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Haldane–Shastry spin chains of BC_N type

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Abstract

We introduce four types of $SU(2M + 1)$ spin chains which can be regarded as the BC_N versions of the celebrated Haldane–Shastry chain. These chains depend on two free parameters and, unlike the original Haldane–Shastry chain, their sites need not be equally spaced. We prove that all four chains are solvable by deriving an exact expression for their partition function using Polychronakos’s “freezing trick”. From this expression we deduce several properties of the spectrum, and advance a number of conjectures that hold for a wide range of values of the spin M and the number of particles. In particular, we conjecture that the level density is Gaussian, and provide a heuristic derivation of general formulas for the mean and the standard deviation of the energy.

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1. Introduction

The Haldane–Shastry (HS) chain [1,2] describes a fixed arrangement of equally spaced spin $1/2$ particles in a circle with pairwise interactions inversely proportional to the square of the chord distance between the particles. The original interest of this model lies in the fact that the $U \rightarrow \infty$ limit of Gutzwiller’s variational wave function for the Hubbard model [3–5], which also coincides with the one-dimensional version of the resonating valence bond state introduced by Anderson [6], is an exact eigenfunction of the HS chain. The

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exact solvability of the HS chain was already proved in the original papers of Haldane and Shastry. A few years later Fowler and Minahan [7] used Polychronakos's exchange-operator formalism [8] to show that this model is also completely integrable. Although the obvious relation of the HS chain with the Sutherland (scalar) model of A_N type [9–11] was already remarked by Shastry, an explicit quantitative connection was first established by Polychronakos through the so-called “freezing trick” [12]. In the latter paper it is shown how to construct an integrable spin chain from a Calogero–Sutherland (CS) model of A_N type with internal degrees of freedom (“spin”) [13–17] by freezing the particles at the classical equilibrium positions of the scalar part of the CS potential. The first integrals of the spin chain are essentially obtained as the large coupling constant limit of the first integrals of the corresponding CS model. Polychronakos applied this technique to the original (rational) Calogero model of A_N type [18], constructing in this way a new integrable spin chain of HS type in which the spin sites were no longer equally spaced. In a subsequent publication [19], the same author gave a heuristic argument based on the freezing trick that relates the spectrum of the integrable spin chain with those of the corresponding scalar and spin dynamical models.

Both the integrability and the spectrum of the Haldane–Shastry and Polychronakos spin chains can thus be obtained from the trigonometric and rational CS spin dynamical models of A_N type. By contrast, the spin chains associated with the spin models of BC_N type [20–26] have received comparatively little attention. This is in part due to the fact that, unlike their A_N counterparts, the BC_N -type spin chains depend nontrivially on free parameters (one in the rational case and two in the trigonometric or hyperbolic cases). The integrability of the spin chain associated with the BC_N rational CS model was established by Yamamoto and Tsuchiya [21] using the Dunkl operator formalism [8,27,28], although, to the best of the authors' knowledge, the spectrum of this model has not been computed so far. The Haldane–Shastry (trigonometric) spin chain of BC_N type was discussed by Bernard, Pasquier, and Serban [29], but only for spin $1/2$ and with the assumption that the sites are equally spaced, which restricts the pair of free parameters in the model to just three particular values. Finkel et al. [25] recently discussed the integrability of the hyperbolic HC spin chain of BC_N type, but did not examine its spectrum.

In this paper we study the BC_N version of the Haldane–Shastry spin chain for arbitrary values of the spin and the coupling constants. It turns out that there are actually four different BC_N spin chains related to the original Haldane–Shastry chain, two of which are ferromagnetic and the other two antiferromagnetic. We prove that these chains are exactly solvable provided that the sites are the coordinates of an equilibrium of a suitable scalar potential, which is the same for all four chains. In particular, for generic values of the coupling constants the sites are not equally spaced. In addition, we rigorously establish the essential uniqueness of the equilibrium point of the scalar potential determining the chain sites. Using Polychronakos's freezing trick, we are able to derive an exact expression for the partition function of the models, thus establishing their solvability. From this expression, which is the main result of this paper, we deduce several interesting general properties of the spectrum. In the first place, the spectrum depends on the coupling constants only through their semisum $\tilde{\beta}$, while for generic values of $\tilde{\beta}$ the degeneracies of the energy levels depend only on the spin M and the number of particles N . Secondly, although the energy levels are in general unequally spaced, for $\tilde{\beta} \gg N$ (and sufficiently

large M) they cluster around an equally spaced set. In the third place, for half-integer spin the spectra of the two types of (anti)ferromagnetic chains are exactly the same, even if their Hamiltonians differ by a nontrivial term.

Apart from the rigorous results just mentioned, the evaluation of the partition function for several values of M and N has led us to several conjectures regarding the spectrum. First of all, our calculations strongly suggest that the clustering of the levels around an equally spaced set when $\tilde{\beta} \gg N$ occurs in fact for all values of the spin M . Secondly, even for moderately large values of N the level density follows a Gaussian distribution with great accuracy. This fact, which is the main conjecture of this paper, is reminiscent of the analogous property of the “embedded Gaussian ensemble” (EGOE) in random matrix theory [30]. It should be noted, however, that the essential requirement defining the EGOE, namely that the ratio of the number of particles to the number of one-particle states tend to zero as both quantities tend to infinity does not hold in our case. If the level density is Gaussian to a very high degree of approximation (for sufficiently large N), it is fully characterized by the mean μ and standard deviation σ of the energy. From a natural conjecture on the dependence of μ and σ on the number of particles, we have derived general formulas expressing these parameters as functions of N and M . We have then checked that these formulas yield the exact values of μ and σ for a wide range of values of the spin and the number of particles. We have also rigorously proved (without making use of the previous conjectures) that the standard deviations for both types of (anti)ferromagnetic chains with integer spin exactly coincide, even if their spectra are essentially different.

The paper is organized as follows. In Section 2 we introduce the Sutherland model of BC_N type with internal degrees of freedom, and outline a proof of its integrability by expressing the Hamiltonian in terms of an appropriate commuting family of self-adjoint Dunkl operators. The spectrum of the latter model is determined in Section 3 by explicit triangularization of the Hamiltonian. In particular, we compute the ground state energy in terms of the parameters of the model. The four types of Haldane–Shastry spin chains of BC_N type, which are the main subject of this paper, are presented in Section 4. Section 5 is devoted to the calculation of the spectrum of the chains introduced in the previous section. We first provide a semi-rigorous detailed justification of the freezing trick, whose key points are the uniqueness of the equilibrium point of the associated scalar potential together with the knowledge of the full spectrum of the corresponding scalar and spin Sutherland models of BC_N type. From the freezing trick we directly obtain an explicit expression for the ground state energy of the chains. We next make use of the freezing trick (which, by itself, does not completely determine the spectrum) to compute in closed form the partition functions of all four types of BC_N spin chains. In the last section we present concrete examples for spin $1/2$ and 1 , which led us to formulate the general conjectures mentioned above. The paper ends with a technical appendix, in which we establish the uniqueness of the equilibrium point of the scalar potential determining the sites of the chains.

2. The spin dynamical models

In this section we shall study the integrability of the *trigonometric Sutherland spin models of BC_N type*. Each of these models describes a system of N identical particles

with internal degrees of freedom (“spin”) moving on a circle, subject to one- and two-body interactions depending on the particles’ spatial and internal coordinates. We shall denote by \mathcal{S} the finite-dimensional Hilbert space corresponding to the spin degrees of freedom spanned by the states $|s_1, \dots, s_N\rangle$, where $-M \leq s_i \leq M$ and M is a half-integer. We shall respectively denote by S_{ij} and S_i ($i, j = 1, \dots, N$) the spin permutation and reversal operators, whose action on the basis of spin states is defined by

$$\begin{aligned} S_{ij}|s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle &= |s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle, \\ S_i|s_1, \dots, s_i, \dots, s_N\rangle &= |s_1, \dots, -s_i, \dots, s_N\rangle. \end{aligned} \quad (1)$$

These operators are represented in \mathcal{S} by $(2M+1)^N$ -dimensional Hermitian matrices. We shall denote by \mathfrak{S} the multiplicative group generated by the operators S_{ij} and S_i , which is isomorphic to the Weyl group of B_N type. We shall also use the customary notation $\tilde{S}_{ij} = S_i S_j S_{ij}$.

The BC_N -type spin dynamical models we shall study in this section are collectively described by a Hamiltonian of the form

$$\begin{aligned} H_{\epsilon\epsilon'}^* &= - \sum_i \partial_{x_i}^2 + a \sum_{i \neq j} [\sin^{-2} x_{ij}^- (a - \epsilon S_{ij}) + \sin^{-2} x_{ij}^+ (a - \epsilon \tilde{S}_{ij})] \\ &\quad + b \sum_i \sin^{-2} x_i (b - \epsilon' S_i) + b' \sum_i \cos^{-2} x_i (b' - \epsilon' S_i), \end{aligned} \quad (2)$$

where $\epsilon, \epsilon' = \pm 1$ are two independent signs, a, b, b' are real parameters greater than $1/2$, and $x_{ij}^\pm = x_i \pm x_j$. Here and in what follows, the sums (and products) run from 1 to N unless otherwise constrained. The potential in (2) possesses inverse-square type singularities at the hyperplanes $x_i \pm x_j = k\pi$, $x_i = k\pi/2$, with $k \in \mathbb{Z}$. In fact, since the nature of these singularities makes it impossible for one particle to overtake another or to cross the singularities at $x_i = k\pi/2$, we can regard the particles as distinguishable and take as configuration space the set

$$\tilde{\mathcal{C}} = \left\{ \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N \mid 0 < x_1 < \dots < x_N < \frac{\pi}{2} \right\}. \quad (3)$$

The Hilbert space of the system may thus be taken as $\mathcal{H} = L_0^2(\tilde{\mathcal{C}}) \otimes \mathcal{S}$, where

$$\begin{aligned} L_0^2(\tilde{\mathcal{C}}) &= \left\{ f \in L^2(\tilde{\mathcal{C}}) \mid \exists \lim_{x_i \pm x_j \rightarrow k\pi} |x_i \pm x_j - k\pi|^{-a} |f(\mathbf{x})|, \exists \lim_{x_i \rightarrow 0} |x_i|^{-b} |f(\mathbf{x})|, \right. \\ &\quad \left. \exists \lim_{x_i \rightarrow \pi/2} |x_i - \pi/2|^{-b'} |f(\mathbf{x})|; k = 0, 1, 1 \leq i \neq j \leq N \right\}. \end{aligned}$$

Note, in particular, that the physical wavefunctions vanish faster than the square root of the distance to the singular hyperplanes in their vicinity.

