

Entropic measures of joint uncertainty: effects of lack of majorization

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We compute Rényi entropies for the statistics of a noisy simultaneous observation of two complementary observables in two-dimensional quantum systems. We find paradoxical results such as that the relative amount of uncertainty between two states depends on the uncertainty measure used, as well as unexpected relations between the entropy of the joint distribution and the product of its marginals for the same state. These results are not reproduced by a more standard duality relation. We show that the striking behaviors coincide with a lack of majorization relation between the corresponding statistics.

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I. INTRODUCTION

Historically, the joint uncertainty of pairs of observables has been mostly addressed in terms of the product of their variances. Nevertheless, there are situations where such formulation is not satisfactory enough [1], thus alternative approaches have been proposed, mainly in terms of diverse entropic measures [2]. In this work we consider in particular Rényi entropies [3] and the corresponding uncertainty relations, for two complementary observables.

Previous works [4, 5] have shown that entropic uncertainty relations may lead to surprising results. For example, the relative amount of uncertainty between two observables depends on the uncertainty measure used. This is an intriguing result that might jeopardize the usefulness of entropy as an uncertainty measure. In this regard, the aim of this work is twofold. On the one hand, we show that these unexpected behaviors can be fully and satisfactorily explained in terms of the lack of majorization relation between the corresponding statistics. This connection holds because entropic measures are monotone with respect to majorization [6–9]. Thus, such surprising entropic results are not tricky features of entropic measures, but may have a deeper meaning that is actually overlooked by more popular measures of uncertainty or complementarity. On the other hand, we extend the application of entropic measures to the statistics of a simultaneous joint observation of two complementary observables in the same system realization [10–12]. This setting of complementarity in practice provides a rich arena to examine the interplay between entropic measures and majorization. The simultaneous measurement provides a true joint classical-like probability distribution that enables alternative assessments of joint uncertainty, different from the ones given by the product of individual statistics, either intrinsic or of operational origin.

For simplicity we address these issues in the sim-

plest quantum system described by a state in a two-dimensional Hilbert space. This comprises very relevant practical situations such as the path–interference complementarity in two-beam interference experiments. This allows us to contrast the performance of entropic measures with more standard assessments of complementarity [13–15].

The paper is organized as follows: in Sec. II we introduce the discussion on statistics of simultaneous measurements for spin observables. Sec. III exhibits results for entropic quantities that appear to be paradoxical, and an explanation for that behavior is given in Sec. IV. In Sec. V, a duality relation is analyzed. Finally, some concluding remarks are outlined in Sec. VI.

II. STATISTICS AND SIMULTANEOUS MEASUREMENTS

We consider two complementary observables represented by the Pauli spin matrices σ_x and σ_z . In practical terms they may represent phase and path, respectively, in two-beam interference experiments. The system state is described by a density matrix operator acting on the Hilbert space \mathcal{H}_S that in Bloch representation acquires the form $\rho = \frac{1}{2}(I + \mathbf{s} \cdot \boldsymbol{\sigma})$, where I is the identity matrix, $\boldsymbol{\sigma}$ represents the three Pauli matrices, and $\mathbf{s} = \text{Tr}(\rho \boldsymbol{\sigma})$ is a three-dimensional Bloch vector with $|\mathbf{s}| \leq 1$. When $|\mathbf{s}| = 1$ the state is pure, $\rho = |\psi\rangle\langle\psi|$, with

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle, \quad (1)$$

where $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z with corresponding eigenvalues $+1$ and -1 , and $s_x = |\mathbf{s}| \sin \theta \cos \varphi$, $s_y = |\mathbf{s}| \sin \theta \sin \varphi$, and $s_z = |\mathbf{s}| \cos \theta$. Since we focus on the observables σ_x and σ_z , we consider for simplicity a one-parameter family S of pure states with \mathbf{s} laying in the xz plane, this is $s_y = 0$ and $\varphi = 0$. In such a case,

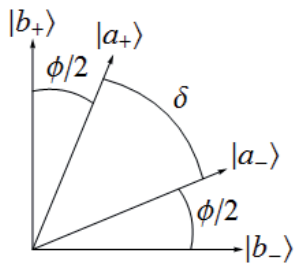


FIG. 1. Schematic representation of the states $|a_{\pm}\rangle$ and $|b_{\pm}\rangle$.

the *intrinsic statistics* for the observables σ_x and σ_z are

$$p_j^X = \frac{1}{2}(1 + j s_x) \quad \text{and} \quad p_k^Z = \frac{1}{2}(1 + k s_z), \quad (2)$$

with $j = \pm 1$ and $k = \pm 1$.

