Geometry and Physics of the Elementary Fermions. 1
(On pride of Jordan Wigner Pauli Weyl Dirac).

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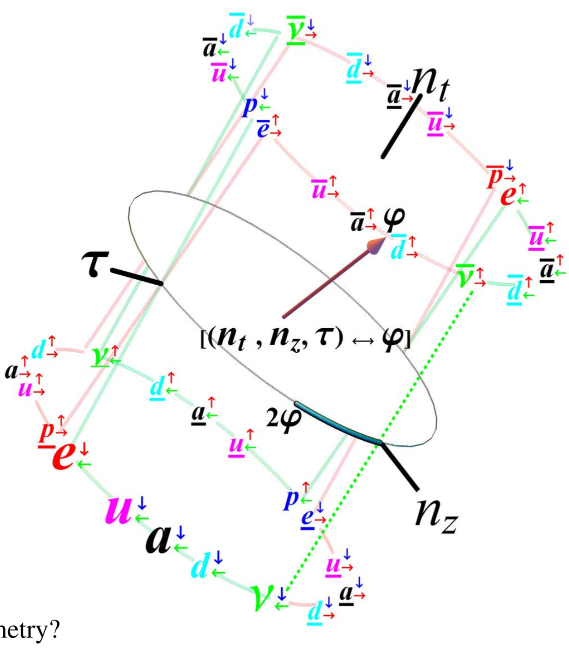
We develop a general formalism for defining distinct creation and annihilation operators for every elementary fermion (leptons and quarks). Spin, vector-spin, chirality and electric charge, are intrinsic to them. Specific values of a discrete angle variable provide the electric charges and the vectors-spin.

The above mentioned formalism consists in a geometrical generalization, using algebraic methods, of the algebraic formalism established by Jordan and Wigner in 1928. The method proposed introduces a second numbering and a product with an intermediary term. Arguments of symmetry are underneath this construction. In this way, the elementary fermions could be viewed as geometrical structures. These contents are a tentative first approach.

Keywords: Elementary fermions (leptons, quarks). Jordan Wigner transformation. Geometry - Algebra.

CONTENTS



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PART I

ALGEBRA.

I. INTRODUCTION.

> “ … . This invention (by Jordan and Wigner [2]), is very useful although its physical meaning appears to be obscure: the sign of an expression in the amplitudes becomes dependent upon the normal modes. ”
>
> Pauli, “Selected Topics in Field Quantization” (page 7). [1]

Let us schematically consider several historical events:

- 1843. Hamilton. The quaternion. “*VIII ON QUATERNIONS; OR ON A NEW SYSTEM OF* **IMAGINARIES** *IN ALGEBRA*”. Hamilton obtained them in the study of the arithmetic of the triples and quadruples. They appeared as deeply involved with the geometry of the **rotations** [2]

- 1897. The electron: an electrically **charge**d particle (J. J. Thomson).

- 1897. The anomalous Zeeman effect. A splitting in certain atomic spectral lines.

- 1922. Stern and Gerlach. Some atoms going through an inhomogeneous magnetic field produced a double result.

- 1925. Kronig, Uhlenbeck and Goudsmit proposed for the electron a property of “**spin**ning” for explaining this behavior. *Dynamical rotations in an* **internal space**, versus rotations as a coordinate transformation. [3]

- 1927. Pauli [4] matrices $\{\sigma^x, \sigma^y, \sigma^z\}$. Jordan reminds to Pauli a “connection with quaternions: $s_x = \mathrm{i}k_1$, $s_y = \mathrm{i}k_2$, $s_z = \mathrm{i}k_3$”.

- 1928. Jordan and Wigner. [5] [6] Their “non-local algebraic transformation”. There exist antisymmetrical operators satisfying anticommutators relations that justify the Pauli exclusion principle. Generic **creation and annihilation operators** independent of the time. They used the Pauli spin matrices. The spin of a particle was not defined in this step.

- 1928. Dirac. [7] An equation with four solutions. Two of them representing spinning electrons. Later on, the other two to represent spinning positrons. We can see the integration *ad hoc* of the creation and annihilation operators for electrons and positrons with spin in the solutions of the equation [8]. The spin incorporated in the creation and annihilation operators without the time and an added four-“spinor” structure in the form of the column solutions. **Chirality**.

- 1928. Heisenberg: the ferromagnetism [9]. He established a relation with quantum mechanics, using a chain of spins, a one dimensional model: the Heisenberg XYZ model. $\{\sigma^x, \sigma^y, \sigma^z\}$ used in an algebraic way [10].

- 1965. Lee, Yang and Madame Wu. Weak interaction: rupture of the parity symmetry. Chirality. **Geometry!**

The solution provided by Kronig, Uhlenbeck, Goudsmit and the development by Pauli of the electrons without structure but at once with (dynamical) **internal** rotations (spinnings) solved the problems arisen by the anomalous Zeeman effect and the Stern and Gerlach experiment. The solution: the spin.

The concept underneath: an added magnetic moment to the electron (in the Faraday's conception) due to a half unit of angular momentum. The angular momentum concept implies, in mechanics, the rotations. The physicists at that time were conscious of the problems raised with this interpretation [11], [12] and [13]. A spinning would oblige a velocity of rotation and an axis of rotation of the electrically charged surface of an electron, as a very small sphere, both prohibited by the relativity theory and the quantum mechanics, respectively. Also, a spinning electron in a state needs of a rotation with a 4π angle to return to that state, with the appearance of a minus sign for a 2π angle [14].

Gerard 't Hooft describes in a crystal-clear way the situation: “Nowadays such objections are simply ignored” [13]. We have a fixed spin but we do not have an axis of rotation for the electron. In popular terms: the electron ‘rotates in an internal (unknown) space‘, curiously, without geometry. We rest upon the matrix theory, the algebra.

On the other hand, Jordan and Wigner defined a transformation using the Pauli matrices and another two matrices, σ^+ and σ^-. The Pauli matrices can represent the vectors of rotation, belonging to $\mathbb{R}^3$, but σ^+ and σ^- can not belong to this $\mathbb{R}^3$. Do they belong to some kind of space? So, as above, we ignore the geometry and we use these matrices as an algebraic instrument. At most, with Gerard 't Hooft we admit the axes of rotation for massless particles in the same direction of their movements [13]. Would this privilege a **local** direction of space? The helicity underneath, particularly with the neutrinos.

Even more, does the chirality select a **local**ly privileged reference frame, with its "right and left handed distinction"?

Here, **local** means: at the level of every isolated elementary fermion. We discuss these concepts in [15].

In this study (**1**) we propose definitions for the creation andthe annihilation operators based in directions in the time space. Previous paragraphs advice us of the difficulties with the geometry (more on this in **2**). Therefore, following the above written historical steps, we consider our generalization of the Jordan and Wigner transformation as an algebraic method, with a guidance in the geometry in an **internal space**. In this sense we will consider discrete values of an angle type variable and related to them discrete values of the time and the space coordinates. This has profound implications in our conception of the time space, including a generalization of the Minkowski metric. [15]

We define distinct specific creation and annihilation operators for every one of the elementary fermions in the Standard Model, with their corresponding anticommutators. For their achievement we define an angle type parameter, named as φ. φ acquires specific values in a discrete set, corresponding a family of fermions to every one of these values. We add a second element: we define a product containing an intermediary term. [16]

The Jordan and Wigner method imposes a numbering (an ordering), $m \in \{2, \ldots, N\}$. This is the reason for the sentences written by Pauli in the starting quoted paragraph. The procedure developed here imposes a second numbering, $\lambda r \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$, and associated to this numbering a certain intermediary product. A consequence of this product is the antisymmetry of our operators and so forth anticommutator relations for them.

The pathway from the Jordan and Wigner method to the method in this study is as follows: look at equations (2.7) and substitute $(\pm\sigma^z) \in \mathbb{R}^3$ by a generic matrix (vector) $(\pm \mathbb{n}) \in \mathbb{R}^3$ written in the form $(\pm \mathbb{n}) = \mathbf{R}_O^{\varphi} \, (\pm\sigma^z) \, \mathbf{R}_O^{-\varphi}$ (see Appendix A) for the 1 to the $m-1$ sites. Apply the same $\mathbf{R}_O^{\pm\varphi}$ matrices to the left and to the right of σ^+ and of σ^- at the m site, multiplied by a certain factor. Clearly these last matrices-vectors do not belong to $\mathbb{R}^3$. **Do they belong to any space?** We introduce some arguments of symmetry with the angle type variable (φ), also related to some type of time coordinate and to some type of space coordinates of an axial vector, an imaginary type vector in the sense of Hamilton. At this point we can forget about the geometry presented in Section III, though in a subtle way it is persistent in the intermediary product, and we get our objective: the creation - annihilation operators and their anticommutators. The creation and annihilation operators for fermions, defined in this way, could be suitable for a quantum field theory, with the newly defined intermediary anticommutators.

Therefore, either as a result of this construction, or starting directly with the algebraic definitions in (4.1), we have creation and annihilation operators for every fermion, understanding with this that every one of them has a definite:

electric charge	(a value: negative - 0 - positive),
spin	(an eigenvalue: $+1$ - -1, or up - down),
vector-spin	(characteristic three dimensional vectors, with 4 different values) and
chirality	(left - right),

driving to 2 x 2 x 4 / 2 = 8 possibilities per family. At some points of the study we will show in what way we restrict these possibilities.

It is remarkable that we set the values of the electric charges in relation to the angle parameter φ, up to a global sign per family, which we fix by looking at the phenomenology of the elementary fermions. Also, the vector-spin depends of the same specific angle parameter 2φ.

The symmetries in the time and the space appear directly related with the physical magnitudes: electric charge, spin, chirality and vector-spin. Not so straightly they can be related to the electro-magneto-weak interatcions.

We suggest some previous readings:

for a historical and critical presentation,	Weinberg [17],
for the construction,	Sakurai [8],
for the chirality, the spinor vectors **u** and **v**,	Griffiths [18],
complementary contextual material with	Wilczek [19] [20].
and a pictorial presentation,	Flip Tanedo [21].

In this study the mathematics is 'simple and repetitive', the physics is 'complex'.

We end up remarking again about the research in the present **1** as an algebraic method, following the ideas developed in the Jordan and Wigner transformation and leaving aside, with the inclusion of an internal space, the geometry.

The geometry now includes four dimensional directions, in a four dimensional complex space? See [15] and **2** (to appear).

II. NOTATION.

The Pauli spin matrices σ^j, with the identity, constitute a basis for the linear space of $2x2$ matrices over the complex field, the **Pauli basis**. It has the form:

$$\left\{ \; \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; \right\} . \tag{2.1}$$

After doing a simple change: $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm \mathrm{i}\,\sigma^y)$, $\overset{\wedge}{\sigma} = \frac{1}{2}(\mathbb{1} + \sigma^z)$, $\overset{\vee}{\sigma} = \frac{1}{2}(\mathbb{1} - \sigma^z)$, we write a new set of $2x2$ matrices, the **canonical set** :

$$\left\{ \; \overset{\wedge}{\sigma} = \sigma^+\sigma^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \overset{\vee}{\sigma} = \sigma^-\sigma^+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; \right\} . \tag{2.2}$$

The matrices $\mathbf{R}_o^{\varphi}$ and $\mathbf{R}_e^{\varphi}$ will be very relevant:

$$\begin{aligned} \mathbf{R}_o^{\varphi}(\phi) &\equiv \begin{pmatrix} \cos\varphi & -\sin\varphi\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\varphi\, \mathrm{e}^{\mathrm{i}\phi} & \cos\varphi \end{pmatrix} = \cos\varphi \;\; \mathbb{1} \; + \; \sin\varphi\, \mathbf{R}_o^{\frac{\pi}{2}} \\ \mathbf{R}_e^{\varphi}(\phi) &\equiv \begin{pmatrix} \cos\varphi & \sin\varphi\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\varphi\, \mathrm{e}^{\mathrm{i}\phi} & -\cos\varphi \end{pmatrix} = \cos\varphi\; \sigma^z + \; \sin\varphi\, \mathbf{R}_e^{\frac{\pi}{2}} \end{aligned} \quad , \tag{2.3}$$

where $\varphi \in [-\pi\,,\,\pi)$ and $\phi \in [\,0\,,\,\pi\,)$. We define these two variables in Section III. The Appendix A contains formulas showing useful relations for the matrices $\mathbf{R}_o^{\varphi}(\phi)$ and $\mathbf{R}_e^{\varphi}(\phi)$. We also define:

$$\mathrm{m}^+ \equiv \mathrm{e}^{\,\mathrm{i}\,(\phi - \frac{\pi}{2})} = -\mathrm{i}\,\mathrm{e}^{\,\mathrm{i}\,\phi} \qquad \text{and} \qquad \mathrm{m}^- \equiv \mathrm{e}^{-\mathrm{i}\,(\phi - \frac{\pi}{2})} = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\phi} = \overline{\mathrm{m}^+} = \mathrm{m}^{+\,-1} . \tag{2.4}$$

Using: $\mathbf{R}_o^{\epsilon\frac{\pi}{2}}\, \sigma^{\pm}\, \mathbf{R}_o^{-\epsilon\frac{\pi}{2}} = -\,\mathrm{e}^{\pm\,2\,\mathrm{i}\,\phi}\, \sigma^{\mp} = \mathrm{m}^{\pm 2}\sigma^{\mp}$, with $\epsilon \in \{+,-\}$, we have:

$$\mathbf{R}_o^{\epsilon\frac{\pi}{2}}(\,\mathrm{m}^{\mp}\,\sigma^{\pm})\,\mathbf{R}_o^{-\epsilon\frac{\pi}{2}} = \mathrm{m}^{\pm}\,\sigma^{\mp} . \tag{2.5}$$

Consider the direct product, or Kronecker product, of N of these linear spaces of 2x2 complex matrices, and denote in this space the following matrices:

$$\begin{array}{llllllllllllll} \mathbf{1} & \equiv \;\; \mathbb{1}_N & = & \mathbb{1} & \otimes & \ldots & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes \; \ldots \; \otimes & \mathbb{1} \\ \lambda\,\Sigma_m & = \;\big(\lambda\,\Sigma\,\big)_m & \equiv & \mathbb{1} & \otimes & \ldots & \otimes & \mathbb{1} & \otimes & \lambda\Sigma & \otimes & \mathbb{1} & \otimes \; \ldots \; \otimes & \mathbb{1} \\ \mathbf{A}_{M-1} & \equiv \;\prod_{k=1}^{m-1} \mathbf{A}_k & = & \mathbf{A} & \otimes & \ldots & \otimes & \mathbf{A} & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes \; \ldots \; \otimes & \mathbb{1} \\ \mathbf{U}_{\mathcal{M}} & \equiv \mathbf{A}_{M-1}\Sigma_m & = & \mathbf{A} & \otimes & \ldots & \otimes & \mathbf{A} & \otimes & \Sigma & \otimes & \mathbb{1} & \otimes \; \ldots \; \otimes & \mathbb{1} \\ & & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\ & & & 1 & & & & m-1 & & m & & & & N \end{array} ,$$

with $2 \le m \le N$. If we put for Σ, σ^j with $j = \{\,x,\; y,\; z\,\}$, we take them as Pauli matrices in the m site of a linear time space chain with N sites. Similarly for the matrices in the canonical set, definition (2.2) with $j = \{+,\,-,\,\wedge,\,\vee\}$, or more general matrices. The matrix $\mathbf{A}$ for $\mathbf{R}_e^{2\varphi}$ or $\mathbf{R}_{\{e,\,o\}}^{\varphi}$, with capital sub-index $M-1$ to denote the same non identity matrix for all the sites, from the 1 to the $m-1$ sites. Mathcal sub-index $\mathcal{M}$ as indicated above. We also verify:

$$\begin{aligned} \big(-\sigma^z\,\big)_{M-1} &\equiv \prod_{k=1}^{m-1}(\,-\sigma^z\,)_k = (-1)^{m-1}\prod_{k=1}^{m-1}\sigma^z_k \;=\; (-1)^{m-1}\,\sigma^z_{M-1} \\ \sigma^z_{M-1} \equiv \big(-\,\sigma^{z\,\prime}\,\big)_{M-1} &\equiv \prod_{k=1}^{m-1}(-\sigma^{z\,\prime}\,)_k = (-1)^{m-1}\prod_{k=1}^{m-1}\sigma^{z\,\prime}_k \;=\; (-1)^{m-1}\,\sigma^{z\,\prime}_{M-1} \end{aligned} \quad . \tag{2.6}$$

Algebraic relationships. The interpretation of these equations is complex and will be considered apart.

The Pauli spin matrices satisfy:

$$\sigma^j_m\;\sigma^k_m = \delta^{jk}\,\mathbb{1}_N + \mathrm{i}\,\epsilon^{jkl}\sigma^l_m\,, \qquad \text{so that :} \qquad \begin{cases} [\,\sigma^j_{m_1}\,,\sigma^k_{m_2}\,] = \mathrm{i}\,2\epsilon^{jkl}\,\delta_{m_1m_2}\,\sigma^l_{m_1} \\ \{\,\sigma^j_m\,,\,\sigma^k_m\,\} = \;2\,\delta^{jk}\;\,\mathbb{1}_N \end{cases} ,$$

with $\{j,k,l\} = \{x,y,z\}$, the commutator $[\,A\;,\;B\,] \equiv A\;B - B\;A$ and the anticommutator $\{A\;,\;B\} \equiv A\;B + B\;A$.

In order to obtain operators obeying the Pauli's exclusion principle, we start transforming the spin matrices σ_m^+ and σ_m^- in the fermion creation and annihilation operators $a_{\mathcal{M}}^\dagger$ and $a_{\mathcal{M}}$ (without spin), by means of a **Jordan-Wigner transformation**, in the following way:

$$\begin{aligned} \mathbf{a}_{\mathcal{M}}^\dagger &\equiv \mathbf{V}_{M-1}^{2}\ \sigma_m^+ \equiv \sigma_m^+\ \mathbf{V}_{M-1}^{2} \\ \mathbf{a}_{\mathcal{M}} &\equiv \mathbf{V}_{M-1}^{2}\ \sigma_m^- \equiv \sigma_m^-\ \mathbf{V}_{M-1}^{2} \end{aligned} \qquad \text{with} \qquad \mathbf{V}_{M-1}^{\pm 2} \equiv (\mp\sigma^z)_{M-1} = \prod_{k=1}^{m-1} \left(\mathrm{e}^{\epsilon \mathrm{i}\frac{\pi}{2}(\sigma^z \pm \mathbb{1})}\right)_k , \tag{2.7}$$

$\mathbf{V}_{M-1}^{2}$ either $\mathbf{V}_{M-1}^{+2}$ or $\mathbf{V}_{M-1}^{-2}$ and $\epsilon \in \{+1,-1\}$. Also, it is $\mathbf{V}_{M-1}^{\pm 4} = \prod_{k=1}^{m-1} (\pm\sigma^z)_k^2 = \mathbb{1}_N$, and $\epsilon\frac{1}{2}(\sigma^z \pm \mathbb{1}) \in \{\pm\hat{\sigma}, \pm\check{\sigma}\}$ related to the number operators.

The essence of the method, the algebraic root for the transformation, consists in:

$$\sigma_m^z\ \sigma_m^\pm\ \sigma_m^z = -\sigma_m^\pm \qquad \text{or} \qquad \sigma_m^z\ \sigma_m^\pm = -\,\sigma_m^\pm\ \sigma_m^z = \pm\,\sigma_m^\pm\ . \tag{2.8}$$

With these elements we write the following commutation relations and the desired anticommutation relations:

$$\begin{array}{ccc} SPIN & & FERMION \\ \left.\begin{aligned} [\sigma_{m_1}^+, \sigma_{m_2}^+] = [\sigma_{m_1}^-, \sigma_{m_2}^-] &= \mathbb{0}_N \\ [\sigma_{m_1}^+, \sigma_{m_2}^-] &= \delta_{m_1 m_2}\, \sigma_{m_1}^z \\ \{\sigma_m^+, \sigma_m^-\} &= \mathbb{1}_N \end{aligned}\right\} & \overset{J.W.}{\longleftrightarrow} & \left\{\begin{aligned} \{\mathbf{a}_{\mathcal{M}_1}^\dagger, \mathbf{a}_{\mathcal{M}_2}^\dagger\} = \{\mathbf{a}_{\mathcal{M}_1}, \mathbf{a}_{\mathcal{M}_2}\} &= \mathbf{0} \\ \{\mathbf{a}_{\mathcal{M}_1}^\dagger, \mathbf{a}_{\mathcal{M}_2}\} &= \delta_{m_1 m_2}\ \mathbf{1} \end{aligned}\right. \end{array} . \tag{2.9}$$

The first line, with $m_1 = m_2 = m$, can be written in either way, as commutators or as anticommutators, as it is:

$$\sigma_m^{+\,2} = \sigma_m^{-\,2} = \begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix}_N = \mathbb{0}_N\ ; \qquad\qquad \mathbf{a}_{\mathcal{M}}^{\dagger\,2} = \mathbf{a}_{\mathcal{M}}^{2} = \mathbf{0}\ . \tag{2.10}$$

We have established the existence of operators satisfying anticommuting relationships. We have operators with commuting relationships, the boson operators, symmetric with a + sign in a product (an ordering), and others, the fermion operators, anti-symmetric with a − sign in a product (other ordering). See Weinberg [17].

In brief, with the method of Jordan and Wigner we construct antisymmetric operators which justify the exclusion principle. After it, in the pathway of the second quantization method, we ignore the "numbering", we incorporate a momentum and we introduce *ad hoc* the spin but not the chirality for the creation and annihilation operators for electrons and positrons with the spin up and down. The particle-antiparticle, spin up-down and chirality right-left concepts form part of the covariant wave Dirac equation.

The corresponding anticommutation relations for the creation and annihilation operators of the charged leptons are:

$$\{\mathbf{b}_{\mathbf{p}}^{s\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{s'\,\dagger}\} = \{\mathbf{b}_{\mathbf{p}}^{s\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{s'}\} = \{\mathbf{b}_{\mathbf{p}}^{s}, \mathbf{d}_{\mathbf{p}'}^{s'\,\dagger}\} = \{\mathbf{b}_{\mathbf{p}}^{s}, \mathbf{d}_{\mathbf{p}'}^{s'}\} = \mathbf{0} \tag{2.11a}$$

$$\{\mathbf{b}_{\mathbf{p}}^{s\,\dagger}, \mathbf{b}_{\mathbf{p}'}^{s'\,\dagger}\} = \{\mathbf{b}_{\mathbf{p}}^{s}, \mathbf{b}_{\mathbf{p}'}^{s'}\} = \{\mathbf{d}_{\mathbf{p}}^{s\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{s'\,\dagger}\} = \{\mathbf{d}_{\mathbf{p}}^{s}, \mathbf{d}_{\mathbf{p}'}^{s'}\} = \mathbf{0} \tag{2.11b}$$

$$\left.\begin{aligned} \{\mathbf{b}_{\mathbf{p}}^{\uparrow\,\dagger}, \mathbf{b}_{\mathbf{p}'}^{\downarrow}\} = \{\mathbf{b}_{\mathbf{p}}^{\downarrow\,\dagger}, \mathbf{b}_{\mathbf{p}'}^{\uparrow}\} &= \{\mathbf{d}_{\mathbf{p}}^{\uparrow\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{\downarrow}\} = \{\mathbf{d}_{\mathbf{p}}^{\downarrow\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{\uparrow}\} = \mathbf{0} \\ \{\mathbf{b}_{\mathbf{p}}^{\uparrow\,\dagger}, \mathbf{b}_{\mathbf{p}'}^{\uparrow}\} = \{\mathbf{b}_{\mathbf{p}}^{\downarrow\,\dagger}, \mathbf{b}_{\mathbf{p}'}^{\downarrow}\} &= \{\mathbf{d}_{\mathbf{p}}^{\uparrow\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{\uparrow}\} = \{\mathbf{d}_{\mathbf{p}}^{\downarrow\,\dagger}, \mathbf{d}_{\mathbf{p}'}^{\downarrow}\} = \delta_{\mathbf{p}\mathbf{p}'}\ \mathbf{1} \end{aligned}\right\} \tag{2.11c}$$

These operators in:

$$\psi(\mathbf{x}, t) \sim \sum_{\mathbf{p}} \left\{ \mathbf{b}_{\mathbf{p}}^{\uparrow}\mathbf{u}^{\uparrow}(\mathbf{p})e^{-\mathrm{i}f_-} + \mathbf{b}_{\mathbf{p}}^{\downarrow}\mathbf{u}^{\downarrow}(\mathbf{p})e^{-\mathrm{i}f_-} + \mathbf{d}_{\mathbf{p}}^{\uparrow\,\dagger}\mathbf{v}^{\uparrow}(\mathbf{p})e^{\mathrm{i}f_-} + \mathbf{d}_{\mathbf{p}}^{\downarrow\,\dagger}\mathbf{v}^{\downarrow}(\mathbf{p})e^{\mathrm{i}f_-} \right\} , \tag{2.12}$$

with $f_- \equiv \frac{1}{\hbar}(E\,t - \mathbf{p}\,.\,\mathbf{x})$, $\mathbf{b}^\dagger$ and $\mathbf{b}$ for electrons, $\mathbf{d}^\dagger$ and $\mathbf{d}$ for positrons, $\mathbf{p}$ and $\mathbf{p}'$ for momentum and $\{s, s'\}$ for spin up $\uparrow$ or down $\downarrow$. The chirality with the spinor vectors $\mathbf{u}$ and $\mathbf{v}$. We have written these equations in this way to emphasize the role of the spin, in order to compare them later on with the ones obtained here. The usual convention in most textbooks is: $\mathbf{b}^\dagger$ with σ^-; in this way the number operators are ordered in the forms $N_{\mathbf{b}} \in \{0, 1\}$ (see Pauli in the Appendix D: cites). The convention used in this study is the opposite (see (2.7) and Bjorken [22]).

$\mathcal{A}$) GENERIC FERMIONS.

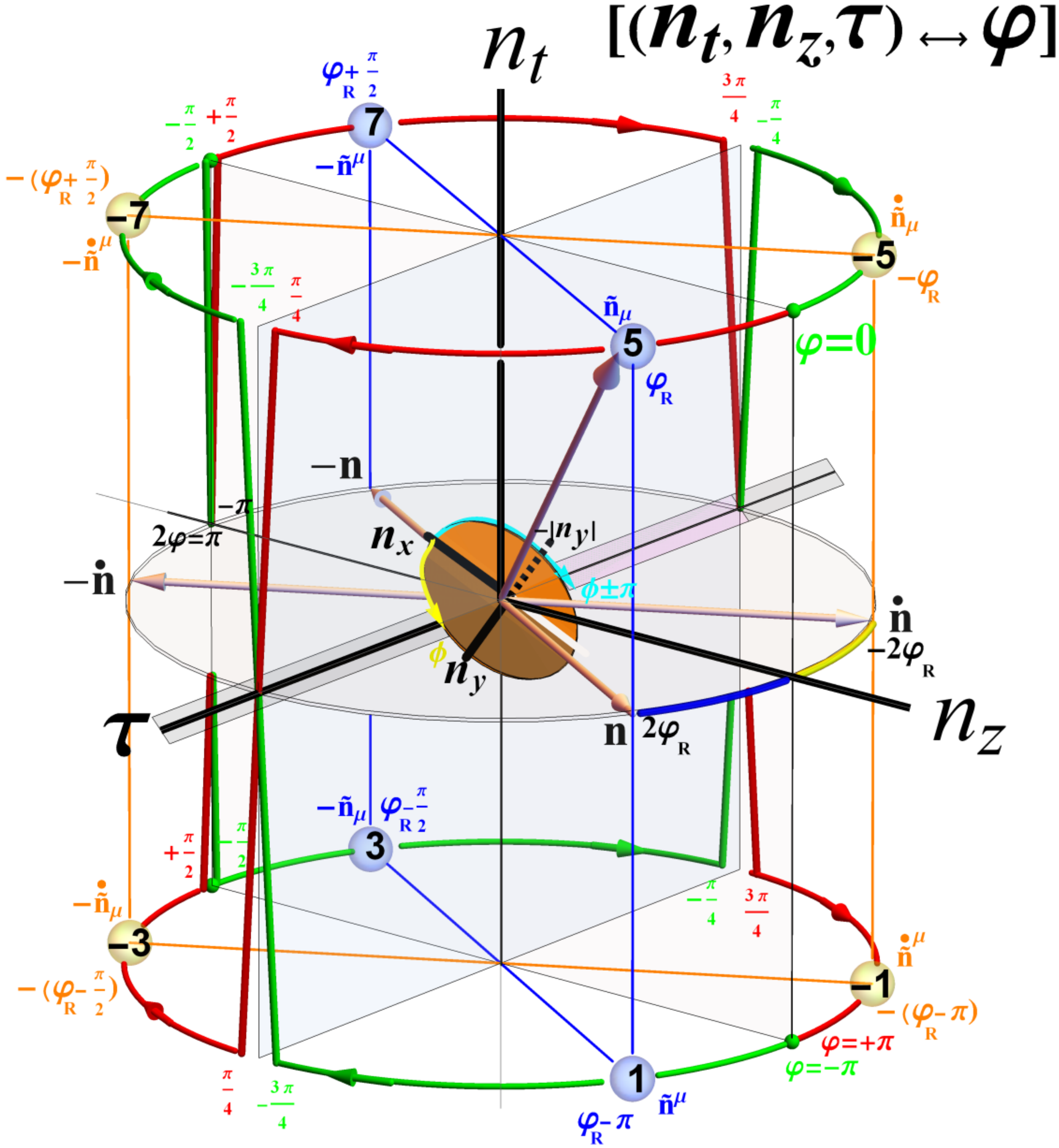
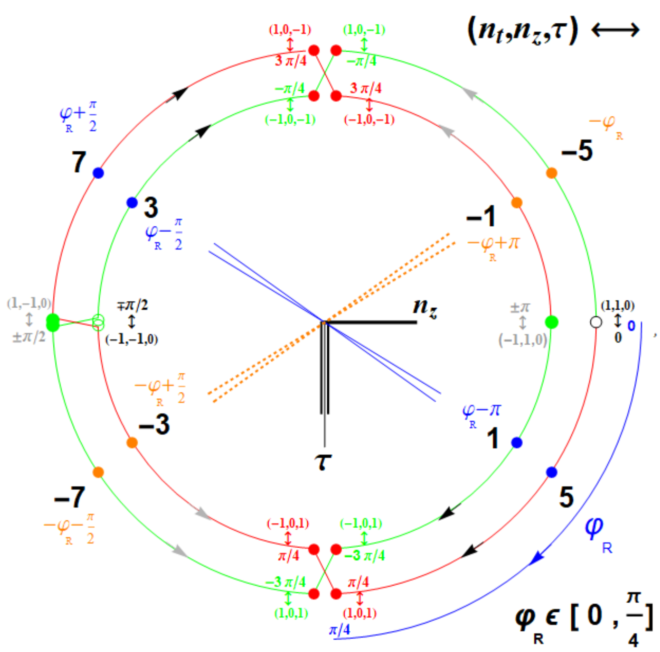


FIG. 1: (1+1+ "2")d representation in time space, in relation to characteristic vectors.

