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Power corrections and renormalons in Transverse Momentum Distributions

Ignazio Scimemi^a and Alexey Vladimirov^b

^a*Departamento de Física Teórica II, Universidad Complutense de Madrid,
Ciudad Universitaria, 28040 Madrid, Spain*

^b*Institut für Theoretische Physik, Universität Regensburg,
D-93040 Regensburg, Germany*

E-mail: ignazios@fis.ucm.es, aleksey.vladimirov@gmail.com

ABSTRACT: We study the power corrections to Transverse Momentum Distributions (TMDs) by analyzing renormalon divergences of the perturbative series. The renormalon divergences arise independently in two constituents of TMDs: the rapidity evolution kernel and the small- b matching coefficient. The renormalon contributions (and consequently power corrections and non-perturbative corrections to the related cross sections) have a non-trivial dependence on the Bjorken variable and the transverse distance. We discuss the consistency requirements for power corrections for TMDs and suggest inputs for the TMD phenomenology in accordance with this study. Both unpolarized quark TMD parton distribution function and fragmentation function are considered.

KEYWORDS: Effective field theories, Perturbative QCD, Renormalization Regularization and Renormalons, Resummation

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1 Introduction

The transverse momentum dependent (TMD) distributions are fundamental non-perturbative objects that appear in many relevant processes at LHC, EIC, and e^+e^- colliders, like Vector Boson Production, Higgs production, Semi-Inclusive Deep Inelastic Scattering, $e^+e^- \rightarrow 2$ hadrons. The factorization theorems which establish the definitions of TMD distributions in QCD and/or in effective field theory have been formulated recently in [1–4], using different regularization schemes.

The perturbative properties of unpolarized TMDs, such as evolution and operator product expansion (OPE) in the regime of small transverse momentum separation, have been deduced by several groups using different frameworks (see e.g. [1, 2, 4–9]). The explicit direct calculation of the TMD evolution function \mathcal{D} at NNLO has been provided in [10, 11] and recently it was obtained at N³LO [12]. Therefore, nowadays the perturbative knowledge of the unpolarized TMDs parton distribution functions (PDFs) and fragmentation functions (FFs) is comprehensive, thanks to the results obtained by various groups [11, 13–19].

On the contrary, the study of the non-perturbative properties of TMDs has been based mainly on phenomenological arguments which combine the perturbative information on TMDs with their perturbatively incalculable part [4, 8, 20–25]. These works have lead to different forms of implementation of TMDs which in general are not easy to compare. For instance, on one hand, the well-known phenomenological considerations of Drell-Yan

by [26] and [27] (the so-called BLNY model) implement an ansatz within the standard CSS approach with b^* -prescription in the impact parameter space (or b -space). They introduce a set of non-perturbative parameters $g_{1,2,3}$ and all these parameters (including the definition of b^* prescription) are fundamental for these fits. The same model is also the core of the RESBOS program package [28] which is widely used in applications. On another hand, the implementation of TMDPDFs by [29] does not use b^* -prescription. They have found that part of the non-perturbative corrections (essentially to the TMD evolution kernel) are negligible. They were able to describe the same data with a different shape of non-perturbative input parameterized by two parameters $\lambda_{1,2}$. Fits by other groups that limited themselves to the analysis of Vector Boson Production and Higgs production are less sensitive to the non-perturbative input (although it is still necessary) [30, 31]. Additional problems arise in the consideration of TMDFFs which are known to have very different and/or incomparable (in comparison to TMDPDFs) non-perturbative input.

This work is devoted to the study of the leading power corrections to TMD distributions. With this aim, we perform an analysis of the leading renormalon structure of TMD distributions. A renormalon analysis of the perturbative series gives an important check of theoretical consistency for any phenomenological ansatz, although it cannot give too stringent restrictions on the fitting parameters. The study of renormalon poles allows to understand the asymptotic behavior of the perturbative series and to deduce the form of the leading non-perturbative corrections [32–35].

An explicit analysis of the renormalon structure for TMDs has never been done to our best knowledge, although assumptions on its structure were used even before the actual field-theoretical definition of TMDs. We refer here, for instance, to the seminal work of [33] about the Sudakov factor in differential cross-section which is usually referred to justify a Gaussian behavior for the non-perturbative part of the TMD evolution kernel [25]. In order to describe this effect in the modern TMD framework, we recall that the definition of TMDs requires the combination of the Soft Function matrix element with the transverse momentum dependent collinear function. As we show in this work, the renormalon divergences arise in the perturbative consideration of both of these functions. These renormalon contributions have different physical meaning and should be treated independently. Firstly, the renormalon divergence of the soft factor results to a power correction within the TMD evolution kernel, which are strictly universal for any TMD due to the universality of the soft factor itself. The leading power correction that we derive here is quadratic. The presence of these corrections has been shown in [36] by the analysis of the corrections to conformal anomaly. Secondly, the renormalon divergences naturally arise within the coefficients of the small- b OPE. A study of those contributions gives access to the next twist corrections of small- b matching and specifies the shape and the general scaling of TMD.

The paper is built as the following. We provide the necessary concepts and definitions in section 2. In section 3 we perform the calculation of various TMD constituents (such as anomalous dimensions and coefficient functions) within the large- β_0 approximation. In the end of this section we provide a collection of the main lessons, that follows from our results. The impact of the renormalon divergences on the perturbative series and renormalon subtracted series are studied in section 4. One of the main outcomes of the study, namely a consistent ansatz for TMDs is presented in section (4.2).

2 Notation and basic concepts

Throughout the paper we follow the notation for TMDs and corresponding functions introduced in [14]. The quark TMDPDFs and TMDFFs are given by the following matrix elements

$$\begin{aligned}
 F_{q\leftarrow N}(x, \mathbf{b}; \zeta, \mu) &= \frac{Z_q(\zeta, \mu)R_q(\zeta, \mu)}{2} \\
 &\times \sum_X \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \langle N | \left\{ T \left[\bar{q}_i \tilde{W}_n^T \right]_a \left(\frac{\xi}{2} \right) | X \rangle \gamma_{ij}^+ \langle X | \bar{T} \left[\tilde{W}_n^{T\dagger} q_j \right]_a \left(-\frac{\xi}{2} \right) \right\} | N \rangle, \\
 \Delta_{q\rightarrow N}(z, \mathbf{b}) &= \frac{Z_q(\zeta, \mu)R_q(\zeta, \mu)}{4zN_c} \\
 &\times \sum_X \int \frac{d\xi^-}{2\pi} e^{-ip^+\xi^-/z} \langle 0 | T \left[\tilde{W}_n^{T\dagger} q_j \right]_a \left(\frac{\xi}{2} \right) | X, N \rangle \gamma_{ij}^+ \langle X, N | \bar{T} \left[\bar{q}_i \tilde{W}_n^T \right]_a \left(-\frac{\xi}{2} \right) | 0 \rangle,
 \end{aligned} \tag{2.1}$$

where R_q and Z_q are rapidity and ultraviolet renormalization constants, q are quark fields and W^T are Wilson lines, and $\xi = \{0^+, \xi^-, \mathbf{b}\}$. The TMDs depend on the Bjorken variables (x for TMDPDFs and z for TMDFFs), the impact parameter \mathbf{b} and the factorization scales ζ and μ . The considerations of the TMDPDF and TMDFF are similar in many aspects. Therefore, in order to keep the description transparent we mostly concentrate on the case of the TMDPDFs, while the results for TMDFFs are presented without derivation.

The dependence on the factorization scales is given by the evolution equations, which are the same for TMDPDF and TMDFF, namely

$$\frac{d}{d\ln\mu^2} F_{q\leftarrow N}(x, \mathbf{b}; \zeta, \mu) = \frac{\gamma^q(\mu, \zeta)}{2} F_{q\leftarrow N}(x, \mathbf{b}; \zeta, \mu), \tag{2.2}$$

$$\frac{d}{d\ln\zeta} F_{q\leftarrow N}(x, \mathbf{b}; \zeta, \mu) = -\mathcal{D}^q(\mu, \mathbf{b}) F_{q\leftarrow N}(x, \mathbf{b}; \zeta, \mu). \tag{2.3}$$

Through the article we consider only the quark TMDs, therefore in the following we suppress the subscript q on the anomalous dimensions. The values for both anomalous dimensions can be deduced from the renormalization constants [14]. Also γ and \mathcal{D} are related to each other by the cross-derivatives

$$\frac{d\mathcal{D}(\mu, \mathbf{b}_T)}{d\ln\mu^2} = -\frac{1}{2} \frac{d\gamma(\mu, \zeta)}{d\ln\zeta} = \frac{\Gamma_{cusp}}{2}, \tag{2.4}$$

where Γ_{cusp} is the honored cusp anomalous dimension.

The solution of the evolution equations eqs. (2.2), (2.3) is

$$F(x, \mathbf{b}; \zeta_f, \mu_f) = \mathcal{R}(\mathbf{b}; \zeta_f, \mu_f, \zeta_i, \mu_i) F(x, \mathbf{b}; \zeta_i, \mu_i), \tag{2.5}$$

where \mathcal{R} is the evolution kernel,

$$\mathcal{R}(\mathbf{b}; \zeta_f, \mu_f, \zeta_i, \mu_i) = \exp \left\{ \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma \left(\alpha_s(\mu), \ln \frac{\zeta_f}{\mu^2} \right) \right\} \left(\frac{\zeta_f}{\zeta_i} \right)^{-\mathcal{D}(\mu_i, \mathbf{b})}. \tag{2.6}$$

The final values of scaling parameters is dictated by the kinematic of the TMD cross-section. The variable $\zeta_f \sim Q^2$ (with Q being a typical hard scale) is the scale of the

rapidity factorization, and the variable μ_f is the scale of hard subprocess factorization. The intriguing point is that the evolution kernel \mathcal{R} is not entirely perturbative, but contains a non-perturbative part. An estimate of the non-perturbative contribution to \mathcal{R} is necessary in order to obtain the cross section in the momentum space where it is usually measured.

