Instituto Complutense de Análisis Económico

UNIVERSIDAD COMPLUTENSE

FACULTAD DE ECONOMICAS

Campus de Somosaguas

28223 MADRID

Teléfono 394 26 11 - FAX 294 26 13

Documento de trabajo

A Class of *Learning by Doing* Model with Multiplicative Uncertainty: Economic Implications

Francisco Alvarez

Emilio Cerdá

No.9723 Diciembre 1997

Instituto Complutense de Análisis Económico

UNIVERSIDAD COMPLUTENSE

A CLASS OF *LEARNING BY DOING* MODELS WITH MULTIPLICATIVE UNCERTAINTY: ECONOMIC IMPLICATIONS

Francisco Alvarez[1]
Departamento de Fundamentos del Análisis Económico II
Universidad Complutense. Madrid. Spain.

Emilio Cerdá
Departamento de Fundamentos del Análisis Económico I
Universidad Complutense. Madrid. Spain.

ABSTRACT

Learning by doing denotes the cost reduction in production that firms achieve with their output. We check if the known properties of deterministic models, concerning the behaviour of the firms, hold under uncertainty. A discrete time and finite horizon model is considered: a monopolist, facing a linear demand, maximizes the expected profit flow, with multiplicative uncertainty on the cost reduction and an upper bound for this reduction. We show analytically that some properties do hold and some others do not.

RESUMEN

Learning by doing denota la reducción de costes de producción que las empresas logran mediante la experiencia. Se analiza si las propiedades conocidas para modelos determinísticos, relativas al comportamiento óptimo de las empresas, se mantienen bajo incertidumbre. Se considera un modelo en tiempo discreto y horizonte finito. Un monopolista, enfrentado a una demanda lineal, maximiza el flujo esperado de beneficios, con incertidumbre multiplicativa en la reducción de costes y un límite superior en esta reducción. Se prueba analíticamente que algunas propiedades del caso determinístico se mantienen bajo incertidumbre y otras no.

JEL: C60 (Mathematical Methods and Programming, General); D21 (Production and Organizations; Firm Behaviour).

[1] Correspondence to: Francisco Alvarez González. Dpto. Análisis Económico II (Economía Cuantitativa) Facultad CC. Económicas y Empresariales. U.C.M. 28223 Somosaguas, Madrid, Spain.
Ph: 34-1-3942342. Fax: 34-1-3942613. Email: eccua17@emducms1.sis.ucm.es

We thank C. Mazón for suggestions. Any error is our responsibility.

1. Introduction

The firms of some industries have reductions in their production cost simply because they accumulate experience by repiting their activity, that is, by means of production. This is known as *learning by doing* and has been observed in industries which are at an early stage of their productive life. The first study which presents some empirical evidence on this effect is by Wright (1936), and it is referred to the aircraft industry. Others are those by Joskow (1993) and Lester (1993) for electric power in nuclear plants, Dick (1991) for microconductors in Japan, or Argote et al. (1990) for the ship industry.

The first work where economic implications of *learning by doing* are studied is by Arrow (1962). After Arrow's work, there is a wide branch of the economic theory literature concerning the behaviour of firms in an industry with *learning by doing* (e.g: Fudenberg and Tirole (1983), Stokey (1986) and Dasgupta and Stiglitz (1988)). In general, all of these works consider deterministic models and they find some properties of the optimal policy of the firms. In this paper, we study if these properties hold when some uncertainty is introduced in the cost reduction process. Uncertainty is introduced in such a way that the resulting problem is still a *learning by doing* problem: cost reduction can only be achieved thorough production. We find that some properties of the deterministic problem do hold and some others do not.

We consider a model in discrete time and with finite horizon. A monopolist, facing no entry and opperating in a market with a linear demand function, maximizes current and expected future profits (risk neutrality is assumed). We take as starting point the deterministic cost reduction equation proposed by Dasgupta and Stiglitz (1988): $c(t+1)=\max\{\tau, c(t)-\beta q(t)\}$, where $c(t)$ and $q(t)$ are, respectively, the unitary cost and output in period t. We introduce uncertainty by letting: $c(t+1)=\max\{\tau, c(t)-\beta(t)q(t)\}$ where $\{\beta(t)\}$ is a i.i.d. sequence of random variables with known probability distribution. This is the so-called multiplicative uncertainty (e.g.: Kendrick (1981)). Note that the problem with multiplicative uncertainty is strictly a *learning by doing* problem in the sense given above.

We divide the analysis in two parts. In the first part, we study how uncertainty changes the optimal output in a given period. In the second part, we study if the known porperties of the optimal output's path of the deterministic model hold under uncertainty. Both parts are different, i.e. it is a known property of the deterministic model that output increases over time. Whether it holds under multiplicative uncertainty or not (second part) is not related to ask if, in a given period, the optimal

output in the deterministic model is greater or lower than in the stochastic model (first part).

We present analytical results showing that the effect of multiplicative uncertainty on the optimal output of a given period depends on the initial cost of the monopolist. Furthermore, the known properties of the optimal output's path for the deterministic case are: i) in every period, output is superior to the miopic case (that in which only current profit is maximized); ii) output increases over time; iii) in some cases (which we determine both for the deterministic and the stochastic case) the monopolist opperates at a loss. We show analytically that, under multiplicative uncertainty: i) holds, ii) does not hold for any set of parameters; iii) holds under similar conditions for the deterministic and the stochastic case.

In the paper, Section 2 describes the model. Section 3 and 4 have, respectively, the first and the second part of the analysis stated above. Section 5 has conclusions and some ideas for further research.

2. The Model

First, we present the model in its deterministic version, and then we extend it to a stochastic case. In both problems we take discrete time and two periods as time horizon (periods 0 and 1)[**], and we consider a monopolist, facing no entry. In the derministic case, the monopolist maximizes the present and future discounted profits. The discount parameter is λ. The demand function is linear and constant over time, in period t the inverse demand function is:

$$p(t) = a - bq(t) \quad t = 0,1 \tag{1}$$

where $p(t)$ and $q(t)$ are, respectively, the price and the output in period t. In that period, the unitary cost is $c(t)$. There are not fixed costs. So, given $q(t)$ and $c(t)$, the present profit in period t is $(a-bq(t)-c(t))q(t)$. The change from $c(0)$ to $c(1)$ is given by: $c(1)=\max\{\tau, c(0)-\beta q(0)\}$. This equation is proposed by Dasgupta and Stigiltz (1988). It assumes that the cost reduction is a linear function on the output of the previous period while the cost remains above a certain value τ, and if the cost takes that value, remains in it forever. Furthermore, β is the cost reduction that a unit of output generates while τ is not reached. Formally, the problem is:

[**] We will extend the model to a time horizon T when necessary. However, we start with a two period model for simplicity and because it also allows to study some of the questions stated before.

Problem I

$$\underset{q(0),q(1)}{MAX} \quad \{ \sum_{t=0}^{1} \lambda^t (a-bq(t)-c(t))q(t) \} \tag{2}$$

subject to:

$$c(1) = \max\{\tau, c(0)-\beta q(0)\} \tag{3}$$

$$q(t) \geq 0 \text{ for every } t \in \{0,1\}.$$

$c(0)$ is given, and a,b,λ,β,τ,T are known.

Other assumptions for the deterministic case are:

$$c(0) > \tau \tag{4}$$

$$a > c(0) \tag{5}$$

$$\tau > 0 \; ; \; \beta > 0 \; ; \; b > 0 \; ; \; \lambda \in (0,1] \tag{6}$$

Assumption **(4)** establishes that in **problem I** there can be cost reduction. Assumption **(5)** ensures that output is positive in every period. Finally, in **(6)** we assume that the lower bound on the cost is positive ($\tau > 0$), that a raise in the output reduces the future cost ($\beta > 0$), that the demand function is decreasing (b>0) and that the discount factor belongs to (0,1].

