Application of Convex Duality to the risk hedging of financial claims.

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Abstract

We work in an incomplete market with finite states. We obtain all the no arbitrage prices of a financial claim associating them to entropy levels. This is done by means of convex programs with an entropy constraint. We apply Fenchel duality to translate these programs to their duals and we obtain two particular cases. One arises when the dual entropy variable is null and represents the superreplicating case giving as solution the super-replicating portfolio at no risk. The other arises when the dual entropy variable is different from zero and stands for the partial replicating case corresponding to the prices in the interior of the no arbitrage prices interval.

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1. Introduction

1.1. Scope and motivation

The topic of this paper is the determination of no arbitrage prices for a financial claim and how to deduce a general statement for their optimal static hedging positions with respect to a given set of benchmarks (assets), in a finite incomplete market in one time period.

The search for a model independent upper (lower) bound for a derivative price based only on the no arbitrage assumption, together with its *super(sub)-hedging*

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position, is a well established subject in Operational Research (OR) literature. In this line of research Bertsimas and Popescu (2002) and Kahalé (2017) find methods based on convex programming in discrete time to calculate the price/hedging positions in terms of other derivatives, and King et al. (2005) study the hedging problem by means of stochastic programming and conjugate duality. This problem of finding a *super(sub)-hedging position* has also been studied for particular options. Pennanen (2011) finds buyer's and seller's prices for American options applying convex optimization, and d'Aspremont and El Ghaoui (2006) find model independent upper/lower bounds for European basket options. Their main achievement is dodging the model risk by obtaining model free seller's and buyer's prices with their corresponding super and sub-replicating portfolios.

In this draft paper we innovate by moving into the interval of no arbitrage prices grouping them in pairs associated with entropy levels by means of convex optimization. On the one hand, our methodology is model free in the sense that it belongs to the family of inverse methods (see Breeden and Litzenberger (1978) and Jackwerth and Rubinstein (1996)), which are able to work with any underlying stochastic model. On the other hand, it enables the search for hedging positions consisting of what we will call *partial replicating portfolios* that do not exactly replicate the claim payoff at expiry. This is done considering any of the programs supplying a no arbitrage price as a primal, then translating its constituents into their Fenchel conjugates and applying convex duality to obtain a dual feasible set consisting of all those partial replicating portfolios. It happens that this dual set is close to the statement made in Dembo and Rosen (1999). Yet our approach adds value by showing how our representation of the hedging positions is deduced in a natural way from a primal entropy pricing program by means of Fenchel duality.

1.2. Literature review

Given a claim payoff at the end of a time period, the Fundamental Asset Pricing Theorem (Avellaneda and Laurence (1999)) states the equivalence between the *no arbitrage* hypothesis and the existence of an Equivalent Martingale Pricing Measure (EMPM) such that the price is the claim payoff expectation discounted at the risk free rate. In a complete market this pricing probability measure is unique, while in an incomplete one there is an infinite family of EMPM giving rise to an infinity of prices.

It is thus possible to determine an interval of prices fulfilling the no arbitrage hypothesis, in accordance with the prices of some given benchmark assets. The upper and lower extremes of this interval are called the *seller's* and *buyer's prices*, respectively. In a one time period and finite state market, these prices can be calculated by means of two linear programs (see for instance Musiela and Rutkowsky

(2005), p. 95). The solutions to their respective dual linear programs are the *super* and *sub-replicating portfolios* made of those benchmarks (see also Luenberger (2002)).

The interpretation of no arbitrage prices in terms of Expected Utility Theory is an important result of Convex Duality Theory applied to Financial Mathematics. For any material on Convex Analysis, Convex Optimization and Fenchel Duality Theory we refer to Rockafellar (1970) and Mordukhovich and Nam (2013). This is an active field of research as evidenced by the work of Carr and Zhu (2018), Miyahara (2012) and all the references therein. In the former reference, in the case of a one period finite market model, they explain how no arbitrage prices are given by different EMPM pointed out by different concave utility functions of investors solving portfolio optimization problems. This approach is rather theoretical and from a practical point of view has the usual drawback of how to determine one particular investor's utility function.

Entropy Pricing Theory (EPT) (Gulko (1999)) is founded on the idea that the asset price should fulfill the *efficient market* hypothesis to its best. This means that an informational efficient price should keep all investors at their maximum degree of uncertainty about the next price change, so that perfectly uncertain common market beliefs prevail. To our knowledge, EPT lacks an extension clarifying how the risk incurred by the claim issuer could be hedged. The core of EPT is the calculation of the maximum entropy price using a set of benchmarks the prices of which calibrate an EMPM. This is done through the maximization of the entropy as in Bose and Murray (2014) and Neri and Schneider (2012). Alternatively, the *Kullback-Leibler relative entropy* of the EMPM relative to some initial probability can be minimized subject to the benchmark prices, as in Frittelli (2000). When the market states are finite, one particular use of Kullback-Leibler relative entropy minimization is to set the uniform probability as the initial one (Avellaneda et al. (2001)), which is known as weighted Monte Carlo method. More generally, any *divergence* (Amari (2016), chapter 3) measuring the informational discrepancy to uniform probability can be used as an objective to minimize. For instance, this is the field explored in Vilar and Peraita (2018) and Vilar and Peraita (2019), where the *variation distance* of Ali and Silvey (1966) (see also Amari (2016), p.58) is plugged into the objective function and a linear goal programming method is applied to calibrate a risk neutral probability. The use of general entropy measures for calculating risk neutral probabilities is also explored in Sheraz and Preda (2015). In this article we work in a finite state market, and we will use the Shanon's entropy to obtain prices depending on their informational efficiency.