The non-perturbative part of the evolution kernel is encoded in the \mathcal{D} -function which can be obtained from the rapidity renormalization constant R_q . The definition of the rapidity renormalization constant differs from scheme to scheme. In this work we use the δ -regularization scheme defined in [10, 14]. In this scheme, the δ -regularization is used to regularize the rapidity divergences, and the dimensional regularization regularizes the rest of divergences. Such a configuration appears to be very effective for the TMD calculus. In particular, the rapidity renormalization factor R_q is expressed via the soft factor S as $R_q = S^{-1/2}$ [14]. In the coordinate space the soft factor is given by the following matrix element

$$\tilde{S}(\mathbf{b}_T) = \frac{\text{Tr}_c}{N_c} \langle 0 | T \left[S_n^{T\dagger} \tilde{S}_n^T \right] (0^+, 0^-, \mathbf{b}) \bar{T} \left[\tilde{S}_n^{T\dagger} S_n^T \right] (0) | 0 \rangle, \quad (2.7)$$

where we explicitly denote the ordering of operators and S^T are Wilson lines, as defined in [10]. Considering the relation between renormalization constants one can show [10], that

$$\mathcal{D} = \frac{1}{2} \frac{d \ln \tilde{S}}{d \mathbf{l}_\delta} \Big|_{\epsilon \rightarrow \text{finite}} \quad (2.8)$$

where $\mathbf{l}_\delta = \ln(\mu^2/|\boldsymbol{\delta}|)$. Eq. (2.8) can be used as the formal definition of the TMD evolution function \mathcal{D} . In this way, a non-perturbative calculation of the SF gives access to the non-perturbative structure of \mathcal{D} . The soft function is perturbatively universal for both Semi Inclusive Deep Inelastic Scattering and Drell-Yan type processes. Therefore, the perturbative part of the anomalous dimension \mathcal{D} is universal for TMDPDF and TMDFF. One can also expect its universality in the non-perturbative regime.

The TMDs are entirely non-perturbative functions. They cannot be evaluated in perturbative QCD, due to the non-perturbative origin of hadron states. The main subject of the paper is the dependence of TMDs on the parameter \mathbf{b} which is generically unrestricted since it is a variable of Fourier transformation. However it is interesting and numerically important to consider the range of small b (here and later $b = \sqrt{\mathbf{b}^2}$). In this range, the TMDs can be matched onto corresponding integrated parton distributions. At the operator level, the small- b matching is given by the leading term of the small- b OPE. The small- b OPE is a formal operator relation, that relates operators with both light-like and space-like field separation to operators with only light-like field separation. It reads

$$O(\mathbf{b}) = \sum_n C_n(\mathbf{b}, \mu_b) \otimes O_n(\mu_b), \quad (2.9)$$

where C_n are Wilson coefficient functions, the μ_b is the scale of small- b singularities factorization or the OPE matching scale (for simplicity we omit in eq. (2.9) other matching scales included in the definitions of each component of this equation). Generally, the operators O_n are all possible operators with proper quantum numbers and can be organized for

instance according to a power expansion, i.e. twists. In this case, the matching coefficients behave as

$$C_n(\mathbf{b}, \mu_b) \sim \left(\frac{b}{B}\right)^n f(\ln(\mathbf{b}^2 \mu_b^2)), \quad (2.10)$$

where f is some function. The value of the parameter B is unknown, and its origin is entirely non-perturbative. In other words, the unknown scale B represents some characteristic transverse size of interactions inside a hadron $B \simeq \mathcal{O}(1 \text{ GeV})$. In practice it is reasonable to consider only the leading term ($n = 0$) of eq. (2.9) for $b \ll B$. In this case, f is an integrated parton distribution (or fragmentation function), and coefficient function is called the matching coefficient. So far, the power suppressed terms in eq. (2.9) has been not considered, to our best knowledge.

For completeness, we recall here the renormalization group properties of the TMD Wilson coefficients that we use in the following sections. The evolution equations for the matching coefficients (at $\mu_b = \mu$) with respect to ζ is

$$\frac{d}{d \ln \zeta} C_{f \leftarrow f'}(x, \mathbf{b}_T; \mu, \zeta) = -\mathcal{D}^f(\mu, \mathbf{b}_T) C_{f \leftarrow f'}(x, \mathbf{b}_T; \mu, \zeta), \quad (2.11)$$

where $f = q, g$ species, $C_{f \leftarrow f'}$ are the matching coefficients on PDFs. It is practically convenient to extract the ζ -dependence from the matching coefficient. We introduce the notation

$$C_{f \leftarrow f'}(x, \mathbf{b}_T; \mu, \zeta) = \exp\left(-\mathcal{D}^f(\mu, \mathbf{b}_T) \mathbf{L}_{\sqrt{\zeta}}\right) \hat{C}_{f \leftarrow f'}(x, \mathbf{L}_\mu). \quad (2.12)$$

Here and further we use the following notation for logarithms

$$\mathbf{L}_X = \ln\left(\frac{X^2 \mathbf{b}^2}{4e^{-2\gamma_E}}\right), \quad \mathbf{l}_X = \ln\left(\frac{\mu^2}{X}\right). \quad (2.13)$$

The ζ -free coefficient function \hat{C} satisfies the following renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} \hat{C}_{f \leftarrow f'}(x, \mathbf{L}_\mu) = \sum_r \int_x^1 \frac{dy}{y} \hat{C}_{f \leftarrow r}\left(\frac{x}{y}, \mathbf{L}_\mu\right) K_{r \leftarrow f'}^f(y, \mathbf{L}_\mu), \quad (2.14)$$

where the kernel K is

$$K_{r \leftarrow f'}^f(x, \mathbf{L}_\mu) = \frac{\delta_{rf'} \delta(1-x)}{2} \left(\Gamma_{cusp}^f \mathbf{L}_\mu - \gamma_V^f \right) - P_{r \leftarrow f'}(x),$$

and $P(x)$ is the splitting function (DGLAP kernel). The matching coefficient for TMDFF $\mathbb{C}_{f \rightarrow f'}$ satisfies the same set of evolution equation with only substitution of PDF splitting function $P(x)$ by the FF ones, $\mathbb{P}(z)/z^2$ [14]. Using these equations one can find the expression for the logarithmic part of the matching coefficients at any given order, in terms of the anomalous dimensions and the finite part of the coefficient functions. The expressions for the anomalous dimensions, the recursive solution of the RGEs and the explicit expressions for the coefficients C and \mathbb{C} can be found, e.g. in [14].

3 TMD in large- β_0 approximation and renormalon divergences

The leading non-perturbative contribution to the perturbative series is commonly associated with renormalons. The renormalon contributions were intensively studied for various matrix elements and in different regimes, for review see [37, 38]. A typical signature of renormalons is the factorial divergence of the perturbative series. These divergences are often discussed in terms of the corresponding singularities in the Borel plane.

The best representative and the only stable way to study the renormalon divergence within perturbative QCD is the large- β_0 approximation. The large- β_0 expression can be obtained from the large- N_f expression through the procedure of “naive Abelianization” [39, 40]. In this section, we present the calculation of large- β_0 correction to TMDs. Since the technique of large- N_f calculus is well-known, we skip the detailed evaluation (redirecting the reader to the related literature) and present only intermediate expressions.

3.1 The soft function in the large- β_0 approximation

The soft function matrix elements is a key structure for the TMD construction and as such it is a good starting point for the renormalon analysis. The large- β_0 calculation of the soft factor runs in parallel to the calculation of the integrated soft factor for Drell-Yan, which is presented in [32] (see section 5.3). Here we present our results of the evaluation.

To begin with, we evaluate the large- N_f contribution to the soft factor, which is given by the “bubble” resummed diagram, shown in figure 1.A. The expression for the (renormalized) diagram with n -bubble insertion is

$$\text{SF}_n = -\frac{4C_F}{\beta_0^f} \left(\frac{a_s \beta_0^f}{-\epsilon} \right)^{n+1} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(-1)^k}{n-k+1} G(-\epsilon, -(n+1-k)\epsilon) (\mathbf{L}_\delta - \psi(-(n-k+1)\epsilon) - \gamma_E), \quad (3.1)$$

where $\beta_0^f = \frac{4}{3}T_r N_f$, $a_s = g^2/(4\pi)^2$, ϵ is the parameter of dimension regularization ($d = 4 - 2\epsilon$), $\delta = |2\delta^+ \delta^-|$ with $\delta^{+(-)}$ the being parameters of rapidity regularization for Wilson lines pointing in $n(\bar{n})$ -direction [14]. The function G is a standard function that appears in the large- β_0 calculation [32, 37, 39, 40], and is given by the expression

$$G(\epsilon, s) = e^{s\gamma_E} \mathbf{B}_\mu^{-s} A_\epsilon^{s/\epsilon-1} \frac{\Gamma(1+s)}{\Gamma(1-s+\epsilon)}, \quad (3.2)$$

with

$$A_\epsilon = \frac{6\Gamma(1+\epsilon)\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}, \quad \mathbf{B}_\mu = \frac{\mathbf{b}^2 \mu^2}{4e^{-2\gamma_E}}.$$

Here, the Euler-Mascheroni constant is a result of the $\overline{\text{MS}}$ scheme. For $n = 1, 2$ this expression agrees with the direct calculation of the soft factor in δ -regularization [10]. We also introduce an additional function for the double-pole part

$$\tilde{G}(\epsilon, s) = -G(\epsilon, s)(\psi(s) + \gamma_E). \quad (3.3)$$

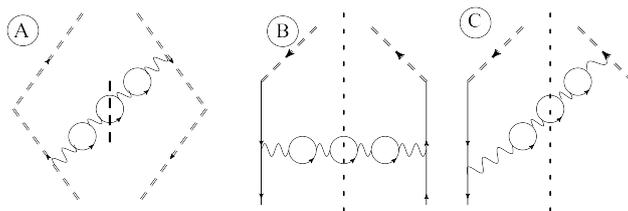


Figure 1. Diagrams contributing to the leading order of large- N_f limit. The diagram A is the contribution to the soft factor. Diagrams B and C are contribution to the matching coefficient. The counter term diagrams are not shown.