In the stochastic case, there is multiplicative uncertainty in the cost reduction equation. More concretely, the change from $c(0)$ to $c(1)$ is given now by $c(1)=\max\{\tau, c(0)-\beta(0)q(0)\}$, where $\beta(0)$ is a random variable with probability distribution: $\text{Prob}(\beta(t)=\beta+\theta)=\text{Prob}(\beta(t)=\beta-\theta)=½$, where β and θ are known. This equation is a natural extension of the deterministic version given above[***]. We also assume that the monopolist is risk neutral. The problem now can be formally expresed as:

[***] If we consider an arbitrary and finite number of realizations for $\beta(0)$, the mathematical treatment is more complex, but the conclusions are the same. So we take the simplest case.

Problem II

$$\underset{q(0),q(1)}{MAX} \quad \{ E\{\sum_{t=0}^{1} \lambda^t (a-bq(t)-c(t))q(t) \; / c(0)\}\} \tag{7}$$

subject to:

$$c(1) = \max\{\tau, c(0)-\beta(0)q(0)\} \tag{8}$$

$$\text{Prob}(\beta(0)=\beta+\theta)=\text{Prob}(\beta(0)=\beta-\theta)=½$$

$$q(t) \geq 0 \text{ for every } t \in \{0,1\}.$$

$c(0)$ is given, and $a,b,\lambda,\beta,\tau,\theta,T$ are known.

For the stochastic case we keep assumptions **(4)** to **(6)** and we add a new one:

$$\theta > 0 \; ; \; \beta-\theta > 0 \tag{9}$$

This assumption implies that unitary cost cannot increase over time. Note also that, at period 0, if $q(0)=0$, the unitary cost remains costant and there is not uncertainty about future cost. That is why **problem II** is still a pure *learning by doing* problem.

Next, we define some notation for the optimal policy of **problems I** and **II**:

Definition 1 $Q_d(x)$ is the optimal output in period 0 for **problem I** when $c(0)=x$.

Definition 2 $Q_e(x)$ is the optimal output in period 0 for **problem II** when $c(0)=x$.

3. Effect of the multiplicative uncertainty in a given period: Marginal and Overall impact

In this Section we study how does uncertainty affect to the optimal level of output in a concrete period. This can be done in two different ways: a) *marginal impact*, given that uncetainty exists (given **problem II**), how does a *small* change in it do change the optimal output?, b) *overall impact*, what is the difference between the deterministic **(problem I)** and the stochastic **(problem II)** case.

In the *marginal impact*, we must define what we mean by a *small change* in the uncertainty. Note that given any two different probability distributions for $\beta(0)$ such that they only differ on the value of θ, the one with higher θ is a *mean preserving spread* of the other, that is, if β is constant, the risk increases with θ. In the marginal impact we study the variation on $Q_e(c(0))$ when there is a variation on $\beta(0)$ in the mean preserving spread sense****. The next result summarizes our findings:

Theorem 1

For **problem II**, if the sufficient condition for global optimality given in **lemma 2** holds (see appendix), then we can define points δ_1, δ_2 and δ_3 such that:

i) $\tau < \delta_1 < \delta_2 < \delta_3 < a$;

ii) if $c(0) \in (\delta_3, a)$ or $c(0) \in (\delta_1, \delta_2)$ then $Q_e(c(0))$ increases with θ;

iii) if $c(0) \in (\delta_2, \delta_3)$ then $Q_e(c(0))$ decreases with θ;

iv) if $c(0) \in (\tau, \delta_1)$ then $Q_e(c(0))$ does not vary with θ.

■

The proofs for all theorems are left to the appendix. This theorem shows that the sign of the change in the optimal output of the first period ($Q_e(c(0))$) when the risk (θ) changes, depends on the initial cost ($c(0)$). This can be explained by the fact that the probability of reaching τ in period 1 under the optimal policy depends on $c(0)$, but also, depending on this probability, the objective function to be maximized in period 0 takes different functional forms*****.

Let us take some examples to illustrate this. First, suppose that $c(0) \in (\tau, \delta_1)$. In this case $c(0)-\tau$ is so small that it is optimal $c(1)=\tau$ w.p.1 (w.p. stands for with probability hereafter), and furthermore this can be done by producing in period 0 the output which maximices current profit in that period, this output does not depend on θ, so $Q_e(c(0))$ does not vary with θ. Now, if $c(0) \in (\delta_1, \delta_2)$ then it is still optimal $c(1)=\tau$ w.p.1, but now the output which maximizes current profit in period 0 is not enough for that, so the monopolist needs to produce more than that, in fact the monopolist produces a quantity which ensures $c(1)=\tau$, and this quantity increases as θ increases. In the other extreme, if $c(0) \in (\delta_3, a)$ then it is optimal $c(1) > \tau$ w.p.1., that is, $c(1)=c(0)-\beta(0)q(0)$ w.p.1 and τ does not play any role. This means that if we replace **(3)** in **problem II** by this linear equation we have an equivalent problem (the same optimal policy and value function). It can be easily shown from **lemma 2** in the appendix that in this equivalent problem the expected profit of period 1 increases with $Q_e(c(0))$ (what is intuitive since $Q_e(c(0))$ diminishes $c(1)$) and this increment is higher as θ is higher.

**** In a two period problem, output on the second period is always the same linear function of the cost for that period, no matter how the risk changes. In fact, it is the same function for **problem I** and **II**. So, we only look at output on the first period.

***** This is formally presented, together with the optimal policy in **lemma 2** in appendix.

The next theorem presents the results related to the *overall impact*, that is, it compares the optimal output of the first period in **problems I** and **II******.

Theorem 2

For **problem I** and **II**, if the sufficient condition for global optimality given in **lemmas 1** and **2** hold (see appendix) then we can define points c_1, c_2, c_3 and c_4 such that:

i) $\tau < c_1 < c_2 < c_3 < c_4 < a$;

ii) $Q_e(c_1) = Q_d(c_1)$;

iii) $Q_e(c_2) > Q_d(c_2)$;

iv) $Q_e(c_3) < Q_d(c_3)$;

v) $Q_e(c_4) > Q_d(c_4)$.

■

As in the *marginal impact* result, this theorem indicates that the sign of the difference between the optimal output in the first period in **problems I** and **II**, that is $Q_d(c(0))-Q_e(c(0))$, depends on $c(0)$. In fact, although the theorem about *marginal impact* presents stronger results than the one about *overall impact* (the first referrs to any value for $c(0)$ in (τ,a) while the second just chooses points within this interval) both results seem to go in the same direction. It seems that overall impact in this models can be considered as a limit case of the marginal impact with similar explanation to the one given for the marginal impact.

4. ¿Do the known properties for the deterministic model hold under multiplicative uncertainty?

For **problem I**, the known results in the literature are (see e.g. Fudenberg and Tirole (1983)): whenever cost reduction exists, i) optimal output in every period is greater than in the miopic case (that in which the monopolist only maximizes current profit), ii) output increases over time, iii) in some periods the monopolist could opperate at a loss. Now we check out if these properties still hold under multiplicative uncertainty.

4.1 Is output greater than in the miopic case?

In a deterministic case, if present output reduces future cost, the monopolist will produce, in

****** See previous footnote.

a given period, a greater output than the one which maximizes current profit (miopic output) to take advantage of this fact. Next result shows that this argument still holds with multiplicative uncertainty.

