The zeta-function of a quasi-ordinary singularity II

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ABSTRACT. We prove that the zeta-function of a hypersurface quasi-ordinary singularity f equals the zeta-function of a plane curve singularity g. If we reorder the coordinates, then g has the form $g = f(x_1, 0, \dots, 0, x_{d+1})$. Moreover, the topological type of g can also be recovered from the set of distinguished tuples of f.

1. Introduction

The main goal of this note is the computation of the zeta-function of a hypersurface quasi-ordinary singularity $f : (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$. The paper generalizes [**McN01**], where the result is proved for an irreducible germ f. The zeta-function formula was conjectured in [**BMcN00**].

The quasi-ordinary assumption means, that in some local coordinates the projection $pr : (F, 0) := (\{f = 0\}, 0) \to (\mathbf{C}^d, 0)$, induced by $(x, x_{d+1}) \mapsto x$ ($x = (x_1, \dots, x_d) \in \mathbf{C}^d$), is finite and its (reduced) discriminant is included in $(\{x_1 \cdots x_d = 0\}, 0)$.

The zeta-function of a hypersurface germ $f : (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ is defined as follows. Fix a sufficiently small closed ball B_r in $\mathbf{C}^{d+1}$ of radius r, and consider the Milnor fiber $F_\epsilon := f^{-1}(\epsilon) \cap B_r$ ($0 < \epsilon \ll r$). By [**Milnor68**], $f^{-1}(\{|w| = \epsilon\}) \cap B_r \to \{|w| = \epsilon\}$ is a fibration with fiber F_ϵ. Let $m_q \in AutH_q(F_\epsilon, \mathbf{R})$ ($q \geq 0$) be the algebraic monodromy operators of this fibration. Then the zeta-function of f is defined by the following rational function:

$$\zeta(f)(t) := \prod_{q \geq 0} \det(I - tm_q)^{(-1)^q}.$$

The most efficient way to determine $\zeta(f)$ is by A'Campo's formula [**A'Campo75**] via the embedded resolution of f. Hence, in the case of those families of singularities whose embedded resolution is well understood, one gets $\zeta(f)$. This is the case for plane curve singularities (see e.g. [**EN85**]) and isolated singularities with non-degenerate Newton boundary ([**Varchenko76**], see also [**MO70**]). For non-isolated singularities the methods of series of singularities provide partial results

2000 *Mathematics Subject Classification.* 14B05, 32S**.

Key words and phrases. Quasi-ordinary singularities, Milnor fiber, monodromy, zeta-function, Newton polyhedron, toric resolution.

The third author is partially supported by the NSF Grant DMS-0088950.

(see e.g. [**Siersma90, Schrauwen90, Némethi93**]) provided that the singular locus is one-dimensional. But, in general, for non-isolated singularities there is no nice, explicit formula of $\zeta(f)$.

We recall that in general the singular locus of quasi-ordinary singularities is large. Our main result for these singularities is the following:

Theorem A. *Assume that $f : (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ $(d \geq 2)$ is a quasi-ordinary singularity (which does not need to be reduced). Then there is a (precisely described) reordering of the coordinates $(x_1, \ldots, x_d)$ such that $\zeta(f) = \zeta(f|_{x_2=\cdots=x_d=0})$.*

This theorem can also be reinterpreted as follows. The quasi-ordinary singularities are generalizations of the plane curve singularities. There is a generalization of Puiseux pairs in general coordinates, called the distinguished tuples, associated to the quasi-ordinary singularity f and the projection pr (see [**Zariski67, Lipman65, Lipman83, Lipman88**]). The distinguished tuples can be organized in a tree generalizing the Egger-Wall weighted tree encoding the topological type of a plane curve singularity (see [**Eggers83, GarcíaB00, PopescuP01**]).

In the irreducible case, using Zariski's result on saturation of local rings, one can prove that the distinguished tuples determine the embedded topological type of f (cf. [**Zariski68**] and [**Lipman88**], or [**Oh93**]). In the general case this fact is still a conjecture. On the other hand, by [**BMcN00**] (cf. also with [**Lê73**]), the zeta-function of any reduced hypersurface singularity depends only on the embedded topological type of the singularity. This shows that in the irreducible case $\zeta(f)$ depends only on the set of distinguished tuples (fact explicitly verified in [**McN01**]). In the general case, a similar result would strongly support the above conjecture.

Theorem A provides exactly this result; for its revised version, see §4.

Theorem A is the consequence of Theorem B:

Theorem B. *Let f be as in Theorem A. Then $\zeta(x^\beta f)(t) = 1$ for any $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{Z}^d_{>0}$, where $x^\beta := x_1^{\beta_1} \cdots x_d^{\beta_d}$.*

The proof of Theorem B occupies all of sections 2, 3, and the first part of 4.

The definition of the distinguished tuples, and some of their properties is given in Section 2. As a general reference, see [**Lipman65, Lipman83, Lipman88**] for the irreducible case, and [**González01, González00a**] in the general case.

In section 3 we recall the construction of a toric modification $\pi(\Sigma)$ associated with the distinguished tuple of f from [**González01**], nevertheless our presentation is different from [loc. cit.] (where even a more general case is treated). Here, we insert the needed new results of [**González01**] in the package used in [**McN01**] (which was based on the thesis of the first author [**González00a**]). For general facts about toric geometry, we recommend e.g. the books [**KKMS73, Oda88**]. For the properties of the toric modification $\pi(\Sigma)$ see also [**Varchenko76**] or [**GT00**].

In the first part of section 4 we will prove Theorem B. The proof uses a result of [**GLM97**], which provides a formula for the zeta-function in the presence of a partial embedded resolution. We will apply this result in the case of $\pi(\Sigma)$. Then we prove Theorem A, as a consequence of Theorem B and the "weak splitting property" of the zeta-function proved in [**McN01**].

Some general properties of the Milnor fibration and smoothing invariants, in particular, of the zeta-function, can be found in the books [**Milnor68**] or [**EN85**].

Finally (and similarly as in [**McN01**]) we notice that in the literature the notation for the parametrization of an irreducible quasi-ordinary singularity is ζ, and for the zeta-function of a hypersurface singularity is $\zeta(f)$. Even if these notations are almost identical, we do not modify them: the meaning of the corresponding notation will also be clear from the context.

2. Quasi-ordinary singularities and their distinguished tuples

In this section we recall the definition of the distinguished tuples associated with a hypersurface quasi-ordinary singularity $(F,0)\subset(\mathbf{C}^{d+1},0)$ (see [**Lipman65, Lipman83, Lipman88**] for the irreducible, and [**González01**] for the non-irreducible case).

2.1. We will use the notation $\bar{x}=(x,x_{d+1})=(x_1,\dots,x_d,x_{d+1})\in\mathbf{C}^d\times\mathbf{C}=\mathbf{C}^{d+1}$ for coordinates in $\mathbf{C}^{d+1}$ or for local coordinates in $(\mathbf{C}^{d+1},0)$. Let $\mathbf{C}\{x\}$ denote the ring of convergent power series in x.

We assume that $(F,0)=(\{f=0\},0)$ for some (not necessarily reduced) analytic germ $f:(\mathbf{C}^{d+1},0)\to(\mathbf{C},0)$ (i.e. F will always be considered reduced, but f not). Then, by the very definition of the hypersurface quasi-ordinary singularities, $(F,0)$ is quasi-ordinary, if there exist local coordinates (x,x_{d+1}) such that f can be expressed as a pseudo-polynomial $f(x,x_{d+1})=x_{d+1}^n+g_1(x)x_{d+1}^{n-1}+\cdots+g_n(x)$ with $g_i\in\mathbf{C}\{x\}$, and the reduced discriminant of $pr:(F,0)\to(\mathbf{C}^d,0)$ induced by $(x,x_{d+1})\mapsto x$ is contained in $(\{x_1\cdots x_d=0\},0)$. Let $\{f^{(i)}\}_{i\in I}$ be the set of irreducible components of f, and set $(F^{(i)},0):=(\{f^{(i)}=0\},0)$ for any $i\in I$. Then each $(F^{(i)},0)$ is an irreducible quasi-ordinary singularity.

