Formality of Donaldson submanifolds

Marisa Fernández and Vicente Muñoz

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Abstract

We introduce the concept of *s–formal minimal model* as an extension of formality. We prove that any orientable compact manifold M, of dimension $2n$ or $(2n-1)$, is formal if and only if M is $(n-1)$–formal. The formality and the hard Lefschetz property are studied for the Donaldson submanifolds of symplectic manifolds constructed in [13]. This study permits us to show an example of a Donaldson symplectic submanifold of dimension eight which is formal simply connected and does not satisfy the hard Lefschetz theorem.

1 Introduction

A symplectic manifold (M, ω) is a pair consisting of a $2n$–dimensional differentiable manifold M with a closed 2–form ω which is non-degenerate (that is, ω^n never vanishes). The form ω is called symplectic. By the Darboux theorem, in canonical coordinates, ω can be expressed as $\omega = \sum_{i=1}^{n} dx^i \wedge dx^{n+i}$. Therefore any symplectic manifold admits an almost complex structure J compatible with the symplectic form, which means that $\omega(X, Y) = \omega(JX, JY)$ for any X, Y vector fields on M.

The simplest examples of symplectic manifolds are Kähler manifolds; for example, the complex projective space $\mathbb{CP}^n$ with the standard Kähler form ω_0 defined by its natural complex structure and the Fubini–Study metric. Gromov [19, 20] and Tischler [43] prove that if M is a compact symplectic manifold, of dimension $2n$, with an integral symplectic form ω, then there is a symplectic embedding $f\colon (M, \omega) \longrightarrow (\mathbb{CP}^{2n+1}, \omega_0)$ such that $f^*(\omega_0) = \omega$.

If (M, ω) is a compact symplectic manifold, then the de Rham cohomology groups $H^{2k}(M)$, $k \leq n$, are non-trivial. (We denote by $H^*(M)$ the real or de Rham cohomology, and by $H^*(M; \mathbb{Z})$ the integral one.) The problem of how compact symplectic manifolds differ topologically from Kähler manifolds led during the last years to the introduction of several geometric methods for constructing symplectic manifolds. They include compact nilmanifolds [9, 11, 42], symplectic blow ups [32], and fiber connected sums [17]. The symplectic manifolds there presented do not admit a Kähler metric since either they are not formal or do not satisfy hard Lefschetz theorem, or they fail both properties of compact Kähler manifolds.

Let (M, ω) be a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ having a lift to an integral cohomology class h. In [13] Donaldson proves the existence of some integer number k_0 such that for any $k \geq k_0$ there exists a symplectic submanifold $Z \hookrightarrow M$ of dimension $2n-2$ that realizes the Poincaré dual of kh, that is, $\mathrm{PD}[Z] = kh \in H^2(M; \mathbb{Z})$. We shall call these manifolds *Donaldson symplectic submanifolds* (or, indistinctly, *Donaldson submanifolds*) of M. Such manifolds satisfy a *Lefschetz theorem on hyperplane sections*. This means that the inclusion $j\colon Z \hookrightarrow M$ is $(n-1)$–connected, i.e., up to homotopy M is constructed out of Z by attaching cells of dimension n and higher. In particular,

- for $i < n-1$ there is an isomorphism $j^*: H^i(M) \to H^i(Z)$ induced by j on cohomology;
- for $i = n-1$ there is a monomorphism $j^*: H^i(M) \hookrightarrow H^i(Z)$.

Our purpose in this note is to study the formality and the hard Lefschetz theorem for Donaldson submanifolds of symplectic manifolds. As a consequence of this study, we prove that a compact simply connected symplectic 8-manifold constructed in [24] is formal, while it does not satisfy the hard Lefschetz theorem.

The description of a minimal model for a Donaldson submanifold of a symplectic manifold can be very complicated even for the degree $n-1$. This is the reason for which we need first to weaken the condition of formal manifold to *s–formal manifold* ($s \geq 0$) as follows. Let $(\bigwedge V, d)$ be a minimal model of a differentiable manifold M (of arbitrary dimension). We say that $(\bigwedge V, d)$ is a *s–formal minimal model*, or M is a *s–formal manifold*, if for each $i \leq s$ the subspace V^i of V, consisting of the generators of degree i, decomposes as a direct sum $V^i = C^i \oplus N^i$ where the spaces C^i and N^i satisfy the three following conditions:

(i) $d(C^i) = 0$,

(ii) the differential map $d: N^i \longrightarrow \bigwedge V$ is injective,

(iii) any closed element in the ideal $I(\bigoplus_{i\leq s} N^i) = N^{\leq s} \cdot (\bigwedge V^{\leq s}))$, generated by $\bigoplus_{i\leq s} N^i$ in $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

Note that if M is $(s+1)$–formal, then M is s–formal. All connected manifolds are obviously 0-formal, and there are examples of non-formal manifolds. For any $s \geq 0$, we show examples of manifolds which are s–formal but not $(s+1)$–formal (see Example 5 in Section 6). The relation between the s–formality and the formality is given in Theorem 3.1 of Section 3. There we prove that any orientable compact connected manifold M, of dimension $2n$ or $(2n-1)$, is formal if and only if M is $(n-1)$–formal. This means that the formality of M is contained in the $(n-1)$ first subspaces V^i $(1 \leq i \leq n-1)$ of the minimal model of M, and so we can ignore the other subspaces.

As a consequence of Theorem 3.1 we get Miller's theorem [34] for the formality of a $(k-1)$–connected compact manifold of dimension less than or equal to $(4k-2)$. We show that any simply connected oriented compact manifold M of arbitrary dimension is 2–formal, as well as if M has dimension 7 or 8, M is formal if and only if is 3–formal.

In Theorem 5.2 we prove that if M is a $(n-2)$–formal compact symplectic manifold of dimension $2n$, and $Z \hookrightarrow M$ is a Donaldson submanifold, then Z is formal. Therefore, it can happen that Z is formal but M is non-formal. Moreover, in Theorem 5.3 we get the conditions on M under which it is possible to state that Z satisfies the hard Lefschetz theorem.

The paper is structured as follows. In Section 2 we establish the concept of s–formal manifold. For such a manifold M we show, in Lemma 2.7, that if $(\bigwedge V, d)$ is the minimal model of M then the minimal model $(\bigwedge W, d)$ of the differential algebra $(H^*(M), d = 0)$ is given by $(\bigwedge V^{\leq s}, d)$ by adding spaces $W^{>s}$ with suitable differentials. The relation between s–formality and Massey products is proved in Lemma 2.9.

In Section 3 we determine the smallest value of s for which the s–formality is equivalent to the formality of M proving Theorem 3.1. Some consequences are discussed; in particular, Miller's theorem [34]. Section 4 is dedicated to compact symplectic manifolds (M, ω) with the s–Lefschetz property ($s \leq (n-1)$), i.e., satisfying that the cup product

$$[\omega]^{n-i} : H^i(M) \longrightarrow H^{2n-i}(M)$$

is an isomorphism for all $i \leq s$. In Section 5 we prove that a Donaldson submanifold $Z \subset M$ is hard Lefschetz if and only if M has the $(n-2)$–Lefschetz property. We also show the Theorem 5.2 previously mentioned on the formality of Donaldson submanifolds.

Finally, Section 6 is devoted to the discussion of examples illustrating the concepts and results of the previous sections. Such examples reveal the existence of Donaldson symplectic submanifolds satisfying one of the following properties: formal and hard Lefschetz; neither formal nor hard Lefschetz; or formal but not hard Lefschetz. Furthermore, some of those are 4–dimensional symplectic manifolds that have neither Kähler metrics nor complex structures.

2 s–formality and real homotopy

In this section, we establish the concept of s–formal minimal model and show some properties for such manifolds. First, we need some definitions and results about minimal models.

Let (A, d) be a *differential graded algebra* (in the sequel, we shall say just a differential algebra), that is, A is a graded commutative algebra over a field K, of characteristic zero, with a differential d which is a derivation, i.e. $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a. Throughout this article all vector spaces and algebras are defined over the field $\mathbb{R}$ of real numbers unless there is indication to the contrary.

A differential algebra (A, d) is said to be *minimal* if:

(i) A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and

(ii) there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ ($\mu < \tau$). This implies that da_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. Given a differential algebra (A, d), we denote by $H^*(A)$ its cohomology. A is *connected* if $H^0(A) = \mathbb{R}$, and A is *one–connected* if, in addition, $H^1(A) = 0$.

We shall say that $(\mathcal{M}, d)$ is a *minimal model* of the differential algebra (A, d) if $(\mathcal{M}, d)$ is minimal and there exists a morphism of differential graded algebras $\rho\colon (\mathcal{M}, d) \longrightarrow (A, d)$ inducing an isomorphism $\rho^*\colon H^*(\mathcal{M}) \longrightarrow H^*(A)$ on cohomology.

In [21] Halperin proved that any connected differential algebra (A, d) has a minimal model unique up to isomorphism. For 1–connected differential algebras, a similar result was proved by Deligne, Griffiths, Morgan and Sullivan [12, 18, 40].

A minimal model $(\mathcal{M}, d)$ is said to be *formal* if there is a morphism of differential algebras $\psi\colon (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}), d = 0)$ that induces the identity on cohomology. The formality of a minimal model can be characterized as follows.

Theorem 2.1 *[12]. A minimal model $(\mathcal{M}, d)$ is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega M, d)$ of differential forms on M. If M is a simply connected manifold, the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i. This relation also happens when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for $j > 1$ (see [12, 18]).

We shall say that M is *formal* if its minimal model is formal or, equivalently, the differential algebras $(\Omega M, d)$ and $(H^*(M), d = 0)$ have the same minimal model. (For details see [12, 18, 27] for example.) Therefore, if M is formal and simply connected, then the real homotopy groups $\pi_i(M) \otimes \mathbb{R}$ are obtained from the minimal model of $(H^*(M), d = 0)$.

From now on, we will consider only connected differentiable manifolds. In order to obtain some information on the formality of a manifold, we introduce the concept of s–formality as follows.

Definition 2.2 *Let $(\mathcal{M}, d)$ be a minimal model. We say that $(\mathcal{M}, d)$ is s–formal ($s \geq 0$) if we can write $\mathcal{M} = \bigwedge V$ such that for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:*

(i) $d(C^i) = 0$,

(ii) the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective,

(iii) any closed element in the ideal $I_s = I(\bigoplus_{i\leq s} N^i)$, generated by the space $\bigoplus_{i\leq s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

In what follows, we shall write $N^{\leq s}$ and $\bigwedge V^{\leq s}$ instead of $\bigoplus_{i\leq s} N^i$ and $\bigwedge(\bigoplus_{i\leq s} V^i)$, respectively. In particular, $I_s = N^{\leq s} \cdot (\bigwedge V^{\leq s})$.

Remark 2.3 *We must note that the conditions in Definition 2.2 are not the same as to ask that $(\bigwedge V^{\leq s}, d)$ is formal, since in the later case, one should ask that every closed element in the ideal $I(N^{\leq s})$ is exact in $\bigwedge V^{\leq s}$. Moreover, Definition 2.2 implies that if $(\bigwedge V, d)$ is formal then it also is s–formal for any $s \geq 0$.*

Definition 2.4 *A differentiable manifold M is s–formal if its minimal model is s–formal (in the sense of the previous Definition).*

Remark 2.5 *The concept of s–formality can be defined for CW–complexes which have a minimal model $(\bigwedge V, d)$. Such a minimal model is constructed as the minimal model associated to the differential complex of piecewise-linear rational polynomial forms [18].*

Let M be an s–formal manifold with minimal model $(\bigwedge V, d)$ as in Definition 2.2. Clearly, the space $\bigwedge V^{\leq s}$ has the decomposition

$$\bigwedge V^{\leq s} = (\bigwedge C^{\leq s}) \oplus N^{\leq s} \cdot (\bigwedge V^{\leq s}). \qquad (1)$$

For degree $i \leq s$ it is clear that $(\bigwedge V^{\leq s})^i = (\bigwedge V)^i$. Then for $i \leq s$ we have a surjection

$$(\bigwedge C^{\leq s})^i \twoheadrightarrow H^i(M).$$

Before going into the study of s–formal manifolds, we show examples of compact connected manifolds which are 0–formal but not 1–formal. The simplest examples are the compact nilmanifolds non-tori. Let G be a connected rational nilpotent Lie group, and denote by Γ a discrete subgroup of G such that the quotient space $M = \Gamma\backslash G$ is compact [29]. Such a manifold M is called a compact nilmanifold. Hasegawa's theorem [22] states that the tori are the only formal compact nilmanifolds. We reformulate that theorem as follows.

