

Laurent sequences, extended Rota algebras and categorical discretization of dynamical systems

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Abstract

We introduce a novel integrability-preserving discretization for a broad class of differential equations with variable coefficients, encompassing both linear and nonlinear cases. The construction is achieved via a categorical approach that enables a unified treatment of continuous and discrete dynamical systems.

Our theoretical framework is grounded on a generalization of G. C. Rota's finite operator calculus, which enables us to extend the theory of basic sequence of polynomials to the setting of Laurent polynomials. Accordingly, we introduce the notion of an *extended Rota algebra*, defined as a Galois differential algebra in which all difference operators act as derivations on the space of Laurent power series with respect to a suitably defined functional product.

The core of our theory relies on the existence of covariant functors between the newly proposed Rota category of Galois differential algebras and suitable categories of abstract dynamical systems.

In this setting, under certain regularity assumptions, a differential equation and its discrete analogues are naturally interpreted as objects of the same category. This perspective enables the construction of a vast class of integrable maps that share with their continuous analogues a wide set of exact solutions, *regular* or *singular* and, in the linear case, the Picard-Vessiot group.

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1. Introduction

1.1. Historical background

The discretization of dynamical systems in an *integrability preserving* way has been the subject of intensive investigation over the past decades [47], [52], owing to its considerable impact across a wide range of fields, including numerical analysis, discrete mathematics, algorithm theory, quantum physics and statistical mechanics. The guiding principle in this line of research is that a meaningful discretization should not merely approximate the continuous dynamics, but rather reproduce its underlying geometric and algebraic structures. In particular, the preservation of first integrals, Poisson or symplectic structures, and Lax representations has emerged as a key requirement for ensuring that the discrete evolution retains the qualitative features of the original system. This viewpoint has led to the development of geometric and structure-preserving integrators, as well as to intrinsically discrete models that are integrable in their own right.

In this framework, discrete differential geometry and its associated algebraic structures have played an increasingly prominent role [1], [5], [6], [14], [32], [23], [11], [15], [24], [31]. A central insight is that integrability can often be reformulated in purely discrete terms, for instance through multidimensional consistency, zero-curvature representations on lattices, or compatibility conditions for discrete connections. These approaches reveal that many integrable hierarchies admit natural discretizations that are not ad hoc approximations, but instead arise from deeper combinatorial or algebraic principles. In particular, the interplay between discrete geometry, Poisson-Lie structures, and algebraic frameworks such as cluster-type transformations has provided a unifying language to describe a broad class of integrable discrete systems.

At the same time, the need for structurally consistent discretizations is strongly motivated by applications in fundamental physics. Lattice formulations of field theories [30], together with modern approaches to quantum mechanics [13] and quantum gravity [3], [19], [44], require discrete counterparts of continuum equations in which key properties, such as gauge invariance, conservation laws, or integrability features, are retained at the discrete level. In such contexts, discretization is not simply a computational device, but a conceptual tool that may encode fundamental aspects of the underlying theory, particularly when a continuum description is expected to emerge only in a suitable limit.

Closely related to these developments is the problem of symmetry-preserving discretization of continuous models, which has also attracted considerable attention (see, e.g., [27] and references therein). Here, the aim is to construct discrete evolutions that exactly inherit the symmetry group of the original equations, thereby ensuring the preservation of invariant manifolds and structural constraints. This perspective is deeply intertwined with integrability, since many integrable systems are characterized by rich symmetry algebras, and their preservation at the discrete level is often essential for maintaining solvability.

Within this general setting, the approach developed in this paper is motivated by the idea that integrability-preserving discretizations can be systematically constructed by merging category theory with the finite operator theory developed by Roman, Rota and collaborators. Our results point towards a framework in which discrete dynamics emerge naturally from structural considerations, allowing one to retain integrability properties in a transparent and conceptually unified way.

The central feature of our discretization scheme is that all operators involved—namely, the delta operators of the Roman-Rota formalism—act as *derivations* on suitable Galois function algebras. Consequently, many structural properties of categorically equivalent equations are preserved. In

particular, a large class of solutions is retained. We emphasize that, to the best of our knowledge, the classes of difference equations arising from our approach are novel, and not related to those obtained with other discretization schemes.

1.2. Statement of the problem

The purpose of this paper is to introduce a unified theoretical framework which allows us to address and solve, in a quite general sense and from a novel perspective, the longstanding problem of the integrability-preserving discretization of a large class of dynamical systems, both *linear and nonlinear*, even in the case of variable coefficients. These coefficients are expressed in terms of either real *analytic* functions (regular case), or Laurent polynomials with *polar singularities* (singular case).

Precisely, the main theorems of this work ensure the integrability-preserving discretization of two classes of differential equations with variable coefficients:

a) Linear ODEs of the form

$$a_N(x) \frac{d^N}{dx^N} y + a_{N-1}(x) \frac{d^{N-1}}{dx^{N-1}} y + \cdots + a_1(x) \frac{d}{dx} y + a_0(x)y + g(x) = 0; \quad (1.1)$$

b) Nonlinear ODEs of the form

$$\frac{d^m}{dx^m} y = c_N(x)y^N + c_{N-1}(x)y^{N-1} + \cdots + c_1(x)y + c_0(x). \quad (1.2)$$

Here $y: \mathbb{R} \rightarrow \mathbb{R}$ is the dependent, scalar variable; $N, m \in \mathbb{N} \setminus \{0\}$, $a_0(x), \dots, a_N(x)$, $g(x)$ and $c_0(x), \dots, c_N(x)$ are real analytic functions, or Laurent polynomials.

For each of the families of equations (1.1) and (1.2), we will provide two distinct discrete analogs, designed to preserve either regular or singular solutions. All resulting equations are defined on a regular mesh of points. In the continuum limit, these difference equations converge to their corresponding differential equations, thereby providing a *discrete approximation* of the original continuous models.

The proposed approach combines the theory of Galois differential algebras and an extension of the theory of finite difference operators in the formulation given by G.-C. Rota [43], and S. Roman [41], with methods from category theory. Within our framework, a continuous dynamical system and its infinitely many discrete counterparts are regarded as distinct representations of an underlying abstract equation, and are naturally interpreted as objects in a category of equations associated with the given system. The morphisms of this category correspond to the various discrete realizations of the abstract equation. Accordingly, the resulting difference equations are said to be *categorically equivalent*.

The theoretical framework developed in the present work allows us to substantially generalize the results of [50], where discretizations were restricted to nonlinear dynamical systems, and with *constant* coefficients only. In particular, our approach provides the first general treatment of discretizations for nonlinear systems with variable coefficients. Moreover, for the novel case of linear dynamical systems, we extend the Frobenius-type theorem presented in [39] – originally formulated for second-order equations – to encompass differential equations of arbitrary order with variable coefficients. Another significant novelty concerns the preservation of *singular solutions* of polar type, not just regular ones as in the previous studies. This property is achieved

through the introduction of suitable *Laurent basic sequences of polynomials*, a concept developed in the present work.

1.3. Extended Rota algebras

We shall introduce the notion of *extended Rota differential algebra* as a pair $(\mathcal{F}_{\mathcal{L}}, \mathcal{Q})$ where $(\mathcal{F}_{\mathcal{L}}, +, \cdot, *)$ is an algebra of formal Laurent series on a mesh $\mathcal{L}$ of points, endowed with an associative and commutative product “ $*$ ”; besides, $\mathcal{Q}$ is a *delta operator* [43] that acts as a derivation with respect to this product. This notion generalizes that of Rota algebra, introduced in [50] and further studied in [39], [40], where standard formal power series were considered.

In particular, this standpoint allows one to consider ordinary difference operators as derivations under suitable function products. The idea of an adapted product of polynomials, ensuring for difference operators the validity of the Leibniz rule, was proposed in the important papers [53], [8].

An analogous polynomial product has appeared, in a different context, in the theory of linear operators acting on polynomial spaces [20]. The collection of all Rota algebras $\mathcal{R}(\mathcal{F})$ represents a subcategory of the category of associative algebras [29].

1.4. A categorical discretization

Given a continuous dynamical system of the form (1.1) or (1.2), we first select a specific difference operator to serve as the discrete derivative, imposing the additional requirement that it be a delta operator $\mathcal{Q}$. The standard forward, backward and symmetric derivatives provide typical examples of discrete delta operators; however, one can construct infinitely many alternatives. Next, we define the Galois differential algebra in which the delta operator $\mathcal{Q}$ acts as a derivation, by constructing the corresponding $*$ -product. The original dynamical system is then defined in the Galois algebra, where it typically assumes the form of a recurrence equation. The resulting discretization procedure is inherently structure-preserving.

Thus, in general, we associate with the family of dynamical systems defined by eq. (1.1) a category of linear equations $\mathcal{K}_{\{a_0, \dots, a_N, g\}}$; similarly, the family (1.2) corresponds to a category of nonlinear equations, denoted by $\mathcal{N}_{\{m; c_0, \dots, c_N\}}$. We also establish the existence of two natural functors $F : \mathcal{R}(\mathcal{F}) \longrightarrow \mathcal{K}_{\{a_0, \dots, a_N, g\}}$ and $G : \mathcal{R}(\mathcal{F}) \longrightarrow \mathcal{N}_{\{m; c_0, \dots, c_N\}}$. Within this algebraic framework, different discretizations of a given continuous dynamical system are represented as morphisms within the same category. In particular, differential equations can be mapped into difference equations *while preserving the underlying differential structure*. Consequently, the integrability properties of a given continuous dynamical system are naturally inherited by its associated discrete counterparts and in particular, solutions (regular or singular) of the continuous system are isomorphically mapped to (regular or singular) solutions of the corresponding discrete equations.

A crucial feature of the integrable maps¹ associated with eqs. (1.1) and (1.2), according to the procedure outlined above, is that they can be interpreted as *discrete analogues of integro-differential equations*. Indeed, these maps are nonlocal recurrences possessing the general form

$$P(\Delta)u(n) = \mathcal{I}[u(n)]. \quad (1.3)$$

¹ Throughout this work, the expressions “map”, “difference equation” or “discrete equation” will be used as synonyms.

Here $P(\Delta)$ is a polynomial in a delta operator (typically representing a discrete derivative of a given order), $u(n)$ is the dependent variable of the map, n is the independent discrete variable, and $\mathcal{I}[u(n)]$ is a functional depending on all values of u ranging from an initial point up to n . This aspect reflects the inherent nonlocality of the equation (1.3), which, in turn, originates from the intrinsic nonlocal nature of the $*$ -product. This property can be interpreted in light of a fundamental no-go theorem proved in [21], which states that it is impossible to define a field theory on an infinite lattice endowed with a nontrivial product rule that simultaneously satisfies the Leibniz rule, translational invariance, and locality. Consequently, the structure-preserving discretization scheme proposed in this work - based on enforcing the Leibniz rule for difference operators - is intrinsically linked to the nonlocal character of the resulting integrable maps.

In this scenario, the most elementary instance arises in the discretization of linear differential equations with constant coefficients. Here, the resulting discrete equations are standard, i.e., *local* difference equations, depending on a finite number of lattice sites determined solely by the order of the discrete derivative. Conversely, for dynamical systems with variable coefficients or nonlinear dependencies, the nonlocal aspects of the theory become unavoidable and explicitly manifest.

1.5. Main results

The general discretization theory we propose relies on the following new results.

I) We prove that the integrable maps representing discrete versions of the dynamical systems (1.1) or (1.2) satisfy the following property: if $\sum_{k=0}^{\infty} \zeta_k x^k$ (or $\sum_{k=1}^s \frac{\zeta_k}{x^k}$) is an analytic (singular) solution of an equation of the form (1.1) (Theorem 2 and Theorem 5) or (1.2) (Theorem 3 and Theorem 6), then $\sum_{k=0}^n \zeta_k p_k(n)$ (or $\sum_{k=-s}^{-1} \zeta_k p_k(n)$), where $p_k(n)$ are the Laurent basic polynomials associated with the chosen delta operator, for each n is a solution of the corresponding discrete map. In other words, under mild assumptions, *analytic (singular) solutions of the continuous systems (1.1) or (1.2) are transformed into regular (singular) exact solutions of the associated integrable maps (1.3)*. Besides, the discrete maps (eqs. (5.5), (5.21) for the regular case and eqs. (8.4) and (8.14) for the singular case) associated with the original continuous systems are derived, through our categorical approach, in a constructive and explicit way.

II) A Galois theory for the integrable maps considered in this work is developed, starting from the novel notion of Rota category. Within this setting, we also establish the existence of an isomorphism between the *Picard–Vessiot group* of a linear differential equation with constant coefficients and that of the corresponding difference equation obtained through our categorical approach.

In the continuum limit, when the lattice step h goes to zero, the site n goes to infinite and the product nh remains constant, these new integrable maps reduce to the original differential equations (as we will show in several examples).

1.6. Discussion

We recall that the time scale calculus, initiated by Hilger in [17] represents a well-established and intriguing approach aimed at unifying discrete and continuous calculus. This unification is achieved by defining a general domain that can be continuous, discrete, or mixed (corresponding to time scales or, more generally, measure chains) and by introducing appropriate jump operators

on this domain. Over the past two decades, this theory has been extensively investigated by many authors (see, e.g., the monograph [7]), resulting in a substantial body of results and applications.

However, the theoretical framework proposed in this article is independent of, and fundamentally distinct from, time scale calculus. Instead, our approach is based on the guiding principle of preserving the Leibniz rule in discrete calculus – through the notion of Rota’s Galois algebra – and on the use of an adapted point mesh for discretization, defined by the zeros of a set of basic polynomials.

We emphasize that our categorical approach, in principle, allows for the reformulation of the main theorems of this work for *arbitrary discrete derivatives*. This can be done by analogy with the case of the forward difference operator, which we have analyzed explicitly due to its prominence in applications. In particular, the extension of our theory to discretizations based on the backward difference operator is entirely straightforward.

