On the loss of mass for the heat equation in an exterior domain with general boundary conditions

Joaquín Domínguez-de-Tena[*,1]
Aníbal Rodríguez-Bernal[*,2]

October 30, 2023

Departamento de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
and
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM[3] , Spain

[1]E-mail: joadomin@ucm.es
[2]E-mail: arober@ucm.es

Key words and phrases: Heat equation, exterior domain, mass loss, asymptotic mass, decay rates, Dirichlet, Neumann, Robin, boundary conditions.
Mathematical Subject Classification 2020: 35B30, 25B40, 35E15, 35K05

Abstract

In this work, we study the decay of mass for solutions to the heat equation in exterior domains, i.e., domains which are the complement of a compact set in $\mathbb{R}^N$. Different homogeneous boundary conditions are considered, including Dirichlet, Robin, and Neumann conditions. We determine the exact amount of mass loss and identify criteria for complete mass decay, in which the dimension of the space plays a key role. Furthermore, the paper provides explicit mass decay rates.

1 Introduction

One of the main properties of the heat equation in the entire space

$$u_t - \triangle u = 0, \qquad x \in \mathbb{R}^N, \quad t > 0,$$

is that the mass of the solution, which is defined as

$$m(t) := \int_{\mathbb{R}^N} u(x,t)dx,$$

*Partially supported by Projects PID2019-103860GB-I00 and PID2022-137074NB-I00, MICINN and GR58/08 Grupo 920894, UCM, Spain
[3]Partially supported by ICMAT Severo Ochoa Grant CEX2019-000904-S funded by MCIN/AEI/ 10.13039/501100011033

is conserved during the temporal evolution. This can be obtained, formally, by integrating the equation in $\mathbb{R}^N$ and assuming the decay at infinity of the solution or, more rigorously, by using the integral representation of the solution using the Gaussian heat kernel. Mass conservation is consistent with various probabilistic and physical interpretations of the equation and reflects the phenomenon of diffusion of u in $\mathbb{R}^N$. Of course, for nonnegative solutions, this property implies the conservation of the $L^1(\mathbb{R}^N)$ norm of the solutions with time.

In bounded domains, the situation changes. For example, if we consider the heat equation in a bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet conditions on $\partial\Omega$,

$$\begin{cases} u_t - \triangle u = 0 & in\ \Omega \times (0,\infty) \\ u = 0 & on\ \partial\Omega \times [0,\infty) \\ u = u_0 & in\ \Omega \times \{0\}, \end{cases}$$

we find that the solutions decay to zero in the $L^1(\Omega)$ and $L^\infty(\Omega)$ norms and, consequently, the mass decays to 0 for all solutions. The same occurs if we impose Robin boundary conditions of the form $\frac{\partial u}{\partial n} + bu = 0$ with $b > 0$. On the other hand, if we impose homogeneous Neumann conditions, i.e. $b = 0$, the mass is conserved during the evolution. The physical reason for this is that mass is lost through the boundary in the case of the first two boundary conditions while there is no flux through the boundary in the latter one. Mathematically, the explanation stems from the sign of the first eigenvalue of the Laplacian, which is positive in the first two cases and is zero in the latter.

However, if the domain is unbounded but has nonempty boundary, we expect to have a flux of mass through the boundary and the Laplacian to have continuous spectrum $(0,\infty)$ so the evolution of the mass is unclear. Actually, integrating the equation in Ω we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega \triangle u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(x,t)dx.$$

So, if $u \geq 0$ and $u_{|\partial\Omega} = 0$ then $\frac{\partial u}{\partial n} \leq 0$ on $\partial\Omega$ and then $\displaystyle\int_\Omega u(x,t)dx$ decreases in time although we have no quantitative estimate of the decay. The same argument holds for Robin boundary conditions $\frac{\partial u}{\partial n} + bu = 0$ with $b > 0$, while for Neumann, $b = 0$, again mass is conserved.

In this paper we consider a connected exterior domain, that is, the complement of a compact set $\mathscr{C}$ that we denote the *hole*, which is the closure of a bounded smooth set; hence, $\Omega = \mathbb{R}^N \backslash \mathscr{C}$. We will assume $0 \in \mathring{\mathscr{C}}$, the interior of the hole, and observe that $\mathscr{C}$ may have different connected components, although Ω is connected.

As we have shown, the phenomenon of loss of mass depends on the boundary conditions in the hole and we are interested in understanding and determining the amount of mass lost for any given solution. We will show then that the answer depends on the dimension. If $N \geq 3$, then there will be a certain remaining mass, while in other cases, all the mass will be lost through the hole. Also, we will show that we can explicitly compute the amount of mass lost for each initial data $u_0 \in L^1(\Omega)$. More precisely we will show that there exists a nonnegative function, Φ, that we denote the asymptotic profile, determined by the domain and boundary conditions alone, such that the amount of mass not lost through the hole by a solution with initial data $u_0 \in L^1(\Omega)$, that is, the *asymptotic mass* of the solution, is given by

$$m_{u_0} = \int_\Omega u_0(x)\Phi(x)\, dx$$

see Proposition 4.2. It is then crucial to understand this function Φ. In this direction we will show that $\Phi \equiv 1$ for Neumman boundary conditions in any dimensions (hence no loss of mass at all for any solution), while for Robin or Dirichlet boundary conditions, if $N \leq 2$ then $\Phi = 0$. That is, all mass is lost through the boundary. On the other hand, if $N \geq 3$, then

$$1 - \frac{C}{|x|^{N-2}} \leq \Phi(x) \leq 1 \qquad x \in \Omega$$

see Theorem 4.9. Also, the dependence of the loss of mass with respect to the boundary conditions is analysed in Proposition 4.4.

Finally in Theorem 4.10 we study the rate of mass loss and prove that, except for Neumann boundary conditions, for $N \geq 3$ all solutions lose mass at a uniform rate, while if $N \leq 2$ there are solutions for which the mass decays to zero as slow as we want.

The paper is organised as follows. In Section 2, we introduce the setting of the problem and the general boundary conditions we consider. We prove the main results regarding the existence and regularity of solutions and some comparison results that will be very useful thereafter. In Section 3, we construct the asymptotic profile for the problem, which is determined by the domain and the boundary conditions. In Section 4 we show that the asymptotic profile allows us to explicitly determine the amount of mass lost by each solution, see Proposition 4.2. We will also provide some estimates on the behavior of the profile that, in particular, will imply the dimension dependent behaviour discussed above, see Theorem 4.9. Appendix A contain some classical Schauder-type estimates for harmonic functions that are used for the main result.

2 The problem and preliminary elliptic and parabolic results

In this section we consider a slightly more general setting than that of an exterior domain, by allowing Ω to be a connected open set with compact boundary. That incudes the case of exterior domains but also bounded ones.

Hence, we will study the heat equation

$$\begin{cases} u_t - \triangle u = 0 & in\ \Omega \times (0,T) \\ B_\theta(u) = 0 & on\ \partial\Omega \times [0,T] \\ u = u_0 & in\ \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where $u_0 \in L^1(\Omega)$ and we consider Dirichlet, Robin or Neumann homogeneous boundary conditions on $\partial\Omega$, written in the form

$$B_\theta(u) := \sin(\frac{\pi}{2}\theta(x))\frac{\partial u}{\partial n} + \cos(\frac{\pi}{2}\theta(x))u, \tag{2.2}$$

where $\theta \in C(\partial\Omega, [0,1])$ satisfies one of the following cases in each connected component of $\partial\Omega$:

(i) Dirichlet conditions: $\theta \equiv 0$
(ii) Mixed Neumann and Robin conditions: $0 < \theta \leq 1$.

In particular, if $\theta \equiv 1$ we recover Neumann boundary conditions. In general, we will refer to these as homogeneous θ-boundary conditions. Note that, by suitably choosing $\theta(x)$, (2.2) includes all boundary conditions of the form $\frac{\partial u}{\partial n} + b(x)u = 0$. The restriction $0 \leq \theta \leq 1$ makes $b(x) \geq 0$ which is the standard dissipative condition. The reason for these notations will be seen in the results below about monotonicity of solutions with respect to θ, see Section 2.3.

As a general notation, for a given function θ as above, we define the Dirichlet part of $\partial\Omega$ as

$$\partial^D\Omega := \{x \in \partial\Omega \ : \ \theta(x) = 0\},$$

the Robin part of $\partial\Omega$ as

$$\partial^R\Omega := \{x \in \partial\Omega \ : \ 0 < \theta(x) < 1\},$$

and the Neumann part of $\partial\Omega$ as

$$\partial^N\Omega := \{x \in \partial\Omega \ : \ \theta(x) = 1\}.$$

The conditions imposed on θ imply that $\partial^D\Omega$ is a union of connected components of $\partial\Omega$, although Neumann and Robin conditions can coexist in the same connected component of $\partial\Omega$.

In general we will use a superscript θ to denote anything related to (2.1). For example, the semigroup of solutions to (2.1) will be denoted by $S^\theta(t)$ and the associated kernel by $k^\theta(x,y,t)$, see Section 2.2. Sometimes, we will add as subscript Ω to indicate the dependence of these objects in the domain.

