AN EXTENSION RESULT FOR MAPS ADMITTING AN ALGEBRAIC ADDITION THEOREM

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Abstract. We prove that if an analytic map $f : U \to \mathbb{C}^n$, where $U \subset \mathbb{C}^n$ is an open neighborhood of the origin, admits an algebraic addition theorem then, there exists a meromorphic map $g : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ admitting an algebraic addition theorem such that each coordinate function of f is algebraic over $\mathbb{C}(g)$ on U (this was proved by K. Weierstrass for $n = 1$). Furthermore, g admits a *rational* addition theorem.

1. Introduction

The aim of this paper is to study maps admitting an algebraic addition theorem, maps whose coordinate functions can be viewed as limitting (degenerate) cases of abelian functions. Let $\mathbb{K}$ be $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{M}_{\mathbb{K},n}$ be the quotient field of $\mathcal{O}_{\mathbb{K},n}$, the ring of power series in n variables with coefficients in $\mathbb{K}$ that are convergent in a neighborhood of the origin.

Definition. Let u and v be variables of $\mathbb{C}^n$. We say $(\phi_1, \dots, \phi_n) \in \mathcal{M}_{\mathbb{K},n}^n$ admits an *algebraic addition theorem* (AAT) if $\phi_1, \dots, \phi_n$ are algebraically independent over $\mathbb{K}$ and if each $\phi_i(u + v)$, $i = 1, \dots, n$, is algebraic over

$$\mathbb{K}(\phi_1(u), \dots, \phi_n(u), \phi_1(v), \dots, \phi_n(v)).$$

The concept of AAT was introduced by K. Weierstrass during his lectures on abelian functions in Berlin in the second half of the 19th century. The statement for *dimension one* of the main result concerning AAT proved by Weierstrass appeared for the first time in [19]:

> *Eine analytische Function $\phi(u)$, für welche ein algebraisches Additions-theorem besteht, ist entweder I. eine algebraische Function von u, oder II., wenn mit ω eine passend gewählte Constante bezeichnet wird, eine algebraische Function der Exponentialfunction $e^{u\pi i}\omega$, oder III., eine algebraische Function einer Function $\wp u = s$, welche, wenn mit g_2 und g_3 zwei passend gewählte Constanten bezeichnet wergden, durch die Differentialgleichung $(\frac{ds}{du})^2 = 4s^3 - g_2 s - g_3$ und die Bedingung $\wp(0) = \infty$ bestimmt werden kann.*

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Most of Weierstrass' lectures were never published. However, several proofs of the above statement can be found in the literature. The first one by E. Phragmen [14] in 1884. There are also proofs by A.R. Forsyth in 1893, by P. Koebe in 1905, and by M. Falk and H. Hancock both in 1910 (see [18] for historical details). All these proofs – despite their differences – go through an *extension result*:

> *the germ of an analytic function admitting an AAT can be transformed algebraically into the germ of a global meromorphic function admitting an AAT.*

The general idea to prove the latter is to first show that any ϕ admitting an AAT can be analytically extended to a multivalued analytic map with a finite number of branches (see [14, p. 40]), and then – making use of the elementary symmetric functions of the branches – the desired global univaluated meromorphic function admitting an AAT is obtained (see [14, p.41]).

The above statemement – in German – has a several variables version, which was also introduced by Weierstrass in his lectures. It did not appeared published until 1894 by P. Painlevé in [12, p. 348] (see also [13, p. 1]):

> *Tout systeme de n fonctions (indépendantes) a n variables qui admet un théoreme d'addition est une combinaison algébrique de n fonctions abéliennes (oú dégénérescences) á n arguments et aux mêmes périodes.*

Painlevé proves the latter statement for systems of analytic functions with a finite number of branches (see [13, § 6]) which is essentially the same than considering only systems of global meromorphic functions. F. Severi in [16] and Y. Abe [1, 2] also give a proof under these hypothesis. We are interested in applying Painlevé's result to Nash groups (see [4]), where the functions to be consider are defined locally, hence we need an extension result in several variables which we think it has interest by its own.

P.J. Myrberg [11] studies a generalization of the statement – in French – in which the number of functions n and that of variables m do not need to coincide. His aim is to reduce the generalization to the known case $n = m$. He first consider systems of analytic functions which satisfy a *rational* addition theorem (*i.e.*, each $\phi_i(u+v) \in \mathbb{K}(\phi_1(u),\dots,\phi_n(u),\phi_1(v),\dots,\phi_n(v))$ in the above definition of AAT) and for these systems he proves an extension result via a theorem of Poincaré concerning systems of functions satisfying a *rational* multiplication theorem. Myrberg claims (see [11, p. 2]) that this Poincaré's theorem also applies to systems of functions satisfying an *algebraic* multiplication theorem. Making use of this, he sketches an argument in [11, § IV] to pass from the hypothesis of satisfying an algebraic addition theorem to a rational addition theorem.