In the Figure $n_t > 0$, $n_z > 0$, $n_x > 0$, $n_y > 0$ and $\tau > 0$. φ is a label for a time space angle. The three dimensional angle is 2φ. The arrows in red and green indicate the increasing value of φ.



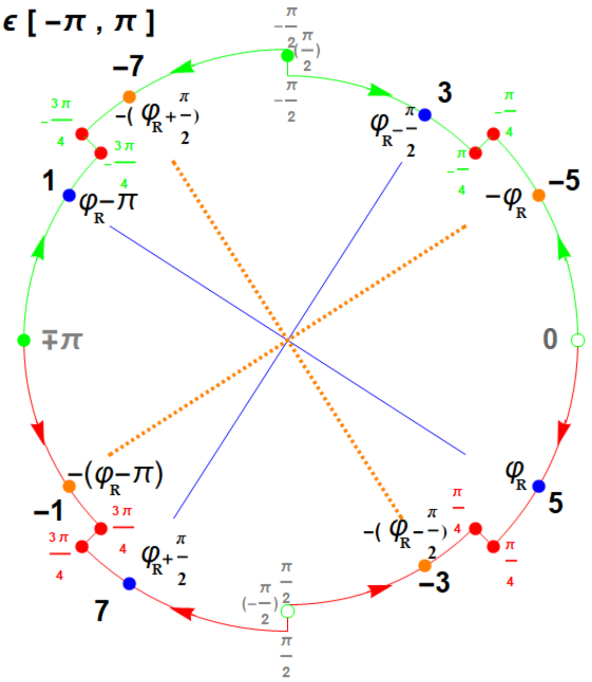


FIG. 2: $(n_t,\ n_z,\ \tau)(\varphi)$. FIG. 3: $\varphi\ \ ((n_t,\ n_z,\ \tau))$.

The arrows indicate the way for the variation of the values of φ as φ_R increases from 0 to $\frac{\pi}{4}$.
The pieces of arc of the interior circumferences for the negative time type coordinate, the exterior ones for the positive ones.

III. GENERALIZED SPIN MATRICES: TIME SPACE SPIN MATRICES. GEOMETRY.

A first approach to the contents of this section in [16].

We assign a triad (n_x, n_y, n_z) of real numbers to a unit vector $\mathbf{n} \in \mathbb{R}^3$. The unit vectors $\mathbf{n}$ satisfy $r^2 \equiv n_z^2+(n_x^2+n_y^2) \equiv n_z^2+\tau^2=1$, which we relate with the definition of angle type parameters, different to the usual ones, in the following way:

$$\begin{cases} n_z = \cos 2\varphi \\ \tau = \sin 2\varphi \end{cases} (r=1)\,, \qquad \begin{matrix} \mathbf{n} = \mathbf{n}_z + \boldsymbol{\tau} \\ 2\varphi \in [-\pi, \pi)\,, \\ \tau \in [-1,1] \end{matrix} \qquad \begin{cases} n_x = \tau \cos\phi \\ n_y = \tau \sin\phi \end{cases}, \qquad \begin{matrix} n_x + \mathrm{i} n_y = \tau\, \mathrm{e}^{\,\mathrm{i}\phi} \\ \phi \in [\,0,\ \pi) \quad (\tau \neq 0) \end{matrix} \,. \tag{3.1}$$

With Pauli [4] and Cartan [23] we associate a matrix $\mathbb{n}$ to the unit vectors $\mathbf{n}$ and we also denote it as "unit vector $\mathbb{n}$": $\mathbb{n} \equiv n_z\,\sigma^z + n_x\,\sigma^x + n_y\,\sigma^y = n_z\,\sigma^z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}} = \mathbf{R}_e^{2\varphi}(\phi) \equiv \mathbb{n}(2\varphi\,,\,\phi)$. Explicitly it is:

$$\mathbb{n} = \begin{pmatrix} n_z & n_x - \mathrm{i} n_y \\ n_x + \mathrm{i} n_y & -n_z \end{pmatrix} = \mathbf{R}_e^{2\varphi}(\phi) = \mathbf{R}_o^{\varphi}(\phi)\ \sigma^z\ \mathbf{R}_o^{-\varphi}(\phi) = \mathbf{R}_o^{\varphi}(\phi)\ \mathrm{e}^{\pm\,\mathrm{i}\,\frac{\pi}{2}\,[-\mathbb{1}+\sigma^z]}\ \mathbf{R}_o^{-\varphi}(\phi)\,. \tag{3.2}$$

See Appendix A.

It is $\mathbb{n}(\phi\pm\pi,\varphi) = \mathbb{n}(\phi,-\varphi)$, so that, if $\mathbb{n}(n_z,n_x,n_y) = \mathbb{n}(n_z,\tau,\phi) = \mathbb{n}(2\varphi,\phi)$ then $\mathbb{n}(n_z,-n_x,-n_y) = \mathbb{n}(n_z,-\tau,\phi) = \mathbb{n}(-2\varphi,\phi)$.

For $n_y \in [-|\tau|, |\tau|]$ it is: $\mathrm{sign}(\tau) \equiv \mathrm{sign}(n_y)$. With this parametrization we have: $\varphi \in \{0, \pm\frac{\pi}{2}, \pm\pi\} \Rightarrow \{\tau = 0\,,\ \forall\phi\} \Rightarrow n_x = n_y = 0$. Also: $\tau = \tau(n_z) = \tau(n_z(2\varphi))$. In general we ignore the angle ϕ, it remains as a hidden angle variable.

Now, we add a time variable, and we change the domain of φ to $\varphi \in [-\pi,\pi)$. Due to various symmetries that will be relevant later, we state: an angle $\varphi_R \in [0, \frac{\pi}{4}]$ as a base for oppositions in the three dimensional space or to opposition in the added time dimension, and with $\lambda \in \{+,-\}$ related to the opposite signs of τ (n_y) and also in relation to opposite values of φ. With this, we write the angle $\varphi \in [-\pi,\pi)$ in the form: $\varphi = \{\, \lambda\varphi_R$ or $;\ \lambda(\varphi_R \pm \frac{\pi}{2})$ or $\lambda(\varphi_R + \pi)$ or $\lambda(\varphi_R - \pi)\,\}$. Finally, $\epsilon = \{+,-\}$, although it is not relevant in the final algebraic results. Limiting values have a special treatments. See Figures 1,2,3. These specifications and the relationship $\mathbb{n} = \mathbf{R}_e^{2\varphi} = \mathbf{R}_o^{\varphi}\,\sigma^z\,\mathbf{R}_o^{-\varphi}$ suggest to write the vectors $\mathbb{n}$ in an arbitrary direction $\mathbf{n}$, and referred to a position m in the chain, as *space-spin matrices*, using the following notation:

$$\mathbb{n} \longrightarrow \begin{cases} s_m^{z\ \lambda\varphi_R} = s_m^{z\ \lambda(\varphi_R - \pi)} = s_m^{z\ \lambda\varphi_R\,\dagger} = \left(\sigma^z s_m^{z\ -\lambda\varphi_R}\,\sigma^z\right)_m = \left(\mathbf{R}_o^{\lambda\varphi_R}\,\sigma^z\,\mathbf{R}_o^{-\lambda\varphi_R}\right)_m = \mathbf{R}_{e_m}^{\lambda 2\varphi_R} \\ s_m^{z\ \lambda(\varphi_R + \epsilon\frac{\pi}{2})} = \mathbf{R}_{e_m}^{\lambda 2(\varphi_R + \epsilon\frac{\pi}{2})} = \left(-\,s^{z\ \lambda\varphi_R}\right)_m = \left(\mathbf{R}_o^{\lambda\varphi_R}\,(-\sigma^z)\,\mathbf{R}_o^{-\lambda\varphi_R}\right)_m = \left(-\,\mathbf{R}_e^{\lambda 2\varphi_R}\right)_m \end{cases}, \tag{3.3}$$

which drives, for the value $\varphi_R = 0$, to the direction z in the three dimensional space with $\pm\sigma^z$.

We also define a generalization for the matrices $\sigma^{\pm}$:

$$\begin{cases} s_m^{\pm\,\lambda\varphi_R} = s_m^{\pm\,\lambda(\varphi_R-\pi)} = s_m^{\mp\,\lambda\varphi_R\,\dagger} = \left(-\sigma^z\, s^{\pm\,-\lambda\varphi_R}\,\sigma^z\right)_m = \left(\mathbf{R}_O^{\lambda\varphi_R}\,\sigma^{\pm}\,\mathbf{R}_O^{-\lambda\varphi_R}\right)_m \\ s_m^{\pm\,\lambda(\varphi_R+\epsilon\frac{\pi}{2})} = \left(\mathfrak{m}^{\pm 2}\,\mathbf{R}_O^{\lambda\varphi_R}\,\sigma^{\mp}\,\mathbf{R}_O^{-\lambda\varphi_R}\right)_m = \left(\mathbf{R}_O^{\lambda\varphi_R}\,\mathbf{R}_O^{\epsilon\frac{\pi}{2}}\,\sigma^{\pm}\,\mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\mathbf{R}_O^{-\lambda\varphi_R}\right)_m \end{cases}, \tag{3.4}$$

The appearance of the factors $\mathfrak{m}^{\pm 2}$ in the previous equation and of $\mathrm{e}^{\pm\,\mathrm{i}\,\frac{\pi}{2}\,\mathbb{1}}$ in equation (3.2) with the $\mathbb{1}$ interpreted as a time dimension, suggests us the definition of new spin type matrices which we denote *time space spin matrices*:

$$\begin{array}{ll} \widetilde{s}^{\,z}_{\lambda 5,m} \equiv s_m^{z\,\lambda\varphi_R}, & \widetilde{s}^{\,\pm}_{\lambda 5,m} \equiv \left(\ \mathfrak{m}^{\mp}\, s^{\pm\,\lambda\varphi_R}\ \right)_m \\ \widetilde{s}^{\,z}_{\lambda 3,m} \equiv s_m^{z\,\lambda(\varphi_R+\epsilon\frac{\pi}{2})}, & \widetilde{s}^{\,\mp}_{\lambda 3,m} \equiv \left(-\,\mathfrak{m}^{\pm}\, s^{\mp\,\lambda\varphi_R}\ \right)_m \\ \widetilde{s}^{\,z}_{\lambda 1,m} \equiv s_m^{z\,\lambda\varphi_R}, & \widetilde{s}^{\,\mp}_{\lambda 1,m} \equiv \left(-\,\mathfrak{m}^{\pm}\, s^{\mp\,\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\right)_m \\ \widetilde{s}^{\,z}_{\lambda 7,m} \equiv s_m^{z\,\lambda(\varphi_R+\epsilon\frac{\pi}{2})}, & \widetilde{s}^{\,\pm}_{\lambda 7,m} \equiv \left(\ \mathfrak{m}^{\mp}\, s^{\pm\,\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\right)_m \end{array}. \tag{3.5}$$

Where we have added a second sub-index $\{1\,,\,3\,,\,5\,,\,7\} \equiv \{r\,,\,s\}$ or also $\{2\,,\,4\,,\,6\,,\,8\} \equiv \{r\,,\,s\}$ (see Figures 1 , 11 - 14).

They mean: the 5 or the 6 sub-indexes with the values of a certain angle parameter φ at two values of φ_R;

similarly, 3 and 4 for $\varphi_R-\frac{\pi}{2}$,

7 and 8 for $\varphi_R+\frac{\pi}{2}$ and

1 and 2 for $\varphi_R-\pi$.

The distinction odd - even in relation to different type of fermions.

$\lambda = -1$ drives φ_R to $-\varphi_R$.

The ± signs previous to $\mathfrak{m}^{\pm}$ are related to the signs of an n_t coordinate, a time type coordinate.

So that we can define a four dimensional vector: $\widetilde{\mathfrak{n}} \equiv \widetilde{\mathfrak{n}}(n_t, \mathfrak{n}) \equiv \widetilde{\mathfrak{n}}(\varphi)$, imposing for n_t one out of the two specific values: $n_t = \pm 1$ (a discretization). These vectors represent the characteristic vectors in our definitions of the creation and of the annihilation operators of the elementary fermions.

It is easy to verify:

$$\begin{array}{c} \widetilde{s}^{\,z}_{\lambda 7,m} = \widetilde{s}^{\,z}_{\lambda 3,m} = -\,\widetilde{s}^{\,z}_{\lambda 1,m} = -\widetilde{s}^{\,z}_{\lambda 5,m} = \left(\mathbf{R}_O^{\lambda\varphi_R}\,(-\,\sigma^z\,)\,\mathbf{R}_O^{-\lambda\varphi_R}\ \right)_m \\ \widetilde{s}^{\,\mp\,\dagger}_{\lambda 1,m} = \widetilde{s}^{\,\pm}_{\lambda 1,m} = \widetilde{s}^{\,\mp}_{\lambda 3,m} = -\widetilde{s}^{\,\pm}_{\lambda 7,m} = -\widetilde{s}^{\,\mp}_{\lambda 5,m} = \left(\mathbf{R}_O^{\lambda\varphi_R}\,(-\mathfrak{m}^{\pm}\sigma^{\mp}\,)\,\mathbf{R}_O^{-\lambda\varphi_R}\ \right)_m \\ \widetilde{s}^{\,z}_{-\lambda r,m} = \sigma^z_m\ \ \widetilde{s}^{\,z}_{\lambda r,m}\ \ \sigma^z_m\,, \qquad \widetilde{s}^{\,\pm}_{-\lambda r,m} = \sigma^z_m\ (-\,\widetilde{s}^{\,\pm}_{\lambda r,m})\ \sigma^z_m \end{array}. \tag{3.6}$$

We establish the following relationship of φ with n_t, n_z and τ (n_y):

$$\mathrm{sign}(\varphi) = \mathrm{sign}(n_t)\,\mathrm{sign}(n_z)\,\mathrm{sign}(n_y) = \begin{cases} -\quad(-\pi \le \varphi < 0\,) & \text{for}\quad \lambda r \in \{\,1, -7, 3, -5\,\} \\ +\quad(\ 0 \le \varphi < \pi\,) & \text{for}\quad \lambda r \in \{\,5, -3, 7, -1\,\} \end{cases}. \tag{3.7}$$

Cases for which

$$\varphi \in \{\,0\,,\ \pm\tfrac{\pi}{4}\,,\ \pm\tfrac{\pi}{2}\,,\ \pm\tfrac{3\pi}{4}\,,\ -\pi\,\},\quad \text{i.e.}\quad \tau = 0\quad\text{or}\quad n_z = 0\,,$$

correspond to leptons and they are treated in their sections.

Although there is not a one to one correspondence between the Cartesian coordinates and the angle parameter φ at these points (except at 0 and $-\pi$), it is possible to define the physical properties (spin, charge, vector-spin and chirality) in relation to them. In particular, for the value $\varphi_R = \frac{\pi}{4}$ consider: $n_t\Big|_{\varphi_R=\frac{\pi}{4}^{\mp}} = -\dfrac{d}{d\varphi}\,|\,\varphi-\varphi_R\,|\,\Big|_{\varphi_R=\frac{\pi}{4}^{\mp}} = \pm 1.$

Also: $\mathrm{sign}(\lambda) = \mathrm{sign}(\sin(4\varphi)\,) = \mathrm{sign}(n_t(\varphi))\,\mathrm{sign}(\varphi) = \mathrm{sign}(n_z)\,\mathrm{sign}(\tau)$. And, $\lambda(\varphi)$ is:

$$\lambda(\varphi) = \begin{cases} +1 \quad \text{for}\ \ \varphi \in (-\pi, -\frac{3\pi}{4}) \cup (-\frac{\pi}{2}, -\frac{\pi}{4}) \cup (0, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{3\pi}{4}), \quad \lambda r\ \in\ \{\,1\,,\ 7\,,\ \ 5,\ \ 3\,\} \\ -1 \quad \text{for}\ \ \varphi \in (-\frac{3\pi}{4}, -\frac{\pi}{2}) \cup (-\frac{\pi}{4}\,,\ 0\,) \cup (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{3\pi}{4}, \pi), \quad \lambda r \in \{-1, -7, -5, -3\,\} \end{cases}. \tag{3.8}$$

The Figures 1,2,3 clarify the meaning of the sub-indexes λr with respect to the time space positions and to the angle φ.

A. The $\overset{\mathbf{I}}{\bullet}$-product. Anticommutators (a unique value m).

With $\{r, s\} = \{1, 3, 5, 7\}$, $\lambda \in \{+1, -1\}$, and $*$ indicating either $+$ or $-$, similarly for $**$, we define the following product:

$$\begin{cases} \widetilde{s}^{\,*}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,**}_{\lambda s,m} \equiv \widetilde{s}^{\,*}_{\lambda r,m}\ \widetilde{s}^{\,**}_{\lambda r,m}\,, & r = s \\ \widetilde{s}^{\,*}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,**}_{\lambda s,m} \equiv \widetilde{s}^{\,*}_{\lambda r,m}\, \mathbf{I}_{\lambda r \lambda s,m}\, \widetilde{s}^{\,**}_{\lambda s,m}\,, & r \neq s \end{cases}, \tag{3.9}$$

where:

$$\mathbf{I}_{\lambda r \lambda s,m} \equiv \mathbf{I}_{rs,m} \equiv \mathbf{R}_{O_m}^{\mathrm{sign}(r-s)\,\widetilde{\epsilon}\frac{\pi}{2}} = \left(\mathrm{sign}(r-s)\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\right)_m = \left(-\,\mathrm{sign}(s-r)\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\right)_m = -\,\mathbf{I}_{\lambda s \lambda r,m}\,, \tag{3.10}$$

and: $\begin{cases} \mathrm{sign}(r-s) = +1 & \text{for} \quad r > s \\ \mathrm{sign}(r-s) = -1 & \text{for} \quad r < s \end{cases}$; also it is: $-\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}} = \mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}}$, $\widetilde{\epsilon} = \{+1, -1\}$.

With this product we get:

a) for $r \neq s$ $\qquad \widetilde{s}^{\,*}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,**}_{\lambda s,m} = -\,\widetilde{s}^{\,**}_{\lambda s,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,*}_{\lambda r,m}$;

where we have used:

$$\sigma^{\pm}\, \mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\, \sigma^{\mp} = \mathbb{0}\,,$$

$$\sigma^{\pm}\, \mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\, \sigma^{\pm} + \sigma^{\pm}\, \mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}}\, \sigma^{\pm} = \mathbb{0}\,.$$

b) for $r = s$

$$\widetilde{s}^{\,+}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,+}_{\lambda s,m} = \widetilde{s}^{\,-}_{\lambda s,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,-}_{\lambda r,m} = \widetilde{s}^{\,+}_{\lambda r,m}\, \widetilde{s}^{\,+}_{\lambda s,m} = \widetilde{s}^{\,-}_{\lambda s,m}\, \widetilde{s}^{\,-}_{\lambda r,m} = \mathbb{0}_N\,, \qquad ({\sigma^{+}}^2 = {\sigma^{-}}^2 = \mathbb{0}),$$

$$\widetilde{s}^{\,+}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,-}_{\lambda s,m} + \widetilde{s}^{\,-}_{\lambda s,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,+}_{\lambda r,m} = \widetilde{s}^{\,+}_{\lambda r,m}\, \widetilde{s}^{\,-}_{\lambda s,m} + \widetilde{s}^{\,-}_{\lambda s,m}\, \widetilde{s}^{\,+}_{\lambda r,m} = \mathbb{1}_N\,, \qquad (\{\sigma^{+}, \sigma^{-}\} = \mathbb{1}).$$

Their anticommutators are:

$$r \neq s \qquad \left\{\widetilde{s}^{\,*}_{\lambda r,m} \overset{\mathbf{I}}{;} \widetilde{s}^{\,**}_{\lambda s,m}\right\} = \left\{\widetilde{s}^{\,**}_{\lambda s,m} \overset{\mathbf{I}}{;} \widetilde{s}^{\,*}_{\lambda r,m}\right\} = \widetilde{s}^{\,*}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,**}_{\lambda s,m} + \widetilde{s}^{\,**}_{\lambda s,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,*}_{\lambda r,m} = \mathbb{0}_N$$

$$r = s \qquad \begin{cases} \left\{\widetilde{s}^{\,\pm}_{\lambda r,m} \overset{\mathbf{I}}{;} \widetilde{s}^{\,\pm}_{\lambda r,m}\right\} \equiv \left\{\widetilde{s}^{\,\pm}_{\lambda r,m}\,,\, \widetilde{s}^{\,\pm}_{\lambda r,m}\right\} = \mathbb{0}_N \\ \left\{\widetilde{s}^{\,+}_{\lambda r,m} \overset{\mathbf{I}}{;} \widetilde{s}^{\,-}_{\lambda r,m}\right\} \equiv \left\{\widetilde{s}^{\,+}_{\lambda r,m}\,,\, \widetilde{s}^{\,-}_{\lambda r,m}\right\} = \mathbb{1}_N \end{cases}, \tag{3.11}$$

With $r = s$ or $r \neq s$, we also write the following products:

$$1) \qquad \widetilde{s}^{\,z}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,z}_{\lambda s,m} \equiv \widetilde{s}^{\,z}_{\lambda r,m}\ \widetilde{s}^{\,z}_{\lambda s,m} \in \left\{+\mathbb{1}_N, -\mathbb{1}_N\right\}, \qquad \text{as it is} \quad {\sigma^{z}}^2 = \mathbb{1}\,, \tag{3.12}$$

$$2) \qquad \begin{cases} \widetilde{s}^{\,\pm}_{\lambda r,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,z}_{\lambda s,m} \equiv \widetilde{s}^{\,\pm}_{\lambda r,m}\ \widetilde{s}^{\,z}_{\lambda s,m} = \mp K\, \widetilde{s}^{\,\pm}_{\lambda r,m} \\ \widetilde{s}^{\,z}_{\lambda s,m} \overset{\mathbf{I}}{\bullet} \widetilde{s}^{\,\pm}_{\lambda r,m} \equiv \widetilde{s}^{\,z}_{\lambda s,m}\ \widetilde{s}^{\,\pm}_{\lambda r,m} = \pm K\, \widetilde{s}^{\,\pm}_{\lambda r,m} \end{cases}, \tag{3.13}$$

thanks to (2.8), and with a common non null factor K.

The anticommutators involving the products in (3.13) are:

$$\left\{\widetilde{s}^{\,\pm}_{\lambda r,m} \overset{\mathbf{I}}{;} \widetilde{s}^{\,z}_{\lambda s,m}\right\} = \left\{\widetilde{s}^{\,\pm}_{\lambda r,m}\,,\, \widetilde{s}^{\,z}_{\lambda s,m}\right\} = \mathbb{0}_N\,. \tag{3.14}$$

IV. CREATION AND ANNIHILATION OPERATORS. PHYSICS.

We generalize the Jordan and Wigner transformation and we propose the following definitions:

$$\begin{aligned}
\mathbf{U}^{\dagger}_{\lambda 5,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 5,k}\Big)\, \widetilde{s}^{\,+}_{\lambda 5,m} = \Big(\ \mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \quad \Big(\mathbf{R}_o^{\lambda\varphi_R}\,(\ \mathrm{m}^-\sigma^+)\ \mathbf{R}_o^{-\lambda\varphi_R}\Big)_m \\
\mathbf{U}_{\lambda 5,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 5,k}\Big)\, \widetilde{s}^{\,-}_{\lambda 5,m} = \Big(\ \mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \quad \Big(\mathbf{R}_o^{\lambda\varphi_R}\,(\ \mathrm{m}^+\sigma^-)\ \mathbf{R}_o^{-\lambda\varphi_R}\Big)_m \\
\mathbf{U}_{\lambda 3,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 3,k}\Big)\, \widetilde{s}^{\,-}_{\lambda 3,m} = \Big(-\mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \quad \Big(\mathbf{R}_o^{\lambda\varphi_R}\,(-\,\mathrm{m}^+\sigma^-)\ \mathbf{R}_o^{-\lambda\varphi_R}\Big)_m \\
\mathbf{U}^{\dagger}_{\lambda 3,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 3,k}\Big)\, \widetilde{s}^{\,+}_{\lambda 3,m} = \Big(-\mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \quad \Big(\mathbf{R}_o^{\lambda\varphi_R}\,(-\,\mathrm{m}^-\sigma^+)\ \mathbf{R}_o^{-\lambda\varphi_R}\Big)_m \\
\mathbf{U}_{\lambda 1,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 1,k}\Big)\, \widetilde{s}^{\,-}_{\lambda 1,m} = \Big(\ \mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \Big(\mathbf{R}_o^{\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\,(-\,\mathrm{m}^+\sigma^-)\ \mathbf{R}_o^{-\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\Big)_m \\
\mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 1,k}\Big)\, \widetilde{s}^{\,+}_{\lambda 1,m} = \Big(\ \mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \Big(\mathbf{R}_o^{\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\,(-\,\mathrm{m}^-\sigma^+)\ \mathbf{R}_o^{-\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\Big)_m \\
\mathbf{U}^{\dagger}_{\lambda 7,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 7,k}\Big)\, \widetilde{s}^{\,+}_{\lambda 7,m} = \Big(-\mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \Big(\mathbf{R}_o^{\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\,(\ \mathrm{m}^-\sigma^+)\ \mathbf{R}_o^{-\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\Big)_m \\
\mathbf{U}_{\lambda 7,\mathcal{M}}(\varphi_R) &\equiv \Big(\prod_{k=1}^{m-1} \widetilde{s}^{\,z}_{\lambda 7,k}\Big)\, \widetilde{s}^{\,-}_{\lambda 7,m} = \Big(-\mathbf{R}_e^{2\lambda\varphi_R}\Big)_{M-1} \Big(\mathbf{R}_o^{\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\,(\ \mathrm{m}^+\sigma^-)\ \mathbf{R}_o^{-\lambda(\varphi_R+\epsilon\frac{\pi}{2})}\Big)_m
\end{aligned} \quad , \qquad (4.1)$$

with $\mathrm{m}^{\pm} \equiv e^{\pm i\,(\phi - \frac{\pi}{2})} = \mp i\, e^{\,\pm i\,\phi}$ (see (2.4)) and $\mathbf{R}_o^{\epsilon\frac{\pi}{2}}(\,\mathrm{m}^{\mp}\,\sigma^{\pm})\,\mathbf{R}_o^{-\epsilon\frac{\pi}{2}} = \mathrm{m}^{\pm}\,\sigma^{\mp}$ (see (2.5)).