Formally, the four Hamiltonians (2) can be represented as a single Hamiltonian $H^* = H_{\epsilon\epsilon'}^*$ for an arbitrary choice of the signs ϵ and ϵ' , provided that the parameters a, b and b' are also allowed to take negative values less than $-1/2$. We have preferred to use the more explicit representation (2) since, as we shall see in the following section, the spectrum of $H_{\epsilon\epsilon'}^*$ depends in an essential way on ϵ and ϵ' . It can be shown that the operator $H_{\epsilon\epsilon'}^*: \mathcal{H} \rightarrow \mathcal{H}$ is equivalent to any of its extensions to spaces of symmetric or antisymmetric functions

(with respect to both permutations and sign reversals) in $L_0^2(C) \otimes \mathcal{S}$, where C is the N -cube $(-\frac{\pi}{2}, \frac{\pi}{2})^N$ and $L_0^2(C)$ is defined similarly to $L_0^2(\tilde{C})$. We shall consider without loss of generality that $H_{\epsilon\epsilon'}^*$ acts in the Hilbert space

$$\mathcal{H}_{\epsilon\epsilon'} = \Lambda_{\epsilon\epsilon'}(L_0^2(C) \otimes \mathcal{S}), \quad (4)$$

where $\Lambda_{\epsilon\epsilon'}$ is the projection operator on states with parity ϵ under simultaneous permutations of spatial coordinates and spins and ϵ' under sign reversals. The latter operator is characterized by the relations

$$K_{ij} \Lambda_{\epsilon\epsilon'} = \epsilon S_{ij} \Lambda_{\epsilon\epsilon'}, \quad K_i \Lambda_{\epsilon\epsilon'} = \epsilon' S_i \Lambda_{\epsilon\epsilon'}, \quad (5)$$

where K_{ij} and K_i respectively denote the spatial coordinates' permutation and sign reversing operators, defined by

$$\begin{aligned} (K_{ij} f)(x_1, \dots, x_i, \dots, x_j, \dots, x_N) &= f(x_1, \dots, x_j, \dots, x_i, \dots, x_N), \\ (K_i f)(x_1, \dots, x_i, \dots, x_N) &= f(x_1, \dots, -x_i, \dots, x_N). \end{aligned}$$

The relations (5) suggest the definition of a mapping $_{\epsilon\epsilon'}^*: \mathfrak{D} \otimes \mathfrak{K} \rightarrow \mathfrak{D} \otimes \mathfrak{S}$, where \mathfrak{D} denotes the algebra of scalar linear differential operators and $\mathfrak{K} \simeq \mathfrak{S}$ is the multiplicative group generated by the operators K_{ij} and K_i , as follows:

$$(DK_{i_1 j_1} \cdots K_{i_r j_r} K_{l_1} \cdots K_{l_s})_{\epsilon\epsilon'}^* = \epsilon^r \epsilon'^s D S_{l_s} \cdots S_{l_1} S_{i_r j_r} \cdots S_{i_1 j_1}, \quad (6)$$

where $D \in \mathfrak{D}$. This determines a linear map $A \mapsto A_{\epsilon\epsilon'}^*$ in $\mathfrak{D} \otimes \mathfrak{K}$, which by Eq. (5) satisfies

$$A \Lambda_{\epsilon\epsilon'} = A_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'}. \quad (7)$$

In particular, each of the physical Hamiltonians $H_{\epsilon\epsilon'}^*$ in (2) is the image under the corresponding star mapping of a *single* operator H , given by

$$\begin{aligned} H = & - \sum_i \partial_{x_i}^2 + a \sum_{i \neq j} [\sin^{-2} x_{ij}^- (a - K_{ij}) + \sin^{-2} x_{ij}^+ (a - \tilde{K}_{ij})] \\ & + b \sum_i \sin^{-2} x_i (b - K_i) + b' \sum_i \cos^{-2} x_i (b' - K_i). \end{aligned} \quad (8)$$

The integrability of the Hamiltonian (2) can be established by the same method applied in Ref. [25] to the hyperbolic version of H_{--}^* , based on the fact that H can be expressed

as the sum of the squares of the commuting Dunkl operators

$$J_k = i\partial_{x_k} + a \sum_{l \neq k} [(1 - i \cot x_{kl}^-) K_{kl} + (1 - i \cot x_{kl}^+) \tilde{K}_{kl}] \\ + [b(1 - i \cot x_k) + b'(1 + i \tan x_k)] K_k - 2a \sum_{l < k} K_{kl}. \quad (9)$$

These operators are related to the hyperbolic Dunkl operators \hat{J}_k of Ref. [25] by $J_k(\mathbf{x}) = -\hat{J}_k(i\mathbf{x})$. The commutativity of the Dunkl operators J_k implies that the operators

$$I_p = \sum_k J_k^{2p}, \quad p = 1, \dots, N, \quad (10)$$

form a complete set of commuting integrals of motion of $H = I_1$. From this fact it follows, as in Ref. [25], that the corresponding operators $(I_p)_{\epsilon\epsilon'}^*$, $p = 1, \dots, N$, act on the Hilbert space $\mathcal{H}_{\epsilon\epsilon'}$ and form a complete set of integrals of motion of the Hamiltonian $H_{\epsilon\epsilon'}^* = (I_1)_{\epsilon\epsilon'}^*$.

To end this section, we shall prove that the integrals of motion $(I_p)_{\epsilon\epsilon'}^*$ are self-adjoint. Note first of all that, unlike the operators \hat{J}_k , the Dunkl operators (9) and hence the integrals of motion (10) are self-adjoint. Since, furthermore, I_p and $\Lambda_{\epsilon\epsilon'}$ are self-adjoint and commute with one another (in fact, I_p commutes with K_{ij} and K_i by Lemma 4 of Ref. [25]), we have

$$(I_p)_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'} = I_p \Lambda_{\epsilon\epsilon'} = (I_p \Lambda_{\epsilon\epsilon'})^\dagger = ((I_p)_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'})^\dagger.$$

On the other hand, since $(I_p)_{\epsilon\epsilon'}^*$ also commutes with $\Lambda_{\epsilon\epsilon'}$ we obtain

$$((I_p)_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'})^\dagger = (\Lambda_{\epsilon\epsilon'} (I_p)_{\epsilon\epsilon'}^*)^\dagger = (I_p)_{\epsilon\epsilon'}^*{}^\dagger \Lambda_{\epsilon\epsilon'},$$

from which it follows that $(I_p)_{\epsilon\epsilon'}^*{}^\dagger = (I_p)_{\epsilon\epsilon'}^*$ by Lemma 1 of Ref. [25].

3. Spectrum of the spin dynamical models

In this section we shall compute the spectrum of the trigonometric Sutherland spin models of BC_N type (2). The results of this section will be used in Section 5 to derive the asymptotic behavior of the partition function of these models in the large coupling constant limit.

The computation of the spectrum of the Hamiltonian (2) is analogous to the corresponding computation for the hyperbolic model studied in Ref. [25], in spite of the fact that the boundary conditions are different. The starting point of this computation is the invariance under the Dunkl operators J_i of the finite-dimensional spaces

$$\mathcal{R}_k = \left\langle \phi(\mathbf{x}) \exp \left(2i \sum_j n_j x_j \right) \middle| n_j = -k, -k+1, \dots, k, \quad j = 1, \dots, N \right\rangle, \quad (11)$$

where

$$\phi(\mathbf{x}) = \prod_{i < j} |\sin x_{ij}^- \sin x_{ij}^+|^a \prod_i |\sin x_i|^b |\cos x_i|^{b'}, \quad (12)$$

for all nonnegative integer values of k . It follows that the operator $H = I_1$ preserves the spaces \mathcal{R}_k for all k . Since H commutes with $\Lambda_{\epsilon\epsilon'}$, Eq. (7) implies that

$$H_{\epsilon\epsilon'}^* [\Lambda_{\epsilon\epsilon'}(\varphi|\sigma)] = \Lambda_{\epsilon\epsilon'} [(H\varphi)|\sigma], \quad (13)$$

for all $\varphi \in L_0^2(C)$ and $|\sigma\rangle \in \mathcal{S}$. Hence the Hamiltonian $H_{\epsilon\epsilon'}^*$ leaves invariant the infinite increasing sequence of finite-dimensional spaces

$$\mathcal{M}_{k,\epsilon\epsilon'} = \Lambda_{\epsilon\epsilon'}(\mathcal{R}_k \otimes \mathcal{S}), \quad k = 0, 1, \dots, \quad (14)$$

and is therefore exactly solvable in the sense of Turbiner [31,32].