2. Algebraic preliminaries. Delta operators

The theory of delta operators, introduced in [43] as a foundational framework for combinatorics, has been extensively developed in the literature (see also [42], [41]). Let $\mathcal{P}$ denote the space of polynomials in a variable $x \in \mathbb{K}$, where $\mathbb{K}$ is a field of characteristic zero and let $\mathbb{N}$ denote the set of non-negative integers.

Let T denote the shift operator, whose action on a function f is given by $Tf(x) = f(x+1)$.

Definition 1. An operator S is said to be *shift-invariant* if it commutes with the shift operator T . A shift-invariant operator $\mathcal{Q}$ is a *delta operator* if $\mathcal{Q}x = \text{const} \neq 0$.

We deduce immediately the following property.

Corollary 1. For every constant $c \in \mathbb{R}$, $\mathcal{Q}c = 0$.

The most common examples of delta operators are provided by the derivative D , the forward discrete derivative $\Delta = T - 1$, the backward derivative $\nabla := 1 - T^{-1}$ and the symmetric operator $\Delta^s = \frac{T - T^{-1}}{2}$.

Given a delta operator $\mathcal{Q}$, a polynomial sequence $\{p_k(x)\}_{k \in \mathbb{N}}$ is said to be the sequence of *basic polynomials* for $\mathcal{Q}$ if the following conditions are satisfied: (1) $p_0(x) = 1$, (2) $p_k(0) = 0$ for all $k > 0$; (3) $\mathcal{Q}p_k(x) = kp_{k-1}(x)$.

Notice that, for a given delta operator $\mathcal{Q}$ the sequence of associated basic polynomials is unique.

2.1. Formal groups and delta operators

In this section, as a new result, we will demonstrate that the theory of delta operators admits a natural interpretation within the framework of formal group theory [37]. Over the past decades, formal group theory has been extensively investigated due to its central role in algebraic topology [9], [25], cobordism theory [36], analytic number theory [18], [49], [51], and related fields.

Following [16], [9], we remind that a commutative one-dimensional formal group law over a commutative, unital ring A is a formal power series $\Phi(x, y) \in A[[x, y]]$ such that

$$(1) \quad \Phi(x, 0) = \Phi(0, x) = x,$$

$$(2) \quad \Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z)).$$

When $\Phi(x, y) = \Phi(y, x)$, the formal group law is said to be commutative.

Let us consider the polynomial ring $\mathbb{Q}[c_1, c_2, \dots]$ and the formal group logarithm $F(u) = u + c_1 \frac{u^2}{2} + c_2 \frac{u^3}{3} + \dots$. Let $G(v)$ be its inverse series, i.e., the formal group exponential

$$G(v) = v - c_1 \frac{v^2}{2} + (3c_1^2 - 2c_2) \frac{v^3}{6} + \dots \quad (2.1)$$

so that $F(G(v)) = v$. The formal group law associated with these series, known as the *Lazard Universal Formal Group*, is given by

$$\Phi(u_1, u_2) = G(F(u_1) + F(u_2)).$$

It is defined over the Lazard ring L , i.e. the subring of $\mathbb{Q}[c_1, c_2, \dots]$ generated by the coefficients of the power series $G(F(u_1) + F(u_2))$. For any commutative one-dimensional formal group law over any ring A , there exists a unique homomorphism $L \rightarrow A$ under which the Lazard group law is mapped into the given group law (*universal property* of the Lazard group).

In algebraic topology, delta operators are related to Thom classes and complex cobordism theory [38]. In order to relate formal groups to difference delta operators, we first describe a simple technique for generating a family of difference delta operators. More precisely, we introduce the difference operators

$$\Delta_p = \frac{1}{h} \sum_{k=l}^m \alpha_k T^k, \quad l, m \in \mathbb{Z}, \quad l < m, \quad m - l = p, \quad \sum_{k=l}^m \alpha_k = 0, \quad \sum_{k=l}^m k \alpha_k = c \quad (2.2)$$

where h can be interpreted as a mesh spacing and α_k are constants with $\alpha_m \neq 0$, $\alpha_l \neq 0$. We choose $c = 1$, to reproduce possibly the derivative D in the continuum limit. A difference operator of the form (2.2) is said to be a delta operator of order p , if it approximates the continuous derivative up to terms of order h^p [26]. Apart from the two constraints in relation (2.2), one can arbitrarily choose $m - l - 1$ more linear conditions to fix all constants α_k . Consequently, by means of the representation $T \sim e^v$ (the “symbol” of the translation operator), each delta operator of the family (2.2) is associated with a group exponential (2.1) and then with a realization of the one-dimensional Lazard universal formal group law. Conversely, with the identification $v \sim D$, there corresponds to each formal group exponential a delta operator (see also [37], [51]).

3. Laurent basic sequences of polynomials

3.1. Definitions

In this section, we develop a novel approach to finite operator theory, by generalizing the classical notion of basic polynomials [43,41] to the case of Laurent polynomials. In the literature, several extensions of umbral calculus (the old denomination of finite operator theory) are available. In particular, in [41], nonclassical umbral calculi are also discussed. Also, in [28] and [4], several extensions to generalized finite differences calculi are reviewed. However, the notions of Laurent basic sequences, dual sequences, twisted Rota algebras, etc. to the best of our knowledge were not proposed before in the literature.

Definition 2. Given a delta operator $\mathcal{Q}$, a Laurent basic sequence is a sequence of rational functions $\{p_k(x)\}_{k \in \mathbb{Z}}$, which reduce to polynomials for $k \in \mathbb{N}$, defined by the following properties:

- 1) $p_0(x) = 1$
- 2) $p_k(0) = 0$ $k \in \mathbb{N} \setminus \{0\}$
- 3) $\mathcal{Q}p_k(x) = kp_{k-1}(x)$, $k \in \mathbb{Z}$.

Thus, a Laurent basic sequence for a delta operator $\mathcal{Q}$ is a sequence of basic polynomials extended for $k < 0$ to a sequence of rational polynomials satisfying a natural “derivation” property.

Let us denote by $\mathcal{P}^L$ the space of Laurent polynomials in a variable $x \in \mathbb{K}$. Let $\mathcal{Q}$ be a delta operator acting on $\mathcal{P}^L$, let $\mathcal{D}$ denote the set of all delta operators, and $\{p_k(x)\}_{k \in \mathbb{Z}}$ be the basic sequence of Laurent polynomials of order k uniquely associated with $\mathcal{Q}$. Let $\mathcal{H}$ denote the algebra of formal Laurent series in x . Since the polynomials $\{p_k(x)\}_{k \in \mathbb{Z}}$ for any choice of $\mathcal{Q}$ form a basis of $\mathcal{H}$, any $f \in \mathcal{H}$ can be expanded as a formal Laurent series of the form $f(x) = \sum_{k=-\infty}^{\infty} a_k p_k(x)$. In this way, we can extend the action of delta operators on functions. Let $\mathcal{L}$ be a set of points on the real line, isomorphic to $\mathbb{Z}$. Denote by $\mathcal{H}_{\mathcal{L}}$ the vector space of the formal power series defined on $\mathcal{L}$. The space $\mathcal{H}$ (and, consequently, $\mathcal{H}_{\mathcal{L}}$) can be endowed with an algebra structure by introducing a new, suitable product.

Definition 3. Given a delta operator $\mathcal{Q}$ and the associated Laurent basic sequence $\{p_k(x)\}_{k \in \mathbb{Z}}$, the $*_{\mathcal{Q}}$ product is defined via the relation

$$p_k(x) *_{\mathcal{Q}} p_j(x) := p_{k+j}(x), \quad k, j \in \mathbb{Z}. \quad (3.1)$$

Remark 1. An analogous product, for the case of the standard basic polynomial sequence associated with the forward difference operator Δ , was proposed in [53] and in [20]. In what follows, we will use the symbol $*$ whenever the choice of the delta operator $\mathcal{Q}$ will be obvious. It can be shown that the space $(\mathcal{H}, +, \cdot, *_{\mathcal{Q}})$, equipped with the usual operations of series addition, scalar multiplication, and the $*$ -product (3.1), forms an associative algebra.

3.2. The natural Laurent sequence

The simplest example of a Laurent basic sequence is given by

$$p_k(x) := x^k, \quad k \in \mathbb{Z} \quad (3.2)$$

for the delta operator $\mathcal{Q} = D$. We shall call it the *natural Laurent basic sequence*. In order to construct other nontrivial examples, we will introduce the notion of dual sequences of basic polynomials.

3.3. Dual sequences

We denote by $p_k^+(x) := x(x-1) \cdots (x-k+1)$, $k \in \mathbb{N}$, the sequence of basic polynomials for the operator $\Delta = T - 1$. Similarly, $p_k^-(x) := x(x+1) \cdots (x+k-1)$ is the basic sequence for $\nabla = 1 - T^{-1}$.

We shall introduce sequences of *rational basic polynomials*. To this aim, let us observe that

$$\Delta \frac{1}{p_k^-(x)} = \frac{1}{p_k^-(x+1)} - \frac{1}{p_k^-(x)} = -\frac{k}{p_{k+1}^-(x)}.$$

Thus, if we define:

$$q_k^+(x) := \frac{1}{p_k^-(x)}, \quad k \in \mathbb{N} \setminus \{0\}, \quad q_0^+ = 1 \quad (3.3)$$

we obtain

$$\Delta q_k^+(x) = -k q_{k+1}^+(x). \quad (3.4)$$

Analogously, for the operator ∇ we have

$$\nabla \frac{1}{p_k^+(x)} = \frac{1}{p_k^+(x)} - \frac{1}{p_k^+(x-1)} = -\frac{k}{p_{k+1}^+(x)}.$$

Consequently, denoting by

$$q_k^-(x) = \frac{1}{p_k^+(x)}, \quad k \in \mathbb{N} \setminus \{0\}, \quad q_0^- = 1 \quad (3.5)$$

the rational functions associated with the basic polynomials for Δ , we obtain that

$$\nabla q_k^-(x) = -k q_{k+1}^-(x). \quad (3.6)$$

Definition 4. The sequences $\{p_k^+(x)\}_{k \in \mathbb{N}}$ and $\{q_k^+(x)\}_{k \in \mathbb{N}}$ are said to be dual for the operator Δ ; similarly, $\{p_k^-(x)\}_{k \in \mathbb{N}}$ and $\{q_k^-(x)\}_{k \in \mathbb{N}}$ are dual sequences for ∇ .

As a consequence of the previous discussion, we have proved the following

Proposition 1. *The sequence*

$$p_k(x) := \begin{cases} x(x-1) \cdots (x-k+1) & k \in \mathbb{N} \setminus \{0\}, \\ \frac{1}{x(x+1) \cdots (x-k-1)} & k \in \mathbb{Z}^-, \\ 1 & k = 0, \end{cases} \quad (3.7)$$

is a Laurent basic sequence for the delta operator Δ .

Open Problem. It would be very interesting to ascertain whether there exist infinitely many dual sequences.

3.4. The Abel-Laurent basic sequence

Consider the Abel delta operator:

$$\mathcal{Q}^A = DT^a \quad (3.8)$$

with $D = \frac{d}{dx}$, $T^a f(x) = f(x + a)$, $a > 0$. The Laurent basic sequence for the Abel operator is given by

$$p_k^A(x) := \begin{cases} x(x - ak)^{k-1} & k \in \mathbb{N} \setminus \{0\}, \\ (x - ak)^k & k \in \mathbb{Z}^- \cup \{0\}. \end{cases} \quad (3.9)$$

3.5. The symmetric Laurent basic sequence

The symmetric delta operator $\Delta^s := \frac{T - T^{-1}}{2}$ is also relevant in several applicative contexts. The basic Laurent polynomials associated with Δ^s are

$$p_k^s(x) := \begin{cases} x \prod_{j=1}^{k-1} (x + k - 2j) & k \in \mathbb{N} \setminus \{0\}, \\ 1 & k = 0, \\ \frac{1}{\prod_{j=0}^{-k-1} (x - k - 1 - 2j)} & k \in \mathbb{Z}^-. \end{cases} \quad (3.10)$$

3.6. Lattices and Laurent basic sequences

Consider the uniform lattice in $\mathbb{R}^+ \cup \{0\}$ with sites at the integers and the Laurent sequence (3.7). Then the values of $p_k(x)$ at the sites $x = n \in \mathbb{N}$, with $k \in \mathbb{Z}$, explicitly read

$$p_k(n) = \begin{cases} \prod_{j=0}^{k-1} (n - j) = \frac{n!}{(n - k)!} & k \geq 0, \\ \frac{1}{\prod_{j=0}^{-k-1} (n + j)} = \frac{(n - 1)!}{(n - 1 - k)!} & k < 0. \end{cases}$$

Hereafter, we shall assume that negative factorials in the denominator make the fraction vanish.

4. Category theory and dynamical systems

4.1. Twisted Rota algebras

The notion of Rota algebra has been introduced in [50], as the natural Galois differential algebra where the discretization procedure is carried out. In this section, this notion will be generalized to the case of Laurent formal power series. We shall also develop a categorical approach.

Definition 5. A twisted Rota differential algebra is a Galois differential algebra $(\mathcal{H}, \mathcal{Q})$, where $(\mathcal{H}, +, \cdot, *_\mathcal{Q})$ is an associative algebra of Laurent formal power series, the product $*_\mathcal{Q}$ is the composition law defined by (3.1), and $\mathcal{Q}$ is a delta operator acting as a derivation on $\mathcal{H}$:

$$i) \quad \mathcal{Q}(a + b) = \mathcal{Q}(a) + \mathcal{Q}(b), \quad \mathcal{Q}(\lambda a) = \lambda \mathcal{Q}(a), \quad \lambda \in \mathbb{K}, \quad (4.1)$$

$$ii) \quad \mathcal{Q}(a * b) = \mathcal{Q}(a) * b + a * \mathcal{Q}(b). \quad (4.2)$$

4.2. The generalized Rota category

Definition 6. The generalized Rota category, denoted by $\mathcal{R}(\mathcal{H})$, is the collection of all Rota algebras $(\mathcal{H}, \mathcal{Q})$, with morphisms defined by

$$\rho_{\mathcal{Q}, \mathcal{Q}'} : (\mathcal{H}, +, \cdot, *_{\mathcal{Q}}) \longrightarrow (\mathcal{H}, +, \cdot, *_{\mathcal{Q}'}) \quad (4.3)$$

which are closed under composition.