1.3. Contributions

Our paper contributes to the existing literature in several respects. Its innovations are related to the interior of the no arbitrage price interval, containing also as particular instances the cases of super (or infra) replication at the endpoints of that interval.

Our first contribution is showing how no arbitrage prices can be clustered by pairs associated to entropy levels. This is done by means of two primal concave (maximizing) and convex (minimizing) programs incorporating an entropy constraint, generalizing the two primal linear programs in Musiela and Rutkowsky (2005) p.94, and also the methodology inspired by EPT. Prices greater (resp. lower) than the most efficient price are solutions to a maximization (resp. minimization) program and we name them as *seller's prices* (resp. *buyer's prices*) *at some entropy level*. We develop our methodology adopting the seller's point of view as this is the role played by an insurance company aiming to sell policies.

Our second contribution is the translation of these primal programs to their duals using Fenchel conjugates. By strong duality, each dual program shares the optimal value with its primal -the no arbitrage price corresponding to an entropy level. The multiplier associated to the entropy constraint discriminates between two main cases. When it is null, the dual becomes the linear program (see Musiela and Rutkowsky (2005)) giving as solution the super-replicating risk free hedge at the seller's price. When the multiplier is different from zero, the dual program has optimal value equal to the no arbitrage price at the chosen entropy level, with that multiplier measuring the price sensitivity to an entropy variation. This last case is not risk free thus we must calculate an optimal solution inside the dual feasible set to get a hedging position. As this set contains all the partial replicating portfolios made of benchmarks, underlying asset and bond, no matter their price, in a first instance we must find our path to the solution by cutting this feasible set by an hyper-plane containing all the partial replicating portfolios at some no arbitrage price (to be set by the decision maker). Then we get a feasible subset consisting of all the hedging positions at that price, plus two deviation vectors: one is non negative representing a remaining claim not replicated by the portfolio at expiry, while the other is a non positive excess vector representing the gains at expiry over the market states. The formulation of this feasible subset is quite natural, and its merit is being deduced applying convex duality to our primal entropy pricing program, bridging the gap between the pricing and hedging steps. Furthermore, through this process we learn how the positions (both in value and in weights) in underlying asset and bond are deduced by duality from the so-called martingale and probability constraints respectively, while the so-called remaining claim (issuer losses) and excess (issuer gains) vectors come from the entropy and non negativity constraints respectively.

1.4. Organization

The rest of the paper is organized as follows. In section 2 we set the notation used throughout the paper. In section 3 we derive our primal pricing programs and discuss their properties. Technical details about the calculations of the Fenchel dual programs (Fenchel conjugates, support and perspective functions) are reported in section4. In section 5 we apply the Fenchel duality to obtain the dual program corresponding to the hedging step, and discuss super and partial replications

2. Notation

We denote random variables and processes with capital letters, their values and realizations are noted with lowercase letters. Random variable samples are vectors written as lowercase bold letters whose components (realizations) are indexed by the time t and the trajectory j. A bold number stands for a vector with the same value in all its coordinates. We account for losses with a positive sign.

Our methodology works in one time period $[0,T]$. Though given that we will exemplify it by means of cliquet guarantees using forward start options as benchmarks, we shall work with yearly periods $t = 1,\ldots,T$ to be able to sample their payoffs.

We consider the continuous stochastic process $S = \{S_t\}_{t\in[0,T]}$ modeling an underlying portfolio over several yearly periods whose initial value is s_0. The yearly values of the underlying portfolio will be needed to sample the claim and benchmarks payoffs, and they are given by the random variables $S_1,S_2,\ldots,S_T$. Using the natural probability $\mathcal{P}$ of the stochastic model we simulate m trajectories of the process S and obtain sample vectors $\boldsymbol{s}_t = (s_t^j)_{j=1}^m$ with $t = 1,2,\ldots T$ years. The financial claim to be priced and hedged has random payoff X at expiry T which depends on the yearly evolution of the underlying S. By the yearly samples $\boldsymbol{s}_t$ we calculate the payoff sample $\boldsymbol{x} = \left(x^j\right)_{j=1}^m \in \mathbb{R}_+^m$ (we omit the time index as the payoff is paid at the expiry T). Looking at the time interval $[0,T]$ as one time period, we thus reduce the continuous setting to a discrete one (both in time and states), Ω_m being the set of the scenarios (market states) at the expiry T and $P(\Omega_m)$ the $\sigma - algebra$ power set of Ω_m. The probability measure Q^m is *an EMPM* on $(\Omega_m, P(\Omega_m))$ and its mass function is noted by $\boldsymbol{q} = \left(q^j\right)_{j=1}^m \in \mathcal{S}^m$, this last symbol standing for the *m-simplex*.