Theorem 3

For **problem II**, if the sufficient condition for global optimality given in **lemma 2** holds, and $c(0) > (2b\tau + (\beta-\theta)a)(2b+\beta-\theta)^{-1}$, then $Q_e(c(0))$ is greater than in the miopic case.

■

In this theorem, we exclude the case where $c(0) \leq (2b\tau + (\beta-\theta)a)(2b+\beta-\theta)^{-1}$ because in this case $c(0)$ is close enough to τ so as to have $c(1)=\tau$ w.p.1 with the miopic output in the first period. A similar case occurs in the deterministic problem (see **lemma 1** in appendix), so this makes no difference between a determinist and a multiplicative uncertainty problem.

4.2 Does output increases over time?

To answer this question we need a model where cost reduction takes several periods. To do that, we extend **problem II** to a case where time horizon takes an arbitrary and finite value, say T, so periods go from 0 to T-1, and also $c(t+1)=\max\{\tau, c(t)-\beta(t)q(t)\}$ for $t=0,\ldots,T$-1 where $\{\beta(t)\}_{t=0}^{T-1}$ is a sequence of random variables i.i.d. such that $\text{Prob}(\beta(t)=\beta+\theta)=\text{Prob}(\beta(t)=\beta-\theta)=½$. In **lemma 4** in appendix we identify, for this problem, when $c(T\text{-}1)>\tau$ w.p.1 under the optimal policy, and we also present the optimal policy in this case. We use this optimal policy to study this question. In order to connect results in previous section to the ones given here, we must note that in the T period problem we identify the optimal policy for the biggest possible values of $c(0)$ (see **lemma 4** in appendix), which are those that, in the two period problem (previous sections) have optimal output in period 0 increasing with θ and greater in the stochastic than in the deterministic case.

Furthermore, under multiplicative uncertainty, the output in every period is a random variable, so we need to define what we mean by *increasing output* over time. This is done in the next definition.

Definition 3 For **problem II** with time horizon T, output is increasing in period $t+1$ if and only if:

$$E\{q_e^*(t+1)/q_e^*(t), c(t)\} > q_e^*(t)$$

where $q_e^*(t)$ denotes optimal output in period t. Also, output is decreasing in period $t+1$ if the reverse holds with stricte inequality.

Next notation is used later in the theorem. Let:

$$K(T,\theta) = 0$$

$$K(t,\theta) = \lambda K(t+1,\theta) + \frac{(1+2\lambda\beta K(t+1,\theta))^2}{4b-4\lambda(\beta^2+\theta^2)K(t+1,\theta)} \qquad t = 0,\ldots,T-1 \tag{10}$$

The sequence defined above has a simple interpretation. Let us consider **problem II** with time horizon T, and such that there is *enough difference* between $c(0)$ and τ so as to have $c(T\text{-}1)>\tau$ w.p.1 under the optimal policy (in **lemma 4** in appendix it is formally presented what *enough difference* is). If in period t we have the unitary cost $c(t)$, then the expected discounted profit flow from t to T-1 (value function in period t) is $K(t)(a-c(t))^2$, for $t=0,\ldots,T$-1.

The results concerning increasing output over time are presented in the next theorem.

Theorem 4

Consider **problem II** with time horizon T, and such that both the global optimality condition and the sufficient condition to have $c(T\text{-}1)>\tau$ w.p.1 under the optimal policy, which are specified in **lemma 4** in appendix, hold. For any $t\in\{0,\ldots,T\text{-}2\}$ satisfying:

$$\frac{1+2\lambda^2\beta K(t+2,\theta)}{2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda)} > \left[1-\frac{\lambda\theta^2}{4b^2}\right]\phi(t+1,\theta) \tag{11}$$

the output is decreasing in period $t+1$.

■

In **lemma 3** in appendix we consider **problem I** (deterministic case) with time horizon T and cost evolution given by $c(t+1)=\max\{\tau, c(t)-\beta q(t)\}$ for $t=0,\ldots,T$-1. We identify when $c(T\text{-}1)>\tau$ under the optimal policy and we present it in this case. As we see there, the sequence $K(t,0)$ for $t=0,\ldots,T$-1, defined in **(10)** taking $\theta=0$, is used to construc the optimal policy for that problem. Since for the deterministic case the optimal output is always increasing, we must have that that **(11)** never holds for $\theta=0$. In effect, note that both sides in the inequality in **(11)** become equal if $\theta=0$*******.

******* Since **(11)** is only sufficient for decreasing output, this argument just shows that our findings are conformable with the known properties for deterministic models rather than being a proof of these properties.

Furthermore, **(11)** holds when β-θ is small and $\lambda \approx 1$[********]. Since, by hypothesis, $\theta \in (0,\beta)$, when β-θ is small we have a big risk in the probability distribution of $\beta(t)$, furthermore $\lambda \approx 1$ means that future periods are very important in current period. In a deterministic case, the output is increasing over time since, in any two consecutive periods with cost reduction, the present output reduces future cost, and in the later period the cost is lower. With multiplicative uncertainty, the cost in the later period still could be smaller, but also, in every period, the variance of the future cost conditioned on present cost and output increases with the present output. This tends to diminish current output from a certain level of conditioned variance of the future cost on: if θ is bigh enough (what means a big conditioned variance) and the ouput is increasing over time, we improve in terms of the objective function by diminishing this variance; this is done by diminishing expected output more in periods with bigger expected output, and this leads to decreasing output[*********].

4.3 When do the monopolist opperate at a loss?

There is an argument in the literature to justify that the monopolist might opperate at a loss in some periods in a deterministic model (e.g. Dasgupta and Stiglitz (1988)): since the monopolist will deviate from miopic output to take advantage of the cost reduction, there might be a loss in the current period. However, in the literature it is not specified when a current loss takes place. So, we must first be able to specify when, both in the deterministic and the stochastic problem, the monopolist will have a current loss, and then to compare the results. We work with **problems I** and **II** with time horizon T as defined in Section 4.2 becuase it allows to study how losses (if exist) are distributed over time when the cost reduction takes several periods.

First, we consider the deterministic problem. The next result indicates a sufficient condition for the monopolist to opperate at a loss in a given period.

Theorem 5

Consider **problem I** with time horizon T, and such that both the global optimality condition

[********] As an example, if $(T,\lambda,\beta,\theta,c(0),\tau,a,b)=(3,0.99,1,0.99,4,1,5,8)$ then the optimal policy for **problem II** is given by **lemma 4** and **(11)** holds for $t=0$, that is, output decreases in period 1 according to our definition. Also, if we set $b=9$ and keep the other parameter values then the optimal policy is also given by **lemma 4**, and the output is still decreasing at $t=0$ (see necesary and sufficient condition given in the proof of the latter theorem) although **(11)** does not longer hold. This emphasizes that **(11)** is only sufficient.

[*********] This does not occur for any value of the variance of the future cost since, from **theorem 2**, when $c(1)>1$ w.p.1 in **problem II** we have a bigger output than in **problem I** when $c(1)>1$. This ambiguous role of that variance (when small it just increases output, and when big it also changes some qualities of the output path) is due to the fact that the monopolist is assumed to be risk neutral.

and the sufficient condition to have $c(T\text{-}1)>\tau$ under the optimal policy, which are specified in **lemma 3** in appendix, hold. Given any period $t \in \{0,\ldots,T\text{-}1\}$, the monopolist opperates at a loss in that period when:

$$\lambda^j\Big(1+\frac{\beta}{2b}\Big)^j > 1+\frac{b}{\beta+b}\Big(1-\frac{2b}{\beta}\frac{1-\lambda}{\lambda}\Big) \qquad \textbf{(12)}$$

if $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}>0$; or:

$$\lambda^j\Big(1+\frac{\beta}{2b}\Big)^j < 1+\frac{b}{\beta+b}\Big(1-\frac{2b}{\beta}\frac{1-\lambda}{\lambda}\Big) \qquad \textbf{(13)}$$

if $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}<0$.

where $j=T\text{-}t$.

