley-MA rule of the MA problem $(N^o, P(N^o), f^o)$ for channels positions is given by

$$\begin{aligned}
\phi_1^1(N, P(N), f) &:= \phi_{1^1}(N^o, P(N^o), f^o) = 60,\\
\phi_1^2(N, P(N), f)) &:= \phi_{1^2}(N^o, P(N^o), f^o) = 5,\\
\phi_2^1(N, P(N), f) &:= \phi_{2^1}(N^o, P(N^o), f^o) = 5,\\
\phi_2^2(N, P(N), f) &:= \phi_{2^2}(N^o, P(N^o), f^o) = 30.
\end{aligned}$$

We can observe the following relation with the Shapley-MA rule when the order is not considered relevant:

$$\phi_1^1(N, P(N), f) + \phi_1^2(N, P(N), f) = 65 = \phi_1(N, P(N), f), \quad (12)$$

$$\phi_2^1(N, P(N), f) + \phi_2^2(N, P(N), f) = 35 = \phi_2(N, P(N), f). \quad (13)$$

Then, we can interpret the attribution to channels 1 and 2 as the sum of the attribution obtained by each channel, when it occupies the first position or the second position in a path. In this particular case, channel 1 contributes much more when it is the first touch-point to conversion, whereas channel 2 contributes much more when it is the last touch-point.

Note that for every MA problem $(N, P(N), f)$ with $r_i = 1$, for all $i \in N$, the Shapley-MA rule of channel i in position $j \in \mathcal{IP}(i)$ can also be obtained by means of the following simplified expression in terms of the original KPI function:

$$\phi_i^j(N, P(N), f) = \sum_{p \in P_i^j(N)} \frac{f(p)}{\ell_p}, \quad (14)$$

where $P_i^j(N) \subseteq P(N)$ is the set of paths in which player i occupies position j.

If no repetition occurs and thus each channel appears only once in every observed path to conversion $p \in P(N)$, the next proposition (which is straightforward to check) shows that relations (12) and (13) generalise.

Definition 12 We say that a MA rule γ verifies **decomposition with respect to positions** if for all MA problem $(N, P(N), f)$ it holds:

$$\gamma_i(N, P(N), f) = \sum_{j \in \mathcal{IP}(i)} \gamma_{i^j}(N^o, P(N^o), f^o), \quad \forall i \in N. \quad (15)$$

Proposition 4 *If no repetition occurs, the Shapley-MA rule verifies decomposition with respect to positions.*

Table 5 KPI function with repetitions for observed combinations

S^{ro}	$f^{ro}(S)$
$\{1^{11}\}$	20
$\{1^{11}, 2^{12}\}$	40
$\{2^{11}, 1^{12}\}$	10
$\{2^{11}, 1^{12}, 2^{23}\}$	30

It is worth highlighting that in Zhao et al. (2018) an intuitive idea regarding the use of a similar approach to deal with the case in which the order is relevant was considered, which they called "ordered Shapley values". However, they did not consider any formalisation of the procedure to obtain these ordered Shapley values, neither did they consider its relation to the case when the order was not relevant.

Following Zhao et al. (2018) we can use the attribution of the Shapley-MA rule of each channel and position to measure the importance of a given fixed position j. Formally:

Definition 13 For any MA problem $(N, P(N), f)$, and any observed position j (i.e., there exists a path $p \in P(N)$ of length $\ell \geq j$), the **pShapley-like** value of position j is defined as:

$$\phi^o_{(j)}(N, P(N), f) := \sum_{\substack{i \in N \\ j \in \mathcal{IP}(i)}} \phi^j_i(N, P(N), f).$$

The pShapley-like value of position j defined above can also be obtained in terms of the KPI function as follows:

$$\phi^o_{(j)}(N, P(N), f) = \sum_{\substack{p \in P(N) \\ \ell_p \geq j}} \frac{f(p)}{\ell_p},$$

for any observed position j in the MA problem $(N, P(N), f)$.

Note that by construction it always holds $\phi^o_{(j)}(N, P(N), f) \leq \phi^o_{(\ell)}(N, P(N), f)$, for every pair of positions $j \leq \ell$.

As for a more general case, in which a channel appears more than once in several paths, we can follow an analogous approach by means of considering as many fictitious channels as necessary, and defining the rShapley-like MA rule of channel i in position $j \in \mathcal{IP}(i)$. In that case, if repetition occurs, the rShapley-like MA rule verifies decomposition with respect to positions, whereas Shapley-MA rule does not.

Example 3 Let us consider the MA problem of Example 1 to illustrate the general case. We should consider fictitious players i^{kj}, where k deals with repetitions and j with positions. Then, the KPI function with repetitions f^{ro} for the positions of the channels defined over combinations in $N^{ro} = \{1^{11}, 1^{12}, 2^{11}, 2^{12}, 2^{23}\}$ is depicted below in Table 5.

Thus, the rShapley-like MA rule of each channel, and the rShapley-like MA rule of each channel and positions are given by:

$$\begin{aligned}
&\phi^r_1(N, P(N), f) = 55 = 40 + 15 = \phi^{r,1}_1(N, P(N), f) + \phi^{r,2}_1(N, P(N), f), \\
&\phi^r_2(N, P(N), f) = 45 = 15 + 20 + 10 = \\
&\qquad = \phi^{r,1}_2(N, P(N), f) + \phi^{r,2}_2(N, P(N), f) + \phi^{r,3}_2(N, P(N), f).
\end{aligned}$$

The pShapley-like rule of positions 1, 2 and 3 is given by $\phi^o(N, P(N), f) = (55, 35, 10)$.

Remark 2 It should be noted that, the purpose of taking into account the number of times a channel appears on each path to conversion, differs from that of taking into account the position each channel occupies on those paths. Taking into consideration repetitions is intended to modify the attribution to each channel, on the basis that the part of the benefit produced by each path attributed to each channel should increase if the number of times the ad is viewed on that channel increases. However, considering positions is intended to endogenously assess the importance of positions in each path for conversion, which is a major issue in marketing attribution. The information on the value obtained from each channel in each of the positions it occupies is not used to alter the attributions it obtained regardless of its positions.

4 The second model: a bankruptcy approach

In this section, we propose a different approach to the multi-channel attribution problem, by considering it a bankruptcy problem. First, we recall some basic preliminaries on bankruptcy. Then, we introduce the basic multi-channel bankruptcy problem on which our proposal is based, and characterise the class of bankruptcy problems that defines it. In particular, we explore the *proportional* (PROP) rule and the *constrained equal-losses* (CEL) rule (Aumann & Maschler, 1985) as multi-channel attribution rules. Proportional rules are always appealing since they can be easily computed and understood by the user (the marketing campaign manager in our context). However, in this framework, additivity is an interesting property for the manager, and moreover, there are other important aspects that are attractive for the manager who would be interested in discarding channels with low contributions. Taking into account that there is no additive bankruptcy rule (see (Bergantiños & Vidal-Puga, 2004)), we propose to rely on the CEL rule, which verifies a weaker version of additivity -which we refer to as *quasi additivity*- and also excludes weaker claimants. We also analyse PROP rule as a benchmark to compare it with. Next, following Sects. 3.4 and 3.5, we rely on the multi-channel attribution problem with repetitions and the multi-channel attribution problem for the positions of the channels and consider, from a bankruptcy perspective more general attribution situations whereby the order in which channels appear in the conversion path, as well as the number of times each channel appears, could play a relevant role.