Lemma 2.6 *[22]. Let $M = \Gamma\backslash G$ be a compact nilmanifold. Obviously M is 0–formal. Moreover, M is 1–formal if and only if M is a torus, and so formal.*

Proof : It is clear that any differentiable manifold is 0–formal. Consider $M = \Gamma\backslash G$ a compact nilmanifold. Then, a minimal model $(\mathcal{M}, d)$ of M is given by $(\bigwedge(\mathfrak{g}^*), d)$, where $\mathfrak{g}$ is the Lie algebra of G, and d is the Chevalley–Eilenberg differential in $\bigwedge(\mathfrak{g}^*)$. Since all the generators of $\mathcal{M}$ have degree 1, according to Definition 2.2 and Definition 2.4, we get that $\mathcal{M}$ is 1–formal if and only if $\mathcal{M}$ is formal, and so M is formal. This completes the proof using Hasegawa's theorem. *QED*

In Section 6 we construct examples of *non-symplectic* manifolds which are s–formal but not $(s+1)$–formal for $s \geq 2$ (see Example 5), and we show examples of compact *symplectic* manifolds which are s–formal but not $(s+1)$–formal for $s = 1$ (see Example 3) and for $s = 3$ (see Example 4).

Next, we show the first properties of s–formal manifolds. For such a manifold, the analogous result to Theorem 2.1 is the following lemma.

Lemma 2.7 *Let M be a manifold with minimal model $(\bigwedge V, d)$. Then M is s–formal if and only if there is a map of differential algebras*

$$\varphi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0),$$

such that the map $\varphi^* : H^*(\bigwedge V^{\leq s}, d) \longrightarrow H^*(M)$ induced on cohomology is equal to the map $\imath^* : H^*(\bigwedge V^{\leq s}, d) \longrightarrow H^*(\bigwedge V, d) = H^*(M)$ induced by the inclusion $\imath : (\bigwedge V^{\leq s}, d) \longrightarrow (\bigwedge V, d)$.

In particular, $\varphi^* : H^i(\bigwedge V^{\leq s}) \longrightarrow H^i(M)$ is an isomorphism for $i \leq s$, and a monomorphism for $i = s + 1$. So, if M is simply connected, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to $V^i = W^i$ for any $i \leq s$, $(\bigwedge W, d)$ being the minimal model of $(H^*(M), d = 0)$.

Proof : Since $(\bigwedge V, d)$ is a minimal model of M, we know that there is a morphism $\rho\colon (\bigwedge V, d) \longrightarrow (\Omega M, d)$ inducing an isomorphism ρ^* on cohomology. Thus to prove the *only if* part, it is sufficient to show that there is a map of differential algebras

$$\psi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(\bigwedge V), d = 0),$$

such that the map $\psi^* : H^*(\bigwedge V^{\leq s}) \longrightarrow H^*(\bigwedge V)$ coincides with the map $\imath^* : H^*(\bigwedge V^{\leq s}) \longrightarrow H^*(\bigwedge V)$ induced by the inclusion $\imath : (\bigwedge V^{\leq s}, d) \longrightarrow (\bigwedge V, d)$. Then the map φ given by $\varphi = \rho^* \circ \psi$ satisfies the conditions that we need.

We define $\psi(x) = [x]$ for $x \in C^i$ and $\psi(x) = 0$ for $x \in N^i$, $i \leq s$. We extend ψ to an algebra map $\psi : \bigwedge V^{\leq s} \longrightarrow H^*(\bigwedge V)$ by multiplicativity. We see that s–formality implies that ψ commutes with the differentials, as follows. Let $x \in \bigwedge V^{\leq s}$. Then, from (1), it follows that dx decomposes $dx = a + b$ with $a \in \bigwedge C^{\leq s}$ and $b \in N^{\leq s} \cdot (\bigwedge V^{\leq s})$. Since a is closed, so is $b = dx - a$ and hence exact by s–formality. Therefore $a = dx - b$ is exact as well, and $\psi(dx) = \psi(a) = [a] = 0$. This shows that $\psi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(\bigwedge V), d = 0)$ is a map of differential algebras.

Moreover, if $x \in \bigwedge V^{\leq s}$ is closed, decompose $x = a + b$ with $a \in \bigwedge C^{\leq s}$, $b \in N^{\leq s} \cdot (\bigwedge V^{\leq s})$. Then $b = x - a$ is closed, hence exact by s–formality. So $\psi^*[x] = [\psi(x)] = \psi(x) = \psi(a) = [a] = [a + b] = [x]$, hence $\psi^* = \imath^*$, as required.

To prove the *if* part, suppose that we have a map $\varphi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0)$ satisfying $\varphi^* = \imath^*$. We want to find an s–formal model for M, i.e., $\bigwedge V = \bigwedge \hat{V}$ such that $\hat{V}^i = \hat{C}^i \oplus \hat{N}^i$ satisfies the conditions of Definition 2.2. Moreover we construct this model in such a way that $\bigwedge V^{\leq i} = \bigwedge \hat{V}^{\leq i}$ for all i. Let us do this by induction on i. Suppose that we have defined $\hat{V}^{<i} = \hat{C}^{<i} \oplus \hat{N}^{<i}$, with $i \leq s$. Then we define

$$N^i = \ker(\varphi : V^i \to H^i(M)/\mathrm{im}(\varphi\colon \bigwedge V^{<i} \to H^i(M))).$$

For $x \in N^i$, let $a_x \in \bigwedge V^{<i}$ be a closed element such that $\varphi(x) = [a_x]$ and set $\hat{x} = x - a_x$. This defines a space $\hat{N}^i$ isomorphic to N^i. Consider the space C^i given by

$$C^i = \ker(d : V^i \to \bigwedge V/d(\bigwedge V^{<i})).$$

Now for $y \in C^i$, let $b_y \in \bigwedge V^{<i}$ such that $dy = db_y$. This b_y is well-defined up to a closed element, so we may suppose that $\varphi(b_y) = 0$. Set $\hat{y} = y - b_y$. This defines a space $\hat{C}^i$ isomorphic to C^i. Now $\varphi(\hat{N}^i) = 0$ and $d(\hat{C}^i) = 0$. If we check that $V^i = C^i \oplus N^i$ then it follows that $\hat{V}^i = \hat{C}^i \oplus \hat{N}^i$ is isomorphic to V^i.

First, if $x \in N^i \cap C^i$ then $\varphi(x - a_x) = 0$ and $d(x - b_x) = 0$. So $x - a_x - b_x$ is closed and $\varphi(x - a_x - b_x) = 0$. Therefore $x - a_x - b_x$ is exact, which contradicts the minimality of the model. Thus $N^i \cap C^i = 0$. Second, if $x \in V^i$ then consider $\varphi(x) = [t]$ where $t \in V^i \oplus \bigwedge V^{<i}$ and $dt = 0$. Decompose $t = t_1 + t_2$ where $t_1 \in V^i$, $t_2 \in \bigwedge V^{<i}$, so that $t_1 \in C^i$. Now $\varphi(x - t) = 0$ so that $x - t_1 \in N^i$. Therefore $x \in C^i \oplus N^i$. The properties of Definition 2.2 are now easy to verify for $\hat{V}^i$.

For the final assertion, let $(\bigwedge W, d)$ be the minimal model of $H^*(M)$. This can be constructed starting with $(\bigwedge V^{\leq s}, d)$, and with the map $\varphi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0)$, and adding subspaces $W^{>s}$ with suitable differentials (see [44]). *QED*

Remark 2.8 *The concept of* 1*–formality appears in [1, Chapter 2] defined by the formulation given in Lemma 2.7. In [1] the* 1*–formality is studied in connection with the fundamental group of the manifold.*

It is well known that all Massey products vanish for any formal manifold. The relation between s–formality and Massey products is given in the following lemma.

Lemma 2.9 *Let M be an s–formal manifold. Suppose that there are cohomology classes $\alpha_i \in H^{p_i}(M)$, $p_i > 0$, $1 \leq i \leq t$, such that the Massey product $\langle \alpha_1, \alpha_2, \dots, \alpha_t \rangle$ is defined. If $p_1 + p_2 + \cdots + p_{t-1} \leq s + t - 2$ and $p_2 + \cdots + p_t \leq s + t - 2$, then $\langle \alpha_1, \alpha_2, \dots, \alpha_t \rangle$ vanishes.*

Proof : Let $(\bigwedge V, d)$ be a minimal model of M. There exists a morphism $\rho\colon (\bigwedge V, d) \longrightarrow (\Omega M, d)$ inducing an isomorphism ρ^* on cohomology. For $1 \leq i \leq t$, denote by $[a_i] \in H^{p_i}(\bigwedge V)$ the cohomology classes such that $\rho^*[a_i] = \alpha_i$. To prove the Lemma we see that the Massey product $\langle [a_1], [a_2], \dots, [a_t] \rangle$ vanishes.

First we show it for triple Massey products. Consider a Massey product $\langle [a_1], [a_2], [a_3] \rangle$, $[a_i] \in H^{p_i}(\bigwedge V)$, with $p_1 + p_2 \leq s + 1$ and $p_2 + p_3 \leq s + 1$. Suppose that $a_1 \cdot a_2 = d\xi_1$ and $a_2 \cdot a_3 = d\xi_2$. Since the degree of ξ_j does not exceed s, we have that $\xi_j \in \bigwedge V^{\leq s}$. Projecting onto the second summand of (1) we can suppose that $\xi_j \in I_s$. By the s–formality,

$$a_1 \cdot \xi_2 + (-1)^{p_1+1} \xi_1 \cdot a_3$$

is exact, and so the triple Massey product $\langle [a_1], [a_2], [a_3] \rangle$ vanishes.

The case of the higher Massey product is similar. Let us first recall the definition (see [26, 30, 41]). If the Massey product $\langle [a_1], [a_2], \dots, [a_t]\rangle$ is defined, then there are elements $a_{i,j}$ of the minimal model $\bigwedge V$ of M, with $1 \le i \le j \le t$, except for the case $(i,j) = (1,t)$, such that $a_{i,i}$ is a cocycle representing $[a_i]$ and $d\, a_{i,j} = \sum_{k=i}^{j-1} \bar{a}_{i,k} \cdot a_{k+1,j}$, where $\bar{a} = (-1)^{\deg(a)} a$. Then the Massey product $\langle [a_1], [a_2], \dots, [a_t]\rangle$ is the set of all possible cohomology classes of degree $p_1 + \cdots + p_t - (t-2)$ whose representatives are $\sum_{k=1}^{t-1} \bar{a}_{1,k} \cdot a_{k+1,t}$. If one of these representatives is exact, then the Massey product $\langle [a_1], [a_2], \dots, [a_t]\rangle$ is zero.

Now, for $k = 1$, $a_{1,1}$ is a closed element representing the cohomology class $[a_1]$ and we know that $\deg(a_1) = p_1 \le s$. For $k = 2$, $d\, a_{1,2} = a_1 \cdot a_2$, that is, $\deg(a_{1,2}) = p_1 + p_2 - 1$ which is $\le s$. For any $3 \le k \le t-1$, $d\, a_{1,k}$ is a representative of the Massey product $\langle [a_1], [a_2], \dots, [a_k]\rangle$. Thus, $\deg(a_{1,k}) = p_1 + \cdots + p_k - (k-1)$ which is less than or equal to s by hypothesis. Hence $a_{1,1} \in \bigwedge C^{\le s}$ and $a_{1,k} \in \bigwedge V^{\le s}$ for $2 \le k \le t-1$. In a similar way we see that $\deg(a_{k+1,t}) \le s$ for $1 \le k \le t-2$; and, for $k = t-1$, the element $a_{k+1,t} = a_{t,t}$ is a representative of $[a_t]$ and so it has degree $\le s$. Therefore, $a_{1,t-1}$, $a_{2,t}$, $a_{1,k}$ and $a_{k+1,t} \in \bigwedge V^{\le s}$ for $2 \le k \le t-2$, and $a_{1,1}$, $a_{t,t} \in \bigwedge C^{\le s}$. Using (1) one can project onto I_s, so we can make choices so that $a_{1,t-1}$, $a_{2,t}$, $a_{1,k}$ and $a_{k+1,t} \in I_s$, for $2 \le k \le t-2$. This implies that $\sum_{k=1}^{t-1} \bar{a}_{1,k} a_{k+1,t}$ is a closed element in the ideal I_s and hence it is exact since $(\bigwedge V, d)$ is s–formal. *QED*

Other properties of s–formal manifolds are given in the following lemmas.