2.1 Some elliptic results

We present some elliptic results based on an L^2 framework. For this we will denote

$$H^1_\theta(\Omega) = \{u \in H^1(\Omega) \ : \ u_{|_{\partial^D\Omega}} \equiv 0\}.$$

which is a closed subspace of $H^1(\Omega)$. Then we have the following standard result, based on Lax-Milgram theorem and the coercivity of the bilinear form in $H^1_\theta(\Omega)$

$$a_\theta(u,\varphi) = \int_\Omega \nabla u \nabla\varphi + \gamma \int_\Omega u\varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) u\varphi.$$

Notice that here we use the fact that $0 < \theta < 1$ on $\partial^R\Omega$ so $0 < \cot(\frac{\pi}{2}\theta) < \infty$ on that set.

Theorem 2.1. *Given Ω a domain with compact boundary and some homogeneous θ-boundary condition and $L \in (H^1_\theta(\Omega))'$, assume $\gamma > 0$ or $\gamma = 0$ and $\theta \not\equiv 1$, that is, except Neumann boundary conditions. Then the problem*

$$\begin{cases} -\triangle u + \gamma u = L & \text{in } \Omega \\ B_\theta(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

has a unique weak solution $u \in H^1_\theta(\Omega)$, that is,

$$\int_\Omega \nabla u \nabla\varphi + \gamma \int_\Omega u\varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) u\varphi = L(\varphi) \qquad \forall \varphi \in H^1_\theta(\Omega) \tag{2.4}$$

and there exists a constant $C > 0$ such that $\|u\|_{H^1(\Omega)} \le C\|L\|_{(H^1_\theta(\Omega))'}$.

In particular, the mapping $(H^1_\theta(\Omega))' \ni L \mapsto u \in H^1_\theta(\Omega)$ is an isomorphism.

When $L \in (H^1_\theta(\Omega))'$ is given by a function $f \in L^2(\Omega)$ in the sense that $L(\varphi) = \int_\Omega f\varphi$, with some modified arguments of the standard theory of regularity we obtain:

Theorem 2.2. *Let u be a weak solution of problem (2.3) with $\gamma > 0$ or $\gamma = 0$ and $\theta \not\equiv 1$, that is, except Neumann boundary conditions and $L(\varphi) = \int_\Omega f\varphi$ with $f \in L^2(\Omega)$. If the boundary $\partial\Omega$ is of class C^2 and $\theta \in C^1(\partial\Omega)$, then $u \in H^2(\Omega)$ and there exists a $C > 0$ independent of f such that*

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.5}$$

In particular, the mapping $f \mapsto u$ defines an isomorphism from $L^2(\Omega)$ into

$$D(\triangle_\theta) = \{u \in H^2(\Omega),\ B_\theta(u) = 0 \text{ on } \partial\Omega\}$$

endowed with the norm of $H^2(\Omega)$, which is dense in $H^1_\theta(\Omega)$. The inverse of this operator is $-\Delta + \gamma I$ on $D(\triangle_\theta)$ and is a closed and selfadjoint operator in $L^2(\Omega)$.

Proof. If Ω is bounded the result is standard and can be found for Dirichlet boundary conditions e.g. in [Eva10] Section 6.3 Theorem 4 and for Neumann and Robin ones in [Mik78] Page 217, Theorem 4 and footnote. If Ω is unbounded, and hence an exterior domain, the proof is not easy to find in the literature, so we give a proof.

In this case, consider $\Omega_R = \Omega \cap B(0,R)$ with R large enough so that $\partial\Omega \subset \partial\Omega_R$. Then, we consider a cut-off function $\chi \in C_c^\infty(\Omega_{2R})$ such that $\chi(\Omega_R) \equiv 1$. Then $\tilde{u} = u\chi$ is a weak solution of

$$\begin{cases} -\triangle \tilde{u} + \gamma \tilde{u} = \tilde{f} & \text{in } \Omega_{2R} \\ B_\theta(\tilde{u}) = 0 & \text{on } \partial\Omega_{2R} \end{cases}$$

with $\tilde{f} = f\chi - \nabla u \nabla \chi - u \triangle \chi$. Hence,

$$\left\|\tilde{f}\right\|_{L^2(\Omega_{2R})} \leq C\|f\|_{L^2(\Omega_{2R})} + C\|u\|_{H^1(\Omega_{2R})} \overset{\text{Thm } 2.1}{\leq} C\|f\|_{L^2(\Omega_{2R})}.$$

Now, from the regularity of the boundary $\partial\Omega$ and θ, we can use classical elliptic regularity results in bounded domains (see for example [Mik78] Page 217, Theorem 4 and footnote) to obtain $\|\tilde{u}\|_{H^2(\Omega_{2R})} \leq C\left\|\tilde{f}\right\|_{L^2(\Omega_{2R})} \leq C\|f\|_{L^2(\Omega_{2R})}$. Thus,

$$\|u\|_{H^2(\Omega_R)} \leq C\|f\|_{L^2(\Omega)}. \tag{2.6}$$

Now, classical interior regularity results (see for example [Eva10] Section 6.3) guarantee that $u \in H^2_{loc}(\Omega)$ and, for any two concentric balls $B(x,r) \subset B(x,2r) \subset \Omega$

$$\|u\|_{H^2(B(x,r))} \leq C(r)(\|f\|_{L^2(B(x,2r))} + \|u\|_{L^2(B(x,2r))}).$$

Then, choosing $r > 0$ sufficiently small and covering $\Omega \backslash \Omega_R$ with a countable family of balls $B(x_i,r) \subset B(x_i,2r) \subset \Omega$) in a way that every point $x \in \Omega$ is contained only in a finite number (m, independent of x) of balls $B(x_i,2r)$ we obtain

$$\begin{aligned} \|u\|_{H^2(\Omega\backslash\Omega_R)} &\leq \sum_i \|u\|_{H^2(B(x_i,r))} \leq \sum_i C(r)(\|f\|_{L^2(B(x_i,2r))} + \|u\|_{L^2(B(x_i,2r))}) \\ &\leq mC(r)(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \overset{\text{Thm } 2.1}{\leq} C\|f\|_{L^2(\Omega)}. \end{aligned} \tag{2.7}$$

Finally, combining (2.6) and (2.7), we obtain (2.5).

Once $u \in H^2(\Omega)$, integrating in parts in the weak formulation (2.4) we easily get $B_\theta(u) = 0$ on $\partial\Omega$. Hence, the description of $D(\Delta_\theta)$ follows. That this space is dense in $H^1_\theta(\Omega)$ is because the isomorphisms in Theorems 2.1 and 2.2 and the fact that $H^1_\theta(\Omega)$ is dense in $L^2(\Omega)$, which implies in turn that $L^2(\Omega)$ is dense in $(H^1_\theta(\Omega))'$. The rest also follows easily. ■

2.2 Semigroup generated by $\triangle_\theta$

Now we present some results about the semigroup of solutions associated to (2.1), that we will denote $S^\theta(t)$, which is the semigroup generated by $\triangle_\theta$. If at some point we want to stress the dependence on the domain, we will denote it by $S^\theta_\Omega(t)$.

We start with the case of initial data in $L^2(\Omega)$.

Theorem 2.3. *Given Ω a domain with compact boundary and some homogeneous θ-boundary conditions, the operator $(\triangle_\theta, D(\triangle_\theta))$ generates an analytic C^0 semigroup of contractions $\{S^\theta(t)\}_{t>0}$ in $L^2(\Omega)$, that is, a family of bounded linear functions from $L^2(\Omega)$ into itself such that:*

(i) *Semigroup property:* $S^\theta(t+s)u_0 = S^\theta(t)S^\theta(s)u_0$ *for every* $0 < s < t$ *and* $u_0 \in L^2(\Omega)$.
(ii) C^0 *property:* $\lim_{t\to 0} S^\theta(t)u_0 = u_0$ *in* $L^2(\Omega)$ *for every* $u_0 \in L^2(\Omega)$.
(iii) *Contraction property:* $\left\| S^\theta(t) \right\|_{\mathcal{L}(L^2(\Omega))} \leq 1$ *for every* $t > 0$.
(iv) *Satisfies the PDE: The semigroup is analytic and therefore for every* $u_0 \in L^2(\Omega)$, $u(t) = S^\theta(t)u_0 \in D(\Delta_\theta)$ *for* $t > 0$ *and satisfies*

$$\begin{cases} \dfrac{d}{dt}u(t) - \triangle_\theta u(t) = 0 & \forall t > 0 \\ B_\theta(u(t)) = 0 & \forall t > 0. \end{cases}$$

(v) *Assume furthermore that the boundary* $\partial\Omega$ *is of class* C^m *and* $\theta \in C^m(\partial\Omega)$ *for* m *large enough. Then for* $u_0 \in L^2(\Omega)$, $u(x,t) = S^\theta(t)u_0(x)$ *is a* $C^{2,1}(\overline{\Omega} \times (0,\infty))$ *solution of the heat equation, that is*

$$\begin{cases} u_t(x,t) - \triangle u(x,t) = 0 & \forall (x,t) \in \Omega \times (0,\infty) \\ B_\theta(u)(x,t) = 0 & \forall x \in \partial\Omega, \ \forall t > 0. \end{cases}$$

Proof. (i)-(iv). Let us see that the operator $\triangle_\theta$ is dissipative. Take $u \in D(\triangle_\theta)$, then using $\cot(\frac{\pi}{2}\theta)u \geq 0$,

$$\int_\Omega u \triangle u = -\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u\frac{\partial u}{\partial n} = -\int_\Omega |\nabla u|^2 - \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta)u^2 \leq 0.$$

In addition, $D(\triangle_\theta)$ is dense in $L^2(\Omega)$ and $(0,\infty) \subset \rho(\triangle_\theta)$, due to Theorem 2.1 and Theorem 2.2. Thus, we can use Lumer-Phillips Theorem (See [Paz10] Chapter 1 Theorem 4.3) to obtain the C^0 semigroup of contractions. The analiticity is a consequence of the selfadjointness of the operator see Theorem 3.2.1 in [CH90].