We will work with n benchmarks with random payoffs $(G_i)_{i=1}^n$ (with maturities $t = 1,\ldots,T$ in our application), paid at the expiry T depending on the evolution of the process S. They are also sampled by n payoff vectors $\boldsymbol{g}_i = (g_i^j)_{j=1}^m \in \mathbb{R}_+^m$ with $i = 1,\ldots,n$. The benchmarks prices are noted with vector $\boldsymbol{c} = (c_i) \in \mathbb{R}_+^n$, and a benchmarks portfolio with $\boldsymbol{\theta} = (\theta_i) \in \mathbb{R}^n$. Short and long positions are noted with negative and positive coordinates, respectively. A portfolio payoff at maturity T is given by $G=\sum_{i=1}^{n}\theta_i G_i$, thus the sampled portfolio payoff is $\boldsymbol{g}=\sum_{i=1}^{n}\theta_i\boldsymbol{g}_i$ with price $\boldsymbol{\theta}'.\boldsymbol{c}=\sum_{i=1}^{n}\theta_i c_i$.

We suppose a constant yearly risk free interest rate R, thus $r=\log(1+R)$ is the instantaneous forward rate and the discounting factor is e^{-rT}. The risk free zero coupon bond that costs $b_0=1$ at $t=0$ has a sampled payoff $\left(e^{rT},...,e^{rT}\right)'\in\mathbb{R}_+^m$ at expiry T. The notation $\boldsymbol{x}\succeq\mathbf{0}$ means that the inequality $\geq$ is satisfied by all the vector components.

Finally, the *Shanon's entropy* will be noted by:

$$\mathrm{H}(\boldsymbol{q})=-\sum_{j=1}^{m}q^j\log\left(q^j\right). \tag{2.1}$$

We consider H continuously prolonged by setting $0\log(0)=0$. Considering the space $(\Omega_m,P(\Omega_m))$, our aim is to calculate pricing measures Q^m calibrated to some benchmarks, satisfying some entropy constraints, to recover all the no arbitrage prices of the claim $\boldsymbol{x}$ at $t=0$ while finding their corresponding optimal hedging positions at $t=T$. We also investigate the behaviors of our solutions when the number of scenarios grows.

3. No arbitrage prices and entropy

In this section we firstly calibrate an EMPM to a chosen benchmark set. Secondly, we price the claim $\boldsymbol{x}$ by two primal programs. Their two solutions are related by the same entropy level H.

Inspired by the programs we find in Avellaneda et al. (2001) and Frittelli (2000), we calculate the maximal entropy H^* that calibrates an EMPM to the benchmark prices $\boldsymbol{c}$ by means of:

$$\begin{cases}\max\limits_{\boldsymbol{q}\in\mathbb{R}_+^m}\mathrm{H}(\boldsymbol{q})\\ \text{s.t.}:\ e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{g}_i-c_i=0,\ i=1,2,\ldots,n\ \text{(Benchmark price constraints)}\\ \qquad e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{s}_T-\boldsymbol{s}_0=0\ \text{(Martingale constraint)}\\ \qquad \boldsymbol{q}'\cdot\mathbf{1}=1\ \text{(Probability constraint)}.\end{cases} \tag{3.1}$$

The feasible set in (3.1) is defined by the intersection of hyper-planes and the m-simplex, thus it is a convex compact subset of $\mathbb{R}^m$. As its objective is a continuous strictly convex function, it reaches its global maximal value $H^*=H(\boldsymbol{q}^*)$ at a unique optimum $\boldsymbol{q}^*$ which is the mass function of a pricing measure Q^{m*} which correctly prices the benchmarks (benchmark price constraints) and the underlying (martingale constraint). The claim $\boldsymbol{x}$ is then priced using the Fundamental Asset Pricing Theorem:

$$\pi^* = e^{-rT}\mathbb{E}^{Q^{m*}}[\boldsymbol{x}] = e^{-rT}\boldsymbol{q}^{*\prime}\cdot\boldsymbol{x}. \tag{3.2}$$

This is the *maximum efficient price* belonging to the interval of no arbitrage prices, as calculated by EPT.

We now build up our two primal programs. Firstly, we set the pricing formula as the objective function:

$$\pi(\boldsymbol{q}) = e^{-rT}\mathbb{E}^{Q^{m}}[\boldsymbol{x}] = e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{x}. \tag{3.3}$$

Secondly, we define the feasible set. We need to replace each equality constraint of (3.1) by an equivalent couple of inequality constraints. We do it so to be able to build the dual program in the next section. Therefore, we substitute the benchmark price constraints by two inequalities:

$$\begin{aligned} f_k(\boldsymbol{q}) &= e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{g}_k - c_k \le 0, \qquad k = 1,2,\ldots,n \\ f_k(\boldsymbol{q}) &= c_{k-n} - e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{g}_{k-n} \le 0, \quad k = n+1,\ldots,2n. \end{aligned} \tag{3.4}$$