The action of the morphism $\rho_{\mathcal{Q}, \mathcal{Q}'}$ on formal power series is defined by

$$\sum_k \zeta_k p_k(x) \longrightarrow \sum_m \zeta_m q_m(x), \quad (4.4)$$

where $\{p_k(x)\}_{k \in \mathbb{N}}$ and $\{q_m(x)\}_{m \in \mathbb{N}}$ are the basic sequences associated with $\mathcal{Q}$ and $\mathcal{Q}'$ respectively. The property of closure under composition is trivial.

4.3. New categories for dynamical systems

We introduce two new definitions of categories for differential equations of the form (1.1) and (1.2). For $N \in \mathbb{N} \setminus \{0\}$, define $u^{*N} := \underbrace{u * \dots * u}_{N\text{-times}}$.

Definition 7. For any choice of the set of functions $\{a_0(x), \dots, a_N(x), g(x)\}$, the category $\mathcal{K}_{\{a_0, \dots, a_N, g\}}$ of linear dynamical systems of order N is the collection of all equations of the form

$$\text{lin}(\mathcal{Q}, u, *_{\mathcal{Q}}) := a_N(x) * \mathcal{Q}^N y + a_{N-1}(x) * \mathcal{Q}^{(N-1)} y + \dots + a_1(x) * \mathcal{Q} y + a_0(x) * y + g(x) = 0. \quad (4.5)$$

The set of correspondences

$$\lambda_{\mathcal{Q}, \mathcal{Q}'} : \mathcal{K}_{\{a_0, \dots, a_N, g\}} \longrightarrow \mathcal{K}_{\{a_0, \dots, a_N, g\}}, \quad (4.6)$$

$$\text{lin}(\mathcal{Q}, y, *) \longrightarrow \text{lin}(\mathcal{Q}', y, *'), \quad (4.7)$$

defines the class of morphisms of the category.

In other words, given a differential equation of the form (1.1), one may define a category whose objects are the equation (1.1) together with all infinitely many of its discretizations, obtained by varying $\mathcal{Q} \in \mathcal{D}$ and constructing the corresponding morphisms (4.7).

Definition 8. For any choice of the set of functions $\{c_0(x), \dots, c_N(x)\}$, the category $\mathcal{N}_{\{m; c_0, \dots, c_N\}}$ of nonlinear dynamical systems of order $m \in \mathbb{N}$ consists of all equations of the form

$$eq(\mathcal{Q}, y, *_{\mathcal{Q}}) := \mathcal{Q}^m y - c_N(x) * y^{*N} - c_{N-1}(x) * y^{*(N-1)} - \dots - c_1(x) * y - c_0(x) = 0. \quad (4.8)$$

The set of correspondences

$$\nu_{\mathcal{Q}, \mathcal{Q}'} : \mathcal{N}_{\{m; c_0, \dots, c_N\}} \longrightarrow \mathcal{N}_{\{m; c'_0, \dots, c'_N\}}, \quad (4.9)$$

$$eq(\mathcal{Q}, y, *) \longrightarrow eq(\mathcal{Q}', y, *'), \quad (4.10)$$

defines the class of morphisms of the category.

The closure of the morphisms $\lambda_{\mathcal{Q}, \mathcal{Q}'}$ and $\nu_{\mathcal{Q}, \mathcal{Q}'}$ under composition is easily verified.

Definition 9. Two objects in the category $\mathcal{K}_{\{a_0, \dots, a_N, k\}}$ (or in $\mathcal{N}_{\{m; c_0, \dots, c_N\}}$) are said to represent two categorically equivalent equations. Alternatively, two equations are categorically equivalent if there exists a morphism of categories $\lambda_{\mathcal{Q}, \mathcal{Q}'}$ (or $\nu_{\mathcal{Q}, \mathcal{Q}'}$) mapping one equation to the other.

A notable property of the categories introduced above is their hierarchical structure via subcategories, with inclusions determined by the natural identifications:

$$\begin{aligned} \mathcal{K}_{\{a_0, \dots, a_k\}} &:= \mathcal{K}_{\{a_0, \dots, a_k, \underbrace{0, \dots, 0}_{(N-k)\text{-times}}\}}, \\ \mathcal{N}_{\{m; c_0, \dots, c_k\}} &:= \mathcal{N}_{\{m; c_0, \dots, c_k, \underbrace{0, \dots, 0}_{(N-k)\text{-times}}\}}, \end{aligned}$$

for $k < N$.

Lemma 1. For any fixed $N \in \mathbb{N}$, and for any choice of the polynomials $\{a_0, \dots, a_N, g\}$, there exists a filtration of subcategories, given by the finite sequences

$$\mathcal{K}_{\{a_0\}} \subset \mathcal{K}_{\{a_0, a_1\}} \subset \dots \subset \mathcal{K}_{\{a_0, \dots, a_k\}} \subset \dots \subset \mathcal{K}_{\{a_0, \dots, a_N, g\}} \quad (4.11)$$

and

$$\mathcal{N}_{\{m; c_0\}} \subset \mathcal{N}_{\{m; c_0, c_1\}} \subset \dots \subset \mathcal{N}_{\{m; c_0, \dots, c_k\}} \subset \dots \subset \mathcal{N}_{\{m; c_0, \dots, c_N\}}, \quad (4.12)$$

respectively.

Proof. For each of the subsets $\mathcal{K}_{\{a_0, \dots, a_k\}}$, the restriction of the morphisms $\lambda_{\mathcal{Q}, \mathcal{Q}'}$ over these subsets still preserves the closure under composition and defines morphisms of subcategories. The same argument holds for the sequence related to $\mathcal{N}_{\{m; c_0, \dots, c_N\}}$. $\square$

As a consequence of the previous construction, one can define functors between the category of Rota differential algebras and the categories of abstract dynamical systems defined above.

Theorem 1. For any choice of the functions $\{a_0(x), \dots, a_N(x), g(x)\}$, the application

$$\begin{aligned} F : \mathcal{R}(\mathcal{H}) &\longrightarrow \mathcal{K}_{\{a_0, \dots, a_N, g\}}, \\ (\mathcal{H}, +, \cdot, *_Q) &\longrightarrow \text{lin}(\mathcal{Q}, y, *), \\ \rho_{Q, Q'} &\longrightarrow \lambda_{Q, Q'}, \end{aligned} \quad (4.13)$$

is a covariant functor.

Proof. A direct verification shows that F preserves the composition of morphisms:

$$F(\rho_{Q'', Q'} \circ \rho_{Q', Q}) = F(\rho_{Q'', Q'}) \circ F(\rho_{Q', Q}).$$

If we denote by $id_Q := \rho_{Q, Q}$ the identity morphism, we also have

$$F(id_Q(\mathcal{A})) = id_Q(F(\mathcal{A})),$$

where $\mathcal{A} \in \mathcal{R}(\mathcal{F})$. $\square$

In the same manner one can prove that the application

$$\begin{aligned} G : \mathcal{R}(\mathcal{H}) &\longrightarrow \mathcal{N}_{\{m; c_0, \dots, c_N\}}, \\ (\mathcal{H}, +, \cdot, *_Q) &\longrightarrow \text{eq}(\mathcal{Q}, y, *), \\ \rho_{Q, Q'} &\longrightarrow \nu_{Q, Q'}, \end{aligned} \quad (4.14)$$

is a covariant functor.

The functors (4.13) and (4.14) capture the essential aspects of the proposed discretization, and provide *functorial Rota correspondences* between continuous and discrete dynamical systems. Moreover, these functors can be naturally extended to the subcategories introduced above.

Corollary 2. The restriction of the functors F and G to the subcategories defined by the filtrations (4.11) and (4.12) respectively remain covariant functors.

5. Main theorems for the regular case

In this section, we present the main results of the article concerning the discretization of ODEs admitting analytical solutions. Precisely, using the new categorical framework developed above, we construct a class of integrable maps associated with the dynamical systems (1.1) and (1.2), respectively. These maps are categorically equivalent to the original continuous systems and represent them in the Galois algebra corresponding to the forward difference operator. The discretization is performed on an *equally spaced* mesh.

Theorem 2 establishes a solution to the problem of the integrability-preserving discretization of linear n -th order ODEs with analytic variable coefficients. Theorem 3 solves the analogous problem for the case of nonlinear first-order ODEs.

Remark 2. In both Theorems 2 and 3, we will define our maps in the Rota algebra $(\mathcal{F}, \mathcal{Q})$, where $\mathcal{F}$ is the space of formal power series, and $p_k(x)_{k \in \mathbb{N}}$ are basic sequences for the delta operator $Q = \Delta$. In section 8, when dealing with singular solutions, this picture will be extended to the case of Laurent formal series, and Laurent basic polynomials.

5.1. Integrable maps from linear ODEs

Theorem 2. Consider the differential equation

$$\text{lin}(\partial, y) := a_N(x) \frac{d^N}{dx^N} y + a_{N-1}(x) \frac{d^{N-1}}{dx^{N-1}} y + \dots + a_1(x) \frac{d}{dx} y + a_0(x)y + g(x) = 0, \quad (5.1)$$

where $a_i(x)$, $i = 0, \dots, N$ and $g(x)$ are real analytic functions at $x = 0$ (with $a_N(0) \neq 0$):

$$a_i(x) = \sum_{k_i=0}^{\infty} \frac{\alpha_{ik_i}}{k_i!} x^{k_i}, \quad g(x) = \sum_{\ell=0}^{\infty} \frac{\gamma_{\ell}}{\ell!} x^{\ell}. \quad (5.2)$$

Assume that

$$y(x) = \sum_{k=0}^{\infty} \zeta_k x^k \quad (5.3)$$

be a real solution of (5.1), analytic at $x = 0$. Then the difference equation

$$\sum_{i=0}^N \sum_{k_i=0}^n \frac{n! \alpha_{ik_i}}{(n-k_i)! k_i!} \Delta^i u_{n-k_i} + \sum_{\ell=0}^n \frac{n! \gamma_{\ell}}{\ell! (n-\ell)!} = 0, \quad (5.4)$$

or, in the equivalent form, the equation

$$\sum_{i=0}^N \sum_{k_i=0}^n \sum_{k=0}^i (-1)^{i-k} \binom{n}{k_i} \binom{i}{k} \alpha_{ik_i} u_{n-k_i+k} + \sum_{\ell=0}^n \frac{n! \gamma_{\ell}}{\ell! (n-\ell)!} = 0, \quad (5.5)$$

defined on a regular set of points $\mathcal{L} \subset \mathbb{R}$ indexed by the variable $n \in \mathbb{N}$, admits as a solution the series

$$u_n = \sum_{k=0}^n \zeta_k \frac{n!}{(n-k)!}. \quad (5.6)$$

Proof. We start by applying the morphism (4.4) to the series (5.3); we obtain the transformation

$$\sum_{k=0}^{\infty} \zeta_k x^k \longrightarrow \sum_{k=0}^{\infty} \zeta_k p_k(x). \quad (5.7)$$

The discrete transform (5.7) is finite whenever $x \in \mathcal{L}$, provided that the points of the mesh $\mathcal{L}$ coincide with the zeros of the basic sequence of polynomials $\{p_k(x)\}_{k \in \mathbb{N}}$. In the following, we

choose our basic sequence to be the set of lower factorial polynomials. Consequently, $\mathcal{L}$ denotes an equally spaced set of points on the real nonnegative axis, indexed by $n \in \mathbb{N}$. Thus, we have

$$p_k(n) = \begin{cases} 0, & \text{if } n < k, \\ \frac{n!}{(n-k)!}, & \text{if } n \geq k. \end{cases} \quad (5.8)$$

Let us introduce the function $u : \mathbb{N} \rightarrow \mathbb{R}$ defined by

$$u_n = \sum_{k=0}^n \frac{n!}{(n-k)!} \zeta_k. \quad (5.9)$$

For its *inverse interpolating transform*, we have

$$\zeta_k = \sum_{l=0}^k \frac{(-1)^{k-l}}{l!(k-l)!} u_l. \quad (5.10)$$

For the purpose of constructing the equation categorically equivalent to eq. (5.1) in $\mathcal{K}_{\{a_0, \dots, a_N, g\}}$, we represent the product $x^r \frac{d^s}{dx^s} y(x)$ on the set $\mathcal{L}$ by the action of the morphism $\rho_{\partial, \Delta} : (\mathcal{F}, +, \cdot) \rightarrow (\mathcal{F}, +, *_\Delta)$ between the standard (Rota) algebra of power series endowed with the pointwise product and that equipped with the forward difference operator. This morphism determines the general (umbral) correspondences

$$x^r \frac{d^s}{dx^s} y(x) \longrightarrow \frac{n!}{(n-r)!} \Delta^s u_{n-r}, \quad r, s \in \mathbb{N} \quad (5.11)$$

which can be specialized to the following cases:

$$y(x) \longrightarrow u_n, \quad (5.12)$$

$$x^r y(x) \longrightarrow \frac{n!}{(n-r)!} u_{n-r}, \quad (5.13)$$

$$\frac{d^s}{dx^s} y(x) \longrightarrow \Delta^s u_n. \quad (5.14)$$

To prove these relations, observe that

$$x^r \frac{d^s}{dx^s} y(x) \longrightarrow p_r(n) * \sum_{k=0}^{n+s} \frac{k!}{(k-s)!} \zeta_k p_{k-s}(n) = \sum_{k=0}^{n+s-r} \frac{k!}{(k-s)!} \zeta_k p_{k-s+r}(n). \quad (5.15)$$

Then, we deduce that

$$\begin{aligned}
\sum_{k=0}^{n+s-r} \frac{k!}{(k-s)!} \zeta_k p_{k-s+r}(n) &= \sum_{k=0}^{n+s-r} \frac{n! k!}{(k-s)!(n-k+s-r)!} \zeta_k \\
&= \sum_{k=0}^{n+s-r} \sum_{j=0}^k \frac{n! k!}{(k-s)!(n-r-(k-s))!} \frac{(-1)^{k-j}}{j!(k-j)!} u_j \\
&= \frac{n!}{(n-r)!} \sum_{j=0}^{n+s-r} \left(\sum_{k=j}^{n+s-r} (-1)^{k-j} \binom{n-r}{k-s} \binom{k}{j} \right) u_j.
\end{aligned} \tag{5.16}$$

It can be shown that (see the Appendix A):

$$\sum_{k=j}^{n+s-r} (-1)^{k-j} \binom{n-r}{k-s} \binom{k}{j} = \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} \delta_{n-r+i,j}. \tag{5.17}$$

Thus,

$$\begin{aligned}
x^r \frac{d^s}{dx^s} y(x) &\longrightarrow \frac{n!}{(n-r)!} \sum_{j=0}^{n+s-r} \left(\sum_{i=0}^s (-1)^{s-i} \binom{s}{i} \delta_{n-r+i,j} \right) u_j \\
&= \frac{n!}{(n-r)!} \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} u_{n-r+i} = \frac{n!}{(n-r)!} \Delta^s u_{n-r}.
\end{aligned} \tag{5.18}$$

By using the previous formulas, we deduce that eq. (5.4) is categorically equivalent to eq. (5.1). Both in turn are representations of the abstract equation (4.5).