Finally, for (v), since the semigroup is analytic, $S(t) : L^2(\Omega) \to D((-\triangle_\theta)^k)$ is continuous for any $k \in \mathbb{N}$. Now, using higher regularity estimates up to the boundary (See [Eva10] Section 6.3, [GT15] Section 6 or [Mik78] Section IV.2), we have that, if the boundary $\partial\Omega$ is of class C^{2k} and $\theta \in C^{2k-1}(\partial\Omega)$, then $D((-\triangle_\theta)^k) \subset H^{2k}(\Omega)$ continuously. Hence, for sufficiently large k, we have that $S(t) : L^2(\Omega) \to C^2(\overline{\Omega})$ is continuous. This and Lemma 3.1 of [ACDRB04] implies that, for any $u_0 \in L^2(\Omega)$, $t \mapsto S(t)u_0 \in C^2(\overline{\Omega})$ is analytic. Therefore, $u(x,t) = S(t)u_0(x)$ belongs to $C^{2,1}(\overline{\Omega} \times (0,\infty))$. ■

Now we show that we can extend the semigroup above to $L^p(\Omega)$ spaces and it has nice properties.

Theorem 2.4. *The semigroup* $\{S^\theta(t)\}_{t>0}$ *above has the following properties:*

(i) *It extends to a semigroup of contractions in* $L^p(\Omega)$ *for* $1 \leq p \leq \infty$ *which is* C^0 *if* $p \neq \infty$ *and analytic if* $1 < p < \infty$.
(ii) $S^\theta(t)u_0 \geq 0$ *for every* $0 \leq u_0 \in L^p(\Omega)$ *with* $1 \leq p \leq \infty$. *That is, the semigroup is order preserving.*
(iii) $S^\theta(t)$ *is selfadjoint in* $L^2(\Omega)$ *and moreover for* $1 \leq p \leq \infty$,

$$\int_\Omega f S^\theta(t) g = \int_\Omega g S^\theta(t) f \qquad \text{for all } f \in L^p(\Omega), \ g \in L^q(\Omega) \tag{2.8}$$

where q *is the conjugate of* p, *that is* $\frac{1}{p} + \frac{1}{q} = 1$.
(iv) *If* $\partial\Omega$ *and* θ *are regular enough, the semigroup has an integral positive kernel, that is* $k^\theta : \Omega \times \Omega \times (0,\infty) \to (0,\infty)$ *such that for all* $1 \leq p \leq \infty$ *and* $u_0 \in L^p(\Omega)$,

$$S^\theta(t)u_0(x) = \int_\Omega k^\theta(x,y,t)u_0(y)dy, \qquad x \in \Omega, \quad t > 0. \tag{2.9}$$

Moreover $k^\theta(x,y,t) = k^\theta(y,x,t)$.

(v) $\{S^\theta(t)\}_{t>0}$ *is an analytic* C^0 *semigroup in* $BUC_\theta(\Omega) = \{u \in BUC(\Omega) : u_{|_{\partial^D\Omega}} \equiv 0\}$ *where* $BUC(\Omega)$ *is the space of bounded uniformly continuous functions.*

Proof. For simplicity in the proof we will not write the superscript θ.
(i)-(ii)-(iii). Observe that from Theorem 2.2 the the quadratic form associated to $-\triangle_\theta$ is given by

$$Q(u) = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega^R} \cot(\frac{\pi}{2}\theta)u^2, \qquad u \in H^1_\theta(\Omega).$$

Since for $u \in H^1_\theta(\Omega)$ we have $|u| \in H^1_\theta(\Omega)$, then from Theorem 1.3.2 in [Dav89] we have that semigroup is order preserving in $L^2(\Omega)$.

Also, if $0 \le u \in H^1_\theta(\Omega)$ we have $v = \max\{u, 1\} \in H^1_\theta(\Omega)$ and

$$Q(v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega^R} \cot(\frac{\pi}{2}\theta)\,|v|^2 \le Q(u).$$

Therefore, Theorem 1.3.3 in [Dav89] implies that we have an order preserving semigroup of contractions in $L^p(\Omega)$, $1 \le p \le \infty$. Additionally, by Theorem 1.4.1 in [Dav89] these semigroups are consistent in the sense that they coincide on $L^p(\Omega) \cap L^q(\Omega)$ for any p, q and satisfy the duality property (2.8). Finally, the analyticity dor $1 < p < \infty$ follows from Theorem 1.4.2 in [Dav89].

(iv) As in Theorem 2.3, since $S(t)$ is an analytic semigroup in $L^2(\Omega)$, for any $t > 0$ and $k \in \mathbb{N}$, if $\partial\Omega$ and θ are sufficiently regular, $S(t) : L^2(\Omega) \to D((-\triangle_\theta)^k) \subset H^{2k}(\Omega)$ continuously.

Thus, taking k large, we obtain that, for any $t > 0$, $S(t) : L^2(\Omega) \to L^\infty(\Omega)$ is continuous and by duality we also have that $S(t) : L^1(\Omega) \to L^2(\Omega)$ is continuous. Therefore, using $S(t) = S(t/2)S(t/2)$ we have that $S(t) : L^1(\Omega) \to L^\infty(\Omega)$ is also continuous. Hence, we can use [AB94] Theorem 4.16, to prove the existence of the kernel, $k^\theta(x, y, t)$, which is positive because $S(t)$ is order preserving.

In particular for $f, g \in C^\infty_c(\Omega)$, (2.8) implies

$$\int_{\Omega\times\Omega} k^\theta(x, y, t)f(x)g(y)dxdy = \int_{\Omega\times\Omega} k^\theta(y, x, t)f(x)g(y)dydx$$

and therefore $k^\theta(x, y, t) = k^\theta(y, x, t)$.

(v) This follows from Theorem 2.4 in [Mor83]. ■

In addition, the positivity of the semigroup gives us a useful consequence.

Corollary 2.5. *With the notations above, for any* $u_0 \in L^p(\Omega)$, $1 \le p \le \infty$,

$$\left|S^\theta(t)u_0(x)\right| \le S^\theta(t)\,|u_0|\,(x), \qquad x \in \Omega,\ t > 0.$$

Proof. Splitting $u_0 = u_0^+ - u_0^-$ and using the linearity and positivity of the semigroup:

$$\begin{aligned}\left|S^\theta(t)u_0(x)\right| &= \left|S^\theta(t)u_0^+(x) - S^\theta(t)u_0^-(x)\right| \le \left|S^\theta(t)u_0^+(x)\right| + \left|S^\theta(t)u_0^-(x)\right| \\ &= S^\theta(t)u_0^+(x) + S^\theta(t)u_0^-(x) = S^\theta(t)\,|u_0|\,(x).\end{aligned}$$

■

2.3 Comparison principles with general boundary conditions

Now we prove some monotonicity results for the solutions of the elliptic and parabolic problems above as well as monotonicity with respect to the function θ.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a domain with compact boundary and let $f_1, f_2 \in L^2(\Omega)$ and $g_1, g_2 \in L^2(\partial\Omega)$. Let $u_1, u_2 \in H^1_\theta(\Omega)$ be two weak solutions of:*

$$\begin{cases} -\triangle u_i + \gamma u_i = f_i & \text{in } \Omega \\ B_\theta(u_i) = g_i & \text{in } \partial\Omega, \end{cases}$$

where $i = 1, 2$, in the sense that $u_i = g_i$ on $\partial^D\Omega$ and for any $\varphi \in H^1_\theta(\Omega)$,

$$\int_\Omega \nabla u_i \nabla \varphi + \gamma \int_\Omega u_i \varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) u_i \varphi = \int_{\partial^R\Omega \cup \partial^N\Omega} \frac{g_i}{\sin(\frac{\pi}{2}\theta)} \varphi + \int_\Omega f_i \varphi.$$

Then, if $f_1 \geq f_2$, $g_1 \geq g_2$ and $\gamma > 0$ or $\gamma = 0$ but $\theta \not\equiv 1$, that is, except for Neumann boundary conditions, we have

$$u_1 \geq u_2 \qquad x \in \Omega.$$

Proof. We take $v = u_2 - u_1$. It satisfies:

$$\int_\Omega \nabla v \nabla \varphi + \gamma \int_\Omega v \varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) v \varphi = \int_{\partial^R\Omega \cup \partial^N\Omega} \frac{g}{\sin(\frac{\pi}{2}\theta)} \varphi + \int_\Omega f \varphi.$$

with $f = f_2 - f_1 \leq 0$ and $g = g_2 - g_1 \leq 0$. Then we take $0 \leq \varphi = v^+ = \max\{v, 0\} \in H^1_\theta(\Omega)$ to get

$$\int_\Omega \left|\nabla v^+\right|^2 + \gamma \int_\Omega \left|v^+\right|^2 + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) |v^+|^2 \leq 0.$$

If $\gamma > 0$ we get $\int_\Omega \left|v^+\right|^2 = 0$ and then $v \leq 0$ in Ω as claimed. On the other hand, if $\gamma = 0$ but $\theta \not\equiv 1$, we obtain

$$\int_\Omega \left|\nabla v^+\right|^2 = 0, \qquad \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta) |v^+|^2 = 0.$$

Hence, from the first term above, v^+ is constant in Ω and from the second we get $v^+ \equiv 0$. Whence $v \leq 0$ again. ■

We will also be able to compare solutions with different types of boundary conditions. The following theorem, as well as its parabolic version (Theorem 2.9, presented later), justifies the parametrization used for the boundary conditions B_θ in (2.2).