The *simultaneous measurement* of noncommuting observables requires involving auxiliary degrees of freedom, usually referred to as apparatus. In our case we consider an apparatus described by a two-dimensional Hilbert space \mathcal{H}_A . The measurement performed in \mathcal{H}_A addresses that of σ_z , while σ_x is measured directly on the system space \mathcal{H}_S . The system–apparatus coupling transferring information about σ_z from the system to the apparatus is arranged via the following unitary transformation acting on $\mathcal{H}_S \otimes \mathcal{H}_A$,

$$U = |0\rangle\langle 0| \otimes U_+ + |1\rangle\langle 1| \otimes U_-, \quad (3)$$

where U_{\pm} are unitary operators acting solely on \mathcal{H}_A . For simplicity the initial state of the apparatus, $|a\rangle \in \mathcal{H}_A$, is assumed to be pure so that the system–apparatus coupling leads to

$$U|\psi\rangle|a\rangle = \cos \frac{\theta}{2} |0\rangle |a_+\rangle + \sin \frac{\theta}{2} |1\rangle |a_-\rangle, \quad (4)$$

where the states $|a_{\pm}\rangle = U_{\pm}|a\rangle \in \mathcal{H}_A$ are not orthogonal in general, with $\cos \delta = \langle a_+ | a_- \rangle$ assumed to be a positive real number without loss of generality. The measurement in \mathcal{H}_A introducing minimum uncertainty is given by projection on the orthogonal vectors $|b_{\pm}\rangle$ (see Fig. 1, and Ref. [16]):

$$\begin{aligned} |b_+\rangle &= \frac{1}{\cos \phi} \left(\cos \frac{\phi}{2} |a_+\rangle - \sin \frac{\phi}{2} |a_-\rangle \right), \\ |b_-\rangle &= \frac{1}{\cos \phi} \left(-\sin \frac{\phi}{2} |a_+\rangle + \cos \frac{\phi}{2} |a_-\rangle \right), \end{aligned} \quad (5)$$

where $\phi = \frac{\pi}{2} - \delta$.

The *joint statistics* for the simultaneous measurement of σ_x acting on \mathcal{H}_S and of σ_z addressed by the orthogonal vectors $|b_{\pm}\rangle$ in \mathcal{H}_A is

$$\tilde{p}_{j,k}^{X,Z} = \frac{1}{4}(1 + j s_x \cos \delta + k s_z \sin \delta), \quad (6)$$

where $j = \pm 1$ represents the outcomes of the σ_x measurement, and $k = \pm 1$ those of the σ_z measurement. The *marginal statistics* for both observables are

$$\tilde{p}_j^X = \frac{1}{2}(1 + j s_x \cos \delta) \quad \text{and} \quad \tilde{p}_k^Z = \frac{1}{2}(1 + k s_z \sin \delta). \quad (7)$$

When contrasted with the intrinsic statistics in (2) we get that the observation of σ_x is exact for $\delta = 0$, while the observation of σ_z is exact for $\delta = \frac{\pi}{2}$. For $\delta = \frac{\pi}{4}$, the extra uncertainty introduced by the unsharp character of the simultaneous observation is balanced between observables.