We also write the creation operators (“daga”) in the form:

$$\begin{aligned}
\mathbf{U}^{\dagger}_{\lambda 5,\mathcal{M}}(\varphi_R) &\equiv \mathbf{R}_{o_M}^{\lambda\varphi_R}\Big[\ \big(\ \sigma^z\big)_{M-1}\ \big(\ \mathrm{m}^-\sigma^+\ \big)_m\ \Big]\,\mathbf{R}_{o_M}^{-\lambda\varphi_R} \\
\mathbf{U}^{\dagger}_{\lambda 3,\mathcal{M}}(\varphi_R) &\equiv \mathbf{R}_{o_M}^{\lambda\varphi_R}\Big[\ \big(-\sigma^z\big)_{M-1}\ \big(-\,\mathrm{m}^-\sigma^+\ \big)_m\ \Big]\,\mathbf{R}_{o_M}^{-\lambda\varphi_R} \\
\mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}}(\varphi_R) &\equiv \mathbf{R}_{o_M}^{\lambda\varphi_R}\Big[\ \big(\ \sigma^z\big)_{M-1}\big(\mathbf{R}_o^{\epsilon\frac{\pi}{2}}(-\mathrm{m}^-\sigma^+)\,\mathbf{R}_o^{-\epsilon\frac{\pi}{2}}\big)_m\ \Big]\,\mathbf{R}_{o_M}^{-\lambda\varphi_R} \\
\mathbf{U}^{\dagger}_{\lambda 7,\mathcal{M}}(\varphi_R) &\equiv \mathbf{R}_{o_M}^{\lambda\varphi_R}\Big[\ \big(-\sigma^z\big)_{M-1}\big(\mathbf{R}_o^{\epsilon\frac{\pi}{2}}(\,\mathrm{m}^-\sigma^+)\,\mathbf{R}_o^{-\epsilon\frac{\pi}{2}}\big)_m\ \Big]\,\mathbf{R}_{o_M}^{-\lambda\varphi_R}
\end{aligned} \quad , \qquad (4.2)$$

In these formulas the actual value of ϵ, $+1$ or -1, does not matter. Due to the form of the action of the $\mathbf{R}_o^{\pm\epsilon\frac{\pi}{2}}$ matrices, to the right and to the left of the $\sigma^{\pm}$ matrices, it is irrelevant to choose anyone of the two values. The value of ϵ is important for a geometrical interpretation.

The set of values that we set for φ_R is:

$$\varphi_R \in \Big\{\ 0\ ,\ \frac{\pi}{12}\ ,\ \frac{\pi}{6}\ ,\ \frac{\pi}{4}\ \Big\} \cup \Big\{\ \frac{\pi}{8}\ \Big\}. \qquad (4.3)$$

One justification for these values is in Appendix B. We can not set discrete values for the ϕ angle variable. Perhaps, there should be three values in correspondence with the three generations of elementary fermions. Neither we know in what way to incorporate the color to the quarks.

The **number operators** are:

$$\mathbf{N}_{\lambda r,m}(\varphi_R) \equiv \mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}}(\varphi_R)\ \mathbf{U}_{\lambda r,\mathcal{M}}(\varphi_R) \ . \tag{4.4}$$

They satisfy

$$\mathbf{N}_{\lambda r,m}\,\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}} = 1\,\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}}\,, \qquad \mathbf{N}_{\lambda r,m}\,\mathbf{U}_{\lambda r,\mathcal{M}} = 0\,\mathbf{U}_{\lambda r,\mathcal{M}}\,, \tag{4.5}$$

and:

$$\mathbf{1}-\mathbf{N}_{\lambda r,m} = \mathbf{U}_{\lambda r,\mathcal{M}}\,\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}}\,, \qquad \mathbf{N}_{\lambda r,m}\,(\mathbf{1} - \mathbf{N}_{\lambda r,m}) = \mathbf{0}\,. \tag{4.6}$$

The $\overset{\mathbf{I}}{\bullet}$**-product:** for these creation and annihilation operators, we generalize the product defined in (3.9):

$$m_1 \neq m_2 \qquad \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1}(\varphi_R) \overset{\mathbf{I}}{\bullet} \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2}(\varphi_R) \equiv \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1}(\varphi_R)\ \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2}(\varphi_R)$$

$$m_1 = m_2 = m \quad \begin{cases} r = s & \mathbf{U}^{*}_{\lambda r,\mathcal{M}}(\varphi_R) \overset{\mathbf{I}}{\bullet} \mathbf{U}^{**}_{\lambda r,\mathcal{M}}(\varphi_R) \equiv \mathbf{U}^{*}_{\lambda r,\mathcal{M}}(\varphi_R)\ \mathbf{U}^{**}_{\lambda r,\mathcal{M}}(\varphi_R) \\ r \neq s & \mathbf{U}^{*}_{\lambda r,\mathcal{M}}(\varphi_R) \overset{\mathbf{I}}{\bullet} \mathbf{U}^{**}_{\lambda s,\mathcal{M}}(\varphi_R) \equiv \mathbf{U}^{*}_{\lambda r,\mathcal{M}}(\varphi_R)\,\mathbf{I}_{\lambda r\lambda s,\,m}\,\mathbf{U}^{**}_{\lambda s,\mathcal{M}}(\varphi_R) \end{cases} \ , \tag{4.7}$$

$*$ or $**$ indicate either creation or annihilation operators (“daga” or “nothing”).

Particles - Antiparticles , Creation - Annihilation. See the second citation in Appendix D.

Observe that the expressions in the definitions (4.1) or (4.2) of $\mathbf{U}^{\dagger}_{\lambda 5,\mathcal{M}}$ as a creation operator of a particle and of $\mathbf{U}_{\lambda 1,\mathcal{M}}$ as an annihilation operator of the corresponding antiparticle are **formally (algebraically)** similar (up to a minus sign); similarly with:

$$\mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}} \text{ and } \mathbf{U}_{\lambda 5,\mathcal{M}}\,, \qquad \mathbf{U}^{\dagger}_{\lambda 7,\mathcal{M}} \text{ and } \mathbf{U}_{\lambda 3,\mathcal{M}}\,, \qquad \mathbf{U}^{\dagger}_{\lambda 3,\mathcal{M}} \text{ and } \mathbf{U}_{\lambda 7,\mathcal{M}}\,.$$

Underneath this there is a time opposition, and in relation to the φ parameter a $\pm\pi$ difference. All of this in a Jordan Wigner type transformation.

Also:

$$\mathbf{U}^{\dagger}_{\lambda 5,\,\mathcal{M}} = (-1)^{m-1}\mathbf{U}_{\lambda 7,\,\mathcal{M}}\,, \qquad \mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}} = (-1)^{m-1}\mathbf{U}_{\lambda 3,\,\mathcal{M}} \qquad \text{and} \qquad \text{their adjoints.}$$

Now, the difference in the interpretation as creation or annihilation operators for particles and the corresponding antiparticles comes from the space opposition in the φ parameter, a $+\frac{\pi}{2}$ or a $-\frac{\pi}{2}$ difference in a Jordan Wigner type transformation. The $(-1)^{m-1}$ factor is more subtle than the previous (-1) factor: it is the one appearing in formulas (2.6). For m odd or even we get a $+$ or a $-$ global algebraic factor.

These relationships can be summarized in:

$$\boxed{\mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}}(\varphi_R) = (-1)^m\,\mathbf{U}^{\dagger}_{\lambda 7,\mathcal{M}}(\varphi_R) = -\,[\,\mathbf{U}_{\lambda 5,\mathcal{M}}(\varphi_R)] = -\,[\,(-1)^m\,\mathbf{U}_{\lambda 3,\mathcal{M}}(\varphi_R)]\ .} \tag{4.8}$$

If we take m odd, we can distinguish them algebraically

$$\mathbf{U}^{\dagger}_{\lambda 1,\mathcal{M}}(\varphi_R) = -\,\mathbf{U}^{\dagger}_{\lambda 7,\mathcal{M}}(\varphi_R) \qquad \text{and} \qquad \mathbf{U}^{\dagger}_{\lambda 5,\mathcal{M}}(\varphi_R) = -\,\mathbf{U}^{\dagger}_{\lambda 3,\mathcal{M}}(\varphi_R)\,,$$

1 with 7 and 5 with 3, through the minus sign, whosoever is the physical meaning of m.

Presented in this way, all these relations are **only algebraic relationships**. The geometry of generalized rotations underlies.

The definition of the $\overset{\mathbf{I}}{\bullet}$-product takes account of the above mentioned underlying geometry; this makes significant the differentiation of these operators. We will refine these comments with the definition of the anticommutator structures in **2**. Relate this to the Pauli’s paragraph in the first cite in the Appendix D.

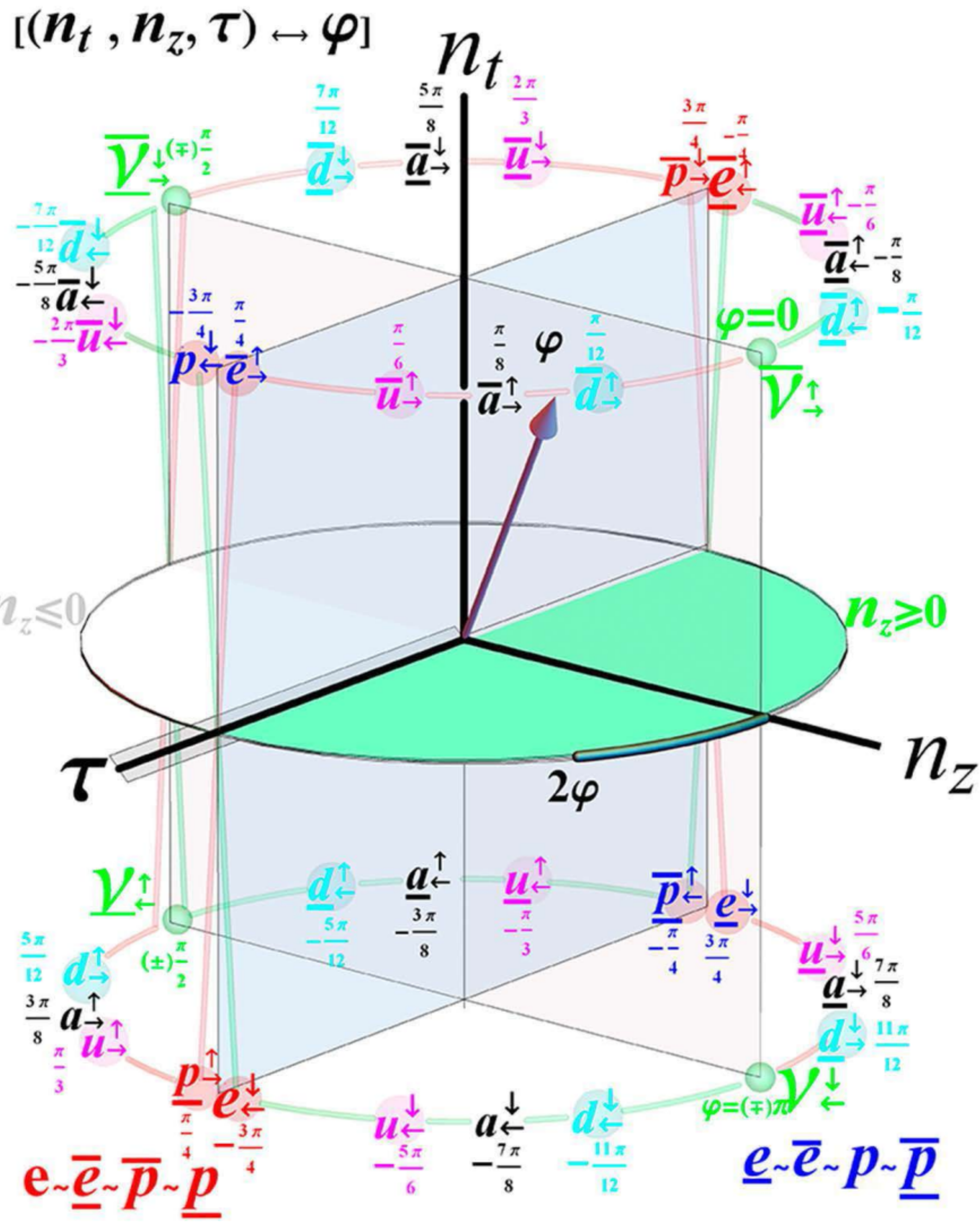


FIG. 4: elementary fermions.

A representation of the elementary fermions in a time - space graphic in relation to four dimensional axes defined with the values of φ_R in (4.3): Leptons ($\varphi_R \in \{0, \frac{\pi}{4}\}$), quarks ($\varphi_R \in \{\frac{\pi}{12}, \frac{\pi}{6}\}$) and an unknown particle ($\varphi_R \in \{\frac{\pi}{8}\}$).

An over-bar indicates an antiparticle in relation to a time opposition, similarly with
an under-bar in relation to an space opposition (except in e, p),
both bars same electrically charged particle.
↑ for up-spin and ↓ for down-spin.
Green, $n_z \geq 0$ indicates weak interaction and Gray, $n_z \leq 0$ without weak interaction; $n_z = 0$ with a special treatment.

A. Spin.

In order to obtain the eigenvalues of an spin operator for the different creation operators $\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}}$, we define:

$$\mathbb{S}_m \equiv \mathbb{S}^{\lambda}_m \equiv s^{z\ \lambda\varphi_R}_m \equiv \mathbf{R}^{\lambda 2\varphi_R}_{e_m} = \left(\mathbf{R}_O^{\lambda\varphi_R}\,\sigma^z\,\mathbf{R}_O^{-\lambda\varphi_R}\right)_m , \tag{4.9}$$

with action in the m sites of the operators in (4.1) or (4.2). This operator for $\varphi_R = 0$ is just σ^z_m. It acts over two types of creation operators:

$$\begin{cases} \mathbb{S}_m\,\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}} = (+1)\ \mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}} , & \lambda r \in \{\pm 3, \pm 5\} : \quad \text{spin up} \ \uparrow \\ \mathbb{S}_m\,\mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}} = (-1)\ \mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}} , & \lambda r \in \{\pm 1, \pm 7\} : \quad \text{spin down} \ \downarrow \end{cases} . \tag{4.10}$$

In brief, this can be stated as:

$$\text{for} \begin{cases} \lambda r \in \{\pm 3, \pm 5\} : & \text{sign}(n_t)\ \text{sign}(n_z) = + , \quad \text{spin up} \ \uparrow \\ \lambda r \in \{\pm 1, \pm 7\} : & \text{sign}(n_t)\ \text{sign}(n_z) = - , \quad \text{spin down} \ \downarrow \end{cases} . \tag{4.11}$$

except for charged leptons ($n_z = 0$ or $\varphi \in \{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$), which can be considered as limiting cases in (4.10). Following formula includes the cases:

$$\text{for} \begin{cases} -\frac{\pi}{2} < \varphi < \frac{\pi}{2} : & \text{sign}(\cos\varphi) = + , \quad \text{spin up} \ \uparrow \\ \frac{\pi}{2} < |\varphi| \leq \pi : & \text{sign}(\cos\varphi) = - , \quad \text{spin down} \ \downarrow \end{cases} . \tag{4.12}$$

for $\varphi = \pm\frac{\pi}{2}$ we apply directly (4.10). These last two values correspond to uncharged leptons.

The value of the spin does not depend of τ. Under opposition in the3-dimensional space or in the z-coordinate or in time, we have opposite spin. It is worth noting that although this spin operator does not depend on n_t, its action produces different results which depend on the n_t value of the creation operator at the m^{th} position.

Formula (4.10) is relevant for different spin values at the same Cartesian points $n_z = 0$, $n_t = \pm 1$, $\tau = \pm 1$, but different values of φ (charged leptons).

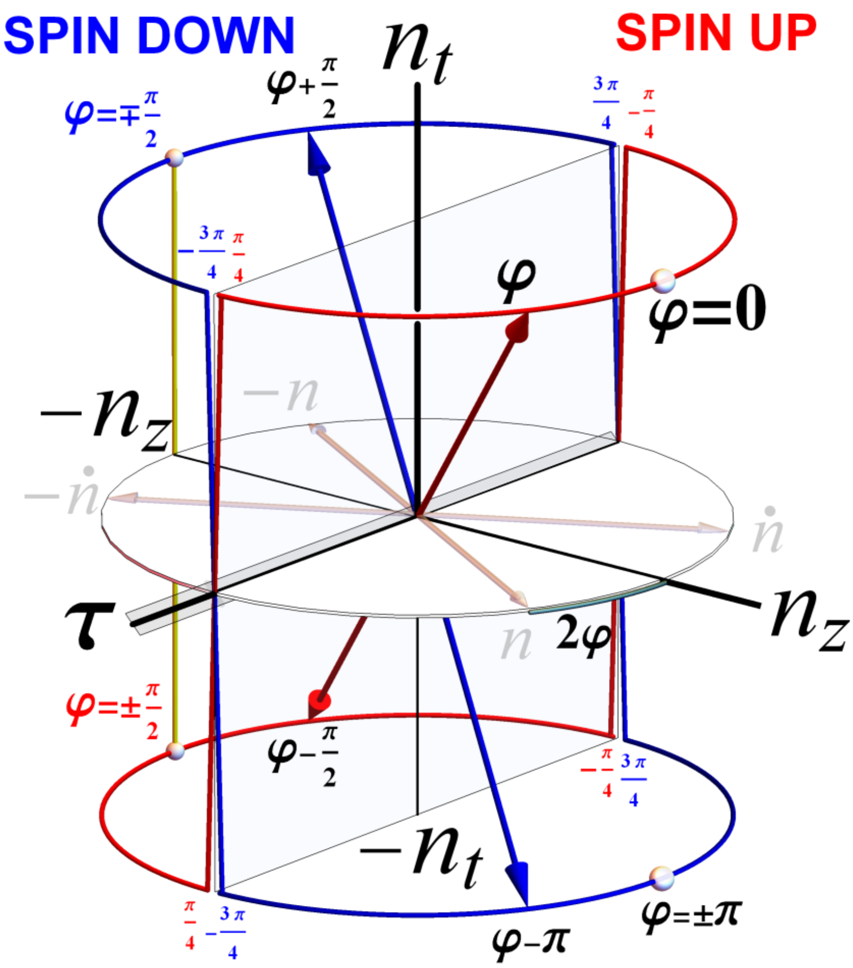


FIG. 5: Spin.

B. Vector-spin.

Timeless; or opposition in space, point-mirror. The vectors-spin have a main role in the magnetism [24].

Let us consider the "tail" part (the terms with 1-to- $m-1$) in the definitions (4.1) or (4.2):

$$\left(\pm\,\mathbb{n}(\lambda 2\varphi_R)\right) \equiv \left(\pm\,\mathbf{R}_e^{\lambda 2\varphi_R}\right) \equiv \left(\mathbf{R}_O^{\lambda\varphi_R}\,(\pm\,\sigma^z)\,\mathbf{R}_O^{-\lambda\varphi_R}\right). \tag{4.13}$$

This formula for a given φ_R provides 4 different values, except for $2\varphi_R = 0$ $(\tau = 0)$ and for $2\varphi_R = \frac{\pi}{2}$ $(n_z = 0)$, for which we obtain only two values (for each one of them). The $\pm$ previous to a three dimensional semi-direction $\mathbb{n}(2\varphi_R)$ gives us a whole axis for a three dimensional rotation. The value $\lambda = -1$ drives to an axis of rotation symmetric in $\mathbf{n}_z$ to the one with $\lambda = +1$. At the same time, it represents a π-rotated axis over $\mathbf{n}_z$ and it provides another two values. Then:

$$\text{opposite}\left.\begin{array}{ll}(\ 1,\ \ 5): & \left(\ \mathbb{n}(\ 2\varphi_R)\right)_{(M-1)} = \left(\ \mathbf{R}_e^{2\varphi_R}\right)_{(M-1)} \\ (\ 7,\ \ 3): & \left(-\,\mathbb{n}(\ 2\varphi_R)\right)_{(M-1)} = \left(\mathbf{R}_e^{2\varphi_R \pm\pi}\right)_{(M-1)}\end{array}\right\} \quad \text{sign}(n_z)\ \text{sign}(\tau) = \text{sign}(\lambda) = +\ , \tag{4.14a}$$

$$\text{opposite}\left.\begin{array}{ll}(-1,-5): & \left(\ \dot{\mathbb{n}}(-2\varphi_R)\right)_{(M-1)} = \left(\ \mathbf{R}_e^{-2\varphi_R}\right)_{(M-1)} \\ (-7,-3): & \left(-\,\dot{\mathbb{n}}(-2\varphi_R)\right)_{(M-1)} = \left(\mathbf{R}_e^{-2\varphi_R \pm\pi}\right)_{(M-1)}\end{array}\right\} \quad \text{sign}(n_z)\ \text{sign}(\tau) = \text{sign}(\lambda) = -\ . \tag{4.14b}$$

Inside each bracket we consider the opposite signs, obtaining the couples $(1\,,7)$, $(5\,,3)$, $(-1\,,-7)$ and $(-5\,,-3)$. With this feature we get the above mentioned distinction. Also, we have added a dependence in the value of the angle parameter 2φ, like the one we have defined with the electric charge.

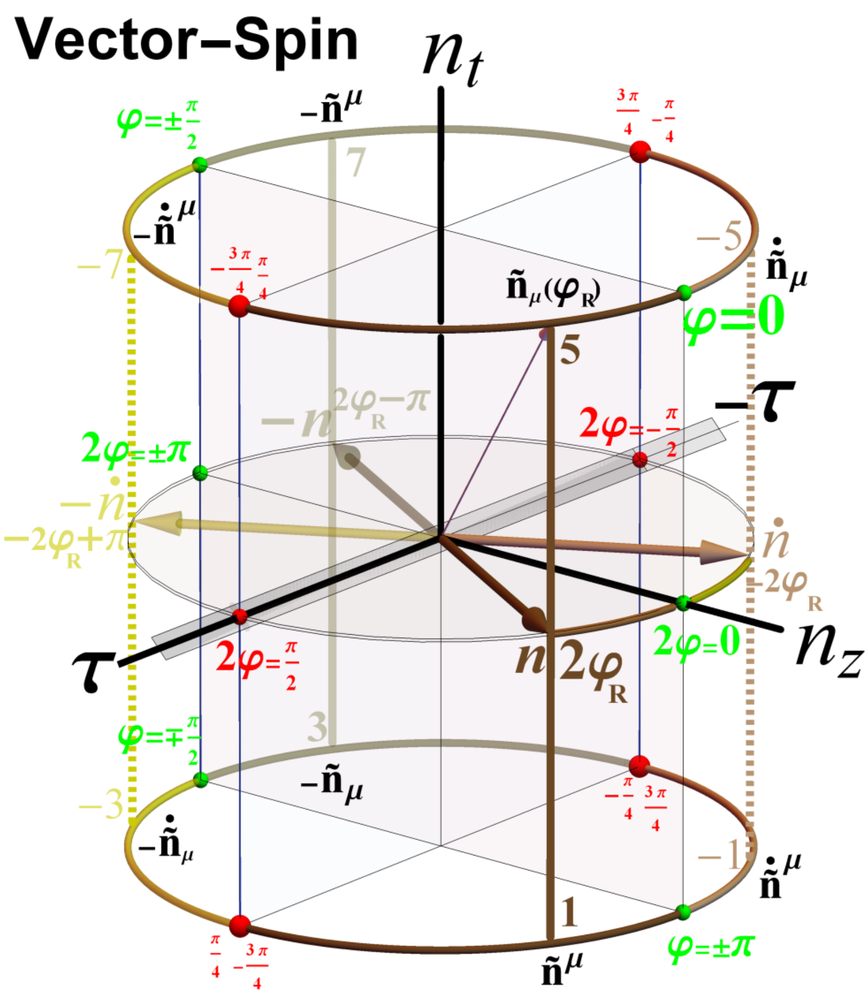


FIG. 6: Vector-Spin.

C. Chirality.

Opposition in time; or opposition in an n_z space coordinate (plane-mirror in $(n_x\,,\,n_y)$); or opposition in τ. Its main role in the weak interaction.

Customarily, chirality is defined in a spinor formalism in quantum relativistic mechanics, with two values, taken as left and right chiral. It is a concept closely related to helicity. Left handed chirality is also left handed helicity for massless particles, or 'almost' for ultrarelativistic particles. Sometimes its exposition is not clearly stated.

Tentatively, we conjecture an assignment for the definition of the chirality as function of sign(φ) (formula (3.7)) in the following way, which we establish as a rule:

$$\mathrm{sign}(\varphi) \;=\; \mathrm{sign}(n_t)\,\mathrm{sign}(n_z)\,\mathrm{sign}(\tau) \;=\; \begin{cases} - \;\Rightarrow\; \text{left chirality} \;\; [\,\mathrm{lc}\;(\leftarrow)\,] \;\; \text{for}\ \{1\,,-7\,,3\,,-5\} \\ + \;\Rightarrow\; \text{right chirality} \;\; [\,\mathrm{rc}\;(\rightarrow)\,] \;\; \text{for}\ \{-1\,,7\,,-3\,,5\} \end{cases} . \tag{4.15}$$

The spin of the particles at the values $\varphi \in \{\pm\frac{\pi}{4}\,,\,\pm\frac{3\pi}{4}\}$ (charged leptons) requires a special treatment. We have a similar situation with the chirality at the values $\varphi \in \{0\,,\,\pm\frac{\pi}{4}\,,\,\pm\frac{\pi}{2}\,,\,\pm\frac{3\pi}{4}\,,\,-\pi\}$ or $\tau = n_z = 0$ (leptons). We impose for them the criteria shown in the Figure 6.

(4.15) implies eight different combinations due to the time and space symmetries, four with the left chirality and another four with the right chirality. We can establish a correspondence of these combinations with the eight different ones obtained with the signs of the charge, the spin and the chirality:

sign(φ) =	sign(τ)	sign(n_t)	sign(n_z)		
$-$	$+$	$-$	$+$	$-\mathbf{q}\ \downarrow\leftarrow$	(1)
$-$	$-$	$+$	$+$	$-\mathbf{q}\ \uparrow\leftarrow$	(-5)
$-$	$-$	$-$	$-$	$\mathbf{q}\ \uparrow\leftarrow$	(3)
$-$	$+$	$+$	$-$	$\mathbf{q}\ \downarrow\leftarrow$	(-7)
$+$	$-$	$-$	$+$	$\mathbf{q}\ \downarrow\rightarrow$	(-1)
$+$	$+$	$+$	$+$	$\mathbf{q}\ \uparrow\rightarrow$	(5)
$+$	$+$	$-$	$-$	$-\mathbf{q}\ \uparrow\rightarrow$	(-3)
$+$	$-$	$+$	$-$	$-\mathbf{q}\ \downarrow\rightarrow$	(7)

We have to pay attention to the fact that each one of the couples $\langle 1\,,-5\rangle$, $\langle 3\,,-7\rangle$, $\langle -1\,,5\rangle$ and $\langle -3\,,7\rangle$ have the same charge and the same chirality, but they are already distinguished by their opposite spins inside every couple. We can relate this with different helicities inside each couple. Include here the pion decay with the electron with left chirality but right helicity.

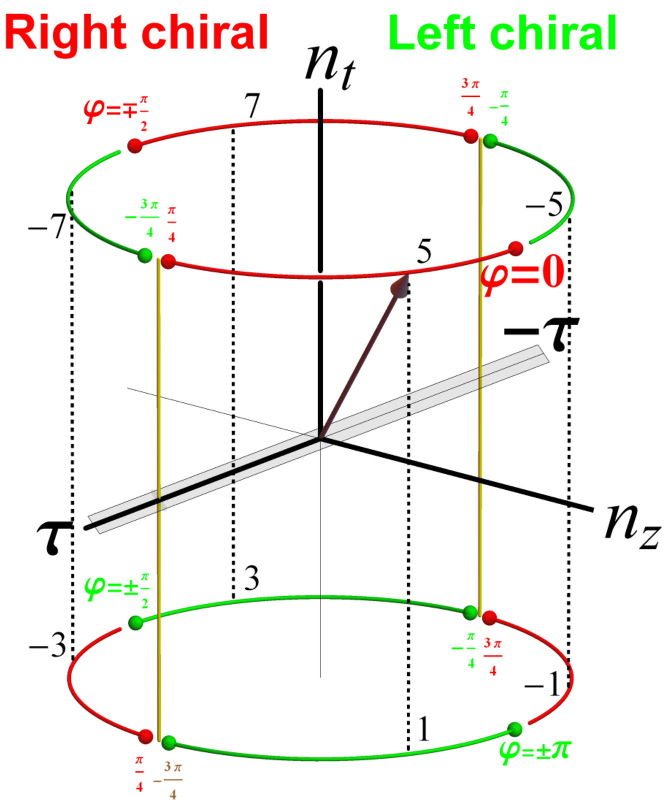


FIG. 7: Chiraity.

D. Electrical Charge.

We consider different particles for different values of φ, or of n_t, n_z and τ (n_y) in a certain local reference frame, related by some symmetries, not the concept of particles in relation to different frames with parity and time inversion transformations.