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain with compact boundary $\partial\Omega$ of class C^2 and $\theta \in C^1(\partial\Omega)$ and let $0 \leq f \in L^2(\Omega)$. Let $u_i \in D(\Delta_{\theta i})$, $i = 1, 2$, be solutions of*

$$\begin{cases} -\triangle u_i + \gamma u_i = f & \text{in } \Omega \\ B_{\theta_i}(u_i) = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \leq \theta_i \leq 1$, $\gamma > 0$ or $\gamma = 0$ but $\theta_i \not\equiv 1$, that is, except for Neumann boundary conditions. Then,

$$\theta_1 \leq \theta_2 \quad \text{implies} \quad u_1 \leq u_2.$$

Proof. From Theorem 2.6, $u_i \geq 0$ and from Theorem 2.2 they satisfy the boundary conditions pointwise in $\partial\Omega$.

Now observe that $\sin(\frac{\pi}{2}\theta_1) \leq \sin(\frac{\pi}{2}\theta_2)$, $\cos(\frac{\pi}{2}\theta_1) \geq \cos(\frac{\pi}{2}\theta_2)$ and since $u_2 \geq 0$ then $\frac{\partial u_2}{\partial n} \leq 0$ on $\partial\Omega$. This is clear in the Dirichlet and Neumann parts of the boundary and in the Robin one is because $0 < \theta_2 < 1$. Therefore,

$$B_{\theta_1}(u_2) = \sin(\frac{\pi}{2}\theta_1)\frac{\partial u_2}{\partial n} + \cos(\frac{\pi}{2}\theta_1)u_2 \geq \sin(\frac{\pi}{2}\theta_2)\frac{\partial u_2}{\partial n} + \cos(\frac{\pi}{2}\theta_2)u_2 = B_{\theta_2}(u_2) = 0.$$

Thus $v = u_2 - u_1$ satisfies $-\Delta v + \gamma v = 0$ in Ω and $B_{\theta_1}(v) \geq 0$ in $\partial\Omega$ and then Theorem 2.6 implies $v \geq 0$. ■

For the parabolic problems, we have the following results.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^N$ be a domain with compact boundary and let $u_{10}, u_{20} \in L^2(\Omega)$, $f_1, f_2 \in L^1((0,T), L^2(\Omega))$ and $g_1, g_2 \in L^1((0,T), L^2(\partial\Omega))$ with $T > 0$. Finally, assume $u_1, u_2 \in C^1((0,T), H^1_\theta(\Omega)) \cap C([0,T], L^2(\Omega))$ are such that they are weak solutions of the problems*

$$\begin{cases} \dfrac{\partial}{\partial t}u_i - \triangle u_i = f_i & \text{in } \Omega \times (0,T) \\ B_\theta(u_i) = g_i & \text{on } \partial\Omega \times (0,T) \\ u_i = u_{i,0} := u_i(0) & \text{in } \Omega \times \{0\}, \end{cases}$$

for $i = 1, 2$, in the sense that $u_i = g_i$ on $\partial^D\Omega \times (0,T)$ and for any $\varphi \in C([0,T], H^1_\theta(\Omega))$,

$$\int_\Omega (u_i)_t\varphi + \int_\Omega \nabla u_i \nabla\varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta)u_i\varphi = \int_{\partial^R\Omega\cup\partial^N\Omega} \frac{g_i}{\sin(\frac{\pi}{2}\theta)}\varphi + \int_\Omega f_i\varphi \qquad t \in (0,T).$$

Then, if $f_1 \geq f_2$, $g_1 \geq g_2$ and $u_{1,0} \geq u_{2,0}$, we have

$$u_1 \geq u_2 \qquad x \in \Omega,\ t \in (0,T).$$

Proof. The function $v := u_2 - u_1$ satisfies $v(0) = u_{2,0} - u_{1,0} \leq 0$ and for any $\varphi \in C([0,T], H^1_\theta(\Omega))$,

$$\int_\Omega v_t\varphi + \int_\Omega \nabla v\nabla\varphi + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta)v\varphi = \int_{\partial^R\Omega\cup\partial^N\Omega} \frac{g}{\sin(\frac{\pi}{2}\theta)}\varphi + \int_\Omega f\varphi$$

with $f = f_2 - f_1 \leq 0$ and $g = g_2 - g_1 \leq 0$.

Now take $0 \leq \varphi = v_+ = \max\{v, 0\} \in C([0,T], H^1_\theta(\Omega))$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_+|^2 + \int_\Omega |\nabla v_+|^2 + \int_{\partial^R\Omega} \cot(\frac{\pi}{2}\theta)|v_+|^2 \leq 0.$$

Therefore, using the continuity up to $t = 0$ in $L^2(\Omega)$, $\int_\Omega |v_+(t)|^2 \leq \int_\Omega |v_+(0)|^2 = 0$ and then $v \leq 0$. ■

As in the elliptic case, we can also compare solutions with different types of boundary conditions in the parabolic framework.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^N$ be a domain with compact boundary and let $S^{\theta_1}(t)$ and $S^{\theta_2}(t)$ be the semigroups in Theorem 2.4 for different θ-boundary conditions. Then, for any $1 \leq p \leq \infty$ and $0 \leq u_0 \in L^p(\Omega)$ we have*

$$0 \leq \theta_1 \leq \theta_2 \leq 1 \quad \text{implies} \quad S^{\theta_1}(t)u_0 \leq S^{\theta_2}(t)u_0 \quad t > 0.$$

In particular, if we denote k^{θ_1} and k^{θ_2} the corresponding heat kernels, we have that:

$$0 < k^{\theta_1}(x,y,t) \leq k^{\theta_2}(x,y,t) \qquad x, y \in \Omega,\ t > 0.$$

Proof. Assume first $0 \le u_0 \in L^2(\Omega)$ and consider $v(t) := S^{\theta_2}(t)u_0 - S^{\theta_1}(t)u_0 = u_2(t) - u_1(t)$. From Theorem 2.8, $u_i(t) \ge 0$ and from the smoothness in Theorem 2.3, v satisfies $v_t - \triangle v = 0$ in $\Omega \times (0, \infty)$, $v(0) = 0$ in Ω and

$$B_{\theta_1}(v(x)) = \sin(\frac{\pi}{2}\theta_1(x))\frac{\partial v}{\partial n} + \cos(\frac{\pi}{2}\theta_1(x))v \ge 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

This is because $\sin(\frac{\pi}{2}\theta_1) \le \sin(\frac{\pi}{2}\theta_2)$, $\cos(\frac{\pi}{2}\theta_1) \ge \cos(\frac{\pi}{2}\theta_2)$ and $u_2(t) \ge 0$ in Ω, $\frac{\partial u_2(t)}{\partial n} \le 0$ on $\partial\Omega$. This is clear in the Dirichlet and Neumann parts of the boundary and in the Robin one is because $0 < \theta_2 < 1$.

Thus, $\sin(\frac{\pi}{2}\theta_1)\frac{\partial u_2(t)}{\partial n} + \cos(\frac{\pi}{2}\theta_1)u_2(t) \ge \sin(\frac{\pi}{2}\theta_2)\frac{\partial u_2(t)}{\partial n} + \cos(\frac{\pi}{2}\theta_2)u_2(t) = B_{\theta_2}(u_2(t)) = 0$. Therefore, we can apply Theorem 2.8 to obtain that $v \ge 0$ as claimed (note that, as v is a classical solution of the heat equation, it is in a particular a weak solution, so we can apply Theorem 2.8).

In particular for any $0 \le \varphi \in C_c^\infty(\Omega)$:

$$\int_\Omega k^{\theta_1}(x, y, t)\varphi(y)dy = S^{\theta_1}(t)\varphi(x) \le S^{\theta_2}(t)\varphi(x) = \int_\Omega k^{\theta_2}(x, y, t)\varphi(y)dy \quad \forall x \in \Omega, \quad \forall t > 0,$$

and therefore $k^{\theta_1}(x, y, t) \le k^{\theta_2}(x, y, t)$ for every $x, y \in \Omega$ and $t > 0$.

Finally, if $0 \le u_0 \in L^p(\Omega)$, the ordering of the kernels above and (2.9) imply $S^{\theta_1}(t)u_0 \le S^{\theta_2}(t)u_0$. ■

As a consequence we get the following important result for exterior domains.