4.1 Preliminaries. Basic concepts on bankruptcy problems

O'Neill (1982) and Aumann and Maschler (1985) introduce the *bankruptcy problem* as a game theoretic problem for solving the question of how to allocate a given estate among the different agents that have rights on a part of it, in the event that the estate is not sufficient to meet all their claims. The reader is referred to Thomson (2003, 2015, 2019a) for a survey on bankruptcy problems, where the most important allocation rules for bankruptcy are described and characterised.

Formally, a bankruptcy problem is given by (N, E, c), where $N = \{1, \ldots, n\}$ is the set of claimants, E is the *estate* and $c = (c_1, \ldots, c_n)$ is the vector of *claims*, being c_i the claim of agent i, such that $0 < E \leq \sum_{i=1}^{n} c_i$. We shall denote by $C = \sum_{i=1}^{n} c_i$ the total quantity that is claimed and by $D = C - E \geq 0$ the deficit. Let $\mathcal{U} = \{1, 2, \ldots\}$ be the universe of all potential claimants, and let $\mathcal{N}$ be the class of all non-empty finite subsets of $\mathcal{U}$. For any element $N \in \mathcal{N}$, let $\mathcal{B}^N$ denote the family of all bankruptcy problems with set of claimants N.

Definition 14 For all $N \in \mathcal{N}$, a **bankruptcy rule** for $\mathcal{B}^N$ is a mapping R that associates with every problem $(N, E, c) \in \mathcal{B}^N$ a unique vector $R(N, E, c) \in \mathbb{R}^N$ such that $0 \leq R_i(N, E, c) \leq c_i$, for all $i \in N$ (*Boundedness*) and $\sum_i R_i(N, E, c) = E$ (*Efficiency*).

The bankruptcy problem can be modelled as a game (O'Neill, 1982; Aumann & Maschler, 1985) and specific game theoretic solutions as the nucleolus (Schmeidler, 1969) or the Shapley value can be derived as bankruptcy rules. However, a direct treatment of the bankruptcy problem could be more intuitive and acceptable from the point of view of the user. To be specific, we focus on the proportional rule and the constrained equal-losses (Aumann & Maschler, 1985) rule.

Definition 15 For all $N \in \mathcal{N}$, and for every bankruptcy problem $(N, E, c) \in \mathcal{B}^N$, the **proportional rule** assigns to each claimant:

$$PROP_i(N, E, c) = \frac{c_i}{C} E, \quad i = 1, \ldots, n.$$

The **CEL (constrained equal-losses rule)** assigns to each agent in a bankruptcy problem $(N, E, c) \in \mathcal{B}^N$ the amount:

$$CEL_i(N, E, c) = \max\{0, c_i - \lambda\},$$

where $\lambda > 0$ verifies $\displaystyle\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

The **proportional rule** assigns to each claimant a part of the estate which is proportional to its claim, whereas The **CEL rule** allocates the deficit D to the claimants as equally as possible.

4.2 A bankruptcy approach. The constrained equal-losses multi-channel attribution rule

For each MA problem $(N, P(N), f)$, we consider an associated bankruptcy problem in which the estate represents the total revenues obtained from the campaign, whereas we consider that each channel claims all profits generated by the conversions of the viewers who have seen the ad on that channel.

We first consider the case when only the presence of the channels is relevant. In this case, given a MA problem, $(N, P(N), f)$, we denote by $(N, B^f, c^f) \in \mathcal{B}^N$ the associated **bankruptcy multi-channel attribution (BMA) problem**, where the estate is given by:

$$B^f = \sum_{S \subseteq N} f(S),$$

and claims are given by

$$c_i^f = \sum_{\substack{S \subseteq N \\ i \in S}} f(S), \quad i \in N.$$

Obviously, the sum of the claims $C^f = \sum_{i \in N} c_i^f$ exceeds the global benefit B^f (the estate), which is precisely the global amount that should be attributed to the channels. Let $D^f \geq 0$ denote the deficit $D^f = C^f - B^f$.

Let us denote by $\mathcal{BA}^N$ the class of all those BMA problems with channel set N, for all $N \in \mathcal{N}$. The next theorem proves that the class of bankruptcy multi-channel attribution problems coincides with the class of *Simple claims* bankruptcy problems introduced in O'Neill (1982)[6], which is a proper subset of $\mathcal{B}^N$.

For convenience, hereafter we will omit the superscript f when, within the context, there is no doubt about which KPI function f is being used.

Theorem 3 *For every finite set $N \in \mathcal{N}$, and any given bankruptcy problem $(N, E, c) \in \mathcal{B}^N$,*

$$\sum_{i \in N} c_i \geq E \geq \max_i c_i \tag{16}$$

is a necessary and sufficient condition for $(N, E, c) \in \mathcal{BA}^N$, i.e., for the existence of a non-negative map $f : 2^N \rightarrow \mathbb{R}_+$ with $f(\emptyset) := 0$, such that

$$E = \sum_{S \subseteq N} f(S) \text{ and } c_i = \sum_{\substack{S \subseteq N \\ i \in S}} f(S), \; \forall i \in N.$$

Proof By definition, if $(N, E, c) \in \mathcal{BA}^N$, there exists a MA problem $(N, P(N), f)$ such that $(N, E, c) = (N, B^f, c^f)$. Thus, clearly condition (16) is satisfied. We will prove that it is also a sufficient condition by induction on the number of claimants.

Let us first prove that, under condition (16), the next set of linear constraints corresponding to a bankruptcy problem with two claimants, has a non-negative feasible solution with $f(\emptyset) := 0$:

$$\begin{aligned} E &= f(\{1\}) + f(\{2\}) + f(\{1, 2\}), \\ c_1 &= f(\{1\}) + f(\{1, 2\}), \\ c_2 &= f(\{2\}) + f(\{1, 2\}). \end{aligned}$$

Trivially, $f(\{1\}) = E - c_2$, $f(\{2\}) = E - c_1$ and $f(\{1, 2\}) = c_1 + c_2 - E$ solves the system and condition (16) assures that all of them are non-negative.

Now, let us suppose by induction hypothesis that given a bankruptcy problem with $n = |N|$ claimants satisfying condition (16) there exists a non-negative map $f(\cdot)$ with $f(\emptyset) = 0$ such that $E = \sum_{S \subseteq N} f(S)$ and $c_i = \sum_{\substack{S \subseteq N \\ i \in S}} f(S)$, for all $i \in N$, and we will prove the existence of a similar function for any bankruptcy problem with $n + 1$ claimants verifying (16).