We shall next construct a (nonorthonormal) basis \mathcal{B} of the Hilbert space $L_0^2(C)$ in which H is represented by a triangular infinite-dimensional matrix, thereby obtaining an exact formula for the spectrum of this operator. To this end, note that the (scaled) exponential monomials

$$f_n(\mathbf{x}) = \phi(\mathbf{x}) \exp\left(2i \sum_j n_j x_j\right), \quad n = (n_1, \dots, n_N), \quad n_j \in \mathbb{Z}, \quad (15)$$

span a dense subspace of the Hilbert space $L_0^2(C)$. We can introduce a partial ordering $<$ in the set of exponential monomials (15) as follows. Given a multiindex $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$, we define the nonnegative and nonincreasing multiindex $[n]$ by

$$[n] = (|n_{i_1}|, \dots, |n_{i_N}|), \quad \text{where } |n_{i_1}| \geq \dots \geq |n_{i_N}|. \quad (16)$$

If $n, n' \in [\mathbb{Z}^N]$ are nonnegative and nonincreasing multiindices, we shall say that $n < n'$ if $n_1 - n'_1 = \dots = n_{i-1} - n'_{i-1} = 0$ and $n_i < n'_i$. For two arbitrary multiindices $n, n' \in \mathbb{Z}^N$, by definition $n < n'$ if and only if $[n] < [n']$. Finally, we shall say that $f_n < f_{n'}$ if and only if $n < n'$. Note that the partial ordering $<$ is preserved by the action of the Weyl group \mathfrak{R} , i.e., if $f_n < f_{n'}$ then $Wf_n < Wf_{n'}$ for all $W \in \mathfrak{R}$.

We can take as the basis \mathcal{B} any ordering of the set of exponential monomials (15) compatible with the partial ordering $<$. This follows from the fact that

$$Hf_n = \sum_i \lambda_{[n],i}^2 f_n + \sum_{\substack{n' \in \mathbb{Z}^N \\ n' < n}} c_n^{n'} f_{n'}, \quad n \in \mathbb{Z}^N, \quad (17)$$

where $\lambda_{[n],i}$ and $c_n^{n'}$ are real numbers (cf. Proposition 2 of Ref. [25]). The numbers $\lambda_{m,i}$ ($m \in [\mathbb{Z}^N]$) are explicitly given by

$$\lambda_{m,i} = \begin{cases} 2m_i + b + b' + 2a(N + i + 1 - \#(m_i) - 2\ell(m_i)), & m_i > 0, \\ -b - b' + 2a(i - N), & m_i = 0, \end{cases} \quad (18)$$

where we have used the following notation:

$$\#(s) = \text{card}\{i \mid m_i = s\}, \quad \ell(s) = \min\{i \mid m_i = s\}.$$

For instance, if $m = (5, 2, 2, 1, 1, 1, 0)$ then $\#(1) = 3$ and $\ell(1) = 4$. It will also be convenient in what follows to take $\ell(s) = +\infty$ if $m_i \neq s$ for all $i = 1, \dots, N$. Eq. (17) implies that the operator H is represented in the basis \mathcal{B} by an upper triangular matrix with diagonal elements

$$E_n = \sum_i \lambda_{[n],i}^2.$$

From the previous formula it is straightforward to deduce the following more compact expression for the eigenvalues E_n of the operator H :

$$E_n = \sum_i (2[n]_i + b + b' + 2a(N - i))^2. \quad (19)$$

Indeed, if $m = [n] \in [\mathbb{Z}^N]$ and $m_{k-1} > m_k = \dots = m_{k+p} > m_{k+p+1} \geq 0$ then $\ell(m_{k+j}) = k$ and $\#(m_{k+j}) = p + 1$ for $j = 0, \dots, p$, so that

$$\begin{aligned} \lambda_{m,k+j} &= 2m_{k+j} + b + b' + 2a(N - k - p + j) \\ &= 2m_{k+p-j} + b + b' + 2a(N - (k + p - j)) \end{aligned}$$

and hence

$$\sum_{i=k}^{k+p} \lambda_{m,i}^2 = \sum_{i=k}^{k+p} (2m_i + b + b' + 2a(N - i))^2. \quad (20)$$

If, on the other hand, $m_{k-1} > m_k = \dots = m_N = 0$, Eq. (20) follows directly from (18). This completes the proof of the formula (19).

Let us now compute the spectrum of the Hamiltonian (2) in $\mathcal{H}_{\epsilon\epsilon'}$. Note, first of all, that the states of the form

$$\Lambda_{\epsilon\epsilon'}(f_n | s_1, \dots, s_N), \quad n \in [\mathbb{Z}^N], \quad (21)$$

span a dense subset of the Hilbert space $\mathcal{H}_{\epsilon\epsilon'}$, by the analogous property of the functions (15) in $L_0^2(C)$. The states (21), however, are not linearly independent (in particular, some of them vanish if ϵ or ϵ' are negative). A (nonorthonormal) basis of $\mathcal{H}_{\epsilon\epsilon'}$ may be obtained from the states (21) by imposing the following conditions on the spin vector $|s_1, \dots, s_N\rangle$ (cf. Proposition 3 of Ref. [25]):

$$(i) \quad s_i - s_j \geq \delta_{-1,\epsilon}, \quad \text{if } n_i = n_j \text{ and } i < j; \quad (22a)$$

$$(ii) \quad s_i \geq \frac{1}{2} \delta_{-1,\epsilon'}, \quad \text{if } n_i = 0, \quad (22b)$$

where δ is Kronecker's delta. Indeed, if $n_i = n_j$ with $i < j$ we can clearly permute the i th and j th particles (if necessary) so that $s_i \geq s_j$, leaving the state (21) invariant up to a sign. If, in addition, $\epsilon = -1$ we must have $s_i > s_j$ by antisymmetry under permutations. Likewise, if $n_i = 0$ we can assume that $s_i \geq 0$ after a possible reversal of the sign of the coordinates of the i th particle, which again preserves the state (21) up to a sign. Moreover, $\epsilon' = -1$ forces $s_i > 0$ by antisymmetry under sign reversals. Note that when $\epsilon = -1$, i.e., when the basis states (21) are antisymmetric with respect to permutations, the first

condition implies the following restriction on the multiindex $n \in [\mathbb{Z}^N]$:

$$\#(n_i) \leq \begin{cases} 2M+1, & \text{if } n_i > 0, \\ M_{\epsilon'}, & \text{if } n_i = 0, \end{cases} \quad (23)$$

where $M_+ = \lfloor M \rfloor + 1$ and $M_- = \lceil M \rceil$. Here $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the integer part of x and the smallest integer greater than or equal to x . Let $\mathcal{B}_{\epsilon\epsilon'}$ be any ordering of the set of states (21)–(23) compatible with the partial ordering $<$. It follows from Eqs. (13) and (17) that the matrix of $H_{\epsilon\epsilon'}^*$ with respect to the basis $\mathcal{B}_{\epsilon\epsilon'}$ is upper triangular, with eigenvalues given by

$$E_{\epsilon\epsilon'}^*(n; s) = \sum_i (2n_i + b + b' + 2a(N - i))^2, \quad n \in [\mathbb{Z}^N], \quad s = (s_1, \dots, s_N). \quad (24)$$

It is worth mentioning at this point that, although a cursory inspection of the previous equation may suggest that the models (2) are isospectral, this is in general not the case. Indeed, condition (23) implies that many eigenvalues of the models with $\epsilon = 1$ are absent from the spectrum of the models with $\epsilon = -1$. Besides, for a fixed value of ϵ , by condition (22b) the degeneracy of the eigenvalues also depends on ϵ' when M is an integer (see Eqs. (50) and (51) in Section 5 for the minimum degeneracy of each level).

Since $E_{\epsilon\epsilon'}^*(n; s)$ is an increasing function of the components of the multiindex n , the ground state of the system is obtained when each component n_i takes the lowest possible value. Thus for $\epsilon = 1$, or $\epsilon = -1$ and $N \leq M_{\epsilon'}$, we have $n = 0$. On the other hand, when $\epsilon = -1$ and $N > M_{\epsilon'}$ condition (23) implies that

$$n = (\overbrace{m_0, \dots, m_0}^r, \overbrace{m_0 - 1, \dots, m_0 - 1}^{2M+1}, \dots, \overbrace{1, \dots, 1}^{2M+1}, \overbrace{0, \dots, 0}^{M_{\epsilon'}}), \quad (25)$$

where $N = M_{\epsilon'} + (m_0 - 1)(2M + 1) + r$ with $r = 1, \dots, 2M + 1$. The ground state energy $E_{\epsilon\epsilon', \min}^*$ is easily computed in this case using Eq. (24) with the multiindex n given in (25). We thus obtain

$$\begin{aligned} E_{\epsilon\epsilon', \min}^* &= \frac{4}{3}a^2N^3 - 2acN^2 + \frac{1}{3}(3c^2 - a^2)N \\ &\quad + \frac{1}{3}\kappa m_0[4m_0^2(1 - a\kappa) + 6cm_0 + a\kappa + 2] \\ &\quad + 2m_0\rho\left[c + m_0(1 - a\kappa) - \frac{1}{2}a\rho\right], \end{aligned} \quad (26)$$

where $c = a - b - b' - 2m_0$ and

$$\kappa = 2M + 1, \quad \rho = \begin{cases} \epsilon', & M = 0, 1, \dots, \\ 0, & M = \frac{1}{2}, \frac{3}{2}, \dots \end{cases} \quad (27)$$

It can be easily shown that Eq. (26) is also valid when $n = 0$ if we take $m_0 = 0$. Thus Eq. (26) yields the ground state energy in all cases provided that m_0 is defined by

$$m_0 = \delta_{\epsilon, -1} \left\lceil \frac{N - M_{\epsilon'}}{2M + 1} \right\rceil. \quad (28)$$