Lemma 2.10 *Let M be a differentiable manifold of dimension m. Then M is formal if and only if M is m–formal.*

Proof : From Theorem 2.1 and Definition 2.2 it follows that if M is formal then is m–formal because M is s–formal for all s.

Conversely, let us suppose that the differentiable manifold M is m–formal and let $(\bigwedge V, d)$ be an m–formal minimal model of M. Because V is a graded vector space, we can define the spaces N^i by $N^i = V^i$ for $i > m$. Denote by N the graded space $N = \bigoplus_{j>0} N^j$. To prove that M is formal we use Theorem 2.1. It is sufficient to show that any closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact. Let η be such an element. There are two possibilities according to $\deg(\eta) \le m$ or $\deg(\eta) > m$. If $\deg(\eta) \le m$, then η lies in the ideal $I_m(N^{\le m})$, and so η is exact because M is m–formal. If $\deg(\eta) > m$, then η defines a cohomology class in the cohomology group $H^{\deg(\eta)}(\bigwedge V)$. Since that M has dimension $m < \deg(\eta)$ and $(\bigwedge V, d)$ is a model of M, the group $H^{\deg(\eta)}(\bigwedge V)$ must be equal to zero, which implies that the cohomology class $[\eta]$ is the trivial class and so η is exact. *QED*

Lemma 2.11 *Let M_1 and M_2 be differentiable manifolds. For any $s \ge 0$, the product manifold $M = M_1 \times M_2$ is s–formal if and only if M_1 and M_2 are s–formal.*

Proof : Denote by $(\bigwedge V_i, d_i)$ the minimal model of M_i. Then the minimal model of M is $(\bigwedge V, d)$ with $\bigwedge V = \bigwedge V_1 \otimes \bigwedge V_2$ and differential $d = d_1 \otimes 1 + 1 \otimes d_2$. Since M_i $(i = 1, 2)$ is s–formal, Lemma 2.7 implies the existence of a map of differential algebras

$$\varphi_i \colon \bigwedge V_i^{\le s} \longrightarrow H^*(M_i)\,,$$

such that the induced map in cohomology φ_i^* equals the map induced by the inclusion $\bigwedge V_i^{\leq s} \hookrightarrow \bigwedge V_i$. Consider $\varphi = \varphi_1 \otimes \varphi_2$. Then

$$\varphi\colon \bigwedge(V_1^{\leq s} \oplus V_2^{\leq s}) \longrightarrow H^*(M_1) \otimes H^*(M_2) = H^*(M)$$

satisfies the conditions of Lemma 2.7, hence M is s–formal.

Suppose now that $M = M_1 \times M_2$ is s–formal. Then by Lemma 2.7, there exists a map of differential algebras

$$\varphi\colon \bigwedge V^{\leq s} \longrightarrow H^*(M),$$

such that $\varphi^*\colon H^*(\bigwedge V^{\leq s}) \to H^*(\bigwedge V) \cong H^*(M)$ is the map induced by the inclusion $\bigwedge V^{\leq s} \hookrightarrow \bigwedge V$.

Define φ_1 as the inclusion $\bigwedge V_1^{\leq s} \hookrightarrow \bigwedge V^{\leq s}$ followed by φ and by the projection $H^*(M) = H^*(M_1) \otimes H^*(M_2) \twoheadrightarrow H^*(M_1)$. This is a map of differential algebras and it is easy to see that it satisfies the conditions of Lemma 2.7. In fact, φ_1^* equals the map induced in cohomology by the composition $\bigwedge V_1^{\leq s} \hookrightarrow \bigwedge V^{\leq s} \hookrightarrow \bigwedge V = \bigwedge(V_1 \oplus V_2) \twoheadrightarrow \bigwedge V_1$, and this map is the inclusion $\bigwedge V_1^{\leq s} \hookrightarrow \bigwedge V_1$. QED

3 Formality and s–formality

The purpose of this section is to prove the following theorem.

Theorem 3.1 *Let M be a connected and orientable compact differentiable manifold of dimension $2n$, or $(2n-1)$. Then M is formal if and only if is $(n-1)$–formal.*

Proof : In one direction the proof is obvious. So we need to show that the $(n-1)$–formality of M implies its formality. First suppose that the theorem holds for any $(n-1)$–formal manifold of dimension $2n$. Now, if M is a $(n-1)$–formal manifold of $\dim M = (2n-1)$, the product manifold $M \times S^1$ is $2n$–dimensional and $(n-1)$–formal according to Lemma 2.11. Our assumption implies that $M \times S^1$ is formal. But a product manifold $M_1 \times M_2$ is formal if and only if each one of the manifolds M_1 and M_2 is formal. Therefore, M must be formal, which proves the theorem for odd-dimensional differentiable manifolds.

To prove the theorem when $\dim M = 2n$ we will show that M is $(n+r-1)$–formal for any $r \geq 0$ proceeding by induction on r. If $r = 0$ then M is $(n-1)$–formal by the hypothesis of the theorem. Let us suppose that M is $(n+r-1)$–formal and we will show that M is $(n+r)$–formal for $r \geq 0$.

Let $(\bigwedge V, d)$ be a $(n+r-1)$–formal minimal model of M. By the induction hypothesis, we know that each one of the spaces $V^{\leq(n+r-1)}$, of generators of degree $\leq (n+r-1)$, satisfies the conditions of Definition 2.2. Since $(\bigwedge V, d)$ is a minimal differential algebra, it is possible to order the generators $\{x_1, x_2, \ldots\}$ of V^{n+r} in such way that $d\,x_j \in \bigwedge(V^{\leq(n+r-1)} \oplus \langle x_1, \ldots, x_{j-1}\rangle)$ for $j \geq 1$. Now, for each generator x_i of V^{n+r}, we define the space V_i by

$$V_i = V^{\leq(n+r-1)} \oplus \langle x_1, \ldots, x_i\rangle.$$

Here we can take $i \geq 0$ and $V_0 = V^{\leq(n+r-1)}$.

We aim to construct a $(n+r)$–formal minimal model of M. For this, for each $x_i \in V^{n+r}$ we shall find $\psi_i \in \bigwedge V_{i-1}$ such that $\hat{x}_i = x_i - \psi_i$ gives a new set of generators, and the space

$$\hat{V}_i = V^{\leq(n+r-1)} \oplus \langle \hat{x}_1, \ldots, \hat{x}_i\rangle$$

satisfies the conditions of Definition 2.2, i.e., $\hat{V}_i$ decomposes as a direct sum $\hat{V}_i = \hat{C}_i \oplus \hat{N}_i$ with $d(\hat{C}_i) = 0$, d injective on $\hat{N}_i$, and that every closed element in the ideal $I(\hat{N}_i)$, generated by $\hat{N}_i$ in $\bigwedge \hat{V}_i$, is exact in $\bigwedge V$. Note that $\bigwedge \hat{V}_i = \bigwedge V_i$ for all i. If we do this, then $\hat{V}^{n+r} = \langle \hat{x}_1, \hat{x}_2, \ldots \rangle$ satisfies the conditions of Definition 2.2, and thus $\bigwedge V^{\leq (n+r)} = \bigwedge (V^{\leq (n+r-1)} \oplus \hat{V}^{n+r})$ is $(n+r)$–formal. We shall proceed by induction on i. It is clear for $i = 0$. Let us suppose that it is true for $i-1$, and we shall show it for i.

To start with, consider the composition

$$V^{n+r} \xrightarrow{d} \bigwedge V^{\leq (n+r)} \longrightarrow \frac{\bigwedge V^{\leq (n+r)}}{d(\bigwedge V^{\leq (n+r-1)})}. \tag{2}$$

We reorder the generators of V^{n+r} as follows. Let $x_1, \ldots, x_p$ be generators of the kernel of (2). Without loss of generality, we may suppose that they are the first p generators of V^{n+r}. Then for x_i, $1 \leq i \leq p$, we have that dx_i lies in $d(\bigwedge V^{\leq (n+r-1)})$, i.e., there is some $\psi_i \in \bigwedge V^{\leq (n+r-1)}$ with $dx_i = d\psi_i$. Put $\hat{x}_i = x_i - \psi_i$, so that $d\hat{x}_i = 0$.

For $1 \leq i \leq p$, define

$$\hat{C}_i = \hat{C}_{i-1} \oplus \langle \hat{x}_i \rangle = C^{\leq (n+r-1)} \oplus \langle \hat{x}_1, \ldots, \hat{x}_i \rangle$$

and

$$\hat{N}_i = \hat{N}_{i-1} = N^{\leq (n+r-1)}.$$

Then, we only must show that any closed element in the ideal $\hat{N}_i \cdot (\bigwedge \hat{V}_i) = \hat{N}_{i-1} \cdot \bigwedge (\hat{V}_{i-1} \oplus \langle \hat{x}_i \rangle)$ is exact. Let $\eta \in \hat{N}_i \cdot (\bigwedge \hat{V}_i)$ be a closed element. Thus, $\eta = \eta_0 + \eta_1 \cdot \hat{x}_i + \cdots + \eta_k \cdot \hat{x}_i^k$ for some $\eta_j \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$. Moreover, $d\eta = 0$ implies that $d\eta_0 + d\eta_1 \cdot \hat{x}_i + \cdots + d\eta_k \cdot \hat{x}_i^k = 0$ in $\bigwedge \hat{V}_i$. Therefore $d\eta_j = 0$ for $0 \leq j \leq k$. From this fact and the induction hypothesis on $\hat{V}_{i-1}$ it follows that each element η_j is exact, and so η also is exact.

Now let $i > p$. Then we put $\hat{C}_i = \hat{C}_{i-1}$. We want to see that there is an element $\psi_i \in \bigwedge \hat{V}_{i-1}$ such that putting $\hat{x}_i = x_i - \psi_i$ and $\hat{N}_i = \hat{N}_{i-1} \oplus \langle \hat{x}_i \rangle$, the decomposition $\hat{V}_i = \hat{C}_i \oplus \hat{N}_i$ satisfies the conditions of Definition 2.2. No matter ψ_i, d is injective in $\hat{N}_i$. This follows from the fact that (2) is injective in $\langle x_{p+1}, \cdots, x_i \rangle$ and that $\hat{x}_j = x_j - \psi_j$ with $\psi_j \in \bigwedge \hat{V}_{j-1}$, for $p+1 \leq j \leq i$.

For the time being, write $N_i = \hat{N}_{i-1} \oplus \langle x_i \rangle$ and let $\eta \in N_i \cdot \bigwedge (\hat{V}_{i-1} \oplus \langle x_i \rangle) = (\hat{N}_{i-1} \oplus \langle x_i \rangle) \cdot \bigwedge (\hat{V}_{i-1} \oplus \langle x_i \rangle)$ be a closed element. Then $\eta = \eta_0 + \eta_1 \cdot x_i + \cdots + \eta_k \cdot {x_i}^k$. We distinguish three cases:

(i) $k = 0$. Now $\eta = \eta_0$ with $\eta_0 \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$. By the induction hypothesis on i we know that η is exact.

(ii) $k \geq 2$. Note that in this case the degree of x_i must be even. Because $\deg x_i \geq n$ we have that $\deg \eta \geq 2n$. If either $k > 2$ or $k = 2$ and $\deg x_i > n$, it happens that $\deg \eta > 2n$. Then η must be exact because $H^{>2n}(M) = 0$ and $(\bigwedge V, d)$ is a minimal model of M. The only remaining possibility is that $k = 2$ and $\deg x_i = n$. In this case $\deg \eta = 2n$ and η has an expression of the form $\eta = \eta_0 + \eta_1 \cdot x_i + \lambda\, {x_i}^2$ where λ is a non-zero real number. Thus $0 = d\eta = (d\eta_0 + \eta_1 \cdot dx_i) + (d\eta_1 + 2\lambda\, dx_i)x_i$ and hence $\eta_1 + 2\lambda\, x_i$ is closed.