In this paper, we prove by different methods that any system of analytic functions admitting a AAT is algebraic over a system admitting a rational addition theorem (Theorem 1). As a consequence (Corollary 2), we obtain a several variable case of the extension result mentioned above.

Besides the mentioned application to Nash groups, the relevance of the AAT transcend the context of elliptic or abelian functions: A.A. Belavin and

V. G. Drinel'd [5] use ideas concerning algebraic addition theorems in the setting of the theory of integrable quantum systems. Moreover, they prove an extension result for maps admitting an AAT in the specific situation given by the Yang-Baxter equations.

Theorem 1 (Extension Theorem)**.** *Let $\phi := (\phi_1, \ldots, \phi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admit an AAT. Then, there exist $\psi := (\psi_1, \ldots, \psi_n) \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT and algebraic over $\mathbb{K}(\phi)$, and an additional meromorphic series $\psi_0 \in \mathcal{M}_{\mathbb{K},n}$ algebraic over $\mathbb{K}(\psi)$ such that,*

(1) *For each $f(u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$,*
   (a) *$f(u+v) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u), \psi_0(v), \ldots, \psi_n(v))$ and*
   (b) *$f(-u) \in \mathbb{K}(\psi_0(u), \ldots, \psi_n(u))$.*

(2) *Each $\psi_0, \ldots, \psi_n$ is the quotient of two convergent power series whose complex domain of convergence is $\mathbb{C}^n$.*

Corollary 2. *Any $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT is algebraic over $\mathbb{K}(\psi)$ for some $\psi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT and whose coordinate functions are the quotient of two convergent power series whose complex domain of convergence is $\mathbb{C}^n$.*

We obtain the rational version in (1a) through the coefficients of the polynomial associated to each $\phi_i(u+v)$. Then, we obtain the extension result of Theorem 1 (2) by considering the rational expression obtained in Theorem 1 (1a). In particular, this shows that any ϕ admitting an AAT can be analytically extended to a multivalued analytic map with a finite number of branches. Thus, we provide a new way of proving Weierstrass' extension result in dimension one, whose classical proofs go the other way around (and do not provide a rational counterpart).

The motivation of the results of this paper is to study abelian locally $\mathbb{K}$-Nash groups, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Charts at the identity of such groups admit an AAT. Locally Nash groups (i.e. for $\mathbb{K} = \mathbb{R}$) were studied by J.J. Madden and C.M. Stanton [10] and M. Shiota [17], mainly in dimension 1. In particular, the Extension Theorem will allow us to reduce the study of simply connected abelian locally Nash groups to those whose charts are restrictions of (global) meromorphic functions admitting an AAT (see [3]). Moreover, the new rational version of the AAT we have obtained in this paper will allow us to compare these groups with the algebraic ones.

The results of this paper are part of the second author's Ph.D. dissertation.

2. The Extension Theorem.

For each $\epsilon > 0$, let $U_{\mathbb{K},n}(\epsilon) := \{a \in \mathbb{K}^n \mid \|a\| < \epsilon\}$. *We will only consider convergence over open subsets of $\mathbb{C}^n$, let $U_n(\epsilon) := U_{\mathbb{C},n}(\epsilon)$.* We say that $(\phi_1, \ldots, \phi_m) \in \mathcal{M}^m_{\mathbb{K},n}$ is *convergent* in $U_n(\epsilon)$ if each $\phi_1, \ldots, \phi_m$ is the quotient of two power series convergent on $U_n(\epsilon)$.

As usual, by the identity principle for analytic functions, we identify $\mathcal{O}_{\mathbb{K},n}$ with the ring of germs of analytic functions at 0, and $\mathcal{M}_{\mathbb{K},n}$ with its quotient field. We will use without mention properties of $\mathcal{O}_{\mathbb{K},n}$, see e.g. R.C. Gunning and H. Rossi [8] and J.M. Ruiz [15].