To conclude the proof, observe that, by means of the action of the functor F , any formal power series y of eq. (5.1) is carried into a solution u_n of the equation

$$\text{lin}(\mathcal{Q}, u, *_{\mathcal{Q}}) = \lambda_{\partial, \Delta}(\text{lin}(\partial, y, \cdot)).$$

In addition, on the mesh $\mathcal{L}$, the sum $\sum_k \zeta_k p_k(n)$ truncates and converts into the finite sum (5.6). $\square$

Remark 3. Theorem 2 can be considerably generalized by choosing an arbitrary different delta operator $\mathcal{Q}$ and the corresponding basic sequence $\{p_k(x)\}_{k \in \mathbb{N}}$. In this case, we define the lattice $\mathcal{L}$ to be the union set of all zeroes of the polynomials of the sequence $\{p_k(x)\}_{k \in \mathbb{N}}$. The action of the morphism $\lambda_{\partial, \mathcal{Q}}$ will provide a different, categorically equivalent representation of eq. (5.1) in $\mathcal{K}_{\{a_0, \dots, a_N, g\}}$.

5.2. Integrable maps from nonlinear ODEs

Theorem 3. Consider a dynamical system of the form

$$\text{eq}(\partial, y) := \frac{d^m}{dx^m} y - \sum_{r=0}^N c_r(x) y^r = 0, \tag{5.19}$$

where $N, m \in \mathbb{N} \setminus \{0\}$, $c_l(x) = \sum_{k_l=0}^{\infty} \beta_{lk_l} x^{k_l}$ are real analytic at $x = 0$, with $\beta_{lk_l} \in \mathbb{R}$, $l = 0, \dots, N$. Assume that

$$y(x) = \sum_{k=0}^{\infty} \zeta_k x^k \quad (5.20)$$

be a real solution of (5.19), analytic at $x = 0$. Then the difference equation

$$\sum_{i=0}^m (-1)^{m-i} \binom{m}{i} u_{n+i} - \sum_{r=0}^N \sum_{s_r, j_1, \dots, j_r=0}^n \beta_{rs_r} \frac{(1-r)^{n-s_r-j_1-\dots-j_r} n!}{(n-s_r-j_1-\dots-j_r)!} \prod_{l=1}^r \frac{u_{j_l}}{j_l!} = 0, \quad (5.21)$$

which represents eq. (5.19) on a regular set of points $\mathcal{L} \subset \mathbb{R}$ indexed by the variable $n \in \mathbb{N}$, admits as a solution the series

$$u_n = \sum_{k=0}^n \zeta_k \frac{n!}{(n-k)!}. \quad (5.22)$$

Proof. We consider the correspondence (5.14)

$$y^{(r)}(x) \rightarrow \Delta^r u_n = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} u_{n+j}, \quad (5.23)$$

where u_n is related to its inverse transform by the reciprocal eqs. (5.9) and (5.10). We will also determine how the morphism $\rho_{\partial, \Delta} : (\mathcal{F}, +, \cdot) \rightarrow (\mathcal{F}, +, *_\Delta)$ acts on the powers of $y(x)$. Let $\mathbf{k}, \mathbf{j}$ be the multiindices: $\mathbf{k} = (k_1, \dots, k_r)$, $\mathbf{j} = (j_1, \dots, j_r)$, $k = \sum_{i=1}^r k_i$, $j = \sum_{i=1}^r j_i$ (see the Appendix B for a detailed proof of eqs. (5.24) and (5.25)):

$$\begin{aligned} y(x)^r &= \left(\sum_{m=0}^{\infty} \zeta_m x^m \right)^r = \sum_{\mathbf{k}=0}^{\infty} \zeta_{k_1} \cdots \zeta_{k_r} x^k \rightarrow \sum_{\mathbf{k}=0}^{\infty} \zeta_{k_1} \cdots \zeta_{k_r} p_{\mathbf{k}}(x) \\ &\rightarrow \sum_{\mathbf{k}=0}^{\infty} \frac{n!}{(n-k)!} \prod_{i=1}^r \left(\sum_{j_i=0}^{k_i} \frac{(-1)^{k_i-j_i}}{j_i! (k_i-j_i)!} u_{j_i} \right) = \sum_{\mathbf{j}=0, j \leq n} \frac{(1-r)^{n-j} n!}{(n-j)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \end{aligned} \quad (5.24)$$

As usual, factorials of negative numbers in the denominators yield vanishing fractions. Then, the last series in eq. (5.24) have a finite number of terms only.

Also, we observe that the products of the form $x^s y^r$ are mapped into the terms

$$\begin{aligned} x^s y^r &= x^s \left(\sum_{m=0}^{\infty} \zeta_m x^m \right)^r = \sum_{\mathbf{k}=0}^{\infty} \zeta_{k_1} \cdots \zeta_{k_r} x^{k+s} \sum_{\mathbf{k}=0}^{\infty} \zeta_{k_1} \cdots \zeta_{k_r} p_{\mathbf{k}+s}(x) \\ &\rightarrow \sum_{\mathbf{k}=0}^{\infty} \frac{n!}{(n-s-k)!} \prod_{i=1}^r \left(\sum_{j_i=0}^{k_i} \frac{(-1)^{k_i-j_i}}{j_i! (k_i-j_i)!} u_{j_i} \right) \end{aligned} \quad (5.25)$$

$$= \sum_{j=0, j \leq n-s} \frac{(1-r)^{n-s-j} n!}{(n-s-j)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}.$$

The product of a function $c(x) = \sum_{s=0}^{\infty} \alpha_s x^s$ times a power of y is mapped into:

$$\begin{aligned} c(x)y^r &= \left(\sum_{s=0}^{\infty} \beta_s x^s \right) \left(\sum_{m=0}^{\infty} \zeta_m x^m \right)^r = \left(\sum_{s=0}^{\infty} \beta_s x^s \right) \left(\sum_{\mathbf{k}=0}^{\infty} \zeta_{k_1} \cdots \zeta_{k_r} x^k \right) \\ &= \sum_{s, \mathbf{k}=0}^{\infty} \beta_s \zeta_{k_1} \cdots \zeta_{k_r} x^{k+s} \rightarrow \sum_{s, \mathbf{k}=0}^{\infty} \beta_s \zeta_{k_1} \cdots \zeta_{k_r} p_{k+s}(x) \\ &\rightarrow \sum_{s, \mathbf{k}=0}^{\infty} \beta_s \zeta_{k_1} \cdots \zeta_{k_r} \frac{n!}{(n-k-s)!} = \sum_{s, j=0}^{\infty} \beta_s \frac{(1-r)^{n-s-j} n!}{(n-s-j)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \end{aligned} \quad (5.26)$$

By combining together all the previous results, we obtain the proof that the difference equation (5.21) is categorically equivalent to the equation

$$\sum_{j=0}^m (-1)^{m-j} \binom{m}{j} u_{n+j} - \sum_{q=0}^N \sum_{s_q, j=0}^{\infty} \beta_{q s_q} \frac{(1-r)^{n-s_q-j} n!}{(n-s_q-j)!} \prod_{i=1}^q \frac{u_{j_i}}{j_i!} = 0 \quad (5.27)$$

defined on $\mathcal{L}$, i.e. is the image of eq. (5.19) under the action of the morphism $v_{\partial, \Delta}$ defined in $\mathcal{N}_{\{m; a_0, \dots, a_N\}}$. In turn, both are realizations of eq. (4.8).

To prove that the series (5.22) is solution of eq. (5.21), observe that the morphism $\rho_{\partial, \Delta}$ provides the correspondence

$$\sum_k \zeta_k x^k \longrightarrow \sum_k \zeta_k p_k(n). \quad (5.28)$$

The action of the functor G will carry any analytic solution y of eq. (5.19), defined on the algebra $(\mathcal{F}, +, \cdot)$, into a solution u_n of the corresponding equation

$$eq(\Delta, u, *_\Delta) = v_{\partial, \Delta}(eq(\partial, y, \cdot)),$$

defined on the Rota algebra $(\mathcal{F}, +, *_\Delta)$. Once represented this equation on the lattice $\mathcal{L}$, it reduces to eq. (5.21). Besides, the series expansion of the solution u_n truncates and converts into the finite sum (5.22). $\square$

The same scheme of Remark 3 can be applied to generalize Theorem 3 to arbitrary objects of $\mathcal{N}_{\{m; c_0, \dots, c_N\}}$.

Remark 4. A direct comparison between the main results of the linear and nonlinear cases can be easily obtained. By putting $a_N(x) = 1$, and $a_1(x) = \dots = a_{n-1}(x) = 0$ into eq. (5.1), it converts into eq. (5.19) for $c_2(x) = \dots = c_N(x) = 0$. We observe that, coherently, eq. (5.5) converts into eq. (5.21).

6. Integrable dynamics in the Fourier space

The integrable maps arising from the preceding construction induce an auxiliary dynamics on the space of their Fourier coefficients, which is of independent interest. We consider here the case of nonlinear equations with constant coefficients, which provides a prototypical illustration of this alternative construction.

Proposition 2. *Consider a dynamical system of the form*

$$\frac{d^m}{dx^m} y = c_N y^N + c_{N-1} y^{N-1} + \dots + c_1 y + c_0, \quad (6.1)$$

where $c_0, \dots, c_N \in \mathbb{R}$. Assume that

$$y = \sum_{k=0}^{\infty} \zeta_k x^k \quad (6.2)$$

be a real solution of (6.1), analytic at $x = 0$. Then ζ_k satisfies the associated equation

$$\begin{aligned} \frac{(n+m)!}{n!} \zeta_{n+m} &= c_N \sum_{\substack{l_1, \dots, l_{N-1}=0 \\ l_1 + \dots + l_{N-1} = n}} \zeta_{l_1} \cdots \zeta_{l_{N-1}} \zeta_{n-l_1-\dots-l_{N-1}} + \dots \\ &+ c_2 \sum_{l_1=0}^n \zeta_{l_1} \zeta_{n-l_1} + c_1 \zeta_n + c_0 \delta_{n,0}. \end{aligned} \quad (6.3)$$

Proof. As a consequence of the definition of basic sequence for $\{p_k\}_{k \in \mathbb{N}}$, one can easily prove that, for each $m \in \mathbb{N} \setminus \{0\}$

$$\Delta^m u_n = \sum_{l=0}^n \frac{n!}{(n-l)!} \frac{(l+m)!}{l!} \zeta_{l+m}. \quad (6.4)$$

Also,

$$u_n^{*p} = \sum_{l_1, \dots, l_p=0}^{\infty} \zeta_{l_1} \cdots \zeta_{l_p} p_{(l_1+\dots+l_p)}(n) = \sum_{l=0}^n \frac{n!}{(n-l)!} \cdot \sum_{\substack{l_1, \dots, l_{p-1}=0 \\ l_1 + \dots + l_{p-1} \leq l}} \zeta_{l_1} \cdots \zeta_{l_{p-1}} \zeta_{l-l_1-\dots-l_{p-1}}. \quad (6.5)$$

Combining the previous expressions yields the stated result. $\square$

Both the linear case and the case of dynamical systems with nonconstant coefficients can be treated analogously. Their detailed analysis is left to the reader.

7. Rota algebras and Picard-Vessiot theory

This section aims to extend certain fundamental results of Galois theory, established for homogeneous linear differential equations with constant coefficients, to their *alter ego* on $\mathcal{L}$ as defined by Theorem 2 (See e.g. [12], [22], [34], [35], [46] for relevant definitions and results).

Let $\{a_0, \dots, a_{N-1}\} \subset \mathbb{R}$. Let $\mathcal{B}(\mathcal{F}, \mathcal{Q})$ be the space of linear operators acting on the Rota differential algebra $(\mathcal{F}, +, \cdot, *_\mathcal{Q})$ over $\mathbb{R}$. We introduce the linear operator $T[\mathcal{Q}] \in \mathcal{B}(\mathcal{F}, \mathcal{Q})$ defined by $T[\mathcal{Q}] := \mathcal{Q}^N + a_{N-1}\mathcal{Q}^{N-1} + \dots + a_1\mathcal{Q} + a_0$. We consider the linear differential equation

$$T[\partial](y) := y^{(N)} + a_{N-1}y^{(N-1)} + \dots + a_1y' + a_0y = 0, \quad (7.1)$$

and its categorically equivalent equation

$$T[\Delta](u) := \Delta^N u + a_{N-1}\Delta^{N-1}u + \dots + a_1\Delta u + a_0u = 0. \quad (7.2)$$

One can show that the rings C_∂ and C_Δ of constants for $\mathcal{R}(\mathcal{F}, \partial)$ and $\mathcal{R}(\mathcal{F}, \Delta)$ coincide. The following result holds.