Now $\eta_1 \in (\bigwedge \hat{V}_{i-1})^n$, so it must be of the form $\eta_1 = a + b$ with $a \in \bigwedge V^{\leq (n-1)}$, $b \in \langle x_1, \ldots, x_{i-1} \rangle$. So $d(2\lambda\, x_i - b) = -da$. This means that $2\lambda\, x_i - b$ is in the kernel of the map (2), which is a contradiction.

(iii) $k = 1$. Thus $\eta = \eta_0 + \eta_1 \cdot x_i$, with $\eta_0 \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$ and $\eta_1 \in \bigwedge \hat{V}_{i-1}$. In this case $d\eta = 0$ implies that $d\eta_1 = 0$. We shall see that we can change x_i to some $\hat{x}_i = x_i - \psi_i$ with an element $\psi_i \in (\bigwedge \hat{V}_{i-1})^{n+r}$ so that any closed element of the form $\eta_0 + \eta_1 \cdot \hat{x}_i$ must be exact in $\bigwedge V$. Note that substituting x_i by $\hat{x}_i$ does not spoil the argument in the previous two cases.

If $\deg \eta > 2n$ and η is closed, one has that η is exact by the same argument as the case $\deg \eta > 2n$ of (ii).

Now, we deal with the case that η has degree $2n$. This implies that η_1 is closed of degree $n-r$. In order to show the exactness of η we proceed as follows. In general, consider the collection of those closed $z_j \in (\bigwedge \hat{V}_{i-1})^{n-r}$ such that there exists $\kappa_j \in \left(\hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})\right)^{2n}$ in such way that the element $z_j \cdot x_i + \kappa_j$ is closed. Hence there is $\xi_j \in \bigwedge V$ satisfying

$$z_j \cdot x_i + \kappa_j = \lambda_j \omega + d\xi_j \tag{3}$$

where λ_j are real numbers and ω is a (fixed) closed element of degree $2n$ generating $H^{2n}(\bigwedge V) \cong \mathbb{R}$. We want to achieve that $\lambda_j = 0$ for all j. First, for a given z_j, suppose that we have two different expressions $z_j \cdot x_i + \kappa_j = \lambda_j \omega + d\xi_j$ and $z_j \cdot x_i + \kappa_j' = \lambda_j' \omega + d\xi_j'$. Then the difference $\kappa_j - \kappa_j' = (\lambda_j - \lambda_j')\omega + d\xi_j - d\xi_j'$ is closed and lives in $\hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$. By induction hypothesis, it is exact and hence $\lambda_j = \lambda_j'$. So if we manage to make $\lambda_j = 0$ we have dealt with any possible expression (3) for z_j.

So we may restrict to a basis of those z_j satisfying (3). If $[z_j] = 0$ (for example, when $H^{n-r}(\bigwedge V) = 0$) then $z_j = d\phi$, with $\phi \in (\bigwedge V)^{n-r-1}$. Clearly one can take $\phi \in N^{\leq (n-r-1)} \cdot \bigwedge V^{\leq (n-r-1)}$. Now

$$z_j \cdot x_i + \kappa_j = d\phi \cdot x_i + \kappa_j = d(\phi \cdot x_i) - (-1)^{n-r-1} \phi \cdot dx_i + \kappa_j,$$

which implies that $\phi \cdot dx_i + (-1)^{n-r} \kappa_j \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$ is closed and hence exact taking into account (3). So $z_j \cdot x_i + \kappa_j$ is exact and $\lambda_j = 0$. This means that if $\eta = \eta_0 + \eta_1 \cdot x_i$ is closed and $[\eta_1] = 0$ then η is exact.

Therefore we may restrict ourselves to a collection of z_j such that $[z_j]$ are a basis of the possible z_j's satisfying (3).

Let $[z_1], \ldots, [z_k] \in H^{n-r}(\bigwedge V)$ be such a basis. By Poincaré duality there is some element $[\psi_i] \in H^{n+r}(\bigwedge V)$ with $[z_j] \cdot [\psi_i] = \lambda_j [\omega]$ for all j. Such a closed element $\psi_i \in (\bigwedge V)^{n+r}$ must lie in $\bigwedge (V^{<(n+r)} \oplus \langle \hat{x}_1, \ldots, \hat{x}_p \rangle)$ since $(\bigwedge V)^{n+r} = (\bigwedge V^{<(n+r)})^{n+r} \oplus \langle \hat{x}_1, \ldots, \hat{x}_p \rangle \oplus \langle x_{p+1}, x_{p+2}, \ldots \rangle$ and, by (2), ψ_i cannot have a component in $\langle x_{p+1}, x_{p+2}, \ldots \rangle$. Now define $\hat{x}_i = x_i - \psi_i$. Then, from (3), it is easy to check that $z_j \cdot \hat{x}_i + \kappa_j$ is exact, i.e., whenever

$$z \cdot \hat{x}_i + \kappa, \quad z \text{ closed in } (\bigwedge \hat{V}_{i-1})^{n-r}, \quad \kappa \in \left(\hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})\right)^{2n},$$

is closed, it is exact.

Put $\hat{N}_i = \hat{N}_{i-1} \oplus \langle \hat{x}_i \rangle$. Now we can check that the conditions of Definition 2.2 hold. It only remains to show that if $\eta = \eta_0 + \eta_1 \cdot \hat{x}_i$ is closed of degree $< 2n$, with $\eta_0 \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$ and $\eta_1 \in \bigwedge \hat{V}_{i-1}$, then η is exact. But $[\eta_1] \in H^p(\bigwedge V)$ with $p < n-r$. If $H^{n-r-p}(\bigwedge V) = 0$, then $H^{n+r+p}(\bigwedge V) = 0$ and thus $[\eta_0 + \eta_1 \cdot \hat{x}_i] = 0$. If $H^{n-r-p}(\bigwedge V) \neq 0$, take an arbitrary $[w] \in H^{n-r-p}(\bigwedge V)$. Now we have

$$[w] \cdot [\eta_0 + \eta_1 \cdot \hat{x}_i] = [w \cdot \eta_0 + w \cdot \eta_1 \cdot \hat{x}_i],$$

with $w \cdot \eta_0 \in \hat{N}_{i-1} \cdot (\bigwedge \hat{V}_{i-1})$ and $w \cdot \eta_1$ closed of degree $n-r$. Then by the construction above, $[w] \cdot [\eta_0 + \eta_1 \cdot \hat{x}_i] = 0$. Since this holds for arbitrary $[w]$, Poincaré duality implies that $[\eta_0 + \eta_1 \cdot \hat{x}_i] = 0$.

QED

Miller's theorem [34] for the formality of a $(k-1)$–connected compact manifold of dimension less than or equal to $(4k-2)$ follows easily from our Theorem 3.1.

Theorem 3.2 *[34] Let M be a $(k-1)$–connected compact manifold of dimension less than or equal to $(4k-2)$, $k>1$. Then M is formal.*

Proof : Since M is $(k-1)$–connected, a minimal model $(\bigwedge V, d)$ of M must satisfy $V^i = 0$ for $i \leq k-1$ and $V^k = C^k$ (i.e., $N^k = 0$). Therefore the first non-zero differential, being decomposable, must be $d : V^{2k-1} \to V^k \cdot V^k$. This implies that $V^j = C^j$ (i.e., $N^j = 0$) for $k \leq j \leq (2k-2)$. Hence M is $2(k-1)$–formal. Now, using Theorem 3.1 we have that M is formal. Note that M is orientable since it is simply connected. QED

Note that this implies in particular that any simply connected compact manifold of dimension less than or equal to 6 is formal, which is a result of [35] previous to Miller's theorem.

Also, as a consequence of Theorem 3.1 we have the corollary following.

Corollary 3.3 *Any simply connected compact manifold, of arbitrary dimension, is 2–formal. Moreover, a simply connected compact manifold M of dimension 7 or 8 is formal if and only if it is 3–formal.*

Remark 3.4 *Theorem 3.1 continues to hold for rational Poincaré duality spaces (see Remark 2.5).*

A symplectic manifold (M,ω) is said to be *symplectically aspherical* if $\omega|_{\pi_2(M)} = 0$, that is,

$$\int_{S^2} f^*\omega = 0$$

for every map $f : S^2 \to M$. Hurewicz's theorem implies that a compact symplectically aspherical manifold always has a non-trivial fundamental group.

Remark 3.5 *We note that, from Theorem 3.1, if M is a 1–formal manifold of dimension 4, then M is formal, so all Massey products are trivial. However, the converse is not true. Actually, Amorós and Kotschick [3] claim to have examples of non-formal manifolds of dimension 4 with all Massey products of length $t \leq K$ vanishing, for any arbitrary large number K, as well as to have examples of non-formal symplectic 4–manifolds which are hard Lefschetz. Their examples are symplectically aspherical (and hence non-simply connected).*

4 Lefschetz property

In this section we introduce the s–Lefschetz property for any compact symplectic manifold, generalizing the hard Lefschetz property. We will study this property for Donaldson submanifolds of symplectic manifolds in the next section.

Definition 4.1 *Let (M,ω) be a compact symplectic manifold of dimension $2n$. We say that M is s–Lefschetz with $s \leq (n-1)$ if*

$$[\omega]^{n-i} : H^i(M) \longrightarrow H^{2n-i}(M)$$

is an isomorphism for all $i \leq s$. By extension, if we say that M is s–Lefschetz with $s \geq n$ then we just mean that M is hard Lefschetz.

Note that M is $(n-1)$–Lefschetz if M satisfies the hard Lefschetz theorem. Also it is said in [33] that M is a Lefschetz manifold meaning that M is 1–Lefschetz. M is 0–Lefschetz if it is cohomologically symplectic.

In Section 6 we present examples of compact symplectic manifolds which are s–Lefschetz but not $(s+1)$–Lefschetz for $s = 0, 1, 2$ (see Examples 1, 3 and 4). However, we do not know examples of symplectic manifolds being s–Lefschetz but not $(s+1)$–Lefschetz for $s \geq 3$.

Proposition 4.2 *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and let $M = M_1 \times M_2$ with the symplectic form $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$, λ_1, λ_2 non-zero real numbers. Then for any s, M is s–Lefschetz if and only if M_1 and M_2 are s–Lefschetz.*

Proof : First note that we may rescale the symplectic forms ω_1 and ω_2 as $\lambda_1\omega_1$ and $\lambda_2\omega_2$, so the coefficients can be supposed equal to one. Also let $2n_1$ and $2n_2$ be the dimensions of M_1 and M_2 respectively. Let $n = n_1 + n_2$.

Suppose first that $H^*(M)$ is s–Lefschetz. Let us see that M_1 is also s–Lefschetz. Take $i \leq s$ with $i \leq n_1 - 1$. Let $a_i \in H^i(M_1)$ and suppose that $[\omega_1]^{n_1-i}a_i = 0$. It is enough to see that $a_i = 0$ because then $[\omega_1]^{n_1-i}\colon H^i(M_1) \to H^{n_1-i}(M_1)$ is injective and hence an isomorphism. But

$$[\omega]^{n-i}(a_i \otimes 1) = \sum_{k\geq 0} \binom{n-i}{n_2-k} [\omega_1]^{n_1-i+k} a_i \otimes [\omega_2]^{n_2-k} = 0$$

and the s–Lefschetz property for M implies that $a_1 \otimes 1 = 0$ and hence $a_i = 0$.

For the converse, the s–Lefschetz property for M_1 implies that we may decompose $H^*(M_1) = (\oplus P_i) \oplus R_1$ in vector subspaces, so that

$$P_i = \langle e_i, [\omega_1]e_i, \ldots, [\omega_1]^{n_1-d_i}e_i \rangle,$$

where $e_i \in H^{d_i}(M_1)$, $d_i \leq s$, and $[\omega_1]^{n_1-d_i+1}e_i = 0$. This is possible thanks to the s–Lefschetz property. The elements e_i are called primitive elements. The subspace R_1 is concentrated in degrees going from $s+1$ up to $2n_1 - s - 1$. Similarly $H^*(M_2) = (\oplus Q_j) \oplus R_2$, where $Q_j = \langle f_j, [\omega_2]f_j, \ldots, [\omega_2]^{n_2-d_j}f_j \rangle$ and R_2 is concentrated in degrees going from $s+1$ up to $2n_2 - s - 1$. Therefore

$$H^*(M_1 \times M_2) = \left(\bigoplus_{i,j} P_i \otimes Q_j \right) \oplus R,$$

where $R = R_1 \otimes H^*(M_2) + H^*(M_1) \otimes R_2$ (not a direct sum). Then R is concentrated in degrees going from $s+1$ up to $2n_1 + 2n_2 - s - 1$. This means that R is irrelevant for checking the s–Lefschetz condition for M.