Let $\epsilon > 0$. Let $\phi := (\phi_1, \ldots, \phi_m) \in \mathcal{M}^m_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$, let $a \in U_{\mathbb{K},n}(\epsilon)$ and let $(u, v) := (u_1, \ldots, u_n, v_1, \ldots, v_n)$ be a $2n$-tuple of variables. We will use the following notation:

$$\begin{aligned}
&\phi_{(u,v)} := (\phi_1(u), \ldots, \phi_m(u), \phi_1(v), \ldots, \phi_m(v)) \in \mathcal{M}^{2m}_{\mathbb{K},2n}.\\
&\phi_{u+v} := (\phi_1(u+v), \ldots, \phi_m(u+v)) \in \mathcal{M}^m_{\mathbb{K},2n}.\\
&\phi_{u+a} := (\phi_1(u+a), \ldots, \phi_m(u+a)) \in \mathcal{M}^m_{\mathbb{K},n}.
\end{aligned}$$

Given $\phi \in \mathcal{M}^n_{\mathbb{K},p}$ and $\psi \in \mathcal{M}^m_{\mathbb{K},p}$ we say that the tuple ϕ is *algebraic* over $\mathbb{K}(\psi) := \mathbb{K}(\psi_1, \ldots, \psi_m)$ if each component, $\phi_1, \ldots, \phi_n$, is algebraic over $\mathbb{K}(\psi)$.

Thus, $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admits an algebraic addition theorem (AAT) *if $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$ and ϕ_{u+v} is algebraic over $\mathbb{K}(\phi_{(u,v)})$.*

Note that if $\phi \in \mathcal{M}_{\mathbb{R},n}$ admits an AAT then ϕ also admits an AAT when considered as an element of $\mathcal{M}_{\mathbb{C},n}$.

We first prove two properties of maps admitting an AAT.

Lemma 3. *Let $\epsilon > 0$ and let $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$. If ϕ admits an AAT then ϕ_{u+a} is algebraic over $\mathbb{K}(\phi)$, for each $a \in U_{\mathbb{K},n}(\epsilon)$.*

Proof. Fix $j \in \{1, \ldots, n\}$ and let $f(u, v) := \phi_j(u + v)$. By hypothesis, there exists $P \in \mathbb{K}[X_1, \ldots, X_{2n}][Y]$ such that $P(\phi(u), \phi(v); Y) \neq 0$ and $P(\phi(u), \phi(v); f(u, v)) = 0$. For any $a \in U_{\mathbb{K},n}(\epsilon)$ such that $P(\phi(u), \phi(a), Y)$ is not identically zero, we clearly obtain that $f(u, a)$ is algebraic over $\mathbb{K}(\phi)$. We have to consider those $a \in U_{\mathbb{K},n}(\epsilon)$ such that $P(\phi(u), \phi(a); Y)$ is identically zero.

We first check that there exists an open dense subset U of $U_{\mathbb{K},n}(\epsilon)$ such that for each $a \in U$, $P(X_1, \ldots, X_n, \phi(a); Y) \in \mathbb{K}[X_1, \ldots, X_n][Y]$ is a non-zero polynomial. Let W be an open dense subset of $U_{\mathbb{K},n}(\epsilon)$ such that

$$W \subset \{a \in U_{\mathbb{K},n}(\epsilon) \mid \phi(a) \in \mathbb{K}^n\}$$

and $\phi : W \to \mathbb{K}^n$ is analytic. Let

$$U := \{a \in W \mid P(X_1, \ldots, X_n, \phi(a); Y) \neq 0\}.$$

Since W is an open dense subset of $U_{\mathbb{K},n}(\epsilon)$, it is enough to show that $W \setminus U$ is closed and nowhere dense in W. Clearly $W \setminus U$ is closed in W because ϕ is continuous in W. To prove the density, we note that if $W \setminus U$ contains an open subset of W then

$$\{a \in U_{\mathbb{K},n}(\epsilon) \mid P(\phi(u), \phi(a); Y) \in \mathcal{M}_{\mathbb{K},n+1} \text{ and } P(\phi(u), \phi(a); Y) = 0\}$$

contains an open subset of $U_{\mathbb{K},n}(\epsilon)$ and therefore $P(\phi(u), \phi(v); Y) = 0$, a contradiction.

To finish the proof we will show that for each $a \in U_{\mathbb{K},n}(\epsilon)$, there exists $Q_a \in \mathbb{K}[X_1, \ldots, X_n][Y]$ such that $Q_a(\phi(u); Y)$ is not identically zero and $Q_a(\phi(u); f(u, a)) = 0$. We follow the proof of [7, Ch. IX. §5. Theorem 5].