III. ENTROPIC UNCERTAINTY ASSESSMENTS

We focus here on Rényi entropies to quantify the uncertainty related to a probability distribution. Let $p = \{p_i\}_{i=1}^N$ be the statistics of some observable with N outcomes, then

$$R_{\alpha}(p) = \frac{1}{1 - \alpha} \ln \left(\sum_{i=1}^N p_i^{\alpha} \right), \quad (8)$$

where $\alpha \geq 0$ is the so-called entropic index. The limiting case $\alpha \rightarrow 1$ is well defined and gives the Shannon entropy $R_1 \equiv -\sum_i p_i \ln p_i$. The minimum entropy $R_{\alpha} = 0$ is achieved when all the probability is concentrated in a single outcome: $p_i = \delta_{i,i'}$ for some i' , whereas maximum entropy $R_{\alpha} = \ln N$ occurs when all the outcomes are equally probable: $p_i = \frac{1}{N}$ for all i .

We notice that the operational entropies of the marginal statistics (7) are always larger than for the intrinsic ones (2): $R_{\alpha}(\tilde{p}^X) \geq R_{\alpha}(p^X)$, and similarly for the probability distributions related to σ_z . Comparing Eqs. (2) and (7), we can appreciate that the observation amounts to a reduction of the absolute values of the x - and z -components of the Bloch vector, and Rényi entropies are decreasing functions of $|s_i|$. Their only extreme holds at the uniform distribution, which is an absolute maximum.

Other relevant property of the Rényi entropies (8) is additivity, that is, for the product of two statistics p and q :

$$R_{\alpha}(pq) = R_{\alpha}(p) + R_{\alpha}(q). \quad (9)$$

Following Refs. [5], it has been seen that the corresponding entropic uncertainty relations for the intrinsic statistics (2) are:

$$R_{\alpha}(p^X p^Z) \geq \begin{cases} \ln 2 & \text{if } 0 \leq \alpha \leq \alpha_I \\ \frac{2 \ln \left[\left(\frac{1 + \frac{1}{\sqrt{2}}}{2} \right)^{\alpha} + \left(\frac{1 - \frac{1}{\sqrt{2}}}{2} \right)^{\alpha} \right]}{1 - \alpha} & \text{if } \alpha > \alpha_I, \end{cases} \quad (10)$$

where $\alpha_I \approx 1.43$. There are two sets of states that compete to be the *minimum uncertainty states* (as well as

those of maximum uncertainty within the set S) depending on the value of entropic index used. We refer to them as *extreme* and *intermediate* states:

- Extreme states are eigenstates of σ_x or σ_z . These are pure states with $\theta = m\frac{\pi}{2}$ for integer m in Eq. (1), then $s_x = \pm 1, s_z = 0$ or $s_x = 0, s_z = \pm 1$. They present full certainty for one of the observables, and complete uncertainty for the other one.
- Intermediate states are eigenstates of $\sigma_x \pm \sigma_z$. These are the pure states with $\theta = (2m + 1)\frac{\pi}{4}$ for integer m in Eq. (1), then $s_x = \pm s_z$ and $s_z = \pm \frac{1}{\sqrt{2}}$. They have essentially the same statistics for both complementary observables so they might be considered as a finite-dimensional counterpart of the Glauber coherent states.

A. Joint uncertainty of σ_x and σ_z

In order to assess the joint uncertainty of σ_x and σ_z , we compute the Rényi entropies of the joint statistics $\tilde{p}^{X,Z}$ (6) and of the product of marginal statistics $\tilde{p}^X \tilde{p}^Z$ (7), for any given value of the entropic index α , as functions of θ within the set S of states (1). These quantities are calculated for balanced measurement, $\delta = \frac{\pi}{4}$. We also compute the Rényi entropies of the product of intrinsic statistics $p^X p^Z$ (2). For the sake of clarity and to simplify comparisons, we mostly focus on normalized quantities of the form

$$R_\alpha^{\text{norm}} = \frac{R_\alpha - R_{\alpha,\text{min}}}{R_{\alpha,\text{max}} - R_{\alpha,\text{min}}}, \quad (11)$$

where $R_{\alpha,\text{max}}$ and $R_{\alpha,\text{min}}$ are the maximum and minimum values of R_α , respectively, within the set S .