A benchmark price c_k, c_{k-n} in (3.4) is *mark-to-market* if it is quoted by the market in which case the decision maker will choose among bid/ask prices. Or it is *mark-to-model* when the benchmark is not traded and it is calculated by means of a model calibrated on market data. The martingale and the probability constraints are also rewritten as two inequalities:

$$f_{2n+1}(\boldsymbol{q}) = e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{s}_T - \boldsymbol{s}_0 \le 0, \;\; f_{2n+2}(\boldsymbol{q}) = \boldsymbol{s}_0 - e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{s}_T \le 0. \tag{3.5}$$

$$f_{2n+3}(\boldsymbol{q}) = \boldsymbol{q}'\cdot\mathbf{1} - 1 \le 0, \;\; f_{2n+4}(\boldsymbol{q}) = 1 - \boldsymbol{q}'\cdot\mathbf{1} \le 0. \tag{3.6}$$

Let us note $\mathfrak{Q}$ for the set of mass distributions satisfying the constraints $f_1,\ldots,f_{2n+4}$. We can now express (3.1) as $\boldsymbol{q}^* = \arg\max_{\mathfrak{Q}} \mathrm{H}(\boldsymbol{q})$.

We include an inequality constraining the feasible solutions to have an entropy level $\mathrm{H}(\boldsymbol{q})$ greater than some fixed $H \le H^*$,

$$f_{2n+5}(\boldsymbol{q}) = -\mathrm{H}(\boldsymbol{q}) + H \le 0, \tag{3.7}$$

and we get our primal feasible set $\mathfrak{Q}_H = \{\boldsymbol{q}\in\mathfrak{Q} : f_{2n+5}(\boldsymbol{q}) \le 0\} \subset \mathbb{R}^m$. $\mathfrak{Q}_H$ is convex ((3.4), (3.5) and (3.6) define closed half-spaces of $\mathbb{R}^m$, and f_{2n+5} is a strictly convex function), and compact (it is defined through the intersection with the m-simplex). Therefore, as the objective function (3.3) is linear, the two programs

$$\min_{\boldsymbol{q}\in\mathfrak{Q}_H}\pi(\boldsymbol{q}) \tag{3.8}$$

$$\max_{\boldsymbol{q}\in\mathfrak{Q}_H}\pi(\boldsymbol{q}) \tag{3.9}$$

reach their optimums $\boldsymbol{q}_H^{min} = \arg\min_{\boldsymbol{q}\in\mathfrak{Q}_H}$, $\boldsymbol{q}_H^{max} = \arg\max_{\boldsymbol{q}\in\mathfrak{Q}_H}$, with global minimal and maximal values satisfying:

$$\pi_H^{min} \leq \pi^* \leq \pi_H^{max},\ \forall H \leq H^*. \tag{3.10}$$

We name the prices π_H^{min}, π_H^{max} as *buyer's price* and *seller's price at entropy level* $H \leq H^*$. In the particular case $H = H^*$, we get $\mathfrak{Q}_H = \{\boldsymbol{q}^*\}$ and the equality in (3.10) is reached:

$$\pi_{H^*}^{min} = \pi^* = \pi_{H^*}^{max}. \tag{3.11}$$

Therefore, while programs (3.8) and (3.9) satisfy the *Slater condition* for any $H < H^*$, where strong duality applies, this is not true for $H = H^*$.

4. Fenchel conjugates, support and perspective functions

In this appendix we write the Fenchel conjugates and the perspective functions necessary to build up the dual programs of (3.8) and (3.9). The Fenchel conjugate of a convex function f with domain $\mathrm{dom} f = \{\boldsymbol{q} \in \mathbb{R}^m : f(\boldsymbol{q}) < \infty\}$, is defined as:

$$\forall \boldsymbol{y} \in \mathbb{R}^m : f^*(\boldsymbol{y}) = \sup_{\boldsymbol{q}\in\mathrm{dom} f} \left\{\boldsymbol{y}' \cdot \boldsymbol{q} - f(\boldsymbol{q})\right\}. \tag{4.1}$$

Consider the set $F = \{\boldsymbol{q} \in \mathbb{R}^m : f(\boldsymbol{q}) \leq 0\}$. The *indicator function* of F is:

$$\delta_F(\boldsymbol{q}) = \begin{cases} 0, & \boldsymbol{q} \in F \\ \infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

The Fenchel conjugate of δ_F is the *support function* δ_F^*. It is linked to the Fenchel conjugate f^* (see Roos et al. (2020) pp. 3, 27), provided that F and its relative interior are non empty sets, by the identity:

$$\forall \boldsymbol{y} \in \mathbb{R}^m : \delta_F^*(\boldsymbol{y}) = \sup_{\boldsymbol{q}\in F} \left\{\boldsymbol{y}' \cdot \boldsymbol{q}\right\} = \min_{u\geq 0} \left\{(uf)^*(\boldsymbol{y})\right\}. \tag{4.3}$$

The perspective function associated with F is:

$$(uf)^*(\boldsymbol{y}) = \sup_{\boldsymbol{q}\in F} \left\{\boldsymbol{y}' \cdot \boldsymbol{q} - uf(\boldsymbol{q})\right\} = uf^*\left(\frac{\boldsymbol{y}}{u}\right). \tag{4.4}$$

When working with inequality constraints, perspective functions are key elements to the calculation of support functions, which allow expression of the Fenchel dual of the primal program in a tractable way.