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain and k^θ its associated heat kernel for some homogeneous $\theta-$boundary conditions. There exists constants $c, C > 0$ such that*

$$0 < k^\theta(x, y, t) \le C\frac{e^{-\frac{|x-y|^2}{4ct}}}{t^{N/2}} \qquad x, y \in \Omega, \quad t > 0. \tag{2.10}$$

In particular,

$$\left\| S^\theta(t)u_0 \right\|_{L^\infty(\Omega)} \le C\frac{\|u_0\|_{L^1(\Omega)}}{t^{N/2}} \quad t > 0. \tag{2.11}$$

Proof. The Gaussian bound (2.10) can be found in [Gyr07] Theorem 1.3.1 for Neumann boundary conditions (see also [GS11] Theorem 3.10), that is for $\theta \equiv 1$. Theorem 2.9 implies the bound for other θ-boundary conditions.

The estimate (2.11) is a consequence of the Gaussian bounds above. ■

3 The asymptotic profile

In this section we get back to the case of an exterior domain, that is, the complement of a compact set $\mathscr{C}$ that we denote the *hole*, which is the closure of a bounded smooth set; hence, $\Omega = \mathbb{R}^N \backslash \mathscr{C}$. We assume $0 \in \mathring{\mathscr{C}}$, the interior of the hole, and observe that $\mathscr{C}$ may have different connected components, although Ω is connected.

We are going to construct the asymptotic profile for problem (2.1), which is a function that depends only on the boundary conditions and the domain, which will characterise the asymptotic mass of solutions as we will see in Section 4. We will show below that the profile can be constructed from parabolic and elliptic arguments and that both procedures give the same function.

We start with the parabolic construction.

Lemma 3.1. *Given an exterior domain $\Omega \subset \mathbb{R}^N$ and some homogenous θ-boundary conditions on $\partial\Omega$, we define its associated θ-parabolic profile as the pointwise monotonically decreasing limit:*

$$\Phi_p^\theta(x) := \lim_{t\to\infty} S^\theta(t)1_\Omega(x) \qquad x \in \Omega, \tag{3.1}$$

where 1_Ω is the characteristic function of Ω. Then, Φ_p^θ is well defined and furthermore

$$0 \le \Phi_p^\theta(x) \le 1 \qquad \forall x \in \Omega.$$

Proof. From Theorem 2.4, the semigroup $S^\theta(t)$ is of contractions in $L^\infty(\Omega)$. Hence in particular, $\left\| S^\theta(t)1_\Omega \right\|_{L^\infty(\Omega)} \le 1$, that is:

$$0 \le S^\theta(t)1_\Omega(x) \le 1_\Omega(x) \qquad \forall x \in \Omega.$$

Since the semigroup is order preserving we obtain:

$$S^\theta(t+s)1_\Omega \le S^\theta(s)1_\Omega \qquad \forall s,t > 0,$$

that is, $S^\theta(t)1_\Omega(x)$ is pointwise monotonically decreasing in t and is bounded below by 0. Therefore, the limit in (3.1) is well defined and $0 \le \Phi_p^\theta \le 1$. ■

Now we perform the elliptic construction. Firstly, for every $R > 0$ we consider the problem

$$\begin{cases} -\triangle\phi_R^\theta(x) = 0 & \forall x \in \Omega_R := \Omega \cap B(0,R) \\ B_\theta(\phi_R^\theta)(x) = 0 & \forall x \in \partial\Omega \\ \phi_R^\theta(x) = 1 & \forall\, |x| = R. \end{cases} \tag{3.2}$$

As for large R, Ω_R is a bounded regular domain, we have a unique solution ϕ_R^θ to this problem (see for example [GT15] Theorem 6.31). Furthermore, we can compare these functions for different R:

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain, $R_1 < R_2$ and $\phi_{R_1}^\theta, \phi_{R_2}^\theta$ be defined as in (3.2). Then*

$$1 \ge \phi_{R_1}^\theta(x) \ge \phi_{R_2}^\theta(x) \ge 0 \qquad \forall x \in \Omega_{R_1}.$$

Proof. First of all, as 1_{Ω_R} is a supersolution of (3.2) for any $R > 0$, using Theorem 2.6, we obtain that $\phi_R(x) \le 1$ in Ω_R.

Now, for $|x| = R_1$, $\phi_{R_2}(x) \le 1 = \phi_{R_1}(x)$, thus ϕ_{R_2} is a subsolution of problem (3.2) with $R = R_1$. Hence, using Theorem 2.6 we obtain the result. ■

With this we construct the elliptic profile as follows.

Lemma 3.3. *Let the θ-elliptic profile Φ_e^θ be defined as the monotonically decreasing limit*

$$0 \le \Phi_e^\theta(x) := \lim_{R\to\infty} \phi_R^\theta(x) \le 1 \qquad x \in \Omega.$$

Then, if $\partial\Omega$ and θ are regular enough, Φ_e^θ is a harmonic function in Ω, $\Phi_e^\theta \in C^2(\overline{\Omega}) \cap C^\infty(\Omega)$ and $B_\theta(\Phi_e^\theta) \equiv 0$ on $\partial\Omega$. In particular, $\Phi_e^\theta \in BUC_\theta(\Omega)$.

Proof. The limit exists because of the monotonicity of $\phi_R^\theta(x)$. Furthermore, for any ball $B \subset \Omega$, for sufficiently large R we have that ϕ_R^θ is harmonic in B. As the monotonic pointwise limit of harmonic functions is harmonic, Φ_e^θ is harmonic in B. As B was arbitrary, Φ_e^θ is harmonic in Ω and hence $C^\infty(\Omega)$.

Finally, to see that Φ_e^θ satisfies the boundary condition, we need to see that the $\{\phi_R^\theta\}$ and its derivatives converge uniformly, close to the boundary $\partial\Omega$. To do this, we consider, for $R > R_0$, the restriction of ϕ_R^θ to Ω_{R_0} that satisfies

$$\begin{cases} -\triangle\phi_R^\theta(x) = 0 & \forall x \in \Omega_{R_0} \\ B_\theta(\phi_R^\theta)(x) = 0 & \forall x \in \partial\Omega \\ \phi_R^\theta(x) \le 1 & \forall\, |x| = R_0. \end{cases}$$

Then, if $\partial\Omega$ is regular and $\theta \in C^{1+\alpha}(\partial\Omega)$ for $0 < \alpha < 1$, we apply Schauder estimates from Theorem A.2 and obtain

$$\left\|\phi_R^\theta\right\|_{C^{2+\alpha}(\overline{\Omega_{R_0}})} \le C.$$

Hence, using the Ascoli-Arzelà theorem, we have uniform convergence of a subsequence of ϕ_R^θ and its derivatives. Therefore, as $B_\theta(\phi_R^\theta) \equiv 0$ in $\partial\Omega$ for any R, then $B_\theta(\Phi_e^\theta) \equiv 0$. In particular, Φ_e^θ is in $BUC(\Omega)$ because it is continuous up to the boundary and it is a bounded harmonic function, so uniformly continuous due to the Schauder estimates of Theorem A.1. ∎

Remark 3.4. *The values of the function $\phi_R^0(x)$ above is denoted in the literature as the harmonic measure for the point x of the set $E = \{y \in \Omega : |y| = R\}$, and is denoted as $\omega_{\Omega_R}^x(E)$ (see e.g. [CKL05]).*

Also, the elliptic profile above when $N \ge 3$ is sometimes referred in the literature as a harmonic profile or réduite. See for example [GS11]. Furthermore, it can be understood as $1-\omega_\Omega^x(\partial\Omega)$ where ω_Ω^x is the harmonic measure in Ω (See [CKL05]).

We now prove that both profiles in fact coincide:

Proposition 3.5. *Let $\Omega = \mathbb{R}^N \backslash \mathscr{C}$ be an exterior domain. Then the elliptic and parabolic profiles coincide, $\Phi_e^\theta = \Phi_p^\theta$, so we denote them Φ^θ. Also $S^\theta(t)\Phi^\theta = \Phi^\theta$ for $t > 0$.*

Proof. (**1**). $\mathbf{\Phi_e^\theta \le \Phi_p^\theta}$: Since, $\Phi_e^\theta \in BUC_\theta(\Omega) \cap C^\infty(\Omega)$ and harmonic, then it is a strict solution of the heat equation (2.1). Therefore, using the uniqueness of strict solutions for C^0 semigroups (See for example, [Paz10] Theorem 1.3.) and part (v) in Theorem 2.4, we obtain that $S^\theta(t)\Phi_e^\theta = \Phi_e^\theta$ for $t > 0$.