The functions G and \tilde{G} have the following Taylor series

$$G(\epsilon, s) = \sum_{j=0}^{\infty} G_j(\epsilon) s^j = \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} g_k^{[j]} \epsilon^k, \quad (3.4)$$

$$\tilde{G}(\epsilon, s) = \sum_{j=0}^{\infty} \tilde{G}_j(\epsilon) s^{j-1} = \sum_{j=0}^{\infty} s^{j-1} \sum_{k=0}^{\infty} \tilde{g}_k^{[j]} \epsilon^k. \quad (3.5)$$

These expressions define the coefficients $g_k^{[j]}$ and G_j . Note, that $g_k^{[0]} = \tilde{g}_k^{[0]}$ and $g_k^{[1]} = \tilde{g}_k^{[1]}$.

The procedure of “naive Abelianization” consists in the replacement of N_f by the corresponding β_0 expression [39], i.e.

$$\beta_0^f = \frac{4}{3} T_r N_f \quad \longrightarrow \quad -\beta_0 = -\frac{11}{3} C_A + \frac{4}{3} T_r N_f. \quad (3.6)$$

In this way, we obtain the large- β_0 expression for the soft factor

$$\begin{aligned} \text{SF} = & - \sum_{n=0}^{\infty} \frac{4C_F c_s^{n+1}}{\beta_0} \left[(-1)^n n! \left(\mathbf{L}_\delta G_{n+1}(-\epsilon) + \tilde{G}_{n+2}(-\epsilon) \right) \right. \\ & \left. + \frac{(-1)^n}{n+1} \left(-\mathbf{L}_\delta \frac{G_0(-\epsilon)}{\epsilon^{n+1}} + \frac{\tilde{G}_0(-\epsilon)}{\epsilon^{n+2}} (\psi(n+2) + \gamma_E) - \frac{\tilde{G}_1(-\epsilon)}{\epsilon^{n+1}} \right) \right], \end{aligned} \quad (3.7)$$

where we have introduced the large- β_0 coupling constant

$$c_s = \beta_0 a_s > 0.$$

Note, that in eq. (3.7) the terms suppressed in ϵ are dropped.

Eq. (3.7) gives access to the anomalous dimension \mathcal{D} , which we study in section 3.3, and to the rapidity renormalization factor R_q . The factor R_q (we recall that it is equal to $R_q = S^{-1/2}$ in the δ -regularization [14]) from the perspective of the large- β_0 approximation has the same perturbative combinatorics as the one-loop-truncated perturbation series. It is given by

$$R_q = 1 - \frac{\text{SF}}{2} \quad (3.8)$$

at $\delta^- = \zeta/p^+$ and SF given in eq. (3.7). This expression is used in the next section to extract the large- β_0 expression of the Wilson coefficients of small- b OPE.

3.2 The TMD in the large- β_0 approximation

To obtain the TMD matching coefficient one should evaluate the diagrams B and C, which are shown in figure 1. The result for the sum of these diagrams and their Hermitian conjugations is

$$\Phi_{q\leftarrow q} = \frac{2C_F}{\beta_0^f} \sum_{n=0}^{\infty} \frac{(a_s\beta_0^f)^{n+1}}{(-\epsilon)^{n+1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(-1)^k G(-\epsilon, -(n-k+1)\epsilon)}{n-k+1} \quad (3.9)$$

$$\left[\bar{x}x^{(n-k)\epsilon}(1-\epsilon)(1+(n-k)\epsilon) + 2\frac{x^{1+(n-k)\epsilon}}{(1-x)_+} - 2\delta(\bar{x})\ln\left(\frac{\delta^+}{p^+}\right) \right],$$

where we have used the same notation as in eq. (3.1) and $\bar{x} = 1 - x$. The last term in square brackets represents the rapidity divergence which appears in the diagram C. For $n = 0, 1$ this expression reproduces the result of explicit calculation made in [13].

Using eq. (3.8) and eq. (3.9) we can complete the result for the large- N_f expression of the TMDPDF,

$$R_q\Phi = \Phi_{q\leftarrow q} - \frac{\text{SF}}{2} = \frac{2C_F}{\beta_0^f} \sum_{n=0}^{\infty} \frac{(a_s\beta_0^f)^{n+1}}{(-\epsilon)^{n+1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{(-1)^k G(-\epsilon, -(n-k+1)\epsilon)}{n-k+1} \quad (3.10)$$

$$\left[\bar{x}x^{(n-k)\epsilon}(1-\epsilon)(1+(n-k)\epsilon) + 2\frac{x^{1+(n-k)\epsilon}}{(1-x)_+} + \delta(\bar{x})(\mathbf{L}_\mu - \mathbf{1}_\zeta - \psi(-(n-k+1)\epsilon) - \gamma_E) \right].$$

Here, we observe the cancellation of the rapidity divergences that leaves the residual $\mathbf{1}_\zeta$ dependence.

In order to extract the matching coefficient of the TMDPDF onto the PDF one has to proceed to the renormalization of eq. (3.10). This is greatly simplified in the δ -regularization scheme, where all virtual graphs and integrated graphs are zero. The only non-zero contribution is the UV counterterm which is a pure ϵ -singularity. The accounting of this part eliminates terms singular in ϵ , leaving the finite part unchanged. The latter provides the coefficient function. Performing the “naive Abelianization” as in eq. (3.6) we obtain the large- β_0 result

$$C_{q\leftarrow q} = \frac{2C_F}{\beta_0} \sum_{n=0}^{\infty} c_s^{n+1} \left\{ \left[\bar{x} + 2\frac{x}{(1-x)_+} \right] \left[\frac{\gamma_{n+1}(x)}{n+1} + (-1)^n n! g_0^{[n+1]}[\mathbf{B}_{\sqrt{\bar{x}\mu}}] \right] \quad (3.11)$$

$$+ \frac{\bar{x}}{n+1} (2\gamma_n(x) + \gamma_{n-1}(x)) - \bar{x}(-1)^n n! g_0^{[n]}[\mathbf{B}_{\sqrt{\bar{x}\mu}}]$$

$$+ \delta(\bar{x})(\mathbf{L}_\mu - \mathbf{1}_\zeta) \left[\frac{g_{n+1}^{[0]}}{n+1} + (-1)^n n! g_0^{[n+1]} \right]$$

$$+ \delta(\bar{x}) \left[\tilde{g}_{n+2}^{[0]} \frac{\psi(n+2) + \gamma_E}{n+1} + \frac{\tilde{g}_{n+1}^{[1]}}{(n+1)} + (-1)^n n! \tilde{g}_0^{[n+2]} \right] \right\},$$

where $\mathbf{B}_{\sqrt{\bar{x}\mu}} = x\mathbf{B}_\mu$, and

$$x^\epsilon G_0(\epsilon) = \sum_{k=0}^{\infty} \gamma_k \epsilon^k.$$

The additional variable in the square brackets for the functions g indicates the modified value of \mathbf{B}_μ to be substituted.

The calculation of TMDFFs matching coefficient proceeds in the same way as for TMDPDFs. The result of the calculation is

$$\begin{aligned}
 z^2 \mathbb{C}_{q \leftarrow q} = & \frac{2C_F}{\beta_0} \sum_{n=0}^{\infty} c_s^{n+1} \left\{ \left[\bar{z} + 2 \frac{z}{(1-z)_+} \right] \left[\frac{\gamma_{n+1}(z^{-1})}{n+1} + (-1)^n n! g_0^{[n+1]}[\mathbf{B}_{\mu/\sqrt{z}}] \right] \right. \\
 & + \frac{\bar{z}}{n+1} (2\gamma_n(z^{-1}) + \gamma_{n-1}(z^{-1})) - \bar{z} (-1)^n n! g_0^{[n]}[\mathbf{B}_{\mu/\sqrt{z}}] \\
 & + \delta(\bar{z}) (\mathbf{L}_\mu - \mathbf{1}_\zeta) \left[\frac{g_{n+1}^{[0]}}{n+1} + (-1)^n n! g_0^{[n+1]} \right] \\
 & + \delta(\bar{z}) \left[\tilde{g}_{n+2}^{[0]} \frac{\psi(n+2) + \gamma_E}{n+1} + \frac{\tilde{g}_{n+1}^{[1]}}{(n+1)} + (-1)^n n! \tilde{g}_0^{[n+2]} \right] \\
 & \left. - \sum_{r=1}^{n+1} \left(\left(\bar{z} + \frac{2z}{1-z} \right) \gamma_{n-r+1}(z) + \bar{z} (2\gamma_{n-r}(z) + \gamma_{n-r-1}(z)) \frac{(-1)^r \ln^r(z^2)}{(n+1)r!} \right) \right\}.
 \end{aligned} \tag{3.12}$$

One can see that the expression for TMDPDF eq. (3.11) is related to the first four lines of the expression for TMDFF eq. (3.12) by the crossing relation $x \rightarrow z^{-1}$. The last line of eq. (3.12) is specific for TMDFF and it is an effect of the expansion of the normalization factor $z^{-2\epsilon}$.

One can check that at $n = 0, 1$ the expressions (3.11) and (3.12) coincide with the one calculated in [14].

3.3 The TMD anomalous dimensions at large- β_0 and renormalon singularities of \mathcal{D}

In the articles [10, 14] it was shown that in the δ -regularization scheme the anomalous dimension \mathcal{D} can be obtained from the rapidity singular part of the soft factor as in (2.8). Considering the eq. (3.7) we obtain the anomalous dimension \mathcal{D} in the large- β_0 approximation

$$\mathcal{D} = -\frac{2C_F}{\beta_0} \sum_{n=0}^{\infty} c_s^{n+1} \left((-1)^n n! g_0^{[n+1]} + \frac{g_{n+1}^{[0]}}{n+1} \right). \tag{3.13}$$

The first term in the brackets of eq. (3.13) behaves $\sim n!$ at large n , and represents the renormalon singularity.