By Jung–Abhyankar Theorem, for each $i\in I$, there exists a parametrization of $(F^{(i)},0)$ by a fractional power series $\zeta^{(i)}=H^{(i)}(x_1^{1/m},\dots,x_d^{1/m})$, where $H^{(i)}(s_1,\dots,s_d)$ is a power series and m is a suitable natural number (see [**Abhyankar55**] Th. 3 and [**Jung08**]). This means that there exists a finite map $(\mathbf{C}^d,0)\to(F^{(i)},0)$ given by $x_{d+1}=H^{(i)}(s_1,\dots,s_d)$, $x_t=s_t^m$ for $t=1,\dots,d$. The number m can be chosen independently of the index $i\in I$, but, obviously, it depends on f. (E.g. $m=n!$ is suitable; for an explicit construction of $\zeta^{(i)}$ one can consult [**González00b**] and [**ALR89**]).

The conjugates of $\zeta^{(i)}$ are obtained by multiplying any of $x_t^{1/m}$ $(t=1,\dots,d)$ by m^{th}-roots of unity; the number of different conjugates $\{\zeta_k^{(i)}\}_k$ of $\zeta^{(i)}$ is precisely the degree $n^{(i)}$ of the covering $pr|_{F^{(i)}}:(F^{(i)},0)\to(\mathbf{C}^d,0)$, and

$$f^{(i)}(x,x_{d+1})=\prod_{k=1}^{n^{(i)}}(z-\zeta_k^{(i)}).$$

We recall that f is not necessarily reduced; write $f_{red}=\prod_i f^{(i)}$ for its reduced germ.

Obviously, the equation of the discriminant of the projection pr is a product of type:

$$\prod(\zeta_k^{(i)}-\zeta_l^{(j)}).$$

There is an important finite subset of the set of exponents $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbf{Q}_{\geq0}^d$ of the terms $x^\lambda:=x_1^{\lambda_1}\cdots x_d^{\lambda_d}$ appearing in $\{\zeta^{(i)}\}_i$ with non-zero coefficient, called the *distinguished tuples* (in some articles they are called characteristic exponents). In the irreducible case they play a role similar to Puiseux pairs for irreducible plane curve singularities. In the case of non-irreducible germs, besides the tuples of the irreducible components there are some additional tuples as well: they measure the order of coincidence of the parametrizations of different components.

By unique factorization of the discriminant one has:

$$\zeta_k^{(i)} - \zeta_l^{(j)} = x^{\lambda_{kl}^{ij}} \cdot \epsilon_{kl}^{ij}(x_1^{1/m}, \ldots, x_d^{1/m}) \text{ with } \epsilon_{kl}^{ij}(0) \neq 0,$$

and $\lambda_{kl,t}^{ij} \in \frac{1}{m}\mathbf{Z}_{\geq 0}$. The set $\Lambda := \{\lambda_{kl}^{ij}\}_{ij,kl} = \{(\lambda_{kl,1}^{ij}, \ldots, \lambda_{kl,d}^{ij})\}_{ij,kl} \subset \frac{1}{m}\mathbf{Z}_{\geq 0}^d$ constitute the set of distinguished tuples.

Fix an irreducible component $(F^{(i)}, 0)$ for some $i \in I$, and let $\zeta^{(i)}$ be its parametrization. Then (by similar argument as above) the distinguished tuples of $(F^{(i)}, 0)$ are $\Lambda^{(i)} := \{\lambda_{kl}^{ii}\}_{kl}$. Set $g^{(i)} := \#\Lambda^{(i)}$.

2.2. In the sequel we will use the following partial ordering. For any $\lambda,\ \mu \in \mathbf{Q}^d$ we say that $\lambda \leq \mu$ if $\lambda_t \leq \mu_t$ for all $t = 1, \ldots, d$. If $\lambda \leq \mu$, but $\lambda \neq \mu$, then we write $\lambda < \mu$.

The point is that for any fixed irreducible component $(F^{(i)}, 0)$, the distinguished tuples $\Lambda^{(i)}$ are totally ordered (cf. [**Lipman65, Zariski67, Lipman83, Lipman88**]). For simplicity, we denote the tuples of $\Lambda^{(i)}$ by $\{\lambda_k^{(i)}\}_{k=1}^{g^{(i)}}$. Hence:

(1) $\quad 0 < \lambda_1^{(i)} < \lambda_2^{(i)} < \cdots < \lambda_{g^{(i)}}^{(i)}$.

$\Lambda^{(i)}$ is empty if and only if $F^{(i)}$ is smooth.

Moreover, let $\alpha_\lambda^{(i)}$ be the coefficient of the term x^λ in $\zeta^{(i)}$ having exponent λ. Then the distinguished tuples $\Lambda^{(i)}$ generate all the exponents appearing in $\zeta^{(i)}$ in the following sense:

(2) $\quad$ If $\alpha_\lambda^{(i)} \neq 0$ then $\lambda \in \mathbf{Z}^d + \sum_{\lambda_k^{(i)} \leq \lambda} \mathbf{Z}(\lambda_k^{(i)})$.

In general, the set Λ is *not* totally ordered. This is a crucial difference between the irreducible and non-irreducible cases, therefore we explain it briefly.

Fix an index $i \in I$ as above. Assume that $\zeta_{k_1}^{(i)}$ is obtained from $\zeta_1^{(i)}$ by multiplying any $x_t^{1/m}$ by an m^{th}-root of unity. Assume that the same Galois action transforms $\zeta_{l_1}^{(i)}$ into $\zeta_{l_1'}^{(i)}$. Then $\lambda_{k_1 l_1}^{ii} = \lambda_{1 l_1'}^{ii}$. Similarly, for another pair (k_2, l_2), one has $\lambda_{k_2 l_2}^{ii} = \lambda_{1 l_2'}^{ii}$ for some l_2'. Since $\zeta_{l_1'}^{(i)} - \zeta_{l_2'}^{(i)} = (\zeta_{l_1'}^{(i)} - \zeta_1^{(i)}) - (\zeta_{l_2'}^{(i)} - \zeta_1^{(i)})$ divides the discriminant of f, one must have $\lambda_{1l_1'}^{ii} \leq \lambda_{1l_2'}^{ii}$ or $\lambda_{1l_2'}^{ii} < \lambda_{1l_1'}^{ii}$. We call this line of argument "the Galois trick".

On the other hand, in the case of Λ, the same argument does not work, since it is impossible to compare λ_{kl}^{ij} and $\lambda_{k'l'}^{i'j'}$ if $\{i, j\} \cap \{i', j'\} = \emptyset$. If this intersection is not empty, then a similar argument as above gives (cf. [**Zariski67**]):

(3) $\quad$ Fix an index $i \in I$. Then $\{\lambda_{kl}^{ij}\}_{j,kl}$ is totally ordered.