Lemma 2. *The morphism $\rho_{\partial, \Delta} : (\mathcal{F}, \partial) \longrightarrow (\mathcal{F}, \Delta)$ maps isomorphically a fundamental system of solutions $\mathcal{S}$ of the linear differential equation (7.1) into a fundamental system $\rho_{\partial, \Delta}(\mathcal{S})$ of the linear equation (7.2).*

Proof. As a consequence of Theorem 2, power series solutions of eqs. (7.1) and (7.2) are in one-to-one correspondence. The morphism $\rho_{\partial, \Delta}$ maps basic sequences into basic sequences, so it preserves linear independence of solutions and the dimension of the associated vector space. Then $\mathcal{S}' = \rho_{\partial, \Delta}(\mathcal{S})$ is a fundamental set for eq. (7.2). $\square$

Our purpose will be to define an extension of a differential subring $\mathcal{F}' \subseteq \mathcal{F}$ for eq. (7.2) by using the categorical approach developed before. To this aim, we construct the *universal solution algebra* $\mathcal{U}$ of eq. (7.2). Let

$$\Delta Y = AY, \quad A \in M_N(\mathbb{R}) \quad (7.3)$$

be its matrix form, where it is understood that $Y_{i+1, j} = \Delta Y_{ij}$. We introduce an $N \times N$ matrix of indeterminates $Y = (Y_{ij})$ and define

$$\mathcal{U}[\Delta] := \mathcal{F}' \left[Y_{ij}, \frac{1}{\det_* Y} \right], \quad 1 \leq i, j \leq N. \quad (7.4)$$

Here we introduce the modified Wronskian

$$\det_* Y := \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1N} \\ \Delta Y_{11} & \Delta Y_{12} & \dots & \Delta Y_{1N} \\ \Delta^2 Y_{11} & \dots & \dots & \Delta^2 Y_{1N} \\ \vdots & & & \vdots \\ \Delta^{N-1} Y_{11} & \dots & \dots & \Delta^{N-1} Y_{1N} \end{vmatrix}. \quad (7.5)$$

Definition 10. A Picard–Vessiot (PV) ring for the matrix equation (7.3) over the Rota algebra $(\mathcal{F}', \Delta)$ is a simple differential ring $\mathcal{O}(+, \cdot, *_\Delta)$ such that

- i) There exists a fundamental matrix $Z \in \mathrm{GL}_N(\mathcal{O})$ for eq. (7.3).
- ii) The ring $\mathcal{O}$ is generated by $\mathcal{F}'$, the entries of Z and $\frac{1}{\det_* Z}$.

To construct a PV-ring, we introduce the notion of *Rota ideal*.

Definition 11. A *Rota ideal* for the derivation Δ is an ideal invariant under Δ .

The following result holds.

Proposition 3. Let $\mathcal{I}$ be a maximal Rota ideal of $\mathcal{U}[\Delta]$. Then the ring $\mathcal{V}[\Delta] := \mathcal{U}[\Delta]/\mathcal{I}$ is a Picard–Vessiot ring for the equation (7.3).

Proof. Because $\mathcal{I}$ is a maximal Rota ideal, the quotient ring $\mathcal{V}[\Delta]$ is a simple differential ring under the induced action of Δ . Then, it suffices to observe that in the quotient ring the Wronskian (7.5) is invertible. $\square$

We shall denote by $\mathcal{U}[\partial]$ the universal solution algebra of the matrix differential equation $Y' = AY$ associated to eq. (7.1), where $Y_{i+1,j} = Y'_{i,j}$. We introduce the notion of a differential Galois group for the previous equations.

Definition 12. Given the Rota algebra $(\mathcal{F}', \Delta)$, let $\mathcal{V}[\Delta]$ be a Picard–Vessiot ring over $\mathcal{F}'$. The differential Galois group of $\mathcal{V}[\Delta]$ over $(\mathcal{F}', \Delta)$, $\mathrm{DGal}(\mathcal{V}/(\mathcal{F}', \Delta))$, is the group of the differential $(\mathcal{F}', \Delta)$ -automorphisms.

Let $\mathcal{M}$ denote a maximal differential ideal of eq. (7.1). The main result of this section is the following

Theorem 4. Let $\mathcal{W}[\partial] := \mathcal{U}[\partial]/\mathcal{M}$ be a PV-ring over $(\mathcal{F}', \partial)$ for the linear homogeneous differential equation (7.1) and $\mathcal{V}[\Delta] = \mathcal{U}[\Delta]/\mathcal{I}$ be the associated PV-ring over $(\mathcal{F}', \Delta)$ for the equation (7.2). Then there exists an isomorphism of groups Φ such that

$$\mathrm{DGal}(\mathcal{W}[\partial]/(\mathcal{F}', \partial)) \xrightarrow{\Phi} \mathrm{DGal}(\mathcal{V}[\Delta]/(\mathcal{F}', \Delta)). \quad (7.6)$$

Proof. According to Lemma 2, given a fundamental set $\mathcal{S}$ of solutions of eq. (7.1), $\mathcal{S}' = \rho_{\partial, \Delta}(\mathcal{S})$ is a fundamental set of solutions of eq. (7.2). This implies that the universal solution algebras $\mathcal{U}[\partial]$ and $\mathcal{U}[\Delta]$ are isomorphic. The morphism $\lambda_{\partial, \Delta}$ maps a (maximal) differential ideal $\mathcal{M}$ into a (maximal) Rota ideal $\mathcal{I}$, so that the associated Picard–Vessiot rings are isomorphic. Then the differential automorphisms of the Galois groups associated with eqs. (7.1) and (7.2) are in one-to-one correspondence. $\square$

Remark 5. An analogous result holds for the equation $T[\nabla](u) = 0$, where $\nabla = 1 - T^{-1}$. However, the previous construction does not obviously extend to an arbitrary delta operator $\mathcal{Q}$. In fact, the fundamental set of eq. $T[\mathcal{Q}](u) = 0$ generates a linear space whose dimension is not necessarily preserved.

8. Main theorems for the singular case

In this Section we shall prove the main theorems of our theory of Laurent polynomial sequences.

In the following Theorems 5 and 6, we will define our maps in the *twisted Rota algebra* $(\mathcal{H}, \mathcal{Q})$, where $\mathcal{H}$ is the space of Laurent formal series, and $p_k(x)_{k \in \mathbb{Z}}$ are *Laurent basic sequences* for the delta operator $Q = \Delta$.

8.1. Linear difference equations

Theorem 5. Consider an ordinary linear differential equation of order $N \in \mathbb{N} \setminus \{0\}$ of the form

$$\text{lin}(\partial, y) := a_N(x) \frac{d^N}{dx^N} y + a_{N-1}(x) \frac{d^{N-1}}{dx^{N-1}} y + \dots + a_1(x) \frac{d}{dx} y + a_0(x)y + g(x) = 0, \quad (8.1)$$

where $a_i(x)$, $i = 0, \dots, N$ and $g(x)$ are real functions admitting at most polar singularities at $x = 0$ of the form:

$$a_i(x) = \sum_{j=-\ell_i}^{r_i} \alpha_{ij} x^j, \quad g(x) = \sum_{j=-\ell}^r \gamma_j x^j. \quad (8.2)$$

Assume that eq. (8.1) has a solution possessing a polar singularity of order s at the origin:

$$y(x) = \sum_{k=1}^s \frac{\zeta_{-k}}{x^k} \quad (8.3)$$

where $s \geq \max\{\ell + r, \ell_0 + r_0, \dots, \ell_N + r_N\}$. Then, the difference equation

$$\sum_{m=0}^N \sum_{j=-\ell_m}^{r_m} \alpha_{mj} \frac{(n-1)!}{(n-j-1)!} \Delta^m u_{n-j} + \sum_{j=-\ell}^r \gamma_j p_j(n) = 0, \quad (8.4)$$

lying in the positive real line, has a solution of the form

$$u_n = \sum_{k=1}^s \frac{(n-1)!}{(n+k-1)!} \zeta_{-k}. \quad (8.5)$$

Proof. Our aim is to construct the equation categorically equivalent to eq. (8.1) in $\mathcal{K}_{\{a_0, \dots, a_N, g\}}$, for the class of functions (8.3). Consequently, we represent the product $x^r \frac{d^s}{dx^s} y(x)$ on the set $\mathcal{L}$ by the action of the morphism $\rho_{\partial, \Delta} : (\mathcal{H}, +, \cdot) \rightarrow (\mathcal{H}, +, *_\Delta)$. This morphism determines the correspondences detailed below.

Let us first consider the case of the higher derivatives

$$y^{(m)} = \sum_{k=-\infty}^{\infty} \frac{k!}{(k-m)!} \zeta_k x^{k-m} \longrightarrow \sum_{k=-\infty}^{\infty} \frac{k!}{(k-m)!} \zeta_k p_{k-m}(n), \quad (8.6)$$

$$\Delta^m u_n = \sum_{k=-\infty}^{\infty} \zeta_k \Delta^m p_k(n) = \sum_{k=-\infty}^{\infty} \frac{k!}{(k-m)!} \zeta_k p_{k-m}(n), \quad (8.7)$$

and then,

$$y^{(m)} \longrightarrow \Delta^m u_n = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} u_{n+i}. \quad (8.8)$$

These results are a natural consequence of the properties of the delta operators. By analogy, we get

$$\begin{aligned} x^r y^{(m)} &\longrightarrow p_r(n) * \sum_{k=-\infty}^0 \frac{k!}{(k-m)!} \zeta_k p_{k-m}(n) = \sum_{k=-\infty}^0 \frac{k!}{(k-m)!} \zeta_k p_{k-m+r}(n) \\ &= \sum_{k=-\infty}^{-1} \frac{k!}{(k-m)!} \zeta_k \frac{(n-1)!(n-r-k+m-1)!}{(n-r-1)!(n-k+m-r-1)!} p_{k-m}(n-r), \\ &= \frac{(n-1)!}{(n-r-1)!} \sum_{k=-\infty}^{-1} \frac{k!}{(k-m)!} \zeta_k p_{k-m}(n-r) \\ &= \frac{(n-1)!}{(n-r)!} \left((n-r) \sum_{k=-\infty}^{-1} \frac{k!}{(k-m)!} \zeta_k p_{k-m}(n-r) \right) \\ &= \frac{(n-1)!}{(n-r-1)!} \Delta^m u_{n-r}. \end{aligned}$$

Therefore, taking into account relations (8.2), we obtain

$$a_m(x) y^{(m)}(x) \rightarrow \sum_{j=-\ell_m}^{r_m} \alpha_{mj} \frac{(n-1)!}{(n-j-1)!} \Delta^m u_{n-j}. \quad (8.9)$$

By combining the previous formulas, we deduce the form of eq. (8.4).

To prove that the series (8.3) is solution of eq. (8.4), observe that the morphism $\lambda_{\partial, \Delta}$ provides the correspondence

$$\sum_k \zeta_{-k} x^{-k} \longrightarrow \sum_k \zeta_{-k} \frac{1}{p_k^-(n)}. \quad (8.10)$$

The action of the functor F carries any solution y of eq. (8.1), defined on the algebra $(\mathcal{H}, +, \cdot)$, into a solution u_n of the corresponding equation

$$\text{lin}(\Delta, u, *_\Delta) = \lambda_{\partial, \Delta}(\text{lin}(\partial, y, \cdot)),$$

defined on the extended Rota algebra $(\mathcal{H}, +, *_\Delta)$. Once represented this equation on the lattice $\mathcal{L}$, any finite Laurent series solution (8.3) of eq. (8.1) is carried into a solution of the form (8.5) of the equation

$$\text{lin}(\mathcal{Q}, u, *_\mathcal{Q}) = \lambda_{\partial, \Delta}(\text{lin}(\partial, y, \cdot)).$$

This equation, according to the previous construction, coincides with eq. (8.4). $\square$

8.2. Nonlinear difference equations

We shall prove a theorem concerning the discretization of nonlinear equations of the form (1.2) which preserves solutions admitting polar singularities.