Figure 2(a) shows $R_\alpha^{\text{norm}}(\tilde{p}^{X,Z})$ for $\alpha = 1$ and 2.5 as functions of θ . We observe that for $\alpha = 1$ the minimum uncertainty states are the intermediate states $\theta = \frac{\pi}{4}$, whereas for $\alpha = 2.5$ the minimum uncertainty states are the extreme states $\theta = 0, \frac{\pi}{2}$. The opposite happens for the product of marginal statistics $R_\alpha(\tilde{p}^X \tilde{p}^Z)$, as illustrated in Fig. 2(b), that is, for $\alpha = 1$ the minimum uncertainty states are the extreme states $\theta = 0, \frac{\pi}{2}$, whereas for $\alpha = 2.5$ the minimum uncertainty states are the intermediate states $\theta = \frac{\pi}{4}$. This result coincides with the conclusions derived from the intrinsic entropies $R_\alpha(p^X p^Z)$ as shown in Fig. 2(c) (see also Refs. [5]).

B. Intermediate versus extreme states

Now, we analyze the intermediate–extreme competition for minimum uncertainty. In Fig. 3, we plot the difference of Rényi entropies between intermediate and extreme states of the joint, product of marginals and

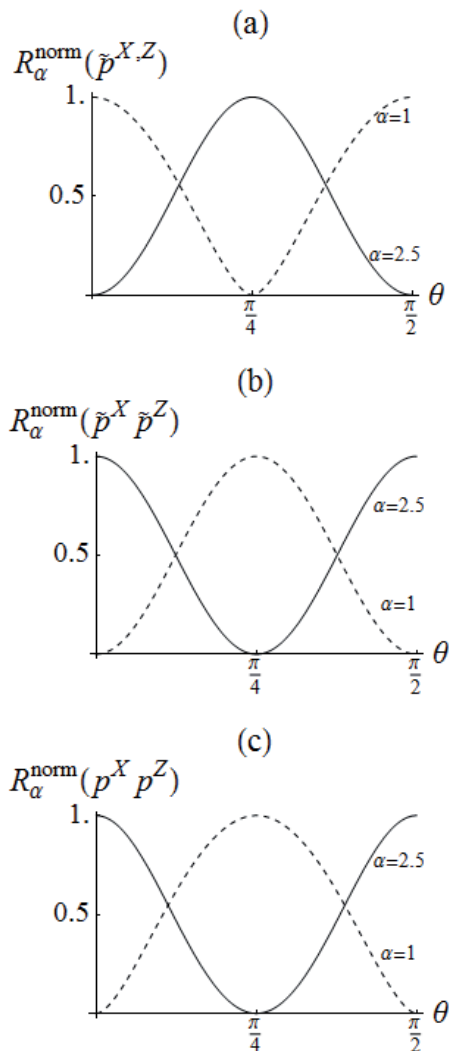


FIG. 2. Normalized Rényi entropies of: (a) the joint statistics $R_\alpha^{\text{norm}}(\tilde{p}^{X,Z})$, (b) the product of marginal statistics $R_\alpha^{\text{norm}}(\tilde{p}^X \tilde{p}^Z)$, and (c) the product of intrinsic statistics $R_\alpha^{\text{norm}}(p^X p^Z)$, for $\alpha = 1$ (dashed line) and $\alpha = 2.5$ (solid line), as functions of θ . In all cases we consider balanced measurement $\delta = \frac{\pi}{4}$.

product of intrinsic statistics, always for balanced measurement $\delta = \frac{\pi}{4}$, namely,

$$\begin{aligned} \Delta R_\alpha(\tilde{p}^{X,Z}) &= R_\alpha(\tilde{p}^{X,Z})|_{\theta=\frac{\pi}{4}} - R_\alpha(\tilde{p}^{X,Z})|_{\theta=0} \\ \Delta R_\alpha(\tilde{p}^X \tilde{p}^Z) &= R_\alpha(\tilde{p}^X \tilde{p}^Z)|_{\theta=\frac{\pi}{4}} - R_\alpha(\tilde{p}^X \tilde{p}^Z)|_{\theta=0} \\ \Delta R_\alpha(p^X p^Z) &= R_\alpha(p^X p^Z)|_{\theta=\frac{\pi}{4}} - R_\alpha(p^X p^Z)|_{\theta=0} \end{aligned} \quad (12)$$

as functions of the entropic index α . $\Delta R_\alpha > 0$ implies that extreme states are of minimum uncertainty while, on the contrary, $\Delta R_\alpha < 0$ implies that intermediate states are the minimum uncertainty ones.