2.3. Definition. Fix two indices $i, j \in I$, $i \neq j$, and set $\Lambda^{(i,j)} := \{\lambda_{kl}^{ij}\}_{kl}$. Since $\Lambda^{(i,j)} \subset \{\lambda_{kl}^{ij}\}_{j,kl}$, by (3) one obtains that $\Lambda^{(i,j)}$ is totally ordered. Let $\lambda^{(i,j)} := \max \Lambda^{(i,j)}$ be its maximal element. $\lambda^{(i,j)}$ is called *the order of coincidence* of the components $f^{(i)}$ and $f^{(j)}$. Obviously, $\Lambda^{(i,j)} = \Lambda^{(j,i)}$, hence $\lambda^{(i,j)} = \lambda^{(j,i)}$ as well.

In fact, $\lambda^{(i,j)}$ carries all the information about the mutual positions of $(F^{(i)},0)$ and $(F^{(j)},0)$ (cf. the next 2.4 (a)). Moreover, one has the following "valuation type properties". (For the proof use (3) and the "Galois trick", cf. also with [**González01**], Lemma 16.)

2.4. Lemma. *Assume that* $i \neq a \neq j \neq i$.
(a) $\Lambda^{(i,j)} = \{\lambda \in \Lambda^{(i)} \cap \Lambda^{(j)}; \lambda < \lambda^{(i,j)}\} \cup \{\lambda^{(i,j)}\}$.
(b) $\min\{\lambda^{(i,a)}, \lambda^{(i,j)}\} \leq \lambda^{(j,a)}$.
(c) If $\lambda^{(i,a)} < \lambda^{(i,j)}$ *then* $\lambda^{(i,a)} = \lambda^{(j,a)}$.

2.5. Definition. Fix an index $i \in I$, and consider the subset $\mathcal{A}(i) := (\{\lambda^{(i,j)}\}_j \cap \mathbf{Z}^d) \cup \{\lambda_1^{(i)}\}$ of $\{\lambda_{kl}^{ij}\}_{j,kl}$. In special cases this subset can be empty; define $I' := \{i|\mathcal{A}(i) \neq \emptyset\}$. If $i \in I'$ then $\mathcal{A}(i)$ is totally ordered by (3). Following [**González01**], we define $\lambda_{\kappa(i)} := \min \mathcal{A}(i)$ for any $i \in I'$. For $i \in I \setminus I'$ we write $\lambda_{\kappa(i)} = +\infty$. (By convention: $\lambda < +\infty$ for any $\lambda \in \mathbf{Q}^d$.)

Notice that using (1), property (2) (cf. 2.2) reads as:

(2*) Assume that $\lambda \notin \mathbf{Z}^d$ satisfies $\lambda_{\kappa(i)} \not\leq \lambda$ for some $i \in I$. Then $\alpha_\lambda^{(i)} = 0$.

The above facts imply the next result.

2.6. Proposition. ([**González01**], Lemma 17)
(a) $\#(I \setminus I') \leq 1$. *If* $\#(I \setminus I') = 1$ *then the number of smooth irreducible component of* F *is exactly one. In particular,* $\Lambda = \emptyset$ *if and only if* F *is irreducible and smooth.*

(b) Assume that $\lambda_{\kappa(i)} \in \mathbf{Z}^d$ *for some* $i \in I'$. *Then* $\lambda_{\kappa(j)} \leq \lambda_{\kappa(i)}$ *for any* $j \in I'$. *In particular, the set* $\{\lambda_{\kappa(i)}\}_{i \in I'}$ *contains at most one element from* $\mathbf{Z}^d$ *(which is its greatest element).*

(c) The set $\{\lambda_{\kappa(i)}\}_{i \in I'}$ *is totally ordered.*

2.7. This proposition together with the next technical result will allow us to modify the coefficients of f by a change of variables in such a way that in the new situation the Newton polyhedron of f_{red} will reflect the set $\{\lambda_{\kappa(i)}\}_i$. The change of variable has the form $x'_{d+1} = x_{d+1} - \varphi(x)$, $x'_t = x_t$ for $t = 1, \ldots, d$, with $\varphi(x)$ of type $\sum_{\lambda \in \mathbf{Z}^d_{\geq 0}} a_\lambda x^\lambda$. This provides automatically a set of new parametrizations $\zeta'^{(i)} := \zeta^{(i)} - \varphi(x)$ $(i \in I)$. Such a change of variables and parametrizations does not modify the set of distinguished tuples.

First, we verify the following lemma.

2.8. Lemma. *Assume that* $\lambda_{\kappa(i)} \not\leq \lambda$ *and* $\lambda_{\kappa(j)} \not\leq \lambda$. *Then* $\alpha_\lambda^{(i)} = \alpha_\lambda^{(j)}$.

Proof. By (2*) we can assume that $\lambda \in \mathbf{Z}^d$. Assume that $\alpha_\lambda^{(i)} \neq \alpha_\lambda^{(j)}$. Then by the definition of $\lambda^{(i,j)}$ one has $\lambda^{(i,j)} \leq \lambda$. This together with $\lambda_{\kappa(i)} \not\leq \lambda$ implies $(*)$ $\lambda_{\kappa(i)} \not\leq \lambda^{(i,j)}$. Therefore, by the definition of $\lambda_{\kappa(i)}$ one has $(**)$ $\lambda^{(i,j)} \notin \mathbf{Z}^d$. Then $(*)$, $(**)$ and (2*) give $\alpha^{(i)}_{\lambda^{(i,j)}} = 0$. Similarly $\alpha^{(j)}_{\lambda^{(i,j)}} = 0$. These two vanishing contradict the definition of $\lambda^{(i,j)}$.

2.9. Proposition/Definition. (cf. [**González01**]) *There is a change of variables as in 2.7 such that the new parametrizations have the following form:*

$$\zeta^{(i)} = \alpha_1^{(i)} x^{\lambda_{\kappa(i)}} + \sum_{\lambda_{\kappa(i)} < \lambda} \alpha_\lambda^{(i)} x^\lambda \quad (\alpha_1^{(i)} \neq 0) \quad (i \in I').$$

If $i \in I \setminus I'$ then $f^{(i)}(x, x'_{d+1}) = x'_{d+1}$ (i.e. $\zeta^{(i)} = 0$).

A coordinate system which admits parametrizations as above is called "good".

Proof. Choose an index $i_0 \in I$ so that $\lambda_{\kappa(i_0)} = \max\{\lambda_{\kappa(i)}\}_{i \in I}$. Then define $\varphi(x)$ by $\sum \alpha_\lambda^{(i_0)} x^\lambda$, where the sum is over $\lambda \in \mathbf{Z}^d_{\geq 0}$ with $\lambda_{\kappa(i_0)} \not\leq \lambda$. (In particular, if $i \in I \setminus I'$ then $\varphi = \zeta^{(i_0)}$.) Then the result follows from lemma 2.8 .

2.10. Corollary. Λ *has a unique minimal element* $\min \Lambda$ *which equals* $\min\{\lambda_{\kappa(i)}\}_i$.

3. The partial toric resolution.

In this section we will assume that f is a quasi-ordinary singularity, and we will use the notations of the previous section. We will consider a toric modification $\pi(\Sigma) : Z(\Sigma) \to \mathbf{C}^{d+1}$ whose restriction above a small ball B_r provides a partial resolution of $(F, 0) := (f^{-1}(0), 0) \subset (\mathbf{C}^{d+1}, 0)$. For simplicity and uniformity of the notation we will use $\mathbf{C}^{d+1}$ instead of B_r, but this will not affect the proofs. For the definition of $\pi(\Sigma)$ associated with an arbitrary fan Σ, see [**KKMS73**] or [**Oda88**]. In the case of some special hypersurface singularities, $\pi(\Sigma)$ is a natural (partial) embedded resolution, see e.g. [**Varchenko76**] or [**GT00**]. For irreducible quasi-ordinary singularities the modification $\pi(\Sigma)$ was constructed in [**González00a**], and [**González01**] contains the reduced case. The presentation here combines [**González00a**], [**González01**] and [**McN01**].