At this point it is convenient to consider the Borel transformation of the result. We define the Borel transformation of a perturbative series in the usual way

$$f(c_s) = \sum_{n=0}^{\infty} f_n c_s^{n+1} \quad \Longrightarrow \quad B[f](u) = \sum_{n=0}^{\infty} f_n \frac{u^n}{n!}. \tag{3.14}$$

A perturbative series is Borel summable if an integral

$$\tilde{f} = \int_0^\infty du e^{-u/c_s} B[f](u), \tag{3.15}$$

exists. Performing the Borel transformation on the \mathcal{D} function and applying eq. (3.15), we find

$$\mathcal{D} = -\frac{2C_F}{\beta_0} \left(\int_0^{c_s} dx \frac{G(x, 0) - 1}{x} - \int_0^\infty du \frac{G(0, -u) - 1}{u} e^{-u/c_s} \right). \quad (3.16)$$

The first term is analytical and reproduces the cusp-anomalous dimension at large- β_0 [32]

$$\Gamma_{cusp}(c_s) = \frac{4C_F c_s}{\beta_0} \frac{\Gamma(4 + 2c_s)}{6\Gamma^2(2 + c_s)\Gamma(1 + c_s)\Gamma(1 - c_s)} = \frac{4C_F c_s}{\beta_0} G(c_s, 0). \quad (3.17)$$

The function which appears in the second term

$$G(0, -u) = \mathbf{B}_\mu^u e^{(\frac{5}{3} - 2\gamma_E)u} \frac{\Gamma(1 - u)}{\Gamma(1 + u)}, \quad (3.18)$$

contains a series of poles at $u = 1, 2, \dots$ which correspond to infrared renormalons. One can check explicitly that the relation eq. (2.4) holds for large- β_0 expression, due to cancellation of the renormalon divergences in the second term of eq. (3.16) between derivative of coupling constant (in the Borel exponent) and derivative of \mathbf{B}_μ (in the function $G(0, -u)$).

There are multiple possibilities to define the sum eq. (3.13), e.g. one can slightly shift the integration contour for eq. (3.16) into the complex plane. The difference between integrals passing from the lower and upper sides of poles is called infrared (IR)-ambiguity and is given by a $(-\pi)$ times the residue at the pole. For the anomalous dimension \mathcal{D} it reads

$$\delta_{IR}\{\mathcal{D}\} = c\mathbf{b}^2 \Lambda^2, \quad (3.19)$$

where

$$c = \frac{\pi C_F}{2\beta_0} e^{\frac{5}{3}} \simeq 1.2. \quad (3.20)$$

The IR-ambiguity represents the typical scale of the error for perturbative series.

The same conclusion, namely the presence of a \mathbf{b}^2 -correction for \mathcal{D} , was made in ref. [36] using different argumentation. In ref. [36] the factorized cross-section has been considered within the soft collinear effective field theory (SCET). It has been shown that the power correction to the soft factor which arises in the next-to-leading term of large- Q^2 OPE, is proportional to the soft factor matrix element. Exponentiating the power correction one obtains the same result as presented here. It is an expected agreement because the renormalon calculation is equivalent to the calculation of the correction term of OPE.

The anomalous dimension γ_V can be extracted from the coefficient function eq. (3.11). We consider the derivative of coefficient function at $\mathbf{l}_\zeta = \mathbf{L}_\mu$

$$\mu^2 \frac{d}{d\mu^2} \hat{C}_{q\leftarrow q}(x, \mathbf{L}_\mu) = \int_x^1 \frac{dy}{y} \hat{C}_{q\rightarrow q}\left(\frac{x}{y}, \mathbf{L}_\mu\right) \left(\frac{1}{2} (\Gamma_{cusp} \mathbf{L}_\mu - \gamma_V) \delta(\bar{y}) - P_{q\leftarrow q}(y) \right), \quad (3.21)$$

where we have dropped the mixing among flavors. The DGLAP kernel at large- β_0 is given by the expression

$$P_{q\leftarrow q}(x) = \frac{2C_F}{\beta_0} \sum_{n=0}^{\infty} c_s^{n+1} \left\{ \frac{1+x^2}{1-x} \gamma_n(x) + \bar{x}(2\gamma_{n-1}(x) + \gamma_{n-2}(x)) \right\}_+. \quad (3.22)$$

Considering the derivative of eq. (3.11) and comparing right and left hand sides of eq. (3.21) we obtain

$$\begin{aligned} \gamma_V = & -\frac{4C_F}{\beta_0} \left\{ c_s \left(\psi(1+c_s) + 2\gamma_E + \frac{3-c_s^2}{(1+c_s)(2+c_s)} \right) G(c_s, 0) \right. \\ & \left. + \ln(G(c_s, 0)\Gamma(1+c_s)) G(c_s, 0) + \int_0^1 \frac{G(xc_s, 0) - G(c_s, 0)}{1-x} dx \right\}. \end{aligned} \quad (3.23)$$

This expression contains no singularity, and hence it is renormalon-free, as it is usually expected for an ultraviolet anomalous dimension.

3.4 TMD matching coefficient at large- β_0

Before the evaluation of the sums in eqs. (3.11)–(3.12) we extract the part related to the anomalous dimension \mathcal{D} to obtain the coefficients \hat{C} defined in eq. (2.12). This procedure is important since the function \mathcal{D} contains its own renormalon singularities, as described in eq. (3.19). The contribution of \mathcal{D} is easily recognized in the third lines of (3.11)–(3.12) (compare with eq. (3.13)).

The result of the Borel transform for the coefficient \hat{C} , eq. (3.11) is

$$\begin{aligned} \hat{C} = & \frac{2C_F}{\beta_0} \int_0^\infty du e^{-u/c_s} \left\{ \left(\bar{x} + \frac{2x}{(1-x)_+} \right) \frac{\gamma(u) - 1}{u} + \bar{x} \int_0^1 dy (2+uy) \gamma(yu) \right. \\ & + \delta(\bar{x}) \left(\frac{\mathbf{G}_1(u) - 1}{u} + \int_0^1 dy \frac{\mathbf{G}'_0(u) - \mathbf{G}'_0(yu)}{u(1-y)} \right) \\ & + \left(\bar{x} + \frac{2x}{(1-x)_+} \right) \left(\frac{1 - G[\mathbf{B}_{\sqrt{x}\mu}](0, -u)}{u} \right) - \bar{x} G[\mathbf{B}_{\sqrt{x}\mu}](0, -u) \\ & \left. + \delta(\bar{x}) \left(\frac{G(0, -u)(\psi(-u) + \gamma_E)}{u} - \frac{1}{u^2} - \frac{\mathbf{L}_\mu + \frac{5}{3}}{u} \right) \right\}, \end{aligned} \quad (3.24)$$

where by bold font we denote the Borel transformed functions,

$$\mathbf{G}_i(u) = \sum_{n=0}^{\infty} g_n^{[i]} \frac{u^n}{n!}, \quad \mathbf{G}'_0(x) = \frac{d}{dx} \mathbf{G}_0(x), \quad \gamma_0(u) = \sum_{n=0}^{\infty} \gamma_n(x) \frac{u^n}{n!}. \quad (3.25)$$

The terms in eq. (3.24) are collected such that every bracket is finite at $u \rightarrow 0$. The expression for TMDFF coefficient function \hat{C} can be obtained using the crossing transformation ($x \rightarrow z^{-1}$) and the addition of the normalization contribution (the last line in eq. (3.12)).

In the last two lines of eq. (3.24) we have the infrared renormalon poles in $u = 1, 2, \dots$. One can see that the third line contains only first order poles, while the last line contains second order poles at $G(0, -u)\psi(-u)$. Considering the infrared ambiguity at $u = 1$ we obtain

$$\delta_{IR}\{\hat{C}\} = -c(x\mathbf{b}^2\Lambda^2) \left\{ 2\bar{x} + \frac{2x}{(1-x)_+} - \delta(\bar{x}) \left(\mathbf{L}_\Lambda + \frac{2}{3} \right) \right\}, \quad (3.26)$$

where constant c is given in eq. (3.20). The x -dependence of this expression *exactly* reproduces the x -dependence of the leading terms of the next power correction in small- b OPE, see detailed description in [41]. The consideration of ambiguities of higher renormalon poles gives access to the higher-power corrections. We obtain

$$\delta_{\text{IR}}^{u=n}\{\hat{C}\} = \frac{\pi C_F}{\beta_0} \frac{\left(-x\mathbf{b}^2\Lambda^2 e^{\frac{5}{3}}\right)^n}{n!n!} \left(\frac{2x}{(1-x)_+} + (n+1)\bar{x} - \delta(\bar{x}) \left(\mathbf{L}_\Lambda - \psi_{n+1} - \gamma_E + \frac{5}{3} \right) \right). \quad (3.27)$$

However, these expressions can be modified by the infrared renormalon contributions of the higher-twist terms. The most important information of the higher-power corrections is that the renormalons scale as $x\mathbf{b}^2$, but not as \mathbf{b}^2 which is a naive assumption. The consequences of this fact are discussed in the next sessions.

The corresponding calculation for TMDFF gives

$$\delta_{\text{IR}}\{z^2\hat{C}\} = -c \left(\frac{\mathbf{b}^2\Lambda^2}{z} \right) \left\{ 2\bar{z} + \frac{2z}{(1-z)_+} - \delta(\bar{z}) \left(\mathbf{L}_\Lambda + \frac{2}{3} \right) \right\}. \quad (3.28)$$

which is the same as eq. (3.26) with the crossing change $x \rightarrow 1/z$. One can see that the difference in normalization which spoils the crossing between TMDPDFs and TMDFFs, disappears in the renormalon contribution. The higher poles ambiguities are provided using the crossing relation $x \rightarrow 1/z$ in eq. (3.27).