For each $a \in U$, where U is as above, let

$$P_a(X_1, \dots, X_n; Y) = \sum_{i,\mu \leq N} b_{i,\mu,a} \ X_1^{\mu_1} \dots X_n^{\mu_n} Y^i$$

denote the polynomial $P(X_1, \dots, X_n, \phi(a); Y)$. We have that U is dense in $U_{\mathbb{K},n}(\epsilon)$ and $P_a \neq 0$ for all $a \in U$. For each $a \in U$, we define

$$E(P_a) := \sum_{i,\mu \leq N} \|b_{i,\mu,a}\|^2.$$

We note that $E(P_a) > 0$, for all $a \in U$. For each $a \in U$, let

$$Q_a(X_1, \dots, X_n; Y) := \sum_{i,\mu \leq N} c_{i,\mu,a} \ X_1^{\mu_1} \dots X_n^{\mu_n} Y^i,$$

where

$$c_{i,\mu,a} := \frac{b_{i,\mu,a}}{\sqrt{E(P_a)}}.$$

Hence, for each $a \in U$, we have that $Q_a(\phi(u); Y)$ is not identically zero, $Q_a(\phi(u); f(u, a)) = 0$ and $E(Q_a) = 1$. We define

$$\vec{v}(a) := (c_{i,\mu,a})_{i,\mu \leq N} \in \{z \in \mathbb{K}^{(N+1)^{(n+1)}} \mid \|z\| = 1\}.$$

Take $a \in U_{\mathbb{K},n}(\epsilon) \setminus U$. Since U is an open dense subset of $U_{\mathbb{K},n}(\epsilon)$, there exists a sequence $\{a_k\}_{k \in \mathbb{N}} \subset U$ that converges to a. For each a_k, the identity $Q_{a_k}(\phi(u); f(u, a_k)) = 0$ holds, therefore

$$\sum_{i,\mu \leq N} c_{i,\mu,a_k} \phi_1(u)^{\mu_1} \dots \phi_n(u)^{\mu_n} f(u, a_k)^i = 0.$$

By hypothesis there are $\alpha, \beta \in \mathcal{O}_{\mathbb{K},2n}$, $\beta \neq 0$, convergent on $U_{2n}(\epsilon)$, such that $f(u, v) = \frac{\alpha(u,v)}{\beta(u,v)}$ and $\beta(u, a) \neq 0$ for all $a \in U_{\mathbb{K},n}(\epsilon)$. In particular

$$\sum_{i,\mu \leq N} c_{i,\mu,a_k} \phi_1(u)^{\mu_1} \dots \phi_n(u)^{\mu_n} \alpha(u, a_k)^i \beta(u, a_k)^{N-i} = 0. \tag{2.1}$$

Since $\{z \in \mathbb{K}^{(N+1)^{(n+1)}} \mid \|z\| = 1\}$ is compact, taking a suitable subsequence we can assume that the sequence $\{\vec{v}(a_k)\}_{k \in \mathbb{N}}$ is convergent. For each $i, \mu \leq N$, we define

$$c_{i,\mu,a} := \lim_{k \to \infty} c_{i,\mu,a_k}.$$

Since α and β are continuous, when k tends to infinity equation (2.1) becomes

$$\sum_{i,\mu \leq N} c_{i,\mu,a} \phi_1(u)^{\mu_1} \dots \phi_n(u)^{\mu_n} \alpha(u, a)^i \beta(u, a)^{N-i} = 0.$$

So dividing by $\beta(u, a)^N$, we also have

$$\sum_{i,\mu \leq N} c_{i,\mu,a} \phi_1(u)^{\mu_1} \dots \phi_n(u)^{\mu_n} f(u, a)^i = 0$$

and hence the polynomial

$$Q_a(X_1, \dots, X_n; Y) := \sum_{i,\mu \leq N} c_{i,\mu,a} X_1^{\mu_1} \dots X_n^{\mu_n} Y^i$$

satisfies $Q_a(\phi(u), f(u,a)) = 0$. We note that $E(Q_a) = \lim_{k\to\infty} E(Q_{a_k}) = 1$, so $Q_a \neq 0$. Since $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$ and $Q_a(X_1, \ldots, X_n, Y) \neq 0$, we have $Q_a(\phi(u), Y)$ is not identically zero. $\square$

Lemma 4. *Let $\phi, \psi \in \mathscr{M}^n_{\mathbb{K},n}$ and suppose that ϕ is algebraic over $\mathbb{K}(\psi)$. If ϕ admits an AAT then ψ admits an AAT. The converse is also true, provided $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$.*