As $S^\theta(t)$ preserves the order (Theorem 2.4), and $0 \le \Phi_e^\theta \le 1$, we have that

$$\Phi_p^\theta = \lim_{t\to\infty} S^\theta(t)1_\Omega \ge \lim_{t\to\infty} S^\theta(t)\Phi_e^\theta = \Phi_e^\theta.$$

(**2**). $\mathbf{\Phi_\Omega^e \ge \Phi_\Omega^p}$: Consider the bounded and uniformly continuous initial data in Ω:

$$0 \le v_0(x) := \begin{cases} \phi_R^\theta(x) & \forall x \in \Omega_R \\ 1 \quad \forall x \in \mathbb{R}^N \backslash B(0,R) \end{cases} \le 1 \quad x \in \Omega,$$

and their evolution in Ω, $v(t) := S^\theta(t)v_0(x)$, $t > 0$. As $v_0 \in BUC_\theta(\Omega)$, we have that v is continuous up to $t = 0$ and $0 \le v(t) \le 1$ in Ω because the semigroup is order preserving and of contractions in

$L^\infty(\Omega)$. Then, restricted to Ω_R, v satisfies

$$\begin{cases} v_t - \triangle v = 0 & \forall x \in \Omega_R, \quad \forall t > 0 \\ v(x,0) = \phi_R^\theta(x) & \forall x \in \Omega_R, \\ B_\theta(v)(x,t) = 0 & \forall x \in \partial\Omega, \quad \forall t > 0 \\ v(x,t) \le 1 & \forall\, |x| = R, \ \forall t > 0. \end{cases}$$

Therefore, if we define $\tilde{v}(t) := v(t)_{|\Omega_R} - \phi_R^\theta$, it satisfies in Ω_R:

$$\begin{cases} \tilde{v}_t - \triangle \tilde{v} = 0 & \forall x \in \Omega_R, \quad \forall t > 0 \\ \tilde{v}(x,0) = 0 & \forall x \in \Omega_R \\ B_\theta(\tilde{v})(x,t) = 0 & \forall x \in \partial\Omega, \quad \forall t > 0 \\ \tilde{v}(x,t) \le 0 & \forall\, |x| = R, \ \forall t > 0 \end{cases}$$

and using Theorem 2.8 in the domain Ω_R, we obtain that $\tilde{v} \le 0$, i.e. $v(t) \le \phi_R^\theta(x)$ in Ω_R.

Now, let us see that $v(t)$ converges to Φ_e^θ when $t \to \infty$. Consider $w_0 = 1_\Omega - v_0 \ge 0$. Then, w_0 has compact support and then $w_0 \in L^1(\Omega)$, so the estimate (2.11), gives $\lim_{t\to\infty} \left\| S^\theta(t) w_0 \right\|_{L^\infty(\Omega)} = 0$. Then in Ω_R we have

$$0 \le S^\theta(t) 1_\Omega(x) = S^\theta(t)(v_0 + w_0)(x) \le \phi_R^\theta(x) + S^\theta(t) w_0.$$

Therefore, taking the limit $t \to \infty$ we have $\Phi_p^\theta(x) = \lim_{t\to\infty} S^\theta(t) 1_\Omega(x) \le \phi_R^\theta(x)$ in Ω_R for any $R > 0$. Hence, taking $R \to \infty$, $\Phi_p^\theta(x) \le \Phi_e^\theta(x)$ as we wanted to prove. ■

4 The asymptotic mass of the solutions

Now we will address the main problem of this paper. Given an initial datum $u_0 \in L^1(\Omega)$, we want to know how much mass is lost through the hole. First, we define the asymptotic mass of a solution:

Definition 4.1. *For a given exterior domain $\Omega \subset \mathbb{R}^N$ and initial datum $u_0 \in L^1(\Omega)$ and some homogeneous θ-boundary conditions, we define the asymptotic mass of the solution with initial data u_0 as*

$$m_{u_0}^\theta := \lim_{t\to\infty} \int_\Omega S^\theta(t) u_0(x) dx.$$

Now, we prove a result that characterises the asymptotic mass of a solution in terms of the initial datum and the asymptotic profile:

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^N$ an exterior domain and $u_0 \in L^1(\Omega)$ an initial datum. Then, if we denote $u(t) = S^\theta(t) u_0$ the solution of the heat equation (2.1) in Ω with homogenous θ-boundary conditions and the initial datum u_0, we have that*

$$m_{u_0}^\theta = \int_\Omega u_0(x) \Phi^\theta(x) dx \tag{4.1}$$

where Φ^θ is the asymptotic profile in Ω. In particular, $\int_\Omega u(t)\Phi^\theta$ is constant in time.

Proof. With the notations above, we have,

$$m_{u_0}^\theta = \lim_{t\to\infty} \int_\Omega S^\theta(t)u_0(x)1_\Omega(x)dx \overset{(2.8)}{=} \lim_{t\to\infty} \int_\Omega u_0(x)S^\theta(t)1_\Omega(x)dx \overset{(3.1)}{=} \int_\Omega u_0(x)\Phi^\theta(x)dx,$$

where we have used the dominated convergence theorem in the last step as $\left|u_0 S^\theta(t)1_\Omega\right| \leq |u_0|$ which is integrable.

To see that $\displaystyle\int_\Omega u(t)\Phi^\theta$ is a conserved quantity, we just use that, for $t, s > 0$,

$$\int_\Omega u(t+s)\Phi^\theta = \int_\Omega S^\theta(s)u(t)\Phi^\theta \overset{(2.8)}{=} \int_\Omega u(t)S^\theta(s)\Phi^\theta = \int_\Omega u(t)\Phi^\theta$$

■

Remark 4.3. *The profile Φ^θ provides an explicit computation of the amount of mass lost for any solution. It also has an interesting interpretation in terms of initial point masses. Actually, since,*

$$S^\theta(t)1_\Omega(x) = \int_\Omega k^\theta(x, y, t)1_\Omega(y)dy,$$

using that k^θ is spatially symmetric, see Theorem 2.4,

$$S^\theta(t)1_\Omega(x) = \int_\Omega k^\theta(y, x, t)dy = \left\|S^\theta(t)\delta(\cdot - x)\right\|_{L^1(\Omega)},$$

where δ is the Dirac distribution. Therefore,

$$\Phi^\theta(x) = \lim_{t\to\infty} S^\theta(t)1_\Omega(x) = \lim_{t\to\infty}\left\|S^\theta(t)\delta(\cdot - x)\right\|_{L^1(\Omega)},$$

which is the remaining mass for a point source in x as initial condition. Therefore (4.1) reflects the contribution of a mass density $u_0(x)$ at each $x \in \Omega$ in the total remaining mass.

Now we give some estimates on the profile Φ^θ. In particular, we prove that for $N = 1, 2$, $\Phi^\theta \equiv 0$ while if $N \geq 3$ then $\Phi^\theta \not\equiv 0$ and it actually converge to 1 as $|x| \to \infty$. Therefore in two dimension all the mass of every solution is lost through the hole, while for higher dimensions there is always some remaining mass.

First, the following result allows us to compare the asymptotic profiles for different boundary conditions:

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain and Φ^{θ_1}, Φ^{θ_2} asymptotic profiles in Ω for different θ-boundary conditions. Then, if $0 \leq \theta_1 \leq \theta_2 \leq 1$ we have that*

$$\Phi^{\theta_1} \leq \Phi^{\theta_2} \quad \text{in } \Omega.$$

In particular, if $0 \leq u_0 \in L^1(\Omega)$ and $0 \leq \theta_1 \leq \theta_2 \leq 1$ the asymptotic masses satisfy

$$m_{u_0}^{\theta_1} \leq m_{u_0}^{\theta_2}.$$

Proof. This is just a consequence of Theorem 2.9:

$$\Phi^{\theta_1} \overset{(3.1)}{=} \lim_{t\to\infty} \int_\Omega S^{\theta_1}(t)1_\Omega \overset{\text{Thm } 2.9}{\leq} \lim_{t\to\infty} \int_\Omega S^{\theta_2}(t)1_\Omega \overset{(3.1)}{=} \Phi^{\theta_2}.$$

The rest is immediate. ■

In the same way, if instead of changing the boundary conditions, we change the domain, for homogeneous Dirichlet boundary conditions we have the following comparison result.

Proposition 4.5. *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ be two exterior domains and consider Dirichlet boundary conditions, that is $\theta = 0$, in both of them. Then we have that their asymptotic profiles satisfy:*

$$\Phi^0_{\Omega_1}(x) \le \Phi^0_{\Omega_2}(x) \qquad \forall x \in \Omega_1.$$

Proof. With the notation in (3.2), we have that $\phi^0_{(\Omega_2)_R} \ge 0$ in $\partial\Omega_1$, while $\phi^0_{(\Omega_1)_R} = 0$ on $\partial\Omega_1$. Therefore, if we denote $v := \phi^0_{(\Omega_1)_R} - \phi^0_{(\Omega_2)_R}$ we have that:

$$\begin{cases} -\triangle v(x) = 0 & \forall x \in (\Omega_1)_R \\ v(x) \le 0 & \forall x \in \partial(\Omega_1)_R. \end{cases}$$

Then, using Theorem 2.6, we obtain that $v \le 0$ in $(\Omega_1)_R$, or equivalently $\phi^0_{(\Omega_1)_R} \le \phi^0_{(\Omega_2)_R}$ in $(\Omega_1)_R$. Taking the limit when $R \to \infty$ we have $\Phi^0_{\Omega_1} \le \Phi^0_{\Omega_2}$ in Ω_1. ■

Now, we will explore briefly what the form of the asymptotic profile Φ^θ is. For this, we will firstly study the case when the domain is for the complement of a ball $D_r = \mathbb{R}^N \backslash B(0,r)$ and θ is a constant.