Theorem 6. Consider an ordinary differential equation of order $m \in \mathbb{N} \setminus \{0\}$ of the form

$$\text{eq}(\partial, y) := \frac{d^m}{dx^m} y - \sum_{r=0}^N c_r(x) y^r = 0, \quad (8.11)$$

where $N, m \in \mathbb{N} \setminus \{0\}$, $c_r(x)$, $r = 0, \dots, N$ are real functions admitting at most polar singularities at $x = 0$ of the form:

$$c_r(x) = \sum_{j=-\ell_r}^{\rho_r} \beta_{rj} x^j. \quad (8.12)$$

Assume that eq. (8.11) has a solution possessing a polar singularity of order s at the origin:

$$y(x) = \sum_{k=1}^s \frac{\zeta_{-k}}{x^k}, \quad (8.13)$$

where $s \geq \max\{\ell_0 + \rho_0, \dots, \ell_N + \rho_N\}$. Then, the difference equation

$$\sum_{i=0}^m (-1)^{m-i} \binom{m}{i} u_{n+i} = \sum_{r=0}^N \sum_{j=-\ell_r}^{\rho_r} \sum_{k_1, \dots, k_r=-\infty}^{-1} \beta_{rj} \zeta_{k_1} \cdots \zeta_{k_r} p_{K_r+j}(n) \quad (8.14)$$

with $K_r = \sum_{i=1}^r k_i$, defined over a uniform lattice in the positive real line, has a solution of the form

$$u_n = \sum_{k=1}^s \frac{(n-1)!}{(n+k-1)!} \zeta_{-k}. \quad (8.15)$$

Proof. The discretization runs as follows. The function $y^r(x)$, is written as a Laurent expansion centred at the origin. Let us write formally

$$y^r \rightarrow \left(\sum_{k_1=-\infty}^{-1} \zeta_{k_1} p_{k_1}(n) \right) * \cdots * \left(\sum_{k_r=-\infty}^{-1} \zeta_{k_r} p_{k_r}(n) \right) = \sum_{k_1, \dots, k_r=-\infty}^{-1} \zeta_{k_1} \cdots \zeta_{k_r} p_{K_r}(n), \quad (8.16)$$

where $K_r = \sum_{i=1}^r k_i$, and $\zeta_{k_i} = 0$ for $k_i = -s - 1, \dots, -\infty$. Thus,

$$\begin{aligned} c_r(x) y^r &\rightarrow \sum_{j=-\ell_r}^{\rho_r} \beta_{rj} p_j(n) * \sum_{k_1, \dots, k_r=-\infty}^{-1} \zeta_{k_1} \cdots \zeta_{k_r} p_{K_r}(n) = \\ &= \sum_{j=-\ell_r}^{\rho_r} \sum_{k_1, \dots, k_r=-\infty}^{-1} \beta_{rj} \zeta_{k_1} \cdots \zeta_{k_r} p_{K_r+j}(n). \end{aligned} \quad (8.17)$$

Then, the discretized function, u_n , is written as

$$u_n = \sum_{k=1}^s \frac{\zeta_{-k}}{p_k^-(n)}. \quad (8.18)$$

The derivatives $y^{(m)}$, $m = 1, \dots, r$, are discretized according to the operator Δ :

$$\Delta^m u_n = \sum_{i=0}^m (-1)^i \binom{m}{i} u_{n+i}. \quad (8.19)$$

The constants ζ_{-k} satisfy a linear algebraic system:

$$\begin{cases} u_{n+1} = \sum_{k=1}^s \frac{\zeta_{-k}}{p_k^-(n+1)} = \sum_{k=1}^s A_{n+1,k} \zeta_{-k}, \\ \vdots \\ u_{n+s} = \sum_{k=1}^s \frac{\zeta_{-k}}{p_k^-(n+s)} = \sum_{k=1}^s A_{n+s,k} \zeta_{-k}, \end{cases} \quad (8.20)$$

where $A_{jk} = \frac{(j-1)!}{(j+k-1)!}$, $k = 1, \dots, s$, $j = n+1, \dots, n+s$. Unlike in the previous theorems, where the corresponding coefficient matrix acquires a triangular form, in the present case the solution of system (8.20) cannot be easily written in a closed form in terms of the variables u_{n+i} . We obtain

$$\zeta_{-k} = \sum_{\lambda=1}^s (A^{-1})_{k,\lambda} u_{n+\lambda}. \quad (8.21)$$

Thus, the discrete equation reads:

$$\begin{aligned} \sum_{i=0}^m (-1)^i \binom{m}{i} u_{n+i} &= \sum_{r=0}^N \sum_{j=-\ell_r}^{\rho_r} \sum_{k_1, \dots, k_r=-\infty}^{-1} \sum_{\lambda_1, \dots, \lambda_r=1}^s \beta_{rj} (A^{-1})_{k_1, \lambda_1} \cdots (A^{-1})_{k_r, \lambda_r} \cdot \\ &\quad \cdot u_{n+\lambda_1} \cdots u_{n+\lambda_r} p_{K_r+j}(n), \end{aligned} \quad (8.22)$$

where

$$p_{K_r+j}(n) = \begin{cases} \frac{n!}{(n - K_r - j)!}, & K_r + j \geq 0, \\ \frac{(n-1)!}{(n - K_r - j - 1)!}, & K_r + j < 0. \end{cases} \quad (8.23)$$

Let us prove that the series (8.15) is solution of eq. (8.14). To this aim, we observe that the morphism $\rho_{\partial, \Delta}$ carries on

$$\sum_{k=1}^s \zeta_{-k} x^{-k} \longrightarrow \sum_{k=1}^s \zeta_{-k} \frac{1}{p_k^-(n)}. \quad (8.24)$$

Thus, the action of the functor F carries any solution y of eq. (8.1), defined on the algebra $(\mathcal{H}, +, \cdot)$, into a solution u_n of the corresponding equation

$$eq(\Delta, u, *_\Delta) = v_{\partial, \Delta}(eq(\partial, y, \cdot)),$$

defined on the extended Rota algebra $(\mathcal{H}, +, *_\Delta)$. According to the previous analysis, we represent this equation on the lattice $\mathcal{L}$. Then, any finite Laurent series solution (8.13) of eq. (8.11) is carried into a solution of the form (8.15) of the equation

$$eq(\mathcal{Q}, u, *_\mathcal{Q}) = v_{\partial, \Delta}(eq(\partial, y, \cdot)).$$

This equation coincides with eq. (8.14). $\square$

Remark 6. We wish to emphasize that, in concrete examples, the applicability of the categorical discretization method proposed in Theorems 5 and 6 can be considerably extended. While generalizing the hypotheses of our theorems may not be straightforward, specific cases can nevertheless be analyzed and solved rigorously.

9. Nonlocal Lie symmetries and integrable maps: some conjectures

The categorical discretization introduced in Theorems 2, 3, 5 and 6 establishes a precise structural correspondence between the space of smooth solutions of the original differential equations and the space of solutions of the associated difference equations. This result ensures that the passage from the continuous to the discrete setting, carried out within this categorical framework, does not merely approximate the original system but preserves its essential solution structure in a rigorous and exact sense.

Building on this observation, it is a natural problem to ascertain to what degree such a correspondence, established at the level of solutions, extends to the symmetry properties of the systems under consideration. Specifically, we hypothesize the existence of a meaningful relationship between the Lie symmetry groups and their corresponding Lie algebras admitted by ordinary differential equations and those admitted by their discrete counterparts. For a comprehensive account of the modern theory of Lie symmetries for continuous systems, see e.g. [33]. If confirmed, this would suggest that the discretization not only preserves solutions but also retains,

in an appropriate sense, the geometric and algebraic structures encoded by symmetries, which play a central role in the qualitative analysis of differential equations.

We first conjecture the existence of symmetry transformations.

Conjecture 1. *The integrable maps (5.4) and (8.4) (resp. (5.21) and (8.14) for the singular case) admit a Lie group $\mathcal{G}$ of nonlocal diffeomorphisms that leave the map invariant and transform solutions into solutions.*

This conjecture relies on the idea that, for the ODEs considered, the determining linear equations generating the Lie symmetries in the continuous case would be preserved by our discretization scheme, as well as their solutions. The nonlocal character of the diffeomorphisms generating these symmetries is suggested by the intrinsic nonlocality of our $*$ -product.

However, in general, objects categorically equivalent do not possess isomorphic fundamental solution sets, with the exception of linear equations with constant coefficients, as discussed in Section 6. Consequently, the full Lie algebras generated by the symmetries postulated in the previous conjecture, are, in general, not isomorphic. The subsequent conjecture establishes a weaker form of correspondence between these algebras.

Conjecture 2. *The Lie algebra of the generators of the Lie group $\mathcal{G}$ of the nonlocal symmetry diffeomorphisms associated with the maps (5.4) and (8.4) (resp. (5.21) and (8.14)) contains a subalgebra which is isomorphic to the Lie algebra of the classical Lie point symmetries of the continuous dynamical system (1.1) (resp. (1.2)).*

10. New integrable maps from ODEs with regular solutions

As an illustration of the usefulness of the categorical approach developed in this work, we propose here some examples of integrable maps constructed according to the main Theorems 2 and 3, concerning differential equations admitting regular solutions. These maps are associated with ODEs that are particularly relevant in the applications.

10.1. A discrete damped harmonic oscillator

The damped harmonic oscillator

$$y''(x) + 2q\omega y'(x) + \omega^2 y(x) = 0 \quad (10.1)$$

in the case $q < 1$ admits the general solution

$$y_{\text{gen}}(x) = Ae^{-q\omega x} \sin\left(\sqrt{1 - q^2}\omega x + \phi\right). \quad (10.2)$$

The discrete damped oscillator reads

$$u_{n+2} + 2(q\omega - 1)u_{n+1} + (\omega^2 - 2q\omega + 1)u_n = 0, \quad (10.3)$$

with the general solution

$$u_n = \sum_{k=0}^n g(k) \frac{n!}{(n-k)!}, \quad g(k) := \frac{1}{k!} \left[\frac{d}{dx^k} y_{\text{gen}} \right]_{x=0}. \quad (10.4)$$

We shall consider as lattice sites the points nh , $h \in \mathbb{R}$. In this mesh, the operator Δ_h is defined as $\Delta_h u_n = \frac{u_{n+1} - u_n}{h}$, with basic polynomials

$$p_k(x) = \prod_{j=0}^{k-1} (x - jh), \quad (10.5)$$

possessing zeros at the sites of the lattice. Thus:

$$p_k(nh) = \begin{cases} 0 & \text{if } n < k, \\ \frac{n!h^k}{(n-k)!} & \text{if } n \geq k \end{cases} \quad (10.6)$$

and the inverse interpolating transform reads

$$u_n = \sum_{k=0}^n \frac{n!h^k}{(n-k)!} \zeta_k, \quad \zeta_k = \frac{1}{h^k} \sum_{j=0}^k \frac{(-1)^{k-j}}{j!(k-j)!} u_j. \quad (10.7)$$

Therefore, the discrete damped oscillator equation is:

$$u_{n+2} + 2(hq\omega - 1)u_{n+1} + (h^2\omega^2 - 2hq\omega + 1)u_n = 0. \quad (10.8)$$

Its continuum limit (when $h \rightarrow 0$, $n \rightarrow \infty$ and nh remains bounded) is the equation (10.1), as can be easily checked.

10.2. A difference equation for the Gaussian function

We discretize the differential equation satisfied by the Gaussian function:

$$y'(x) = -xy(x). \quad (10.9)$$

The solution (up to a factor) has the following Taylor expansion:

$$y(x) = e^{-\frac{x^2}{2}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k}. \quad (10.10)$$

According to Theorem 2, in a uniform lattice of step size h the associated integrable map reads

$$u_{n+1} - u_n + h^2 n u_{n-1} = 0. \quad (10.11)$$

It admits the solution ($u_0 = 1$):

$$u_n = \sum_{k=0}^{\infty} \frac{n! h^k \zeta_k}{(n-k)!} = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! h^{2k}}{2^k k! (n-2k)!}, \quad (10.12)$$

where the coefficients ζ_k are given by the Taylor expansion in (10.10). Here $[n/2]$ denotes the integer part of $n/2$.

Remark 7. The purpose of this simple analysis is to illustrate, in a transparent way and using a basic example, the non-standard form of the difference equations obtained through our categorical discretization. In fact, a classical discretization of eq. (10.9) would lead to the difference equation

$$\hat{u}_{n+1} - \hat{u}_n + h^2 n \hat{u}_n = 0. \quad (10.13)$$

However, it is straightforward to verify that this equation does not admit a solution of the form (10.12), which arises directly from the Gaussian function. The equation (10.13) admits an exact solution which can be easily obtained ($\hat{u}_0 = 1$):

$$\hat{u}_n = \prod_{k=0}^{n-1} (1 - k h^2), \quad n \geq 1 \quad (10.14)$$

In both cases (categorical and standard discretizations), the resulting difference equations (10.11) and (10.13) tend to the differential equation (10.10) in a straightforward way (we divide by h and take the continuum limit, $h \rightarrow 0$, $n \rightarrow \infty$, keeping nh constant). It can also be verified that the discretized solutions provide a very good approximation of the exponential function.

10.3. Hypergeometric equation

The real hypergeometric differential equation

$$x(1-x) \frac{d^2 y}{dx^2} + [c - (a+b+1)x] \frac{dy}{dx} - aby = 0, \quad (10.15)$$

where $a, b, c \in \mathbb{R}$, possesses three regular singular points $\{0, 1, \infty\}$ [2]. By way of example, we shall restrict to the singularity at $x = 0$. Around this point, eq. (10.15) admits two algebraically independent solutions,

$$s_1 := {}_2F_1(a, b; c; x), \quad (10.16)$$

and a second solution s_2 whose explicit form depends on the particular values of the parameters. Theorem 2 provides the following discrete version of eq. (10.15)

$$\begin{aligned} (n+c)u_{n+1} - (n^2 + (a+b+2)n + ab + c)u_n \\ + n(a+b+2n)u_{n-1} - n(n-1)u_{n-2} = 0. \end{aligned} \quad (10.17)$$

If we introduce the finite Gauss sum

$$G(a, b; c; n) := \sum_{k=0}^n \frac{(a)_k (b)_k}{(c)_k} \frac{n!}{(n-k)!}, \quad (10.18)$$

the solution of eq. (10.18) corresponding to s_1 is provided by

$$u_n^{(1)} := G(a, b; c; n). \quad (10.19)$$

A numerical analysis of this solution shows that this solution is in excellent agreement with the continuous one.

Theorem 2 allows us to introduce integrable maps which discretize the differential equations characterizing families of orthogonal polynomials. Here we shall focus on the nontrivial case of the equation defining the classical family of Jacobi.

10.4. A discrete Jacobi equation

Applying Theorem 2, the equation

$$(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + m(m + \alpha + \beta + 1)y(x) = 0, \quad (10.20)$$

can be discretized in the form

$$u_{n+2} - (\alpha - \beta + 2)u_{n+1} + \left(m^2 - n^2 + (m - n)(\alpha + \beta + 1) + \alpha - \beta + 1\right)u_n + n(\alpha + \beta + 2n)u_{n-1} - n(n - 1)u_{n-2} = 0. \quad (10.21)$$

This equation admits as a solution the family of polynomials $\{P_m^{(\alpha, \beta)}(n)\}_{m \in \mathbb{N}}$, where

$$P_m^{(\alpha, \beta)}(n) := \frac{\Gamma(\alpha + m + 1)}{m! \Gamma(\alpha + \beta + m + 1)} \times \sum_{k=0}^m \binom{m}{k} \frac{\Gamma(\alpha + \beta + m + k + 1)}{\Gamma(\alpha + k + 1)} \frac{n!}{2^k} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^{k-i}}{(n-i)!}.$$

Virtually all classes of orthogonal polynomials, including nonclassical and Sobolev-type ones, can be associated with integrable maps in an analogous way. The analysis of other similar cases is left to the reader.