■

If $\lambda=1$ assumption **(12)** becomes $(1+\beta(2b)^{-1})^j>1+b(\beta+b)^{-1}$, which always holds for j big enough. This means that when $\lambda=1$ and there is enough difference between $c(0)$ and τ, the monopolist opperates at a loss in the first period. Furthermore, for any $\lambda \neq 0$, the hypothesis holds more easily as b gets smaller in terms of λ and β.

When neither **(12)** nor **(13)** hold, this result says nothing, since it only gives a sufficient condition. So, the question which remains is: when b is big enough in terms of λ and β (neither **(12)** nor **(13)** hold), does the monopolist opperate at a loss? The next result gives an answer. It shows that, precisely when b is big enough in terms of λ and β, the monopolist will never opperate at a loss, no matters how big the difference between $c(0)$ and τ is (if $c(0)\text{-}\tau$ is big enough so as not to reach τ in T periods).

Theorem 6

Consider **problem I** with time horizon T, and such that both the global optimality condition and the sufficient condition to have $c(T\text{-}1)>\tau$ under the optimal policy, which are specified in **lemma 3** in appendix, hold. If:

$$\frac{\lambda\beta}{2b} < \frac{1-\lambda}{\beta}\Big[b-\sqrt{b^2-\frac{\lambda\beta}{1-\lambda}(2b+\beta)}\ \Big]-\lambda < \frac{b}{b+\beta} \qquad \textbf{(14)}$$

the monopolist will never opperate at a loss.

■

As we have previously indicated, **(14)** holds when b is big enough in terms of λ and β. If it occurs, the root is real, and then it can be easily shown that the first inequality always holds. Also, the second inequality holds more easily as b gets bigger in terms of λ and β (if it occurs, the left side of the inequality gets closer to 0, and the right side gets closer to 1).

So **theorems 5** and **6** say; first, that the monopolist opperates at a loss in the first period when the slope of the inverse demand function (b) is small enough in terms of the discount parameter (λ) and the cost reduction that a unit of output generates (β); and second, that the monopolist *never* opperates at a loss when b is big enough in terms of λ and β. Now we ask how do things change if we introduce multiplicative uncertainty. The next two results show that, basically, the conclusion are the same as in the deterministic case. This is so even although the periods where the monopolist opperate at a loss are not exactly the same for a deterministc and a stochastic problem.

As in the deterministic case, first we give a condition under which the monopolist opperates at a loss in period t. This is done in the next theorem.

Theorem 7

Consider **problem II** with time horizon T, and such that both the global optimality condition and the sufficient condition to have $c(T\text{-}1)>\tau$ w.p.1 under the optimal policy, which are specified in **lemma 4** in appendix, hold. Given any period $t\in\{0,\dots,T\text{-}1\}$, the monopolist opperates at a loss in that period when:

$$\lambda^j\left(1+\frac{\beta}{2b}\right)^j > 1+\frac{b}{\beta+b+\frac{\theta^2}{\beta}}\left(1-\frac{2b}{\beta}\frac{1-\lambda}{\lambda}\right) \tag{15}$$

if $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}>0$; or:

$$\lambda^j\left(1+\frac{\beta}{2b}\right)^j < 1+\frac{b}{\beta+b+\frac{\theta^2}{\beta}}\left(1-\frac{2b}{\beta}\frac{1-\lambda}{\lambda}\right) \tag{16}$$

if $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}<0$.
where $j=T\text{-}t$.

■

Now, we present a sufficient condition under which the monopolist never opperates at a loss.

Theorem 8

Consider **problem II** with time horizon T, and such that both the global optimality condition and the sufficient condition to have $c(T\text{-}1)>\tau$ w.p.1 under the optimal policy, which are specified in **lemma 4** in appendix, hold. If:

$$\frac{1}{4b} < \frac{1-\lambda}{2\lambda(\beta^2+(1-\lambda)\theta^2)}\left[b-\sqrt{b^2-\frac{\lambda}{1-\lambda}(\beta(2b+\beta)+\theta^2)}\ \right]-\frac{\beta}{2(\beta^2(1-\lambda)\theta^2))} < \frac{b}{2\lambda(\beta(b+\beta)+\theta^2))} \tag{17}$$

the monopolist never opperates at a loss.

■

Note the analogy between **theorem 5** and **6** and **theorem 7** and **8**. This suggests, as we have indicated, that the monopolist opperates at a loss under similar conditions in the deterministic and in the stochastic case.

6. Conclusions and further research

The firms of an *infant* industry (Stokey (1986)) have reductions in their production cost over time because they accumulate experience. This is known as *learning by doing*, and it has been extensively studied in deterministic models. We ask if the known properties of the behaviour of firms with *learning by doing* in a deterministic model hold when some uncertainty is introduced in the cost reduction process.

A discrete time and finite time horizon model is considered, in which a monopolist, who faces no entry and opperates in a market with linear demand, maximizes current and discounted future profits. We take as starting point a deterministic cost reduction as in Dasgupta and Stiglitz (1988): $c(t+1)=\max\{\tau,c(t)\text{-}\beta q(t)\}$, where $q(t)$ and $c(t)$ are, respectively, output and unitary cost in period t, and introduce multiplicative uncertainty in the later equation by setting $c(t+1)=\max\{\tau,c(t)\text{-}\beta(t)q(t)\}$, where $\{\beta(t)\}$ is a i.i.d. sequence of random variables with known probability distribution. An interesting fact of this form of uncertainty is that the model is still a *learning by doing* model: $\text{Prob}(c(t+1)<c(t)/c(t),q(t))>0$ only if $q(t)>0$, also $\text{Prob}(c(t+1)>c(t)/c(t),q(t))=0$. When uncertainty is introduced, the monopolist is assumed to be risk neutral.

Our analysis has two parts. In the first part, we study the influence of the multiplicative uncertainty on the optimal output in a given period. In the second part, we study if the known properties of the optimal output's path of the deterministic model hold under uncertainty.

We have analytical results showing that: 1) concerning the first part, to introduce uncertainty, or to alter it in a *mean preserving spread* sense, changes the level of output, and the sign of this change depends on the initial cost; 2) concerning the second part, not all of the known properties for a deterministic model hold under uncertainty, as we show in the next table.

deterministic properties	current output greater than the one which maximizes current profit	output increases over time	monopolist opperates at a loss in the initial periods when b small in terms of λ and β	monopolist never opperates at a loss when b big in terms of λ and β
multiplicative uncertainty properties	holds	does not hold when b small in terms of λ and θ	holds writting: ... λ, β and θ.	holds writting: ... λ, β and θ.

notes: b is the slope of the inverse demand function, λ is the discount parameter, and θ is the standard deviation of $\beta(t)$. Also, in the multiplicative uncertainty case, β is the mean of $\beta(t)$.