We observe that, for the joint statistics, $\Delta R_\alpha(\tilde{p}^{X,Z})$ is positive for $\alpha \in (2, 3)$, thus there are two critical values of the entropic index at which the minimizer changes. On

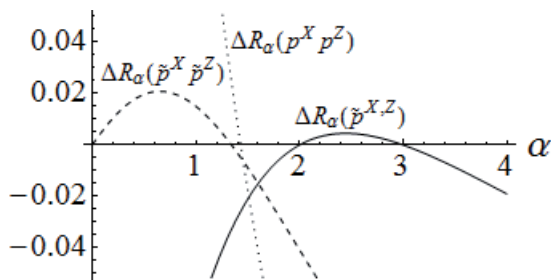


FIG. 3. Differences between the entropies for intermediate and extreme states, Eqs. (12), for the joint statistics $\Delta R_\alpha(\tilde{p}^{X,Z})$ (solid line), the product of marginals $\Delta R_\alpha(\tilde{p}^X \tilde{p}^Z)$ (dashed line), and the product of intrinsic statistics $\Delta R_\alpha(p^X p^Z)$ (dotted line), as functions of α for balanced joint measurement $\delta = \frac{\pi}{4}$. A positive value of ΔR_α means that extreme states give the minimum uncertainty, while a negative value corresponds to minimizing intermediate states.

the other hand, for the product of marginal and intrinsic statistics, $\Delta R_\alpha(\tilde{p}^X \tilde{p}^Z)$ and $\Delta R_\alpha(p^X p^Z)$ change their sign at one critical value: $\alpha_M \approx 1.34$ in the former case and $\alpha_I \approx 1.43$ in the latter; in both situations, the relative difference changes from positive to negative with increasing entropy index.

C. Entropy of the joint statistics versus entropy of the product of marginal and of intrinsic statistics

We report other behaviors regarding entropy-related quantities. In particular, we compute the Rényi entropy of the joint statistics $\tilde{p}^{X,Z}$, of the product of its marginals $\tilde{p}^X \tilde{p}^Z$ and of the product of intrinsic statistics $p^X p^Z$. The differences

$$R_\alpha(\tilde{p}^X \tilde{p}^Z) - R_\alpha(\tilde{p}^{X,Z}) \quad (13)$$

$$R_\alpha(p^X p^Z) - R_\alpha(\tilde{p}^{X,Z}) \quad (14)$$

are plotted in Fig. 4 as functions of α for $\theta = \frac{\pi}{4}$ (intermediate states). The fact that the difference of entropies (13) is negative for some entropic indices as can be seen in the solid curve plotted in Fig. 4 is not surprising, due to it is known that the Rényi entropies do not satisfy the subadditivity property [17]. On the other hand, the negativity of the difference (14) as showing in the dotted curve plotted in Fig. 4 reveals the paradoxical result that the entropy of the joint distribution $\tilde{p}^{X,Z}$ can be larger than the entropy of the product of the intrinsic distributions $p^X p^Z$. Differences between the entropy of joint statistics and of the product of marginals have also been found in Ref. [12] for Shannon entropy. We will see in Sec. IV that these behaviors can be interpreted in a more general framework given by the majorization theory.

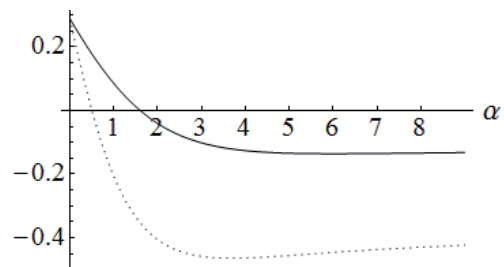


FIG. 4. Difference of Rényi entropies (13) (solid line) and (14) (dotted line) as function of α for $\theta = \frac{\pi}{4}$ (intermediate states).