3.5 Lessons from large- β_0

The eqs. (3.19), (3.26), (3.28) are one of the main results of this work. These expressions represent the leading power correction to the small- b regime, where all perturbative properties of TMDs are derived. These expressions give access to a general structure of the next-to-small- b regime. The practical implementation of results eqs. (3.19), (3.26), (3.28) is given in the next section, while here we collect the most important observation that follows from the large- β_0 calculation and which should be taken into account for TMD phenomenology.

The first, and the most obvious, observation is that the leading power corrections are $\sim \mathbf{b}^2$. It implies that an exponential decay of the TMDs that is sometimes suggested in phenomenological studies (e.g. [42, 43]) can in no way affect the small- b region. Indeed, it would imply the corrections $\sim \sqrt{\mathbf{b}^2}$ to the small- b OPE, that cannot appear without extra scaling parameter. Nonetheless, exponential corrections can occur in the large- b regime, which is inaccessible by perturbative considerations.

Second, one can see that the renormalon corrections to TMDPDFs matching coefficient scales like $x\mathbf{b}^2$, and not as simply \mathbf{b}^2 (as it is usually assumed), nor as $x^2\mathbf{b}^2$ (as suggested by Laguerre polynomial decomposition [9]). Therefore, the contributions of higher-twist terms in small- b OPE for TMDPDF are largely functions of $x\mathbf{b}^2$. Correspondingly, TMDFFs matching coefficients are a function of \mathbf{b}^2/z . This is important in respect of the phenomenological implementation of the TMDs. For instance, the b^* -prescription

which is often adopted does not respect this scaling and so, in this sense, it is not fully consistent with the estimated higher twist effects.

Third, the renormalon contributions to the anomalous dimension \mathcal{D} and to matching coefficients have different physical origins and do not mix with each other. In fact, the anomalous dimension \mathcal{D} is an universal object that is the same for all regimes of b and for TMDs of different quantum numbers [25]. Thus, the renormalon contribution to \mathcal{D} represents a generic universal non-perturbative contribution, alike in the case of heavy quark masses. On the other hand, the (infrared) renormalon divergences within the matching coefficients are to be canceled by the corresponding (ultraviolet) renormalon contributions of higher twists. Therefore, while eq. (3.19) represents a size of a universal non-perturbative contribution, eqs. (3.26), (3.28) give the form of the twist-four contribution to small- b OPE. In other words, eqs. (3.26), (3.28) estimate very accurately the x -behavior of subleading correction to small- b OPE.

The consideration of the anomalous dimension \mathcal{D} for gluon distributions is identical to those of quarks (apart of trivial replacing of the common factor C_F by C_A). Contrary, the calculation of the renormalon contribution for gluon and quark-gluon matching coefficient is much more complicated than the one presented here and is beyond the scope of this paper. In general, we can expect a non-trivial dependence of the renormalon contribution on the Bjorken variables. At present, we cannot find arguments which suggest a location for the renormalon poles and an $x\mathbf{b}^2$ scaling different from that of quarks.

4 Renormalon subtraction and power corrections

Our analysis is limited to the quark TMDs only. Nonetheless, we can advance some considerations on possible inputs, which are consistent with our findings and evaluate their impact on the non-perturbative structure of TMDs. The suggested ansatz for TMDs does not pretend to be unique and moreover is inspired by other popular models. We postpone to a future publication a more dedicated study on the subject.

We recall here the form of the TMDPDFs which emerges at small- b is

$$F_{q\leftarrow N}^{pert}(x, \mathbf{b}; \zeta_f, \mu_f) = \mathcal{R}(\mathbf{b}, \zeta_f, \mu_f; \zeta_b, \mu) \sum_j \int_x^1 \frac{dy}{y} \hat{C}_{q\leftarrow j} \left(\frac{x}{y}, \mathbf{b}; \mu \right) f_{j\leftarrow N}(y, \mu), \quad (4.1)$$

where the evolution kernel \mathcal{R} is given in eq. (2.6). The argument ζ_b of \mathcal{R} is collected from the combination of two exponents: the original factor \mathcal{R} (2.6) and the exponential prefactor of \hat{C} (2.12), and it takes the value

$$\zeta_b = \frac{4e^{-2\gamma_E}}{\mathbf{b}^2}.$$

The analogue equation for TMDFFs is obtained replacing consistently the PDF $f_{j\leftarrow N}$ by the fragmentation function $d_{j\rightarrow N}$ and the coefficient function $C_{q\leftarrow j}$ by $\mathbb{C}_{q\rightarrow j}$, while the evolution kernel remains the same. This expression is usually taken as an initial ansatz for TMD phenomenology.

As we pointed earlier there are two places where the non-perturbative effects arise. The first one is the evolution kernel \mathcal{D} which is a part of the evolution prefactor \mathcal{R} , and it is common for all TMDs (TMDPDFs and TMDFFs of various polarizations). The second one is the higher twist corrections to the small- b OPE. These non-perturbative contributions are of essentially different origin and should not be mixed. In particular it is important to realize that the non-perturbative contribution of \mathcal{D} enters eq. (4.1) as a prefactor, while the higher order terms of OPE are added to the convolution integral. Therefore, the structure of non-perturbative corrections to TMD that we keep in mind is the following

$$F_{q\leftarrow N}(x, \mathbf{b}; \zeta_f, \mu_f) = \exp \left\{ \int_{\mu}^{\mu_f} \frac{d\mu'}{\mu'} \gamma(\mu', \zeta_f) \right\} \left(\frac{\zeta_f}{\zeta_b} \right)^{-\mathcal{D}(\mu, \mathbf{b}) - \mathcal{D}^{NP}(\mathbf{b})} \times \quad (4.2)$$

$$\left(\sum_j \int_x^1 \frac{dy}{y} \hat{C}_{q\leftarrow j} \left(\frac{x}{y}, \mathbf{b}; \mu \right) f_{j\leftarrow N}(y, \mu) + f_{q\leftarrow N}^{NP}(x, \mathbf{b}; \mu) \right).$$

Here, \mathcal{D}^{NP} is the non-perturbative addition to the anomalous dimension \mathcal{D} , and f^{NP} is the cumulative effect of the higher twist corrections to the small- b OPE. At small (perturbative) \mathbf{b} , the non-perturbative parts should turn to zero, such that eq. (4.2) reproduces eq. (4.1). In the following subsections we construct a minimal non-contradicting ansatz for TMD distributions that respect the study of large- β_0 approximation.

4.1 Non-perturbative corrections to the anomalous dimension \mathcal{D}

The non-perturbative part of the anomalous dimension \mathcal{D} is one of the most studied in the literature and the one for which a general consensus is achieved. Usually, the anomalous dimension \mathcal{D} is assumed to have quadratic behavior in the non-perturbative region. As we show in eq. (3.19) the quadratic behavior is also suggested by the large- β_0 approximation. A more subtle issue concerns the amount of non-perturbative correction to \mathcal{D} , which can be very different depending on the implementation of the TMDs. A check of the renormalon contribution, as provided in this section, gives an estimate of such correction and it is so useful for practical implementations.

Let us present the perturbative series for \mathcal{D} in the form

$$\mathcal{D}(\mu, \mathbf{b}) = \frac{C_F}{\beta_0} \sum_{n=1}^{\infty} (\beta_0 a_s(\mu))^n (d_n(\mathbf{L}_\mu) + \delta_n(\mathbf{L}_\mu)), \quad (4.3)$$

where $d_n \sim n!g_0^{[n+1]}$ can be obtained from eq. (3.13) and δ_n is the large- β_0 suppressed part. The numerical comparison of the large- β_0 expression eq. (3.16) and the exact expression for \mathcal{D} is given in the table 1. One can see that generally *the large- β_0 expression overestimates the exact numbers*, which is typical for this approximation.

In order to study the properties of the large- β_0 series we introduce a function for its partial sum

$$M_N(\mu, \mathbf{b}) = \frac{1}{\beta_0} \sum_{n=1}^N (\beta_0 a_s(\mu))^n d_n(\mathbf{L}_\mu). \quad (4.4)$$

n	$d_n + \delta_n$	d_n	δ_n
1	$2\mathbf{L}_\mu$	$2\mathbf{L}_\mu$	0
2	$\mathbf{L}_\mu^2 + 2.03\mathbf{L}_\mu^2 - 1.31$	$\mathbf{L}_\mu^2 + 3.33\mathbf{L}_\mu + 3.11$	$-1.30\mathbf{L}_\mu - 4.42$
3	$0.67\mathbf{L}_\mu^3 + 2.82\mathbf{L}_\mu^2 + 0.24\mathbf{L}_\mu - 2.41$	$0.67\mathbf{L}_\mu^3 + 3.33\mathbf{L}_\mu^2 + 5.56\mathbf{L}_\mu + 7.67$	$-0.51\mathbf{L}_\mu^2 - 5.32\mathbf{L}_\mu - 10.$

Table 1. Numerical comparison of the large- β_0 component of the anomalous dimension \mathcal{D} to the exact expression. The coefficients d_n and δ_n are defined in eq. (4.3).

$\mu=10$ GeV	$b = 0.2$	$b = 1.5$	$b = 3.0$
M_1	0.032	0.145	0.184
M_2	0.047	0.228	0.304
M_3	0.051	0.277	0.388
M_4	0.053	0.310	0.455
M_5	0.054	0.223	0.513
M_6	0.054	0.354	0.567
M_7	0.055	0.372	0.622
$M_\infty \pm \delta M$	0.055 ± 0.001	0.376 ± 0.072	0.577 ± 0.267

Table 2. The values of partial sums M_N at several values of b . The estimate converge value M_∞ and its error band δM are obtained as described in the text.