10.5. A new discrete Painlevé I equation

Let us discuss the interesting case of the discretization of the classical Painlevé I equation. Let us start with the differential equation

$$\frac{d^2 y}{dy^2} - c_2(x)y^2 - c_1(x)y - c_0(x) = 0. \quad (10.22)$$

According to Theorem 3, for $m = 2$, and $N = 2$, the associated difference equation is:

$$\begin{aligned}
& u_{n+2} - 2u_{n+1} + u_n \\
& - \sum_{s, j_1, j_2=0}^n \beta_{2s} \frac{(-1)^{n-s-j_1-j_2} n!}{(n-s-j_1-j_2)!} \frac{u_{j_1} u_{j_2}}{j_1! j_2!} \\
& - \sum_{s, j_1=0}^n \beta_{1s} \frac{n!}{(n-s-j_1)!} \frac{u_{j_1}}{j_1!} - \sum_{s=0}^n \beta_{0s} \frac{n!}{(n-s)!} = 0.
\end{aligned} \tag{10.23}$$

In particular, for the Painlevé I equation

$$y'' = 6y^2 + x, \tag{10.24}$$

we have

$$c_0(x) = x, \quad c_1(x) = 0, \quad c_2(x) = 6. \tag{10.25}$$

As is well-known, the Painlevé I equation admits an analytic solution around $x = 0$, with a local expansion given by

$$y(x) = \zeta_0 + \zeta_1 x + 3\zeta_0^2 x^2 + \left(2\zeta_0 \zeta_1 + \frac{1}{6}\right) x^3 + \left(\frac{1}{2} \zeta_1^2 + 3\zeta_0^3\right) x^4 + \dots \tag{10.26}$$

Here $\zeta_0 = y(0)$ and $\zeta_1 = y'(0)$.

Consequently, by applying Theorem 2, we can introduce the novel *discrete Painlevé I equation*

$$u_{n+2} - 2u_{n+1} + u_n - 6 \sum_{j_1, j_2=0}^n \frac{(-1)^{n-j_1-j_2} n!}{(n-j_1-j_2)!} \frac{u_{j_1} u_{j_2}}{j_1! j_2!} - n = 0. \tag{10.27}$$

To check the continuum limit of eq. (10.27), we introduce in the semi-axis $x \geq 0$ a uniform mesh $\mathcal{L}_h$ of points

$$x_n = nh, \quad n \in \mathbb{N}, \tag{10.28}$$

where $h > 0$. The continuum limit is defined by $h \rightarrow 0$, $n \rightarrow \infty$ and nh bounded.

By using formulas (10.5)–(10.7), we rewrite eq. (10.27) in terms of the mesh amplitude h . We get

$$\frac{1}{h^2} (u_{n+2} - 2u_{n+1} + u_n) - 6 \sum_{j_1, j_2=0}^n \frac{(-1)^{n-j_1-j_2} n!}{(n-j_1-j_2)!} \frac{u_{j_1} u_{j_2}}{j_1! j_2!} - nh = 0. \tag{10.29}$$

A straightforward analysis shows that in the double limit $h \rightarrow 0$, $n \rightarrow \infty$ (nh bounded), eq. (10.29) reduces to the Painlevé I equation (10.24).

The first equations $n = 0, 1, 2, 3$ are explicitly given by:

$$\begin{aligned} -6h^2u_0^2 + u_0 - 2u_1 + u_2 &= 0, \\ -h^3 - 6h^2u_0(2u_1 - u_0) + u_1 - 2u_2 + u_3 &= 0, \\ -2h^3 - 6h^2(u_0^2 + (2u_2 - 4u_1)u_0 + 2u_1^2) + u_2 - 2u_3 + u_4 &= 0, \\ -3h^3 + 6h^2(u_0^2 - 2(3u_1 - 3u_2 + u_3)u_0 + 6u_1(u_1 - u_2)) + u_3 - 2u_4 + u_5 &= 0. \end{aligned} \quad (10.30)$$

An exact solution of this equation is:

$$u_n = \sum_{k=0}^n \zeta_k p_k(n) = n! \sum_{k=0}^n \frac{h^k \zeta_k}{(n-k)!}. \quad (10.31)$$

Since $\zeta_0 = u_0$, $\zeta_1 = (u_1 - u_0)/h$, we have

$$\begin{aligned} u_2 &= 6h^2u_0^2 - u_0 + 2u_1, \\ u_3 &= h^3 + 6h^2u_0(u_0 + 2u_1) - 2u_0 + 3u_1, \\ u_4 &= 72h^4u_0^3 + 4h^3 + 12h^2u_1(2u_0 + u_1) - 3u_0 + 4u_1. \end{aligned}$$

One can easily check that these functions satisfy the system (10.30).

11. New integrable maps admitting singular solutions

We shall construct several new maps obtained as an application of the results proposed in Theorems 5 and 6.

11.1. A linear model with two polar singularities

The differential equation

$$y'' + \frac{3}{x}y' + \frac{1}{x}y = \frac{1}{x^2} \quad (11.1)$$

admits a general solution which can be written in terms of Bessel functions, together with a particular solution of the inhomogeneous equation, which may be chosen as:

$$y(x) = \frac{1}{x} + \frac{1}{x^2}. \quad (11.2)$$

The corresponding difference equation can be easily obtained from Theorem 5. We have

$$u_{n+2} - 2u_{n+1} + u_n + \frac{3}{n}(u_{n+2} - u_{n+1}) + \frac{1}{n}u_{n+1} - \frac{1}{n(n+1)} = 0. \quad (11.3)$$

It admits the solution

$$u_n = \frac{n+2}{n(n+1)}, \quad (11.4)$$

constructed directly from solution (11.2):

$$\frac{1}{x} + \frac{1}{x^2} \rightarrow p_{-1}(n) + p_{-2}(n) = \frac{1}{n} + \frac{1}{n(n+1)}. \quad (11.5)$$

If we introduce a step h for the lattice, the equation is:

$$(n+1)[(n+3)u_{n+2} - (2n+3-h)u_{n+1} + nu_n] = 1, \quad (11.6)$$

with solution

$$u_n = \frac{(n+1)h+1}{n(n+1)h^2}. \quad (11.7)$$

11.2. A nonlinear dynamical system admitting a triple-pole solution

Consider the nonlinear equation:

$$(ax+b)^2 y'' - 6x(ax+2b)y^2 = 0, \quad a, b \in \mathbb{R} \quad (11.8)$$

which admits the solution

$$y(x) = \frac{a}{x^2} + \frac{b}{x^3}. \quad (11.9)$$

For sake of clarity, we shall derive its discrete version in a thorough way. This solution has the discrete counterpart given by

$$y(x) \rightarrow u_n = \sum_{k=-3}^{-2} \zeta_k p_k(n). \quad (11.10)$$

Thus, we deduce the following relations.

$$\begin{aligned} xy^2 &\rightarrow p_1(n) * \left(\sum_{k=-3}^{-2} \zeta_k p_k(n) \right) * \left(\sum_{k=-3}^{-2} \zeta_k p_k(n) \right) \\ &= \zeta_{-2}^2 p_{-3}(n) + 2\zeta_{-2}\zeta_{-3} p_{-4}(n) + \zeta_{-3}^2 p_{-5}(n), \\ x^2 y^2 &\rightarrow p_2(n) * \left(\sum_{k=-3}^{-2} \zeta_k p_k(n) \right) * \left(\sum_{k=-3}^{-2} \zeta_k p_k(n) \right) \\ &= \zeta_{-2}^2 p_{-2}(n) + 2\zeta_{-2}\zeta_{-3} p_{-3}(n) + \zeta_{-3}^2 p_{-4}(n), \\ xy'' &\rightarrow p_1(n) * \sum_{k=-3}^{-2} k(k-1)\zeta_k p_{k-2}(n) = (n-1)(u_{n+1} - 2u_n + u_{n-1}), \end{aligned}$$

$$x^2 y'' \rightarrow p_2(n) * \sum_{k=-3}^{-2} k(k-1) \zeta_k p_{k-2}(n) = (n-1)(n-2)(u_n - 2u_{n-1} + u_{n-2}).$$

The coefficients ζ_{-2} and ζ_{-3} can be written in terms of u_n, u_{n+1} solving the linear system:

$$\begin{cases} u_n = \zeta_{-3} p_{-3}(n) + \zeta_{-2} p_{-2}(n) \\ u_{n+1} = \zeta_{-3} p_{-3}(n+1) + \zeta_{-2} p_{-2}(n+1) \end{cases}, \quad p_k(n) = \frac{(n-1)!}{(n-k-1)!}, \quad k < 0 \quad (11.11)$$

whose solution reads

$$\begin{cases} \zeta_{-2} = -n(n+1)(n+2)u_n + (n+1)(n+2)(n+3)u_{n+1} \\ \zeta_{-3} = n(n+1)(n+2)(n+3)u_n - (n+1)(n+2)^2(n+3)u_{n+1}. \end{cases} \quad (11.12)$$

Then, the equation (11.8) admits the following categorically equivalent discretization:

$$\begin{aligned} & b^2 u_{n+2} + \frac{6(n+1)(n+2)(n+3)(a(n+2)(n+4) + 2bn)}{n(n+4)} u_{n+1}^2 + 2b(a(n-1) - b)u_{n+1} \\ & - \frac{12(n+1)(n+2)(a(n+3)(n+4) + 2b(n+2))}{n+4} u_n u_{n+1} \\ & + \frac{6n(n+1)(n+2)(a(n+4) + 2b)}{n+4} u_n^2 \\ & + (a(n-1)(a(n-2) - 4b) + b^2)u_n - 2a(n-1)(a(n-2) - b)u_{n-1} \\ & + a^2(n-2)(n-1)u_{n-2} = 0. \end{aligned}$$

It is straightforward to prove that it admits the categorical counterpart of solution (11.9):

$$u_n = \frac{a}{n(n+1)} + \frac{b}{n(n+1)(n+2)}. \quad (11.13)$$

12. Future perspectives

The categorical discretization approach developed in this article is primarily intended to provide an efficient method for constructing integrable maps from integrable continuous systems. However, it should be emphasized that, although the general framework introduced here finds its natural origin in the theory of integrable systems, it is not restricted to this specific field.

In fact, it can, in principle, be regarded as a general discretization approach that remains applicable even when exact solutions of the underlying ODEs are not explicitly known. In such cases, the resulting difference equations may be interpreted as discrete approximations of the continuous models, and can thus be used to construct approximate solutions.

In this article, we have focused on the case of scalar ODEs as a first essential step toward further generalizations. At present, we do not foresee any technical obstacles to extending the framework to vector-valued ODEs, and we intend to pursue this direction in future work. Extending the categorical approach to partial differential equations also constitutes one of the main objectives of our ongoing research. This fundamental problem could in principle be addressed using the same ideas, *mutatis mutandis*.

The relation between the present approach and the long-standing problem of the symmetry-preserving discretization of differential equations constitutes an intriguing and worthwhile line of research.

A thorough investigation of this relationship will be carried out in future research, encompassing both ordinary differential equations (ODEs) and partial differential equations (PDEs) (for instance, by extending some of the results of [48] to the categorical framework) in order to assess its theoretical implications as well as its practical applicability.

An interesting direction for future research is the extension of the present theory to the case of q -difference equations. A coherent Galois approach to this class of equations has been developed in [10], [45]. We hypothesize that a functorial correspondence can also be envisaged for this case, allowing a unified treatment of the Galois theory for differential equations and their categorically equivalent q -difference equations.

As a broader objective of our work, we conjecture that the study of quantum field theories on a lattice - one of the most significant developments in high-energy physics in recent decades - could be further enhanced through a categorical approach. Work along these research directions is currently underway.

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Appendix A. Some combinatorial relations

The following identities are used in the proofs of the main theorems:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \delta_{n0}, \quad \sum_{k=0}^r \binom{s}{k} \binom{m}{r-k} = \binom{m+s}{r}, \quad (\text{A.1})$$

$$\sum_{k=0}^s (-1)^{s-k} \binom{s}{k} \binom{n+k}{m+s} = \binom{n}{m}, \quad (\text{A.2})$$

$$\sum_{k=j}^{n+s-r} (-1)^{k-j} \binom{n-r}{k-s} \binom{k}{j} = \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} \delta_{n-r+i,j}. \quad (\text{A.3})$$

Appendix B. The inverse transform

We revise here the construction of the inverse transform. Consider the Δ operator and its sequence of basic polynomials computed in the lattice $\mathcal{L}$. If

$$u_n = \sum_{k=0}^n \zeta_k p_k^+(n) = \sum_{k=0}^n \frac{n! \zeta_k}{(n-k)!}, \quad (\text{B.1})$$

then the Fourier coefficients ζ_k are given by [50]:

$$\zeta_k = \sum_{j=0}^k \frac{(-1)^{k-j} u_j}{j!(k-j)!}. \quad (\text{B.2})$$

A crucial point in the proof of this property is that the expression of u_n in terms of the sequence of basic polynomials has a finite number of addends in the sum, since these polynomials have their zeros at the lattice points. However, when we consider a series of Laurent polynomials, the expansions have generically an infinite number of terms (in the Laurent part):

$$u_n = \sum_{k=1}^{\infty} \frac{\zeta_{-k}}{p_k^-(n)} + \sum_{k=0}^n \zeta_k p_k^+(n). \quad (\text{B.3})$$

This formula cannot be inverted as in the case of eq. (B.2) and, consequently, the coefficients ζ_{-k} do not have a general expression as functions of u_n .

Appendix C. Categorical correspondences

Here we show explicitly some technical details, relevant in the proofs of the main theorems, of the way the categorical correspondence acts on several algebraic and differential structures.