Suppose that for a fixed value φ_R we have a particle with an electric charge $\mathbf{q}(\varphi_\mathbf{R})$ and an **up**-spin ↑. We apply the formulas (4.1) or (4.2), with the comments preceding formula (4.8). In this way we obtain the two separate columns that follow. Also, for a given spin value we need both types of charges, positive and negative, neutrinos later; this motivates the opposite charges in the corresponding row:

$$\left\{\begin{array}{lllll|llll} \mathbf{5} & \text{reference} & & \mathbf{q}(\varphi_\mathbf{R}) & \uparrow & \uparrow & -\mathbf{q}(\varphi_\mathbf{R}) = \mathbf{q}(-\varphi_\mathbf{R}) & \text{opposite } \tau & -5 \\ 3 & \text{opposite } n_t n_z \tau & \mathbf{q}(\varphi_\mathbf{R} - \frac{\pi}{2}) = & \mathbf{q}(\varphi_\mathbf{R}) & \uparrow & \uparrow & -\mathbf{q}(\varphi_\mathbf{R}) = \mathbf{q}(-\varphi_\mathbf{R} + \frac{\pi}{2}) & \text{opposite } n_t n_z & -3 \\ 1 & \text{opposite } n_t & \mathbf{q}(\varphi_\mathbf{R} - \pi) = & -\mathbf{q}(\varphi_\mathbf{R}) & \downarrow & \downarrow & \mathbf{q}(\varphi_\mathbf{R}) = \mathbf{q}(-\varphi_\mathbf{R} + \pi) & \text{opposite } n_t \tau & -1 \\ 7 & \text{opposite } n_z \tau & \mathbf{q}(\varphi_\mathbf{R} + \frac{\pi}{2}) = & -\mathbf{q}(\varphi_\mathbf{R}) & \downarrow & \downarrow & \mathbf{q}(\varphi_\mathbf{R}) = \mathbf{q}(-\varphi_\mathbf{R} - \frac{\pi}{2}) & \text{opposite } n_z & -7 \end{array}\right\} . \tag{4.16}$$

We establish a rule: we assign a **minus sign** when we do a 3-dimensional opposition, and departing with a **+ sign** as coincident with the **sign(τ)**, taken equal to **sign(3d)**, we write the following relations:

$$\kappa(\varphi_R)\,\text{sign}(\mathbf{q}(\varphi_R)) = \text{sign}(n_t)\left\{\begin{matrix}\text{sign(3d)}\\ \text{sign}(\tau)\end{matrix}\right\} = \text{sign}(n_z)\,\text{sign}(\varphi) = \text{sign}(\cos(2\varphi))\,\text{sign}(\varphi)\,. \tag{4.17}$$

Therefore

$$\kappa(\varphi_R)\,\text{sign}(\mathbf{q}(\varphi_R)) = \left\{\begin{array}{ll} \text{a sign}, & \varphi \in (-\frac{3\pi}{4}, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, -\frac{\pi}{4}) \cup (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)\,,\ \text{or}\ \lambda r \in \{5, -7, 3, -1\} \\ \text{an opposite sign}, & \varphi \in (-\pi, -\frac{3\pi}{4}) \cup (-\frac{\pi}{4},\ 0) \cup (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})\,,\ \text{or}\ \lambda r \in \{-5, 7, -3, 1\} \end{array}\right. , \tag{4.18}$$

with $\kappa(\varphi_R) = \{+1, -1\}$ as represented in Figure 8 ($q = q = -q = 1$). In the points:

$\varphi \in \{0, \pm\frac{\pi}{2}, \pm\pi\}$ or $\tau = 0$ (neutrinos) the value is $\mathbf{q}(\varphi_\mathbf{R} = \mathbf{0}) = 0$,

$\varphi \in \{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$ or $n_z = 0$ (charged leptons) a value $|\mathbf{q}(\varphi_\mathbf{R} = \frac{\pi}{4})| = 1$, and,

for $\varphi_R = \frac{\pi}{8}$, we consider the value $\mathbf{q}(\varphi_\mathbf{R} = \frac{\pi}{8}) = 0$.

There are more details regarding the value of the charges, with a formula for them, in Appendix B.

The following relations for various oppositions in time and space are implicit in this construction:

$$\begin{array}{lcl} (\ 1,\ 7) & \downarrow & \mathbf{q}(-n_t, n_z, \tau) = \mathbf{q}(n_t, -n_z, -\tau) = \\ (-5, -3) & \uparrow & = \mathbf{q}(n_t, n_z, -\tau) = \mathbf{q}(-n_t, -n_z, \tau) = \\ (\ 5,\ 3) & \uparrow & = -\mathbf{q}(n_t, n_z, \tau) = -\mathbf{q}(-n_t, -n_z, -\tau) = \\ (-1, -7) & \downarrow & = -\mathbf{q}(-n_t, n_z, -\tau) = -\mathbf{q}(n_t, -n_z, \tau)\ . \end{array} \tag{4.19}$$

Now, we have to distinguish inside the couples $(1, 7)$, $(5, 3)$, $(-1, -7)$ and $(-5, -3)$. We do so by applying either one of the following two concepts: vector-spin or chirality.

We start with the 1 or 2 positions for the reference electric charge. Writing with the symbol of the elementary fermion an "over bar", we will associate an opposite charge in relation to time opposition, and an "under bar" will indicate an opposite electric charge in relation to opposite space or to opposite τ. If both "bars" are present, there is no change in the sign of the charge with respect to the one without them. With neutrinos this notation will refer (imply) to opposite spins; for them there is no opposition in τ (τ=0). Also with the ones at $\varphi_R = \frac{\pi}{8}$. For the charged leptons we add a rename: "positrons".

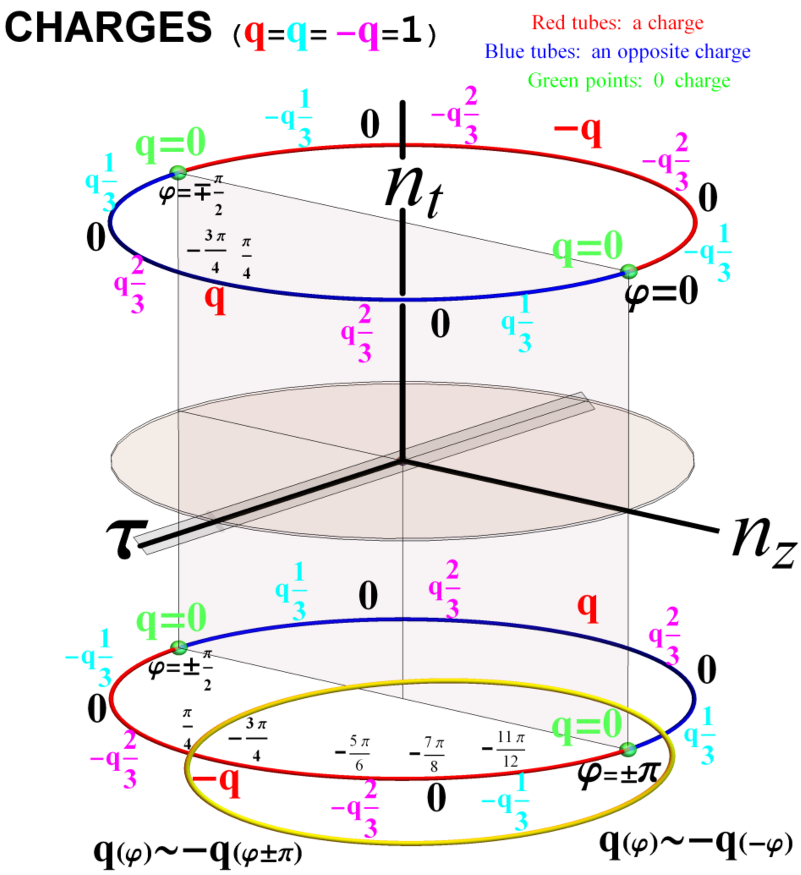


FIG. 8: Electrical charges.

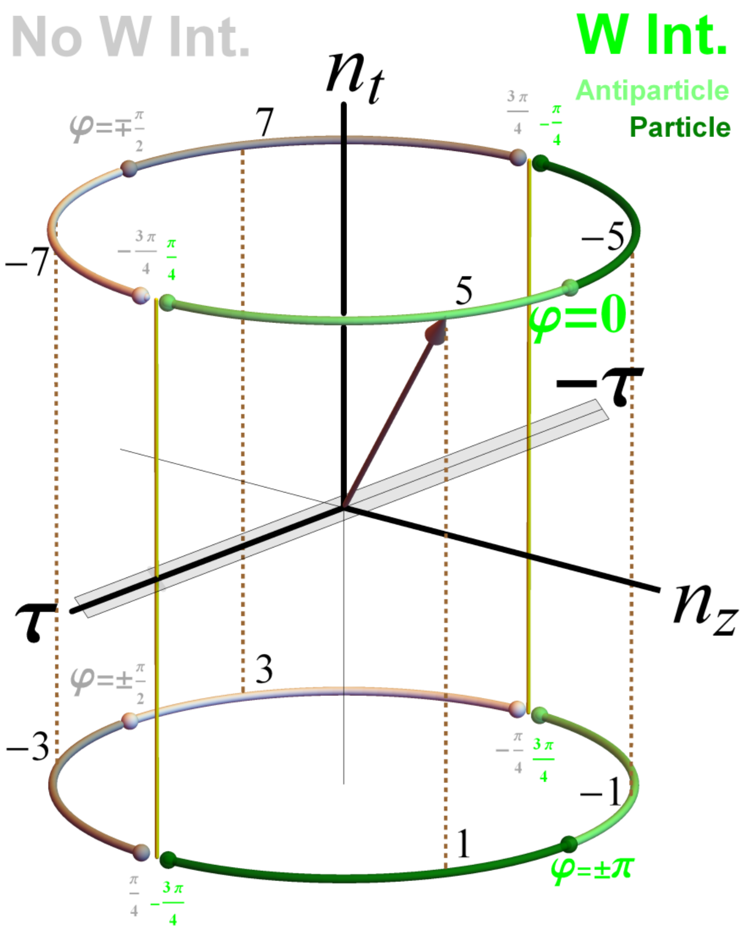


FIG. 9: Weak interaction.

E. Weak interaction.

We consider the elementary fermions as weakly interacting particles under the following rule:

$$\begin{cases} 0 \le |2\varphi| < \frac{\pi}{2} \quad [\,n_z > 0\,], & \text{with weak interaction} \\ \frac{\pi}{2} < |2\varphi| \le \pi \quad [\,n_z > 0\,], & \text{without weak interaction} \end{cases} . \tag{4.20}$$

With the special treatment for $|2\varphi| = \frac{\pi}{2}$ $(n_z = 0)$ (charged leptons).

Or with the vector-spin with $0 \le |2\varphi| < \frac{\pi}{2}$ $(\text{m}(2\varphi_R)$ and $\text{m}(-2\varphi_R))$. This is depicted in the Figures 7 and 10.

We also present the following summary:

		Angle	Charge	Spin	$\mathbf{n_t}$ Chirality	vector-spin	W− Interaction
5	**reference**	φ_R	$\mathbf{q}(\varphi_\mathbf{R})$	↑	+ rc	$2\varphi_R$	W
3	opposite $n_t n_z \tau$	$\varphi_\mathbf{R} - \frac{\pi}{2}$	$\mathbf{q}(\varphi_\mathbf{R})$	↑	− lc	$2\varphi_R - \pi$	−
−1	opposite $n_t \tau$	$-\varphi_\mathbf{R} + \pi$	$\mathbf{q}(\varphi_\mathbf{R})$	↓	− rc	$-2\varphi_R$	W
−7	opposite n_z	$-\varphi_\mathbf{R} - \frac{\pi}{2}$	$\mathbf{q}(\varphi_\mathbf{R})$	↓	+ lc	$-2\varphi_R + \pi$	−
1	opposite n_t	$\varphi_\mathbf{R} - \pi$	$-\mathbf{q}(\varphi_\mathbf{R})$	↓	− lc	$2\varphi_R$	W
7	opposite $n_z\,\tau$	$\varphi_\mathbf{R} + \frac{\pi}{2}$	$-\mathbf{q}(\varphi_\mathbf{R})$	↓	+ rc	$2\varphi_R - \pi$	−
−5	opposite τ	$-\varphi_R$	$-\mathbf{q}(\varphi_\mathbf{R})$	↑	+ lc	$-2\varphi_R$	W
−3	opposite $n_t n_z$	$-\varphi_\mathbf{R} + \frac{\pi}{2}$	$-\mathbf{q}(\varphi_\mathbf{R})$	↑	− rc	$-2\varphi_R + \pi$	−

(4.21)

With the developed elements we can not fix the positiveness or negativeness of the value $\mathbf{q}(\varphi_\mathbf{R})$. We have fixed the relative values. We arbitrarily fix the sign of $\mathbf{q}(\varphi_\mathbf{R} - \pi)$ with the left chiral character at $n_t < 0$, $n_z \ge 0$ and $\tau \ge 0$ of our particles, neutrino, electron, u-quark and d-quark. In order to establish this criteria, we have considered the massive neutrino, as a highly relativistic particle, with its left helicity to have also left chirality, and the corresponding criteria for the antiparticles. Beneath, semidirection of momentum coincident with the positive n_z semidirection. This last assertion implies the local structure of the time space, which we discuss elsewhere.

An interesting fact: for the ones with same charge and spin, we have a vector-spin and a chirality and their opposites:

$$\begin{array}{llll} -\mathbf{q}(\varphi_\mathbf{R}) \,,\, \downarrow \,, & \left\{ \left(\mathbf{R}_e^{\,2\varphi_R}\right)_{M-1} \text{lc} \;;\; \left(-\mathbf{R}_e^{\,2\varphi_R}\right)_{M-1} \text{rc} \right\} & & (\;1,\;\;7) \\ -\mathbf{q}(\varphi_\mathbf{R}) \,,\, \uparrow \,, & \left\{ \left(\mathbf{R}_e^{\,-2\varphi_R}\right)_{M-1} \text{lc} \;;\; \left(-\mathbf{R}_e^{\,-2\varphi_R}\right)_{M-1} \text{rc} \right\} & & (-5,-3) \\ \mathbf{q}(\varphi_\mathbf{R}) \,,\, \uparrow \,, & \left\{ \left(\mathbf{R}_e^{\,2\varphi_R}\right)_{M-1} \text{rc} \;;\; \left(-\mathbf{R}_e^{\,2\varphi_R}\right)_{M-1} \text{lc} \right\} & & (\;5,\;\;3) \\ \mathbf{q}(\varphi_\mathbf{R}) \,,\, \downarrow \,, & \left\{ \left(\mathbf{R}_e^{\,-2\varphi_R}\right)_{M-1} \text{rc} \;;\; \left(-\mathbf{R}_e^{\,-2\varphi_R}\right)_{M-1} \text{lc} \right\} & & (-1,-7) \end{array} . \tag{4.22}$$

For every value of φ_R, considering the symmetries in $\{\, n_t , n_z , \tau \,\}$, we have the 8 possibilities depicted below (4.15):

$$\underline{\textit{2 with } n_t \quad \times \quad \textit{2 with } n_z \quad \times \quad \textit{2 with } \tau \;,}$$

except for $\tau = 0$ (neutrinos) and for $n_z = 0$ (electrons and positrons), although in a different way. For $n_z = 0$, still remain the 8 possibilities. And only 4 for $\tau = 0$. If we characterize every particle related to a value of φ_R by:

the charge (+ ,−) × the spin (↑ , ↓) × the chirality (← , →) ,

we also have 8 possibilities (except for $\varphi_R = 0$). And, with the type of interaction:

{ electro − magneto , weak } interaction .

In brief, in the present framework, we relate:

the symmetries of time space - the three physical magnitudes - the type of interaction .

In regards to the electric charge, a particle and an antiparticle are different types of particles. In a similar way, regarding the spin, a particle with a charge and a spin up is different to a particle with the same charge and spin down. The electric charge and the spin are invariants for a particle. The treatments of the vector-spin or of the chirality are different.

After these concepts, an important point is the distinction matter - antimatter, or (elementary) particle(s) - anti-particle(s) for the neutrino family. The discussion on this subject is postponed to the corresponding section and to Part II.

If instead of chirality we look at vector-spin, $\pm\text{m}(\varphi_R)$, $\pm\text{m}(-\varphi_R)$), we would have 2 x 2 x 4 =**16** possibilities. But after the conditions in (4.16) for the charge, which are related to the values of the spin (4.10) and of the vector-spin (4.14), they get a reduction by a factor 2 (see the comment at the end of the vector-spin). This would not apply to $\varphi_R = 0$ (neutrinos), which is reduced by a factor 4.

F. Elementary fermions.

We suggest the following families of particles:

1) for every chosen value of φ_R in (4.3) we have a different family of fermions:

a) Leptons: i) uncharged, $\varphi_R = 0$, neutrinos (1x2x2=4)

ii) charged ∓ 1, $\varphi_R = \frac{\pi}{4}$, electrons-positrons (2x2x2=8)

b) Quarks: i) charged $\mp\frac{1}{3}$, $\varphi_R = \frac{\pi}{12}$, d-type (2x2x2=8)

ii) charged $\pm\frac{2}{3}$, $\varphi_R = \frac{\pi}{6}$, u-type (2x2x2=8)

c) Unknowns: uncharged, $\varphi_R = \frac{\pi}{8}$, a-type neutrinons(?) ((1+1)x2x2=2x4)

$2 + 2 + 1 = \mathbf{5}$ types of fermions (4 in the standard model , 1 unknown) ,

2) to include the spin charge, up or down, drives to multiply this previous number by 2, a total number of 10. The addition of the electrical charge, + or −, multiplies this number by 2 again, except for the neutrinos with their zero electric charge. Finally, with the inclusion either of the chirality, left or right, or either of the vector-spin (these properties overlap), there is another 2 factor. With these properties we get a total of 36 different creation operators,

3) defining an elementary fermion as a particle constituted by the creation operators with the same charge and spin but combining opposite chiralities or opposite signs in two vectors-spin, we get a total number of distinct elementary fermions (same charge and spin):

$$(1 + 2) \times 2 + (2 + 2) \times 2 + (1 + 1) \times 2 = 14 + 4 = 18 \quad \text{(14 in the standard model, 4 unknowns).}$$

Up to this point, it participates only the electro-magneto-weak interaction. With the introduction of the QCD, color, we have to multiply the number of quarks by 3. And adding gravity, again by 3, for all of them. Therefore, we have a total number of distinct elementary fermions:

$$[(1 + 2) \times 2 \times 1 + (2 + 2) \times 2 \times 3 + (1 + 1) \times 2 \times 1] \times 3 = 90 + 12 = 102 \quad \text{(90 in the standard model, 12 unknowns),}$$

At this stage it is not possible to consider the flavor. Neither we deal with the color.

4) we consider these fermions as "flipping structures" as they evolve ($\downharpoonleft\upharpoonright$ and $\upharpoonleft\downharpoonright$) in the time space:

$$\begin{array}{lcl}
\langle\ \mathbf{U}^{\dagger}_{1,\mathcal{M}}\ \upharpoonleft\downharpoonright\,,\ \downharpoonleft\upharpoonright\ \mathbf{U}^{\dagger}_{7,\mathcal{M}}\ \rangle & \text{with} & \langle\ \overset{\overset{2\varphi_R}{\longleftarrow}}{1\downarrow}\ \upharpoonleft\downharpoonright\,,\ \downharpoonleft\upharpoonright\ \overset{\overset{2\varphi_R-\pi}{\longrightarrow}}{7\ \downarrow}\ \rangle_{\mathcal{M}} \\
\langle\ \mathbf{U}^{\dagger}_{-5,\mathcal{M}}\ \downharpoonleft\upharpoonright\,,\ \upharpoonleft\downharpoonright\ \mathbf{U}^{\dagger}_{-3,\mathcal{M}}\ \rangle & \text{with} & \langle\ \overset{\overset{-2\varphi_R}{\longrightarrow}}{-5\ \uparrow}\ \downharpoonleft\upharpoonright\,,\ \upharpoonleft\downharpoonright\ \overset{\overset{-2\varphi_R+\pi}{\longleftarrow}}{-3\ \uparrow}\ \rangle_{\mathcal{M}} \\
\langle\ \mathbf{U}^{\dagger}_{5,\mathcal{M}}\ \downharpoonleft\upharpoonright\,,\ \upharpoonleft\downharpoonright\ \mathbf{U}^{\dagger}_{3,\mathcal{M}}\ \rangle & \text{with} & \langle\ \overset{\overset{2\varphi_R}{\longrightarrow}}{5\uparrow}\ \downharpoonleft\upharpoonright\,,\ \upharpoonleft\downharpoonright\ \overset{\overset{2\varphi_R-\pi}{\longleftarrow}}{3\ \uparrow}\ \rangle_{\mathcal{M}} \\
\langle\ \mathbf{U}^{\dagger}_{-1,\mathcal{M}}\ \upharpoonleft\downharpoonright\,,\ \downharpoonleft\upharpoonright\ \mathbf{U}^{\dagger}_{-7,\mathcal{M}}\ \rangle & \text{with} & \langle\ \overset{\overset{-2\varphi_R}{\longleftarrow}}{-1\ \downarrow}\ \upharpoonleft\downharpoonright\,,\ \downharpoonleft\upharpoonright\ \overset{\overset{-2\varphi_R+\pi}{\longrightarrow}}{-7\ \downarrow}\ \rangle_{\mathcal{M}}
\end{array} \quad . \qquad (4.23)$$

This scheme has to be modified for neutrinos, $\varphi_R = 0$.

In order to prove the statement in 4) we need two more ingredients (there are elements of this in **2**):

i) to introduce the action of rotations and boosts over these operators, and

ii) to develop a formalism based in γ-type matrices.

Even so, in the following sections we will name *particle* to every creation or annihilation operator,

5) *weak interaction.* They interact under it with the part of the different particles for which $n_z > 0$, with a special treatment for the charged leptons $n_z = 0$,

6) *anomaly cancellation:*

$$\mathbf{q}_{(\varphi_{\mathbf{R}}=\mathbf{0})} + 3 \times (\mathbf{q}_{(\varphi_{\mathbf{R}}=\frac{\pi}{12})} + \mathbf{q}_{(\varphi_{\mathbf{R}}=\frac{\pi}{6})}) + \mathbf{q}_{(\varphi_{\mathbf{R}}=\frac{\pi}{4})} + (\mathbf{q}_{(\varphi_{\mathbf{R}}=\frac{\pi}{8})}) = 0 + 3\,(\tfrac{1}{3} + \tfrac{-2}{3}) + 1 + (0) = 0.$$

Similarly for the other 7 sectors in the variable φ:

$$-\pi \le \varphi \le \tfrac{-3\pi}{4} \;,\quad \tfrac{-3\pi}{4} \le \varphi \le -\tfrac{\pi}{2} \;,\quad -\tfrac{\pi}{2} \le \varphi \le \tfrac{-\pi}{4} \;,\quad \tfrac{-\pi}{4} \le \varphi < 0 \;,\quad \tfrac{\pi}{4} \le \varphi \le \tfrac{\pi}{2} \;,\quad \tfrac{\pi}{2} \le \varphi \le \tfrac{3\pi}{4} \;,\quad \tfrac{3\pi}{4} \le \varphi < \pi \,.$$

Besides the electrical charge, there are global cancellations in the spin, in the vector-spin and in the chirality. All of this due to the symmetries in the time space underlying our construction of the creation operators of the elementary fermions.

G. Anticommutators for the creation and annihilation operators.

With the product defined in (4.7), the anticommutators are:

$$\left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} \right\} \equiv \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{\bullet} \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} + \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} \overset{\mathbf{I}}{\bullet} \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} , \tag{4.24}$$

which they are, for:

$$m_1 \neq m_2 ; \quad \begin{matrix} r = s \\ r \neq s \end{matrix} ; \quad \left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} \right\} \equiv \left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} , \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} \right\} , \tag{4.25a}$$

$$m_1 = m_2 = m ; \quad r = s ; \quad \left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}} \overset{\mathbf{I}}{;} \mathbf{U}^{**}_{\lambda r,\mathcal{M}} \right\} \equiv \left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}} , \mathbf{U}^{**}_{\lambda r,\mathcal{M}} \right\} , \tag{4.25b}$$

$$m_1 = m_2 = m ; \quad r \neq s ; \quad \left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}} \overset{\mathbf{I}}{;} \mathbf{U}^{**}_{\lambda s,\mathcal{M}} \right\} \equiv \mathbf{U}^{*}_{\lambda r,\mathcal{M}} \mathbf{I}_{\lambda r\lambda s,m} \mathbf{U}^{**}_{\lambda s,\mathcal{M}} + \mathbf{U}^{**}_{\lambda s,\mathcal{M}} \mathbf{I}_{\lambda s\lambda r,m} \mathbf{U}^{*}_{\lambda r,\mathcal{M}} . \tag{4.25c}$$

The values of these anticommutators can be expressed in a similar form to (2.11):

$$\begin{cases} \left\{ \mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}_{\lambda s,\mathcal{M}_2} \right\} = \delta_{m_1 m_2}\, \delta_{rs}\ \mathbf{1} \\ \left\{ \mathbf{U}^{\dagger}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}^{\dagger}_{\lambda s,\mathcal{M}_2} \right\} = \left\{ \mathbf{U}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}_{\lambda s,\mathcal{M}_2} \right\} = \mathbf{0} \end{cases} , \tag{4.26}$$

$\lambda \in \{+,-\}$, $\{r,s\} = \{1,3,5,7\}$, or $\{r,s\} = \{2,4,6,8\}$ and $\{m_1, m_2\} = \{2,\ldots,N\}$.

To obtain these values we have used the results of subsection III-A, where in all of them we apply (3.12), and by using (3.14) we get:

$$\left\{ \mathbf{U}^{*}_{\lambda r,\mathcal{M}_1} \overset{\mathbf{I}}{;} \mathbf{U}^{**}_{\lambda s,\mathcal{M}_2} \right\} = \mathbf{0} , \quad \text{with} \quad m_1 \neq m_2 .$$

The ones with $m_1 = m_2 = m$ are implicit in (3.11). These results follow the sketch:

$$\begin{array}{l} \text{for r and s equal or different (any } m_1 \neq m_2 ,\ \forall\{r,s\}): \\ \alpha) \qquad \sigma^z_k\ \sigma^z_k = (-\sigma^z_k)(-\sigma^z_k) = \mathbb{1}_N , \qquad \sigma^z_k\,(-\sigma^z_k) = (-\sigma^z_k)\,\sigma^z_k = -\,\mathbb{1}_N \\ \beta) \qquad \sigma^z_m\ \sigma^\pm_m = -\,\sigma^\pm_m\,\sigma^z_m = \pm\,\sigma^\pm_m \quad \Rightarrow \quad \{\sigma^z_m , \sigma^\pm_m\} = \pm\sigma^\pm_m \mp \sigma^\pm_m = \mathbb{0}_N \\ \text{and for } (m_1 = m_2 = m): \\ \gamma)\ r = s \quad a) \quad (\sigma^\pm_m)^2 = \mathbb{0}_N \quad \Rightarrow \quad \{\sigma^\pm_m , \sigma^\pm_m\} = \mathbb{0}_N \\ \qquad\qquad\quad b) \quad \{\sigma^+_m , \sigma^-_m\} = \mathbb{1}_N \\ \delta)\ r \neq s \quad a) \quad \sigma^\pm_m\ \mathbf{I}_{\lambda r\lambda s,m}\ \sigma^\mp_m = \mathrm{sign}(r-s)\ \sigma^\pm_m\ \mathbf{R}^{\widetilde{\epsilon\frac{\pi}{2}}}_{O_m}\ \sigma^\mp_m = \mathbb{0}_N \\ \qquad\qquad\quad b) \quad \sigma^\pm_m\ \mathbf{I}_{\lambda r\lambda s,m}\ \sigma^\pm_m = -\,\sigma^\pm_m\ \mathbf{I}_{\lambda s\lambda r,m}\ \sigma^\pm_m \quad \Rightarrow \quad \sigma^\pm_m\,\mathbf{I}_{\lambda r\lambda s,m}\,\sigma^\pm_m + \sigma^\pm_m\,\mathbf{I}_{\lambda s\lambda r,m}\,\sigma^\pm_m = \mathbb{0}_N \end{array} , \tag{4.27}$$

$\alpha)$, $\beta)$ and $\gamma)$ correspond to the Jordan and Wigner transformation.

Comparing the anticommutator relationships satisfied by the operators in (2,11) (beneath is (2.9)) and these new ones in (4.26) (up to the introduction of the momentum **p**, and dropping the m), we observe certain parallelism between them: with both we can interpret the existence of particles and antiparticles and spin up and down. And, important differences. (4.26) with previous concepts (charge, spin, vector-spin and chirality) add the following **intrinsic** novel features to the creation and annihilation operators of the particles involved:

1) **anticommutators** with an *intermediary* product,

2) **electrical charges** get their discrete values with specific values of a geometrical parameter,

3) **spin:** with only two values, denoted as **up** and **down**, which are related to time and space values.,

4) **vector-spin**, also depends of the specific value of a parameter. We can relate it with the chirality. Both of them manifest a sense of opposition, taken as (left , right) and as (+ , −) as a (partial) convention, and with

5) relations (4.23) set our elementary fermions as having both vectors-spin and chiralities "at once", although with $n_t = -1$ and $n_t = +1$. Anticommutators have to be modified to include a formulation with the γ matrices,

6) the neutrino family satisfies the Pauli's exclusion principle. Similarly the charged leptons. Also the rest of the elementary particles, but in two separated sub-families (with $\lambda = \pm$). In the standard model we impose the antisymmetry of the operators of all the fermions, and then, the Pauli's exclusion principle is guaranteed.