Objective function (3.3): It is the same for both primal programs (3.8) and (3.9). For any $\boldsymbol{y}_0, \boldsymbol{q} \in \mathbb{R}^m$ its Fenchel conjugate π^* is:

$$\pi^*(\boldsymbol{y}_0) = \sup_{\boldsymbol{q}} \left\{ \boldsymbol{y}_0{}' \cdot \boldsymbol{q} - \pi\big(\boldsymbol{q}\big) \right\} = \begin{cases} 0, & \boldsymbol{y}_0 = e^{-rT} \boldsymbol{x} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.5}$$

The Fenchel conjugate $(-\pi_0)^*$ of the opposite function is:

$$(-\pi)^*(\boldsymbol{y}_0) = \sup_{\boldsymbol{q}} \left\{ \boldsymbol{y}_0{}' \cdot \boldsymbol{q} + \pi\big(\boldsymbol{q}\big) \right\} = \begin{cases} 0, & \boldsymbol{y}_0 = -e^{-rT} \boldsymbol{x} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.6}$$

Benchmark price constraints (3.4): As $F_k = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) = e^{-rT} \boldsymbol{q}' \cdot \boldsymbol{g}_k - c_k \leq 0 \right\}$ for $k = 1, 2, \ldots, n$ and applying (4.1) and (4.4), the Fenchel conjugate f_k^* and the perspective function $(u_k f_k)^*$ are:

$$\forall \boldsymbol{y}_k \in \mathbb{R}^m: \;\; f_k^*(\boldsymbol{y}_k) = \begin{cases} c_k, & \boldsymbol{y}_k = e^{-rT} \boldsymbol{g}_k \\ \infty, & \text{otherwise.} \end{cases} \tag{4.7}$$

$$\forall u_k \geq 0: \;\; (u_k f_k)^*(\boldsymbol{y}_k) = u_k f_k^* \left(\frac{\boldsymbol{y}_k}{u_k} \right) = \begin{cases} u_k c_k, & \boldsymbol{y}_k = e^{-rT} u_k \boldsymbol{g}_k \\ \infty, & \text{otherwise.} \end{cases} \tag{4.8}$$

The reversed inequalities correspond to $F_k = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) = c_{k-n} - e^{-rT} \boldsymbol{q}' \cdot \boldsymbol{g}_{k-n} \leq 0 \right\}$ for $k = n+1, \ldots, 2n$, thus:

$$\forall \boldsymbol{y}_k \in \mathbb{R}^m: \;\; f_k^*(\boldsymbol{y}_k) = \begin{cases} -c_k, & \boldsymbol{y}_k = -e^{-rT} \boldsymbol{g}_k \\ \infty, & \text{otherwise} \end{cases} \tag{4.9}$$

$$\forall u_k \geq 0: \;\; (u_k f_k)^*(\boldsymbol{y}_k) = u_k f_k^* \left(\frac{\boldsymbol{y}_k}{u_k} \right) = \begin{cases} -u_k c_k, & \boldsymbol{y}_k = -e^{-rT} u_k \boldsymbol{g}_k \\ \infty, & \text{otherwise.} \end{cases} \tag{4.10}$$

Martingale constraints (3.5): The two inequalities define the subsets:

$$F_{2n+1} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+1}(\boldsymbol{q}) = e^{-rT} \boldsymbol{s}_T \cdot \boldsymbol{q} - s_0 \leq 0 \right\} \tag{4.11a}$$

$$F_{2n+2} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+2}(\boldsymbol{q}) = s_0 - e^{-rT} \boldsymbol{s}_T \cdot \boldsymbol{q} \leq 0 \right\}. \tag{4.11b}$$

Applying (4.1) and (4.4) we obtain:

$$(u_{2n+1}f_{2n+1})^*(\boldsymbol{y}_{2n+1}) = \begin{cases} s_0 u_{2n+1}\,, & \boldsymbol{y}_{2n+1} = u_{2n+1}e^{-rT}\boldsymbol{s}_T \\ \infty\,, & \text{otherwise} \end{cases} \tag{4.12a}$$

$$(u_{2n+2}f_{2n+2})^*(\boldsymbol{y}_{2n+2}) = \begin{cases} -s_0 u_{2n+2}\,, & \boldsymbol{y}_{2n+2} = -u_{2n+2}e^{-rT}\boldsymbol{s}_T \\ \infty\,, & \text{otherwise.} \end{cases} \tag{4.12b}$$

for any $\boldsymbol{y}_{2n+1}, \boldsymbol{y}_{2n+2} \in \mathbb{R}^m$ and for any $u_{2n+1}, u_{2n+2} \geq 0$.
Probability constraints (3.6): We have two sets:

$$F_{2n+3} = \left\{\boldsymbol{q} \in \mathbb{R}^m : f_{2n+3}(\boldsymbol{q}) = \boldsymbol{q}' \cdot \mathbf{1} - 1 \leq 0\right\} \tag{4.13a}$$

$$F_{2n+4} = \left\{\boldsymbol{q} \in \mathbb{R}^m : f_{2n+4}(\boldsymbol{q}) = 1 - \boldsymbol{q}' \cdot \mathbf{1} \leq 0\right\}. \tag{4.13b}$$