Differentials and products

$$\begin{aligned} x^r \frac{d^s}{dx^s} y(x) &= \sum_{k=0}^{\infty} \frac{k!}{(k-s)!} \zeta_k x^{k+r-s} \longrightarrow \sum_{k=0}^{n-r+s} \frac{k!}{(k-s)!} \zeta_k p_{k+r-s}(nh) \\ &= \sum_{k=0}^{n-r+s} \frac{n! k! h^{k+r-s} \zeta_k}{(k-s)!(n-k-r+s)!}. \end{aligned} \quad (\text{C.1})$$

Using the identity (A.2) we get

$$\begin{aligned} x^r \frac{d^s}{dx^s} y(x) &\longrightarrow \frac{n! h^{r-s}}{(n-r)!} \sum_{k=0}^{n-r+s} \sum_{i=0}^s (-1)^{s-i} k! \binom{s}{i} \binom{n-r+i}{k} \zeta_k \\ &= \frac{n! h^{r-s}}{(n-r)!} \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} \sum_{k=0}^{n-r+i} \frac{(n-r+i)!}{(n-r+i-k)!} \zeta_k \\ &= \frac{n! h^r}{(n-r)!} \frac{1}{h^s} \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} u_{n-r+i} = \frac{n! h^r}{(n-r)!} \Delta_h^s u_{n-r}. \end{aligned} \quad (\text{C.2})$$

Proof of equations (5.24) and (5.25) in Theorem 3.

Let us write the term $y(x)^r$ as a multiple series. We get

$$y(x)^r = \sum_{k_1=0}^{\infty} x^{k_1} \sum_{k_2=0}^{k_1} \zeta_{k_1-k_2} \sum_{k_3=0}^{k_2} \zeta_{k_2-k_3} \cdots \sum_{k_{r-1}=0}^{k_{r-2}} \zeta_{k_{r-2}-k_{r-1}} \sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r}. \quad (\text{C.3})$$

We will express the coefficients ζ_k in terms of u_i by means of the inverse transforms of eqs. (10.7), suitably iterated. First, we observe that there is no dependence from the mesh spacing h , since the terms involving h satisfy the identity

$$h^{k_1} \frac{1}{h^{k_1-k_2}} \frac{1}{h^{k_2-k_3}} \cdots \frac{1}{h^{k_{r-2}-k_{r-1}}} \frac{1}{h^{k_{r-1}-k_r}} \frac{1}{h^{k_r}} = 1. \quad (\text{C.4})$$

Starting from the last sum in (C.3), we get

$$\begin{aligned} & \sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r} \\ \rightarrow & \sum_{k_r=0}^{k_{r-1}} \sum_{j_{r-1}=0}^{k_{r-1}-k_r} \sum_{j_r=0}^{k_r} \frac{(-1)^{k_{r-1}-j_{r-1}-k_r}}{j_{r-1}!(k_{r-1}-j_{r-1}-k_r)!} \frac{(-1)^{k_r-j_r}}{j_r!(k_r-j_r)!} u_{j_{r-1}} u_{j_r} \\ = & \sum_{j_{r-1}=0}^{k_{r-1}} \sum_{k_r=0}^{k_{r-1}-j_{r-1}} \sum_{j_r=0}^{k_r} \frac{(-1)^{k_{r-1}-j_{r-1}-j_r}}{j_{r-1}! j_r! (k_{r-1}-j_{r-1}-k_r)! (k_r-j_r)!} u_{j_{r-1}} u_{j_r} \\ = & \sum_{j_{r-1}=0}^{k_{r-1}} \sum_{j_r=0}^{k_{r-1}-j_{r-1}} \frac{(-1)^{k_{r-1}-j_{r-1}-j_r} u_{j_{r-1}} u_{j_r}}{j_{r-1}! j_r!} \\ \times & \sum_{k_r=j_r}^{k_{r-1}-j_{r-1}} \frac{1}{(k_{r-1}-j_{r-1}-k_r)!(k_r-j_r)!}. \end{aligned} \quad (\text{C.5})$$

Since

$$\sum_{k_r=j_r}^{k_{r-1}-j_{r-1}} \frac{1}{(k_{r-1}-j_{r-1}-k_r)!(k_r-j_r)!} = \frac{2^{k_{r-1}-j_{r-1}-j_r}}{(k_{r-1}-j_{r-1}-j_r)!}, \quad (\text{C.6})$$

then we have

$$\sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r} \rightarrow \sum_{j_{r-1}=0}^{k_{r-1}} \sum_{j_r=0}^{k_{r-1}-j_{r-1}} \frac{(-1)^{k_{r-1}-j_{r-1}-j_r} 2^{k_{r-1}-j_{r-1}-j_r} u_{j_{r-1}} u_{j_r}}{j_{r-1}! j_r! (k_{r-1}-j_{r-1}-j_r)!}. \quad (\text{C.7})$$

The sums in the next term

$$\begin{aligned} & \sum_{k_{r-1}=0}^{k_{r-2}} \sum_{j_{r-2}=0}^{k_{r-2}-k_{r-1}} \frac{(-1)^{k_{r-2}-j_{r-2}-k_{r-1}}}{j_{r-2}!(k_{r-2}-j_{r-2}-k_{r-1})!} u_{j_{r-2}} \\ \times & \sum_{j_{r-1}=0}^{k_{r-1}} \sum_{j_r=0}^{k_{r-1}-j_{r-1}} \frac{(-1)^{k_{r-1}-j_{r-1}-j_r} 2^{k_{r-1}-j_{r-1}-j_r} u_{j_{r-1}} u_{j_r}}{j_{r-1}! j_r! (k_{r-1}-j_{r-1}-j_r)!} \end{aligned} \quad (\text{C.8})$$

can be appropriately ordered in the following way;

$$\sum_{k_{r-1}=0}^{k_{r-2}} \sum_{j_{r-2}=0}^{k_{r-2}-k_{r-1}} \sum_{j_{r-1}=0}^{k_{r-1}} \sum_{j_r=0}^{k_{r-1}-j_{r-1}} = \sum_{j_{r-2}=0}^{k_{r-2}} \sum_{j_{r-1}=0}^{k_{r-2}-j_{r-2}} \sum_{j_r=0}^{k_{r-2}-j_{r-2}-j_{r-1}} \sum_{k_{r-1}=j_r+j_{r-1}}^{k_{r-2}-j_{r-2}}.$$

Thus, we can rewrite the expression (C.8) as

$$\begin{aligned} & \sum_{j_{r-2}=0}^{k_{r-2}} \sum_{j_{r-1}=0}^{k_{r-2}-j_{r-2}} \sum_{j_r=0}^{k_{r-2}-j_{r-2}-j_{r-1}} \frac{(-1)^{k_{r-2}-j_{r-2}-j_{r-1}-j_r} u_{j_{r-2}} u_{j_{r-1}} u_{j_r}}{j_{r-2}! j_{r-1}! j_r!} \\ & \times \sum_{k_{r-1}=j_r+j_{r-1}}^{k_{r-2}-j_{r-2}} \frac{2^{k_{r-1}-j_{r-1}-j_r}}{(k_{r-1}-j_{r-1}-j_r)!(k_{r-2}-j_{r-2}-k_{r-1})!}. \end{aligned} \quad (\text{C.9})$$

The last sum can be easily computed:

$$\sum_{k_{r-1}=j_r+j_{r-1}}^{k_{r-2}-j_{r-2}} \frac{2^{k_{r-1}-j_{r-1}-j_r}}{(k_{r-1}-j_{r-1}-j_r)!(k_{r-2}-j_{r-2}-k_{r-1})!} = \frac{3^{k_{r-2}-j_{r-2}-j_{r-1}-j_r}}{(k_{r-2}-j_{r-2}-j_{r-1}-j_r)!} \quad (\text{C.10})$$

and substituting it into eq. (C.9), we obtain the term:

$$\sum_{j_{r-2}=0}^{k_{r-2}} \sum_{j_{r-1}=0}^{k_{r-2}-j_{r-2}} \sum_{j_r=0}^{k_{r-2}-j_{r-2}-j_{r-1}} \frac{(-1)^{k_{r-2}-j_{r-2}-j_{r-1}-j_r} 3^{k_{r-2}-j_{r-2}-j_{r-1}-j_r} u_{j_{r-2}} u_{j_{r-1}} u_{j_r}}{j_{r-2}! j_{r-1}! j_r! (k_{r-2}-j_{r-2}-j_{r-1}-j_r)!}. \quad (\text{C.11})$$

The comparison between the relations (C.7) and (C.11) allows us to deduce, by analogy, the general relation

$$\begin{aligned} & \sum_{k_l=0}^{k_{l-1}} \zeta_{k_{l-1}-k_l} \cdots \sum_{k_{r-1}=0}^{k_{r-2}} \zeta_{k_{r-2}-k_{r-1}} \sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r} = \\ & = \sum_{j_{l-1}=0}^{k_{l-1}} \sum_{j_l=0}^{k_{l-1}-j_{l-1}} \cdots \sum_{j_{r-1}=0}^{k_{l-1}-\sum_{i=l-1}^{r-2} j_i} \sum_{j_r=0}^{k_{l-1}-\sum_{i=l-1}^{r-1} j_i} \frac{(l-r-2)^{k_{l-1}-\sum_{i=l-1}^r j_i}}{(k_{l-1}-\sum_{i=l-1}^r j_i)!} \prod_{i=l-1}^r \frac{u_{j_i}}{j_i!}. \end{aligned} \quad (\text{C.12})$$

In particular, for $l = 2$, we obtain

$$\sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_1-j_1} \cdots \sum_{j_{r-1}=0}^{k_1-\sum_{i=1}^{r-2} j_i} \sum_{j_r=0}^{k_1-\sum_{i=1}^{r-1} j_i} \frac{(-r)^{k_1-\sum_{i=1}^r j_i}}{(k_1-\sum_{i=1}^r j_i)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \quad (\text{C.13})$$

By moving the first sum to the end, we arrive at the correspondence

$$y(x)^r \rightarrow \sum_{k_1=0}^n \frac{n!}{(n-k_1)!} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_1-j_1} \cdots \sum_{j_{r-1}=0}^{k_1-\sum_{i=1}^{r-2} j_i} \sum_{j_r=0}^{k_1-\sum_{i=1}^{r-1} j_i} \frac{(-r)^{k_1-\sum_{i=1}^r j_i}}{(k_1-\sum_{i=1}^r j_i)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \quad (\text{C.14})$$

Finally, since

$$\begin{aligned} & \sum_{k_1=0}^n \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_1-j_1} \sum_{j_3=0}^{k_1-(j_1+j_2)} \cdots \sum_{j_{r-1}=0}^{k_1-\sum_{i=1}^{r-2} j_i} \sum_{j_r=0}^{k_1-\sum_{i=1}^{r-1} j_i} = \\ & = \sum_{j_1=0}^n \sum_{j_2=0}^{n-j_1} \sum_{j_3=0}^{n-(j_1+j_2)} \cdots \sum_{j_{r-1}=0}^{n-\sum_{i=1}^{r-2} j_i} \sum_{j_r=0}^{n-\sum_{i=1}^{r-1} j_i} \sum_{k_1=\sum_{i=1}^r j_i}^n \end{aligned} \quad (\text{C.15})$$

we are led to the sum

$$\sum_{k_1=\sum_{i=1}^r j_i}^n \frac{(-r)^{k_1-\sum_{i=1}^r j_i}}{(n-k_1)!(k_1-\sum_{i=1}^r j_i)!} = \frac{(1-r)^{n-\sum_{i=1}^r j_i}}{(n-\sum_{i=1}^r j_i)!}. \quad (\text{C.16})$$

Consequently, collecting together all the intermediate steps, our final result is:

$$y(x)^r \rightarrow \sum_{j_1=0}^n \sum_{j_2=0}^{n-j_1} \sum_{j_3=0}^{n-(j_1+j_2)} \cdots \sum_{j_{r-1}=0}^{n-\sum_{i=1}^{r-2} j_i} \sum_{j_r=0}^{n-\sum_{i=1}^{r-1} j_i} \frac{n!(1-r)^{n-\sum_{i=1}^r j_i}}{(n-\sum_{i=1}^r j_i)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \quad (\text{C.17})$$

By following the same technique, the series expansion of the function $x^m y(x)^r$ can be written as:

$$x^m y(x)^r = \sum_{k_1=0}^{\infty} x^{k_1+m} \sum_{k_2=0}^{k_1} \zeta_{k_1-k_2} \sum_{k_3=0}^{k_2} \zeta_{k_2-k_3} \cdots \sum_{k_{r-1}=0}^{k_{r-2}} \zeta_{k_{r-2}-k_{r-1}} \sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r}. \quad (\text{C.18})$$

Its categorically equivalent is:

$$\begin{aligned} x^m y(x)^r & \rightarrow \sum_{k_1=0}^{n-m} \frac{n! h^{k_1+m}}{(n-m-k_1)!} \sum_{k_2=0}^{k_1} \zeta_{k_1-k_2} \sum_{k_3=0}^{k_2} \zeta_{k_2-k_3} \cdots \\ & \sum_{k_{r-1}=0}^{k_{r-2}} \zeta_{k_{r-2}-k_{r-1}} \sum_{k_r=0}^{k_{r-1}} \zeta_{k_{r-1}-k_r} \zeta_{k_r}. \end{aligned} \quad (\text{C.19})$$

By using the previous results, we get:

$$\begin{aligned} & x^m y(x)^r \rightarrow \\ & \rightarrow h^m \sum_{k_1=0}^{n-m} \frac{n!}{(n-m-k_1)!} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_1-j_1} \cdots \sum_{j_{r-1}=0}^{k_1-j_1} \sum_{j_r=0}^{k_1-j_1} \frac{(-r)^{k_1-\sum_{i=1}^r j_i}}{(k_1-\sum_{i=1}^r j_i)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}. \end{aligned} \quad (\text{C.20})$$

If we set $n-m$ as the n in the previous section, as a final result we obtain

$$x^m y(x)^r \rightarrow \begin{cases} 0, & n < m \\ h^m \sum_{j_1=0}^{n-m} \sum_{j_2=0}^{n-m} \cdots \sum_{j_r=0}^{n-m} \frac{n!(1-r)^{n-m-\sum_{i=1}^r j_i}}{(n-m-\sum_{i=1}^r j_i)!} \prod_{i=1}^r \frac{u_{j_i}}{j_i!}, & n \geq m. \end{cases} \quad (\text{C.21})$$

Data availability

No data was used for the research described in the article.

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