Point 1) implies a subtle distinction involving the geometry and the particle - antiparticle interpretation (equation (4.8)).

$\mathcal{B}$) *SPECIFIC FERMIONS.*

V. I-1) LEPTONS I: NEUTRINOS. $\varphi_R = 0,$

$$\{ 2\varphi_R \} \rightarrow \{ 0 , \pi \} . \quad (\tau = 0 , \ \{ n_t , n_z \} \in \{ +1 , -1 \}).$$

Creation and annihilation operators (Figure 11 in Appendix E):

We define the following neutrino creation and annihilation operators:

$$\begin{array}{ll}
\mathbf{V}\overset{\overset{0}{\leftarrow}}{\downarrow} & \mathbf{V}^{\dagger}_{\lambda 1,\mathcal{M}} \equiv \left(\ \sigma^z \ \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (-\mathrm{m}^- \sigma^+) \, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \\
\underline{\overline{\mathbf{V}}}\overset{\overset{\pi}{\rightarrow}}{\downarrow} & \mathbf{V}^{\dagger}_{\lambda 7,\mathcal{M}} \equiv \left(-\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (\ \mathrm{m}^- \sigma^+) \, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \\
\underline{\overline{\mathbf{V}}}\overset{\overset{0}{\rightarrow}}{\uparrow} & \mathbf{V}^{\dagger}_{\lambda 5,\mathcal{M}} \equiv \left(\ \sigma^z \right)_{M-1} \left(\qquad \mathrm{m}^- \sigma^+ \qquad \right)_m \\
\underline{\mathbf{V}}\overset{\overset{\pi}{\leftarrow}}{\uparrow} & \mathbf{V}^{\dagger}_{\lambda 3,\mathcal{M}} \equiv \left(-\sigma^z \right)_{M-1} \left(\quad -\mathrm{m}^- \sigma^+ \qquad \right)_m
\end{array} \quad . \tag{5.1}$$

With the no dependence on λ we have another important consequence: we do not need to define the neutrino or the antineutrino in the sense of opposite electrical charge, as they are determined by the spin. Therefore, we have two distinct particles, but not characterized by their equal or different electric charge, which is zero, but by their opposite spins.

We summarize the particles obtained with this value $\varphi_R = 0$ in the following table:

λr	φ	Charge (φ)	Spin (φ)	vector-spin (2φ)	Chirality (φ)	*Particle*		
± 1	π	0	$\downarrow$	0	$\leftarrow$	$\mathbf{V}\overset{\overset{0}{\leftarrow}}{\downarrow}$	0-mi lc	down neutrino
± 7	$\mp\frac{\pi}{2}$	0	$\downarrow$	π	$\rightarrow$	$\underline{\overline{\mathbf{V}}}\overset{\overset{\pi}{\rightarrow}}{\downarrow}$	π-mi rc	
± 5	0	0	$\uparrow$	0	$\rightarrow$	$\overline{\mathbf{V}}\overset{\overset{0}{\rightarrow}}{\uparrow}$	0-mi rc	up antineutrino
± 3	$\pm\frac{\pi}{2}$	0	$\uparrow$	π	$\leftarrow$	$\underline{\mathbf{V}}\overset{\overset{\pi}{\leftarrow}}{\uparrow}$	π-mi lc	

,

in brief: $\mathbf{V}, \underline{\overline{\mathbf{V}}}$ for neutrino (spin down $\downarrow$) , $\overline{\mathbf{V}}, \underline{\mathbf{V}}$ for antineutrino (spin up $\uparrow$)

0 electric charge; 0 , π for the vectors-spin; $\leftarrow$, $\rightarrow$ for a left or a right chirality .

It is interesting to pay attention to:

$$\left\{ \begin{array}{ll}
\text{a vector-spin } \mathbf{R}^0_{e,M-1} = \left(\ \sigma^z \ \right)_{M-1} = \prod_{k=1}^{m-1} \sigma^z_k \, , & \text{both chiralities,} \\
\text{a vector-spin } \mathbf{R}^{\pi}_{e,M-1} = \left(-\sigma^z \ \right)_{M-1} = \prod_{k=1}^{m-1} (-\sigma^z)_k \, , & \text{both chiralities,} \\
\text{a chirality both vectors-spin, one and its opposed one,} & \\
\text{the value of the operators do not depend of } \lambda : \text{ the } \lambda \text{ in the sub-index can be dropped ,} & \\
\mathbf{V}^{\dagger}_{1,\mathcal{M}} = (-1)^m \, \mathbf{V}^{\dagger}_{7,\mathcal{M}} = -[\mathbf{V}_{5,\mathcal{M}}] = -[(-1)^m \, \mathbf{V}_{3,\mathcal{M}}] \, . &
\end{array} \right. \tag{5.2}$$

The fifth line in (5.2) is a particularization of the formula (4.8), with the substitution of the "**U**" by the "**V**"; in this way the comments previous to (4.8) are pertinent here. But with some modifications, which we will treat in **2**.

The *number operators* are :

$$\mathbf{N}_{r,m} \equiv \mathbf{V}^{\dagger}_{r,\mathcal{M}}\,\mathbf{V}_{r,\mathcal{M}} = \begin{cases} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}}\sigma^+\sigma^-\mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\right)_m = \left(\sigma^-\sigma^+\right)_m = \left(\overset{\vee}{\sigma}\right)_m, & r=\{1,7\} \\ \left(\sigma^+\sigma^-\right)_m = \left(\overset{\wedge}{\sigma}\right)_m, & r=\{3,5\} \end{cases} . \tag{5.3}$$

they satisfy

$$\begin{aligned} \mathbf{N}_{r,m}\,\mathbf{V}^{\dagger}_{r,\mathcal{M}} &= 1\,\mathbf{V}^{\dagger}_{r,\mathcal{M}}, & \mathbf{N}_{\lambda r,m}\,\mathbf{V}_{r,\mathcal{M}} &= 0\,\mathbf{V}_{r,\mathcal{M}}, \\ \mathbf{1}-\mathbf{N}_{r,m} &= \mathbf{V}_{\lambda r,\mathcal{M}}\,\mathbf{V}^{\dagger}_{r,\mathcal{M}}; & \mathbf{N}_{r,m}\,(\mathbf{1}-\mathbf{N}_{r,m}) &= \mathbf{0}. \end{aligned} \tag{5.4}$$

We denote $\mathbf{V}_{r,\mathcal{M}}$ $\begin{cases} \text{with } r\in\{1,7\} \text{ as neutrino, and we represent them with the symbols } \mathbf{V},\ \underline{\overline{\mathbf{V}}},\ \text{and spin down} \\ \text{with } r\in\{5,3\} \text{ as antineutrino, with the symbols } \overline{\mathbf{V}},\ \underline{\mathbf{V}},\ \text{and spin up} \end{cases}$.

Anti $\equiv$ opposite spin, in the sense of opposition $\begin{cases} \text{in space, represented by an } \textbf{under-bar} \\ \text{in time, represented by an } \textbf{over-bar} \end{cases}$.

The elementary fermions obtain the *anti* character in terms of oppositions either in time n_t or either in $\mathbb{R}^3$ $(\tau(n_y)\cup n_z)$; these two also imply opposition in spin. A third way with the opposition only in $\tau(n_y)$; under this one the spin does not change. Neutrinos do not have this last possibility.

Anticommutators for the creation and annihilation neutrino operators.

Using (4.27), these anticommutators, can be written in a similar form to (2.11):

$$\text{for:}\quad \overset{\overset{0}{\leftarrow}}{\mathbf{V}\downarrow} \qquad \overset{\overset{\pi}{\rightarrow}}{\underline{\overline{\mathbf{V}}}\downarrow} \qquad \overset{\overset{0}{\rightarrow}}{\overline{\mathbf{V}}\uparrow} \qquad \overset{\overset{\pi}{\leftarrow}}{\underline{\mathbf{V}}\uparrow}$$

$$\left\{\mathbf{V}^{\dagger}_{1,\mathcal{M}},\mathbf{V}_{1,\mathcal{M}}\right\} = \left\{\mathbf{V}^{\dagger}_{7,\mathcal{M}},\mathbf{V}_{7,\mathcal{M}}\right\} = \left\{\mathbf{V}^{\dagger}_{5,\mathcal{M}},\mathbf{V}_{5,\mathcal{M}}\right\} = \left\{\mathbf{V}^{\dagger}_{3,\mathcal{M}},\mathbf{V}_{3,\mathcal{M}}\right\} = \mathbf{1}, \tag{5.5}$$

Any other anticommutator is zero:

$$\begin{aligned} &\left\{\mathbf{V}^{*}_{r,\mathcal{M}} \,\vdots\, \mathbf{V}^{*}_{r,\mathcal{M}}\right\} = \left\{\mathbf{V}^{*}_{r,\mathcal{M}}\ ,\ \mathbf{V}^{*}_{r,\mathcal{M}}\right\} = \mathbf{0}, && (r=s\,,\ m_1=m_2=m)\,, \\ &\left\{\mathbf{V}^{*}_{r,\mathcal{M}} \,\vdots\, \mathbf{V}^{**}_{s,\mathcal{M}}\right\} = \mathbf{0}, && (r\neq s\,,\ m_1=m_2=m)\,, \\ &\left\{\mathbf{V}^{*}_{r,M_1} \,\vdots\, \mathbf{V}^{**}_{s,M_2}\right\} = \left\{\mathbf{V}^{*}_{r,M_1}\ ,\ \mathbf{V}^{**}_{s,M_2}\right\} = \mathbf{0}, && (m_1\neq m_2)\,, \end{aligned} \tag{5.6}$$

$\{*\,,**\}\in\{``\dagger"\ ,\ ``\ \ "\}$, $\{r\,,\,s\}\in\{1\,,3\,,5\,,7\}$, $\{m\,,\,m_1\,,\,m_2\}\in\{2\,,\ldots,\,N\}$.

Departing from the comments after (4.27), we write three specific points for neutrinos:

1) $\mathbf{V}$, $\underline{\overline{\mathbf{V}}}$ operators with only spin down, and $\overline{\mathbf{V}}$, $\underline{\mathbf{V}}$ operators with only spin up.
The *spin is <u>intrinsic</u> (incorporated) in a <u>singular</u> way*,

2) depending on a specific angle parameter, we have here:

vector-spin $(2\varphi_R)$:	0	for $\tau=0,\ n_z=\ 1,\ n_t=\pm1$,	and
	π	for $\tau=0,\ n_z=-1,\ n_t=\pm1$,	

3)

chirality:	left	for $\tau=0,\ n_z=\pm1,\ n_t=-1$,	and
	right	for $\tau=0,\ n_z=\pm1,\ n_t=\ 1$.	

And a presumed:

4) even with $\tau=0$, there could be non null ϕ´s in our definitions (in $\mathfrak{m}^{\pm}$ and in $\mathbf{R}_O^{\pm\frac{\pi}{2}}$ in (5.1)), with a reminiscent role of the one with the φ angle in $\{(\pm\frac{\pi}{4},\frac{3\pi}{4}),\pm\frac{\pi}{2}\}$, but now with expected 3 values due to the 3 generations.

We suggest: $\overset{\overset{0}{\leftarrow}}{\mathbf{V}\downarrow}$ and $\overset{\overset{0}{\rightarrow}}{\overline{\mathbf{V}}\uparrow}$ interact under the electroweak interaction. For them $\tau=0$, $n_z=1$. And:

$$\begin{aligned} &\langle\,\mathbf{V}^{\dagger}_{1,\mathcal{M}}\Downarrow,\ \Updownarrow\mathbf{V}^{\dagger}_{7,\mathcal{M}}\,\rangle && \text{with} && \langle\,\overset{\overset{0}{\leftarrow}}{\mathbf{V}\downarrow}\ \Downarrow,\ \Uparrow\ \overset{\overset{\pi}{\rightarrow}}{\underline{\overline{\mathbf{V}}}\downarrow}\,\rangle_{\mathcal{M}} \\ &\langle\,\mathbf{V}^{\dagger}_{5,\mathcal{M}}\Uparrow,\ \Downarrow\ \mathbf{V}^{\dagger}_{3,\mathcal{M}}\,\rangle && \text{with} && \langle\,\overset{\overset{0}{\rightarrow}}{\overline{\mathbf{V}}\uparrow}\ \Uparrow,\ \Downarrow\ \overset{\overset{\pi}{\leftarrow}}{\underline{\mathbf{V}}\uparrow}\,\rangle_{\mathcal{M}} \end{aligned} , \tag{5.7}$$

so that in a first attempt, this could represent neutrinos (spin down) and antineutrinos (spin up), each with both chiralities and vectors-spin, evolving in time space. In **2** we will explore the reason for observing only a particular helicity for each one, even with both chiralities.

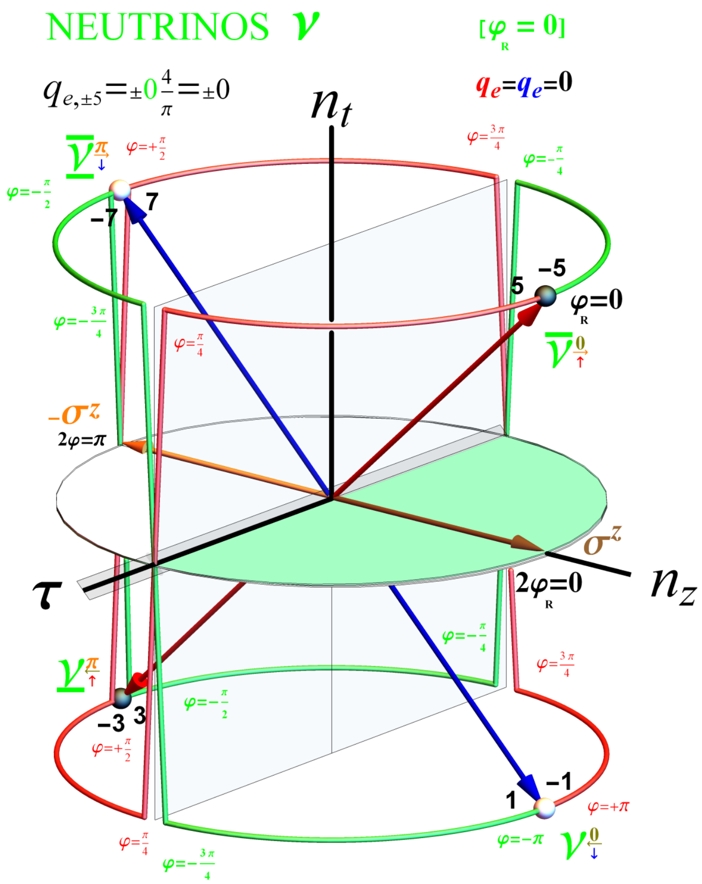


FIG. 9: Neutrinos.

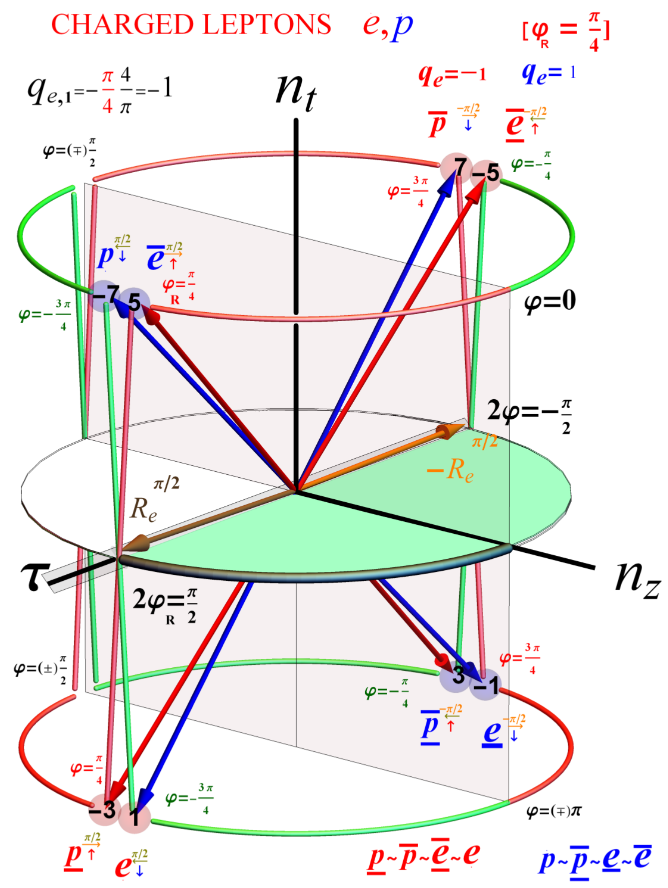


FIG. 9: Charged Leptons.

VI. I-2) LEPTONS II: ELECTRONS AND POSITRONS. $\varphi_R = \frac{\pi}{4}$,

$$\{\, 2\varphi_R \,\} \rightarrow \{\, +\tfrac{\pi}{2}, -\tfrac{\pi}{2} \,\} . \quad (\; n_z = 0 \,,\; \{n_t, \tau\} \in \{+1, -1\} \;) .$$

Creation and annihilation operators (Figure 12 in Appendix E):

$$\begin{aligned}
e\,\overset{\overset{\pi/2}{\leftarrow}}{\downarrow} \qquad & l^{\dagger}_{1,\mathcal{M}} \equiv \mathbf{R}_{OM}^{\frac{\pi}{4}} \Big[\big(\; \sigma^{z} \big)_{M-1} \Big(\mathbf{R}_{O}^{\epsilon\frac{\pi}{2}} (-\mathfrak{m}^{-}\, \sigma^{+})\, \mathbf{R}_{O}^{-\epsilon\frac{\pi}{2}} \Big)_{m} \Big] \mathbf{R}_{OM}^{-\frac{\pi}{4}} \\
\underline{p}\,\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow} \qquad & l^{\dagger}_{7,\mathcal{M}} \equiv \mathbf{R}_{OM}^{\frac{\pi}{4}} \Big[\big(-\sigma^{z} \big)_{M-1} \Big(\mathbf{R}_{O}^{\epsilon\frac{\pi}{2}} (\, \mathfrak{m}^{-}\, \sigma^{+})\, \mathbf{R}_{O}^{-\epsilon\frac{\pi}{2}} \Big)_{m} \Big] \mathbf{R}_{OM}^{-\frac{\pi}{4}} \\
\overline{e}\,\overset{\overset{\pi/2}{\rightarrow}}{\uparrow} \qquad & l^{\dagger}_{5,\mathcal{M}} \equiv \mathbf{R}_{OM}^{\frac{\pi}{4}} \Big[\big(\; \sigma^{z} \big)_{M-1} \qquad \big(\; \mathfrak{m}^{-}\, \sigma^{+} \big)_{m} \qquad \Big] \mathbf{R}_{OM}^{-\frac{\pi}{4}} \\
\underline{\overline{p}}\,\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow} \qquad & l^{\dagger}_{3,\mathcal{M}} \equiv \mathbf{R}_{OM}^{\frac{\pi}{4}} \Big[\big(-\sigma^{z} \big)_{M-1} \qquad \big(-\mathfrak{m}^{-}\, \sigma^{+} \big)_{m} \qquad \Big] \mathbf{R}_{OM}^{-\frac{\pi}{4}}
\end{aligned} \qquad (6.1)$$

$$\begin{aligned}
\underline{\overline{e}}\,\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow} \qquad & l^{\dagger}_{-5,\mathcal{M}} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{4}} \Big[\big(\; \sigma^{z} \big)_{M-1} \qquad \big(\; \mathfrak{m}^{-}\, \sigma^{+} \big)_{m} \qquad \Big] \mathbf{R}_{OM}^{\frac{\pi}{4}} \\
\overline{p}\,\overset{\overset{\pi/2}{\rightarrow}}{\uparrow} \qquad & l^{\dagger}_{-3,\mathcal{M}} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{4}} \Big[\big(-\sigma^{z} \big)_{M-1} \qquad \big(-\mathfrak{m}^{-}\, \sigma^{+} \big)_{m} \qquad \Big] \mathbf{R}_{OM}^{\frac{\pi}{4}} \\
\underline{e}\,\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow} \qquad & l^{\dagger}_{-1,\mathcal{M}} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{4}} \Big[\big(\; \sigma^{z} \big)_{M-1} \Big(\mathbf{R}_{O}^{\epsilon\frac{\pi}{2}} (-\mathfrak{m}^{-}\, \sigma^{+})\, \mathbf{R}_{O}^{-\epsilon\frac{\pi}{2}} \Big)_{m} \Big] \mathbf{R}_{OM}^{\frac{\pi}{4}} \\
p\,\overset{\overset{\pi/2}{\leftarrow}}{\downarrow} \qquad & l^{\dagger}_{-7,\mathcal{M}} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{4}} \Big[\big(-\sigma^{z} \big)_{M-1} \Big(\mathbf{R}_{O}^{\epsilon\frac{\pi}{2}} (\, \mathfrak{m}^{-}\, \sigma^{+})\, \mathbf{R}_{O}^{-\epsilon\frac{\pi}{2}} \Big)_{m} \Big] \mathbf{R}_{OM}^{\frac{\pi}{4}}
\end{aligned} \qquad (6.2)$$

Observe that p is a label for $\overline{e}$. Therefore we relate $\left\{\begin{array}{l}\text{positron}\\ \text{electron}\end{array}\right. \longleftrightarrow$ anti-$\left\{\begin{array}{l}\text{electron}\\ \text{positron}\end{array}\right. \longleftrightarrow$ (anti-anti)-$\left\{\begin{array}{l}\text{positron}\\ \text{electron}\end{array}\right.$. The electrons and the positrons have opposite electrical charges with a geometrical correspondence, in the sense of time or space oppositions; this last one meaningful in $\mathbb{R}^3\ (\tau \cup n_z)$ and also in $\mathbb{R}^2\ (\tau)$.

We summarize the particles obtained with this value $\varphi_R = \frac{\pi}{4}$ in the following table (particle - antiparticle):

λr	φ	Charge (φ)	Spin (φ)	vector-spin (2φ)	Chirality (φ)	*Particle*			
1	$-\frac{3\pi}{4}$	-1	$\downarrow$	$\frac{\pi}{2}$	$\leftarrow$	$e\,\overset{\overset{\pi/2}{\leftarrow}}{\downarrow}$	$\frac{\pi}{2}$-mi lc	down	electron
7	$\frac{3\pi}{4}$	-1	$\downarrow$	$-\frac{\pi}{2}$	$\rightarrow$	$\underline{p}\,\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow}$	$-\frac{\pi}{2}$-mi rc (anti-positron)		
-5	$-\frac{\pi}{4}$	-1	$\uparrow$	$-\frac{\pi}{2}$	$\leftarrow$	$\underline{\overline{e}}\,\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow}$	$-\frac{\pi}{2}$-mi lc	up	
-3	$\frac{\pi}{4}$	-1	$\uparrow$	$\frac{\pi}{2}$	$\rightarrow$	$\overline{p}\,\overset{\overset{\pi/2}{\rightarrow}}{\uparrow}$	$\frac{\pi}{2}$-mi rc (anti-positron)		
5	$\frac{\pi}{4}$	1	$\uparrow$	$\frac{\pi}{2}$	$\rightarrow$	$\overline{e}\,\overset{\overset{\pi/2}{\rightarrow}}{\uparrow}$	$\frac{\pi}{2}$-mi rc (anti-electron)	up	positron
3	$-\frac{\pi}{4}$	1	$\uparrow$	$-\frac{\pi}{2}$	$\leftarrow$	$\underline{\overline{p}}\,\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow}$	$-\frac{\pi}{2}$-mi lc		
-1	$\frac{3\pi}{4}$	1	$\downarrow$	$-\frac{\pi}{2}$	$\rightarrow$	$\underline{e}\,\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow}$	$-\frac{\pi}{2}$-mi rc (anti-electron)	down	
-7	$-\frac{3\pi}{4}$	1	$\downarrow$	$\frac{\pi}{2}$	$\leftarrow$	$p\,\overset{\overset{\pi/2}{\leftarrow}}{\downarrow}$	$\frac{\pi}{2}$-mi lc		

In brief $\left\{\begin{array}{l} e\,,\ \underline{\overline{e}} \text{ for electrons },\quad \overline{p}\,,\ \underline{p} \text{ for anti-positrons },\quad \text{with negative electric charge} \\ \overline{e}\,,\ \underline{e} \text{ for anti-electrons },\quad p\,,\ \underline{\overline{p}} \text{ for positrons },\quad \text{with positive electric charge} \end{array}\right.$,

$\uparrow, \downarrow$ for up or down spin ; $\frac{\pi}{2}, -\frac{\pi}{2}$ for vector-spin ; $\leftarrow, \rightarrow$ for a left or a right chirality.

Anti: either an **under-bar** or an **over-bar** to represent an opposite electrical charge.

After (3.1) and (3.3) with the value $\varphi_R = \frac{\pi}{4}$ and also with (4.8):

$$\begin{cases} \mathbf{R}_{e,M-1}^{\frac{\pi}{2}} = \left(\mathbf{R}_O^{\frac{\pi}{4}}\,\sigma^Z\,\mathbf{R}_O^{-\frac{\pi}{4}}\right)_{M-1} = \prod_{k=1}^{m-1}\left(\mathbf{R}_O^{\frac{\pi}{4}}\,\sigma^Z\,\mathbf{R}_O^{-\frac{\pi}{4}}\right)_k = \mathbf{R}_{O,M-1}^{\frac{\pi}{4}}\left(\prod_{k=1}^{m-1}\sigma_k^Z\right)\mathbf{R}_{O,M-1}^{-\frac{\pi}{4}} = \left(-\mathbf{R}_e^{-\frac{\pi}{2}}\right)_{M-1}, & \text{both chiralities,} \\ \mathbf{R}_{e,M-1}^{-\frac{\pi}{2}} = \left(\mathbf{R}_O^{-\frac{\pi}{4}}\,\sigma^Z\,\mathbf{R}_O^{\frac{\pi}{4}}\right)_{M-1} = \prod_{k=1}^{m-1}\left(\mathbf{R}_O^{-\frac{\pi}{4}}\,\sigma^Z\,\mathbf{R}_O^{\frac{\pi}{4}}\right)_k = \mathbf{R}_{O,M-1}^{\frac{\pi}{4}}\left(\prod_{k=1}^{m-1}(-\sigma^Z)_k\right)\mathbf{R}_{O,M-1}^{-\frac{\pi}{4}} = \left(-\mathbf{R}_e^{\frac{\pi}{2}}\right)_{M-1}, & \text{both chiralities,} \\ \text{for a chirality there are both vectors-spin, one and its opposed, (here, only two vectors-spin),} & \\ l^{\dagger}_{\lambda 1,\mathcal{M}} = (-1)^m\, l^{\dagger}_{\lambda 7,\mathcal{M}} = -[\,l_{\lambda 5,\mathcal{M}}] = -[\,(-1)^m\, l_{\lambda 3,\mathcal{M}}], \quad l^{\dagger}_{\lambda 5,\mathcal{M}} = l^{\dagger}_{-\lambda 7,\mathcal{M}}, \quad l^{\dagger}_{\lambda 1,\mathcal{M}} = l^{\dagger}_{-\lambda 3,\mathcal{M}}, & \text{(algebraic equalities).} \end{cases} \tag{6.3}$$

We apply comments in the same line to the ones previous ones to (4.8) or as we did after (5.2): the expressions of $l^{\dagger}_{\lambda 5,\mathcal{M}}$ and of $l_{\lambda 1,\mathcal{M}}$ are formally the same (up to the minus sign), also with: $l^{\dagger}_{\lambda 1,\mathcal{M}}$ and $l_{\lambda 5,\mathcal{M}}$, $l^{\dagger}_{\lambda 7,\mathcal{M}}$ and $l_{\lambda 3,\mathcal{M}}$, $l^{\dagger}_{\lambda 3,\mathcal{M}}$ and $l_{\lambda 7,\mathcal{M}}$. This is in accordance with an interpretation as creation and annihilation operators for particles and antiparticles involving opposite signs for the time part in the corresponding Jordan Wigner type transformations. Similarly, for the terms which differ in a factor $(-1)^{m-1}$, under opposition in space.