Let (N, E, c) be such a bankruptcy problem, and let us assume without loss of generality $c_1 \leq c_2 \leq \cdots \leq c_n \leq c_{n+1}$. Then, let us define

$$f(\{n + 1\}) := \max\{c_{n+1} - c_n, E - \sum_{i=1}^{n} c_i\} \tag{17}$$

Now, let us consider two cases:

1. If $f(\{n + 1\}) = c_{n+1} - c_n$, then the reduced problem $(N, E', (c_1, \ldots, c_n))$ with $E' := E - c_{n+1} + c_n$ is a bankruptcy problem satisfying condition (16). Note that $c_n = \max_{i=1,\ldots,n} c_i \leq E - c_{n+1} + c_n = E'$ since $E \geq c_{n+1}$, and $\sum_{i=1}^{n} c_i \geq E - c_{n+1} + c_n = E'$,

[6] O'Neill (1982) defines the class of *simple claims problems* as those ones".. *defined by an estate of given size, n heirs and n corresponding wills each specifying a bequest for that heir, which total at least as much as the total estate, with each bequest non-negative and less than or equal to the total estate, will be called a simple claims problem.*". Since the restriction of being $c_i \leq E$, for all claimant $i \in N$, does not appear in other general definition of bankruptcy problem, we have adopted the terminology *simple claims* as an adjective to refer to those bankruptcy problems that verify this condition.

since $c_{n+1} - c_n \geq E - \sum_{i=1}^{n} c_i$. Thus, there exists a non-negative function $f'(\cdot)$ such that:

$$E - c_{n+1} + c_n = \sum_{S \subseteq N} f'(S), \tag{18}$$

$$c_i = \sum_{\substack{S \subseteq N \\ i \in S}} f'(S),\ i = 1, \ldots, n. \tag{19}$$

Now, let us define function $f : 2^{N \cup \{n+1\}} \to \mathbb{R}_+$ as follows: $f(\{n+1\}) := c_{n+1} - c_n$; $f(S) := f'(S)$ and $f(S \cup \{n, n+1\}) := f'(S \cup \{n\})$, for every $S \subseteq \{1, \ldots, n-1\}$; and being $f(S) := 0$ for the remaining S containing only one of the agents n or $n+1$.
Trivially, f verifies

$$E = \sum_{S \subseteq N \cup \{n+1\}} f(S)$$

and

$$c_i = \sum_{\substack{S \subseteq N \cup \{n+1\} \\ i \in S}} f(S),\ \text{for all } i \in N \cup \{n+1\}.$$

2. If $f(\{n+1\}) = E - \sum_{i=1}^{n} c_i$, then $0 \leq c_{n+1} - c_n \leq E - \sum_{i=1}^{n} c_i$, and therefore the reduced problem $(N, E', (c_1, \ldots, c_n))$ with $E' := E - f(\{n+1\}) = \sum_{i \in N} c_i$ is a trivial bankruptcy problem which also satisfies condition (16).
The function $f'(\{i\}) := c_i$ for all $i = 1, \ldots, n$, and $f'(S) = 0$ otherwise, is a non-negative function that trivially verifies

$$E' := \sum_{i=1}^{n} c_i = \sum_{S \subseteq N} f'(S) \text{ and } c_i = \sum_{\substack{S \subseteq N \\ i \in S}} f'(S),\ \forall\, i \in N = \{1, \ldots, n\}.$$

Now, let us define the function f as follows:

$$f(\{n+1\}) := E - \sum_{i=1}^{n} c_i,$$

$$f(\{i\}) := f'(\{i\}) = c_i,\ \ i = 1, \ldots, n-1,$$

$$f(\{n\}) := f'(\{n\}) - D = c_n - \left(\sum_{i=1}^{n+1} c_i - E\right) \geq 0,$$

$$f(\{n, n+1\}) := D = \sum_{i=1}^{n+1} c_i - E \geq 0,$$

$$f(S) := 0,\ \text{otherwise}.$$

Note that $f(\{n\}) \geq 0$ since $c_{n+1} - c_n \leq E - \sum_{i=1}^{n} c_i$. Trivially, f verifies

$$E = \sum_{S \subseteq N \cup \{n+1\}} f(S) \text{ and } c_i = \sum_{\substack{S \subseteq N \cup \{n+1\} \\ i \in S}} f(S), \ \forall i \in N \cup \{n+1\}.$$

□

As we have mentioned before, we adopt the CEL bankruptcy rule as the proposed attribution mechanism based on this second approach. We base our selection on the grounds of its properties. Excluding weaker claimants helps the manager to concentrate on the most powerful channels, whereas quasi-additivity (which we introduce in this paper) can help the manager to aggregate campaigns. Here, we also consider the PROP rule as a matter of comparison.

Definition 16 A Bankruptcy multi-channel attribution (BMA) rule is a MA rule that associates with each MA problem $(N, P(N), f)$ a vector given by $\gamma_i(N, P(N), f) = R_i(N, B^f, c^f)$, for all $i \in N$, being R a given bankruptcy rule.

Definition 17 The PROP-MA rule is the BMA rule that assigns to each MA problem $(N, P(N), f)$ the vector $PROP_i(N, B^f, c^f), \forall i \in N$, for all $N \in \mathcal{N}$.
The **CEL-MA rule** is the BMA rule that assigns to each MA problem $(N, P(N), f)$ the vector $CEL_i(N, B^f, c^f), \forall i \in N$, for all $N \in \mathcal{N}$.

Remark 3 It is worthy to relate our proposal with the different allocation rules for the museum pass problem based on its analysis as a bankruptcy problem. The first paper which proposes to model the museum pass problem using a bankruptcy approach is (Estévez-Fernández et al., 2010, 2012). Since then, other authors have considered this approach, Casas-Méndez et al. (2011) and Bergantiños and Moreno-Ternero (2015). However, the bankruptcy approach followed by these authors is not valid for analysing the multi-channel allocation problem, since they rely on the *admission fees* of each museum, as well as the number of visitors that have visited each museum without a pass, that have no counterpart in the problem under consideration. To be specific, in Estévez-Fernández et al. (2010, 2012), the claims equal the global amount that each museum would receive if they had charged regular fees for the services provided to pass-holders. Casas-Méndez et al. (2011) considered a weighted bankruptcy model to allocate the pass price, the estate, in which the claims are the admission fees and the weights are given by the number of pass-holders that have visited each museum. Bergantiños and Moreno-Ternero (2015) consider three different kinds of proportional rules according to the informational bases they assume. The simplest case, in which the authors assume that they only know the total number of pass holders that visited each museum, coincides with our approach. Nevertheless, the axiomatisation of the proportional rule they derive is based on a *compatibility* property, which involves admission fees and the profile of visits without a pass, and thus is not valid in this context.