4. The spin chains

The Hamiltonian of the HS spin chains of BC_N type associated with the spin dynamical models (2) discussed in the previous sections is given by

$$\begin{aligned} h_{\epsilon\epsilon'} = & \sum_{i \neq j} [\sin^{-2} \xi_{ij}^- (1 - \epsilon S_{ij}) + \sin^{-2} \xi_{ij}^+ (1 - \epsilon \tilde{S}_{ij})] \\ & + \sum_i (\beta \sin^{-2} \xi_i + \beta' \cos^{-2} \xi_i) (1 - \epsilon' S_i), \end{aligned} \quad (29)$$

where β and β' are positive real parameters, $\xi_{ij}^\pm = \xi_i \pm \xi_j$, and $\xi = (\xi_1, \dots, \xi_N)$ is the unique equilibrium point in the set \tilde{C} of the classical potential

$$U(\mathbf{x}) = \sum_{i \neq j} (\sin^{-2} x_{ij}^- + \sin^{-2} x_{ij}^+) + \sum_i (\beta^2 \sin^{-2} x_i + \beta'^2 \cos^{-2} x_i). \quad (30)$$

It is important to note that the classical potential (30) is independent of ϵ and ϵ' , and therefore the sites of the four chains (29) are the same. The existence of a minimum of U in \tilde{C} for all values of β and β' is a consequence of the positivity and continuity of U in \tilde{C} and the fact that it tends to infinity at the boundary of this set. The uniqueness of this minimum is proved in [Appendix A](#). Note that, in contrast, the corresponding potential for the hyperbolic spin chain of BC_N type treated in [25] admits an equilibrium point only for a certain range of values of β and β' .

The chains (29) with $\epsilon = -1$ (respectively $\epsilon = 1$) are of antiferromagnetic (respectively ferromagnetic) type. Note also that the spin chain Hamiltonians $h_{-\epsilon, -\epsilon'}$ and $-h_{\epsilon\epsilon'}$ are related by

$$h_{-\epsilon, -\epsilon'} = -h_{\epsilon\epsilon'} + 2V(\xi), \quad (31)$$

where

$$V(\mathbf{x}) = \sum_{i \neq j} (\sin^{-2} x_{ij}^- + \sin^{-2} x_{ij}^+) + \beta \sum_i \sin^{-2} x_i + \beta' \sum_i \cos^{-2} x_i. \quad (32)$$

On the other hand, for a fixed ϵ the two chains $h_{\epsilon, \pm}$ are essentially different. From Eq. (29) it immediately follows that the eigenvalues $e_{\epsilon\epsilon', j}$ of the Hamiltonian $h_{\epsilon\epsilon'}$ are nonnegative. Moreover, in the ferromagnetic case ($\epsilon = 1$) the ground state energy clearly vanishes, since states symmetric under permutations with parity ϵ' under spin reversals are annihilated by the Hamiltonian. Eq. (31) implies that the maximum energy of the antiferromagnetic chains is $2V(\xi)$.

The integrability of the spin chain (29) with $M = 1/2$ and $\epsilon = \epsilon' = -1$ was proved in Ref. [29] only for the special values $(3/2, 1/2)$, $(3/2, 3/2)$, and $(1/2, 1/2)$ of the pair (β, β') , for which the corresponding sites

$$\frac{i\pi}{2N+1}, \quad \frac{i\pi}{2N+2}, \quad \left(i - \frac{1}{2}\right) \frac{\pi}{2N}, \quad i = 1, \dots, N,$$

are *equally spaced*, as in the original Haldane–Shastry chain [1,2]. As we shall see below, the discussion of the integrability of the HS spin chains of BC_N type (29) for arbitrary

values of the parameters β and β' is completely analogous to that of Ref. [25] for the hyperbolic version of \mathfrak{h}_{--} .

Let us begin by defining the operators J_i ($i = 1, \dots, N$) and I_p ($p \in \mathbb{N}$) by

$$J_i = i\partial_{x_i} + aJ_i, \quad I_p = \sum_i J_i^{2p}.$$

The operators $(I_p)_{\epsilon\epsilon'}^*$ clearly commute with one another, since $[(I_p)_{\epsilon\epsilon'}^*, (I_q)_{\epsilon\epsilon'}^*]$ is the coefficient of $a^{2(p+q)}$ in the expansion in powers of a of the identity $[(I_p)_{\epsilon\epsilon'}^*, (I_q)_{\epsilon\epsilon'}^*] = 0$. Hence the operators

$$(I_p)_{\epsilon\epsilon'}^{*0} \equiv (I_p)_{\epsilon\epsilon'}^*|_{\mathbf{x}=\boldsymbol{\xi}}, \quad p \in \mathbb{N}, \quad (33)$$

also commute with one another. We shall now prove that the operators (33) form a commuting family of integrals of motion for the spin chain Hamiltonian $\mathfrak{h}_{\epsilon\epsilon'}$. Note that this result does not follow trivially from the previous assertions since, in contrast with the dynamical case, $(I_1)_{\epsilon\epsilon'}^{*0} = U(\boldsymbol{\xi})$ is a constant and therefore does not coincide with $\mathfrak{h}_{\epsilon\epsilon'}$.

The starting point in the proof of the commutativity of $\mathfrak{h}_{\epsilon\epsilon'}$ and $(I_p)_{\epsilon\epsilon'}^{*0}$ is the following expansion of H in powers of a :

$$H = - \sum_i \partial_{x_i}^2 - aH + a^2 U(\mathbf{x}), \quad (34)$$

where

$$H = \sum_{i \neq j} [\sin^{-2} x_{ij}^- K_{ij} + \sin^{-2} x_{ij}^+ \tilde{K}_{ij}] + \sum_i (\beta \sin^{-2} x_i + \beta' \cos^{-2} x_i) K_i, \quad (35)$$

U is defined in Eq. (30), and we have set

$$\beta = \frac{b}{a}, \quad \beta' = \frac{b'}{a}. \quad (36)$$

Note that $\mathfrak{h}_{\epsilon\epsilon'} = -H_{\epsilon\epsilon'}^{*0} + V(\boldsymbol{\xi})$, so that the integrability of $\mathfrak{h}_{\epsilon\epsilon'}$ follows from that of $H_{\epsilon\epsilon'}^{*0}$. Arguing as in Ref. [25] it is straightforward to show that

$$[H_{\epsilon\epsilon'}^*, (I_p)_{\epsilon\epsilon'}^*] = \sum_i \frac{\partial U}{\partial x_i} (C_{p,i})_{\epsilon\epsilon'}^*,$$

for certain operators $C_{p,i}$ in $\mathfrak{D} \otimes \mathfrak{K}$. Setting $\mathbf{x} = \boldsymbol{\xi}$ in the previous identity, it follows that $(I_p)_{\epsilon\epsilon'}^{*0}$ commutes with $\mathfrak{h}_{\epsilon\epsilon'}$. Note finally that the first integrals $(I_p)_{\epsilon\epsilon'}^{*0}$ are clearly self-adjoint, since they are equal to the coefficient of a^{2p} in the corresponding self-adjoint operators $(I_p)_{\epsilon\epsilon'}^*$ evaluated at the equilibrium point $\boldsymbol{\xi}$.

5. Partition function and spectrum of the spin chains

In this section we shall compute the partition function of the HS spin chains of BC_N type (29) by using Polychronakos's freezing trick [12,19] applied to the spin dynamical

models discussed in the previous sections. We shall first provide a detailed heuristic justification of the freezing trick in the present context. Our calculation relies on the computation of the large coupling constant limit of the partition functions of the spin dynamical models (2) and the scalar Sutherland model of BC_N type

$$H_s = - \sum_i \partial_{x_i}^2 + a(a-1) \sum_{i \neq j} (\sin^{-2} x_{ij}^- + \sin^{-2} x_{ij}^+) \\ + b(b-1) \sum_i \sin^{-2} x_i + b'(b'-1) \sum_i \cos^{-2} x_i \quad (37)$$

acting on the Hilbert space $L_0^2(\tilde{C})$. Using the definition (36) of β and β' we obtain

$$H_s = - \sum_i \partial_{x_i}^2 + a^2 U(\mathbf{x}) - a V(\mathbf{x}), \quad (38)$$

with $U(\mathbf{x})$ and $V(\mathbf{x})$ respectively given by Eqs. (30) and (32). From Eqs. (34) and (38) it follows that

$$H_{\epsilon\epsilon'}^* = - \sum_i \partial_{x_i}^2 - a H_{\epsilon\epsilon'}^* + a^2 U(\mathbf{x}) = H_s + a(V(\mathbf{x}) - H_{\epsilon\epsilon'}^*), \quad (39)$$

where $H_{\epsilon\epsilon'}^*$ is assumed to act in the Hilbert space $\mathcal{H} = L_0^2(\tilde{C}) \otimes \mathcal{S}$. Let $\{\psi_i(\mathbf{x})\}_{i \in \mathbb{N}}$ be a basis of eigenfunctions of H_s , and let $\{|\sigma_{\epsilon\epsilon',j}\rangle\}_{j=1}^d$, with $d = (2M+1)^N$, be a basis of eigenfunctions of $h_{\epsilon\epsilon'}$, so that

$$H_s \psi_i(\mathbf{x}) = E_i \psi_i(\mathbf{x}), \quad h_{\epsilon\epsilon'} |\sigma_{\epsilon\epsilon',j}\rangle = e_{\epsilon\epsilon',j} |\sigma_{\epsilon\epsilon',j}\rangle.$$

The set $\{\psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle\}_{i \in \mathbb{N}, 1 \leq j \leq d}$ is thus a basis of the Hilbert space \mathcal{H} , and

$$H_s(\psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle) = E_i \psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle,$$

since H_s does not act on the spin variables.