3.1. The germs $f_1^{(i)}$. Assume that f is represented in a "good" coordinate system, which admits parametrizations as in 2.9 .

Define $\zeta_1^{(i)} := \alpha_1^{(i)} x^{\lambda_{\kappa(i)}}$ for $i \in I'$ and $\zeta_1^{(i)} := \zeta^{(i)} = 0$ otherwise. Obviously, $\zeta_1^{(i)}$ determines an irreducible quasi-ordinary singularity $f_1^{(i)}$. If L denotes the fraction field of $\mathbf{C}\{x\}$, then $n^{(i)} = [L(\zeta^{(i)}) : L]$ is the degree of $f^{(i)}$, and $n_1^{(i)} = [L(x^{\lambda_{\kappa(i)}}) : L]$ is the degree of $f_1^{(i)}$. Since the field extension $[L(\zeta^{(i)}) : L]$ is generated over L by the distinguished tuples of $f^{(i)}$ ([**Lipman88**], Lemma 5.7), 2.2 (2) implies that $n_1^{(i)}$ divides $n^{(i)}$. Set $e^{(i)} := n^{(i)}/n_1^{(i)} \in \mathbf{Z}_{>0}$. (If $i \notin I'$ then $n^{(i)} = n_1^{(i)} = e^{(i)} = 1$.) Obviously

$$f_1^{(i)} = \prod_{j=1}^{n_1^{(i)}} (x_{d+1} - \omega_j \alpha_1^{(i)} x^{\lambda_{\kappa(i)}}) = x_{d+1}^{n_1^{(i)}} - \gamma^{(i)} x^{n_1^{(i)} \lambda_{\kappa(i)}}, \quad \text{with } \gamma^{(i)} := (\alpha_1^{(i)})^{n_1^{(i)}},$$

where $\{\omega_j\}_j$ are the $n_1^{(i)}$-roots of unity. Notice also that if $\kappa(i) = \kappa(j)$ and $\gamma^{(i)} = \gamma^{(j)}$ then $f_1^{(i)} = f_1^{(j)}$. The importance of the germ $f_1^{(i)}$ is given by the fact that its Newton polyhedron determines the Newton polyhedron of $f^{(i)}$ (see below).

3.2. The Newton polyhedron of f. Given $f_{red} = \sum_v c_v \bar{x}^v \in \mathbf{C}\{\bar{x}\}$, its Newton polyhedron $\mathcal{N}(f_{red})$ is the convex hull of the set $\sum_{c_v \neq 0} v + \mathbf{R}^{d+1}_{\geq 0}$. Any vector $u \in (\mathbf{R}^{d+1})^*_{\geq 0}$ defines the face $\mathcal{F}_u$ of $\mathcal{N}(f_{red})$ by

$$\mathcal{F}_u := \{v \in \mathcal{N}(f_{red}) :\ \langle u, v\rangle = \inf_{v' \in \mathcal{N}(f_{red})} \langle u, v'\rangle\}.$$

$\mathcal{N}(f^{(i)})$ has only two vertices, namely $(0, \dots, 0, n^{(i)})$ and $(n^{(i)}\lambda_{k(i),1}, \dots, n^{(i)}\lambda_{k(i),d}, 0)$ provided that $i \in I'$. If $i \notin I'$ then $\mathcal{N}(f^{(i)})$ has only one vertex, namely $(0, \dots, 0, 1)$. Finally, the Newton polyhedron $\mathcal{N}(f_{red})$ of f_{red} is the Minkowski sum of the polyhedrons $\mathcal{N}(f^{(i)})$ $(i \in I)$. Recall that the set $\{\lambda_{\kappa(i)}\}_{i\in I}$ is totally ordered, hence the compact faces of $\mathcal{N}(f_{red})$ constitute a "monotone polygonal path" (using the terminology of [**González00b**] and [**McDonald95**]). Let $\mathcal{F}_{co}$ be the union of all compact faces of $\mathcal{N}(f_{red})$. Then

$$f_{red}|_{\mathcal{F}_{co}} = \prod_{i\in I} (f_1^{(i)})^{e^{(i)}}.$$

Above $f_{red}|_{\mathcal{F}}$ denotes the (symbolic) restriction of f_{red} to $\mathcal{F}$, namely $f_{red}|_{\mathcal{F}} = \sum_{v\in\mathcal{F}} c_v \bar{x}^v$.

3.3. The fan $\Sigma(\mathcal{N}(f_{red}))$ and its subdivision Σ in $(\mathbf{R}^{d+1})^*_{\geq 0}$. We say that two vectors in $(\mathbf{R}^{d+1})^*_{\geq 0}$ are related if they define the same face in $\mathcal{N}(f_{red})$. The fan $\Sigma(\mathcal{N}(f_{red}))$ is defined in such a way that the classes of the above relation are the relative interiors of the cones of $\Sigma(\mathcal{N}(f_{red}))$. In fact, the fan $\Sigma(\mathcal{N}(f_{red}))$ is obtained by subdividing the cone $(\mathbf{R}^{d+1})^*_{\geq 0}$ with the linear subspaces $l^{(i)}$ $(i \in I')$ given by the equations

$$l^{(i)}(v) := \sum_{t=1}^{d} \lambda_{k(i),t} v_t - v_{d+1} = 0 \quad (i \in I').$$

Since $\{\lambda_{\kappa(i)}\}_{i\in I'}$ is totally ordered, for any $i, j \in I'$, $\kappa(i) \neq \kappa(j)$, one has

$$(*) \qquad l^{(i)} \cap l^{(j)} \cap (\mathbf{R}^{d+1})^*_{>0} = \emptyset.$$

In general, $\Sigma(\mathcal{N}(f_{red}))$ is not nonsingular (for the terminology, see [**Oda88**] 1.4; in some articles "regular" is used for "nonsingular"). We will use the notation Σ for a nonsingular (simplicial) subdivision of $\Sigma(\mathcal{N}(f_{red}))$ supported by $(\mathbf{R}^{d+1})^*_{\geq 0}$. For the existence of such a fan Σ, see Theorem 11 (page 32) of [**KKMS73**].

For an arbitrary Σ it is *not true* that $\pi(\Sigma) : Z(\Sigma) \to \mathbf{C}^{d+1}$ is an isomorphism above $\mathbf{C}^{d+1} \setminus Sing\, F$ (a property sometimes required for an embedded resolution). But, with a good choice of Σ, one has a slightly weaker property (which is still sufficient for us, cf. 4.2). More precisely, there exists a nonsingular fan Σ so that $\pi(\Sigma)$ is an isomorphism above $\mathbf{C}^{d+1} \setminus F$. For the proof see (3.15) of [**McN01**], a similar property was also used in [**Varchenko76**] and [**González01**].

In fact, in [**González01**], the first author considered those nonsingular subdivisions Σ which have the following additional property: any nonsingular cone of $\Sigma(\mathcal{N}(f_{red}))$ is a cone of Σ. The existence of such a fan is proved in [**Cox00**], Theorem 5.1. If Σ satisfies this property then $\pi(\Sigma)$ is an isomorphism above $\mathbf{C}^{d+1} \setminus F$ (cf. [**McN01**]), and the restriction of $\pi(\Sigma)$ to the strict transform $\tilde{F}$ of F is an isomorphism above $F \setminus Sing\, F$ (fact shown in [**González00a, González01**]).

In the sequel, we will assume that Σ satisfies the above property.