For $N \rightarrow \infty$ the sum is divergent, as discussed in section 3.3. In order to define M_∞ we consider the Borel transform of M_N as in section 3.3. To define the Borel integral in eq. (3.16), we shift the integration contour, slightly above the real axis. The real part of the integral (i.e. the principal value integral) gives M_∞ , while the imaginary part represents the errorband for this estimation. The explicit expression for the latter is

$$\delta M(\mu, \mathbf{b}) = \frac{2\pi}{\beta_0} \left[J_0 \left(\sqrt{\mu^2 \mathbf{b}^2} e^{\frac{5}{6} - \frac{1}{2\beta_0 a_s(\mu)}} \right) - 1 \right], \tag{4.5}$$

and the leading behavior at small- b for δM is given by the infrared ambiguity eq. (3.19).

We investigate the convergence of the partial sums of M_N to its Borel resummed value M_∞ , in order to find the scale at which the non-perturbative corrections associated with renormalons become important. The numerical values of partial sums at $\mu = 10$ GeV and at several values of \mathbf{b} are presented in table 2. The graphical representation of these values is shown in figure 2. The convergence of the series is perfect (in the sense that it converges at M_7 that is far beyond the scope of modern perturbative calculations) for the range of $b \lesssim 2 \text{ GeV}^{-1}$, it becomes weaker at $b \sim 3 \text{ GeV}^{-1}$, and it is completely lost at $b \gtrsim 4 \text{ GeV}^{-1}$. These are the characteristic scales for switching the perturbative and non-perturbative regimes in \mathcal{D} . In other words, the perturbative series can be trustful at $b \lesssim 2 \text{ GeV}^{-1}$, but completely loses its prediction power for $b \gtrsim 4 \text{ GeV}^{-1}$. The number N at which convergence is lost depends on the value of μ , however the interval of convergence in b is μ -independent, e.g. at $\mu = 50 \text{ GeV}$ the series converges to M_8 in the region $b \lesssim 2 \text{ GeV}^{-1}$, but again loses stability at $\sim 4 \text{ GeV}^{-1}$.

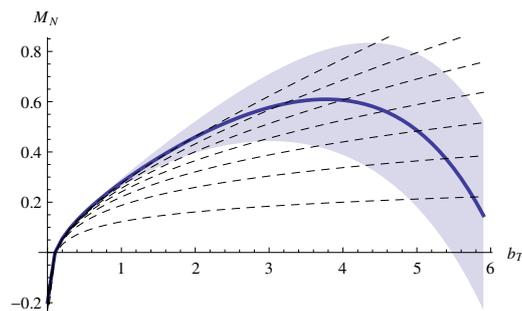


Figure 2. The dependence of partial sums M_N on b (in GeV^{-1}). The dashed lines represent M_N from $N = 1$ (bottom line) till $N = 7$ (top line). The bold line is the value of M_∞ . The shaded area is the error band of M_∞ given by δM .

In order to proceed to an estimate of the non-perturbative part of \mathcal{D} we write it in the form

$$\mathcal{D}(\mu, \mathbf{b}) = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_{cusp}(\mu) + \mathcal{D}^{PT}(\mu_0, \mathbf{b}) + \mathcal{D}^{NP}(\mu_0, \mathbf{b}), \quad (4.6)$$

where \mathcal{D}^{PT} is given by the perturbative expression at μ_0 scale, \mathcal{D}^{NP} encodes the non-perturbative part. The parameter μ_0 depends on \mathbf{b} and should be selected such that $a_s(\mu_0)$ is a reasonably small number. The non-perturbative part \mathcal{D}^{NP} is independent on μ (since the evolution part of \mathcal{D} is renormalon-free) but depends on the choice of μ_0 .

In principle, the best value of the parameter μ_0 can be extracted from the large- β_0 calculation. Indeed, the resummation of bubble-diagrams modifies the coupling in the interaction vertex, such that a loop integral appears to be naturally regularized in the infrared region. Practically, the effect of such resummation can be presented as a freezing of the coupling constant at large b . Particularly popular is the b^* prescription [44] defined as

$$\mu_0 = \mu_b = \frac{C_0}{b^*(\mathbf{b})}, \quad b^*(\mathbf{b}) = \frac{\sqrt{\mathbf{b}^2}}{\sqrt{1 + \mathbf{b}^2/b_{\max}^2}}, \quad C_0 = 2e^{-\gamma}. \quad (4.7)$$

At large b the parameter μ_0 approaches C_0/b_{\max} , which should be chosen much less than Λ , i.e. $b_{\max} \ll C_0/\Lambda \sim 4 \text{ GeV}^{-1}$.

For large- b (say $b \gtrsim 3 \text{ GeV}$) the non-perturbative part of \mathcal{D} dominates the perturbative one. The large- β_0 calculation allows to estimate the leading contribution (from the side of small- b 's) to \mathcal{D}^{NP} from the infrared ambiguity eq. (3.19),

$$\mathcal{D}^{NP}(\mathbf{b}, \mu_0) = c\Lambda^2 \mathbf{b}^2 g_D(b, \mu_0), \quad (4.8)$$

the function g_D should be of order of unity at small- b and it depends on the choice of the scale μ_0 . Here Λ^2 is the position of Landau pole and it is expected to be of order $\mathcal{O}(\Lambda_{QCD}) \sim 250 \text{ MeV}$, which implies

$$c\Lambda^2 = \frac{\pi C_F e^{5/3}}{2\beta_0} \Lambda^2 \sim 0.075 \text{ GeV}^2. \quad (4.9)$$

Since the large- β_0 approximation overestimates the exact values this number can be considered as an upper bound for non-perturbative input.

In order to estimate the parameters of the \mathcal{D} more accurately, we consider a kind of renormalon subtraction scheme for the anomalous dimension \mathcal{D} . We construct a renormalon subtracted expression $\mathcal{D}(\mu, \mathbf{b}) = \mathcal{D}^{RS}(\mu, \mathbf{b})$ by explicitly summing the large- β_0 contribution in eq. (4.3)

$$\mathcal{D}^{RS}(\mu, \mathbf{b}) = M_\infty(\mu, \mathbf{b}) + \frac{C_F}{\beta_0} \sum_{n=1}^{\infty} (\beta_0 a_s)^n \delta_n(\mathbf{L}_\mu). \quad (4.10)$$

The scale μ here should be chosen such that the logarithm \mathbf{L}_μ is reasonably small, otherwise the large- β_0 expansion is significantly violated. Using the model eq. (4.10) we fit the parameters of eq. (4.6) at $\mu = 10 \text{ GeV}$ in the range $b < 3 \text{ GeV}$, with $g_D = \text{constant} \equiv g_K$, at all known perturbative orders. It appears that the result is very stable with respect to b_{max} whose best value we find to be

$$b_{\text{max}} \simeq (1.2 \pm 0.1) \text{ GeV}^{-1}. \quad (4.11)$$

Concerning the non-perturbative part, it appears to be lower than the crude estimation eq. (4.9) and actually consistent with 0,

$$g_K \simeq (0.01 \pm 0.03) \text{ GeV}^2. \quad (4.12)$$

This value is generally smaller than the typical values presented in the literature, e.g. ref. [8] quotes $g_K \simeq 0.17 \text{ GeV}^2$, ref. [45] quotes $g_K \simeq 0.045 \pm 0.005 \text{ GeV}^2$. But ref. [29] finds g_K consistent with 0, which agrees with the present findings. However, one should take into account that contrary to standard fits, the present considerations are purely theoretical. Moreover in fits with experimental data, one should consider the extra non-perturbative part of the TMD distribution itself (which is discussed in the next section).

Finally, we comment on the possibility of a more sophisticated renormalon subtraction scheme as in the MSR scheme of [46]. In this scheme, one provides a subtraction of the renormalon from a perturbative series which depend on an additional scale μ_R . The new renormalon subtraction scale can result into large logarithms which, in turn, should be resummed. Such a consideration can result in more accurate restrictions on parameters.

4.2 Renormalon consistent ansatz for TMDs

The non-perturbative corrections to the matching coefficients are necessary for all analysis which include low energy data. These corrections have not been deeply studied in QCD theory and up to now, only a phenomenological treatment has been provided. In this section, we present a consistent ansatz that interpolates the perturbative small- b part of a TMD distribution with an entirely Gaussian exponent at large- b . The presented ansatz takes into account the lessons learned from the study of renormalon singularities and formulated in section 3.5.

The renormalon contribution accounts the leading power correction (see detailed explanation e.g. in [38, 41, 49]). Thus, the small- b expansion of the TMD distribution, that includes this power correction, has a form

$$\hat{F}_{q\leftarrow N}(x, \mathbf{b}; \mu) = \sum_j \int_x^1 \frac{dy}{y} \left(\hat{C}_{q\leftarrow j}(y, \mathbf{b}; \mu) + yg_q^{\text{in}} \mathbf{b}^2 C_{q\leftarrow j}^{\text{ren}}(y, \mathbf{b}) \right) f_{j\leftarrow N} \left(\frac{x}{y}, \mu \right) + \mathcal{O}(\mathbf{b}^4), \quad (4.13)$$

$$\hat{D}_{q\rightarrow N}(z, \mathbf{b}; \mu) = \sum_j \int_z^1 \frac{dy}{y} \left(\hat{C}_{q\rightarrow j}(y, \mathbf{b}; \mu) + \frac{g_q^{\text{out}} \mathbf{b}^2}{y} C_{q\rightarrow j}^{\text{ren}}(y, \mathbf{b}) \right) d_{j\rightarrow N} \left(\frac{z}{y}, \mu \right) + \mathcal{O}(\mathbf{b}^4), \quad (4.14)$$

where the LO coefficient function of the renormalon contribution was calculated in section 3.4 and reads

$$C_{q\leftarrow q}^{\text{ren}}(x, \mathbf{b}) = C_{q\rightarrow q}^{\text{ren}}(x, \mathbf{b}) = 2\bar{x} + \frac{2x}{(1-x)_+} - \delta(\bar{x}) \left(\mathbf{L}_\Lambda + \frac{2}{3} \right). \quad (4.15)$$

The constants $g_q^{\text{in,out}}$ are of order $c\Lambda^2$ within the large- β_0 approximation, however the actual value should be estimated from data. The non-perturbative scale Λ is the same as in the case of the evolution kernel. The contribution presented here is at LO, and as such has not μ -dependence. The μ -dependence of higher perturbative orders can in principle be calculated, using the evolution equation for TMD and the related integrated distribution.