From Appendix A it follows that the classical potential $a^2 U(\mathbf{x}) - a V(\mathbf{x})$ has a unique equilibrium point (actually, a minimum) $\chi(a)$ in the set \tilde{C} provided that $a > \max(1/\beta, 1/\beta', 1)$. The freezing trick is based on the fact that for $a \gg 1$ the eigenfunctions of H_s are all sharply peaked around the equilibrium $\chi(a)$. Since $\chi(a) = \xi + O(a^{-1})$, for $a \gg 1$ we have

$$\begin{aligned} & [V(\mathbf{x}) - H_{\epsilon\epsilon'}^*](\psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle) \\ &= \psi_i(\mathbf{x})[V(\mathbf{x}) - H_{\epsilon\epsilon'}^*]|\sigma_{\epsilon\epsilon',j}\rangle \simeq \psi_i(\mathbf{x})[V(\xi) - H_{\epsilon\epsilon'}^{*0}]|\sigma_{\epsilon\epsilon',j}\rangle \\ &= \psi_i(\mathbf{x})(h_{\epsilon\epsilon'}|\sigma_{\epsilon\epsilon',j}\rangle) = e_{\epsilon\epsilon',j} \psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle. \end{aligned}$$

Therefore, for $a \gg 1$ the Hamiltonian $H_{\epsilon\epsilon'}^*$ is approximately diagonal in the basis $\{\psi_i(\mathbf{x})|\sigma_{\epsilon\epsilon',j}\rangle\}_{i \in \mathbb{N}, 1 \leq j \leq d}$, with eigenvalues approximately given by

$$E_{\epsilon\epsilon',ij}^* \simeq E_i + a e_{\epsilon\epsilon',j}, \quad i \in \mathbb{N}, 1 \leq j \leq d, a \gg 1. \quad (40)$$

Taking into account that $e_{\epsilon\epsilon',j}$ is independent of a , we immediately obtain the following exact expression for the eigenvalues $e_{\epsilon\epsilon',j}$ of the spin chain Hamiltonian (29):

$$e_{\epsilon\epsilon',j} = \lim_{a \rightarrow \infty} \frac{1}{a} (E_{\epsilon\epsilon',ij}^* - E_i). \quad (41)$$

Using Eq. (40), we can easily derive the ground state energy $e_{\epsilon\epsilon',\min}$ of the spin chains (29). Indeed, clearly the ground state energy $E_{\epsilon\epsilon',\min}^*$ given by Eq. (26) is achieved when both E_i and $e_{\epsilon\epsilon',j}$ in Eq. (40) attain their minimum values E_{\min} and $e_{\epsilon\epsilon',\min}$. From Eqs. (24) and (36) it follows that both $E_{\epsilon\epsilon',ij}^*$ and E_i are polynomials of the second degree in a with the same leading coefficient. Hence

$$e_{\epsilon\epsilon',\min} = \lim_{a \rightarrow \infty} \frac{1}{a} (E_{\epsilon\epsilon',\min}^* - E_{\min}). \quad (42)$$

Since E_{\min} is obtained by setting $n = 0$ in Eq. (24), the coefficient of a in E_{\min} vanishes. From the previous equation, it follows that $e_{\epsilon\epsilon',\min}$ is the coefficient of a in $E_{\epsilon\epsilon',\min}^*$, namely (cf. Eq. (26))

$$e_{\epsilon\epsilon',\min} = m_0 \left[4N^2 + 4(2\bar{\beta} - 1)N + \frac{\kappa}{3}(\kappa - 2m_0(6\bar{\beta} + 2m_0\kappa - 3)) - 2\rho(2\bar{\beta} + m_0\kappa - 1) - \rho^2 \right], \quad (43)$$

where κ , ρ and m_0 are defined in Eqs. (27) and (28), and we have set

$$\bar{\beta} = \frac{1}{2}(\beta + \beta'). \quad (44)$$

Note that (as remarked in the previous section) the ferromagnetic ground state energy vanishes, since $m_0 = 0$ when $\epsilon = 1$.

We emphasize that Eq. (41) cannot be used directly to compute in full the spectrum of $h_{\epsilon\epsilon'}$, since it is not clear a priori which eigenvalues of $H_{\epsilon\epsilon'}^*$ and H_s can be combined to yield an eigenvalue of $h_{\epsilon\epsilon'}$. The importance of Eq. (41) lies on the fact that it can be used as the starting point for the exact computation of the partition function of the spin chain $h_{\epsilon\epsilon'}$, which in turn completely determines the spectrum.

Let us denote by Z_s , $Z_{\epsilon\epsilon'}^*$ and $Z_{\epsilon\epsilon'}$ the partition functions of the scalar Sutherland Hamiltonian (37), the Sutherland spin dynamical model (2), and the spin chain Hamiltonian (29), respectively. From Eq. (40) it follows that $Z_{\epsilon\epsilon'}^*(T) \simeq Z_s(T)Z_{\epsilon\epsilon'}(T/a)$, and hence

$$Z_{\epsilon\epsilon'}(T) = \lim_{a \rightarrow \infty} \frac{Z_{\epsilon\epsilon'}^*(aT)}{Z_s(aT)}. \quad (45)$$

Recall that the spectrum of the Hamiltonian $H_{\epsilon\epsilon'}^*$ of the spin dynamical model is given by Eq. (24), where $n \in [\mathbb{Z}^N]$ is a nonnegative nonincreasing multiindex (satisfying conditions (23) if $\epsilon = -1$), and $s = (s_1, \dots, s_N)$, $-M \leq s_i \leq M$, satisfies (22a)–(22b). The leading terms in the expansion of $E_{\epsilon\epsilon'}^*(n; s)$ are therefore

$$E_{\epsilon\epsilon'}^*(n; s) \simeq a^2 E_0 + 8a \sum_i n_i (\bar{\beta} + N - i), \quad (46)$$

where $E_0 = 4 \sum_i (\bar{\beta} + N - i)^2$ is a constant independent of n . The eigenvalues $E(n)$ of the scalar Sutherland Hamiltonian H_s are given by the right-hand side of Eq. (24), where the multiindex $n \in [\mathbb{Z}^N]$ is now unrestricted. Thus $E(n)$ also satisfies Eq. (46) for $a \gg 1$.

Let us start by computing the large a limit of the denominator in Eq. (45). Note, first of all, that the constant E_0 can be dropped from both $E_{\epsilon\epsilon'}^*(n; s)$ and $E(n)$ without affecting

the value of $Z_{\epsilon\epsilon'}(T)$. Using the asymptotic expansion of $E(n)$ and setting

$$q = e^{-8/(k_B T)} \quad (47)$$

we immediately obtain

$$Z_s(aT) \simeq \sum_{n \in [\mathbb{Z}^N]} q^{\sum_i n_i (\bar{\beta} + N - i)}. \quad (48)$$

Defining $p_i = n_i - n_{i+1}$, $1 \leq i \leq N-1$, and $p_N = n_N$ we have

$$\prod_i q^{n_i (\bar{\beta} + N - i)} = \prod_{i \leq j} q^{p_j (\bar{\beta} + N - i)} = \prod_j q^{p_j \sum_{i=1}^j (\bar{\beta} + N - i)} = \prod_j q^{j p_j (\bar{\beta} + N - \frac{1}{2}(j+1))}$$

and hence

$$\begin{aligned} Z_s(aT) &\simeq \sum_{p_1, \dots, p_N \geq 0} \prod_i q^{i p_i (\bar{\beta} + N - \frac{1}{2}(i+1))} = \prod_i \sum_{p_i \geq 0} q^{i p_i (\bar{\beta} + N - \frac{1}{2}(i+1))} \\ &= \prod_i [1 - q^{i(\bar{\beta} + N - \frac{1}{2}(i+1))}]^{-1}. \end{aligned} \quad (49)$$

Let us compute next the partition function $Z_{\epsilon\epsilon'}^*(aT)$ of the Sutherland spin dynamical model (2) for $a \gg 1$. To this end, it is convenient to represent the multiindex $n \in [\mathbb{Z}^N]$ appearing in Eq. (46) as

$$n = (\overbrace{m_1, \dots, m_1}^{k_1}, \overbrace{m_2, \dots, m_2}^{k_2}, \dots, \overbrace{m_r, \dots, m_r}^{k_r}), \quad (50)$$

where $m_1 > m_2 > \dots > m_r \geq 0$, and $k_i = \#(m_i) \in \mathbb{N}$ satisfies $k_1 + \dots + k_r = N$ (together with condition (23), if $\epsilon = -1$). Thus

$$\begin{aligned} E_{\epsilon\epsilon'}^*(n; s) &\simeq 8a \sum_{i=1}^r m_i \sum_{j=k_1+\dots+k_{i-1}+1}^{k_1+\dots+k_{i-1}+k_i} (\bar{\beta} + N - j) \\ &= 8a \sum_{i=1}^r m_i k_i \left(\bar{\beta} + N - \frac{1}{2} - \frac{k_i}{2} - \sum_{j=1}^{i-1} k_j \right) \equiv 8a \sum_{i=1}^r m_i v_i. \end{aligned}$$

Let $k = (k_1, \dots, k_r)$, and denote by $d_{\epsilon\epsilon'}(k, m_r)$ the cardinal of the set of spin quantum numbers s satisfying conditions (22a)–(22b) for the multiindex (50), namely

$$d_{\epsilon\epsilon'}(k, m_r) = \binom{2M+1+\delta_{1,\epsilon}(k_1-1)}{k_1} \dots \binom{2M+1+\delta_{1,\epsilon}(k_r-1)}{k_r}, \quad m_r > 0; \quad (51a)$$

$$\begin{aligned} d_{\epsilon\epsilon'}(k, 0) &= \binom{2M+1+\delta_{1,\epsilon}(k_1-1)}{k_1} \dots \binom{2M+1+\delta_{1,\epsilon}(k_{r-1}-1)}{k_{r-1}} \\ &\quad \times \binom{M_{\epsilon'}+\delta_{1,\epsilon}(k_r-1)}{k_r}. \end{aligned} \quad (51b)$$