Lemma 4.6. *Let $D_r^R = B(0,R)\backslash B(0,r) \subset \mathbb{R}^N$ and $0 \le \theta < 1$ a constant. Then the solution to the problem*

$$\begin{cases} -\triangle \phi(x) = 0 & \forall x \in \Omega_R \\ B_\theta(\phi)(x) = 0 & \forall\, |x| = r \\ \phi(x) = 1 & \forall\, |x| = R \end{cases}$$

is given by

$$\Phi_{r,R}(x) = \begin{cases} \frac{|x| - C_\theta r}{R - C_\theta r} & N = 1 \\ \frac{\log(|x|) - C_\theta \log(r)}{\log(R) - C_\theta \log(r)} & N = 2 \\ \frac{|x|^{2-N} - C_\theta r^{2-N}}{R^{2-N} - C_\theta r^{2-N}} & N \ge 3, \end{cases}$$

where

$$C_\theta := \begin{cases} 1 + \tan(\pi\theta/2) & N \le 2 \\ 1 + (N-2)\tan(\pi\theta/2) & N \ge 3. \end{cases}$$

If $\theta = 1$, then $\Phi_{r,R} \equiv 1$.

Proof. A radial harmonic function satisfies

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial f}{\partial r}\right) = 0,$$

and then when $N \ge 3$, $f(r) = \frac{A}{r^{N-2}} + B$. Choosing the constants to fit the boundary conditions we get the result. The cases $N = 1, 2$ follow with slight changes. ■

Now, letting $R \to \infty$, we get the explicit form of the asymptotic profile.

Lemma 4.7. *Let $0 \le \theta \le 1$ be a constant. The asymptotic profile for the exterior domain $D_r := \mathbb{R}^N \backslash \overline{B(0,r)}$ with θ-boundary conditions has the following explicit form:*

$$\Phi^\theta(x) = \begin{cases} 0 & \theta \ne 1 \quad N \le 2 \\ 1 - \frac{r^{N-2}}{C_\theta |x|^{N-2}} & \theta \ne 1 \quad N \ge 3 \\ 1 & \theta = 1 \text{ in any dimension} \end{cases} \qquad x \in D_r := \mathbb{R}^N \backslash \overline{B(0,r)}$$

where C_θ is as in Lemma 4.6.

As we see, if $N \leq 2$, the asymptotic profile is the zero function for Dirichlet or Robin conditions. Furthermore, for $N \geq 3$ the asymptotic profile tends to 1 when $|x| \to \infty$. This also happens for an arbitrary exterior domain and general θ-boundary conditions, as we will show below.

First, for a general exterior domain, we analyse the particular case of homogeneous Dirichlet boundary conditions:

Proposition 4.8. *Let $\Omega = \mathbb{R}^N \backslash \mathscr{C}$ be an exterior domain with Dirichlet boundary conditions, that is, $\theta \equiv 0$. Then the asymptotic profile satisfies:*

(i) $\Phi^0 = 0$ *if* $N \leq 2$.

(ii) $\Phi^0 \geq 0$ *and* $\Phi^0(x) \to 1$ *when* $|x| \to \infty$ *if* $N \geq 3$. *In fact, there exists a constant* $C > 0$ *depending on* Ω *such that:*

$$1 - \frac{C}{|x|^{N-2}} \leq \Phi^0(x) \leq 1 \qquad \forall x \in \Omega.$$

Proof. Let us take $r > 0$ such that $B(0,r) \subset \mathscr{C}$, which implies $\Omega \subset D_r = \mathbb{R}^N \backslash B(0,r)$. Then, using Proposition 4.5, we obtain that $\Phi^0_\Omega \leq \Phi^0_{D_r}$ in Ω. This automatically implies that, for $N \leq 2$, $\Phi^0_\Omega \equiv 0$.

Now for $N \geq 3$ choose $r > 0$ such that $\mathscr{C} \subset B(0,r)$, which implies $D_r \subset \Omega$. Then Proposition 4.5 gives now $\Phi^0_{D_r} \leq \Phi^0_\Omega \leq 1$ in D_r and by Lemma 4.7 we get the estimate in D_r. Taking a larger constant, we get the estimate in Ω. ■

Now for other θ-boundary conditions we have the following.

Theorem 4.9. *Assume $0 \leq \theta \leq 1$, not necessarily constant. Then, if $\partial\Omega$ and θ are sufficiently regular, the asymptotic profile Φ^θ satisfies:*

(i) *If* $N \geq 3$,

$$1 - \frac{C}{|x|^{N-2}} \leq \Phi^\theta(x) \leq 1 \qquad \forall x \in \Omega.$$

(ii) *If* $N \leq 2$ *and* $\theta \not\equiv 1$ *then* $\Phi^\theta \equiv 0$.

If $\theta \equiv 1$, that is, for Neumann boundary conditions, then $\Phi^1 \equiv 1$ in any dimensions.

Proof. For (i) it is enough to use Proposition 4.8 jointly with Proposition 4.4.

For (ii) take $M = \max_{\partial\Omega} \Phi^\theta$ and $u := \Phi^\theta - M$. Then, for $R > 0$ large enough, u satisfies

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega_R \\ u \leq 0 & \text{in } \partial\Omega \\ u \leq 1 & \text{in } \partial B(0,R). \end{cases}$$

Therefore, u is a subsolution of (3.2) for $\theta = 0$, that is, for the Dirichlet problem. Therefore, using Theorem 2.6, we obtain that:

$$u(x) \leq \phi^0_R(x) \qquad \forall x \in \Omega_R$$

and since $R > 0$ is arbitrary, we get

$$u(x) \leq \Phi^0(x) \qquad x \in \Omega.$$

Since $N \leq 2$, from Proposition 4.8, $\Phi^0 \equiv 0$ and therefore $0 \leq \Phi^\theta \leq M$ in Ω.

But then, at a point $x_0 \in \partial\Omega$ of maximum, that is, such that $\Phi^\theta(x_0) = M$, by Hopf Lemma we have $\frac{\partial \Phi^\theta_e}{\partial n}(x_0) > 0$ or Φ^θ is constant. In the latter case, since $\theta \not\equiv 1$, we have $\Phi^\theta \equiv 0$. In the former case, if $x_0 \in \partial^D\Omega$ then $M = 0$ and $\Phi^\theta \equiv 0$. If $x_0 \in \partial^R\Omega$

$$0 = B_\theta(\Phi^\theta)(x_0) = \sin(\frac{\pi}{2}\theta(x_0))\frac{\partial \Phi^\theta}{\partial n}(x_0) + \cos(\frac{\pi}{2}\theta(x_0))M \geq \cos(\frac{\pi}{2}\theta(x_0))M$$

and then $M = 0$ and thus $\Phi^\theta \equiv 0$. ■

Then we have the following result about the rate of mass loss. Observe that, except for Neumann boundary conditions, for $N \geq 3$ all solutions lose mass at a uniform rate, while if $N \leq 2$ there are solutions for which the mass decays as slow as we want.

Theorem 4.10. *Assume Ω is an exterior domain and $0 \leq \theta \leq 1$, not necessarily constant, with $\theta \not\equiv 1$, that is, except for Neumann boundary conditions. For every $0 \leq u_0 \in L^1(\Omega)$ denote*

$$m^\theta_{u_0}(t) := \int_\Omega S^\theta(t)u_0(x)dx, \quad t > 0$$

and the asymptotic mass $m^\theta_{u_0} := \lim_{t\to\infty} m^\theta_{u_0}(t)$. Then

(i) *If $N \geq 3$, there exists a constant c_θ such that*

$$|m^\theta_{u_0}(t) - m^\theta_{u_0}| \leq c_\theta \frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N-2}{2}}}.$$

(ii) *If $N \leq 2$, let $g : [0,\infty) \to (0,1]$ a monotonically decreasing continuous function such that $\lim_{t\to\infty} g(t) = 0$.*
Then, there exist an initial value $0 \leq u_0 \in L^1(\Omega)$ with $\|u_0\|_{L^1(\Omega)} = 1$ and $T > 0$ such that

$$m^\theta_{u_0}(t) \geq g(t) \qquad \forall t \geq T. \tag{4.2}$$

Proof. (i) As we have $m^\theta_{u_0} = \int_\Omega u_0\Phi^\theta$ then

$$m^\theta_{u_0}(t) - m^\theta_{u_0} = \int_\Omega u_0\big(S^\theta(t)1_\Omega - \Phi^\theta\big) = \int_\Omega u_0 S^\theta(t)\big(1_\Omega - \Phi^\theta\big)$$

since $S^\theta(t)\Phi^\theta = \Phi^\theta$. From Theorem 4.9, since $N \geq 3$, $0 \leq 1_\Omega(x) - \Phi^\theta(x) \leq \frac{C}{|x|^{N-2}}$ in Ω and then $1_\Omega - \Phi^\theta \in L^p(\Omega)$ for any $p > \frac{N}{N-2}$. Then the Gaussian bounds (2.10) imply

$$S^\theta(t)\big(1_\Omega - \Phi^\theta\big)(x) \leq C\int_\Omega \frac{e^{-\frac{|x-y|^2}{4ct}}}{t^{N/2}} \frac{1}{|y|^{N-2}}\,dy = \left\{ \begin{array}{c} z = \frac{x-y}{\sqrt{t}} \\ y = x - \sqrt{t}z \end{array} \right\} \leq C\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4c}} \frac{1}{\left|x - \sqrt{t}z\right|^{N-2}}\,dz$$

$$= \frac{C}{t^{\frac{N-2}{2}}}\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4c}} \frac{1}{\left|\frac{x}{\sqrt{t}} - z\right|^{N-2}}\,dz = \frac{C}{t^{\frac{N-2}{2}}} F(\frac{x}{\sqrt{t}})$$

with

$$F(w) = \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}}\,dz = \int_{|w-z|\leq a} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}}\,dz + \int_{|w-z|\geq a} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}}\,dz$$

for any $a > 0$. Now

$$F_1(w) = \int_{|w-z|\geq a} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}}\,dz \leq \frac{1}{a^{N-2}} \int_{|w-z|\geq a} e^{-\frac{|z|^2}{4c}}\,dz \leq \frac{1}{a^{N-2}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4c}} = C$$

and

$$F_2(w) = \int_{|w-z|\leq a} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}}\,dz = \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4c}} \frac{1}{|w-z|^{N-2}} \chi_{B(w,a)}\,dz.$$