On the other hand,

$$P_i \otimes Q_j = \langle e_i \otimes f_j, \ldots, [\omega_1]^a e_i \otimes [\omega_2]^b f_j, \ldots, [\omega_1]^{n_1-d_i} e_i \otimes [\omega_2]^{n_2-d_j} f_j \rangle$$

satisfies the hard Lefschetz condition with respect to $\omega = \omega_1 + \omega_2$. Therefore M is s–Lefschetz.

QED

5 Donaldson submanifolds of symplectic manifolds

In this section we study the conditions under which Donaldson symplectic submanifolds are formal and/or satisfy the hard Lefschetz theorem.

Let (M,ω) be a compact symplectic manifold of dimension $2n$ with $[\omega] \in H^2(M)$ admitting a lift to an integral cohomology class. In [13] Donaldson constructs symplectic submanifolds $Z \hookrightarrow M$ of dimension $2n-2$ whose Poincaré dual $\mathrm{PD}[Z] = k[\omega]$ for any large multiple of $[\omega]$. Moreover, these submanifolds satisfy a *Lefschetz theorem in hyperplane sections*, meaning that the inclusion $j\colon Z \hookrightarrow M$ is $(n-1)$–connected. In particular, the map there $j^*\colon H^i(M) \to H^i(Z)$ is an isomorphism for $i < n-1$ and a monomorphism for $i = n-1$.

More in general, let X and Y be compact manifolds. We say that a differentiable map $f\colon X \to Y$ is a *homology s–equivalence* ($s \geq 0$) if it induces isomorphisms $f^*\colon H^i(Y) \xrightarrow{\cong} H^i(X)$ on cohomology for $i < s$, and a monomorphism $f^*\colon H^s(Y) \hookrightarrow H^s(X)$ for $i = s$. Therefore $Z \hookrightarrow M$ is a homology $(n-1)$–equivalence.

In [12] it is proved that if $F\colon B_1 \to B_2$ is a morphism of differential algebras inducing an isomorphism on cohomology, and $\rho_i\colon A_i \to B_i$ is a minimal model for B_i ($i = 1,2$), then F induces $\hat{F}\colon A_1 \longrightarrow A_2$, unique up to homotopy, subject to the condition $F \circ \rho_1 = \rho_2 \circ \hat{F}$. For a homology s–equivalence we have:

Proposition 5.1 *Let X and Y be compact manifolds and let $f\colon X \to Y$ be a homology s–equivalence. Then there exist minimal models $(\bigwedge V_X, d)$ and $(\bigwedge V_Y, d)$ of X and Y, respectively, such that f induces a morphism of differential algebras $F\colon (\bigwedge V_Y^{\leq s}, d) \to (\bigwedge V_X^{\leq s}, d)$ where $F: V_Y^{<s} \xrightarrow{\cong} V_X^{<s}$ is an isomorphism and $F: V_Y^s \hookrightarrow V_X^s$ is a monomorphism.*

Proof : We can do this by induction on s, being evident for $s = 0$. So we can suppose that if f is a homology $(s-1)$–equivalence, there exist minimal models $(\bigwedge V_X, d)$ and $(\bigwedge V_Y, d)$ for X and Y, respectively, and a morphism $F\colon (\bigwedge V_Y^{<s}, d) \longrightarrow (\bigwedge V_X^{<s}, d)$ such that $F\colon V_Y^{<(s-1)} \xrightarrow{\cong} V_X^{<(s-1)}$ is an isomorphism and $F\colon V_Y^{s-1} \hookrightarrow V_X^{s-1}$ is a monomorphism. Then, we shall prove the Proposition for a homology s–equivalence f. So, f^* induces: $H^{s-1}(Y) \cong H^{s-1}(X)$ and $H^s(Y) \hookrightarrow H^s(X)$. We have $H^j(Y) \cong H^j(\bigwedge V_Y)$ and $H^j(X) \cong H^j(\bigwedge V_X)$ for any j. Hence f^* induces: $H^{s-1}(\bigwedge V_Y) \cong H^{s-1}(\bigwedge V_X)$ and $H^s(\bigwedge V_Y) \hookrightarrow H^s(\bigwedge V_X)$.

For convenience, we shall denote by $\hat{\mu}$ the element $\hat{\mu} = F(\mu)$, for $\mu \in \bigwedge V_Y^{\leq(s-1)}$. First, we prove that $F\colon V_Y^{s-1} \xrightarrow{\cong} V_X^{s-1}$ is an isomorphism. Let us order the generators of V_X^{s-1} so that

$$V_X^{s-1} = F(V_Y^{s-1}) \oplus \langle x_1, x_2, \ldots \rangle,$$

where $dx_i \in \bigwedge(F(V_Y^{\leq(s-1)}) \oplus \langle x_1, \ldots, x_{i-1} \rangle)$. We shall show that there is no second summand above by showing that $x_1 \in F(V_Y^{\leq(s-1)})$.

We have $dx_1 \in \bigwedge(F(V_Y^{\leq(s-1)}))$. Set $\hat{\eta} = dx_1 = F(\eta)$ for some $\eta \in (\bigwedge V_Y^{\leq(s-1)})^s$. So $d\eta = 0$ and hence $[\eta] \in H^s(\bigwedge V_Y)$. Under the monomorphism $f^*\colon H^s(\bigwedge V_Y) \hookrightarrow H^s(\bigwedge V_X)$, we get that $f^*[\eta] = [\hat{\eta}]$. Since $[\hat{\eta}] = 0$ in $H^s(\bigwedge V_X)$, we have $[\eta] = 0$ in $H^s(\bigwedge V_Y)$. This guarantees the existence of some $\alpha \in (\bigwedge V_Y)^{s-1} \hookrightarrow (\bigwedge V_X)^{s-1}$ such that $\eta = d\alpha$. Therefore $d(x_1 - \hat{\alpha}) = 0$ and so there is a well-defined cohomology class $[x_1 - \hat{\alpha}]$ living in $H^{s-1}(\bigwedge V_X)$. But this space is isomorphic by f^* to $H^{s-1}(\bigwedge V_Y)$. Hence there exist a cohomology class $[\mu] \in H^{s-1}(\bigwedge V_Y)$ such that $[\hat{\mu}] = [x_1 - \hat{\alpha}]$. This implies that there are a closed element $\mu \in (\bigwedge V_Y)^{s-1}$ and $\xi \in (\bigwedge V_Y)^{s-2} \cong (\bigwedge V_X)^{s-2}$ such that

$$x_1 - \hat{\alpha} = \hat{\mu} + d\hat{\xi},$$

which means that $x_1 \in F(\bigwedge V_Y^{\leq(s-1)})$. So it must be $V_X^{s-1} \cong V_Y^{s-1}$.

Now let us see that f induces a map $F : V_Y^s \hookrightarrow V_X^s$. Write $V_Y^s = \langle y_1, y_2, \ldots \rangle$ with $dy_i \in \bigwedge(V_Y^{\leq(s-1)} \oplus \langle y_1, \ldots, y_{i-1} \rangle)$. Now $F(y_i) \in (\bigwedge V_X)^s = (\bigwedge V_X^{<s})^s \oplus V_X^s$. Since we already have

that $\bigwedge V_X^{<s} \cong \bigwedge V_Y^{<s}$, we may modify y_i by adding a suitable element in $\bigwedge V_Y^{<s}$ (and keep on denoting it by y_i) so that $F(y_i) \in V_X^s$. Now we can assume that $\langle y_1, \ldots, y_{r-1}\rangle$ injects into V_X^s but $\hat{y}_r = F(y_r) = 0$. Then, on the one hand, we have $d\hat{y}_r = 0$, thus $[\hat{y}_r]$ is the zero class in $H^s(\bigwedge V_X)$. On the other hand,

$$dy_r = P(y_1, \ldots, y_{r-1}) \in \bigwedge(V_Y^{\leq(s-1)} \oplus \langle y_1, \ldots, y_{r-1}\rangle), \tag{4}$$

that is, dy_r is a polynomial in previous generators.

Applying F to (4), we have $0 = P(\hat{y}_1, \ldots, \hat{y}_{r-1})$ in the free algebra $\bigwedge(V_X^{\leq(s-1)} \oplus \langle \hat{y}_1, \ldots, \hat{y}_{r-1}\rangle)$. Therefore $dy_r = 0$, so $[y_r]$ is a cohomology class in $H^s(\bigwedge V_Y)$. As $H^s(\bigwedge V_Y) \hookrightarrow H^s(\bigwedge V_X)$ and $[\hat{y}_r] = 0$ in $H^s(\bigwedge V_X)$, we get that $[y_r] = 0$, i.e., $y_r = d\eta$ for $\eta \in \bigwedge V_Y^{\leq(s-1)} \cong \bigwedge V_X^{\leq(s-1)}$. Applying F to $y_r = d\eta$, we have that $0 = d\hat{\eta}$ in $\bigwedge V_X$, so $[\hat{\eta}] \in H^{s-1}(\bigwedge V_X) \cong H^{s-1}(\bigwedge V_Y)$. Therefore there are a closed element $g \in (\bigwedge V_Y)^{s-1}$ and $\xi \in (\bigwedge V_Y)^{s-2} \cong (\bigwedge V_X)^{s-2}$ such that

$$\eta = g + d\xi.$$

Hence $d\eta = 0$ in $\bigwedge V_Y$ and $y_r = 0$. This is a contradiction since y_r is a generator. So it must be that $\langle y_1, \ldots, y_r\rangle \hookrightarrow V_X^s$. This implies that $V_Y^s \hookrightarrow V_X^s$. QED

Theorem 5.2 *(i) Let X and Y be compact manifolds, and let $f\colon X \to Y$ be a homology s–equivalence. If Y is $(s-1)$–formal then X is $(s-1)$–formal.*

(ii) Let M be a compact symplectic manifold of dimension $2n$ and let $Z \hookrightarrow M$ be a Donaldson submanifold. For each $s \leq n-2$, if M is s–formal then Z is s–formal. In particular, Z is formal if M is $(n-2)$–formal.

Proof : Let $(\bigwedge V_X, d)$ and $(\bigwedge V_Y, d)$ be the minimal models of X and Y, respectively, constructed in Proposition 5.1. For $i < s$, decompose $V_Y^i = C_Y^i \oplus N_Y^i$ satisfying the conditions of Definition 2.2. Then, taking into account Proposition 5.1, we set $V_X^i = C_X^i \oplus N_X^i$ under the natural isomorphism $F\colon V_Y^i \cong V_X^i$, $i < s$. Consider a closed element $F(\eta) = \hat{\eta} \in N_X^{<s} \cdot (\bigwedge V_X^{<s})$. Hence η is a closed element in $N_Y^{<s} \cdot (\bigwedge V_Y^{<s})$ and, by the $(s-1)$–formality of Y, it is exact, i.e., $\eta = d\xi$, for $\xi \in \bigwedge V_Y$. Take the image $\hat{\eta} = d(F(\xi))$ in $\bigwedge V_X$. This proves (i). Now (ii) follows from (i) and using that the inclusion $j\colon Z \hookrightarrow M$ is a homology $(n-1)$–equivalence. QED

Theorem 5.3 *Let M be a compact symplectic manifold of dimension $2n$, and let $Z \hookrightarrow M$ be a Donaldson submanifold. Then, for each $s \leq n-2$, M is s–Lefschetz if and only if Z is s–Lefschetz.*

Proof : For any differential form x on M, we shall denote by $\hat{x}$ the differential form on Z given by $\hat{x} = j^*(x)$. Let $p = 2(n-1) - i$, where $i \leq (n-2)$, and consider the restriction map $j^*\colon H^p(M) \to H^p(Z)$. Then, for $[z] \in H^p(M)$, we claim that

$$j^*[z] = 0 \iff [z] \cup [\omega] = 0. \tag{5}$$

This can be seen via Poincaré duality. Clearly $j^*[z] = 0$ if and only if for any $a \in H^i(Z)$ we have $j^*[z] \cdot a = 0$. We know that there is an isomorphism $H^i(Z) \cong H^i(M)$ $(i \leq n-2)$, thus we can assume that there is a closed i–form x on M with $[x|_Z] = [\hat{x}] = a$. So

$$j^*[z] \cdot [\hat{x}] = \int_Z \hat{z} \wedge \hat{x} = \int_M z \wedge x \wedge k\omega,$$

since $[Z] = k\mathrm{PD}[\omega]$. Hence $j^*[z] = 0$ if and only if $[z \wedge \omega] \cdot [x] = 0$ for all $[x] \in H^i(M)$, from where the claim follows.