The partition function $Z_{\epsilon\epsilon'}^*(aT)$ is therefore given by

$$\begin{aligned} Z_{\epsilon\epsilon'}^*(aT) &\simeq \sum_{k \in \mathcal{P}_N} \sum_{m_1 > \dots > m_r \geq 0} d_{\epsilon\epsilon'}(k, m_r) \prod_{i=1}^r q^{m_i v_i} \\ &= \sum_{k \in \mathcal{P}_N} \sum_{m_1 > \dots > m_r > 0} d_{\epsilon\epsilon'}(k, m_r) \prod_{i=1}^r q^{m_i v_i} \\ &\quad + \sum_{k \in \mathcal{P}_N} \sum_{m_1 > \dots > m_{r-1} > 0} d_{\epsilon\epsilon'}(k, 0) \prod_{i=1}^{r-1} q^{m_i v_i}, \end{aligned}$$

where we have denoted by \mathcal{P}_N the set of partitions of the positive integer N . Since

$$\begin{aligned} \sum_{m_1 > \dots > m_s > 0} \prod_{i=1}^s q^{m_i v_i} &= \sum_{p_1, \dots, p_s > 0} \prod_{i=1}^s q^{v_i \sum_{j=i}^s p_j} = \sum_{p_1, \dots, p_s > 0} \prod_{i=1}^s \prod_{j=i}^s q^{p_j v_i} \\ &= \sum_{p_1, \dots, p_s > 0} \prod_{j=1}^s q^{p_j \sum_{i=1}^j v_i} = \prod_{j=1}^s \sum_{p_j > 0} q^{p_j \sum_{i=1}^j v_i} \\ &= \prod_{j=1}^s \frac{q^{N_j}}{1 - q^{N_j}}, \end{aligned}$$

where

$$N_j = \sum_{i=1}^j v_i = \left(\sum_{i=1}^j k_i \right) \left(\bar{\beta} + N - \frac{1}{2} - \frac{1}{2} \sum_{i=1}^j k_i \right), \quad (52)$$

using Eqs. (51) we finally obtain

$$\begin{aligned} Z_{\epsilon\epsilon'}^*(aT) &\simeq \sum_{(k_1, \dots, k_r) \in \mathcal{P}_N} \left\{ \left[\binom{M_{\epsilon'} + \delta_{1,\epsilon}(k_r - 1)}{k_r} + \binom{2M + 1 + \delta_{1,\epsilon}(k_r - 1)}{k_r} \frac{q^{N_r}}{1 - q^{N_r}} \right] \right. \\ &\quad \times \left. \prod_{j=1}^{r-1} \left[\binom{2M + 1 + \delta_{1,\epsilon}(k_j - 1)}{k_j} \frac{q^{N_j}}{1 - q^{N_j}} \right] \right\}. \quad (53) \end{aligned}$$

Eqs. (45), (49) and (53) yield the following exact formula for the partition function of the HS spin chain (29):

$$\begin{aligned} Z_{\epsilon\epsilon'}(T) &= \prod_{i=1}^N \left[1 - q^{i(\bar{\beta} + N - \frac{1}{2}(i+1))} \right] \\ &\quad \times \sum_{(k_1, \dots, k_r) \in \mathcal{P}_N} \left\{ \left[\binom{M_{\epsilon'} + \delta_{1,\epsilon}(k_r - 1)}{k_r} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \binom{2M+1+\delta_{1,\epsilon}(k_r-1)}{k_r} \frac{q^{N_r}}{1-q^{N_r}} \Big] \\
& \times \prod_{j=1}^{r-1} \left[\binom{2M+1+\delta_{1,\epsilon}(k_j-1)}{k_j} \frac{q^{N_j}}{1-q^{N_j}} \right] \Big\}. \quad (54)
\end{aligned}$$

From the previous formula, which is in fact the main result of this paper, one can infer several remarkable properties of the spectrum of the spin chain (29) that we shall now discuss. First of all, for half-integer M the partition function (54) does not depend on ϵ' , since in this case $M_{\pm} = M + 1/2$. Hence the spectrum of the spin chain (29) is independent of ϵ' when M is a half-integer, a property that is not immediately apparent from the expression of the Hamiltonian (29). Secondly, all the denominators $1 - q^{N_k}$, $1 \leq k \leq r$, appearing in the second line of Eq. (54) are included as factors in the product in the first line. Hence the partition function (54) can be rewritten as

$$Z_{\epsilon\epsilon'}(T) = \sum_{\delta \in \{0,1\}^N} d_{\epsilon\epsilon',\delta}(M) q^{\epsilon\delta}, \quad (55)$$

where $\epsilon\delta$ is given by

$$\epsilon\delta = \sum_{i=1}^N \delta_i i \left(\bar{\beta} + N - \frac{1}{2}(i+1) \right), \quad \delta = (\delta_1, \dots, \delta_N), \quad (56)$$

and the degeneracy factor $d_{\epsilon\epsilon',\delta}(M)$ is a polynomial of degree N in M . Therefore, for all values of ϵ , ϵ' and M , the spectrum of the spin chain is contained in the set of 2^N numbers $8\epsilon\delta$, $\delta \in \{0,1\}^N$. Moreover, for generic (sufficiently large) values of the spin M , the spectrum *exactly* coincides with the above set of numbers, the values of ϵ , ϵ' and M affecting only the degeneracy $d_{\epsilon\epsilon',\delta}(M)$ of each level. In particular, from the previous observation and Eqs. (31) and (56) we immediately obtain the following exact expression for the constant $V(\xi)$ (i.e., half the maximum energy of the antiferromagnetic chain):

$$V(\xi) = 4 \sum_{i=1}^N i \left(\bar{\beta} + N - \frac{1}{2}(i+1) \right) = \frac{2}{3} N(N+1)(2N+3\bar{\beta}-2). \quad (57)$$

From Eq. (56) it follows that the energies of the spin chains (29) are of the form $8(j\bar{\beta} + k)$, with j, k nonnegative integers. Since the coefficients of the powers of q appearing in Eq. (53) are independent of $\bar{\beta}$, it follows that for generic¹ values of $\bar{\beta}$ the degeneracy of the levels depends only on the spin M and the number of particles N .

6. Discussion and conjectures

In this section we shall present several concrete examples in which we apply the formula (54) to compute the spectrum of the spin chains (29) for certain values of N and M . These

¹ More precisely, for all real values of $\bar{\beta}$ except for a finite (possibly empty) set of rationals.

examples strongly suggest a number of conjectures that shall be discussed in detail at the end of this section. We shall restrict ourselves to the antiferromagnetic chains $h_{-, \pm}$, the properties of their ferromagnetic counterparts following easily from the relation (31).

Example 1. The structure of Eq. (54) makes it straightforward to compute the spectrum of the spin chains for any *fixed* number of particles as a function of the spin. For instance, for $N = 3$ sites and integer M the energies (divided by 8) of the spin chain h_{--} are $0, \bar{\beta} + 2, 2\bar{\beta} + 3, 3\bar{\beta} + 3, 3\bar{\beta} + 5, 4\bar{\beta} + 5, 5\bar{\beta} + 6, 6\bar{\beta} + 8$, with respective degeneracies

$$\begin{aligned} & \frac{1}{6}M(M-1)(M-2), & \frac{5}{6}M(M^2-1), & \frac{1}{6}M(M+1)(11M-2), \\ & \frac{1}{6}M(M+1)(7M-4), & \frac{1}{6}M(M+1)(7M+11), \\ & \frac{1}{6}M(M+1)(11M+13), & \frac{5}{6}M(M+1)(M+2), \\ & \frac{1}{6}(M+1)(M+2)(M+3). \end{aligned}$$

Note that in this case all energy levels $8\epsilon_\delta$, with ϵ_δ given by (56), are attained for $M \geq 3$, in agreement with the general discussion of the previous section. For $M = 1$ and a few values of β and β' , we have numerically computed the spectrum of the spin chain (29) by representing the operators S_{ij} and S_i as 27×27 matrices. The results obtained are in complete agreement with those listed above.

For a fixed value of the spin M , we have not been able to find an explicit formula expressing the energies and their degeneracies as functions of the number of particles N . However, if we fix M the spectrum can be straightforwardly computed from Eq. (54) for any given value of N . We shall next present two concrete examples for the cases $M = 1/2$ and $M = 1$.