At larger values of b eq. (4.14) is corrected by the higher orders of the OPE, and at a particular scale B (which defines the convergence radius of small- b OPE eq. (2.10)) it is replaced by a single and entirely non-perturbative function. It is commonly assumed that at large- b the TMD distribution has Gaussian behavior. This is also supported by the phenomenological studies of low-energy data (see e.g. ref. [50] for a study dedicated to this issue). The interpolation of a Gaussian with the small- b matching eqs. (4.13)–(4.14) should take into account the previously formulated demands on the power corrections. In particular, we have the following two guidelines:

- (i) In order to be consistent with the general structure of OPE, the interpolation should be done under the convolution integral.
- (ii) According to the structure of renormalon singularities, the powers of \mathbf{b}^2 should be always supplemented by x (for PDF) and z^{-1} (for FF).

A viable model, which takes into account both these points, can have the form

$$\hat{F}_{q\leftarrow N}(x, \mathbf{b}; \mu) = \sum_j \int_x^1 \frac{dy}{y} e^{-g_b y \mathbf{b}^2} \left(\hat{C}_{q\leftarrow j}(y, \mathbf{b}; \mu) + yg_q \mathbf{b}^2 \left(C_{q\leftarrow j}^{\text{ren}}(y, \mathbf{b}) + \delta(\bar{y}) \frac{g_b}{g_q} \right) \right) f_{j\leftarrow N} \left(\frac{x}{y}, \mu \right), \quad (4.16)$$

$$\hat{D}_{q\rightarrow N}(z, \mathbf{b}; \mu) = \sum_j \int_z^1 \frac{dy}{y} e^{-g_b \mathbf{b}^2 / y} \left(\hat{C}_{q\rightarrow j}(y, \mathbf{b}; \mu) + \frac{g_q \mathbf{b}^2}{y} \left(C_{q\rightarrow j}^{\text{ren}}(y, \mathbf{b}) + \delta(\bar{y}) \frac{g_b}{g_q} \right) \right) d_{j\rightarrow N} \left(\frac{z}{y}, \mu \right). \quad (4.17)$$

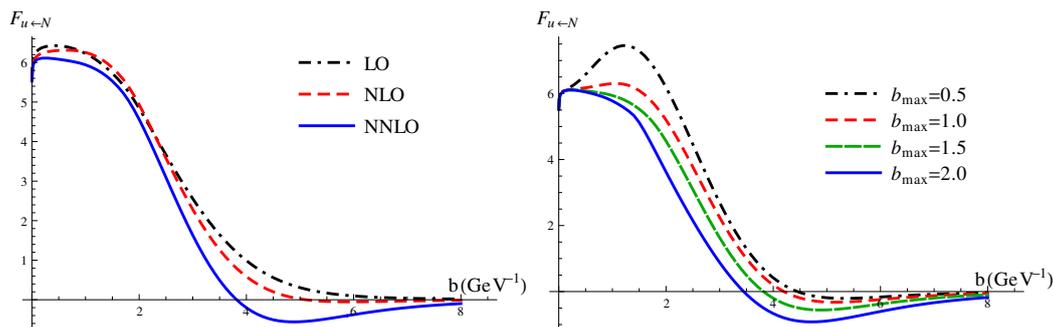


Figure 3. The TMDPDF $\hat{F}_{u\leftarrow p}(x, \mathbf{L}_\mu)$ as in the model of eq. (4.16) (the up-quark PDF is taken from MSTW [47, 48], at $x = 0.1$, $\Lambda = 0.25$ GeV, $N_f = 3$, $g_b = .2$ GeV $^{-2}$, $g_q = 0.01$ GeV $^{-2}$, $\mu = C_0/b^*$) as a function of the parameter b in GeV $^{-1}$ units. On the left panel we show consequently curves for LO, NLO and NNLO matching coefficients ($b_{\max} = 1.5$ GeV $^{-1}$ is used). On the right panel we present NNLO curve at several values of b_{\max} in units of GeV $^{-1}$.

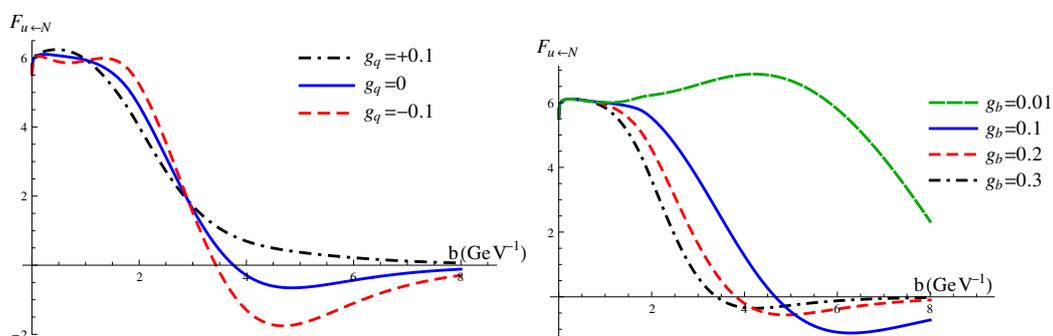


Figure 4. The TMDPDF $\hat{F}_{u\leftarrow p}(x, \mathbf{L}_\mu)$ as in the model of eq. (4.16) at NNLO (PDF from MSTW [47, 48], and with $x = 0.1$, $\Lambda = 0.25$ GeV, $N_f = 3$, $\mu = C_0/b^*$ with $b_{\max} = 1.5$ GeV $^{-1}$) as a function of the impact parameter b in GeV $^{-1}$ units. On the left panel we show several possible choices of g_q in GeV 2 at fixed $g_b = .2$ GeV 2 . On the right panel we show several possible choices of g_b in GeV 2 at fixed $g_q = .01$ GeV 2 . All curves are at NNLO.

The inclusion of the perturbative and power corrections modifies the Gaussian shape differently for PDF and FF kinematics.

In the figures 3–5 we illustrate several features of the renormalon consistent ansatz that we propose. In all the plots we fix the μ scale at the value $\mu = \mu^* = C_0/b^*$. In figure 3-left we show that the change of \hat{F} with respect to the perturbative order of matching coefficient. On the right hand side of figure 3 we show the dependence on the choice of the scale b_{\max} , which we find very mild for $1 \text{ GeV}^{-1} \lesssim b_{\max} \lesssim 2 \text{ GeV}^{-1}$.

The shape of the TMDs can strongly depend of the values of the non-perturbative constants $g_{b,q}$ for $b \geq 2 \text{ GeV}^{-1}$ as shown in figure 4–5. The values used in plots parameters are inspired by the fit in [29]. However, they can also change in a real fit with the present model. For $b \leq 1 \text{ GeV}^{-1}$ the non-perturbative model does not really affect the x -behavior of the TMD. In figure 5 we show instead that for instance at $b \sim 1.5 \text{ GeV}^{-1}$ the model parameter can start to have their impact.

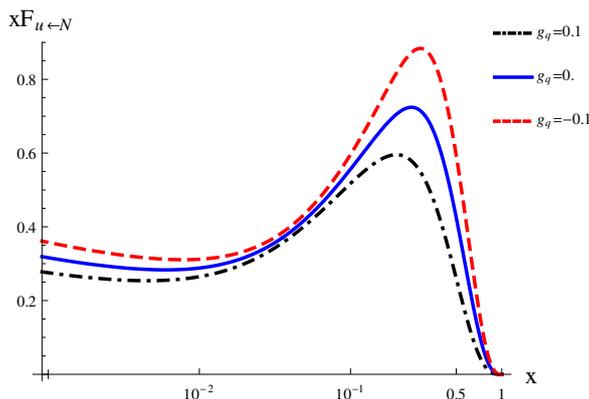


Figure 5. The function $x\hat{F}_{u \leftarrow p}(x, \mathbf{L}_\mu)$ as in the model of eq. (4.16) at NNLO (PDF from MSTW [47, 48]), as a function of x . The other inputs are fixed as $\Lambda = 0.25 \text{ GeV}$, $N_f = 3$, $\mu = C_0/b^*$, $b = b_{\text{max}} = 1.5 \text{ GeV}^{-1}$ and $g_b = 0.2 \text{ GeV}^2$. We show the curves at different values of g_q .

The cross-section built from TMDs in the form (4.16)–(4.17) and the evolution kernel (4.6) is dependent on the parameters g_K , g_b and g_q . While, the parameter g_K is strongly universal, the parameters g_b and g_q are separate for TMDPDFs and TMDFFs, as well as, different for different flavors. Within the cross-section the dependence on these parameters is smoothed to a more-or-less similar shape (especially for parameters g_b and g_K). However, the dependence on these parameters is clearly distinguishable at different energies. As an example, we show the Drell-Yan cross-section in figure 6 and the Z-boson cross section in figure 7 with some typical values of the experimental energies. While the corrections to the Z-boson production are dominated by g_K , at low energies all parameters can compete. In actual experiments the Z-boson production is only minimally affected by non-perturbative effects, so in actual fits it may happen that the value of g_K is compatible with zero, while the other parameters provide the expected minimal correction (this is for instance the case of the fit in ref. [29]). This yields that an estimate of the nature of the TMDs non-perturbative part cannot be done just using the Z-boson production, but needs also data from low energy physics. We postpone to a future work a comparison with data of the model that we have presented here.