IV. MAJORIZATION

Entropic uncertainty relations have a deep connection with the majorization of statistical distributions [7, 8], which has been already applied to examine uncertainty of thermal states [6]. (This is closely related to the idea of mixing character in Ref. [9].) In this section we show that majorization is compatible with the surprising results found above.

The statistics p majorizes the statistics p' , denoted as $p' \prec p$, if after forming with p an N -dimensional vector with components in decreasing order $p_1 \geq p_2 \geq \dots \geq p_N$ and similarly with p' , we get $\sum_{j=1}^k p'_j \leq \sum_{j=1}^k p_j$ for all $k = 1, 2, \dots, N-1$ and $\sum_{j=1}^N p'_j = \sum_{j=1}^N p_j = 1$.

If $p' \prec p$, then $R_\alpha(p') \geq R_\alpha(p)$ for any $\alpha \geq 0$, that is Rényi entropies are order-preserving (or Schur-concave) functions. However, majorization is a relation of partial order, so that there are distributions that cannot be compared. Next we see that this is the case for the extreme and intermediate states, and that the contradictions reported above hold when the two statistics cannot be compared by majorization.

A. Lack of majorization relation between the statistics

Let us call $\lambda = \tilde{p}_{\text{ex}}^{X,Z}$ and $\mu = \tilde{p}_{\text{in}}^{X,Z}$ the four-dimensional vectors obtained by arranging the values of $\tilde{p}^{X,Z}$ in decreasing order, for extreme and intermediate states, respectively:

$$\lambda = \frac{1}{4\sqrt{2}} \left(\sqrt{2} + 1, \sqrt{2} + 1, \sqrt{2} - 1, \sqrt{2} - 1 \right)$$

and

$$\mu = \frac{1}{4} (2, 1, 1, 0).$$

Thus while $\mu_1 > \lambda_1$, we have $\mu_1 + \mu_2 < \lambda_1 + \lambda_2$, so that neither $\lambda \prec \mu$ nor $\mu \prec \lambda$. Similar results happen if one considers the extreme and intermediate states of the products of marginals and intrinsic statistics. This explains the change of sign of ΔR_α reported in Fig. 3.

Moreover, majorization order relation is absent for the joint and product statistics of the same intermediate state, explaining the change of sign in the solid curve reported in Fig. 4. More specifically the decreasingly-ordered vector ν associated to the product statistics $\tilde{p}^X \tilde{p}^Z$ for the intermediate states is

$$\nu = \frac{1}{16} (9, 3, 3, 1),$$

so that while $\nu_1 > \mu_1$ and $\nu_1 + \nu_2 = \mu_1 + \mu_2$, we also get $\nu_1 + \nu_2 + \nu_3 < \mu_1 + \mu_2 + \mu_3$, so that neither $\nu \prec \mu$ nor $\mu \prec \nu$. This situation can be extended also to all states within the subset S except the extreme states for which $\tilde{p}^{X,Z} = \tilde{p}^X \tilde{p}^Z$.

Similarly, there is no majorization order relation between μ and the corresponding decreasingly-ordered vector ξ associated to the product of intrinsic statistics $p^X p^Z$ for the intermediate states:

$$\xi = \frac{1}{8} (3 + 2\sqrt{2}, 1, 1, 3 - 2\sqrt{2}).$$

Thus $\xi_1 > \mu_1$ and $\xi_1 + \xi_2 > \mu_1 + \mu_2$, whereas $\xi_1 + \xi_2 + \xi_3 < \mu_1 + \mu_2 + \mu_3$, so that neither $\xi \prec \mu$ nor $\mu \prec \xi$. This explains the change of sign of the dotted curve plotted in Fig. 4.

Therefore, the striking situations reported in Figs. 3 and 4 are of a rather fundamental character and not an artifact of the particular measures of uncertainty employed.