Now, there is something genuine, the expressions of $l^{\dagger}_{\lambda 5,\mathcal{M}}$ and of $l^{\dagger}_{-\lambda 7,\,\mathcal{M}}$ are formally (algebraically) the same, but the distinction here is based in the spin and in the chirality. Both of them have "$n_t = +1$ and $n_z = 0$", but different values of the angle variable φ: $\begin{cases} \lambda\frac{\pi}{4}\big|_{\lambda=\mp} & (\ \lambda 5) \\ -\lambda\frac{3\pi}{4}\big|_{\lambda=\pm} & (-\lambda 7) \end{cases}$ (geometry), different spin: $\begin{cases} \uparrow & (\ \lambda 5) \\ \downarrow & (-\lambda 7) \end{cases}$ and different chirality $\begin{cases} \rightarrow\ \leftarrow \\ \leftarrow\ \rightarrow \end{cases}$ (physics). Same reason for the values at $(3,-1)$ and $(-3,1)$ $(l^{\dagger}_{\lambda 1,\mathcal{M}} = l^{\dagger}_{-\lambda 3,\,\mathcal{M}})$. For details in the spin see (4.9)-(4.12). This will permit us to obtain more anti-commuting relationships.

The *number operators* are :

$$\mathbf{N}_{\lambda r,m} \equiv l^{\dagger}_{\lambda r,\mathcal{M}}\, l_{\lambda r,\mathcal{M}} = \begin{cases} \left(\mathbf{R}_O^{\lambda(\frac{\pi}{4}+\epsilon\frac{\pi}{2})}\sigma^+\sigma^-\mathbf{R}_O^{-\lambda(\frac{\pi}{4}+\epsilon\frac{\pi}{2})}\right)_m = \left(\mathbf{R}_O^{\lambda\frac{\pi}{4}}\sigma^-\sigma^+\mathbf{R}_O^{-\lambda\frac{\pi}{4}}\right)_m = \left(\mathbf{R}_O^{\lambda\frac{\pi}{4}}\,\check{\sigma}\,\mathbf{R}_O^{-\lambda\frac{\pi}{4}}\right)_m, & r = \{1,7\} \\ \left(\mathbf{R}_O^{\lambda\frac{\pi}{4}}\sigma^+\sigma^-\mathbf{R}_O^{-\lambda\frac{\pi}{4}}\right)_m = \left(\mathbf{R}_O^{\lambda\frac{\pi}{4}}\,\hat{\sigma}\,\mathbf{R}_O^{-\lambda\frac{\pi}{4}}\right)_m, & r = \{3,5\} \end{cases}, \tag{6.4}$$

they satisfy

$$\begin{aligned} &\mathbf{N}_{\lambda r,m}\, l^{\dagger}_{\lambda r,\mathcal{M}} = 1\, l^{\dagger}_{\lambda r,\mathcal{M}}, && \mathbf{N}_{\lambda r,m}\, l_{\lambda r,\mathcal{M}} = 0\, l_{\lambda r,\mathcal{M}}, \\ &\mathbf{1} - \mathbf{N}_{\lambda r,m} = l_{\lambda r,\mathcal{M}}\, l^{\dagger}_{\lambda r,\mathcal{M}}, && \mathbf{N}_{\lambda r,m}\,(\mathbf{1} - \mathbf{N}_{\lambda r,m}) = \mathbf{0}. \end{aligned} \tag{6.5}$$

Anticommutators for the creation and annihilation charged lepton operators.

Using (4.26):

$$\text{for:}\quad \overset{\pi/2}{\overleftarrow{e\ \downarrow}} \qquad \overset{-\pi/2}{\overrightarrow{\underline{p}\ \downarrow}} \qquad \overset{\pi/2}{\overrightarrow{\overline{e}\ \uparrow}} \qquad \overset{-\pi/2}{\overleftarrow{\underline{\overline{p}}\ \uparrow}}$$
$$\left\{l^{\dagger}_{1,\mathcal{M}},\, l_{1,\mathcal{M}}\right\} = \left\{l^{\dagger}_{7,\mathcal{M}},\, l_{7,\mathcal{M}}\right\} = \left\{l^{\dagger}_{5,\mathcal{M}},\, l_{5,\mathcal{M}}\right\} = \left\{l^{\dagger}_{3,\mathcal{M}},\, l_{3,\mathcal{M}}\right\} = \mathbf{1}, \tag{6.6}$$

$$\text{and for:}\quad \overset{-\pi/2}{\overrightarrow{\underline{e}\ \downarrow}} \qquad \overset{\pi/2}{\overleftarrow{p\ \downarrow}} \qquad \overset{-\pi/2}{\overleftarrow{\underline{\overline{e}}\ \uparrow}} \qquad \overset{\pi/2}{\overrightarrow{\overline{p}\ \uparrow}}$$
$$\left\{l^{\dagger}_{-1,\mathcal{M}},\, l_{-1,\mathcal{M}}\right\} = \left\{l^{\dagger}_{-7,\mathcal{M}},\, l_{-7,\mathcal{M}}\right\} = \left\{l^{\dagger}_{-5,\mathcal{M}},\, l_{-5,\mathcal{M}}\right\} = \left\{l^{\dagger}_{-3,\mathcal{M}},\, l_{-3,\mathcal{M}}\right\} = \mathbf{1}, \tag{6.7}$$

The other anticommutators are zero:

$$\begin{aligned} &\left\{l^{*}_{\lambda r,\mathcal{M}} \,\vdots\, l^{*}_{\lambda r,\mathcal{M}}\right\} = \left\{l^{*}_{\lambda r,\mathcal{M}},\, l^{*}_{\lambda r,\mathcal{M}}\right\} = \mathbf{0}, && (r = s\,,\ m_1 = m_2 = m) \\ &\left\{l^{*}_{\lambda r,\mathcal{M}} \,\vdots\, l^{**}_{\lambda s,\mathcal{M}}\right\} = \mathbf{0}, && (r \neq s\,,\ m_1 = m_2 = m)\ , \\ &\left\{l^{*}_{\lambda r,\mathcal{M}_1} \,\vdots\, l^{**}_{\lambda s,\mathcal{M}_2}\right\} = \left\{l^{*}_{\lambda r,\mathcal{M}_1},\, l^{**}_{\lambda s,\mathcal{M}_2}\right\} = \mathbf{0}, && (m_1 \neq m_2) \end{aligned} \tag{6.8}$$

$\{*\,,**\} \in \{\text{“}\dagger\text{”}\ ,\ \text{“ ”}\}$, $\lambda \in \{+\,,-\}$, $\{r\,,s\} \in \{1\,,3\,,5\,,7\}$, $\{m\,,m_1\,,m_2\} \in \{2\,,\ldots\,,N\}$.

We still have to evaluate the anticommutators of the operators with λr (positive - negative) with the operators with $\lambda' s$ (negative - positive) $(\lambda' = -\lambda)$:

$$\left\{ l^{*}_{\lambda r,\mathcal{M}_1} \,\vdots\, l^{**}_{-\lambda s,\mathcal{M}_2} \right\} \equiv l^{*}_{\lambda r,\mathcal{M}_1}\, \mathbf{I}_{\lambda r(-\lambda s),m}\, l^{**}_{-\lambda s,\mathcal{M}_2} + l^{**}_{-\lambda s,\mathcal{M}_2}\, \mathbf{I}_{(-\lambda s)\lambda r,m}\, l^{*}_{\lambda r,\mathcal{M}_1}, \tag{6.9}$$

both, $\{r,s\}$, $\{m_1,m_2\}$ equal or different, and $\mathbf{I}_{\lambda r(-\lambda s),m}$ generalizing (3.10), verifying: $\text{sign}\,(\lambda r-(-\lambda s)) = -\text{sign}\,((-\lambda s)-\lambda r)$.

With the last algebraic equalities in (6.3): $l^{\dagger}_{-\lambda 7,\mathcal{M}} = l^{\dagger}_{\lambda 5,\mathcal{M}}$, $l^{\dagger}_{-\lambda 3,\mathcal{M}} = l^{\dagger}_{\lambda 1,\mathcal{M}}$, we only need to calculate the anticommutators: $\left\{ l^{*}_{\lambda 5,\mathcal{M}_1} \,\vdots\, l^{**}_{-\lambda 7,\mathcal{M}_2} \right\}$ and $\left\{ l^{*}_{\lambda 1,\mathcal{M}_1} \,\vdots\, l^{**}_{-\lambda 3,\mathcal{M}_2} \right\}$ with the definition (6.9). We also use the following algebraic equalities:

$$\begin{cases} \mathbf{R}_o^{\mp\lambda\frac{\pi}{4}}\,\sigma^z\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{4}} = \mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{2}}\,\sigma^z\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{2}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} = -\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\sigma^z\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} = -\,\sigma^z \\ \mathbf{R}_o^{\mp\lambda\frac{\pi}{4}}\,\mathrm{m}^-\sigma^+\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{4}} = \mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{2}}\,\mathrm{m}^-\sigma^+\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{2}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} = \mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\mathrm{m}^+\sigma^-\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} \\ \mathbf{R}_o^{\mp\lambda\frac{\pi}{4}}\,\mathrm{m}^+\sigma^-\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{4}} = \mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{2}}\,\mathrm{m}^+\sigma^-\,\mathbf{R}_o^{\pm\lambda\frac{\pi}{2}}\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} = \mathbf{R}_o^{\pm\lambda\frac{\pi}{4}}\,\mathrm{m}^-\sigma^+\,\mathbf{R}_o^{\mp\lambda\frac{\pi}{4}} \end{cases} .$$

Similarly to sections III.A and IV.G, we obtain:

$$\left\{ l^{*}_{\lambda r,\mathcal{M}_1} \,\vdots\, l^{**}_{-\lambda s,\mathcal{M}_2} \right\} \equiv \left\{ l^{*}_{\lambda r,\mathcal{M}_1} \,,\, l^{**}_{-\lambda s,\mathcal{M}_2} \right\} = \mathbf{0}\,, \qquad m_1 \neq m_2\,. \tag{6.10}$$

$$\left\{ l^{*}_{\lambda r,\mathcal{M}} \,\vdots\, l^{**}_{-\lambda s,\mathcal{M}} \right\} \equiv \mathbf{0}\,, \qquad \begin{cases} r = s \\ r \neq s \end{cases} . \tag{6.11}$$

Again, comparing the anticommutator relationships satisfied by the operators in (2,11) and these new ones in (6.6) - (6.11), we observe a close parallelism between them. Now the **b** operators with the ones with $\{e, \overline{\underline{e}}, \overline{p}, \underline{p}\}$ and the **d** operators with the ones with $\{\overline{e}, \underline{e}, p, \overline{\underline{p}}\}$ (without the momentum **p**). Also, with the considerations after (4.27), we point out here:

1) as different to the neutrino case, we have for spin up two different charged particles (positive and negative) and the same for the spin down; although the usual way is to consider that every charged particle can be presented in two different forms, one with spin up and another with spin down,

2) vectors-spin, with the specific angle parameters:

$\frac{\pi}{2}$ for $\varphi_R = \frac{\pi}{4}$ and $(\varphi_R - \pi) = -\frac{3\pi}{4}$, with both chiralities, and also

$-\frac{\pi}{2}$ for $\varphi_R - \frac{\pi}{2} = -\frac{\pi}{4}$ and $(\varphi_R + \pi) = \frac{3\pi}{4}$, with both chiralities,

3) charged leptons are unique in the following sense: they have anticommutators for operators defined after $\varphi_R = \frac{\pi}{4}$ with the ones after $\varphi = -\varphi_R = -\frac{\pi}{4}$ (up and down spins). No need of this for neutrinos: they do not depend on λ.

We **suggest:** $\left\{ e\overset{\overset{\pi/2}{\leftarrow}}{\downarrow}, e\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow} \right\}$ and $\left\{ \overline{e}\overset{\overset{\pi/2}{\rightarrow}}{\uparrow}, \underline{e}\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow} \right\}$ interact under the weak interaction. And:

$$\begin{array}{lcl} & & \text{electron , anti-positron} \\ \langle\, l^{\dagger}_{1,\mathcal{M}} \;⥮\,,\,⥯\; l^{\dagger}_{7,\mathcal{M}} \,\rangle & \text{with} & \langle\, e\overset{\overset{\pi/2}{\leftarrow}}{\downarrow} \;⥮\,,\,⥯\; \underline{p}\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow} \,\rangle_{\mathcal{M}} \\ \langle\, l^{\dagger}_{-5,\mathcal{M}} \;⥯\,,\,⥮\; l^{\dagger}_{-3,\mathcal{M}} \,\rangle & \text{with} & \langle\, \overline{\underline{e}}\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow} \;⥯\,,\,⥮\; \overline{p}\overset{\overset{\pi/2}{\rightarrow}}{\uparrow} \,\rangle_{\mathcal{M}} \\ & & \text{anti-electron , positron} \\ \langle\, l^{\dagger}_{5,\mathcal{M}} \;⥯\,,\,⥮\; l^{\dagger}_{3,\mathcal{M}} \,\rangle & \text{with} & \langle\, \overline{e}\overset{\overset{\pi/2}{\rightarrow}}{\uparrow} \;⥯\,,\,⥮\; \overline{\underline{p}}\overset{\overset{-\pi/2}{\leftarrow}}{\uparrow} \,\rangle_{\mathcal{M}} \\ \langle\, l^{\dagger}_{-1,\mathcal{M}} \;⥮\,,\,⥯\; l^{\dagger}_{-7,\mathcal{M}} \,\rangle & \text{with} & \langle\, \underline{e}\overset{\overset{-\pi/2}{\rightarrow}}{\downarrow} \;⥮\,,\,⥯\; p\overset{\overset{\pi/2}{\leftarrow}}{\downarrow} \,\rangle_{\mathcal{M}} \end{array} , \tag{6.12}$$

represent electrons and positrons with different spin (each couple with the same spin), flipping their vectors-spin and chiralities as they evolve in time space.

VII. II-1) QUARKS I: *d*-QUARKS. $\varphi_R = \frac{\pi}{12}$,

$$\{ 2\varphi_R \} \rightarrow \{ [+\tfrac{\pi}{6}, -\tfrac{5\pi}{6}], [-\tfrac{\pi}{6}, +\tfrac{5\pi}{6}] \}.$$

Creation and annihilation operators (Figure 12 in Appendix E):

$$
\begin{aligned}
& d \ \overset{\pi/6}{\longleftarrow}\ \downarrow && q^{\dagger}_{1,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{12}} \Big[\left(\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (-\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \Big] \mathbf{R}_{OM}^{-\frac{\pi}{12}}, \\
& \underline{\bar{d}} \ \overset{-5\pi/6}{\longrightarrow}\ \downarrow && q^{\dagger}_{7,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{12}} \Big[\left(-\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \Big] \mathbf{R}_{OM}^{-\frac{\pi}{12}} \\
& \bar{d} \ \overset{\pi/6}{\longrightarrow}\ \uparrow && q^{\dagger}_{5,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{12}} \Big[\left(\sigma^z \right)_{M-1} \left(\mathrm{m}^- \sigma^+ \right)_m \Big] \mathbf{R}_{OM}^{-\frac{\pi}{12}} \\
& \underline{d} \ \overset{-5\pi/6}{\longleftarrow}\ \uparrow && q^{\dagger}_{3,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{12}} \Big[\left(-\sigma^z \right)_{M-1} \left(-\mathrm{m}^- \sigma^+ \right)_m \Big] \mathbf{R}_{OM}^{-\frac{\pi}{12}}
\end{aligned}
\qquad (7.1)
$$

.

$$
\begin{aligned}
& \underline{\bar{d}} \ \overset{-\pi/6}{\longleftarrow}\ \uparrow && q^{\dagger}_{-5,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{12}} \Big[\left(\sigma^z \right)_{M-1} \left(\mathrm{m}^- \sigma^+ \right)_m \Big] \mathbf{R}_{OM}^{\frac{\pi}{12}} \\
& d \ \overset{5\pi/6}{\longrightarrow}\ \uparrow && q^{\dagger}_{-3,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{12}} \Big[\left(-\sigma^z \right)_{M-1} \left(-\mathrm{m}^- \sigma^+ \right)_m \Big] \mathbf{R}_{OM}^{\frac{\pi}{12}} \\
& \underline{d} \ \overset{-\pi/6}{\longrightarrow}\ \downarrow && q^{\dagger}_{-1,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{12}} \Big[\left(\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (-\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \Big] \mathbf{R}_{OM}^{\frac{\pi}{12}} \\
& \bar{d} \ \overset{5\pi/6}{\longleftarrow}\ \downarrow && q^{\dagger}_{-7,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{12}} \Big[\left(-\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \Big] \mathbf{R}_{OM}^{\frac{\pi}{12}}
\end{aligned}
\qquad (7.2)
$$

And, attending the particle - antiparticle scheme:

λr	φ	Charge (φ)	Spin (φ)	vector-spin (2φ)	Chirality (φ)	*Particle*		
1	$-\frac{11\pi}{12}$	$-\frac{1}{3}$	$\downarrow$	$\frac{\pi}{6}$	$\leftarrow$	$d\ \overset{\pi/6}{\leftarrow}\ \downarrow$ $\frac{\pi}{6}$-mi lc	down	d-quark
7	$\frac{7\pi}{12}$	$-\frac{1}{3}$	$\downarrow$	$-\frac{5\pi}{6}$	$\rightarrow$	$\underline{\bar{d}}\ \overset{-5\pi/6}{\rightarrow}\ \downarrow$ $-\frac{5\pi}{6}$-mi rc	down	d-quark
−5	$-\frac{\pi}{12}$	$-\frac{1}{3}$	$\uparrow$	$-\frac{\pi}{6}$	$\leftarrow$	$\underline{\bar{d}}\ \overset{-\pi/6}{\leftarrow}\ \uparrow$ $-\frac{\pi}{6}$-mi lc	up	d-quark
−3	$\frac{5\pi}{12}$	$-\frac{1}{3}$	$\uparrow$	$\frac{5\pi}{6}$	$\rightarrow$	$d\ \overset{5\pi/6}{\rightarrow}\ \uparrow$ $\frac{5\pi}{6}$-mi rc	up	d-quark
5	$\frac{\pi}{12}$	$\frac{1}{3}$	$\uparrow$	$\frac{\pi}{6}$	$\rightarrow$	$\bar{d}\ \overset{\pi/6}{\rightarrow}\ \uparrow$ $\frac{\pi}{6}$-mi rc	up	d-antiquark
3	$-\frac{5\pi}{12}$	$\frac{1}{3}$	$\uparrow$	$-\frac{5\pi}{6}$	$\leftarrow$	$\underline{d}\ \overset{-5\pi/6}{\leftarrow}\ \uparrow$ $-\frac{5\pi}{6}$-mi lc	up	d-antiquark
−1	$\frac{11\pi}{12}$	$\frac{1}{3}$	$\downarrow$	$-\frac{\pi}{6}$	$\rightarrow$	$\underline{d}\ \overset{-\pi/6}{\rightarrow}\ \downarrow$ $-\frac{\pi}{6}$-mi rc	down	d-antiquark
−7	$-\frac{7\pi}{12}$	$\frac{1}{3}$	$\downarrow$	$\frac{5\pi}{6}$	$\leftarrow$	$\bar{d}\ \overset{5\pi/6}{\leftarrow}\ \downarrow$ $\frac{5\pi}{6}$-mi lc	down	d-antiquark

with anti $\equiv$ opposition with opposite electrical charge: $\begin{cases} \text{red color } (d, \underline{\bar{d}}) \text{ for negative charge d-quark} \\ \text{blue color } (\bar{d}, \underline{d}) \text{ for positive charge d-antiquark} \end{cases}$,

$(\frac{\pi}{6}, -\frac{5\pi}{6})$, $(-\frac{\pi}{6}, \frac{5\pi}{6})$ for the vector-spin. We start with the 1 position for the reference d-quark , these quarks are separated in two families, the ones with $\lambda = +1$ and the ones with $\lambda = -1$.

In a parallel way to previous sections:

$$\begin{cases} \mathbf{R}_{e,M-1}^{\frac{\pi}{6}} = \left(\mathbf{R}_o^{\frac{\pi}{12}}\,\sigma^z\,\mathbf{R}_o^{-\frac{\pi}{12}}\right)_{M-1} = \mathbf{R}_{o,M-1}^{\frac{\pi}{12}}\left(\prod_{k=1}^{m-1}\sigma_k^z\right)\mathbf{R}_{o,M-1}^{-\frac{\pi}{12}}, & \text{both chiralities,}\\ \mathbf{R}_{e,M-1}^{-\frac{5\pi}{6}} = \left(-\mathbf{R}_e^{\frac{\pi}{6}}\right)_{e,M-1} = \prod_{k=1}^{m-1}\left(\mathbf{R}_o^{\frac{\pi}{12}}\,(-\sigma^z)\,\mathbf{R}_o^{-\frac{\pi}{12}}\right)_k = \mathbf{R}_{o,M-1}^{\frac{\pi}{12}}\left(\prod_{k=1}^{m-1}(-\sigma^z)_k\right)\mathbf{R}_{o,M-1}^{-\frac{\pi}{12}}, & \text{both chiralities,}\\ \mathbf{R}_{e,M-1}^{-\frac{\pi}{6}} = \left(\mathbf{R}_o^{-\frac{\pi}{12}}\,\sigma^z\,\mathbf{R}_o^{\frac{\pi}{12}}\right)_{M-1} = \mathbf{R}_{o,M-1}^{-\frac{\pi}{12}}\left(\prod_{k=1}^{m-1}\sigma_k^z\right)\mathbf{R}_{o,M-1}^{\frac{\pi}{12}}, & \text{both chiralities,}\\ \mathbf{R}_{e,M-1}^{\frac{5\pi}{6}} = \left(-\mathbf{R}_e^{-\frac{\pi}{6}}\right)_{e,M-1} = \prod_{k=1}^{m-1}\left(\mathbf{R}_o^{-\frac{\pi}{12}}\,(-\sigma^z)\,\mathbf{R}_o^{\frac{\pi}{12}}\right)_k = \mathbf{R}_{o,M-1}^{-\frac{\pi}{12}}\left(\prod_{k=1}^{m-1}(-\sigma^z)_k\right)\mathbf{R}_{o,M-1}^{\frac{\pi}{12}}, & \text{both chiralities,}\\ \text{for each chirality we have the four vectors-spin,} & \\ q_{\lambda 1,\mathcal{M}}^{\dagger} = (-1)^m\, q_{\lambda 7,\mathcal{M}}^{\dagger} = -[\,q_{\lambda 5,\mathcal{M}}] = -[\,(-1)^m\, q_{\lambda 3,\mathcal{M}}]\,. & \end{cases} \tag{7.3}$$

The *number operators* are :

$$\mathbf{N}_{\lambda r,m} \equiv q_{\lambda r,\mathcal{M}}^{\dagger}\, q_{\lambda r,\mathcal{M}} = \begin{cases} \left(\mathbf{R}_o^{\lambda(\frac{\pi}{12}+\epsilon\frac{\pi}{2})}\sigma^+\sigma^-\mathbf{R}_o^{-\lambda(\frac{\pi}{12}+\epsilon\frac{\pi}{2})}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{12}}\sigma^-\sigma^+\mathbf{R}_o^{-\lambda\frac{\pi}{12}}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{12}}\,\check{\sigma}\,\mathbf{R}_o^{-\lambda\frac{\pi}{12}}\right)_m, & r=\{1,7\}\\ \left(\mathbf{R}_o^{\lambda\frac{\pi}{12}}\,\sigma^+\sigma^-\,\mathbf{R}_o^{-\lambda\frac{\pi}{12}}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{12}}\,\hat{\sigma}\,\mathbf{R}_o^{-\lambda\frac{\pi}{12}}\right)_m, & r=\{3,5\} \end{cases}. \tag{7.4}$$

They satisfy

$$\begin{aligned} &\mathbf{N}_{\lambda r,m}\, q_{\lambda r,\mathcal{M}}^{\dagger} = 1\, q_{\lambda r,\mathcal{M}}^{\dagger}\,; && \mathbf{N}_{\lambda r,m}\, q_{\lambda r,\mathcal{M}} = 0\, q_{\lambda r,\mathcal{M}}\,;\\ &\mathbf{1}-\mathbf{N}_{\lambda r,m} = q_{\lambda r,\mathcal{M}}\, q_{\lambda r,\mathcal{M}}^{\dagger}\,; && \mathbf{N}_{\lambda r,m}\,(\mathbf{1}-\mathbf{N}_{\lambda r,m}) = \mathbf{0}\,. \end{aligned} \tag{7.5}$$

Anticommutators for the creation and annihilation d-quarks operators.

With (4.26):

$$\text{for: } \quad d\overset{\overset{\pi/6}{\leftarrow}}{\downarrow} \qquad \bar{d}\overset{\overset{\pi/6}{\rightarrow}}{\uparrow} \qquad \bar{d}\overset{\overset{-5\pi/6}{\rightarrow}}{\downarrow} \qquad \underline{d}\overset{\overset{-5\pi/6}{\leftarrow}}{\uparrow} \tag{7.6}$$

$$\{q_{1,\mathcal{M}}^{\dagger}, q_{1,\mathcal{M}}\} = \{q_{5,\mathcal{M}}^{\dagger}, q_{5,\mathcal{M}}\} = \{q_{7,\mathcal{M}}^{\dagger}, q_{7,\mathcal{M}}\} = \{q_{3,\mathcal{M}}^{\dagger}, q_{3,\mathcal{M}}\} = \mathbf{1}\,,$$

$$\text{and for: } \quad \bar{d}\overset{\overset{-\pi/6}{\leftarrow}}{\uparrow} \qquad \underline{d}\overset{\overset{-\pi/6}{\rightarrow}}{\downarrow} \qquad d\overset{\overset{5\pi/6}{\rightarrow}}{\uparrow} \qquad \bar{d}\overset{\overset{5\pi/6}{\leftarrow}}{\downarrow} \tag{7.7}$$

$$\{q_{-5,\mathcal{M}}^{\dagger}, q_{-5,\mathcal{M}}\} = \{q_{-1,\mathcal{M}}^{\dagger}, q_{-1,\mathcal{M}}\} = \{q_{-3,\mathcal{M}}^{\dagger}, q_{-3,\mathcal{M}}\} = \{q_{-7,\mathcal{M}}^{\dagger}, q_{-7,\mathcal{M}}\} = \mathbf{1}\,.$$

These other anticommutators are zero:

$$\begin{aligned} &\{q_{\lambda r,\mathcal{M}}^{*} \,\vdots\, q_{\lambda r,\mathcal{M}}^{*}\} = \{q_{\lambda r,\mathcal{M}}^{*}\,,\, q_{\lambda r,\mathcal{M}}^{*}\} = \mathbf{0}\,, && (r = s\,,\; m_1 = m_2 = m)\\ &\{q_{\lambda r,\mathcal{M}}^{*} \,\vdots\, q_{\lambda s,\mathcal{M}}^{**}\} = \mathbf{0}\,, && (r \neq s\,,\; m_1 = m_2 = m)\;.\\ &\{q_{\lambda r,\mathcal{M}_1}^{*} \,\vdots\, q_{\lambda s,\mathcal{M}_2}^{**}\} = \{q_{\lambda r,\mathcal{M}_1}^{*}\,,\, q_{\lambda s,\mathcal{M}_2}^{**}\} = \mathbf{0}\,, && (m_1 \neq m_2) \end{aligned} \tag{7.8}$$

The main differences with previous cases are:

1) given a chirality (sign of φ), the vector-spin of the quark with a spin is different to the vectors-spin of the quarks with the other spin, also opposed to the vector-spin of the other quark with same spin. Similarly for the antiquarks. Four values for the vector-spin, two values for the chirality.

2) vectors-spin, with the specific angle parameters:

$\frac{\pi}{6}$ for $\varphi \in \{\frac{\pi}{12} = \varphi_R, -\frac{11\pi}{12}\}$, and $\frac{5\pi}{6}$ for $\{-\frac{7\pi}{12}, \frac{5\pi}{12}\}$, with a {rc,lc} chirality, and also
$-\frac{\pi}{6}$ for $\varphi \in \{-\frac{\pi}{12}, \frac{11\pi}{12}\}$, and $-\frac{5\pi}{6}$ for $\{\frac{7\pi}{12}, -\frac{5\pi}{12}\}$, with a {rc,lc} chirality.