We think of two possible interesting ways to continue this work. First, to consider different kinds of industrial structures, with special emphasis on duopoly and a case where the probability distribution of the perturbation ($\beta(t)$) is different for every firm. It it a well known property for a deterministic model than asimetries in a dupoly (i.e. different initial costs) within a deterministic scheme can lead to a monopoly, and we could ask if this is still true when the differences are introduced only in the perturbation term. Second, to consider other cases of uncertainty, specially a case where there are unknown parameters in the distribution of probability of $\beta(t)$. If this is so, there are two kinds of *learning*: one about an unkown parameter, and one reflecting the cost reduction itself. The key question is: How do they interact?

REFERENCES

ALVAREZ, F. and CERDA, E. (1997a), "A Solution Method for a Class of *Learning by Doing* Models", mimeo.

ALVAREZ, F. and CERDA, E. (1997b), "A Solution Method for a Class of *Learning by Doing* Models with Multiplicative Uncertainty", mimeo.

ARGOTE, L, BECKMAN, S. L. Y APPLE, D. (1990), "The Persistence and Transfer of Learning in Industrial Settings", Management Science, Vol. 36, No. 2, pp. 140-154.

ARROW, K. J. (1962), "The Economic Implications of Learning by Doing", Review of Economic Studies, 29, 155-173.

DASGUPTA, P. and STIGLITZ, J. (1988), "Learning-by-Doing, Market Structure and Industrial and Trade Policies", Oxford Economic Papers, 40, 246-268.

DICK, A. R. (1991), "Learning by Doing and Dumping in the Semiconductor Industry", Journal of Law & Economics, vol. XXXIV, pp. 133-159.

FUDENBERG, D. and TIROLE, J. (1983), "Learning-by-Doing and Market Perfomance", Bell Journal of Economics, 14, 522-530.

JOSKOW, P. L. y ROZANSKI, G. A. (1979), "The effects of Learning by Doing on Nuclear Plant Operating Reability", The Review of Economics and Statistic, Vol. LXI, No. 2, pp. 161-168.

LESTER, R. K. y McCABE M. J. (1993), "The Effect of Industrial Structure on Learning by Doing in Nuclear Power Plant Operation", RAND Journal of Economics, Vol. 24, No. 3, pp. 418-438.

KENDRICK, D. (1981), "Stochastic Control for Economic Models", McGraw-Hill, New York.

STOCKEY, N. (1986), "The Dynamics of Industry-wide Learning", en W. P. Heller, R. M. Starr and D.A. Starrett, Eds, "Equilibrium Analysis: Essays in Honour of Kenneth J. Arrow ", Vol II, Cambrigde University Press.

WRIGHT, T. P. (1936), "Factors Affecting the Costs of Airplanes", Journal of Auronautical Science, 3, pp. 122-128.

APPENDIX

The next two lemmas give the optimal policy for **problems I** and **II** respectively. For the proof of these lemmas see Alvarez and Cerdá (1997a) and (1997b). Previously some notation is introduced. Let:

$$K = \frac{1}{4b}$$

$$g(x,y) = \frac{\tau+(\beta+y)xa}{1+(\beta+y)x}$$

Lemma 1

For **problem I**, if $b>\lambda\beta^2K$ then:

$$Q_d(c(0)) = \begin{cases} \phi_{d,1}(a-c(0)) & \text{if } g(\phi_{d,1},0) < c(0) \\ \frac{1}{\beta}(c(0)-\tau) & \text{if } g(\phi_{d,2},0) < c(0) \le g(\phi_{d,1},0) \\ \phi_{d2}(a-c(0)) & \text{if } c(0) \le g(\phi_{d2},0) \end{cases}$$

where $\phi_{d,1}=(1+2\lambda\beta K)/(2b-2\lambda\beta^2K)$, $\phi_{d,2}=(2b)$-1, and K and $g(x,y)$ are defined as above. ■

Lemma 2

For **problem II**, if $b>\lambda(\beta^2+\theta^2)K$ then:

$$Q_e(c(0)) = \begin{cases} \phi_{e,1}(a-c(0)) & \text{if} \quad g(\phi_{e,1},\theta) < c(0) \\ \dfrac{1}{\beta+\theta}(c(0)-\tau) & \text{if} \quad g(\phi_{e,2},\theta) < c(0) \le g(\phi_{e,1},\theta) \\ \phi_{e,2}(a-c(0)) & \text{if} \quad g(\phi_{e,2},-\theta) < c(0) \le g(\phi_{e,2},\theta) \\ \dfrac{1}{\beta-\theta}(c(0)-\tau) & \text{if} \quad g(\phi_{e,3},-\theta) < c(0) \le g(\phi_{e,2},-\theta) \\ \phi_{e,3}(a-c(0)) & \text{if} \quad c(0) \le g(\phi_{e,3},-\theta) \end{cases}$$

where $\phi_{e,1}=(1+2\lambda\beta K)/(2b-2\lambda(\beta^2+\theta^2)K)$, $\phi_{e,2}=(1+2\lambda(\beta-\theta)K)/(2b-2\lambda(\beta-\theta)^2)K)$, $\phi_{e,3}=(2b)$-1, and K and g(x,y) are defined as before.

■

Proof of theorem 1

We define $\delta_1=g(\phi_{e,3},-\theta)$, $\delta_2=g(\phi_{e,2},-\theta)$ and $\delta_3=g(\phi_{e,1},\theta)$, with $g(x,y)$ defined as before. The part i follows from the fact that $\tau<a$ and so $g(x,y)$ is and increasing function in each argument, and we have $\phi_{e,3}<\phi_{e,2}<\phi_{e,1}$. Now, since the hypothesis given in **lemma 2** hold, $Q_e(c(0))$ is given by that lemma. Next we show ii). First, $c(0)\in(\delta_3,a)\Leftrightarrow c(0)>g(\phi_{e,1},\theta)$ and then $Q_e(c(0))=\phi_{e,1}(a-c(0))$, also $\partial\phi_{e,1}/\partial\theta=4\lambda\theta K\phi_{e,1}{}^2(1+2\lambda\beta K)^{-1}$ (where $\partial f/\partial\theta$ denotes partial derivative of f respect to θ), so we have $\partial\phi_{e,1}/\partial\theta>0$, and so $Q_e(c(0))$ increases with θ inthis case. Second, $c(0)\in(\delta_1,\delta_2)\Leftrightarrow g(\phi_{e,3},-\theta)<c(0)<g(\phi_{e,2},-\theta)$ and then $Q_e(c(0))=(\beta-\theta)^{-1}(c(0)-\tau)$, since $(\beta-\theta)^{-1}$ increases with θ also does $Q_e(c(0))$. To proove iii note that $c(0)\in(\delta_2,\delta_3)\Leftrightarrow c(0)\in(g(\phi_{e,2},-\theta),g(\phi_{e,1},\theta))$ and then $Q_e(c(0))=(\beta+\theta)^{-1}(c(0)-\tau)$ or $Q_e(c(0))=\phi_{e,2}(a-c(0))$; the first expresion in strictly decreasing in θ, the second also is, since: $\partial\phi_{e,2}/\partial\theta=-\lambda K(2(b+\beta-\theta)+\lambda(\beta-\theta)^2K)(2b-\lambda(\beta-\theta)^2K)^{-2}$, and so $\partial\phi_{e,2}/\partial\theta<0$ and, in any case, $Q_e(c(0))$ decreases with θ, hence iii holds. To show iv, note that $c(0)\in(\tau,\delta_1)\Leftrightarrow c(0)\le g(\phi_{e,3},-\theta)$, then $Q_e(c(0))=\phi_{e,3}(a-c(0))$ and $\phi_{e,3}$ does no depend on θ, and so iv holds.