The perspective functions are:

$$\forall u_{2n+3} \geq 0 : (u_{2n+3}f_{2n+3})^*(\boldsymbol{y}_{2n+3}) = \begin{cases} u_{2n+3}\,, & \boldsymbol{y}_{2n+3} = u_{2n+3}\mathbf{1} \\ \infty\,, & \text{otherwise} \end{cases} \tag{4.14a}$$

$$\forall u_{2n+4} \geq 0 : (u_{2n+4}f_{2n+4})^*(\boldsymbol{y}_{2n+4}) = \begin{cases} -u_{2n+4}\,, & \boldsymbol{y}_{2n+4} = u_{2n+4}(-\mathbf{1}) \\ \infty\,, & \text{otherwise.} \end{cases} \tag{4.14b}$$

Entropy constraint (3.7): This defines the set:

$$F_{2n+5} = \left\{\boldsymbol{q} \in \mathbb{R}^m_+ : f_{2n+5}(\boldsymbol{q}) = \sum_{j=1}^{m} q^j \log\left(q^j\right) + H \leq 0\right\}.$$

For any $\boldsymbol{y}_{2n+5}, \boldsymbol{q} \in \mathbb{R}^m_+$ we calculate the Fenchel conjugate of f_{2n+5}:

$$\begin{aligned} f^*_{2n+5}(\boldsymbol{y}_{2n+5}) &= \sup_{\boldsymbol{q}} \left\{\boldsymbol{y}'_{2n+5} \cdot \boldsymbol{q} - f_{2n+5}(\boldsymbol{q})\right\} \\ &= \sup_{\boldsymbol{q}} \left\{\sum_{j=1}^{m} y^j_{2n+5} q^j - \sum_{j=1}^{m} q^j \log\left(q^j\right) - H\right\} \\ &= \sum_{j=1}^{m} e^{y^j_{2n+5} - 1} - H. \end{aligned} \tag{4.15}$$

For any $\boldsymbol{y}_{2n+5} \in \mathbb{R}^m_+$ the perspective function is (see 4.4):

$$u_{2n+5} > 0 : (u_{2n+5} f_{2n+5})^* (\boldsymbol{y}_{2n+5}) = u_{2n+5} \left(\sum_{j=1}^{m} e^{\frac{y^j_{2n+5}}{u_{2n+5}} - 1} - H \right) \tag{4.16a}$$

$$u_{2n+5} = 0 : (u_{2n+5} f_{2n+5})^* (\boldsymbol{y}_{2n+5}) = \begin{cases} 0, & \boldsymbol{y}_{2n+5} = \boldsymbol{0} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.16b}$$

Non negativity constraints: They define the subset $F_{2n+6} = \{\boldsymbol{q} \succeq \boldsymbol{0}\} = \mathbb{R}^m_+$. Its support function is (Roos et al. (2020), p. 8):

$$\delta^*_{F_{2n+6}} (\boldsymbol{y}_{2n+6}) = \begin{cases} 0, & \boldsymbol{y}_{2n+6} \in \mathbb{R}^m_- \\ \infty, & \text{otherwise.} \end{cases} \tag{4.17}$$

5. Partial replication

In this section we find the partial replicating portfolios associated with no arbitrage prices π_H^{max} by translating programs (3.9) to their dual programs applying Fenchel duality. We will only consider the seller's primal program (3.9). The buyer's case (3.8) can be treated in a similar way.

5.1. Fenchel dual programs

We follow the general procedure explained in Roos et al. (2020). The feasible set F of program (3.9) is:

$$F = \bigcap_{k=1}^{2n+6} F_k, \tag{5.1}$$

where

$$F_k = \{\boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) \leq 0\}, \quad k = 1, \ldots, 2n+5, \tag{5.2}$$

$$F_{2n+6} = \{\boldsymbol{q} \succeq \boldsymbol{0}\} = \mathbb{R}^m_+. \tag{5.3}$$

Program (3.9) can be rewritten as:

$$\text{Sup}\{\pi(\boldsymbol{q}) : \boldsymbol{q} \in F\} = -\text{Inf}\{-\pi(\boldsymbol{q}) : \boldsymbol{q} \in F\}. \tag{5.4}$$

Let δ_F be the *indicator function* of the feasible set F (see Mordukhovich and Nam (2013), p. 34):

$$\forall \boldsymbol{q} \in \mathbb{R}^m : \ \delta_F(\boldsymbol{q}) = \sum_{k=1}^{2n+6} \delta_{F_k}(\boldsymbol{q}) = \begin{cases} 0, & \boldsymbol{q} \in F \\ \infty, & \text{otherwise.} \end{cases} \tag{5.5}$$