Now suppose that M is s–Lefschetz, so $[\omega]^{n-i} : H^i(M) \to H^{2n-i}(M)$ is an isomorphism for $i \leq s$. We want to check that the map $[\omega_Z]^{n-1-i} : H^i(Z) \to H^{2n-2-i}(Z)$ is injective. Let $[\hat{z}] \in H^i(Z) \cong H^i(M)$ and extend it to $[z] \in H^i(M)$. Then, $[\omega_Z]^{n-1-i}[\hat{z}] = 0$ implies that $j^*[\omega^{n-1-i} \wedge z] = 0$, which by (5) is equivalent to $[\omega^{n-1-i} \wedge z \wedge \omega] = 0$. Using the s–Lefschetz property of M, we get $[z] = 0$ and thus $[\hat{z}] = 0$.

The converse is easy. If Z is s–Lefschetz and we take $[z] \in H^i(M)$ such that $[\omega^{n-i} \wedge z] = 0$, from (5) it follows that $j^*[\omega^{n-1-i} \wedge z] = 0$, i.e., $[\omega_Z^{n-1-i} \wedge z|_Z] = 0$. Hence $[\hat{z}] = 0$ in $H^i(Z)$ and so $[z] = 0$ since $i \leq n-2$. QED

Corollary 5.4 *Let M be a compact symplectic manifold of dimension $2n$, and let $Z \hookrightarrow M$ be a Donaldson submanifold. We have that if M is hard Lefschetz, Z is also hard Lefschetz. Moreover, M is $(n-2)$–Lefschetz (but not necessarily hard Lefschetz) if and only if Z is hard Lefschetz.*

In the Example 3 of Section 6 we exhibit examples of 6–dimensional compact symplectic manifolds which are 1–Lefschetz but not 2–Lefschetz (i.e., not hard Lefschetz), so its Donaldson symplectic submanifolds are 1–Lefschetz and thus hard Lefschetz.

Corollary 5.5 *Under the conditions of Theorem 5.3, for each $p = 2(n-1) - i$ with $i \leq s$, there is an isomorphism*

$$H^p(Z) \cong \frac{H^p(M)}{\ker([\omega] : H^p(M) \twoheadrightarrow H^{p+2}(M))}.$$

Proof : From (5), we know that there is an inclusion

$$\frac{H^p(M)}{\ker([\omega] : H^p(M) \to H^{p+2}(M))} \hookrightarrow H^p(Z).$$

Furthermore, the map $H^p(M) \to H^{p+2}(M)$ is surjective since the s–Lefschetz property guarantees an isomorphism $H^i(M) \to H^{p+2}(M)$. Computing dimensions, we have $b^p(M) - (b^p(M) - b^{p+2}(M)) = b^{p+2}(M) = b^i(M) = b^i(Z) = b^p(Z)$, which completes the proof. QED

Remark 5.6 *We must note that under the conditions of Theorem 5.3, if M has a non-zero Massey product $\langle [\alpha_1], [\alpha_2], \ldots, [\alpha_t] \rangle \subset H^r(M)$, with $r \leq (n-1)$, then it defines a non-zero Massey product $\langle [\hat{\alpha}_1], [\hat{\alpha}_2], \ldots, [\hat{\alpha}_t] \rangle$ of Z, where $\hat{\alpha}_i = j^*(\alpha_i)$, $1 \leq i \leq t$. If $r \geq n$ and the cohomology classes $[\hat{\alpha}_i]$, $1 \leq i \leq t$, are non-trivial in Z then, from Corollary 5.5, it follows that $\langle [\hat{\alpha}_1], [\hat{\alpha}_2], \ldots, [\hat{\alpha}_t] \rangle$ is non-zero if and only if the cup product of $[\omega]$ by any representative of $\langle [\alpha_1], [\alpha_2], \ldots, [\alpha_t] \rangle$ is a non-trivial cohomology class of M. However, as we show in Section 6, it can happen that a Massey product $\langle [\hat{\alpha}_1], [\hat{\alpha}_2], \ldots, [\hat{\alpha}_t] \rangle$ is defined on Z but $\langle [\alpha_1], [\alpha_2], \ldots, [\alpha_t] \rangle$ is not defined on M.*

6 Examples

We shall apply the previous results to study the s–formality and the s–Lefschetz property of some compact symplectic manifolds and their Donaldson submanifolds. Five examples will be developed.

The first one is the well known Kodaira–Thurston manifold KT; it is the simplest nontrivial example of a compact symplectic manifold with no Kähler metric. Example 2 is the Iwasawa manifold I_3, any Donaldson symplectic submanifold Z of I_3 is neither formal nor hard Lefschetz; moreover $Z \hookrightarrow I_3$ has no complex structures. Example 4 allows us to show a Donaldson symplectic submanifold of dimension eight which is formal simply connected but not hard Lefschetz. Example 3 is a 6–dimensional compact symplectically aspherical manifold M which is 1–formal but not 2–formal, it has the 1–Lefschetz property but not the 2–Lefschetz property; any Donaldson symplectic submanifold of M is formal and hard Lefschetz but it does not carry Kähler metrics. In Example 5 we show that there are compact oriented smooth manifolds which are s–formal but not $(s+1)$–formal, for any $s \geq 2$.

Example 1 *The Kodaira–Thurston manifold.* Let H be the Heisenberg group, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. Then a global system of coordinates x, y, z for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that a basis for the left invariant 1–forms on H consists of

$$\{dx, dy, dz - xdy\}.$$

Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. So the quotient space $M^3 = \Gamma\backslash H$ is compact. Hence the forms dx, dy, $dz - xdy$ descend to 1–forms α, β, γ on M^3.

The Kodaira–Thurston manifold KT is the product $KT = M^3 \times S^1$. Then, there are 1–forms $\alpha, \beta, \gamma, \eta$ on KT such that

$$d\alpha = d\beta = d\eta = 0, \quad d\gamma = -\alpha \wedge \beta,$$

and such that at each point of KT, $\{\alpha, \beta, \gamma, \eta\}$ is a basis for the 1–forms on KT. Moreover, it is easy to use Nomizu's theorem [36] to compute the real cohomology of KT

$$\begin{aligned}
H^0(KT) &= \langle 1 \rangle, \\
H^1(KT) &= \langle [\alpha], [\beta], [\eta] \rangle, \\
H^2(KT) &= \langle [\alpha \wedge \gamma], [\beta \wedge \gamma], [\alpha \wedge \eta], [\beta \wedge \eta] \rangle, \\
H^3(KT) &= \langle [\alpha \wedge \gamma \wedge \eta], [\beta \wedge \gamma \wedge \eta], [\alpha \wedge \beta \wedge \gamma] \rangle, \\
H^4(KT) &= \langle [\alpha \wedge \beta \wedge \gamma \wedge \eta] \rangle.
\end{aligned}$$

Using again Nomizu's theorem, the minimal model of KT is the differential graded algebra $(\mathcal{M}, d)$, where $\mathcal{M}$ is the free algebra $\mathcal{M} = \bigwedge(a_1, a_2, a_3, a_4)$ with all the generators of degree 1, and d is given by $da_i = 0$ for $i = 1, 2, 4$ and $da_3 = -a_1 \cdot a_2$. The morphism $\rho \colon \mathcal{M} \to \Omega(KT)$, inducing an isomorphism on cohomology, is defined by $\rho(a_1) = \alpha$, $\rho(a_2) = \beta$, $\rho(a_3) = \gamma$, $\rho(a_4) = \eta$.

Now, according to Definition 2.2, $C^1 = \langle a_1, a_2, a_4 \rangle$ and $N^1 = \langle a_3 \rangle$. Since the element $a_1 \cdot a_3 \in N^1 \cdot V^1$ is closed but not exact, we conclude that $(\mathcal{M}, d)$ is not 1–formal, and by Theorem 3.1 it is not formal. This fact is also a consequence of Lemma 2.6. (The non-formality of KT was proved in [11] seeing the existence of non-trivial Massey products, and in [22] proving that tori are the only compact nilmanifolds with a formal minimal model.)

A symplectic form ω on KT is $\omega = \beta \wedge \gamma + \alpha \wedge \eta$. Since $[\omega] \cup [\beta] = 0$ in $H^3(KT)$, we get that KT does not have the 1–Lefschetz property. (In [9] it is proved that tori are the only compact symplectic nilmanifolds satisfying the 1–Lefschetz property.)

The Kodaira-Thurston manifold can be also defined as a T^2–bundle over T^2 [42], and the symplectic form ω defines an integral cohomology class. It is clear that any Donaldson submanifold Z of KT is a symplectic manifold of dimension 2. Hence Z *is a Kähler manifold* and thus formal and hard Lefschetz. This result is true in general for the Donaldson submanifolds of any 4–dimensional compact symplectic manifold.

For simplicity we shall denote by the same symbols the differential forms on KT and the ones induced in Z. From Corollary 5.5 it follows that the cohomology classes $[\alpha\wedge\gamma]$ and $[\beta\wedge\eta]$ of KT define the zero class in Z. Moreover, $[\alpha\wedge\eta]$ and $[\beta\wedge\gamma]$ restrict to the same cohomology class in Z. Therefore, the real cohomology of Z is

$$\begin{aligned} H^0(Z) &= \langle 1\rangle, \\ H^1(Z) &= \langle [\alpha],[\beta],[\eta],[e_k]\rangle, \\ H^2(Z) &= \langle [\alpha\wedge\eta]\rangle, \end{aligned}$$

where $[e_k]$ are a finite number of cohomology classes lying in Z but not in KT.

Example 2 *The Iwasawa manifold.* Consider the complex Heisenberg group, that is, the complex nilpotent Lie group G of complex matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The Iwasawa manifold is the compact complex parallelizable nilmanifold obtained as $I_3 = \Gamma\backslash G$, where Γ is the discrete subgroup of G consisting of those matrices whose entries are Gaussian integers. The (complex) differential forms dx, dy, $dz - xdy$ on G are left invariant and descend to holomorphic 1–forms α, β, γ on I_3 such that

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha\wedge\beta.$$

Denote by $\alpha_1 = \mathrm{Re}(\alpha)$, $\alpha_2 = \mathrm{Im}(\alpha)$, $\beta_1 = \mathrm{Re}(\beta)$, $\beta_2 = \mathrm{Im}(\beta)$, $\gamma_1 = \mathrm{Re}(\gamma)$, $\gamma_2 = \mathrm{Im}(\gamma)$. Then, $\{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2\}$ is a basis for the 1–forms on I_3 such that

$$\begin{aligned} d\alpha_i &= d\beta_i = 0, \\ d\gamma_1 &= -\alpha_1\wedge\beta_1 + \alpha_2\wedge\beta_2, \\ d\gamma_2 &= -\alpha_1\wedge\beta_2 - \alpha_2\wedge\beta_1. \end{aligned}$$