■

Proof of theorem 2

The proof has two steps. In the first step, we take $c_1\in(\tau,g(\phi_{e,3},-\theta))$ and $c_4\in(\tau,g(\phi_{e,1},\theta))$, and we show that these values verify ii and v and also $\tau<c_1<c_4<a$. In the second step, we show that there exist values c_2 and c_3 which verify iii, iv and $g(\phi_{e,3},-\theta)<c_3<c_4<g(\phi_{e,1},\theta)$ (and so i holds).

First step. From definitions for g(x,y), $\phi_{e,3}$, and $\phi_{d,2}$ we have:

$g(\phi_{e,3},-\theta)<g(\phi_{d,2},0)$, hence, if $c_1<g(\phi_{e,3},-\theta)$ then $Q_d(c_1)=\phi_{d,2}(a-c_1)$ and $Q_e(c_1)=\phi_{e,3}(a-c_1)$ follow from **lemmas 1** and **2**, but it also follows: $\phi_{d,2}=\phi_{e,3}$, and so ii holds. Also, from definitions of $g(x,y)$, $\phi_{e,1}$ and $\phi_{d,1}$ we have: $g(\phi_{e,1},\theta)>g(\phi_{d,1},0)$, hence if $c_4>g(\phi_{e,1},\theta)$ then from **lemmas 1** and **2**: $Q_d(c_4)=\phi_{d,1}(a-c_4)$ and $Q_e(c_4)=\phi_{e,1}(a-c_4)$. Moreover, from definitions of $\phi_{d,1}$ and $\phi_{e,1}$ we have: $\phi_{d,1}<\phi_{e,1}$, and so v holds. Furthermore, since $g(x,y)$ is increasing in its arguments and $\phi_{e,3}<\phi_{e,1}$ we have $\tau<c_1<c_4<a$.

Second step. To proove iii note that either it occurs a) $g(\phi_{d,2},0)\le g(\phi_{e,2},-\theta)$, or b) $g(\phi_{d,2},0)>g(\phi_{e,2},-\theta)$. If a) occurs then, it exists $c_2\in[g(\phi_{d,2},0),g(\phi_{e,2},-\theta)]$ such that: $Q_d(c_2)=\beta^{-1}(c_2-\tau)$ and $Q_e(c_2)=(\beta-\theta)^{-1}(c_2-\tau)$, hence $Q_e(c_2)>Q_d(c_2)$. If b) occurs, then it exists $c_2\in(g(\phi_{e,2},-\theta),g(\phi_{d,2},0))$ such that: $Q_d(c_2)=\phi_{d,2}(a-c_2)$ and $Q_e(c_2)=\phi_{e,2}(a-c_2)$, and since $\phi_{d,2}<\phi_{e,2}$ we have $Q_e(c_2)>Q_d(c_2)$. To prove iv note that either it ocurrs c) $g(\phi_{d,1},0)\le g(\phi_{e,2},\theta)$, or d) $g(\phi_{d,1},0)>g(\phi_{e,2},\theta)$. If c) occurs then it exits $c_3\in(g(\phi_{d,1},0),g(\phi_{e,2},\theta))$ such c_3 is greater than any c_3 choosen previously, and furthermore, from **lemmas 1** and **2**, c_3 verifies: $Q_d(c_3)=\phi_{d,1}(a-c_3)$ and $Q_e(c_3)=\phi_{e,2}(a-c_3)$, and from definitions of $\phi_{d,1}$ and $\phi_{e,2}$ we have: $\phi_{d,1}>\phi_{e,2}$ and hence $Q_e(c_3)<Q_d(c_3)$. If d) occurs then it exists $c_3\in[g(\phi_{e,2},\theta),g(\phi_{d,1},0)]$ such that c_3 is greater that any c_3 previously choosen, and furthermore, from **lemmas 1** and **2**, c_3 verifies: $Q_d(c_3)=\beta^{-1}(c_3-\tau)$ and $Q_e(c_3)=(\beta+\theta)^{-1}(c_3-\tau)$, hence $Q_e(c_3)<Q_d(c_3)$. Finally, note that c_2 and c_3 always satisfy $c_2<c_3$. Moreover $c_2\ge\min\{g(\phi_{d,2},0),g(\phi_{e,2},-\theta)\}>g(\phi_{e,3},-\theta)$ and $c_3\le\min\{g(\phi_{d,1},0),g(\phi_{e,2},\theta)\}>g(\phi_{e,1},\theta)$.

■

Proof of theorem 3

Let q_m be miopic output in the first period, we have $q_m=(2b)^{-1}(a-c(0))$. Suppose that $Q_e(c(0))=\phi_{e,1}(a-c(0))$, since $\phi_{e,1}>(2b)^{-1}$ we have $Q_e(c(0))>q_m$. Analogously, if $Q_e(c(0))=\phi_{e,2}(a-c(0))$, since $\phi_{e,2}>(2b)^{-1}$ we have $Q_e(c(0))>q_m$. Now, if $Q_e(c(0))=(\beta+\theta)^{-1}(c(0)-\tau)$, then: $Q_e(c(0))>q_m\Leftrightarrow(\beta+\theta)^{-1}(c(0)-\tau)>(2b)^{-1}(a-c(0))\Leftrightarrow$ $c(0)>\gamma a+(1-\gamma)\tau$, where $\gamma=(\beta+\theta)(2b+\theta+\beta)^{-1}$. The lowest value of $c(0)$ such that $Q_e(c(0))=(\beta+\theta)^{-1}(c(0)-\tau)$ is: $(\tau+(\beta+\theta)\phi_{e,2}a)(1+(\beta+\theta)\phi_{e,2})^{-1}$, hence, $Q_e(c(0))>q_m$ if $(\tau+(\beta+\theta)\phi_{e,2}a)(1+(\beta+\theta)\phi_{e,2})^{-1}\ge\gamma a+(1-\gamma)\tau$ holds. Both sides of the later inequality are convex linear combinations of a and τ, and since $a>\tau$, the inequality holds if and only if $(1+(\beta+\theta)\phi_{e,2})^{-1}\le1-\gamma$, or $\phi_{e,2}\ge(2b)^{-1}$, which is true. If $Q_e(c(0))=(\beta-\theta)^{-1}(c(0)-\tau)$, then: $Q_e(c(0))>q_m\Leftrightarrow(\beta-\theta)^{-1}(c(0)-\tau)>(2b)^{-1}(a-c(0))\Leftrightarrow c(0)>\sigma a+(1-\sigma)\tau$ where $\sigma=(\beta-\theta)(2b-\theta+\beta)^{-1}$. But $Q_e(c(0))=(\beta-\theta)^{-1}(c(0)-\tau)$ only if $c(0)>(\tau+(\beta-\theta)\phi_{e,3}a)(1+(\beta-\theta)\phi_{e,3})^{-1}$, hence, $Q_e(c(0))>q_m$ if: $(\tau+(\beta-\theta)\phi_{e,3}a)(1+(\beta-\theta)\phi_{e,3})^{-1}\ge\sigma a+(1-\sigma)\tau$. Both sides of the later inequality are convex linear combinations of a and τ, and since $a>\tau$, the inequality holds if only if:

$(1+(\beta-\theta)\phi_{e,3})^{-1} \leq 1-\sigma$, or $\phi_{e,3} \geq (2b)^{-1}$, which is true.