3.4. The fan Σ supported by $(\mathbf{R}^{d+1})^*_{\geq 0}$ defines a modification $\pi(\Sigma) : Z(\Sigma) \to \mathbf{C}^{d+1}$. The variety $Z(\Sigma)$ is smooth, and it is covered by affine spaces $\{Z(\sigma)\}_{\dim \sigma = d+1}$ (i.e. each $Z(\sigma) \approx \mathbf{C}^{d+1}$). The inclusion $\sigma \subset (\mathbf{R}^{d+1})^*_{\geq 0}$ induces a morphism $\pi(\sigma) : Z(\sigma) \to \mathbf{C}^{d+1}$. If $\sigma = \langle a^1, \dots, a^{d+1} \rangle$ and a^k has coordinates $(a^k_1, \dots, a^k_{d+1})$, then $\pi(\sigma)$ has the form:

$$x_1 = u_1^{a_1^1} \cdots u_{d+1}^{a_1^{d+1}}, \ \dots \ , x_{d+1} = u_1^{a_{d+1}^1} \cdots u_{d+1}^{a_{d+1}^{d+1}}.$$

The total transform of $f = \sum_v c_v \bar{x}^v$ by $\pi(\sigma)$ is

$$f \circ \pi(\sigma) = \sum_v c_v u_1^{\langle a^1, v\rangle} \cdots u_{d+1}^{\langle a^{d+1}, v\rangle} = u_1^{m(a^1)} \cdots u_{d+1}^{m(a^{d+1})} \cdot \tilde{f}_\sigma,$$

where $m(a) := \inf_{v \in \mathcal{N}(f_{red})} \langle a, v \rangle$ and $\tilde{f}_\sigma$ defines the equation of the strict transform of f in the chart $Z(\sigma) \subset Z(\Sigma)$.

3.5. The divisors $D(a)$ in $Z(\Sigma)$. To any primitive vector a of $\Sigma^{(1)}$(:= 1-skeleton of Σ), we associate a divisor $D(a)$ in $Z(\Sigma)$. In any chart $Z(\sigma)$, associated with a cone $\sigma = \langle a^1, \dots, a^{d+1} \rangle$, $D(a^k)$ is given by $\{u_k = 0\}$. If $b \notin \{a^1, \dots, a^{d+1}\}$, then $D(b) \cap Z(\sigma) = \emptyset$. The divisor $D := \cup_{a \in \Sigma^{(1)}} D(a)$ in the smooth variety $Z(\Sigma)$ is a normal crossing divisor with smooth irreducible components.

By 3.4 one gets that $\pi(\Sigma)(D(a^k))$ is the coordinate subspace defined by $x_t = 0$ for all t with $a^k_t \neq 0$. If $e^1, \dots, e^{d+1}$ denote the canonical basis on $\mathbf{Z}^{d+1}$, then $\pi(\Sigma)(D(e^k)) = \{x_k = 0\}$, but for any $a \in \Sigma^{(1)} \setminus \{e^1, \dots, e^{d+1}\}$ one has $\dim \pi(\Sigma)(D(a)) \leq d-1$. The critical locus (exceptional divisor) of $\pi(\Sigma)$ is exactly the union of these divisors:

$$\cup_{a \in \Sigma^{(1)} \setminus \{e^k\}_k} D(a).$$

By the above discussion, $D(a)$ is compact (i.e. $D(a) \subset \pi(\Sigma)^{-1}(0)$) if and only if a is an interior point of $(\mathbf{R}^{d+1})^*_{\geq 0}$. More generally, let $A = \{a^1, \dots, a^s\}$ be a non-empty subset of $\Sigma^{(1)}$, and write $D_A := \cap_{a \in A} D(a)$. The $D_A \neq \emptyset$ if and only if $\{a^1, \dots, a^s\}$ forms a cone σ_A in Σ. In this case D_A is compact if and only if $\sigma_A \cap (\mathbf{R}^{d+1})^*_{>0} \neq \emptyset$. In fact

$$\pi(\Sigma)^{-1}(0) = \bigcup_{\sigma_A \cap (\mathbf{R}^{d+1})^*_{>0} \neq \emptyset} D_A.$$

3.6. The strict transform $\tilde{F}$ of F. Set $F^{(i)} = \{f^{(i)} = 0\}$ and for any space Y write $\tilde{Y}$ for its strict transform via $\pi(\Sigma)$.

We start with the following remark. If we fix an index $i \in I'$, then all the properties of the germs $f^{(i)}$, $f_1^{(i)}$ and their strict transforms follow from section 3 of [**McN01**], where the irreducible case is treated. Indeed, our fan Σ considered here satisfies all the restrictions required in [loc. cit.] (even some more, since it is compatible with all the linear spaces $l^{(j)}$). In the next paragraph we list those properties of $\tilde{F}^{(i)}$ which will be needed later. For proofs see [**McN01**], [**González00a**] or [**González01**] (but, for the convenience of the reader, we provide some hints as well).

For a fixed $i \in I'$ the following facts hold:

(1) For any $A = \{a^1, \dots, a^s\} \subset \Sigma^{(1)}$, if $D_A \cap \tilde{F}^{(i)} \neq \emptyset$, then $\{a^1, \dots, a^s\}$ should be a cone σ_A in $\Sigma \cap l^{(i)}$. In this case, $D_A \cap \tilde{F}^{(i)}$ is compact if and only if

$\sigma_A \cap l^{(i)} \cap (\mathbf{R}^{d+1})^*_{>0} = \sigma_A \cap (\mathbf{R}^{d+1})^*_{>0} \neq \emptyset$. In particular, $D_A \cap \tilde{F}^{(i)}$ is compact if and only if D_A is compact (cf. 3.5).

(2) We say that the "natural stratification" of $D \cap \tilde{F}^{(i)}$ is given by strata of type:

$$(\cap_{a\in A} D(a) \setminus \cup_{a\notin A} D(a)) \cap \tilde{F}^{(i)}, \quad (A \subset \Sigma^{(1)}).$$

Then for any stratum Ξ of $D\cap\tilde{F}^{(i)}$ with $\dim \Xi > 0$, one has $\chi(\Xi) = 0$. (This follows from the fact that Ξ is contined in an orbit of the toric variety $Z(\Sigma)$ represented in some chart $Z(\sigma)$ by an equation u_{d+1} =a non-zero constant.)

Next, we fix $i, j \in I'$ with $i \neq j$. Then the assumption about Σ (cf. 3.3) and the above properties give:

(3) If $\kappa(i) \neq \kappa(j)$, then $\pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(i)} \cap \tilde{F}^{(j)} = \emptyset$. (If $\pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(i)} \cap \tilde{F}^{(j)}$ is non empty, there is $D_A \subset \pi(\Sigma)^{-1}(0)$ such that $D_A \cap \tilde{F}^{(i)} \cap \tilde{F}^{(j)}$ is non empty, and $\sigma_A \cap (\mathbf{R}^{d+1})^*_{>0} \neq \emptyset$, cf. with the last formula of 3.5 . Then by (1) we have that $\sigma_A \subset \Sigma \cap l^{(i)} \cap l^{(j)}$. This implies that $\sigma_A \cap (\mathbf{R}^{d+1})^*_{>0} \subset l^{(i)} \cap l^{(j)} \cap (\mathbf{R}^{d+1})^*_{>0}$ is empty by 3.3 $(*)$, a contradiction.)