Example 2 (*spin* $1/2$). In this case we have computed the partition function $Z_{-, \pm}$ for up to 20 particles (recall that for half-integer M the chains with $\epsilon' = \pm 1$ have the same spectrum). For instance, for $N = 6$ the antiferromagnetic spin chain energies (divided by 8) and their corresponding degeneracies (denoted by subindices) are given by

$$\begin{aligned} & (9\bar{\beta} + 32)_2, & (10\bar{\beta} + 36)_2, & (11\bar{\beta} + 38)_2, \\ & (11\bar{\beta} + 41)_4, & (12\bar{\beta} + 38)_1, & (12\bar{\beta} + 43)_6, \\ & (13\bar{\beta} + 43)_3, & (13\bar{\beta} + 46)_6, & (14\bar{\beta} + 46)_4, \\ & (14\bar{\beta} + 50)_4, & (15\bar{\beta} + 47)_2, & (15\bar{\beta} + 50)_3, \\ & (15\bar{\beta} + 55)_6, & (16\bar{\beta} + 51)_2, & (16\bar{\beta} + 55)_5, \\ & (17\bar{\beta} + 53)_1, & (17\bar{\beta} + 56)_4, & (18\bar{\beta} + 58)_3, \\ & (19\bar{\beta} + 61)_2, & (20\bar{\beta} + 65)_1, & (21\bar{\beta} + 70)_1. \end{aligned}$$

The number of levels increases rapidly with the number of particles N . For example, if $N = 10$ the number of levels (for generic values of $\bar{\beta}$) is 136, while for $N = 20$ this

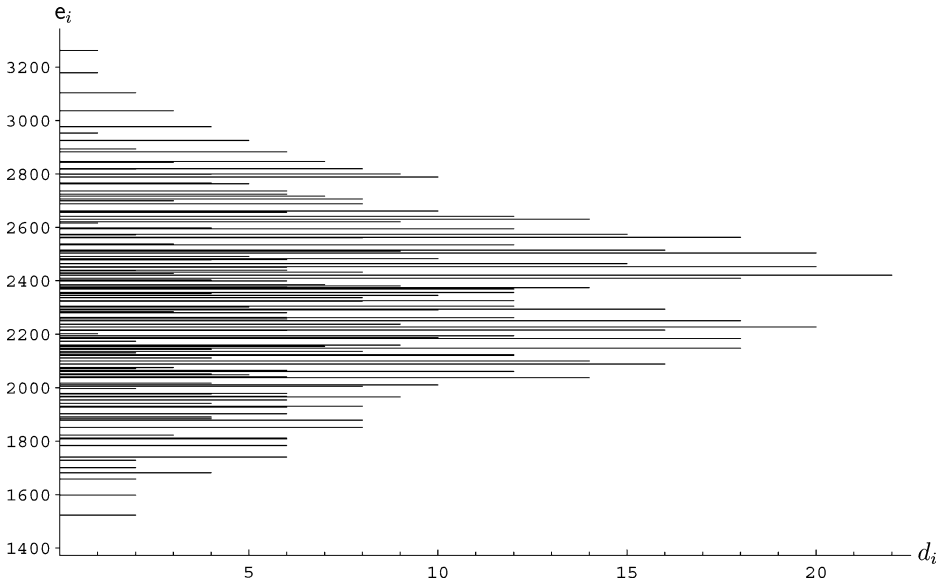


Fig. 1. Energy levels e_i and degeneracies d_i of the antiferromagnetic spin 1/2 chain $h_{-, \pm}$ for $N = 10$ particles and $\tilde{\beta} = \sqrt{2}$.

number becomes 7756. It is therefore convenient to plot the energy levels e_i and their corresponding degeneracies d_i , as is done in Fig. 1 for $N = 10$ particles. Note that Eq. (56) implies that when $\tilde{\beta} \gg N$ the levels cluster around integer multiples of $8\tilde{\beta}$. In fact, for all N up to 20 we have observed that these integers take *all* values in a certain range $j_0, j_0 + 1, \dots, N(N + 1)/2$; for example, in the case $N = 6$ presented above $j_0 = 9$.

Example 3 (spin 1). We have computed the partition functions $Z_{-, \pm}$ of the spin chains $h_{-, \pm}$ with spin $M = 1$ for up to 15 particles. As remarked in the previous section, for integer M the partition functions $Z_{-, \pm}$ are expected to be essentially different. This is immediately apparent from Fig. 2, where we have compared graphically the energy spectra of the even and odd spin chains $Z_{-, \pm}$ with $\tilde{\beta} = \sqrt{2}$ for $N = 10$ particles. However, we shall prove in what follows that the standard deviation of the energy is *exactly* the same for both chains. This rather unexpected result will be relevant in the ensuing discussion of the level density (see Conjecture 2). We also note that, just as for spin 1/2, for N up to (at least) 15 and $\tilde{\beta} \gg N$ the energy levels cluster around an equally spaced set of nonnegative integer multiples of $8\tilde{\beta}$.

The previous examples for spin 1/2 and 1 suggest several conjectures on the spectrum of the (antiferromagnetic) HS spin chains of BC_N type that we shall now present and discuss in detail.

Conjecture 1. For $\tilde{\beta} \gg N$, the energies cluster around an equally spaced set of levels of the form $8j\tilde{\beta}$, with $j = j_0, j_0 + 1, \dots, N(N + 1)/2$.

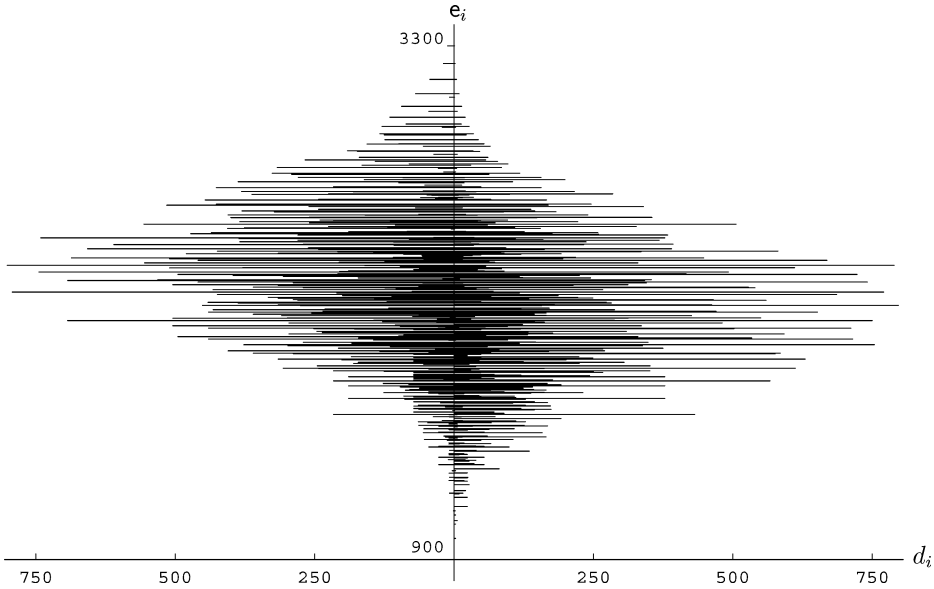


Fig. 2. Comparison of the energy levels e_i and degeneracies d_i of the antiferromagnetic spin 1 chains h_{--} (left) and h_{-+} (right) for $N = 10$ particles and $\beta = \sqrt{2}$.

In fact, for sufficiently large values of the spin M this assertion (with $j_0 = 0$) follows directly from Eq. (56). Our calculations for a wide range of values of N and M fully corroborate the above conjecture.

Conjecture 2. For $N \gg 1$, the level density follows a Gaussian distribution.

More precisely, the number of levels (counting their degeneracies) in an interval I is approximately given by

$$(2M + 1)^N \int_I \mathcal{N}(e; \mu, \sigma) de, \quad (58)$$

where

$$\mathcal{N}(e; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(e-\mu)^2}{2\sigma^2}} \quad (59)$$

is the normal (Gaussian) distribution with parameters μ and σ respectively equal to the mean and standard deviation of the energy spectrum of the spin chain. Although the shape of the plots in Figs. 1 and 2 make this conjecture quite plausible, for its precise verification it is preferable to compare the distribution function

$$F_N(e) = \int_{-\infty}^e \mathcal{N}(t; \mu, \sigma) dt \quad (60)$$

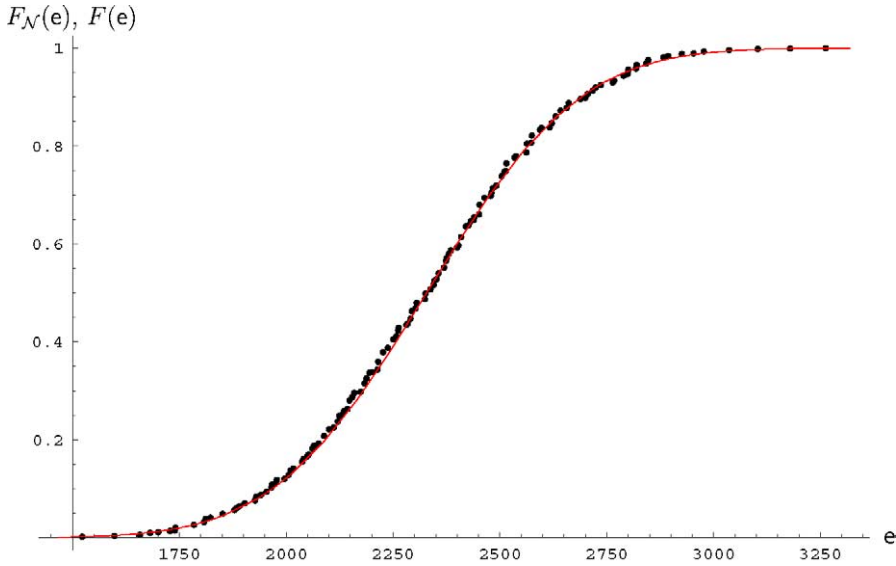


Fig. 3. Distribution functions $F_N(e)$ (continuous line) and $F(e)$ (at its discontinuity points) for $\bar{\beta} = \sqrt{2}$, $M = 1/2$, and $N = 10$.

of the Gaussian probability density with its discrete analogue

$$F(e) = (2M + 1)^{-N} \sum_{i; e_i \leq e} d_i, \quad (61)$$

where d_i denotes the degeneracy of the energy level e_i . Indeed, our computations for a wide range of values of M and $N \gtrsim 10$ are in total agreement with the latter conjecture for all four chains (29). This is apparent, for instance, in the case $\bar{\beta} = \sqrt{2}$, $M = 1/2$, and $N = 10$ presented in Fig. 3. The agreement between the distribution functions (60) and (61) improves dramatically as N increases. In fact, their plots are virtually indistinguishable for $N \gtrsim 15$.