Next, we show that for any $N \in \mathcal{N}$, the subclass $\mathcal{BA}^N$ is closed under all the operations involved in the properties of equal treatment of equals, path independence, and composition from minimal rights, which characterise the CEL rule on $\mathcal{B}^N$ (Herrero & Villar, 2001; Herrero, 2003). Therefore, this CEL axiomatisation is still a valid one for the CEL-MA rule. With respect to the PROP-MA rule, the characterisation of the proportional rule on the subclass of simple claims bankruptcy problems given by O'Neill (1982) is meaningful and valid in this context.

Definition 18 Let R be a bankruptcy rule for $\mathcal{BA}^N$, for all $N \in \mathcal{N}$. It is said to satisfy:

- *ETE equal treatment of equals*, if for all $(N, B, c) \in \mathcal{BA}^N$ and for all $i, j \in N$, $c_i = c_j$ implies $R_i(N, B, c) = R_j(N, B, c)$.
- *PIN path independence*, if for all $(N, B, c) \in \mathcal{BA}^N$ and for all $B' > B$, $R(N, B, c) = R(N, B, R(N, B', c))$.
- *CMR Composition from minimal rights*, if for all N, all $(N, B, c) \in \mathcal{BA}^N$, $R(N, B, c) = m(N, B, c) + R(N, B - \sum_{i \in N} m_i(N, B, c), c - m(N, B, c))$, where $m_i(N, B, c) = \max\{0, B - \sum_{j \neq i} c_j\}$ is the minimal right of claimant i.

Note that the ETE property for BMA problems is stronger than the property of Equal Treatment of Equals, considered in Sect. 3.3, in terms of the MA problem. Two channels can have the same claims but they can differ in the KPI values of the combinations to which they belong. With respect to the PIN property, it is clearly a valid property on $\mathcal{BA}^N$: for every BMA problem $(N, B, c) \in \mathcal{BA}^N$, then $(N, B, R(N, B', c)) \in \mathcal{BA}^N$. On the contrary, in the case of the CMR property, it is not obvious that the derived bankruptcy problem $(N, B - \sum_{i \in N} m_i(N, B, c), c - m(N, B, c)) \in \mathcal{BA}^N$. Let us prove it.

Proposition 5 *If* $(N, B, c) \in \mathcal{BA}^N$, *then* $(N, B - \sum_{i \in N} m_i(N, B, c), c - m(N, B, c)) \in \mathcal{BA}^N$.

Proof First, it is obvious that $\sum_{i \in N}(c_i - m_i(N, B, c)) \geq B - \sum_{i \in N} m_i(N, B, c)$.

Second, we must prove that $\max_{i \in N} c_i - m_i(N, B, c) \leq B - \sum_{i \in N} m_i(N, B, c)$. Assume, without loss of generality that claims are ordered in decreasing order, $c_1 \geq c_2 \geq \cdots \geq c_n$, therefore $c_1 = \max_i c_i$. As

$$\sum_{i \neq 1} c_i \leq \sum_{i \neq j} c_i, \quad \forall j \in N,$$

this implies that m_i is a non-increasing function in i. Then, if there exists $m_i \neq 0$, $m_1 \neq 0$.

If $m_j > 0$, $c_j - m_j = c_j - B + \sum_{i \neq j} c_i = D - B$. If for a given j, $m_j = 0$, we must check that $c_1 - m_1 \geq c_j$. Assume $m_1 > 0$ (otherwise is trivial), $c_1 - m_1 = D - B = \sum_{i \in N} c_i - B \geq c_j$ because $m_j = 0$ implies that $B - \sum_{i \neq j} c_i \leq 0$.

Therefore, we must only check that $B - \sum_{i \in N} m_i \geq c_1 - m_1$. Let k be such that $m_i > 0$ for all $i = 1, .., k$, and $m_i = 0$ for $i > k$. The cases $k = 0, 1$ are trivial. For $k \geq 2$ some simple calculations allows us to obtain that:

$$B - \sum_{i \in N} m_i = (k-1)D + \sum_{j=k+1}^{n} c_j$$

Then, $B - \sum_{i \in N} m_i - (c_1 - m_1) = (k-1)D + \sum_{j=k+1}^{n} c_j - D = (k-2)D + \sum_{j=k+1}^{n} \geq 0$. □

Herrero (2003) characterizes the CEL bankruptcy rule by means of ETE, PIN and CMR. All these properties are still valid on $\mathcal{BA}^N$, for all $N \in \mathcal{N}$, and as we will show next, they also characterise CEL-MA rule.

Theorem 4 *CEL-MA is the only BMA rule verifying ETE, PIN and CMR.*

Proof Clearly, since ETE, PIN and CMR are still valid on $\mathcal{BA}^N$, for all $N \in \mathcal{N}$, CEL-MA verifies all of them. Now, in order to prove the uniqueness of the CEL-MA rule, we will follow the lines of Dagan (1996) proof of the CEA rule characterisation.

Let R be a BMA rule verifying ETE, PIN and CMR, and let N and $c \in \mathbb{R}_+^N$ be two arbitrary finite set of agents and vector of claims, respectively. We will prove that $R_i(N, B, c) = CEL_i(N, B, c)$, for every estate $c_n \leq B \leq \sum_{i \in N} c_i$, where we assume without loss of generality that $c_1 \leq c_2 \leq \cdots \leq c_n$.

Let us start with the case in which $B = \alpha_1 + c_2 + \cdots + c_n$, $0 \leq \alpha_1 \leq c_1$. Then, the minimal right of claimant $i \in N$ is given by:

$$m_i = \alpha_1 + c_2 + \cdots + c_n - \sum_{\substack{j=1 \\ j \neq i}}^{n} c_i = \begin{cases} \alpha_1, & i = 1, \\ (c_i - c_1) + \alpha_1, & 2 \leq i \leq n. \end{cases} \tag{20}$$

Since R verifies CMR, then $R_i(N, B, c) = m_i + R(N, B - \sum_{i=1}^{n} m_i, c - m)$. Note that $(N, B - \sum_{i=1}^{n} m_i, c - m) \in \mathcal{BA}^N$ (Proposition 5). Now, taking into account that $c_i - m_i = c_1 - \alpha_1 \geq 0$, for all $i \in N$, by ETE, it holds:

$$R_i(N, B - \sum_{i=1}^{n} m_i, c - m) = \frac{B - \sum_{i=1}^{n} m_i}{n} = \frac{(n-1)(c_1 - \alpha_1)}{n}.$$

Thus,

$$R_i(N, B, c) = c_i - \tfrac{c_1 - \alpha_1}{n}, \ \forall i \in N, \tag{21}$$

which equals $CEL(N, B, c)$.