Then calling $g(\boldsymbol{q}) = -\pi(\boldsymbol{q}) + \delta_F(\boldsymbol{q})$, we can write (5.4) as an unconstrained program:

$$-\operatorname{Inf}\{-\pi(\boldsymbol{q}) : \boldsymbol{q} \in F\} = -\operatorname{Inf}_{\boldsymbol{q}}\{g(\boldsymbol{q})\}. \tag{5.6}$$

The Fenchel conjugate g^* of a function g with domain $\operatorname{dom} g = \{\boldsymbol{q} \in \mathbb{R}^m : g(\boldsymbol{q}) < \infty\}$ is given by (see Mordukhovich and Nam (2013), p. 77):

$$g^*(\boldsymbol{y}) = \sup_{\boldsymbol{q} \in \operatorname{dom} g}\left\{\boldsymbol{y}' \cdot \boldsymbol{q} - g(\boldsymbol{q})\right\} \quad \forall \boldsymbol{y} \in \mathbb{R}^m,$$

thus we obtain program (5.6) by substituting $\boldsymbol{y} = \boldsymbol{0}$:

$$g^*(\boldsymbol{0}) = \sup_{\boldsymbol{q} \in \operatorname{dom} g}\{-g(\boldsymbol{q})\} = -\inf_{\boldsymbol{q} \in \operatorname{dom} g}\{g(\boldsymbol{q})\}. \tag{5.7}$$

Now, there remains to calculate $g^*(\boldsymbol{0})$. As g is the sum of two functions, we can write its Fenchel conjugate (see Roos et al. (2020), p. 29 Table 2) as:

$$g^*(\boldsymbol{y}) = \min_{\boldsymbol{y}_0, \boldsymbol{y}_1}\left\{(-\pi)^*(\boldsymbol{y}_0) + \delta_F^*(\boldsymbol{y}_1) : \boldsymbol{y}_0 + \boldsymbol{y}_1 = \boldsymbol{y} \in \mathbb{R}^m\right\} \ \forall \boldsymbol{y}_0, \boldsymbol{y}_1 \in \mathbb{R}^m, \tag{5.8}$$

thus:

$$g^*(\boldsymbol{0}) = \min_{\boldsymbol{y}_0}\left\{(-\pi)^*(\boldsymbol{y}_0) + \delta_F^*(-\boldsymbol{y}_0)\right\}. \tag{5.9}$$

Applying the same argument as in (5.9), the Fenchel conjugate of the indicator function δ_F in (5.5) is:

$$\delta_F^*(-\boldsymbol{y}_0) = \min_{(\boldsymbol{y}_k)_{k=1}^{2n+6} \in \mathbb{R}^m}\left\{\sum_{k=1}^{2n+6} \delta_{F_k}^*(\boldsymbol{y}_k) : \sum_{k=1}^{2n+6} \boldsymbol{y}_k = -\boldsymbol{y}_0\right\}. \tag{5.10}$$

As each conjugate $\delta_{F_k}^*$ $(k = 1, 2, \ldots, 2n+5\ )$ is equal to its perspective function $(u_k f_k)^*$ (see (4.3)) and $\delta_{F_{2n+6}}^*$ is equal to (4.17), we write (5.10) as:

$$\delta_F^*(-\boldsymbol{y}_0) = \min_{(\boldsymbol{y}_k)_{k=1}^{2n+6}, \boldsymbol{u} \succeq \boldsymbol{0}}\left\{\sum_{k=1}^{2n+5} (u_k f_k)^*(\boldsymbol{y}_k)\ : \sum_{k=0}^{2n+6} \boldsymbol{y}_k = \boldsymbol{0},\, \boldsymbol{y}_{2n+6} \preceq \boldsymbol{0}\right\}. \tag{5.11}$$

Thus $g^*(\boldsymbol{0})$ in (5.9) can be rewritten as:

$$\min_{(\boldsymbol{y}_k)_{k=0}^{2n+6}, \boldsymbol{u} \succeq \boldsymbol{0}}\left\{(-\pi)^*(\boldsymbol{y}_0) + \sum_{k=1}^{2n+5} (u_k f_k)^*(\boldsymbol{y}_k) : \sum_{k=0}^{2n+6} \boldsymbol{y}_k = \boldsymbol{0},\, \boldsymbol{y}_{2n+6} \preceq \boldsymbol{0}\right\}, \tag{5.12}$$

where $(u_k)_{k=1}^{2n+5}$ and $(\boldsymbol{y}_k)_{k=0}^{2n+6}$ are dual variables and vectors, respectively. Expression (5.12) is the dual program of (3.9).