(4) If $\kappa(i) = \kappa(j)$, then $\pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(i)} = \pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(j)}$ provided that $\gamma^{(i)} = \gamma^{(j)}$; otherwise $\pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(i)} \cap \tilde{F}^{(j)} = \emptyset$. (This follows again by the irreducible case, where the intersection $\pi(\Sigma)^{-1}(0) \cap \tilde{F}^{(i)}$ is described in terms of $f_1^{(i)}$, cf. [**McN01**], 3.20 and 3.21. See also the discussion in 3.7 , or Proposition 24 of [**González01**].)

(5) $D(e^{d+1}) \cap \tilde{F}^{(j)} = \emptyset$ for any $j \in I'$. Since $\tilde{F}^{(i)} = D(e^{d+1})$ for $i \in I \setminus I'$, one obtains that for such an i, $\tilde{F}^{(i)} \cap \tilde{F}^{(j)} = \emptyset$ for any $j \in I'$. (This follows from $e^{d+1} \notin \cup_{i\in I'} l^{(i)}$ and part (1).)

3.7. The total transform of $x^\beta f$. The divisor $D \cap \tilde{F}$ has a "natural stratification" given by strata of type

$$(\cap_{a\in A} D(a) \setminus \cup_{a\notin A} D(a)) \cap \tilde{F}, \quad (A \subset \Sigma^{(1)}).$$

We are interested only in those strata which are situated in $\pi(\Sigma)^{-1}(0)$. They are contained in compact intersections of type D_A.

Next, we fix $\beta \in \mathbf{Z}^d_{>0}$, and consider the germ $x^\beta f$. Notice that $\pi(\Sigma)^{-1}(\{x_t = 0\}) = D(e^t)$ (for $t = 1, \ldots, d$), hence $(D \cup \{\text{the total transform of } \{x^\beta f = 0\}\}) = (D \cup \tilde{F})$, and both of them have the same "natural stratification" (i.e. by introducing the factor x^β, we do not create new strata).

Consider such a strata Ξ with $\dim \Xi > 0$. We claim that the total transform of $x^\beta f$ along Ξ is an equisingular family of singularities. The argument is similar as in [**González00a**] and [**González01**] where it is proved that the total transform of f is equisingular along Ξ.

(By this we mean the following. The strict transform along the stratum Ξ defines a family of hypersurface quasi-ordinary singularities in such a way that each member of the family has the same distinguished tuples. The situation locally is described in [**Lipman88**] Appendix 7, where "natural stratification" of an irreducible quasi-ordinary hypersurface is studied. The important point is that this family is equiresolvable, a resolution is obtained from the embedded resolution of the total transform of f, see [**González01**]).

Now we concentrate our discussion on the zero-dimensional strata $\{\Xi\}_{\dim \Xi = 0}$ of $D \cap \tilde{F}$.

We fix an arbitrary d-cone $\sigma' = \langle a^1, \ldots, a^d \rangle$ of $\Sigma \cap l^{(i)}$ $(i \in I')$, and we consider the *unique* $(d+1)$-cone $\sigma = \langle a^1, \ldots, a^d, a^{d+1} \rangle$ of Σ with a primitive vector a^{d+1} satisfying $l^{(i)}(a^{d+1}) > 0$. Then, in the chart $Z(\sigma)$ one can find a zero-dimensional stratum $O_{\sigma'} := \tilde{F} \cap D(a^1) \cap \cdots \cap D(a^d)$ of $D \cap \tilde{F}$. In fact, if $I = I'$, then there is a one-to-one correspondence between the zero-dimensional strata of $D \cap \tilde{F}$ and the points $O_{\sigma'}$ where σ' runs over the d-cones of $\Sigma \cap l^{(i)}$ for all $i \in I'$. If $I \neq I'$ then $D \cap \tilde{F}$ has some additional zero-dimensional strata on $D(e^{d+1})$ as well.

Using the above notations, in the chart $Z(\sigma)$, the point $O_{\sigma'}$ is given by the equations $u_1 = \cdots = u_d = 0$ and $w := 1 - \gamma^{(i)} u_{d+1} = 0$. The first author in [**González01**] has shown that the strict transform $\tilde{f}_\sigma$ of f in the local coordinates $(u_1, \ldots, u_d, w)$ is a (not necessarily reduced) quasi-ordinary singularity. Moreover, the germ $((\tilde{f}_\sigma)_{red}, O_{\sigma'})$ has smaller complexity than f_{red} (e.g. $\#\Lambda$ is smaller, or its Eggers-Wall diagram is simpler).

In the proof of Theorem B, we will use an inductive step, which replaces the germ $x^\beta f$ by its total transform at the points $\{O_{\sigma'}\}_{\sigma'}$. Obviously, the total transform of $x^\beta f$ at the point $O_{\sigma'}$ has similar form, namely $u^\eta \tilde{f}_\sigma$ for some η (by 3.4). The point is that if $\beta \in \mathbf{Z}^d_{>0}$ then $\eta \in \mathbf{Z}^d_{>0}$ as well. Indeed, let us write $\langle a^k, \beta \rangle$ for $\sum_{t=1}^d a^k_t \beta_t$. Then by 3.4 one gets that $\eta_k = m(a^k) + \langle a^k, \beta \rangle$ for any $k = 1, \ldots, d$. Notice that $a^k \neq e^{d+1}$ since $e^{d+1} \notin l^{(i)}$. Hence $\langle a^k, \beta \rangle > 0$ for each k.

4. The proof of Theorems A and B

4.1. The proof of Theorem B.. The proof is based on the facts listed in sections 2 and 3 and on the following result proved in [**GLM97**]. We mention, that in this paper, stratification means a (semi-analytic) pre-stratification, without any regularity assumption.

4.2. Theorem. *Let $g : (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ be the germ of an analytic function, and B_r a sufficiently small (Milnor) ball of it. Let $\phi : X \to B_r$ be a birational modification such that ϕ is an isomorphism above the complement of $\{g = 0\}$. Let $\mathcal{S}$ be a stratification of $\phi^{-1}(0)$ such that along each stratum Ξ of $\mathcal{S}$ the zeta-function of the germ $(g \circ \phi, x)$ at $x \in \Xi$ does not depend on $x \in \Xi$. Denote this rational function by $\zeta_\Xi(t)$. Then:*

$$\zeta(g)(t) = \prod_{\Xi \in \mathcal{S}} \zeta_\Xi(t)^{\chi(\Xi)}.$$

In order to prove Theorem B we will apply 4.2 for $\phi = \pi(\Sigma)$, where the fan Σ satisfies all the properties 3.3 . Since the image of the critical locus of $\pi(\Sigma)$ is included in F, 4.2 can be applied for the germ $g = x^\beta f$. The divisor D has a natural stratification with the following strata:

$$\{(\cap_{a \in A} D(a) \setminus \cup_{a \notin A} D(a)) \setminus \tilde{F}\}_{A \subset \Sigma^{(1)}} \;,\; \{(\cap_{a \in A} D(a) \setminus \cup_{a \notin A} D(a)) \cap \tilde{F}\}_{A \subset \Sigma^{(1)}}.$$

This stratification of D induces a stratification on $\pi(\Sigma)^{-1}(0)$ which will be denoted by $\mathcal{S}$. Each stratum Ξ of $\mathcal{S}$ is included in some compact intersection of type $\cap_{a \in A} D(a)$ (cf. 3.6 (1)). By the very definition, the stratification is compatible with $\tilde{F}$. Denote by $\mathcal{S} \cap \tilde{F}$ (resp. by $\mathcal{S} \setminus \tilde{F}$) the collection of those strata which are in $\tilde{F}$ (resp. are in $\pi(\Sigma)^{-1}(0) \setminus \tilde{F}$). The stratification $\mathcal{S}$ satisfies the assumptions of 4.2 by the results of 3.6 and 3.7 .