Now, we will prove the same coincidence for every estate B between $B_2 := (c_2 + \cdots + c_n) - c_1(1 - \frac{1}{n})$ and $B_1 := c_2 + \cdots + c_n$. If $n = 2$, then we have already proved that $R_i(N, B, c) = CEL_i(N, B, c)$ for all estate B such that $(N, B, c) \in \mathcal{BA}^N$, since B must satisfy $c_2 \leq B \leq c_1 + c_2$. Otherwise ($n \geq 3$), $B_2 \geq c_n$, and thus $(N, B, c) \in \mathcal{BA}^N$, for all estate B such that $B_2 \leq B \leq B_1$.

Let $B = B_2 + \alpha_2$, $0 \leq \alpha_2 \leq c_1(1 - \frac{1}{n})$. Since R verifies PIN, then $R(N, B, c) = R(N, B, R(N, B_1, c))$. Now, taking into account expression (21) above for $\alpha_1 = 0$, after some calculations, we obtain the following vector of minimal rights in the new bankruptcy problem $(N, B, R(N, B_1, c)) \in \mathcal{BA}^N$:

$$m_i = \begin{cases} \alpha_2, & i = 1, \\ (c_i - c_1) + \alpha_2, & 2 \leq i \leq n. \end{cases} \tag{22}$$

Again, as R verifies CMR, then

$$R_i(N, B, R(N, B_1, c)) = m_i + R_i(N, B - \sum_{i=1}^{n} m_i, R(N, B_1, c) - m), \ \forall i \in N,$$

where $R_i(N, B_1, c) - m_i = \frac{n-1}{n} c_1 - \alpha_2 \geq 0$, for all $i \in N$. Thus, by ETE, it holds:

$$R_i(N, B - \sum_{i=1}^{n} m_i, R(N, B_1, c) - m) = \frac{B - \sum_{i=1}^{n} m_i}{n} = \frac{(n-1)(c_1 - \alpha_2) - \frac{n-1}{n} c_1}{n},$$

for all $i \in N$. Therefore,

$$R_i(N, B, c) = c_i - \frac{1}{n}\big(c_1 + c_1(1 - \frac{1}{n}) - \alpha_2\big), \ \forall i \in N,$$

which equals $CEL(N, B, c)$.

Now, we can repeat the same argument finitely many times considering $B_t = C - c_1 \sum_{j=0}^{t-1}(1-\frac{1}{n})^j$, to show that $R(N,B,c) = CEL(N,B,c)$ for every $B_t \leq B \leq B_{t-1}$. Since R and CEL-MA satisfy PIN, R_i and CEL_i are continuous in B for all $i \in N$ (see Herrero & Villar, 2001), then it follows that $R(N,B,c) = CEL(N,B,c)$, for every $B \geq C - nc_1 = \lim_{t\to\infty} B_t$. Note that in case $C - nc_1 < c_n$, we can always stop the argument in a previous step $B = c_n$.

The whole argument may be applied to show that $R(N,B,c) = CEL(N,B,c)$ for every estate B such that $C - nc_1 - (n-1)(c_2 - c_1) \leq B \leq C - nc_1$. This may be repeated considering the following consecutive intervals:

$$C - \sum_{j=0}^{k}(n-j+1)(c_j - c_{j-1}) \leq B \leq C - \sum_{j=0}^{k-1}(n-j+1)(c_j - c_{j-1}),$$

where $c_0 = c_{-1} = 0$, until all possible estates $B \in [c_n, C]$ are covered. □

PROP-MA also verifies ETE and PIN properties, however it fails to verify CMR.

Next, we introduce other interesting properties. Exclusion (EXC) and additivity (ADD) are classic properties of bankruptcy rules, whereas, to our knowledge, Quasi Additivity (QAD) and Irrelevant claimants property (ICL) have not been considered before.

We say that $i \in N$ is an **irrelevant claimant** for the rule R in (N,B,c) if $R_i(N,B,c) = 0$.

Definition 19 Let R be a bankruptcy rule for $\mathcal{BA}^N$, for all $N \in \mathcal{N}$. It is said to satisfy:

- *EXC Exclusion*, if for all $(N,B,c) \in \mathcal{BA}^N$, if $c_i \leq D/n$ then $R_i(N,B,c) = 0$.
- *ICL irrelevant claimants property*, if for all $(N,B,c) \in \mathcal{BA}^N$, and any **irrelevant** claimant $k \in N$ for the rule R, the attribution $R(N \setminus \{k\}, B, c_{-k})$ of the reduced problem without claimant k, verifies $R_i(N \setminus \{k\}, B, c_{-k}) = R_i(N,B,c)$, for all $i \in N \setminus \{k\}$.
- *ADD additivity*, if for all $(N,B,c), (N,B',c') \in \mathcal{BA}^N$. $R(N, B+B', c+c') = R(N,B,c) + R(N,B',c')$.
- *QAD quasi additivity*, if for all $(N,B,c), (N,B',c') \in \mathcal{BA}^N$ with no irrelevant claimants for R, $R(N, B+B', c+c') = R(N,B,c) + R(N,B',c')$.

EXC and QAD properties are appealing to the user: EXC assures that the attribution concentrates on the channels with the highest values, and QAD guaranties a restrictive additivity of the campaigns, since unfortunately there is no bankruptcy rule verifying ADD (Bergantiños & Vidal-Puga, 2004).

It is worth pointing out that all the properties considered above are well defined for the subclass $\mathcal{BA}^N$, as well as for the general class of bankruptcy problems $\mathcal{B}^N$, for all $N \in \mathcal{N}$. Moreover, the next results regarding PROP-MA and CEL-MA as BMA rules generalises for PROP and CEL as general bankruptcy rules.

Remark 4 If a claimant $i \in N$ is *excludable* (i.e., $c_i \leq D/n$), then it is irrelevant for the CEL-MA rule, but the converse is not true in general (apart from the case in which there is a unique irrelevant claimant).

Proposition 6 *CEL-MA rule verifies EXC, ICL and QAD properties. PROP-MA rule verifies ICL.*

Proof EXC is a typical property of the CEL rule.

With respect to ICL, we first prove that the property is correctly defined for CEL-MA, i.e. the reduced problem $(N \setminus \{k\}, B, c_{-k}) \in \mathcal{BA}^{N\setminus\{k\}}$. Obviously, $B \geq \max_{i\in N} c_i \geq$

$\max_{i \neq k \in N} c_i$. We must check that $C_{-k} = C - c_k \geq B$: If $CEL_k(N, B, c) = 0$, then $c_k \leq (C - B)/n = (C_{-k} + c_k - B)/n$ holds. Thus $C_{-k} - B \geq (n-1)c_k \geq 0$, for all $n \geq 1$.

Now, we will show that the remaining non-irrelevant claimants receive the same in both problems. Let $\lambda > 0$ be the solution of the original problem defining $CEL(N, B, c)$, i.e. $CEL_i(N, B, c) = \max\{0, c_i - \lambda\}$, for all $i \in N$ and

$$B = \sum_{i \in N} \max\{0, c_i - \lambda\} = \sum_{\substack{i \in N \\ CEL_i(N,B,c)>0}} \max\{0, c_i - \lambda\}.$$

Thus, λ is also the solution for the reduced problem defining $CEL(N \setminus \{k\}, B, c_{-k})$ and therefore, the whole set of irrelevant claimants can be removed without changing the proposed attribution.