Nomizu's theorem [36] implies that the real cohomology of I_3 is

$$\begin{aligned} H^0(I_3) &= \langle 1\rangle, \\ H^1(I_3) &= \langle [\alpha_1],[\alpha_2],[\beta_1],[\beta_2]\rangle, \\ H^2(I_3) &= \langle [\alpha_1\wedge\alpha_2],[\alpha_1\wedge\beta_1],[\alpha_1\wedge\beta_2],[\beta_1\wedge\beta_2],[\alpha_1\wedge\gamma_2+\alpha_2\wedge\gamma_1],[\alpha_1\wedge\gamma_1-\alpha_2\wedge\gamma_2], \\ &\qquad [\beta_1\wedge\gamma_2+\beta_2\wedge\gamma_1],[\beta_1\wedge\gamma_1-\beta_2\wedge\gamma_2]\rangle, \\ H^3(I_3) &= \langle [\alpha_1\wedge\alpha_2\wedge\gamma_1],[\alpha_1\wedge\alpha_2\wedge\gamma_2],[\beta_1\wedge\beta_2\wedge\gamma_1],[\beta_1\wedge\beta_2\wedge\gamma_2],[\alpha_1\wedge\beta_1\wedge\gamma_2], \\ &\qquad [\alpha_1\wedge\beta_2\wedge\gamma_1],[\alpha_2\wedge\beta_1\wedge\gamma_1],[\alpha_1\wedge\beta_1\wedge\gamma_1-\alpha_1\wedge\beta_2\wedge\gamma_2], \\ &\qquad [\alpha_1\wedge\beta_2\wedge\gamma_2-\alpha_2\wedge\beta_1\wedge\gamma_2],[\alpha_2\wedge\beta_1\wedge\gamma_2+\alpha_2\wedge\beta_2\wedge\gamma_1]\rangle, \\ H^4(I_3) &= \langle [\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\gamma_1],[\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\gamma_2],[\alpha_1\wedge\alpha_2\wedge\gamma_1\wedge\gamma_2],[\alpha_1\wedge\beta_1\wedge\beta_2\wedge\gamma_1], \\ &\qquad [\alpha_1\wedge\beta_1\wedge\beta_2\wedge\gamma_2],[\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2],[\alpha_1\wedge\beta_1\wedge\gamma_1\wedge\gamma_2+\alpha_2\wedge\beta_2\wedge\gamma_1\wedge\gamma_2], \\ &\qquad [\alpha_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2-\alpha_2\wedge\beta_1\wedge\gamma_1\wedge\gamma_2]\rangle, \end{aligned}$$

$$
\begin{aligned}
H^5(I_3) &= \langle [\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\gamma_1\wedge\gamma_2], [\alpha_1\wedge\alpha_2\wedge\beta_2\wedge\gamma_1\wedge\gamma_2], [\alpha_1\wedge\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2],\\
&\quad [\alpha_2\wedge\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2]\rangle,\\
H^6(I_3) &= \langle [\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\beta_2\wedge\gamma_1\wedge\gamma_2]\rangle.
\end{aligned}
$$

From Lemma 2.6 we know that I_3 is not 1–formal. Independently, one can check that the minimal model of I_3 is the differential graded algebra $(\mathcal{M}, d)$, where $\mathcal{M}$ is the free algebra $\mathcal{M} = \bigwedge(a_1, a_2, b_1, b_2, c_1, c_2)$ with all the generators of degree 1, and d is given by $da_i = db_i = 0$ for $1 \le i \le 2$, $dc_1 = -a_1 \cdot b_1 + a_2 \cdot b_2$ and $dc_2 = -a_1 \cdot b_2 - a_2 \cdot b_1$. The morphism $\rho\colon \mathcal{M} \to \Omega(I_3)$, inducing an isomorphism on cohomology, is defined by $\rho(a_i) = \alpha_i$, $\rho(b_i) = \beta_i$, $\rho(c_i) = \gamma_i$, $(i = 1, 2)$.

Now, according to Definition 2.2, $C^1 = \langle a_1, a_2, b_1, b_2\rangle$ and $N^1 = \langle c_1, c_2\rangle$. Since the element $c_1 \cdot a_1 \cdot a_2$ in the ideal generated by N^1 in $\bigwedge V^1$ is closed but not exact, we conclude that $(\mathcal{M}, d)$ is not 1–formal, and by Theorem 3.1 it is not formal. Therefore, I_3 is not 1–formal, and thus non-formal.

A symplectic form ω on I_3 is given by $\omega = \alpha_1 \wedge \gamma_2 + \alpha_2 \wedge \gamma_1 + \beta_1 \wedge \beta_2$. It is easy to show that $[\omega]^2 \cup [\alpha_1] = 0$, so I_3 does not have the 1–Lefschetz property.

The Iwasawa manifold can be also defined as a T^2–bundle over T^4, and the symplectic form ω defines an integral cohomology class. Let $Z \hookrightarrow I_3$ be a Donaldson submanifold of I_3. Then Z is a 4–dimensional symplectic manifold. For simplicity we shall denote by the same symbols the differential forms on I_3 and the ones induced in Z. Using Corollary 5.5 one can check that the real cohomology of Z is

$$
\begin{aligned}
H^0(Z) &= \langle 1\rangle,\\
H^1(Z) &= \langle [\alpha_1], [\alpha_2], [\beta_1], [\beta_2]\rangle,\\
H^2(Z) &= \langle [\alpha_1\wedge\alpha_2], [\alpha_1\wedge\beta_1], [\alpha_1\wedge\beta_2], [\beta_1\wedge\beta_2], [\alpha_1\wedge\gamma_2+\alpha_2\wedge\gamma_1], [\alpha_1\wedge\gamma_1-\alpha_2\wedge\gamma_2],\\
&\quad [\beta_1\wedge\gamma_2+\beta_2\wedge\gamma_1], [\beta_1\wedge\gamma_1-\beta_2\wedge\gamma_2], [e_k]\rangle,\\
H^3(Z) &= \langle [\beta_1\wedge\beta_2\wedge\gamma_1], [\beta_1\wedge\beta_2\wedge\gamma_2], [\alpha_1\wedge\beta_1\wedge\gamma_2], [\alpha_1\wedge\beta_1\wedge\gamma_1-\alpha_1\wedge\beta_2\wedge\gamma_2]\rangle,\\
H^4(Z) &= \langle [\alpha_1\wedge\alpha_2\wedge\gamma_1\wedge\gamma_2]\rangle,
\end{aligned}
$$

where $[e_k]$ are a finite number of cohomology classes of Z that are not defined in I_3.

Proposition 6.1 *Any Donaldson submanifold $Z \hookrightarrow I_3$ is a 4–dimensional symplectic manifold not formal and not hard Lefschetz. Moreover, Z does not carry complex structures.*

Proof : First let us note that, as a consequence of Theorem 5.3, any Donaldson submanifold Z of I_3 does not satisfy the hard Lefschetz theorem. By Proposition 5.1, the minimal model of Z is of the form $(\mathcal{M}_Z, d)$, where $\mathcal{M}_Z = \bigwedge\left((a_1, a_2, b_1, b_2, c_1, c_2) \oplus V^{\ge 2}\right)$. Setting $C^1 = \langle a_1, a_2, b_1, b_2\rangle$ and $N^1 = \langle c_1, c_2\rangle$, the element $c_1 \cdot a_2 + c_2 \cdot a_1 \in N^1 \cdot \bigwedge V^1$ is closed but not exact in $H^*(Z)$. Therefore by Z is not 1–formal and hence it is not formal.

To show that Z has no complex structures, we use Kodaira's theorem [25] that states that *a complex surface is a deformation of an algebraic surface if and only if its first Betti number is even.* Suppose Z with first Betti number $b_1(Z) = 4$ has a complex structure. Then Kodaira's theorem implies that Z possesses a Kähler metric, and hence Z would be formal according to a result of [12]. But this is impossible. *QED*

Example 3 *The manifold of [16].* Let G be the connected completely solvable Lie group of dimension 6 consisting of matrices of the form

$$a = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $t, x, y_i, z_i \in \mathbb{R}$ $(i = 1, 2)$. Then a global system of coordinates t, x, y_1, y_2, z_1, z_2 for G is defined by $t(a) = t$, $x(a) = x$, $y_i(a) = y_i$, $z_i(a) = z_i$, and a standard calculation shows that a basis for the left invariant 1–forms on G consists of

$$\{dt, dx, e^{-t}dy_1 - xe^{-t}dz_1, e^t dy_2 - xe^t dz_1, e^{-t}dz_1, e^{-t}dz_2\}.$$

Let Γ be a discrete subgroup of G such that the quotient space $M = \Gamma\backslash G$ is compact. (Such a subgroup exists, see [16].) Hence the forms $dt, dx, e^{-t}dy_1 - xe^{-t}dz_1, e^t dy_2 - xe^t dz_1, e^{-t}dz_1, e^{-t}dz_2$ descend to 1–forms α, β, γ_1, γ_2, δ_1, δ_2 on M such that

$$d\alpha = d\beta = 0, \quad d\gamma_1 = -\alpha\wedge\gamma_1 - \beta\wedge\delta_1, \quad d\gamma_2 = \alpha\wedge\gamma_2 - \beta\wedge\delta_2, \quad d\delta_1 = -\alpha\wedge\delta_1, \quad d\delta_2 = \alpha\wedge\delta_2,$$

and such that at each point of M, $\{\alpha, \beta, \gamma_i, \delta_i\}$ is a basis for the 1–forms on M. Using Hattori's theorem [23] we compute the real cohomology of M:

$$\begin{aligned}
H^0(M) &= \langle 1\rangle, \\
H^1(M) &= \langle [\alpha], [\beta]\rangle, \\
H^2(M) &= \langle [\alpha\wedge\beta], [\delta_1\wedge\delta_2], [\gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1]\rangle, \\
H^3(M) &= \langle [\alpha\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2], [\beta\wedge(\gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1)], [\alpha\wedge(\gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1)]\rangle, \\
H^4(M) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2], [\alpha\wedge\beta\wedge\gamma_1\wedge\delta_2], [\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle, \\
H^5(M) &= \langle [\alpha\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2], [\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle, \\
H^6(M) &= \langle [\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]\rangle.
\end{aligned}$$

The minimal model of M must be a differential graded algebra $(\mathcal{M}, d)$, being $\mathcal{M}$ the free algebra of the form $\mathcal{M} = \bigwedge(a_1, a_2) \otimes \bigwedge(b_1, b_2, b_3, b_4) \otimes \bigwedge V^{\geq 3}$ where the generators a_i have degree 1, the generators b_j have degree 2, and d is given by $da_i = db_1 = db_2 = 0$, $db_3 = -a_2 \cdot b_1$ and $db_4 = a_2 \cdot b_3$. The morphism $\rho\colon \mathcal{M} \to \Omega(M)$, inducing an isomorphism on cohomology, is defined by $\rho(a_1) = \alpha$, $\rho(a_2) = \beta$, $\rho(b_1) = \delta_1\wedge\delta_2$, $\rho(b_2) = (1/2)(\gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1)$, $\rho(b_3) = (1/2)(\gamma_2\wedge\delta_1 - \gamma_1\wedge\delta_2)$ and $\rho(b_4) = (1/2)(\gamma_1\wedge\gamma_2)$.

According to Definition 2.2, we get $C^1 = \langle a_1, a_2\rangle$ and $N^1 = 0$, thus M *is 1–formal*. We see that M *is not 2–formal*. In fact, the element $b_4 \cdot a_2 \in N^2 \cdot V^1$ is closed but not exact, which implies that $(\mathcal{M}, d)$ is not 2–formal, and by Theorem 3.1 not formal.

Consider the symplectic form ω on M given by $\omega = \alpha\wedge\beta + \gamma_1\wedge\delta_2 + \gamma_2\wedge\delta_1$. Then (M, ω) is a 1*–formal symplectically aspherical manifold but not* 2*–formal*.

Moreover, $[\omega] \cup [\delta_1\wedge\delta_2] = 0$ in $H^4(M)$, which means that M does not have the 2–Lefschetz property. But a simple computation shows that the cup product by $[\omega]^2$ is an isomorphism between $H^1(M)$ and $H^5(M)$. Therefore, M has the 1–Lefschetz property.

Alternatively, the manifold M can be also defined as a T^4–bundle over T^2 (see [16]), and the symplectic form ω defines an integral cohomology class. Let $Z \hookrightarrow I_3$ be a Donaldson submanifold. For simplicity we shall denote by the same symbols the differential forms on M

and the ones induced in Z. From Corollary 5.5 it follows that the cohomology classes $[\alpha\wedge\delta_1\wedge\delta_2]$, $[\beta\wedge\gamma_1\wedge\gamma_2]$, $[\alpha\wedge\beta\wedge\gamma_1\wedge\gamma_2]$ and $[\alpha\wedge\beta\wedge\gamma_1\wedge\delta_2]-[\gamma_1\wedge\gamma_2\wedge\delta_1\wedge\delta_2]$ of M define the zero class in Z. Therefore, the real cohomology of Z is

$$\begin{array}{rcl} H^0(Z) & = & \langle 1\rangle, \\ H^1(Z) & = & \langle[\alpha],[\beta]\rangle, \\ H^2(Z) & = & \langle[\alpha\wedge\beta],[\delta_1\wedge\delta_2],[\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1],[e_k]\rangle, \\ H^3(Z) & = & \langle[\beta\wedge(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)],[\alpha\wedge(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)]\rangle, \\ H^4(Z) & = & \langle[\alpha\wedge\beta\wedge\gamma_1\wedge\delta_2]\rangle, \end{array}$$

where $[e_k]$ are a finite number of cohomology classes of Z that are not defined in M.