B. Majorization uncertainty relations

Majorization provides a rather neat form for uncertainty relations in terms of suitable constant vectors majorizing the statistics of two or more observables associated for every system state [7, 8]. In our case these are

$$\tilde{p}^{X,Z} \prec \tilde{\omega}, \quad \tilde{p}^X \tilde{p}^Z \prec \tilde{\omega}' \quad \text{and} \quad p^X p^Z \prec \omega, \quad (15)$$

where in the left-hand sides we understand the result of arranging the corresponding statistics into four-dimensional vectors, and $\tilde{\omega}$, $\tilde{\omega}'$, and ω are constant vectors. By readily applying the procedure outlined in Ref. [8] we get

$$\begin{aligned} \tilde{\omega} &= \frac{1}{4} (2, \sqrt{2}, 2 - \sqrt{2}, 0), \\ \tilde{\omega}' &= \frac{1}{16\sqrt{2}} (9\sqrt{2}, 8 - \sqrt{2}, 7\sqrt{2} - 8, \sqrt{2}), \\ \omega &= \frac{1}{8} (3 + 2\sqrt{2}, 5 - 2\sqrt{2}, 0, 0). \end{aligned} \quad (16)$$

Then the expected uncertainty relations holds, for example for the joint distribution, as $R_\alpha(\tilde{p}^{X,Z}) \geq R_\alpha(\tilde{\omega})$.

It is worth noting that there is a definite majorization relation between ω and the other two vectors, that is

$$\tilde{\omega} \prec \omega \quad \text{and} \quad \tilde{\omega}' \prec \omega. \quad (17)$$

These two relations are quite natural expressing that the uncertainty lower bound is larger for the statistics derived from simultaneous joint measurement, either $\tilde{p}^{X,Z}$ or $\tilde{p}^X \tilde{p}^Z$, than for the exact intrinsic statistics. This is the majorization relation counterpart of the well-known result that the variance-based lower bound for operational position–momentum uncertainty is at least four times the intrinsic one [10].

However, there is no majorization relation between $\tilde{\omega}$ and $\tilde{\omega}'$ since while $\tilde{\omega}'_1 > \tilde{\omega}_1$, we have also $\tilde{\omega}'_1 + \tilde{\omega}'_2 + \tilde{\omega}'_3 < \tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3$. This might be related with the general lack of majorization relation between $\tilde{p}^{X,Z}$ and $\tilde{p}^X \tilde{p}^Z$ found above and illustrated in Fig. 4.

Finally let us show that there is no system state ρ leading to statistics equating the distribution (16). To this end we can use Eqs. (2), (6), and (7) to determine the values of s_x and s_z that would lead to $\tilde{p}^{X,Z}$, $\tilde{p}^X \tilde{p}^Z$ and $p^X p^Z$, equating $\tilde{\omega}$, $\tilde{\omega}'$ and ω , respectively. Without loss of generality we consider s_x and s_z to be positive. For the joint statistics, the null component in $\tilde{\omega}$ implies that $s_x = s_z = \frac{1}{\sqrt{2}}$. Thus according to Eq. (6) the other values for $\tilde{p}^{X,Z}$ should be $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$, which are not equal to the corresponding values in $\tilde{\omega}$. For the product of marginals $\tilde{p}^X \tilde{p}^Z$ the sum of the maximum and minimum of $\tilde{\omega}'$ imply that $s_x = s_z = \frac{1}{\sqrt{2}}$. Thus after Eq. (7) the other values for $\tilde{p}^{X,Z}$ should be both $\frac{3}{16}$, which are not equal to the corresponding values in $\tilde{\omega}'$. For the intrinsic statistics, we have that the two zeros of ω imply that either $s_x = 0$ or $s_z = 0$. In any case Eq. (2) would then imply that the other $p^X p^Z$ values should be both $\frac{1}{2}$, which is different from the corresponding values in ω .

V. DUALITY RELATION

Following the approach in Ref. [15] we may compare these entropic results with some other assessments of joint uncertainty or complementarity. Among them, one of the most studied is the duality relation between path knowledge and visibility of interference in a Mach–Zehnder interferometric setting [13]. This fits with our approach by regarding $|\pm\rangle$ as representing the internal paths of a Mach–Zehnder interferometer, while $|a_\pm\rangle$ represent the states of the apparatus monitoring the path followed by the interfering particle.