The function $g(z) = \frac{1}{|w-z|^{N-2}}\chi_{B(w,a)}$ is in $L^q(\mathbb{R}^N)$ for $q < \frac{N}{N-2}$ with a norm independent of w so Hölder's inequality implies

$$F_2(w) \leq C.$$

Hence $F \in L^\infty(\mathbb{R}^N)$ and therefore

$$\|S^\theta(t)\big(1_\Omega - \Phi^\theta\big)\|_{L^\infty(\Omega)} \leq \frac{c}{t^{\frac{N-2}{2}}}$$

and we get the result.
(ii) Observe that we have $\Phi^\theta = 0$ and then $m^\theta_{u_0} = 0$, for any initial data in $L^1(\Omega)$. Now consider $\lambda > 0$ the first Dirichlet eigenvalue for the Laplacian operator in the unit ball, B, and its associated positive eigenfunction $\psi \geq 0$ (normalized such that $\|\psi\|_{L^1(B)} = 1$).

Now we choose $t_n \to \infty$ such that $g(t_n) = \frac{1}{2^{n+2}}$. Then we consider the following initial datum made up by rescaled copies of ψ in disjoint balls with large radius and far away centres:

$$u_0(x) = \sum_{n=1}^{\infty} \frac{1}{2^n R_n^N} \psi\left(\frac{x - x_n}{R_n}\right) \chi_{B(x_n,R_n)}(x),$$

where $\chi_{B(x_n,R_n)}$ is the characteristic function of the $B(x_n,R_n)$ and R_n and x_n are chosen so that

(1) $e^{-\frac{\lambda}{R_n^2}t_n} \geq 1/2$ (This is possible taking R_n large enough)
(2) $B(x_n,R_n) \subset \Omega$ (This is possible taking $|x_n|$ large enough)

Therefore, $\|u_0\|_{L^1(\Omega)} = 1$ and

$$\int_\Omega S^\theta(t_n)u_0(x)dx \overset{\text{Thm 2.9}}{\geq} \int_\Omega S^0(t_n)u_0(x)dx \geq \frac{1}{2^n R_n^N} \int_\Omega S^0(t_n)\left(\psi\left(\frac{\cdot - x_n}{R_n}\right)\chi_{B(x_n,R_n)}\right)(x)dx. \tag{4.3}$$

Now observe that for $0 \leq \varphi$ in Ω, as $B(x_n,R_n) \subset \Omega$, we have $S^0(t)\varphi \geq 0$ in $\partial B(x_n,R_n)$ and then Theorem 2.8 implies $S^0(t)\varphi \geq S^0_{B(x_n,R_n)}\varphi$, that is, the heat semigroup in $B(x_n,R_n)$ with Dirichlet boundary conditions. Therefore,

$$\begin{aligned}\frac{1}{2^n R_n^N}\int_\Omega S^0(t_n)\left(\psi\left(\frac{\cdot - x_n}{R_n}\right)\chi_{B(x_n,R_n)}\right)(x)dx &\geq \frac{1}{2^n R_n^N}\int_{B(x_n,R_n)} S^0_{B(x_n,R_n)}(t_n)\psi\left(\frac{\cdot - x_n}{R_n}\right)(x)dx \\ &= \frac{e^{-\frac{\lambda}{R_n^2}t_n}}{2^n R_n^N}\int_{B(0,R_n)}\psi(x)dx \overset{(1)}{\geq} \frac{1}{2^{n+1}}.\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) we obtain

$$m^\theta_{u_0}(t_n) = \int_\Omega S^\theta(t_n)u_0(x)dx \geq \frac{1}{2^{n+1}}. \tag{4.5}$$

Now, take $T = t_1$ and $t \geq T$. Then, there exists $n \in \mathbb{N}$ such that $t \in [t_n, t_{n+1})$. As g and the mass of the solutions are monotonically decreasing we obtain

$$m_{u_0}^{\theta}(t) \geq m_{u_0}^{\theta}(t_{n+1}) \overset{(4.5)}{\geq} \frac{1}{2^{n+2}} = g(t_n) \geq g(t)$$

which is (4.2). ■

Appendix A Schauder estimates

Here we present some elliptic Schauder estimates, which allow us to estimate the derivatives of a solution of the Laplace equation just with the L^∞ norm of the solutions. These are classical results which can be found, for example, in [GT15] Theorem 4.6:

Theorem A.1. *Let $\Omega \subset \mathbb{R}^N$ and $u \in C^2(\Omega)$ such that*

$$\triangle u(x) = 0 \quad \forall x \in \Omega.$$

Then, for $x_0 \in \Omega$ and any two concentric balls $B_1 := B(x_0, R)$ and $B_2 := B(x_0, 2R) \subset\subset \Omega$, we have

$$R \, |Du|_{B_1} + R^2 \left| D^2 u \right|_{B_1} \leq C \|u\|_{L^\infty(B_2)},$$

where we denote $|Du|_{B_1} = \max_i \|D_i u\|_{L^\infty(B_1)}$, $\left|D^2 u\right|_{B_1} = \max_{i,j} \left\|D_{ij} u\right\|_{L^\infty(B_1)}$ and C is a constant independent on u, x_0 and R.

In the case of homogeneous boundary conditions we also have Schauder estimates. The following result can be found in [GT15] Theorem 6.30 for Robin and Neumann boundary conditions, and in [GT15] Theorem 6.6 for Dirichlet boundary conditions.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. Let $u \in C^{2+\alpha}(\Omega)$ be a function such that:*

$$\begin{cases} -\triangle u = f & \Omega \\ B_\theta(u) = \psi & \partial\Omega. \end{cases}$$

with $\theta \in C^{1+\alpha}(\partial\Omega)$. Then,

$$\|u\|_{C^{2+\alpha}(\overline{\Omega})} \leq C(\|u\|_{C^0(\overline{\Omega})} + \|f\|_{C^\alpha(\overline{\Omega})} + \|\psi\|_{C^{1+\alpha}(\partial\Omega)})$$

where $\|\psi\|_{C^{1+\alpha}(\partial\Omega)} = \inf\{\|\varphi\|_{C^{1+\alpha}(\overline{\Omega})} : \varphi \in C^{1+\alpha}(\overline{\Omega}),\ \varphi \equiv \psi$ on $\partial\Omega\}$.

References

[AB94] W. Arendt and A. Bukhvalov. Integral representations of resolvents and semigroups. *Forum Mathematicum*, 6:111–135, January 1994. Cited ↑ in page: 7

[ACDRB04] J. M. Arrieta, J. W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Linear parabolic equations in locally uniform spaces. *Math. Models Methods Appl. Sci.*, 14(2):253–293, 2004. Cited ↑ in page: 6

[CH90] T. Cazenave and A. Haraux. *Introduction aux problèmes d'évolution semi-linéaires.* Ellipses, 1990. Cited ↑ in page: 6

[CKL05] L. Capogna, C. E Kenig, and L. Lanzani. *Harmonic Measure: Geometric and Analytic Points of View*, volume 35. American Mathematical Soc., 2005. Cited ↑ in page: 12

[Dav89] E. B. Davies. *Heat Kernels and Spectral Theory*. Number 92. Cambridge University Press, 1989. Cited ↑ in page: 7

[Eva10] L. C. Evans. *Partial Differential Equations*, volume 19. American Mathematical Soc., 2010. Cited ↑ in page: 5, 6

[GS11] P. Gyrya and L. Saloff-Coste. *Neumann and Dirichlet Heat Kernels in Inner Uniform Domains*. Number 336 in Astérisque. Société mathématique de France, 2011. Cited ↑ in page: 10, 12

[GT15] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224. Springer, 2015. Cited ↑ in page: 6, 11, 19

[Gyr07] P. Gyrya. *Heat Kernel Estimates for Inner Uniform Subsets of Harnack-type Dirichlet Space*. PhD thesis, Cornell University, 2007. Cited ↑ in page: 10

[Mik78] V.P. Mikhailov. *Partial Differential Equations*. MIR Publishers, 1978. Cited ↑ in page: 5, 6

[Mor83] X. Mora. Semilinear Parabolic Problems Define Semiflows on Ck Spaces. *Transactions of the American Mathematical Society*, 278(1):21–55, 1983. Cited ↑ in page: 7

[Paz10] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Number 44 in Applied Mathematical Sciences. Springer, New York, NY, 3.[print] edition, 2010. Cited ↑ in page: 6, 12