Proposition 6.2 *Any Donaldson submanifold $Z\hookrightarrow M$ is a 4–dimensional formal symplectic manifold that satisfies the hard Lefschetz property. Moreover, Z does not admit complex structures and, in particular, Z does not possess Kähler metrics.*

Proof : Since M is 1–formal and has the 1–Lefschetz property, Z is formal and hard Lefschetz. Suppose that Z has no Kähler metrics. Using Kodaira's theorem and $b_1(Z)=2$, a similar argument to the one given in Proposition 6.1 implies that Z has no complex structures if Z has no Kähler metrics.

In order to show that Z does not admit Kähler metrics, recall that $\Gamma=\pi_1(Z)\cong\pi_1(M)$, which is a semi-direct product $\mathbb{Z}^2\ltimes\mathbb{Z}^4$, so Γ is 2–step solvable. Moreover, its rank is is 6 by additivity. We shall see that Γ cannot be the fundamental group of any compact Kähler manifold.

Assume now that $\Gamma=\pi_1(X)$, where X is a compact Kähler manifold. According to Arapura–Nori's theorem (see Theorem 3.3 of [4]), there exists a chain of normal subgroups

$$0=D^3\Gamma\subset Q\subset P\subset\Gamma,$$

such that Q is torsion, P/Q is nilpotent and Γ/P is finite. Since Γ has no torsion, $Q=0$. As Γ/P is torsion, we have $\operatorname{rank}P=\operatorname{rank}\Gamma=6$. Now, the finite inclusion $P\subset\Gamma$ defines a finite cover $p:Y\to X$ that is also compact Kähler and it has fundamental group P. By Corollary 3.8 of [10], as P is Kähler, nilpotent and has $\operatorname{rank}P=6<9$, it has to be abelian. This is impossible since any pair of non-zero elements $e\in\mathbb{Z}^2\subset\Gamma=\mathbb{Z}^2\ltimes\mathbb{Z}^4$, $f\in\mathbb{Z}^4\subset\Gamma$ do not commute (see [16, page 22]). *QED*

Remark 6.3 *In [16] it is proved that M is not formal showing that the quadruple Massey product $\langle[\delta_1\wedge\delta_2],[\beta],[\beta],[\beta]\rangle$ is non-trivial since any representative is of the form $[\beta\wedge\gamma_1\wedge\gamma_2]+[\delta_1\wedge\delta_2]\cup[u_1]+[\beta]\cup[u_2]+\lambda[\beta\wedge(\gamma_1\wedge\delta_2+\gamma_2\wedge\delta_1)]$, where λ is a real number, and $[u_i]\in H^i(M)$ for $i=1,2$. In particular, a representative of that Massey product is $[\beta\wedge\gamma_1\wedge\gamma_2]$. Now, let $Z\hookrightarrow M$ be a Donaldson submanifold of (M,ω). Using Corollary 5.5, one can check that $[\beta\wedge\gamma_1\wedge\gamma_2]$ defines the zero class in $H^3(Z)$, and so the quadruple Massey product $\langle[\delta_1\wedge\delta_2],[\beta],[\beta],[\beta]\rangle$ vanishes in Z. Moreover, the Massey product $\langle[\alpha],[\alpha],[\delta_1\wedge\delta_2]\rangle$ is trivial in Z but it is not defined in M.*

Example 4 *The manifold V.* Let us show an example of a Donaldson symplectic submanifold which is formal and not hard Lefschetz.

In [24] the authors present an example of a simply connected compact symplectic manifold V with all Massey products (of all orders) vanishing, and such that V does not satisfy the hard Lefschetz theorem. Now, our purpose is to prove that the manifold V is formal.

Recall shortly the definition of V. Let (M, ω) be a 4–dimensional compact symplectic manifold whose first Betti number is $b_1(M) = 1$. Such a manifold exists as consequence of the results of Gompf [17]. Without loss of generality we can assume that the symplectic form on M is integral and therefore, by Gromov and Tischler theorem [19, 43], there exists a symplectic embedding of M in the complex projective space $\mathbb{CP}^5$ with its standard Kähler form.

Denote by X the blow up of $\mathbb{CP}^5$ along M. Then X is a simply connected compact symplectic manifold of dimension 10 whose third Betti number $b_3(X) = 1$. Define V as a Donaldson symplectic submanifold of X. Then V is an eight dimensional simply connected compact symplectic manifold. For $i < 4$ the de Rham cohomology groups $H^i(X)$ and $H^i(V)$ are isomorphic, and there is a monomorphism $H^4(X) \hookrightarrow H^4(V)$.

Lemma 6.4 *[24] The de Rham cohomology group $H^2(X)$ is generated by two elements ρ and σ satisfying that the cup product $\rho^2 \cup \sigma^2$ is a nonzero cohomology class in $H^8(X)$.*

Theorem 6.5 *X is 3–formal but not 4–formal, and it is not 3–Lefschetz. Therefore V is formal and not hard Lefschetz.*

Proof : Since $H^3(X)$ and $H^3(V)$ are isomorphic, $b_3(X) = b_3(V) = 1$. It is well known and easy to see that $b_{2i+1}(M)$ is even for any hard Lefschetz $2n$–manifold M, since otherwise the product bilinear form $H^{2i+1}(M) \otimes H^{2n-2i-1}(M) \to H^{2n}(M)$ is degenerate (see for example [24]). So, X and V are not hard Lefschetz.

From Corollary 3.3, we know that X and V are 2–formal. To prove that V is formal, first we show that X is 3–formal. It is clear that the three cohomology classes ρ^2, σ^2 and $\rho \cup \sigma$ must be non-trivial because $\rho^2 \cup \sigma^2$ is a non-trivial class. This means that, at the level of the minimal model $(\bigwedge V_X, d)$ of X, the subspace N^3 (consisting of the non-closed generators of V_X of degree 3) is the zero space. Then, taking into account Definition 2.2, we get that $(\bigwedge V_X, d)$ is 3–formal and so X is 3–formal. Now, we obtain the formality of V from Theorem 5.2. *QED*

Example 5 *Manifolds which are s–formal but not $(s + 1)$–formal for any s.* To finish this section, we are going to show that the notion of s–formality is not vacuous, by giving some examples of compact oriented manifolds which are s–formal but not $(s + 1)$–formal, for any value of $s \geq 0$. First note that Example 1 covers the case $s = 0$ and Example 3 covers the case $s = 1$ (in the case $s = 1$, the manifold must be non-simply connected, since otherwise Theorem 3.2 implies that M is 2–formal). So we restrict to $s \geq 2$. (By the way, one can check the manifold X defined in Example 4 is 3–formal but not 4–formal.)

The examples that we are going to construct follow the pattern of [14]. They are not symplectic; but maybe with a little bit more work one could obtain symplectic examples.

First we deal with the case $s = 2i$ even. Consider a wedge of three spheres $S^{i+1} \vee S^{i+1} \vee S^{i+1} \subset \mathbb{R}^{i+2}$ and let $\gamma \in \pi_{3i+1}(S^{i+1} \vee S^{i+1} \vee S^{i+1})$ be the element represented by the iterated Whitehead product $[[\iota_1, \iota_2], \iota_3]$ where ι_j is the image of the generator of $\pi_{i+1}(S^{i+1})$ for the inclusion of S^{i+1} as the j–th factor in $S^{i+1} \vee S^{i+1} \vee S^{i+1}$. Let C be the cone of $\gamma : S^{3i+1} \longrightarrow S^{i+1} \vee S^{i+1} \vee S^{i+1}$. By [14] there is a PL embedding $C \subset \mathbb{R}^{(3i+1)+(i+2)+1} = \mathbb{R}^{4i+3}$. Note that C is i–connected and that the only non-vanishing homology groups are $H^{i+1}(C)$ and $H^{3i+2}(C)$. Let $\alpha_j \in H^2(C)$ be the cohomology class evaluating 1 on the j–th copy of S^{i+1} and 0 on the other two. Therefore the Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{3i+2}(C)/([\alpha_1] \cup H^{2i+1}(C) + H^{2i+1}(C) \cup [\alpha_3]) = H^{3i+2}(C)$ is non-zero (see Lemma 7 in [45]).

Let W be a closed regular neighborhood of C and let $Z = \partial W$ be its boundary. We can arrange easily that Z is a *smooth* manifold of dimension $4i + 2$. There is an exact sequence

$$\cdots \to H^k(C) = H^k(W) \to H^k(Z) \to H^{k+1}(W, Z) = H_{4i+3-(k+1)}(W) = H_{4i+2-k}(C) \to \cdots$$

This implies that $H^k(Z) = 0$ for $k \leq i$. Also $\pi_1(Z) = \pi_1(W - C) \cong \pi_1(W) = \pi_1(C) = 1$. Therefore Z is i–connected. By Theorem 3.2, Z is $2i$–formal. Let us prove that Z is not $(2i+1)$–formal. If it were then by Theorem 3.1 it would be formal. Let us see that Z has a non-vanishing Massey product. Let $j : Z \hookrightarrow W$ be the inclusion. Now use that $j_* : H^{3i+2}(C) = H^{3i+2}(W) \to H^{3i+2}(Z)$ is injective (because $H_i(C) = 0$) and $H^{2i+1}(Z) = 0$. It follows that the Massey product $\langle j^*\alpha_1, j^*\alpha_2, j^*\alpha_3\rangle \in H^{3i+2}(Z)$ is non-zero.

To cover the case $s = 2i-1$, $i \geq 2$, we start with $S^i \vee S^{i+1} \vee S^{i+1} \subset \mathbb{R}^{i+2}$ let $\gamma = [[\iota_1, \iota_2], \iota_3] \in \pi_{3i}(S^i \vee S^{i+1} \vee S^{i+1})$, where ι_1 is the image of $\pi_i(S^i)$ and ι_2, ι_3 are the images of the generators of $\pi_{i+1}(S^{i+1})$ for the second and third factors. Let $C \subset \mathbb{R}^{4i+2}$ be the cone of γ as before, and $\alpha_1 \in H^i(C)$, $\alpha_2, \alpha_3 \in H^{i+1}(C)$ be the obvious cohomology classes dual to ι_j. The Massey product $\langle \alpha_1, \alpha_2, \alpha_3\rangle \in H^{3i+1}(C)/([\alpha_1] \cup H^{2i+1}(C) + H^{2i}(C) \cup [\alpha_3]) = H^{3i+1}(C)$ is non-zero.

Again, let W be a closed regular neighborhood of C and let $Z = \partial W$ be its boundary, which is taken to be a smooth manifold of dimension $4i+1$. Let $j : Z \hookrightarrow W$ be the inclusion. Then the Massey product $\langle j^*\alpha_1, j^*\alpha_2, j^*\alpha_3\rangle \in H^{3i+1}(Z)$ is non-zero, as before. So Z is not $2i$–formal. On the other hand, Z is $(i-1)$-connected and $H^i(Z) = \mathbb{R}$. So the minimal model $\mathcal{M}_Z$ of Z has $V^{<i} = 0$ and $V^i = \mathbb{R}$. As in the proof of Theorem 3.2 this implies that $N^{<(2i-1)} = 0$. Let ξ be a generator of V^i. If i is odd then $\xi \cdot \xi = 0$ so also $N^{2i-1} = 0$. If i is even and $\xi \cdot \xi$ is not exact then again $N^{2i-1} = 0$. If i is even and $[\xi \cdot \xi] = 0$ then N^{2i-1} is 1–dimensional and generated by an element v. Any element in $I_{2i-1}(N^{\leq(2i-1)})$ is of the form $v \cdot z$, with $z \in \bigwedge V^{\leq(2i-1)}$. This cannot be closed unless $z = 0$. So Z is $(2i-1)$–formal.

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M. Fernández: Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain. *E-mail:* mtpferol@lg.ehu.es

V. Muñoz: Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain. *E-mail:* vicente.munoz@uam.es