One of most used duality expression is [14]

$$D^2 + V^2 \leq 1, \quad (18)$$

where $D = \text{Tr}_A \left(|w_+ \rho_A^{(+)} - w_- \rho_A^{(-)}| \right)$ is the so-called distinguishability. Regarding our particular case where the system and apparatus are in pure states, we have $\rho_A^{(\pm)} = |a_\pm\rangle\langle a_\pm|$ and $w_+ = 1 - w_- = \cos^2 \frac{\theta}{2}$, so that

$$D = \sqrt{1 - 4w_+w_-} |\langle a_+ | a_- \rangle|. \quad (19)$$

This represents the knowledge available about the path followed by the particle, which is *grosso modo* inversely

proportional to path uncertainty. On the other hand, the interference is assessed by the standard fringe visibility V obtained when the relative phase φ is varied in Eq. (1),

$$V = 2\sqrt{w_+w_-} |\langle a_+|a_- \rangle|. \quad (20)$$

This roughly speaking represents the phase uncertainty, the counterpart of the uncertainty of σ_x in our approach. Note that in these duality relations path and interference are not treated symmetrically, contrary to the approach developed here in terms of entropic measures.

After Eqs. (19) and (20) we can appreciate that $D^2 + V^2 = 1$ whenever the system and apparatus are in pure states. This is to say that this duality relation is blind to the differences between extreme and intermediate states, in sharp contrast to the more complete picture provided by the entropic measures with equal entropic indices. This was already shown in Ref. [15] regarding its intrinsic counterpart $P^2 + V^2 \leq 1$, where $P = |w_+ - w_-|$ is the predictability. Nevertheless, an equivalence with the duality relation is obtained, using different entropic indices that lead to the so-called min-max entropies, as was recently shown in Ref. [18].

Since the duality relation does not discriminate between pure states it may be interesting to complete the duality analysis by examining the states of maximum D or V , as well as those states with $D = V$.

From Eq. (19) the maximum distinguishability, $D = 1$, holds either when $w_+ = 0$, $w_- = 0$, or $\langle a_+|a_- \rangle = 0$. These are all the cases where the particle actually follows just a single path, or when the apparatus can provide full information about the path followed. On the other hand, after Eq. (20), the maximum visibility, $V = |\langle a_+|a_- \rangle|$, holds when both paths are equally probable $w_+ = w_- = \frac{1}{2}$. Furthermore the maximum visibility reaches unity, $V = 1$ when $|a_+\rangle$ is proportional to $|a_-\rangle$. This is when both paths are equally probable and the apparatus provides no information about the path. Within the set S , the extreme states $s_z = \pm 1$ satisfy the requirements for extreme distinguishability, while those with $s_x = \pm 1$ reach maximum visibility. This agrees with

the case of unobserved duality [15].

On the other hand, $D = V$ holds provided that $w_+w_- |\langle a_+|a_- \rangle|^2 = \frac{1}{8}$. For balanced detection, $|\langle a_+|a_- \rangle| = \frac{1}{\sqrt{2}}$ so that $w_+w_- = \frac{1}{4}$ and then $w_+ = w_- = \frac{1}{2}$. Within the set S this is satisfied by the extreme states being eigenstates of σ_x . Contrary to what happens for the unobserved duality relation, the intermediate states do not satisfy $D = V$. The extreme $s_x = \pm 1$ can reach both maximum visibility and $D = V$ since for balanced joint detection we get $D \geq \frac{1}{\sqrt{2}} \geq V$ for all states.

VI. CONCLUDING REMARKS

We have presented several examples of application of Rényi entropies as measures of quantum uncertainty. We have explored those situations leading to unexpected or contradicting predictions for different entropies and states. We have shown that all the striking behaviors found derive from an underlying lack of majorization relation between statistics. Moreover, we have shown that none of the paradoxical features is reproduced by the most popular measure of complementarity. Thus, majorization emerges as a powerful tool to understand fundamental aspects of quantum uncertainty and complementarity in the most complete and simple form.

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