We **suggest:** $\{ d\overset{\overset{\pi/6}{\leftarrow}}{\downarrow}, \bar{d}\overset{\overset{-\pi/6}{\leftarrow}}{\uparrow} \}$ and $\{ \bar{d}\overset{\overset{\pi/6}{\rightarrow}}{\uparrow}, \underline{d}\overset{\overset{-\pi/6}{\rightarrow}}{\downarrow} \}$ $(n_z > 0)$ interact under the weak interaction. And that:

$$\begin{aligned} &\langle\, q_{1,\mathcal{M}}^{\dagger} \;\Downarrow\!\Uparrow,\; \Uparrow\!\Downarrow\; q_{7,\mathcal{M}}^{\dagger} \,\rangle && \text{with} && \langle\, d\overset{\overset{\pi/6}{\leftarrow}}{\downarrow} \;\Downarrow\!\Uparrow,\; \Uparrow\!\Downarrow\; \bar{d}\overset{\overset{-5\pi/6}{\rightarrow}}{\downarrow} \,\rangle_{\mathcal{M}}\\ &\langle\, q_{-5,\mathcal{M}}^{\dagger} \;\Uparrow\!\Downarrow,\; \Downarrow\!\Uparrow\; q_{-3,\mathcal{M}}^{\dagger} \,\rangle && \text{with} && \langle\, \bar{d}\overset{\overset{-\pi/6}{\leftarrow}}{\uparrow} \;\Uparrow\!\Downarrow,\; \Downarrow\!\Uparrow\; d\overset{\overset{5\pi/6}{\rightarrow}}{\uparrow} \,\rangle_{\mathcal{M}}\\ &\langle\, q_{5,\mathcal{M}}^{\dagger} \;\Uparrow\!\Downarrow,\; \Downarrow\!\Uparrow\; q_{3,\mathcal{M}}^{\dagger} \,\rangle && \text{with} && \langle\, \bar{d}\overset{\overset{\pi/6}{\rightarrow}}{\uparrow} \;\Uparrow\!\Downarrow,\; \Downarrow\!\Uparrow\; \underline{d}\overset{\overset{-5\pi/6}{\leftarrow}}{\uparrow} \,\rangle_{\mathcal{M}}\\ &\langle\, q_{-1,\mathcal{M}}^{\dagger} \;\Downarrow\!\Uparrow,\; \Uparrow\!\Downarrow\; q_{-7\mathcal{M}}^{\dagger} \,\rangle && \text{with} && \langle\, \underline{d}\overset{\overset{-\pi/6}{\rightarrow}}{\downarrow} \;\Downarrow\!\Uparrow,\; \Uparrow\!\Downarrow\; \bar{d}\overset{\overset{5\pi/6}{\leftarrow}}{\downarrow} \,\rangle_{\mathcal{M}} \end{aligned} \;, \tag{7.9}$$

represent d-quarks and d-antiquarks with different spin (each one with the same spin). Also each one with both chiralities and two opposite vectors-spin. With different spin different vectors-spin for both particles and antiparticles.

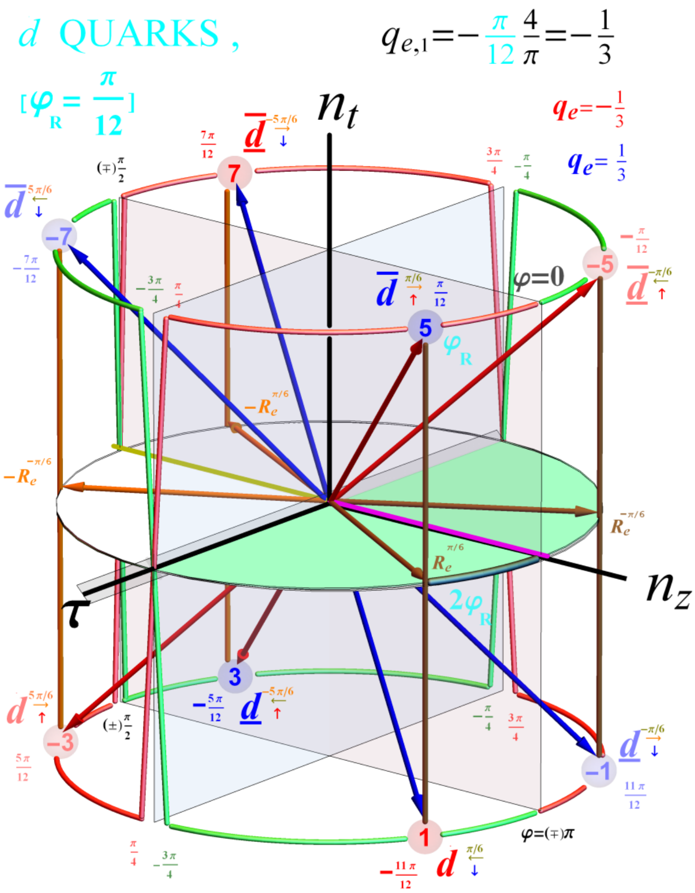


FIG. 12: d quarks.

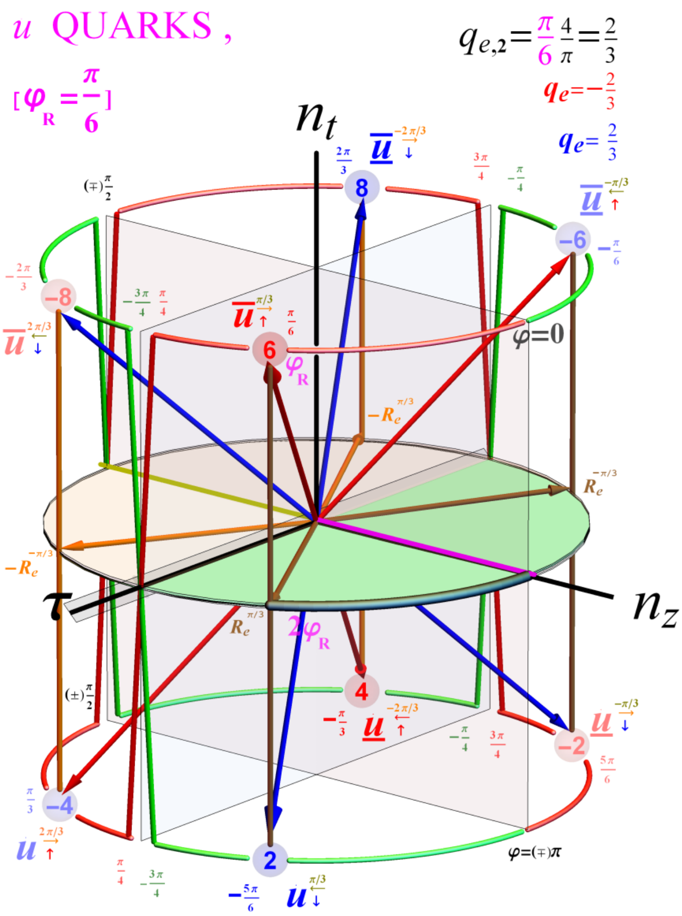


FIG. 13: u quarks.

VIII. II-2) QUARKS II: *u*-QUARKS. $\varphi_R = \frac{\pi}{6}$,

$$\{2\varphi_R\} \to \{[+\tfrac{\pi}{3}, -\tfrac{2\pi}{3}], [-\tfrac{\pi}{3}, +\tfrac{2\pi}{3}]\}.$$

Creation and annihilation operators (Figure 13 in Appendix E):

$$
\begin{aligned}
&u \; \overset{\overset{\pi/3}{\longleftarrow}}{\downarrow} && q^{\dagger}_{2,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{6}} \left[\left(\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (-\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \right] \mathbf{R}_{OM}^{-\frac{\pi}{6}} \\
&\underline{\overline{u}} \; \overset{\overset{-2\pi/3}{\longrightarrow}}{\downarrow} && q^{\dagger}_{8,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{6}} \left[\left(-\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \right] \mathbf{R}_{OM}^{-\frac{\pi}{6}} \\
&\overline{u} \; \overset{\overset{\pi/3}{\longrightarrow}}{\uparrow} && q^{\dagger}_{6,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{6}} \left[\left(\sigma^z \right)_{M-1} \left(\mathrm{m}^- \sigma^+ \right)_m \right] \mathbf{R}_{OM}^{-\frac{\pi}{6}} \\
&\underline{u} \; \overset{\overset{-2\pi/3}{\longleftarrow}}{\uparrow} && q^{\dagger}_{4,M} \equiv \mathbf{R}_{OM}^{\frac{\pi}{6}} \left[\left(-\sigma^z \right)_{M-1} \left(-\mathrm{m}^- \sigma^+ \right)_m \right] \mathbf{R}_{OM}^{-\frac{\pi}{6}}
\end{aligned}
\qquad (8.1)
$$

.

$$
\begin{aligned}
&\underline{\overline{u}} \; \overset{\overset{-\pi/3}{\longleftarrow}}{\uparrow} && q^{\dagger}_{-6,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{6}} \left[\left(\sigma^z \right)_{M-1} \left(\mathrm{m}^- \sigma^+ \right)_m \right] \mathbf{R}_{OM}^{\frac{\pi}{6}} \\
&u \; \overset{\overset{2\pi/3}{\longrightarrow}}{\uparrow} && q^{\dagger}_{-4,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{6}} \left[\left(-\sigma^z \right)_{M-1} \left(-\mathrm{m}^- \sigma^+ \right)_m \right] \mathbf{R}_{OM}^{\frac{\pi}{6}} \\
&\underline{u} \; \overset{\overset{-\pi/3}{\longrightarrow}}{\downarrow} && q^{\dagger}_{-2,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{6}} \left[\left(\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (-\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \right] \mathbf{R}_{OM}^{\frac{\pi}{6}} \\
&\overline{u} \; \overset{\overset{2\pi/3}{\longleftarrow}}{\downarrow} && q^{\dagger}_{-8,M} \equiv \mathbf{R}_{OM}^{-\frac{\pi}{6}} \left[\left(-\sigma^z \right)_{M-1} \left(\mathbf{R}_O^{\epsilon\frac{\pi}{2}} (\mathrm{m}^- \sigma^+)\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}} \right)_m \right] \mathbf{R}_{OM}^{\frac{\pi}{6}} .
\end{aligned}
\qquad (8.2)
$$

And, attending the particle - antiparticle scheme:

λr	φ	Charge (φ)	Spin (φ)	vector-spin (2φ)	Chirality (φ)	*Particle*			
2	$-\frac{5\pi}{6}$	$\frac{2}{3}$	$\downarrow$	$\frac{\pi}{3}$	$\leftarrow$	$u\ \overset{\overset{\pi/3}{\longleftarrow}}{\downarrow}$	$\frac{\pi}{3}$-mi lc	down	u-quark
8	$\frac{2\pi}{3}$	$\frac{2}{3}$	$\downarrow$	$-\frac{2\pi}{3}$	$\rightarrow$	$\underline{\overline{u}}\ \overset{\overset{-2\pi/3}{\longrightarrow}}{\downarrow}$	$-\frac{2\pi}{3}$-mi rc		
-6	$-\frac{\pi}{6}$	$\frac{2}{3}$	$\uparrow$	$-\frac{\pi}{3}$	$\leftarrow$	$\underline{\overline{u}}\ \overset{\overset{-\pi/3}{\longleftarrow}}{\uparrow}$	$-\frac{\pi}{3}$-mi lc	up	
-4	$\frac{\pi}{3}$	$\frac{2}{3}$	$\uparrow$	$\frac{2\pi}{3}$	$\rightarrow$	$u\ \overset{\overset{2\pi/3}{\longrightarrow}}{\uparrow}$	$\frac{2\pi}{3}$-mi rc		
6	$\frac{\pi}{6}$	$-\frac{2}{3}$	$\uparrow$	$\frac{\pi}{3}$	$\rightarrow$	$\overline{u}\ \overset{\overset{\pi/3}{\longrightarrow}}{\uparrow}$	$\frac{\pi}{3}$-mi rc	up	u-antiquark
4	$-\frac{\pi}{3}$	$-\frac{2}{3}$	$\uparrow$	$-\frac{2\pi}{3}$	$\leftarrow$	$\underline{u}\ \overset{\overset{-2\pi/3}{\longleftarrow}}{\uparrow}$	$-\frac{2\pi}{3}$-mi lc		
-2	$\frac{5\pi}{6}$	$-\frac{2}{3}$	$\downarrow$	$-\frac{\pi}{3}$	$\rightarrow$	$\underline{u}\ \overset{\overset{-\pi/3}{\longrightarrow}}{\downarrow}$	$-\frac{\pi}{3}$-mi rc	down	
-8	$-\frac{2\pi}{3}$	$-\frac{2}{3}$	$\downarrow$	$\frac{2\pi}{3}$	$\leftarrow$	$\overline{u}\ \overset{\overset{2\pi/3}{\longleftarrow}}{\downarrow}$	$\frac{2\pi}{3}$-mi lc		

with anti ≡ opposition with opposite electrical charge: $\begin{cases} \text{blue color } (u, \underline{\overline{u}}) \text{ for positive charge u-quark} \\ \text{red color } (\overline{u}, \underline{u}) \text{ for negative charge u-antiquark} \end{cases}$,

$(\frac{\pi}{3}, -\frac{2\pi}{3}), (-\frac{\pi}{3}, \frac{2\pi}{3})$ for the vector-spin. We start with the 2 position for the reference u-quark; they are separated in two families, the ones with $\lambda = +1$ and the ones with $\lambda = -1$.

In a parallel way to previous sections:

$$\begin{cases} \mathbf{R}_{e,M-1}^{\frac{\pi}{3}} = \left(\mathbf{R}_o^{\frac{\pi}{6}}\, \sigma^z\, \mathbf{R}_o^{-\frac{\pi}{6}}\right)_{M-1} = \mathbf{R}_{o,M-1}^{\frac{\pi}{6}} \left(\prod_{k=1}^{m-1} \sigma_k^z \right) \mathbf{R}_{o,M-1}^{-\frac{\pi}{6}} , & \text{both chiralities,} \\ \mathbf{R}_{e,M-1}^{-\frac{2\pi}{3}} = \left(- \mathbf{R}_e^{\frac{\pi}{3}} \right)_{e,M-1} = \prod_{k=1}^{m-1} \left(\mathbf{R}_o^{\frac{\pi}{6}}(-\sigma^z)\, \mathbf{R}_o^{-\frac{\pi}{6}}\right)_k = \mathbf{R}_{o,M-1}^{\frac{\pi}{6}} \left(\prod_{k=1}^{m-1} (-\sigma^z)_k \right) \mathbf{R}_{o,M-1}^{-\frac{\pi}{6}}, & \text{both chiralities,} \\ \mathbf{R}_{e,M-1}^{-\frac{\pi}{3}} = \left(\mathbf{R}_o^{-\frac{\pi}{6}}\, \sigma^z\, \mathbf{R}_o^{\frac{\pi}{6}}\right)_{M-1} = \mathbf{R}_{o,M-1}^{-\frac{\pi}{6}} \left(\prod_{k=1}^{m-1} \sigma_k^z \right) \mathbf{R}_{o,M-1}^{\frac{\pi}{6}} , & \text{both chiralities,} \\ \mathbf{R}_{e,M-1}^{\frac{2\pi}{3}} = \left(- \mathbf{R}_e^{-\frac{\pi}{3}} \right)_{e,M-1} = \prod_{k=1}^{m-1} \left(\mathbf{R}_o^{-\frac{\pi}{6}}(-\sigma^z)\, \mathbf{R}_o^{\frac{\pi}{6}}\right)_k = \mathbf{R}_{o,M-1}^{-\frac{\pi}{6}} \left(\prod_{k=1}^{m-1} (-\sigma^z)_k \right) \mathbf{R}_{o,M-1}^{\frac{\pi}{6}}, & \text{both chiralities,} \\ \text{for each chirality we have the four vectors-spin,} & \\ q^{\dagger}_{\lambda 2,\mathcal{M}} = (-1)^m\, q^{\dagger}_{\lambda 8,\mathcal{M}} = -\left[q_{\lambda 6,\mathcal{M}} \right] = -\left[(-1)^m\, q_{\lambda 4,\mathcal{M}} \right] . & \end{cases} \tag{8.3}$$

The *number operators* are :

$$\mathbf{N}_{\lambda r,m} \equiv q^{\dagger}_{\lambda r,\mathcal{M}}\, q_{\lambda r,\mathcal{M}} = \begin{cases} \left(\mathbf{R}_o^{\lambda(\frac{\pi}{6}+\epsilon\frac{\pi}{2})} \sigma^+\sigma^- \mathbf{R}_o^{-\lambda(\frac{\pi}{6}+\epsilon\frac{\pi}{2})}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{12}} \sigma^-\sigma^+ \mathbf{R}_o^{-\lambda\frac{\pi}{6}}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{6}}\, \check{\sigma}\, \mathbf{R}_o^{-\lambda\frac{\pi}{6}}\right)_m, & r = \{2,8\} \\ \left(\mathbf{R}_o^{\lambda\frac{\pi}{6}}\, \sigma^+\sigma^-\, \mathbf{R}_o^{-\lambda\frac{\pi}{12}}\right)_m = \left(\mathbf{R}_o^{\lambda\frac{\pi}{6}}\, \hat{\sigma}\, \mathbf{R}_o^{-\lambda\frac{\pi}{6}}\right)_m, & r = \{4,6\} \end{cases} . \tag{8.4}$$

They satisfy

$$\begin{aligned} \mathbf{N}_{\lambda r,m}\, q^{\dagger}_{\lambda r,\mathcal{M}} &= 1\, q^{\dagger}_{\lambda r,\mathcal{M}} ; & \mathbf{N}_{\lambda r,m}\, q_{\lambda r,\mathcal{M}} &= 0\, q_{\lambda r,\mathcal{M}} ; \\ \mathbf{1} - \mathbf{N}_{\lambda r,m} &= q_{\lambda r,\mathcal{M}}\, q^{\dagger}_{\lambda r,\mathcal{M}} ; & \mathbf{N}_{\lambda r,m} \left(\mathbf{1} - \mathbf{N}_{\lambda r,m} \right) &= \mathbf{0} . \end{aligned} \tag{8.5}$$

Anticommutators for the creation and annihilation u-quarks operators.

With (4.26):

for: $u\,\overset{\overset{\pi/3}{\leftarrow}}{\downarrow}$ $\quad \overline{u}\,\overset{\overset{\pi/3}{\rightarrow}}{\uparrow}$ $\quad \underline{\overline{u}}\,\overset{\overset{-2\pi/3}{\rightarrow}}{\downarrow}$ $\quad \underline{u}\,\overset{\overset{-2\pi/3}{\leftarrow}}{\uparrow}$

$$\left\{ q^{\dagger}_{2,\mathcal{M}} , q_{2,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{6,\mathcal{M}} , q_{6,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{8,\mathcal{M}} , q_{8,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{4,\mathcal{M}} , q_{4,\mathcal{M}} \right\} = \mathbf{1} , \tag{8.6}$$

and for: $\underline{\overline{u}}\,\overset{\overset{-\pi/3}{\leftarrow}}{\uparrow}$ $\quad \underline{u}\,\overset{\overset{-\pi/3}{\rightarrow}}{\downarrow}$ $\quad u\,\overset{\overset{2\pi/3}{\rightarrow}}{\uparrow}$ $\quad \overline{u}\,\overset{\overset{2\pi/3}{\leftarrow}}{\downarrow}$

$$\left\{ q^{\dagger}_{-6,\mathcal{M}} , q_{-6,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{-2,\mathcal{M}} , q_{-2,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{-4,\mathcal{M}} , q_{-4,\mathcal{M}} \right\} = \left\{ q^{\dagger}_{-8,\mathcal{M}} , q_{-8,\mathcal{M}} \right\} = \mathbf{1} . \tag{8.7}$$

These other anticommutators are zero:

$$\begin{aligned} \left\{ q^{*}_{\lambda r,\mathcal{M}} \,;\, q^{*}_{\lambda r,\mathcal{M}} \right\} &= \left\{ q^{*}_{\lambda r,\mathcal{M}} \,,\, q^{*}_{\lambda r,\mathcal{M}} \right\} = \mathbf{0} , & (r = s \,,\; m_1 = m_2 = m) \\ \left\{ q^{*}_{\lambda r,\mathcal{M}} \,;\, q^{**}_{\lambda s,\mathcal{M}} \right\} &= \mathbf{0} , & (r \neq s \,,\; m_1 = m_2 = m) \\ \left\{ q^{*}_{\lambda r,\mathcal{M}_1} \,;\, q^{**}_{\lambda s,\mathcal{M}_2} \right\} &= \left\{ q^{*}_{\lambda r,\mathcal{M}_1} \,,\, q^{**}_{\lambda s,\mathcal{M}_2} \right\} = \mathbf{0} , & (m_1 \neq m_2) \end{aligned} . \tag{8.8}$$

We assume 1) above (7.9), and we add:

1) vectors-spin, with the specific angle parameters:

$\frac{\pi}{3}$ for $\varphi \in \{\frac{\pi}{6} = \varphi_R, -\frac{5\pi}{6}\}$, and $\frac{2\pi}{3}$ for $\{-\frac{2\pi}{3}, \frac{\pi}{3}\}$, with a {rc,lc} chirality, and also

$-\frac{\pi}{3}$ for $\varphi \in \{ -\frac{\pi}{6} , \frac{5\pi}{6}\}$, and $-\frac{2\pi}{3}$ for $\{\frac{2\pi}{3}, -\frac{\pi}{3}\}$, with a {rc,lc} chirality.

2) the set of values $''2\varphi''$ are included in the set of values $''\varphi''$. Meanwhile for the d-quarks the set of their values $''2\varphi''$ are included in the set of values $''\varphi''$ of the u-quark. It is possible that this has implications for the gluon structure.

We **suggest:** $\left\{ u\,\overset{\overset{\pi/3}{\leftarrow}}{\downarrow} , \underline{\overline{u}}\,\overset{\overset{-\pi/3}{\leftarrow}}{\uparrow} \right\}$ and $\left\{ \overline{u}\,\overset{\overset{\pi/3}{\rightarrow}}{\uparrow} , \underline{u}\,\overset{\overset{-\pi/3}{\rightarrow}}{\downarrow} \right\}$ $(n_z > 0)$ interact under the weak interaction. And that:

$$\begin{aligned} &\langle\, q^{\dagger}_{2,\mathcal{M}} \upharpoonleft\!\downharpoonright , \downharpoonleft\!\upharpoonright q^{\dagger}_{8,\mathcal{M}} \,\rangle && \text{with} && \langle\, u\,\overset{\overset{\pi/3}{\leftarrow}}{\downarrow} \upharpoonleft\!\downharpoonright , \downharpoonleft\!\upharpoonright \underline{\overline{u}}\,\overset{\overset{-2\pi/3}{\rightarrow}}{\downarrow} \,\rangle_{\mathcal{M}} \\ &\langle\, q^{\dagger}_{-6,\mathcal{M}} \downharpoonleft\!\upharpoonright , \upharpoonleft\!\downharpoonright q^{\dagger}_{-4,\mathcal{M}} \,\rangle && \text{with} && \langle\, \underline{\overline{u}}\,\overset{\overset{-\pi/3}{\leftarrow}}{\uparrow} \downharpoonleft\!\upharpoonright , \upharpoonleft\!\downharpoonright u\,\overset{\overset{2\pi/3}{\rightarrow}}{\uparrow} \,\rangle_{\mathcal{M}} \\ &\langle\, q^{\dagger}_{6,\mathcal{M}} \downharpoonleft\!\upharpoonright , \upharpoonleft\!\downharpoonright q^{\dagger}_{4,\mathcal{M}} \,\rangle && \text{with} && \langle\, \overline{u}\,\overset{\overset{\pi/3}{\rightarrow}}{\uparrow} \downharpoonleft\!\upharpoonright , \upharpoonleft\!\downharpoonright \underline{u}\,\overset{\overset{-2\pi/3}{\leftarrow}}{\uparrow} \,\rangle_{\mathcal{M}} \\ &\langle\, q^{\dagger}_{-2,\mathcal{M}} \upharpoonleft\!\downharpoonright , \downharpoonleft\!\upharpoonright q^{\dagger}_{-8,\mathcal{M}} \,\rangle && \text{with} && \langle\, \underline{u}\,\overset{\overset{-\pi/3}{\rightarrow}}{\downarrow} \upharpoonleft\!\downharpoonright , \downharpoonleft\!\upharpoonright \overline{u}\,\overset{\overset{2\pi/3}{\leftarrow}}{\downarrow} \,\rangle_{\mathcal{M}} \end{aligned} , \tag{8.9}$$

with similar comments as the corresponding ones for the d-quarks.

IX. III) ?-NEUTRINONS. UNKNOWN ELEMENTARY FERMIONS. $\varphi_R = \frac{\pi}{8}$,

$$\{2\varphi_R\} \to \{[+\tfrac{\pi}{4}, -\tfrac{3\pi}{4}], [-\tfrac{\pi}{4}, +\tfrac{3\pi}{4}]\}.$$

Creation and annihilation operators.

In order to avoid even more repetitions, we assume section VIII, with the substitutions:

$$u \longrightarrow a \quad \begin{cases} \varphi: & \pm\frac{\pi}{6} \longrightarrow \pm\frac{\pi}{8}, \quad \pm\frac{\pi}{3} \longrightarrow \pm\frac{3\pi}{8}, \quad \pm\frac{2\pi}{3} \longrightarrow \pm\frac{5\pi}{8}, \quad \pm\frac{5\pi}{6} \longrightarrow \pm\frac{7\pi}{8}, \\ 2\varphi: & \pm\frac{\pi}{3} \longrightarrow \pm\frac{\pi}{4}, \quad \pm\frac{2\pi}{3} \longrightarrow \pm\frac{3\pi}{4}. \end{cases}$$

Most of the comments in section VIII can be written here.

We summarize the particles obtained with this value $\varphi_R = \frac{\pi}{8}$ in the following table:

λr	φ	Charge (φ)	Spin (φ)	vector-spin (2φ)	Chirality (φ)		*Particle*	
2	$-\frac{7\pi}{8}$	0	$\downarrow$	$\frac{\pi}{4}$	$\leftarrow$	$\overset{\overset{\pi/4}{\leftarrow}}{a\ \downarrow}$	$\frac{\pi}{4}$-mi lc	down a – neutrinon
8	$\frac{5\pi}{8}$	0	$\downarrow$	$-\frac{3\pi}{4}$	$\rightarrow$	$\overset{\overset{-3\pi/4}{\rightarrow}}{\underline{\overline{a}}\ \downarrow}$	$-\frac{3\pi}{4}$-mi rc	
6	$\frac{\pi}{8}$	0	$\uparrow$	$\frac{\pi}{4}$	$\rightarrow$	$\overset{\overset{\pi/4}{\rightarrow}}{\overline{a}\ \uparrow}$	$\frac{\pi}{4}$-mi rc	up a – antineutrinon
4	$-\frac{3\pi}{8}$	0	$\uparrow$	$-\frac{3\pi}{4}$	$\leftarrow$	$\overset{\overset{-3\pi/4}{\leftarrow}}{\underline{a}\ \uparrow}$	$-\frac{3\pi}{4}$-mi lc	
-6	$-\frac{\pi}{8}$	0	$\uparrow$	$-\frac{\pi}{4}$	$\leftarrow$	$\overset{\overset{-\pi/4}{\leftarrow}}{\underline{\overline{a}}\ \uparrow}$	$-\frac{\pi}{4}$-mi lc	up a – neutrinon
-4	$\frac{3\pi}{8}$	0	$\uparrow$	$\frac{3\pi}{4}$	$\rightarrow$	$\overset{\overset{3\pi/4}{\rightarrow}}{a\ \uparrow}$	$\frac{3\pi}{4}$-mi rc	
-2	$\frac{7\pi}{8}$	0	$\downarrow$	$-\frac{\pi}{4}$	$\rightarrow$	$\overset{\overset{-\pi/4}{\rightarrow}}{\underline{a}\ \downarrow}$	$-\frac{\pi}{4}$-mi rc	down a – antineutrinon
-8	$-\frac{5\pi}{8}$	0	$\downarrow$	$\frac{3\pi}{4}$	$\leftarrow$	$\overset{\overset{3\pi/4}{\leftarrow}}{\overline{a}\ \downarrow}$	$\frac{3\pi}{4}$-mi lc	

Two separated families, one depicted with a more solid color ($\lambda = +1$) and the other one with a stumped color ($\lambda = -1$).

We start with the 2 position for the reference a-neutrinon. We separate them into two families, the ones with $\lambda = +1$ and the ones with $\lambda = -1$. Also, with the criteria of colors: $\lambda = \pm 1$ and τ positive or negative. For these particles, we define anti ≡ opposition with opposite ′spin charge′ in each family. For a value of the spin we have the two opposed values of the chirality and also the two opposed values of the vector-spin inside each family. Over-bar and under-bar represent these new oppositions with the spin:

$(\frac{\pi}{4}, -\frac{3\pi}{4})$ vectors-spin $\begin{cases} \text{blue color } (a, \underline{\overline{a}}) \text{ with the spin down} \\ \text{red color } (\overline{a}, \underline{a}) \text{ with the spin up} \end{cases}$,

$(-\frac{\pi}{4}, \frac{3\pi}{4})$ vectors-spin $\begin{cases} \text{stumped blue color } (a, \underline{\overline{a}}) \text{ with the spin up} \\ \text{stumped red color } (\overline{a}, \underline{a}) \text{ with the spin down} \end{cases}$,

every couple with a $\leftarrow$, $\rightarrow$ for a left or a right chirality.

It has been assumed a zero value for the electrical charge, because the other possible values $\pm\frac{1}{2}$ (see Appendix B), in case of existence, they would have been found experimentally. Also, they should interact weakly, including a **W** boson. More important, the anomaly cancellation would not be verified. Or, including electrical charges $\pm\frac{1}{6}$ and $\pm\frac{5}{6}$, is there a non interacting universe with ours, except for the gravity (dark matter), that overlaps with ours?. It seems fantasy.