■

The next two lemmas give the optimal policy for **problems I** and **II** respectively, with time horizon T and cost evolution given by $c(t+1)=\max\{\tau, c(t)-\beta q(t)\}$ for **problem I** and $c(t+1)=\max\{\tau, c(t)-\beta(t)q(t)\}$ for **problem II**. For the proof of these lemmas see Alvarez and Cerdá (1997a) and (1997b). Previously some notation is introduced. Let:

$$\phi(t,\theta) = \frac{1+2\lambda\beta K(t+1,\theta)}{2b-2\lambda(\beta^2+\theta^2)K(t+1,\theta)} \quad ; \quad t = 0,\ldots,T-1$$

$$R(T-1,\theta) = \tau$$

$$R(t,\theta) = \frac{R(t+1,\theta)+\beta\phi(t,\theta)a}{1+\beta\phi(t,\theta)} \quad ; \quad t = 0,\ldots,T-2$$

with $K(t,\theta)$ as defined in **(10)**.

Lemma 3

For **problem I** with time horizon T, if $b > \lambda\beta^2 K(t+1,0)$ for $t \in \{0,\ldots,T-1\}$ (global optimality condition) and $c(0) > R(0,0)$ hold, then the optimal output in period t is $\phi(t,0)(a-c(t))$ for $t \in \{0,\ldots,T-1\}$, and also $c(T-1) > \tau$ under the optimal policy.

■

Lemma 4

For **problem II** with time horizon T, if $b > \lambda(\beta^2+\theta^2)K(t+1,\theta)$ for $t \in \{0,\ldots,T-1\}$ (global optimality condition) and $c(0) > R(0,\theta)$ hold, then the optimal output in period t is $\phi(t,\theta)(a-c(t))$ for $t \in \{0,\ldots,T-1\}$, and also $c(T-1) > \tau$ w.p.1 under the optimal policy.

■

The condition $c(0) > R(0,\theta)$ ensures $c(T-1) > \tau$ w.p.1 under the optimal policy (and so does $c(0) > R(0,0)$ for the deterministic case).

Proof of theorem 4

If the conditions in **lemma 4** hold then the optimal policy for **problem II** is given by that lemma. Taking expectations, and notting that $1+\beta\phi(t,\theta)>0$, we have: $E\{q(t+1)/q(t),c(t)\} > q(t) \Leftrightarrow \phi(t,\theta)(1+\beta\phi(t,\theta))^{-1} < \phi(t+1,\theta)$. Furthermore: $\phi(t,\theta)(1+\beta\phi(t,\theta))^{-1} = (1+2\lambda\beta K(t+1,\theta))(2b+\beta-2\lambda\theta^2 K(t+1,\theta))^{-1} = (1+2\lambda\beta(\lambda K(t+2,\theta)+ \frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))(2b+\beta-2\lambda\theta^2(\lambda K(t+2,\theta)+\frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))^{-1}$, where the first equallity follows from the expression for $\phi(\theta,t)$ and the second from the expression for $K(\theta,t)$. Then, we have: $E\{q(t+1)/q(t),c(t)\} > q(t) \Leftrightarrow (1+2\lambda\beta(\lambda K(t+2,\theta)+$

$\frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))(2b+\beta-2\lambda\theta^2(\lambda K(t+2,\theta)+\frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))^{-1} < \phi(t+1,\theta)$. We can rewrite the right hand side of the later implication as: $(1+2\lambda\beta(\lambda K(t+2,\theta)+\frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))(2b+\beta-2\lambda\theta^2(\lambda K(t+2,\theta)+ \frac{1}{2}(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)))^{-1} < \phi(t+1,\theta) \Leftrightarrow$ $1+2\lambda^2\beta K(t+2,\theta)+\lambda\beta\phi(t+1,\theta)+2\lambda^2\beta^2 K(t+2,\theta)\phi(t+1,\theta) <$ $(2b+\beta-2\lambda^2\theta^2 K(t+2,\theta))\phi(t+1,\theta)-\lambda\theta^2(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)^2 \Leftrightarrow$ $1+2\lambda^2\beta K(t+2,\theta) < (2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))\phi(t+1,\theta)-\lambda\theta^2(1+2\lambda\beta K(t+2,\theta))\phi(t+1,\theta)^2 \Leftrightarrow$ $(1+2\lambda^2\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1} <$ $\phi(t+1,\theta)-\lambda\theta^2(1+2\lambda\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1}\phi(t+1,\theta)^2$. So, we can write: $E\{q(t+1)/q(t),c(t)\} > q(t) \Leftrightarrow (1+2\lambda^2\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1} <$ $(1-\lambda\theta^2\phi(t+1,\theta)(1+2\lambda\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1})\phi(t+1,\theta)$.

So if the inequallity on the right hand side of the later implication does not hold, we have decreasing output in $t+1$. Now we show that **(11)** is a sufficient condition for this to occur. In effect: $(4b^2)^{-1}=(2b)^{-1}(2b)^{-1} < (1+2\lambda\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1}\phi(t+1,\theta)$, since: $(2b)^{-1} < (1+2\lambda\beta K(t+2,\theta))(2b-2\lambda^2(\beta^2+\theta^2)K(t+2,\theta)+\beta(1-\lambda))^{-1}$; y $(2b)^{-1} < \phi(t+1,\theta)$, hence, the right hand side in **(11)** is greater than the right hand side of the inequallity on the right side of the later implication.

■

Next result is used in the following proofs. It gives a necessary and sufficient condition for the monopolist to opperate at a loss in a given period, within a T period problem. It can be applied for both the stochastic and the deterministic case.

Lemma 5

For **problem II** with time horizon T as defined before and such that the optimal policy is given by **lemma 4**, the monopolist opperates at a loss in period t if and only if:

$$K(t+1,\theta) > \frac{b}{2\lambda(\beta(b+\beta)+\theta^2)} \qquad \textbf{(22)}$$

This condition also stands for the deterministic case (**problem I** with time horizon T and optimal policy given by **lemma 3**) by setting $\theta=0$.

■

Proof of Lemma 5

We make the proof for the stochastic case, the deterministic case is similar. From the optimal policy given by **lemma 2** we have: $p(t) < c(t) \Leftrightarrow a-b\phi(t,\theta)(a-c(t)) < c(t)$; and this can be rewriten as:

$p(t) < c(t) \Leftrightarrow \phi(t,\theta) > b^{-1}$. Now, by considering the value for $\phi(t,\theta)$, we have the hypothesis given in this lemma.

■

Proof of theorem 5

Let the difference equation $x(i+1)=z(x(i))$, for every $i=0,1,\ldots$ with $x(0)=0$; where $z(x)=\lambda x+(1+2\lambda\beta x)^2(4b-4\lambda\beta^2 x)^{-1}$ for every $x\in\{\mathbb{R}:4b>4\lambda\beta^2 x\}$. Note that $x(i)=K(T\text{-}i,0)$. The proof is as follows: from **lemma 5**, the monopolist will opperate at a loss in period $T\text{-}j\text{-}1$ if and only if $K(T\text{-}j,0)$, that is $x(j)$, is greater a certain critical value; to identify a period $T\text{-}j+1$ where this occurs, we find a linear difference equation, say $y(i+1)=g(y(i))$, satisfying: i) $x(i)\geq y(i)$ for every $i=0,1,\ldots,$ and ii) the hypothesis of the theorem is sufficient for $y(j)$ to be greater than the critical value given in **Lemma 5**.

Let $g(x)=(4b)^{-1}+\lambda(1+\beta(2b)^{-1})x$ for every $x\in\{\mathbb{R}:4b>4\lambda\beta^2 x\}$. Note that $z(x)\geq g(x)$. We show, by finite induction on i, that the sequence given by $y(i+1)=g(y(i))$ for every $i=0,1,\ldots$ with $y(i)=0$ satisfies $x(i)\geq y(i)$ for every $i=0,1,\ldots$ In effect for $i=0$ later inequallity holds; now suppose it holds for i, that is $x(i)\geq y(i)$, then $x(i+1)\geq y(i+1)$ since $z(x)\geq g(x)$ for every x in the domain, and so: $x(i+1)=z(x(i))\geq g(x(i))>g(y(i))=y(i+1)$; where later inequallity follows from the fact that g is an increasing function.