In order to prove quasi-additivity of CEL-MA, let $(N, B, c), (N, B', c') \in \mathcal{BA}^N$ be two bankruptcy problems with no irrelevant claimants for CEL-MA, then $c_i > D/n$, $\forall i \in N$ and $CEL_i(N, B, c) = c_i - D/n$, $\forall i \in N$. Analogously, $CEL_i(N, B', c') = c'_i - D'/n$, $\forall i \in N$. Thus, since $D + D'$ is the deficit of the sum problem $(N, B + B', c + c')$, it has no irrelevant claimants for CEL-MA, which implies that $CEL_i(N, B + B', c + c') = c_i + c'_i - (D + D')/n = CEL_i(N, E, c) + CEL_i(N, E', c')$, $\forall i \in N$.

Trivially, PROP-MA verifies ICL, considering that $PROP_i(N, B, c) = 0$ if, and only if, $c_i = 0$. □

On the contrary, it is clear that the PROP-MA rule does not verify neither EXC nor QAD. Following O'Neill (1982), the PROP-MA rule can be characterised by *symmetry* (which is ETE), *continuity on* c_i, *independence of the inclusion of claimants with no claim* and *strategy-proof* (if some claimants merge by adding their claims together, they should receive the same global amount).

4.3 Case when the number of times a channel is visited is relevant

Now we consider the case in which the number of times each channel appears in a path is also regarded as relevant information. We follow an approach similar to that in Sect. 3.4, by means of defining the associated multi-channel attribution problem with repetitions. We will show that both BMA rules, rCEL-like MA and rPROP-like MA, verifies monotonicity with respect to channel repetition.

Definition 20 The *rCEL-like MA* rule, associates to each $(N, P(N), f)$ MA problem, the allocation $CEL^r(N, P(N), f)$ in $\mathbb{R}^N$ given by the following sum:

$$CEL^r_i(N, P(N), f) := CEL_{i^1}(N^r, B, c^r) + \cdots + CEL_{i^{r_i}}(N^r, B, c^r), \; i \in N, \quad (23)$$

where $(N^r, B, c^r) \in \mathcal{BA}^{N^r}$ is the BMA problem associated to the MA problem with repetitions $(N^r, P(N^r), f^r)$.

Analogously, the *rPROP-like MA* rule is defined. Trivially, both rules are MA rules.

Proposition 7 *The rCEL-like MA and rPROP-like MA rules verify monotonicity with respect to channel repetition.*

Proof Let $(N, P(N), f)$ be a MA problem. Without loss of generality, we can prove monotonicity by only considering the case in which there exists a unique channel $i \in N$ that appears twice in $(N, P^{+i}(N), f^{+i})$. That is, there is a new path $p^{+i} \in P^{+i}(N)$ that substitutes path

$p \in P(N)$ by repeating once player $i \in p$ without changing its value $f^{+i}(p^{+i}) = f(p)$, and being $r_j = 1$, for all $j \in N$ in the original problem $(N, P(N), f)$. Let $(N^r, P^{+i}(N^r), f^{+i,r})$ be the MA problem with repetitions for $(N, P^{+i}(N), f^{+i})$.

Let $(N, B, c) \in \mathcal{BA}^N$ be the BMA problem associated with $(N, P(N), f)$, then the bankruptcy problem associated to $(N^r, P^{+i}(N^r), f^{+i,r})$ is (N^r, B, c^r), where $N^r = N \setminus \{i\} \cup \{i_1, i_2\}$, $c^r_j = c_j$, for all $j \in N^r$, $j \neq i_2$ and $c^r_{i_2} = f^{+i}(p^{+i}) = f(p)$.

We first prove the monotonicity of the CEL-MA rule. That is, we must prove that

$$CEL_i(A) =: CEL^r_i(N, P(N), f) \leq CEL^r_i(N, P^{+i}(N), f^{+i}) := CEL_{i_1}(A^r) + CEL_{i_2}(A^r), \tag{24}$$

where, for the sake of brevity throughout the proof A and A^r will denote the BMA problems (N, B, c) and (N^r, B, c^r), respectively. We will distinguish two cases:

- Case 1: If $CEL_{i_2}(A^r) = 0$ then, by ICL property, $CEL_{i_1}(A^r) = CEL_i(A)$ holds.
- Case 2: Otherwise, let us first prove that $IC(A^r) \subseteq IC(A)$, where $IC(\cdot)$ stands for the set of irrelevant claimants for CEL-MA of the given BMA problem. We will show that every irrelevant claimant in A is also irrelevant in A^r and it is excluded at the same stage of the distribution process. Since $CEL_{i_2}(A^r) > 0$, then $c^r_{i_2} = f(p) > D^r/(n+1)$, where the deficit D^r for the new bankruptcy problem is $D^r = D + f(p)$. Thus, $f(p) > D/n$ and $\frac{D+f(p)}{n+1} \geq \frac{D}{n}$. Therefore, $c_j \geq \frac{D+f(p)}{n+1} \geq \frac{D}{n}$, and this implies that every excluded claimant in A (at the first stage) is also an excluded claimant (at the first stage) in A^r. Now, let $E(A) \subseteq N$ be the set of excluded claimants (at the first stage) in A. Then, $c^r_{i_2} = f(p) > (D^r - \sum_{j \in E(A)} c_j)/(|N \setminus E(A)| + 1)$, because $CEL_{i_2}(A^r) > 0$. Then,

$$c_k \geq \frac{D^r - \sum_{j \in E(A)} c_j}{(|N \setminus E(A)| + 1)} > \frac{D - \sum_{j \in E(A)} c_j}{|N \setminus E(A)|}.$$

  Thus, every excluded claimant at the second stage in A is also an excluded claimant at the second stage in A^r. Repeated application of the above reasoning yields $IC(A) \subseteq IC(A^r)$. We are now in a position to prove that (24) holds. Let λ and λ^r be the solutions to the problems defining $CEL(A)$ and $CEL(A^r)$, and let $RC(\cdot)$ stands for the set of relevant claimants for CEL-MA and the given BMA problem. We consider, yet again, two different cases:

  (i) If there exists a claimant $k \in RC(A)$ which turns out to be irrelevant in A^r, then $c_k - \lambda > 0 \geq c_k - \lambda^r$. Thus, $\lambda^r > \lambda$ and therefore $CEL_j(A) = c_j - \lambda \geq \max\{0, c_j - \lambda^r\} = CEL_j(A^r)$, for all $j \in RC(A)$ and condition (24) holds taking into account that

$$\sum_{\substack{j \in RC(A) \\ j \neq i}} CEL_j(A) + CEL_i(A) = B =$$

$$= \sum_{\substack{j \in RC(A) \\ j \neq i}} CEL_j(A^r) + CE_{i_1}(A^r) + CEL_{i_2}(A^r).$$

  (ii) Otherwise, the set $RC(A^r) = RC(A) \setminus \{i\} \cup \{i_1, i_2\}$. We will prove that $\lambda^r \geq \lambda$. Let us suppose that $\lambda^r < \lambda$, then it holds:

$$B = \sum_{\substack{j \in RC(A) \\ j \neq i}} CEL_j(A) + CEL_i(A) < \sum_{\substack{j \in RC(A) \\ j \neq i}} CEL_j(A^r) + CE_{i_1}(A^r) \leq A,$$

Table 6 Campaign data

Path p	KPI value $f(p)$
(1)	20
(1, 3)	40
(3, 1, 2)	30
(2, 3)	10

which is a contradiction. Therefore, $\lambda^r \geq \lambda$ and the reasoning of the previous case applies.