A family ($\lambda = +1$) would not weakly interact with the other family ($\lambda = -1$), as this would require **W** bosons with zero electric charge. Or, perhaps they do under a double **W** bosons interaction, one positively and the other negatively electrically charged. This differentiates them with the neutrinos and charged leptons as they weakly interact through **W** bosons. Something similar with the d and u quarks. Or, there are other uncharged bosons related to a $\Delta\varphi$ of the $\frac{\pi}{4}$ type (this does not seem to be acceptable).

There is an important difference with charged leptons. With a-neutrinons we do not have anticommutators for the operators defined after $\varphi_R = \frac{\pi}{8}$ with the operators defined after $-\varphi_R = -\frac{\pi}{8}$.

They would interact weakly, via the **Z** bosons, and electromagnetically (they have spin). The magnetic part is weaker than the other interactions (strong and electrical) and the electric charge is zero; gravity apart. They would be like neutrinos, but without the possibility of the interaction via **W** bosons. As a result of this, every one of the two families is isolated.

They would have mass (arguments for this elsewhere); and guided by the suggestive idea of the dark matter, these hypothetical particles would have very large masses.

PART III

APPENDIXES.

Appendix A: C-Rotation and C-Reflection matrices.

$$\mathbf{R}_o^{\varphi}(\phi) = \begin{pmatrix} \cos\varphi & -\sin\varphi\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\varphi\, \mathrm{e}^{\mathrm{i}\phi} & \cos\varphi \end{pmatrix}, \qquad \mathbf{R}_e^{\varphi}(\phi) = \begin{pmatrix} \cos\varphi & \sin\varphi\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\varphi\, \mathrm{e}^{\mathrm{i}\phi} & -\cos\varphi \end{pmatrix}.$$

$$\left\{ \mathbb{1} = \mathbf{R}_o^{0}(\phi)\,, \quad \sigma^z = \mathbf{R}_e^{0}(\phi)\,, \quad \sigma^x = \mathbf{R}_e^{\frac{\pi}{2}}(0)\,, \quad \sigma^y = \mathbf{R}_e^{\frac{\pi}{2}}(\tfrac{\pi}{2}) \right\} \qquad \text{(Pauli basis).}$$

$$\begin{cases} \mathbf{R}_o^{\varphi\dagger} = \mathbf{R}_o^{\varphi-1} = \mathbf{R}_o^{-\varphi}\,, \\ \mathbf{R}_e^{\varphi} = \mathbf{R}_o^{\varphi}\sigma^z = \sigma^z\mathbf{R}_o^{-\varphi}\,, \\ \sigma^z\, \mathbf{R}_o^{\varphi}\, \sigma^z = \mathbf{R}_o^{-\varphi}\,, \\ \det[\,\mathbf{R}_o^{\varphi}\,] = 1\,, \end{cases} \qquad \begin{matrix} \mathbf{R}_e^{\varphi\dagger} = \mathbf{R}_e^{\varphi-1} = \mathbf{R}_e^{\varphi} \\ \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{\varphi} = \mathbf{R}_o^{-\varphi}\,\mathbf{R}_e^{\varphi} = \sigma^z \\ \sigma^z\,\mathbf{R}_e^{\varphi}\,\sigma^z = \mathbf{R}_e^{-\varphi} \\ \det[\,\mathbf{R}_e^{\varphi}\,] = -1 \end{matrix}\,.$$

$$\begin{cases} \mathbf{R}_o^{\varphi_1}\,\mathbf{R}_o^{\varphi_2} = \mathbf{R}_o^{\varphi_1+\varphi_2} \\ \mathbf{R}_e^{\varphi_1}\,\mathbf{R}_e^{\varphi_2} = \mathbf{R}_o^{\varphi_1-\varphi_2} \\ \mathbf{R}_o^{\varphi_1}\,\mathbf{R}_e^{\varphi_2} = \mathbf{R}_e^{\varphi_1+\varphi_2} \\ \mathbf{R}_e^{\varphi_1}\,\mathbf{R}_o^{\varphi_2} = \mathbf{R}_e^{\varphi_1-\varphi_2} \end{cases}, \qquad \begin{cases} \mathbf{R}_o^{\varphi 2} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_e^{-\varphi} = \mathbf{R}_o^{2\varphi} \\ \mathbf{R}_e^{\varphi 2} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_o^{0} = \mathbb{1} \\ \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{\varphi} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{2\varphi} \\ \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{\varphi} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{-\varphi} = \mathbf{R}_e^{0} = \sigma^z \end{cases}.$$

It is clear that:

$$\mathbf{R}_o^{\frac{\pi}{2}} \equiv \mathbf{R}_o^{\varphi=\frac{\pi}{2}} = \begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} = -\mathbf{R}_o^{-\frac{\pi}{2}}\,, \qquad \mathbf{R}_e^{\frac{\pi}{2}} \equiv \mathbf{R}_e^{\varphi=\frac{\pi}{2}} = \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} = -\mathbf{R}_e^{-\frac{\pi}{2}}\,, \qquad \text{and}$$

$$\mathbf{R}_o^{0} = -\mathbf{R}_o^{\pm\pi} = -\{\mathbf{R}_o^{\pm\frac{\pi}{2}}\}^2 = \mathbb{1}\,, \qquad \{\mathbf{R}_e^{\pm\frac{\pi}{2}}\}^2 = \mathbb{1}\,, \qquad \mathbf{R}_e^{0} = -\mathbf{R}_e^{\pm\pi} = \sigma^z\,, \qquad \mathbf{R}_e^{\frac{\pi}{2}}\,\mathbf{R}_o^{\frac{\pi}{2}} = -\mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{R}_e^{\frac{\pi}{2}} = \sigma^z\,,$$

so that:

$$\begin{matrix} \mathbf{R}_o^{\varphi}(\phi) \equiv \cos\varphi\,\mathbf{R}_o^{0} + \sin\varphi\,\mathbf{R}_o^{\frac{\pi}{2}} = \cos\varphi\;\mathbb{1} + \sin\varphi\,\mathbf{R}_o^{\frac{\pi}{2}} = \mathrm{e}^{\,\varphi\mathbf{R}_o^{\frac{\pi}{2}}} \\ \mathbf{R}_e^{\varphi}(\phi) \equiv \cos\varphi\,\mathbf{R}_e^{0} + \sin\varphi\,\mathbf{R}_e^{\frac{\pi}{2}} = \cos\varphi\,\sigma^z + \sin\varphi\,\mathbf{R}_e^{\frac{\pi}{2}} = \mathbf{R}_o^{\varphi}(\phi)\;\sigma^z \end{matrix}\,.$$

Acting over diagonal and antidiagonal matrices:

$$\begin{cases} \mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{D}\,\mathbf{R}_o^{-\frac{\pi}{2}} = \mathbf{R}_o^{-\frac{\pi}{2}}\,\mathbf{D}\,\mathbf{R}_o^{\frac{\pi}{2}} = \begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\phi} \\ -\mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}; \\ \mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{A}\,\mathbf{R}_o^{-\frac{\pi}{2}} = -\mathbf{R}_e^{\frac{\pi}{2}}\,\mathbf{A}\,\mathbf{R}_e^{\frac{\pi}{2}} = \begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{m}^- c \\ \mathrm{m}^+ d & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\phi} \\ -\mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{m}^- d \\ \mathrm{m}^+ c & 0 \end{pmatrix} \end{cases},$$

in particular

$$\begin{cases} \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\sigma^z\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = \mathbf{R}_e^{\pm\frac{\pi}{2}}\,\sigma^z\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = \mathbf{R}_e^{\pm\pi} = -\,\sigma^z \\ \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\mathrm{m}^-\sigma^+\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = -\mathbf{R}_e^{\pm\frac{\pi}{2}}\,\mathrm{m}^-\sigma^+\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = -\,\mathrm{e}^{\,\mathrm{i}2\phi}\,\mathrm{m}^-\sigma^- = \mathrm{m}^+\sigma^- \\ \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\mathrm{m}^+\sigma^-\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = -\mathbf{R}_e^{\pm\frac{\pi}{2}}\,\mathrm{m}^+\sigma^-\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = -\,\mathrm{e}^{-\mathrm{i}2\phi}\,\mathrm{m}^+\sigma^+ = \mathrm{m}^-\sigma^+ \end{cases}.$$

$$\begin{cases} \mathbf{R}_o^{\pm\pi}\,\mathbf{D}\,\mathbf{R}_o^{\mp\pi} = (-\mathbb{1})\,\mathbf{D}\,(-\mathbb{1}) = \mathbf{R}_e^{\pm\pi}\,\mathbf{D}\,\mathbf{R}_e^{\pm\pi} = (-\sigma^z)\,\mathbf{D}\,(-\sigma^z) = \mathbf{D}\,; \\ \mathbf{R}_o^{\pm\pi}\,\mathbf{A}\,\mathbf{R}_o^{\mp\pi} = (-\mathbb{1})\,\mathbf{A}\,(-\mathbb{1}) = -\,\mathbf{R}_e^{\pm\pi}\,\mathbf{A}\,\mathbf{R}_e^{\pm\pi} = -\,(-\sigma^z)\,\mathbf{A}\,(-\sigma^z) = \mathbf{A} \end{cases}.$$

σ^z diagonalizes $\mathbf{R}_e(2\varphi) = \mathbb{n} = n_z\sigma^z + n_x\sigma^x + n_y\sigma^y = n_z\,\sigma^z + \tau\,\mathbf{R}_e^{\frac{\pi}{2}}$:

$$\mathbf{R}_o^{\varphi}\,\sigma^z\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{\varphi}\,\sigma^z\,\mathbf{R}_e^{\varphi} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{\varphi} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{2\varphi} = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin 2\varphi\,\mathrm{e}^{\mathrm{i}\phi} & -\cos 2\varphi \end{pmatrix} = \mathbb{n}\,.$$

Appendix B: Fermions and various values of φ_R. Electric charges.

According to to the content of this study we propose the following classification of elementary fermions:

Lepton families: $\varphi_R = \{\ 0\ ,\ \frac{\pi}{4}\ \}$ $\quad \begin{cases} (\ \nu : \{0, \pm\frac{\pi}{2}, \pi\}\ ), \\ (\ e, p : \{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}\ ) \end{cases}$.

Quark families: $\varphi_R = \{\ \frac{\pi}{12}\ ,\ \frac{\pi}{6}\ \}$ $\quad \begin{cases} (\ d_+ : \{-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}\},\ d_- : \{-\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}\}\ ), \\ (\ u_+ : \{-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}\},\ u_- : \{-\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}\}\ ) \end{cases}$.

***a*-neutrinon families ?**: $\varphi_R = \{\ \frac{\pi}{8}\ \}$ $\quad \{\ [\ a_+ : \{-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}\}\ ],\ [\ a_- : \{-\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}\}\ ]$.

With these specific values of $\varphi(n_t, n_z, \tau)$ (and $n_t(\varphi), n_z(\varphi), \tau(\varphi)$), we write the following diagram:

	τ	n_z	φ	2φ	$\longleftrightarrow$	$\mathfrak{m}$	
ν	0	$\{\pm 1\}$	$\{\underline{\underline{0}}, \underline{\pm\frac{\pi}{2}}, \underline{\underline{\pi}}\}$	$\{\underline{\underline{0, \pi}}\}$		$\{\pm\sigma^Z\}$	**Pauli Jordan Wigner**
e, p	$\{\pm 1\}$	0	$\{\underline{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}}\}$	$\{\underline{\pm\frac{\pi}{2}}\}$		$\{\pm \mathbf{R}_e^{\frac{\pi}{2}}\}$	
a	$\{\pm\frac{\sqrt{2}}{2}\}$	$\{\pm\frac{\sqrt{2}}{2}\}$	$\{\pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \pm\frac{7\pi}{8}\}$	$\{\underline{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}}\}$		$\{\ \pm\frac{\sqrt{2}}{2}[\sigma^Z \pm \mathbf{R}_e^{\frac{\pi}{2}}]\ \}$	
d	$\{\pm\frac{1}{2}\}$	$\{\pm\frac{\sqrt{3}}{2}\}$	$\{\pm\frac{\pi}{12}, \pm\frac{5\pi}{12}, \pm\frac{7\pi}{12}, \pm\frac{11\pi}{14}\}$	$\{\overline{\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}}\}$		$\{\ \pm[\frac{\sqrt{3}}{2}\sigma^Z \pm \frac{1}{2}\ \mathbf{R}_e^{\frac{\pi}{2}}]\ \}$	
u	$\{\pm\frac{\sqrt{3}}{2}\}$	$\{\pm\frac{1}{2}\}$	$\{\overline{\pm\frac{\pi}{6}}, \overline{\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}}, \overline{\pm\frac{5\pi}{6}}\}$	$\{\overline{\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}}\}$		$\{\ \pm[\frac{1}{2}\sigma^Z \pm \frac{\sqrt{3}}{2}\ \mathbf{R}_e^{\frac{\pi}{2}}]\ \}$	

For ν and u the sets $\{2\varphi\}$ are included in the sets $\{\varphi\}$.

Another way of viewing previous results is obtained considering a fundamental angle $\psi = \frac{\pi}{24} = 0.13\text{rad}.$ and its multiples from -24 to 24, for a total 2π−interval:

0	1	2	3	4	5	6	7	8	9	10	11	12	
0			1			2			3			4	Multiples of 3, with $\frac{\pi}{8}$
0	–	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	–	$\frac{\pi}{4}$	–	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	–	$\frac{\pi}{2}$	
$\bar{\nu}$	–	d_+	a_+	u_+	–	(p, e)	–	u_-	a_-	d_-	–	$(\underline{\nu}, \underline{\bar{\nu}})$	

13	14	15	16	17	18	19	20	21	22	23	24	
		5			6			7			8	Multiples of 3, with $\frac{\pi}{8}$
–	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	–	$\frac{3\pi}{4}$	–	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	–	π	
–	d_+	a_+	u_+	–	(e, p)	–	u_-	a_-	d_-	–	ν	

Similarly for the negative values.

Instead of the value $\psi = \frac{\pi}{24}$, we can depart from a fundamental angle $\psi' = \frac{\pi}{12} = 0.26$ rad. , and all its multiples from -12 to 12, for the total 2π−interval. Clearly this classification is in better agreement with the standard model. Afterwards it could be added the value $\varphi_R = \frac{\pi}{8}$. We write:

0	1	2	3	4	5	6	7	8	9	10	11	12	
0			1			2			3			4	Multiples of 3, with $\frac{\pi}{4}$
0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π	
$\bar{\nu}$	d_+	u_+	(p, e)	u_-	d_-	$(\underline{\nu}, \underline{\bar{\nu}})$	d_+	u_+	(e, p)	u_-	d_-	ν	

and a similar construction for the negative values. In the Figure 10 with $\psi = \frac{\pi}{24}$:

leptons with the even multipliers of $3\psi = \frac{\pi}{8}$ (0 included): $k\ 3\psi = k\ \frac{\pi}{8},\ \ k \in \{0, \pm 2, \pm 4, \pm 6, 8\}$.
something still not clear about its existence or meaning: $k'3\psi = k'\frac{\pi}{8},\ \ k' \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$.
It seems that there are no elementary particles for the prime multipliers of ψ, except for 2 and 3. The other prime multiplier values can be introduced in a similar way as for the d and u particles, though they do not have the beauty of the symmetry of the ones with the $\frac{\pi}{8}$ value. And, overall, no experimental background for them, there have never been observed $\pm\frac{1}{6}$ and $\pm\frac{5}{6}$ electric charges. For the rest of the values we consider the quarks. Again, the d related to some odd multipliers of $\frac{\pi}{12}$, and the u. to some even ones.

Electric charges.

Let us consider the angle parameter φ with the specified values: $\varphi_k = k\,\frac{\pi}{12}$, with $k \in \{0, 1, 2, 3\}$.

Then: $0 \le \varphi_k \le \frac{\pi}{4}$ and $\varphi_R \in \{\varphi_k\}$, except $\varphi_R = \frac{\pi}{8}$.

Also with $\lambda \in \{-1, 1\}$ and $n \in \{-2, -1, 0, 1\}$, we define the angles:

$$\{\lambda\varphi_{k,n}\} \equiv \{\lambda(\varphi_k + \tfrac{\pi}{2} n)\} = \{\varphi_k - \pi,\ -\varphi_k - \tfrac{\pi}{2},\ \varphi_k - \tfrac{\pi}{2},\ -\varphi_k,\ \varphi_k,\ -\varphi_k + \tfrac{\pi}{2},\ \varphi_k + \tfrac{\pi}{2},\ -\varphi_k + \pi\}.$$

Definition of the values of the electric charges:

For $\varphi_R \in \{\varphi_k\} = \{0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\}$: $q_k \equiv (-1)^{k+1}\varphi_k\,\frac{\mathbf{4}}{\pi} = (-1)^{k+1}\,k\,\frac{\pi}{12}\,\frac{\mathbf{4}}{\pi} = (-1)^{k+1}\,\frac{k}{3}$, which correspond to:

$$q_{k=0} = q_{\bar{\nu}} = 0, \qquad q_{k=1} = q_{\bar{d}} = \frac{1}{3}, \qquad q_{k=2} = q_{\bar{u}} = -\frac{2}{3}, \qquad q_{k=3} = q_{\bar{e}} = 1.$$

General q:

$$\boxed{q(\varphi) \equiv q(\lambda\varphi_{k,n}) \equiv q_{\lambda,k,n} \equiv \lambda\ \mathrm{sign}(\mathrm{i}^{-n})\ (-1)^{k+1}\ \frac{k}{3},}$$

with $\mathrm{sign}(\mathrm{i}^{-(0)}) \equiv \mathrm{sign}(\mathrm{i}^{-(-1)}) \equiv +$ and $\mathrm{sign}(\mathrm{i}^{-(-2)}) \equiv \mathrm{sign}(\mathrm{i}^{-(+1)}) \equiv -$. Also:

$$\boxed{q(k, n_t, \tau) = \mathrm{sign}(n_t(\varphi))\ \ \mathrm{sign}(\tau(\varphi))\ \ (-1)^{k+1}\ \frac{k}{3}\ .}$$

$\mathrm{sign}(\tau) = \mathrm{sign}(n_y)$. $\lambda(\varphi_k + \frac{\pi}{2} n)$ implies the values of $\mathrm{sign}(n_t)$ and of $\mathrm{sign}(\tau)$. $\tau = 0$ corresponds with $k = 0$.
We do not apply these formulas to what we have denoted as a-neutrinons ($\varphi_R = \frac{\pi}{8}$).

One more classification of the elementary fermions, with the value of the electric charge, in the following way:

$\bar{\nu}, \nu$) $\varphi_\nu \equiv \varphi_{k=0} = 0$, $\quad q_\nu = \frac{4}{\pi}\ 0 = 0$ $\quad \{-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}\}$,

e, p) $\varphi_{cl} \equiv \varphi_{k=3} = \frac{\pi}{4}$, $\quad |q_{cl}| = \frac{4}{\pi}\,\frac{\pi}{4} = 1$ $\quad \{-\frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4}\}$,

d) $\varphi_d \equiv \varphi_{k=1} = \frac{\pi}{12}$, $\quad |q_d| = \frac{4}{\pi}\,\frac{\pi}{12} = \frac{1}{3}$ $\quad \{-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}\} \cup \{-\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}\}$,

u) $\varphi_u \equiv \varphi_{k=2} = \frac{\pi}{6}$, $\quad |q_u| = \frac{4}{\pi}\,\frac{\pi}{6} = \frac{2}{3}$ $\quad \{-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}\} \cup \{-\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}\}$,

a) $\varphi_a \equiv \varphi_R = \frac{\pi}{8}$, $\quad |q_a| = 0$ $\quad \{-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}\} \cup \{-\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}\}$.

Blue color for positive electric charges and red color for negative charges. Black color for zero electric charge.

Remember that the spin of a particle does not change sign under parity, but here under 3d-opposition we have the antiparticle of the initial one with opposite spin; this is not the case for opposition in n_z (mirror in the plane $[X, Y]$).
Finally, we can establish the following rule for the appearance of the elementary fermions in the time space by relating a sign for the particles with the positions in the time space and the symmetries in a local reference frame. We write the formula (4.17) in the form $\kappa(\varphi_R) = \mathrm{sign}(\mathbf{q}(n_t, \tau))\ \mathrm{sign}(n_t) \begin{Bmatrix}\mathrm{sign(3d)} \\ \mathrm{sign}(\tau)\end{Bmatrix}$. With our construction we have that [change $\mathrm{sign}(\mathbf{q}(n_t, \tau))$]$x$[change $\mathrm{sign}(n_t)$]x[change sign $\begin{Bmatrix}(3\mathrm{d}) \\ (\tau)\end{Bmatrix}$] does not change our elementary particle (the charge and the spin). So that there can be two changes at once exclusively. Neutrinos have a different treatment. More, later in **2**.
Care should be taken that in this study we are not applying the $\mathcal{CPT}$ symmetry concept. We are not studying the interactions in relation to the combination of $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$. For this, see Perkins [25] or Bettini [26] with the interesting case $\mathcal{CP} = -1$. Here we interpret, for any elementary fermion, the necessity to have a $\mathbf{C} = -1$ or $\mathbf{P} = -1$, but not both at once, if at the same time it is $\mathbf{T} = -1$, for one elementary fermion in its local reference frame.

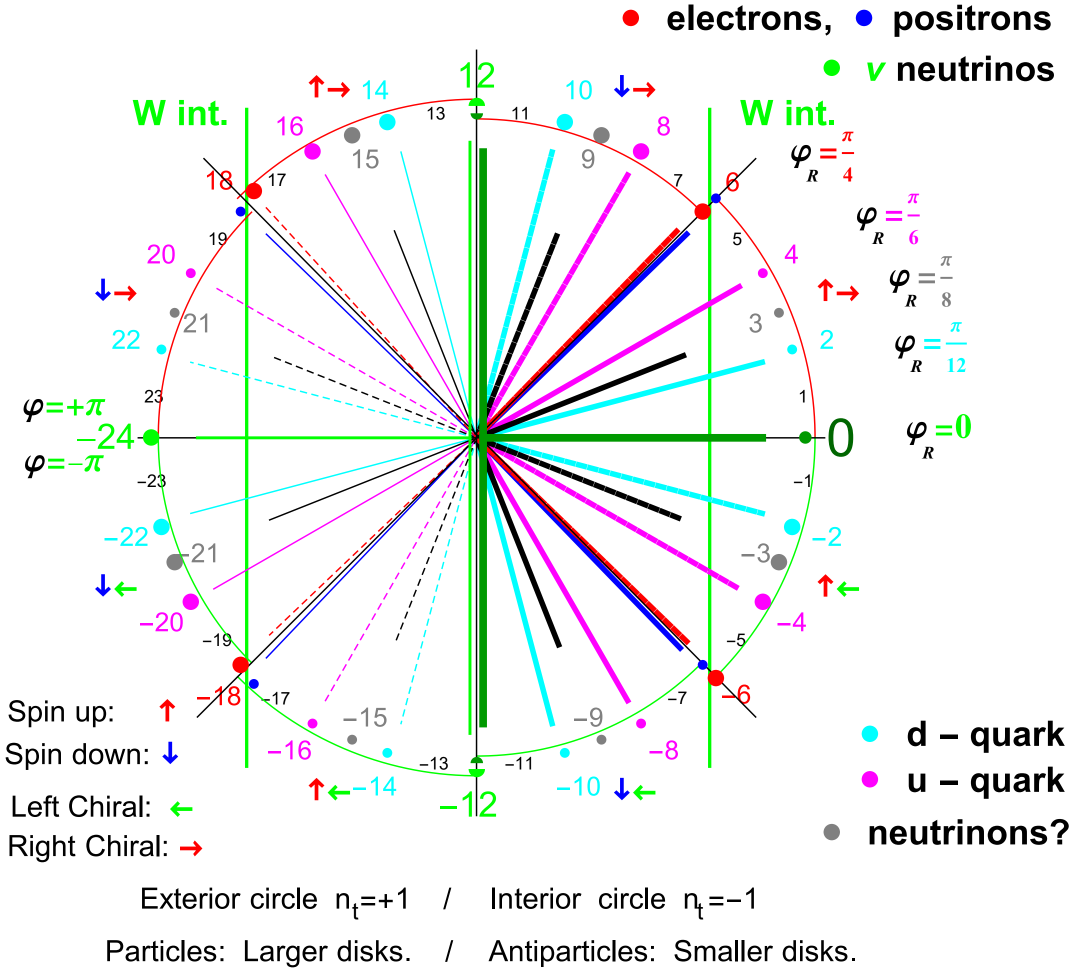


FIG. 9: Charges.

Appendix C: Vector-spin , chirality and weak interaction.

φ_R	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	Vectors of the 3D-rotations		
							Weak Interaction.	
$2[\ \varphi_R]$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\mathfrak{m} = \mathbf{R}_e^{2\varphi_R}$	[right chiral]	A
$2[\ (\varphi_R - \pi)]$							[left chiral]	P
$2[-\varphi_R]$		$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$	$\dot{\mathfrak{m}} = \mathbf{R}_e^{-2\varphi_R}$	[left chiral]	P
$2[-(\varphi_R - \pi)]$							[right chiral]	A
							without Weak Interaction.	
$2[\ (\varphi_R + \frac{\pi}{2})]$	π	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\mathfrak{m} = \mathbf{R}_e^{2\varphi_R - \pi}$	[right chiral]	P
$2[\ (\varphi_R - \frac{\pi}{2})]$							[left chiral]	A
$2[-(\varphi_R + \frac{\pi}{2})]$		$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$-\dot{\mathfrak{m}} = \mathbf{R}_e^{-2\varphi_R + \pi}$	[left chiral]	A
$2[-(\varphi_R - \frac{\pi}{2})]$							[right chiral]	P

M i r r o r i t y $\{2\varphi_R\}$

P ≡ particle, A ≡ antiparticle.
Green color: fundamental elementary fermions as particles.
Red color: fundamental elementary fermions as antiparticles.

Appendix D: cites.

About the Jordan Wigner transformation. Pauli's "Selected Topics in Field Quantization". (**1950-51**) (page 7).

" We will now quantize the amplitudes and, indeed, since we know empirically that electrons satisfy the exclusion principle (there is also a theoretical basis for this [2]), we will follow the scheme given at the end of Section 2. This invention (by Jordan and Wigner [3]), is very useful although its physical meaning appears to be obscure: the sign of an expression in the amplitudes becomes dependent upon the normal modes. "

About the creation and annihilation operators. Pauli's "Selected Topics in Field Quantization". (**1950-51**) (page 4).

" Formally, if one introduces

$$\{\mathbf{a},\mathbf{a}^*\} \equiv \mathbf{a}\mathbf{a}^* + \mathbf{a}^*\mathbf{a} = \mathbf{1}, \quad \mathbf{a}^2 = 0, \quad \mathbf{a}^{*2} = 0, \qquad [2.10]$$

then one obtains as solution

$$\mathbf{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{a}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad [2.11]$$

Furthermore, if we set

$$\mathbf{N} \equiv \mathbf{a}^*\mathbf{a} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [2.12]$$

then

$$\mathbf{1} - \mathbf{N} \equiv \mathbf{a}\mathbf{a}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{N}(\mathbf{1} - \mathbf{N}) = \mathbf{0}. \qquad [2.13]$$

This corresponds exactly to the exclusion principle. Note that here, in contrast with Bose-Einstein statistics , complete symmetry exists between $\mathbf{a}$ and $\mathbf{a}^*$, and $\mathbf{N}$ and $\mathbf{1}-\mathbf{N}$, respectively. "

Appendix E: Program of the *studies* containing this research.

On the fermionization of the XYZ spin Heisenberg chain (algebra).
(2022) https://eprints.ucm.es/id/eprint/72882/ *Study -2)*

The JordanWigner transformations and the fermionization of the XYZ spin Heisenberg chain. Algebra, geometry and physics?
(2022) https://eprints.ucm.es/id/eprint/74550/ *Study -1)*

A tentative model of creation and annihilation operators for neutrinos.
(2021) https://eprints.ucm.es/id/eprint/65151/ *Study 0)*

Expression of the 3- and 4-dimensional vectors in total polar exponential form.
(2021) https://eprints.ucm.es/id/eprint/65825/ *Study I,1)*

Vectors. Dimensions 4 and 8. (2023) https://eprints.ucm.es/id/eprint/76327/ *Study I,2)*

Geometry of the time and the space. *Study I)*

Geometry of the symmetries in dimension 4=(1+[1)+"2"], and general Time-Space-Spin vectors.
(2023) https://eprints.ucm.es/id/eprint/76328/ *Study II)*

Geometry and Physics of the Elementary Fermions. (On pride of Jordan Wigner Pauli Weyl Dirac). **1**.
(This study). (2021) https://eprints.ucm.es/id/eprint/69295/ *Study III)*

Geometry and Physics of the Elementary Fermions. **2**. *Study III)*

Axial vector magnetic charge and magnetic moment. Maxwell's equations and Lorentz force law.
(2021) https://eprints.ucm.es/id/eprint/69294/ *Study IV)*

Addenda. *Study V)*

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