Now we show that the hypothesis of the theorem is sufficient for $y(j)$ to be greater than the critical value given in **lemma 5**. The particular solution to the linear equation $y(i+1)=g(y(i))$ is $y(i)=(4b(1\text{-}\lambda)\text{-}2\lambda\beta)^{-1}(1\text{-}\lambda^i(1+\beta(2b)^{-1})^i)$. Hence, $y(j)$ is greater than the critical value given by **lemma 5** if: $b(2\lambda\beta(b+\beta))^{-1}<(4b(1\text{-}\lambda)\text{-}2\lambda\beta)^{-1}(1\text{-}\lambda^j(1+\beta(2b)^{-1})^j)$. Later inequallity can be rewriten as:$-b(b+\beta)^{-1}>(1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1})^{-1}(1\text{-}\lambda^j(1+\beta(2b)^{-1})^j)$; and depending on the sign of $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}$ we get the hypothesis of the theorem.

■

Proof of theorem 6

Let the difference equation $x(i+1)=z(x(i))$, for every $i=0,1,\ldots$ be defined as in the proof of **theorem 5**. From **lemma 5**, the monopolist will never opperate at a loss if $x(i)$ remains lower than the critical value given in that lemma for any $i=0,1,\ldots$ We show, in two steps, that the hypothesis of the theorem is sufficient for that.

First step. We show that if the hypothesis holds, then there is a value, say x^*, such that it is a solution to the equation: $x=z(x)$ and also $z(0)<x^*<b(2\lambda\beta(b+\beta))^{-1}$ (later term is the critical value given in **lemma 5**). We also proove that z is an increasing function. In effect, the two possible solutions to $x=z(x)$ are:

$$x^* = \frac{1-\lambda}{2\lambda\beta^2}\left[b\mp\sqrt{b^2-\frac{\lambda\beta}{1-\lambda}(2b+\beta)}\;\right]-\frac{1}{2\beta} \qquad \textbf{(23)}$$

Since $z(0)=(4b)^{-1}$; $z(0)<x^*<b(2\lambda\beta(b+\beta))^{-1}$ becomes the hypothesys of the theorem by multipliying by $2\lambda\beta$ and by taking as x^* the lowest possible solution to $x=z(x)$. Moreover: $z'(x)=\lambda+4\lambda\beta(1+2\lambda\beta x)(4b+\beta\text{-}2\lambda\beta^2 x)(4b\text{-}4\lambda\beta^2 x)^{-2}$, (where prime denotes derivatives) and so $z'(x)>0$ for any x in the domain of z.

Second step. We show that if x^* is a solution to $x=z(x)$, such that $z(0)<x^*<b(2\lambda\beta(b+\beta))^{-1}$, then: $x(i)<x^*$ and so $x(i)<b(2\lambda\beta(b+\beta))^{-1}$ for any $i\geq 0$. In effect: for $i=0$ it holds. Let $x(i)<x^*$ hold for i, then $x(i+1)=z(x(i))<z(x^*)=x^*$, where the inequality follows from the fact that z is an increasing function.

■

Proof of theorem 7

Let the difference equation $x(i+1)=m(x(i))$, for every $i=0,1,\ldots$ with $x(0)=0$; where $m(x)=\lambda x+(1+2\lambda\beta x)^2(4b\text{-}4\lambda(\beta^2+\theta^2)x)^{-1}$ for every $x\in\{\mathbb{R}:4b>4\lambda(\beta^2+\theta^2)x\}$. Note that $m(i)=K(T\text{-}i,\theta)$. The proof is similar to **theorem 5**: the monopolist opperates at a loss in period $T\text{-}j\text{-}1$ if and only if $K(T\text{-}j,\theta)$, that is $x(j)$, is greater than the critical value given in **lemma 5**, to identify a period $T\text{-}j+1$ where this occurs, we find a linear difference equation, say $y(i+1)=g(y(i))$, satisfying: i) $x(i)\leq y(i)$ for every $i=0,1,\ldots,$ and ii) the hypothesis of the theorem is sufficient for $y(j)$ to be greater than the critical value given in **lemma 5**.

Let $g(x)=(4b)^{-1}+\lambda(1+\beta(2b)^{-1})x$, for every $x\in\{\mathbb{R}:4b>4\lambda(\beta^2+\theta^2)x\}$. For this linear difference equation, we have, in a similar way to the proof of **theorem 5**, that $x(i)\geq y(i)$ for every $i=0,1,\ldots$ is satisfied. Also, the particular solution to this equation is: $y(i)=(4b(1\text{-}\lambda)\text{-}2\lambda\beta)^{-1}(1\text{-}\lambda^i(1+\beta(2b)^{-1})^i)$. Hence, $y(j)$ is greater than the critical value given by **lemma 5** if: $b(2\lambda(\beta(b+\beta)+\theta^2))^{-1}\leq(4b(1\text{-}\lambda)\text{-}2\lambda\beta)^{-1}(1\text{-}\lambda^j(1+\beta(2b)^{-1})^j)$, and from last inequality, depending on the sign of $1\text{-}(1\text{-}\lambda)2b(\lambda\beta)^{-1}$, we have the hypothesis of the theorem.

■

Proof of theorem 8

Let the difference equation $x(i+1)=m(x(i))$, for every $i=0,1,\ldots$ be defined as in the proof of **theorem 7**. From **lemma 5**, the monopolist never opperates at a loss if $x(i)$ is lower than the critical value given in that lemma for any $i=0,1,\ldots$We show, in two steps, in a similar way to the proof of **theorem 6**, that the hypothesis of the theorem is sufficient for that.

First step. We show that if the hypothesis holds then it exists x^* such that $x^*=m(x^*)$ and also $m(0)<x^*<b(2\lambda(\beta(b+\beta)+\theta^2))^{-1}$ (later term is the critical value given in **lemma 5**). We also show that m is an increasing function. In effect, the two possible solutions to $x=m(x)$ are:

$$x^* = \frac{1-\lambda}{2\lambda(\beta^2+(1-\lambda)\theta^2)}[b \mp \sqrt{b^2-\frac{\lambda}{1-\lambda}(\beta(2b+\beta)+\theta^2)}\ ]-\frac{\beta}{2(\beta^2+(1-\lambda)\theta^2)} \tag{24}$$

Since $m(0)=(4b)^{-1}$; $m(0)<x^*<b(2\lambda(\beta(b+\beta)+\theta^2))^{-1}$ becomes the hypothesys of the theorem by multipliying by $2\lambda\beta$ and by taking as x^* the lowest possible solution to $x=m(x)$. Moreover: $m'(x)=\lambda+(1+2\lambda\beta x)(2\lambda\beta(2b-\lambda(\beta^2+\theta^2))+\lambda(\beta^2+\theta^2))(2b-2\lambda(\beta^2+\theta^2)x)^{-2}$, and so $m'(x)>0$ for any x in the domain of m.

Second step. We show that for x^* given in the previous step: $x(i)<x^*$ and so $x(i)<b(2\lambda(\beta(b+\beta)+\theta^2))^{-1}$ holds for any $i\geq 0$. In effect: for $i=0$ it holds. Let $x(i)<x^*$ hold for i, then $x(i+1)=m(x(i))<m(x^*)=x^*$, where the inequality arises because m is an increasing function.

■