In order to prove $PROP_i(A) \leq PROP_{i_1}(A^r) + PROP_{i_2}(A^r)$, it is only necessary to check that:

$$\frac{c_i}{C}B \leq \frac{c_i}{C + f(p)}B + \frac{f(p)}{C + f(p)}B,$$

which clearly holds since $c_i \leq C$ □

4.4 Case when the position of a channel in a path to conversion is relevant

Now, we consider the case in which a channel has a different impact on the final conversion, depending on its position in the path to that conversion. We follow an approach similar to that in Sect. 3.5, by means of defining the associated multi-channel attribution problem for the positions of the channels assuming there are no repetitions. In this case, the decomposition result obtained for the Shapley-MA rule is still valid for the PROP-MA rule, but CEL-MA does not verify decomposition with respect to positions in general.

Formally, let $(N, P(N), f)$ be a MA problem with $r_i = 1$, for all $i \in N$. Then the **CEL-MA attribution of channel** $i \in N$ **in position** $j \in \mathcal{IP}(i)$ is given by $CEL_i^j(N, P(N), f) := CEL_{i^j}(N^o, B, c^o)$, for all $i \in N$, where (N^o, B, c^o) is the BMA problem associated with $(N^o, P(N^o), f^o)$. Analogously, it is defined the **PROP-MA attribution of channel** $i \in N$ **in position** $j \in \mathcal{IP}(i)$.

Example 4 Given the campaign data in Table 6, CEL-MA and PROP-MA attributions for each channel appear in Table 7. Tables 8 and 9 show the corresponding data for the MA problem for channels positions, and the CEL-MA and PROP-MA attributions of each channel $i \in N = \{1, 2, 3\}$ in its observed positions $\mathcal{IP}(i)$, respectively. Thus, we can check that PROP-MA verifies decomposition with respect positions whereas CEL-MA does not:

$$CEL_1(N, B, c) = 160/3 > CEL_{1^1}(N^o, B, c^o) + CEL_{1^2}(N^o, B, c^o) = 50$$
$$CEL_2(N, B, c) = 10/3 < CEL_{2^1}(N^o, B, c^o) + CEL_{2^3}(N^o, B, c^o) = 10$$
$$CEL_3(N, B, c) = 130/3 > CEL_{3^1}(N^o, B, c^o) + CEL_{3^2}(N^o, B, c^o) = 40$$

In this example, the main reason for this result is that channel 2 is the one that least divides its forces among its respective position channels to the extent that one of them becomes irrelevant, as we will see later.

It should be noticed that, when introducing position channels (without repetitions) in bankruptcy problems, two properties are preserved:

- The claims of the position channels i^j, $j \in \mathcal{IP}(i)$, corresponding to channel $i \in N$ verify: $\sum_{j \in \mathcal{IP}(i)} c_{i^j} = c_i$. Therefore, the deficit D^o does not change, i.e. $D^o = D$. This is the main difference with respect to the case with repetitions.

Table 7 Bankruptcy solutions without order

Channels	1	2	3
CEL-MA	160/3	10/3	130/3
PROP-MA	900/21	400/21	800/21

Table 8 Campaign data with channels positions

Path p	KPI value $f(p)$
(1^1)	20
$(1^1, 3^2)$	40
$(3^1, 1^2, 2^3)$	30
$(2^1, 3^2)$	10

Table 9 Bankruptcy solutions for channels and positions

Channels and positions	1^1	1^2	2^1	2^3	3^1	3^2
CEL-MA	40	10	0	10	10	30
PROP-MA	600/21	300/21	100/21	300/21	300/21	500/21

– The extended BMA problem with channels positions is also a BMA problem: $(N^o, B, c^o) \in \mathcal{BA}^{N^o}$

Then, analysing the effects when considering the different positions a channel has occupied in the observed conversion paths, is an analysis about the splitting effects for a general bankruptcy problem. First, note that previous analysis on manipulation by splitting (see, for instance, Ju, 2003; Ju et al., 2007; Moreno-Ternero, 2007) only deals with the case in which there is only one claimant that splits her claims. Here, we recall the common definition of a *non-manipulable by splitting* rule (Moreno-Ternero, 2007).

Definition 21 A rule R is non-manipulable by splitting (NMS) if for all $(N, E, c), (N', E, c')$, with $N \subsetneq N'$, and such that there is some $i \in N$ such that $c_i = c'_i + \sum_{j \in N' \setminus N} c'_j$ and for each $j \in N \setminus \{i\}$, $c'_j = c_j$ then $R_i(N, E, c) \geq R_i(N', E, c') + \sum_{j \in N' \setminus N} R_j(N', E, c')$.

PROP and CEL are NMS rules (Moreno-Ternero, 2007). However, in the case we are interested in, more than one channel (claimant) may simultaneously split, and therefore, questions regarding how simultaneous splits (or, equivalently, decomposition with respect positions) affect the global attribution of each channel arise naturally.

For the PROP-MA rule is straightforward to prove the next result.

Proposition 8 *If no repetition occurs, the PROP-MA rule verifies decomposition with respect to positions.*

CEL-MA rule does not verify decomposition with respect positions in general. However, some results about splitting effects when claimants split simultaneously and there are not repetitions, can be given:

Proposition 9 *Let $(N, B, c) \in \mathcal{BA}^N$ be a BMA problem such that all claimants in N, as well as all their corresponding position claimants in the BMA problem (N^o, B, c^o) for channels* *positions, are relevant. Then*

$$CEL_i(N,B,c) \leq \sum_{j\in\mathcal{IP}(i)} CEL_{ij}(N^o,B,c^o) \iff p_i \leq \frac{\sum_{\ell\in N} p_\ell}{n}.$$

Proof Since all claimants in N, as well as all their corresponding position claimants in the BMA problem for channels positions (N^o, B, c^o), are relevant, what we have is that $CEL_i(N,B,c) = c_i - D/n$ and $CEL_{ij}(N^o,B,c^o) = c_{ij} - D/\sum_{\ell\in N} p_\ell$, for all $j \in \mathcal{IP}(i)$. Thus, taking into account that $c_i = \sum_{j\in\mathcal{IP}(i)} c_{ij}$, it follows that $CEL_i(N,B,c) \leq \sum_{j\in\mathcal{IP}(i)} CEL_{ij}(N^o,B,c^o)$ if, and only if, $p_i \leq \sum_{\ell\in N} p_\ell/n$. □

That is, channel i benefits from position decomposition if its number of corresponding positions is below the average.