It is well known in this respect that a Gaussian level density is a characteristic feature of the “embedded Gaussian ensemble” (EGOE) in random matrix theory [30]. It should be noted, however, that the EGOE applies to a system of N particles with up to n -body interactions ($n < N$) in the high dilution regime $N \rightarrow \infty$, $\kappa \rightarrow \infty$ and $N/\kappa \rightarrow 0$, where κ is the number of one-particle states. Since in our case $\kappa = 2M + 1$ is fixed, the fact that the level density is Gaussian does not follow from the above general result. A study of the energy spectrum of the spin chains (29) in the framework of random matrix theory is nonetheless worth undertaking, and will be the subject of a subsequent publication.

If Conjecture 2 is true, the level density for large N is completely characterized by the parameters μ and σ through the Gaussian law (59). It is therefore of great interest to compute these parameters in closed form as functions of N and M . To this end, let us write

$$h_{-, \pm} = \sum_{i \neq j} [h_{ij}(1 + S_{ij}) + \tilde{h}_{ij}(1 + \tilde{S}_{ij})] + \sum_i h_i(1 \mp S_i), \quad (62)$$

Table 1
Traces of products of the spin operators

Operator	Trace (integer M)	Trace (half-integer M)
S_i	$(2M+1)^{N-1}$	0
S_{ij}, \tilde{S}_{ij}	$(2M+1)^{N-1}$	$(2M+1)^{N-1}$
$S_i S_j$	$(2M+1)^{N-2+2\delta_{ij}}$	$(2M+1)^N \delta_{ij}$
$S_{ij} S_k, \tilde{S}_{ij} S_k$	$(2M+1)^{N-2}$	0
$S_{ij} \tilde{S}_{kl}$	$(2M+1)^{N-2}$	$(2M+1)^{N-2} (1 - \delta_{ik} \delta_{jl}) (1 - \delta_{il} \delta_{jk})$
$S_{ij} \tilde{S}_{kl}, \tilde{S}_{ij} \tilde{S}_{kl}$	$(2M+1)^{N-2+2\delta_{ik} \delta_{jl} + 2\delta_{il} \delta_{jk}}$	$(2M+1)^{N-2+2\delta_{ik} \delta_{jl} + 2\delta_{il} \delta_{jk}}$

where the constants h_{ij} , \tilde{h}_{ij} and h_i can be easily read off from Eq. (29). We shall begin by computing the average energy $\mu_{--} \equiv \mu_-$ of the odd antiferromagnetic spin chain h_{--} for integer spin. Using the formulas for the traces of the spin operators given in Table 1 we immediately obtain

$$\begin{aligned} \mu_{-} &= (2M+1)^{-N} \text{tr } h_{--} = \frac{2(M+1)}{2M+1} \left[\sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \sum_i h_i \right] \\ &= \frac{2(M+1)}{2M+1} V(\xi) = \frac{4(M+1)}{3(2M+1)} N(N+1)(2N+3\bar{\beta}-2), \quad M \in \mathbb{N}, \end{aligned} \quad (63)$$

where we have used the explicit expression (57) for $V(\xi)$. On the other hand, the average energy $\mu_{-+} \equiv \mu_+$ of the even chain h_{-+} is given by

$$\begin{aligned} \mu_{+} &= (2M+1)^{-N} \text{tr } h_{-+} = \frac{2(M+1)}{2M+1} \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \frac{2M}{2M+1} \sum_i h_i \\ &= \frac{2}{2M+1} [(M+1)V(\xi) - \Sigma_1], \quad M \in \mathbb{N}, \end{aligned} \quad (64)$$

where $\Sigma_1 \equiv \sum_i h_i$ is obviously independent of the spin M . Similarly, for half-integer spin the formulas for the traces of the spin operators in Table 1 yield the following expression for the mean energy $\mu_{-, \pm} \equiv \mu_{\pm}$:

$$\mu_{\pm} = \frac{1}{2M+1} [2(M+1)V(\xi) - \Sigma_1], \quad M = \frac{1}{2}, \frac{3}{2}, \dots \quad (65)$$

Let us turn now to the (squared) standard deviation of the energy, given by

$$\sigma_{-, \pm}^2 \equiv \sigma_{\pm}^2 = \frac{\text{tr}(h_{-, \pm}^2)}{(2M+1)^N} - \frac{(\text{tr } h_{-, \pm})^2}{(2M+1)^{2N}}.$$

For integer spin, a long but straightforward calculation using the formulas in Table 1 yields

$$\sigma_{\pm}^2 = \frac{4M(M+1)}{(2M+1)^2} \left[2 \sum_{i \neq j} (h_{ij}^2 + \tilde{h}_{ij}^2) + \sum_i h_i^2 \right] \equiv \frac{4M(M+1)}{(2M+1)^2} \Sigma_2, \quad M \in \mathbb{N}. \quad (66)$$

Since Σ_2 does not depend on M , the above equation completely determines the dependence of σ_{\pm} on the spin. An important consequence of the previous formula is the equality of

the standard deviation of the energy for the even and odd antiferromagnetic chains (for half-integer spin, this follows trivially from the fact that the even and odd chains have the same spectrum). This result is quite surprising, since for integer spin the energy spectra of the chains $h_{-, \pm}$ are essentially different, cf. Fig. 2. For half-integer spin, an analogous calculation yields the expression

$$\sigma_{\pm}^2 = \frac{4M(M+1)}{(2M+1)^2} \left[\Sigma_2 + \frac{\Sigma_3}{M(M+1)} \right], \quad M = \frac{1}{2}, \frac{3}{2}, \dots, \quad (67)$$

where

$$\Sigma_3 = \frac{1}{4} \sum_i h_i^2 - \sum_{i \neq j} h_{ij} \tilde{h}_{ij} \quad (68)$$

is independent of the spin. As before, Eq. (67) fixes the dependence of σ_{\pm} on the spin.

We still need to evaluate $\Sigma_1(N)$, $\Sigma_2(N)$ and $\Sigma_3(N)$ in order to determine the dependence on N of μ_{\pm} and σ_{\pm} in all cases. Although we have not been able to compute these quantities in closed form, in view of Eq. (63) it is natural to formulate the following conjecture:

Conjecture 3. *The average energy μ_{\pm} and its squared standard deviation σ_{\pm}^2 depend polynomially on N .*

In fact, since $e_{-, \pm; \max} = 2V(\xi)$ is a polynomial of degree 3 in N by Eq. (57), it follows that the degrees in N of μ_{\pm} and σ_{\pm}^2 cannot exceed 3 and 6, respectively. The latter conjecture and this fact allow us to determine the quantities $\Sigma_i(N)$ by evaluating μ_{\pm} and σ_{\pm}^2 for $N = 2, \dots, 8$ and $M = 1/2, 1$ using the exact formula (54) for the partition function (cf. Eqs. (64), (66) and (67)). The final result is

$$\begin{aligned} \Sigma_1 &= \Sigma_3 = 2N(2\bar{\beta} + N - 1), \\ \Sigma_2 &= \frac{4N}{9} \left[2(2N^2 + 3N + 13)\bar{\beta}^2 + (N - 1)(5N^2 + 7N + 20)\bar{\beta} \right. \\ &\quad \left. + \frac{1}{5}(N - 1)(8N^3 + 3N^2 + 13N - 12) \right]. \end{aligned} \quad (69)$$

These expressions, together with Eqs. (63)–(67), completely determine μ_{\pm} and σ_{\pm} for *all* values of M and N . We have checked that the resulting formulas yield the exact values of μ_{\pm} and σ_{\pm} computed from the partition function (54) for a wide range of values of M and N . This provides a very solid confirmation of Conjecture 3. Let us mention, in closing, that formulas analogous to (63)–(67) expressing the mean and standard deviations of the energy for the ferromagnetic chains $h_{+, \pm}$ can be immediately deduced from the previous expressions and Eq. (31).

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Appendix A. Uniqueness of the equilibrium of the classical potential

In this appendix we shall prove that the classical potential (30) has exactly one equilibrium point in the set \tilde{C} . We have already seen in Section 4 that U has at least one minimum in \tilde{C} . We shall now prove that the Hessian of U is positive-definite in \tilde{C} , which implies that all the critical points of U in \tilde{C} must be minima. This implies that U has exactly one critical point (a minimum) in the set \tilde{C} .

If $f(t) = \sin^{-2} t$, we can express the second partial derivatives of U as follows

$$\begin{aligned} \frac{\partial^2 U}{\partial x_i^2} &= 2 \sum_{j \neq i} [f''(x_{ij}^-) + f''(x_{ij}^+)] + \beta^2 f''(x_i) + \beta'^2 f''\left(\frac{\pi}{2} - x_i\right), \\ \frac{\partial^2 U}{\partial x_i \partial x_j} &= 2[f''(x_{ij}^+) - f''(x_{ij}^-)]. \end{aligned} \quad (\text{A.1})$$

Note that $f''(t) = 2 \csc^2 t (1 + 3 \cot^2 t)$ is strictly positive for all values of t , and therefore $(\partial^2 U)/(\partial x_i^2) > 0$ for all i . By Gerschgorin's theorem [33, 15.814], the eigenvalues of the Hessian of U lie in the union of the intervals

$$\left[\frac{\partial^2 U}{\partial x_i^2} - \gamma_i, \frac{\partial^2 U}{\partial x_i^2} + \gamma_i \right], \quad \text{where } \gamma_i = \sum_{j \neq i} \left| \frac{\partial^2 U}{\partial x_i \partial x_j} \right|, \quad i = 1, \dots, N.$$

Since

$$\frac{\partial^2 U}{\partial x_i^2} - \gamma_i \geq \beta^2 f''(x_i) + \beta'^2 f''\left(\frac{\pi}{2} - x_i\right) > 0,$$

all the eigenvalues of the Hessian of U are strictly positive. This establishes our claim.

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