We recall that the divisor $D = \cup_{a\in\Sigma^{(1)}} D(a)$ consists of the exceptional divisors $\{D(a)\}_a$ corresponding to $a \in \Sigma^{(1)} \setminus \{e^k\}_{1\leq k\leq d+1}$, and the strict transforms of the coordinate hyperplanes $\{D(e^k)\}_{1\leq k\leq d+1}$. On the other hand, the divisors $\{D(e^k)\}_{1\leq k\leq d}$ are components of the total transform of g. The divisor $D(e^{d+1})$ will require a special care. Therefore, we will distinguish two cases: (Case A) $I \neq I'$ (i.e. x_{d+1} is a component of f); and (Case B) $I = I'$ (i.e. x_{d+1} is not a component of f).

We will analyze the contribution $\zeta_\Xi(t)^{\chi(\Xi)}$ for each stratum Ξ.

(1) First assume that $\Xi \in \mathcal{S} \setminus \tilde{F}$ is of dimension $\dim \Xi < d$.

Recall that the total transform of $x^\beta f$ outside of $\tilde{F}$ is a normal crossing divisor. Moreover, the zeta-function of a germ $(u_1, \ldots, u_{d+1}) \mapsto u_1^{\rho_1} \cdots d_{d+1}^{\rho_{d+1}}$ is 1, provided that at least two ρ_k's are strictly positive.

In (Case A), along each Ξ there are at least two coordinate hyperplanes in the zero-set of the total transform of g. Hence the zeta-function $\zeta_\Xi(t) = 1$. (Cf. also with the classical case [**A'Campo75**] when ϕ is an embedded resolution.)

In (Case B) $D(e^{d+1}) \cap \tilde{F} = \emptyset$ (cf. 3.6 (5)). Therefore, any stratum Ξ for which the argument presented in (Case A) is not valid, is an orbit of the toric variety $Z(\Sigma)$ of dimension $d-1$ and type $D(e^{d+1}) \cap D(a) \setminus \cup_b D(b)$. So $\chi(\Xi) = 0$.

(2) Assume that $\Xi \in \mathcal{S} \setminus \tilde{F}$ and $\dim \Xi = d$. Then there is a unique stratum Ξ' of D of type $D(a) \setminus (\cup_{b\neq a} D(b))$ such that $\Xi \subset \Xi'$. Moreover, Ξ is obtained from Ξ' by eliminating its intersection with $\tilde{F}$, which is a union of $(d-1)$-dimensional strata of type $\Xi'' \in \mathcal{S} \cap \tilde{F}$. Now, Ξ' is an orbit of the toric variety $Z(\Sigma)$, hence $\chi(\Xi') = 0$. But for each Ξ'', $\chi(\Xi'') = 0$ as well, because of 3.6 (2). Hence $\chi(\Xi) = 0$.

(3) Now, assume that $\Xi \in \mathcal{S} \cap \tilde{F}$ of dimension $\dim \Xi > 0$. Then $\chi(\Xi) = 0$ by 3.6 (2).

(4) Finally, consider the zero-dimensional strata of $\Xi \in \mathcal{S} \cap \tilde{F}$. By 3.7 they corresponds exactly to the set of points $O_{\sigma'}$ described in 3.7 , provided that $I = I'$. Otherwise there are some additional points on $D(e^{d+1})$, but in these points the total transform of g is a normal crossing divisor with zeta-function $= 1$.

Therefore, steps (1)-(4) give the inductive formula:

$$\zeta(x^\beta f)(t) = \prod \zeta(\text{total transform of } x^\beta f \text{ at } O_{\sigma'})(t),$$

where the product is over the d-cones $\sigma' \subset \Sigma \cap (\cup_{i\in I'} l^{(i)})$.

In 3.7 we verified that total transform at $O_{\sigma'}$ has the form $u^\eta \tilde{f}_\sigma$ satisfying all the inductive properties: $\tilde{f}_\sigma$ is quasi-ordinary in local coordinates (u, w) with smaller complexity than f, and $\eta \in \mathbf{Z}^d_{>0}$. Therefore Theorem B follows by induction. (After a finite number of toric modifications, all the total transforms $u^\eta \tilde{f}_\sigma$ will have the form $u^\eta w^{\eta_0}$ with zeta-function 1.)

4.3. Now we will revise Theorem A. First we recall that $\Lambda = \emptyset$ if and only if f is irreducible and smooth. For such a germ f, Theorem A is obviously true (with $\zeta(f)(t) = 1 - t$). Therefore, in the sequel, we will assume that $\Lambda \neq \emptyset$. We reorder the variables $(x_1, \ldots, x_d)$ in such a way that the first entry of $\min \Lambda$ is non-zero.

For any index set $K \subset \{1, \ldots, d\}$ we define $f|_{x_K=0}$ by $f|_{x_k=0 \text{ for all } k\in K}$.

4.4. Theorem A′. *Fix the above notations. Then for any index set* $K \subset \{1, \ldots, d\}$ *with* $1 \notin K$ *one has:* $\zeta(f) = \zeta(f|_{x_K=0})$. *In particular, for* $K = \{2, \ldots, d\}$, *one gets that* $\zeta(f) = \zeta(g)$*, where* g *is the plane curve singularity* $(x_1, x_{d+1}) \mapsto f(x_1, 0, \ldots, 0, x_{d+1})$.

In fact, if $\min \Lambda$ *has at least two non-zero entries, then* $\zeta(f)(t) = 1 - t^{\deg f}$.

Proof of Theorem A'. Obviously we can assume that f is represented in a good coordinate system (cf. 2.9). The proof is based on the following fact.

The "weak splitting property" of the zeta-function. [**McN01**] *Assume that for two germs* $f, g : (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ *and any* $k \geq 1$ *one has* $\zeta(g^k f) = 1$*. Then* $\zeta(f) = \zeta(f|_{g=0})$.

Using this and Theorem B we obtain that $\zeta(f) = \zeta(f|_{x_1 \cdots x_d = 0})$. By a Mayer–Vietoris argument

$$\zeta(f|_{x_1 \cdots x_d = 0}) = \prod_K \zeta(f|_{x_K = 0})^{(-1)^{\# K - 1}},$$

where the product is over the non-empty subsets K of $\{1, \ldots, d\}$. Hence

$$(*_1) \qquad \zeta(f) = \prod_K \zeta(f|_{x_K = 0})^{(-1)^{\# K - 1}}.$$

Notice that for any non-empty K, the restriction $f|_{x_K = 0}$ is a quasi-ordinary germ, so if $\# K \leq d - 2$ one can apply $(*_1)$ repeatedly for $f|_{x_K=0}$. Therefore, by repeated induction, one obtains that

$$(*_2) \qquad \zeta(f) = \frac{\prod_{K : \# K = d-1} \zeta(f|_{x_K = 0})}{(\zeta(f|_{x_1 = \cdots = x_d = 0}))^{d-1}}.$$

Now, if $1 \in K$, then $f|_{x_K = 0} = x_{d+1}^{\deg f}$ (cf. 2.9), hence $(*_2)$ becomes $(*_3)$ $\zeta(f) = \zeta(f|_{x_2 = \cdots = x_d = 0})$. The first statement of the theorem follows from $(*_3)$ applied for both f and $f|_{x_K=0}$. Finally, again by 2.9 , if $\min \Lambda$ has at least two non-zero entries, then $\zeta(f|_{x_2 = \cdots = x_d = 0}) = x_{d+1}^{\deg f}$, hence Theorem A' follows.