As we have seen in Example 4, CEL-MA does not verify decomposition with respect to positions property in general. However, in some particular cases, it can hold as an immediate corollary of the above proposition:

Corollary 1 *Under the same conditions of Proposition 9, if $p_i = p_\ell$ for all $i, \ell \in N$, CEL-MA verifies decomposition with respect to positions.*

The next proposition gives a result on how the splitting affects a channel with an irrelevant position.

Proposition 10 *Let $(N,B,c) \in \mathcal{BA}^N$ be a BMA problem such that all claimants in N, as well as all their corresponding position claimants in the BMA problem (N^o,B,c^o) for channels positions, are relevant except a unique position claimant ℓ^k corresponding to claimant $\ell \in N$ with $p_\ell \leq \frac{\sum_{i\in N} p_i}{n}$, then it holds:*

$$CEL_\ell(N,B,c) \leq \sum_{j\in\mathcal{IP}(i)} CEL_{\ell j}(N^o,B,c^o).$$

Proof We shall denote by $n' = \sum_{i\in N} p_i$. Since ℓ^k is the only irrelevant claimant in (N^o,B,c^o), then $c_{\ell k} \leq D/n'$, $CEL_{\ell k}(N^o,B,c^o) = 0$, and $CEL_{\ell j}(N^o,B,c^o) = c_{\ell j} - (D - c_{\ell k})/(n'-1)$, for all $j \in \mathcal{IP}(i)$, $j \neq k$. Analogously, since all the original claimants are relevant in (N,B,c), then we have $CEL_\ell(N,B,c) = c_\ell - D/n$. Then we shall prove that under the proposition conditions:

$$c_\ell - \frac{D}{n} \leq \sum_{j\neq k}\left(c_{\ell j} - \frac{D - c_{\ell k}}{n'-1}\right),$$

that is equivalent to prove

$$c_{\ell k} - \frac{D}{n} \leq -(p_\ell - 1)\frac{D - c_{\ell k}}{n'-1}$$

since $c_\ell = \sum_j c_{\ell j}$.

After some algebra we obtain that the above condition is equivalent to

$$c_{\ell k}(1-q) \leq D\left(\frac{1}{n} - q\right)$$

where $q = (p_\ell - 1)/(n'-1)$.

But $c_{\ell^k}(1-q) \leq D(1-q)/n'$ since ℓ^k is the only irrelevant player. Then we shall prove that

$$\frac{D(1-q)}{n'} \leq D\Big(\frac{1}{n} - q\Big).$$

if and only if

$$q \leq \frac{n'-n}{n(n'-1)}$$

After substituting q by its expression, we obtain that the above condition holds if and only if

$$p_\ell \leq \frac{n'}{n}.$$

□

To sum up, what Propositions 9 and 10 say is that a channel can benefit from the attribution to its position channels the less it splits its claim. Finally, it should be noted that the above propositions are valid for a general bankruptcy problem.

Remark 5 Despite the fact that in some cases CEL-MA does not verify decomposition with respect to positions, we recall (Remark 2) that the main objective of the valuation of the positions is not to change the attribution to the different channels, but to get some endogenously obtained insight about position importance in the paths to conversion which is an important issue in practical marketing.

5 Conclusions

In this paper, we have addressed the attribution problem that arises when the total benefits obtained by a marketing campaign must be attributed to the different advertising channels involved in the campaign, which is nowadays a cornerstone of any multi-channel marketing strategy. Essentially, We have analysed two kinds of attribution mechanisms, one of them based on the Shapley value of an appropriate game, while the other one is based on bankruptcy problems.

Morales (2016) y Zhao et al. (2018) suggested the idea of using the sum game as the model and the Shapley value as a possible attribution mechanism without developing this approach thoroughly. We have formalised the model, studied its properties and analysed its relation with museum pass games and labeled network games. We also extend this model to the case in which the position or the number of times a channel appears in each path to conversion is relevant. The proposed attribution mechanisms are based on the Shapley value of these games. We have developed a thorough analysis of their properties as multi-channel attribution rules, providing axiomatic characterisations, focusing on monotonicity and decomposition properties when repetition and positions, respectively, are taken into account. From a practical point of view, this approach has many advantages: (1) The characteristic function of the games are conceptually well defined and have good properties. Moreover, their definitions are simple and easy to understand and explain; (2) The proposed basic attribution is very simple and also helps to understand the method: the value attributed to a channel is the sum of the aliquot part of the value of each combination to which it belongs; (3) Its additivity allows to jointly manage a batch of related campaigns; (4) Monotonicity property states that if a channel is more frequently seen its attribution cannot decrease; and (5) Decomposition with respect to positions allows us to endogenously obtain information about the relative importance of the positions.

The approach based on bankruptcy problems is, as far as we know, new. We have proposed an appropriate bankruptcy problem to deal with attribution problems, which has an intuitive and easy to explain definition. We have formalised the model, studied its properties and analysed its relation with museum pass problems studied as bankruptcy problems (Estévez-Fernández et al., 2010, 2012; Casas-Méndez et al., 2011; Bergantiños & Moreno-Ternero, 2015). It is remarkable that the class of bankruptcy problems that arises is the proper subclass of simple claims bankruptcy problems (O'Neill, 1982). We also extend this model to the case in which the position or the number of times a channel appears in each path to conversion is relevant. Among the existing bankruptcy rules, it is the CEL rule that establishes an alternative view which is different to the Shapley value of the sum game. In this case, the exclusion property is fundamental: it gives zero value to very weak channels, if any, and tends to concentrate the attribution in the channels that belong to the highest valued combinations. As it happens with the Shapley-MA rule, the CEL rule is also simple to calculate. Although additivity is lost, the weaker property of quasi additivity holds for CEL. We have provided an axiomatic characterisation of the CEL-MA rule, focusing on monotonicity and decomposition properties when repetition and positions, are respectively taken into account. In particular, we have obtained several results for general bankruptcy problems about the behaviour of the CEL rule when more than one claimant simultaneously split their claims.

Other important issues that deserve further analysis are to study other rules for the proposed models and to tackle certain problems that can appear in practical situations. For instance, taking into account that value $f(\{i\})$ is only obtained by channel i, it would be interesting to consider extended bankruptcy problems in which each of the agents have an *objective entitlement* besides their claims. These models are analysed in Pulido et al. (2002, 2008). Among the practical situations to be addressed, we can point out the existence of spontaneous conversions, the incomplete information about the KPI function and how to use the positional decomposed values of each channel value in order to obtain a general valuation of the positions in any path to conversion.

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