To conclude this section, we observe that in the literature we have not found any non-perturbative input for TMDs fully consistent with the demands dictated by the power analysis presented here. For instance the b^* -prescription which is used in many phenomenological analysis [8, 25, 28] is inconsistent with eq. (4.13). Within the b^* -prescription the higher-twist corrections are simulated by replacing $b \rightarrow b^*$, and including an additional non-perturbative factor as

$$\hat{F}_{q \leftarrow N}^{b^* \text{-presc.}}(x, \mathbf{b}; \mu) = \sum_j e^{g_{j/N}(x, \mathbf{b})} \int_x^1 \frac{dy}{y} \hat{C}_{q \leftarrow j}(y, b^*; \mu) f_{j \leftarrow N} \left(\frac{x}{y}, \mu \right), \quad (4.18)$$

and similarly for TMDFF. This expression violates both guidelines formulated before

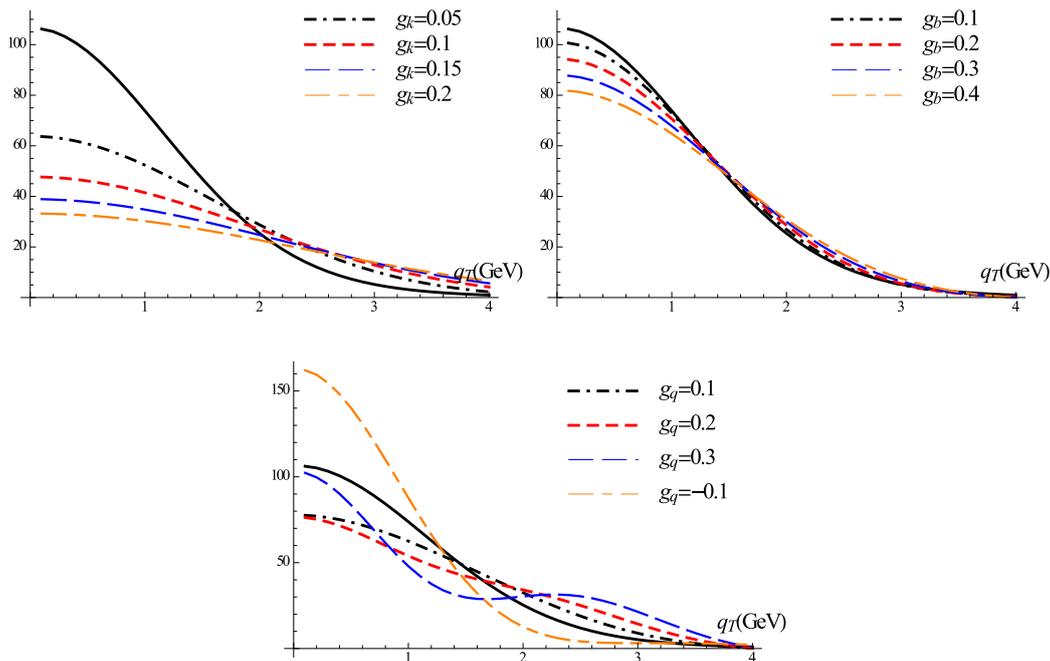


Figure 6. The plots of Drell-Yan cross-section $p + p \rightarrow \gamma + X$ $d\sigma/dQ^2 dy dq_T^2$ at $\sqrt{s} = 100$ GeV, $Q = 10$ GeV and $y = 0$, evaluated using the renormalon ansatz. The impact of different parameters is demonstrated. The black line is the reference curve with all parameters set to 0. The other inputs are fixed as $\Lambda = 0.25$ GeV, $N_f = 3$, $\mu_0 = C_0/b^*$, $b_{\max} = 1.5$ GeV $^{-1}$. All curves are at NNLO.

eq. (4.16). Considering the small- b expansion of $\hat{C}(b^*)$ in eq. (4.18),

$$\hat{F}_{q \leftarrow N}^{b^* \text{-presc.}}(x, \mathbf{b}; \mu)|_{\text{small-}b} \simeq \sum_j \int_x^1 \frac{dy}{y} \left[\hat{C}_{q \leftarrow j}(y, \mathbf{b}; \mu) + \frac{a_s(\mu) C_F \mathbf{b}^2}{b_{\max}^2} \left(\frac{2y}{(1-y)_+} + \bar{y} - \delta(\bar{y}) \left(\mathbf{L}_\mu - \frac{3}{2} \right) \right) + \delta(\bar{y}) \mathbf{b}^2 g''_{j/N}(x, 0) \right] f_{j \leftarrow N} \left(\frac{x}{y}, \mu \right), \quad (4.19)$$

one does not reproduce eq. (4.13). The main difference comes from the general power scaling, $x\mathbf{b}^2$ vs. \mathbf{b}^2 , see point (ii). The point (i) is violated by the non-perturbative exponent that is generally x -dependent and positioned outside of convolution integral (although, we should appreciate that in most application it is taken x -independent).

5 Conclusion

In this work, we have studied the non-perturbative properties associated with renormalons for the soft function and unintegrated matrix elements. With this aim, we have evaluated all constituents of TMD distributions (soft factor, matching coefficient and anomalous dimensions) within the large- β_0 approximation. The (factorial) divergences of the large- β_0 series are associated with the renormalon contribution and allow to estimate the leading non-perturbative contributions. We have found two independent renormalon structures in the perturbative description of TMD: the soft function and small- b matching coefficients.

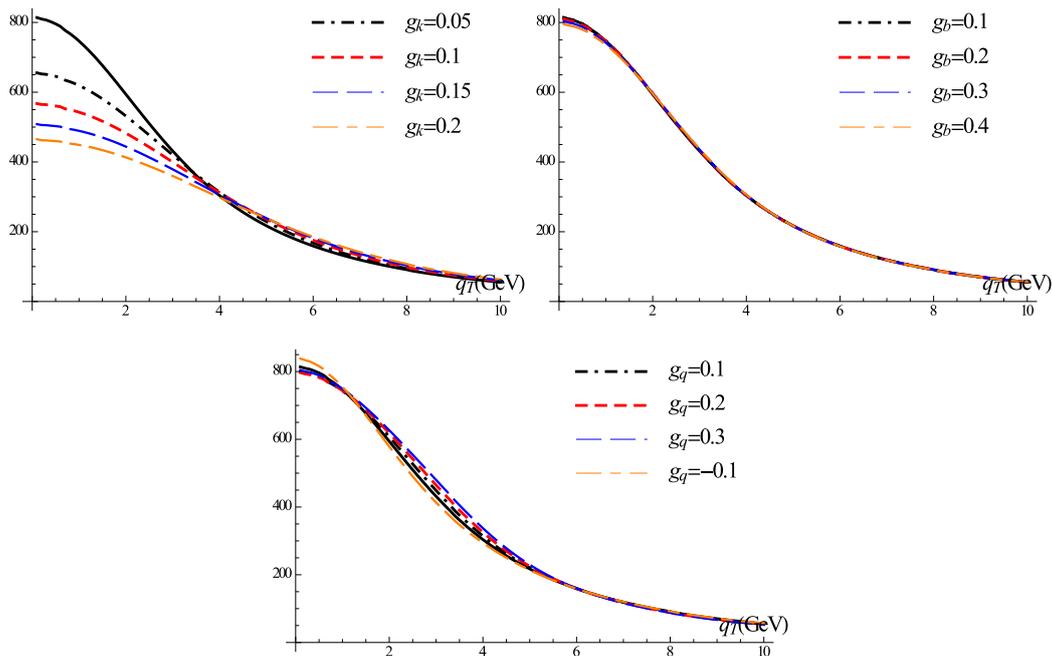


Figure 7. The plots of Z-boson production cross-section $p + p \rightarrow \gamma + X$ $d\sigma/dQ^2 dy dq_T^2$ at $\sqrt{s} = 1.96\text{TeV}$, $Q = M_Z = 91.18\text{ GeV}$ and $y = 0$, evaluated using the renormalon ansatz. The impact of different parameters is demonstrated. The black line is the reference curve with all parameters set to 0. The other inputs are fixed as $\Lambda = 0.25\text{ GeV}$, $N_f = 3$, $\mu_0 = C_0/b^*$, $b_{\text{max}} = 1.5\text{ GeV}^{-1}$. All curves are at NNLO.

The consideration of the soft function allows to fix the power behavior of the evolution kernel of TMDs. We show the evidence of infrared renormalons at $u = 1, 2, \dots$ (u being the Borel parameter). Our results agree with the analysis of the power corrections to factorized cross-section made in [36]. It also supports the popular assumption about a quadratic power correction to the TMD evolution kernel. However, the impact of the non-perturbative corrections is estimated to be not very significant for experiments where TMDs are evaluated at scales higher than a few GeV.

The nature of the renormalon contribution to the evolution kernel is peculiar, in the sense that it is generated by the non-perturbative part of a matrix element. In some aspects, this is very similar to the renormalon contribution to heavy quark masses. We have discussed also an ansatz which implements a consistent renormalon subtraction for the TMD evolution kernel, which can be useful for phenomenology.

The most promising conclusion of the paper comes from the analysis of the renormalon contribution to the small- b expansion of TMDs. The discussion of these results can be found in section 3.5. We demonstrate that the power corrections to small- b behave as a function of $x\mathbf{b}^2$ for TMDPDFs and as \mathbf{b}^2/z for TMDFFs. This observation should have a significant impact on the joined TMDPDF–TMDFF phenomenology. Additionally, the large- β_0 computation unveils the form of x -dependence for the leading power correction to the small- b matching. This behavior should be incorporated in realistic and consistent models for TMDs.

We have discussed and formulated the demands on a phenomenological ansatz to incorporate all collected information. We find that typical models for the non-perturbative part of TMDs, discussed in the literature, are inconsistent with our conclusions, mainly, due to the naive assumption that the combined powers corrections are largely functions of \mathbf{b}^2 (contrary to $x\mathbf{b}^2$). In eqs. (4.16)–(4.17) we construct a simple ansatz that interpolates the Gaussian low-energy model for TMDs with the perturbative small- b regime accounting formulated demands. We postpone to a future work the fit of available data using the presented results.

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