4.5. Final remarks. (1) Assume that $\min \Lambda$ has only one non-zero entry. Then the set of elements of Λ with only one non-zero entry determine completely the topological type of g. This follows from the fact that the Eggers-Wall diagram (i.e. Λ) of a plane curve singularity determines its topological type. For details, see e.g. [**GarcíaB00**]. For a formula for $\zeta(g)$ in terms of the topology of g, see [**EN85**].

(2) $\zeta(f)$ forgets almost all the information about the distinguished tuples Λ of f_{red}. This behaviour is very different compared with the case of irreducible plane curve singularities where $\zeta(f)$ is a complete topological invariant.

(3) Theorem B of the present paper (applied for an irreducible f) is slightly weaker than Theorem B of the article [**McN01**]. Nevertheless, we decided to present this very version since its proof is simpler, more conceptual, and its statement still implies Theorem A.

(The exact analog of Theorem B [**McN01**] is the following: Assume that $\beta \in \mathbf{Z}^d_{\geq 0}$ is non-zero, and $\min \Lambda + \beta$ has at least two non-zero entries. Then $\zeta(x^\beta f)(t) = 1$. Our conjecture is that this statement is true, but we did not verify all the details of its proof.)

References

[A'Campo75] A'Campo, N.: La fonction zeta d'une monodromie, *Comment. Math. Helvetici*, **50** (1975), 233-248.

[Abhyankar55] Abhyankar, S.S.: On the ramification of algebraic functions, *Amer. J. Math.*, **77**, (1955), 575-592.

[ALR89] Alonso, M.E., Luengo, I. Raimondo, M.: An Algorithm on Quasi-Ordinary Polynomials, *LNCS (Proc. AAECC-6)* **357**. Springer-Verlag, (1989), 59-73.

[BMc00] Ban, C. and McEwan, L. J.: Canonical resolution of a quasi-ordinary surface singularity, *Canadian Journal of Mathematics*, Vol. **52**(6) (2000), 1149-1163.

[BMcN00] Ban, C, McEwan, L. J. and Némethi, A.: On the Milnor fiber of a quasi-ordinary surface singularity, to appear in *Canadian Journal of Mathematics.*

[Cox00] Cox, D.: Toric Varieties and Toric Resolutions, *Resolution of Singularities.* Progress in Mathematics No. 181, Birkhäuser-Verlag, 2000, 259-283.

[Eggers83] Eggers, H.: Polarinvarianten und die Topologie von Kurvensingularitaten, *Bonner Mathematische Schriften* **147**, Universität Bonn, 1983.

[EN85] Eisenbud, D, and Neumann, W.: Three-dimensional link theory and invariants of plane curve singularities, *Ann. of Math. Studies* 110, Princeton Univ. Press (1985).

[GarcíaB00] García Barroso, E. R.: Sur les courbes polaires d'une courbe plane réduite, *Proc. London Math. Soc.*, **81**, Part 1, (2000), 1-28.

[Gau88] Gau, Y.-N.: Embedded topological classification of quasi-ordinary singularities, *Memoirs of the AMS*, **388** (1988).

[GT00] Goldin, R. and Teissier, B.: Resolving singularities of plane analytic branches with one toric morphism, *Resolution of singularities*, Progress in Mathematics No 181, Birkhäuser-Verlag 2000, 315-340.

[González00a] González Pérez, P. D.: Quasi-ordinary singularities via toric geometry, Thesis, University of La Laguna, 2000.

[González00b] González Pérez, P. D.: Singularités quasi-ordinaires toriques et polyédre de Newton du discriminant, *Canad. J. Math.* , **Vol. 52** (2), (2000), 348-368.

[González01] González Pérez, P. D.: Toric embedded resolutions of quasi-ordinary singularities, preprint 2001 June.

[GZDC99] Gusein-Zade, S. M., Delgado, F, and Campillo, A.: On the monodromy of a Plane Curve Singularity and the Poincaré Series of the Ring of Functions on the Curve, *Funct. Anal. and its Appl.*, **33** (1) (1999), 56-57.

[GLM97] Gusein-Zade, S. M., Luengo, I. and Melle- Hernández, A.: Partial resolutions and the zeta-function of a singularity, *Comment. Math. Helv.*, **72** (1997), 244-256.

[Jung08] Jung, H.W.E.: Darstellung der Funktionen eines algebraischen Körpers zweier unabhaängigen Veränderlichen x, y in der Umgebung einer stelle $x = a$, $y = b$, *J.Reine Angew. Math.*, **133** (1908), 289-314.

[KKMS73] Kempf, G., Knudsen, F., Mumford, D. and Saint-Donat, B.: *Toroidal Embeddings I*, , Lecture Notes in Math., **339**, Springer-Verlag 1973.

[Lê73] Lê Dũng Tráng: Topologie des singularités des hypersurfaces complexes, *Astérisque*, **7-8** (1973), 171-182.

[Lipman65] Lipman, J.: Quasi-ordinary singularities of embedded surfaces, *Thesis*, Harvard University (1965).

[Lipman83] Lipman, J.: Quasi-ordinary singularities of surfaces in $\mathbf{C}^3$, *Proc. of Symp. in Pure Math.*, **40**, Part 2 (1983), 161-172.

[Lipman88] Lipman, J.: Topological invariants of quasi-ordinary singularities, *Memoirs of the AMS*, **388**, 1988.

[McDonald95] McDonald, J.: Fiber Polytopes and fractional power series, *Journal of Pure and Applied Algebra.*, **104** (1995), 213-233.

[McN01] McEwan, L. J. and Némethi, A.: The zeta-function of a quasi-ordinary singularity, Ohio State Math. Research Inst. Preprints, 01-25, May 2001; to appear in *Compositio Math.*

[Merle77] Merle, M.: Invariants polaires des courbes planes, *Invent. math.*, **41** (1977), 103-111.

[Milnor68] Milnor, J.: Singular points of complex hypersurfaces, *Annals of Math. Studies*, **61**, Princeton Univ. Press, 1968.

[MO70] Milnor, J. and Orlik, P.: Isolated singularities defined by weighted homogeneous polynomials, *Topology*, **9** (1970), 385-393.

[Némethi91] Némethi, A.: The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$, *Compositio Math.*, **79** (1991), 63-97.

[Némethi93] Némethi, A.: The zeta function of singularities, *J. of Algebraic Geometry*, **2** (1993), 1-23.

[Oda88] Oda, T.: Convex Bodies and Algebraic Geometry, Ergebnisse der Mathematic und ihner Grenzgebiete 15, Springer-Verlag, 1988.

[Oh93] Oh, K.: Topological types of quasi-ordinary singularities, *Proc. Amer. Math. Soc.*, **117**, Number 1 (1993), 53-59.

[PopescuP01] Popescu-Pampu, P.: Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles, Thesis, University of Paris 7, 2001.

[Schrauwen90] Schrauwen, R.: Topological series of isolated plane curve singularities, *Eiseignement Mathématique*, **36** (1990), 115-141.

[Siersma90] Siersma, D.: The monodromy of a series of hypersurface singularities, *Comment. Math. Helvetici* **65** (1990), 181-197.

[Varchenko76] Varchenko, A. N.: Zeta function of monodromy and Newton diagram, *Invent. Math.*, **37** (1976), 253-262.

[Zariski67] Zariski, O.: Exceptional Singularities of an Algebroid Surface and their Reduction, *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, 8 **43** (1967), 135-146; reprinted in *Collected papers*, vol. 1 (1979).

[Zariski68] Zariski, O.: Studies in equisingularity, III. Saturation of local rings and equisingularity, *Amer. J. Math.*, **90** (1968), 961-1023; reprinted in *Collected papers*, vol. 4, 96-158.

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