Looking at the perspective function for $k = 2n+5$ (see (4.16a) and (4.16b)), we must distinguish between two cases consisting of what happens at the interior of the no arbitrage prices interval or at its upper endpoint (i.e. the seller's price).
When $u_{2n+5} > 0$, we substitute $(u_{2n+5} f_{2n+5})^*$ by (4.16a), $(-\pi)^*$ by (4.6), the perspective functions $(u_k f_k)^*$ and dual vectors $\boldsymbol{y}_k$ by (4.8), (4.10), (4.12) and (4.14), respectively. We obtain the first particular case of (5.12), $\forall u_{2n+5} > 0$:

$$\begin{cases} \min\limits_{\boldsymbol{y}_{2n+5},\boldsymbol{y}_{2n+6},\boldsymbol{u}} \sum_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) + \\ \qquad\qquad + u_{2n+5} \left(\sum_{j=1}^{m} e^{\frac{y^j_{2n+5}}{u_{2n+5}} - 1} - H \right) \\ \text{s.t.}: \quad \sum_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT} (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad\qquad + e^{rT} (\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6}) = \boldsymbol{x} \\ \qquad \boldsymbol{y}_{2n+5} \succeq \mathbf{0},\ \boldsymbol{y}_{2n+6} \preceq \mathbf{0},\ \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{5.13}$$

When $u_{2n+5} = 0$ we make the same substitutions except for $(u_{2n+5} f_{2n+5})^*$, now replaced by (4.16b). This the second case of (5.12),$u_{2n+5} = 0$:

$$\begin{cases} \min\limits_{\boldsymbol{y}_{2n+6},\boldsymbol{u}} \sum_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) \\ \text{s.t.}: \quad \sum_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT} (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad\qquad + e^{rT} \boldsymbol{y}_{2n+6} = \boldsymbol{x} \\ \qquad \boldsymbol{y}_{2n+6} \preceq \mathbf{0},\ \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{5.14}$$

The linear program (5.14) outputs the super-hedging portfolio with price equal to the upper bound of the no arbitrage price interval. This portfolio is made of benchmarks, underlying asset and bonds. The excess vector $e^{rT} \boldsymbol{y}_{2n+6}$ in (5.14) depicts the issuer's gains over the scenarios at expiry T.
Referring now to program (5.13), we know by strong duality that its optimal value is equal to π_H^{max}, the optimal value of the primal (3.9). Furthermore, the sensitivity of π_H^{max} with respect to the entropy level H is given by the optimal value of the dual variable $u_{2n+5} > 0$.
Let us focus now on the dual feasible set in (5.13). The equality constraint is valued at T and makes the sum of three vectors equal to the claim payoff $\boldsymbol{x}$. These vectors

are:
• The payoff of a portfolio $\boldsymbol{u} = \{u_1, u_2, \dots, u_{2n}, u_{2n+1}, \dots, u_{2n+4}\} \in \mathbb{R}_+^{2n+4}$, made of benchmarks, underlying asset and bonds, where the dual variables stand for short/long positions. The benchmark portfolio is $\boldsymbol{\theta}$ where $\theta_i = u_i - u_{n+i}$ is the net short/long position on asset $i = 1, 2, \dots, n$. The quantities $u_S = u_{2n+1} - u_{2n+2}$ and $u_b = u_{2n+3} - u_{2n+4}$ are the net positions on underlying asset and bonds, respectively. Observe that the bond value and weights in the dual come from the probability constraint in the primal (3.9), and the underlying asset value and weights comes from the martingale constraint.
• The vector $e^{rT}\boldsymbol{y}_{2n+5} \succeq \mathbf{0}$, coming from the entropy constraint in the primal program, quantifying the part of the claim $\boldsymbol{x}$ not replicated by the portfolio $\boldsymbol{u}$. We call it the *remaining claim* describing the losses over the scenarios.
• The vector $e^{rT}\boldsymbol{y}_{2n+6} \preceq \mathbf{0}$ depicting the gains over the scenarios that we call *excess vector*. It comes from the non-negativity constraints in the primal program.
•The vector $e^{rT}(\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6})$ quantifies the mismatch over the scenarios between the claim and the partial replicating portfolio payoffs. Observe that both $\boldsymbol{y}_{2n+5}$ and $\boldsymbol{y}_{2n+6}$ are calculated by programs (5.13) and (5.14) at $t = 0$.
In summary, the feasible set in (5.13) consists of all the portfolios partial replicating the claim $\boldsymbol{x}$, at *any* no arbitrage price. We also conclude that the primal pricing program (3.9) contains all the elements to build, by means of convex duality, the feasible set of partial replicating portfolios and a pricing formula.

To calculate an optimal portfolio, it is natural to ask for its price to be equal to the claim price. This is done introducing one additional constraint fixing the price, for each $H \leq H^*$:

$$\begin{cases} \sum_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) = \pi_H^{max} \\ \sum_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT}\,(u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad\qquad + e^{rT}(\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6}) = \boldsymbol{x} \\ \boldsymbol{y}_{2n+5} \succeq \mathbf{0},\ \boldsymbol{y}_{2n+6} \preceq \mathbf{0},\ \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{5.15}$$

Now, for each entropy level we can optimize a risk measure on the feasible set (5.15).

6. Conclusions

In this draft we show how it is possible, to obtain each no arbitrage price of a financial claim by means of an entropy constrained pricing program. Considering this as a primal we calculate its Fenchel conjugates and perspective functions to deduce its dual program. Slicing the dual feasible set by the no arbitrage prices we obtain all the partial replicating portfolios associated to each no arbitrage price. This way, the problem of hedging the remaining risk is reduced to the optimization of some risk measure on one of those *slices* of the dual feasible set.

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