Proof. Assume that ϕ admits an AAT, hence $\psi_1, \ldots, \psi_n$ are algebraically independent over $\mathbb{K}$ because ϕ is algebraic over $\mathbb{K}(\psi)$. To check that ψ_{u+v} is algebraic over $\mathbb{K}(\psi_{(u,v)})$ it is enough to show that ψ_{u+v} is algebraic over $\mathbb{K}(\phi_{u+v})$, ϕ_{u+v} is algebraic over $\mathbb{K}(\phi_{(u,v)})$ and $\phi_{(u,v)}$ is algebraic over $\mathbb{K}(\psi_{(u,v)})$. The three conditions above are trivially satisfied because ϕ admits an AAT and both ϕ is algebraic over $\mathbb{K}(\psi)$ and ψ is algebraic over $\mathbb{K}(\phi)$. The converse follows by symmetry because if $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{K}$ then ψ is algebraic over $\mathbb{K}(\phi)$. $\square$

Now, we adapt to our context a result on AAT due to H.A.Schwarz, see [9, Ch. XXI. Art. 389] for details.

Lemma 5. *Let $\epsilon > 0$ and let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ such that it admits an AAT. Then, there exist a finite subset $\mathcal{C} \subset U_{\mathbb{K},n}(\epsilon)$, with $0 \in \mathcal{C}$ and $\mathcal{C} = -\mathcal{C}$, and $\epsilon' \in (0, \epsilon]$ satisfying: each element of $\mathbb{K}(\phi_{u+a} \mid a \in \mathcal{C})$ is convergent on $U_n(2\epsilon')$, and there exist $A_0, \ldots, A_N \in \mathbb{K}(\phi_{(u+a,v+a)} \mid a \in \mathcal{C})$ convergent on $U_{2n}(2\epsilon')$ such that ϕ_{u+v} is algebraic over $\mathbb{K}(A_0, \ldots, A_N)$ and, for each $j \in \{0, \ldots, N\}$,*

$$A_j(u,v) = A_j(u+a, v-a), \text{ for all } a \in U_{\mathbb{K},n}(\epsilon'). \tag{2.2}$$

Proof. Fix $i \in \{1, \ldots, n\}$. Let $\mathcal{S}_0 := \{0\}$ and $\mathbb{K}_0 := \mathbb{K}(\phi_{(u,v)})$. Let

$$P_0(X) = X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}(u,v) X^j$$

be the minimal polynomial of $\phi_i(u+v)$ over $\mathbb{K}_0$. If each $A_{0,j}$ satisfies equation (2.2) for $\epsilon' = 2^{-1}\epsilon$ then we are done for this i letting $\epsilon' := 2^{-1}\epsilon$, $\mathcal{C} := \mathcal{S}_0$ and $A_j := A_{0,j}$, for each $0 \leq j \leq \ell_0$. Otherwise, there exists $a_1 \in U_{\mathbb{K},n}(2^{-1}\epsilon)$ such that

$$Q_0(X) := X^{\ell_0+1} + \sum_{j=0}^{\ell_0} A_{0,j}(u,v) X^j - X^{\ell_0+1} - \sum_{j=0}^{\ell_0} A_{0,j}(u+a_1, v-a_1) X^j$$

is not zero. Since $u + v = (u + a_1) + (v - a_1)$, we deduce that $\phi_i(u + v)$ is a root of $Q_0(X)$. Let $\mathcal{S}_1 := \mathcal{S}_0 \cup \{a_1, -a_1\}$ and $\mathbb{K}_1 := \mathbb{K}(\phi_{u+a,v+a} \mid a \in \mathcal{S}_1)$. By definition $\mathbb{K}_0 \subset \mathbb{K}_1$. Let

$$P_1(X) = X^{\ell_1+1} + \sum_{j=0}^{\ell_1} A_{1,j}(u,v) X^j$$

be the minimal polynomial of $\phi_i(u+v)$ over $\mathbb{K}_1$. We note that the elements of $\mathbb{K}_1$ are convergent on $U_{2n}(2^{-1}\epsilon)$. If each $A_{1,j}$ satisfies equation (2.2) for $\epsilon' = 2^{-2}\epsilon$ then we are done for this i letting $\epsilon' := 2^{-2}\epsilon$, $\mathcal{C} := \mathcal{S}_1$ and $A_j := A_{1,j}$, for each $0 \leq j \leq \ell_1$. Otherwise, we can repeat the process to obtain sets $\mathcal{S}_2$, $\mathcal{S}_3$ and so on, where the set $\mathcal{S}_k$ is obtained from the set $\mathcal{S}_{k-1}$ as

$$\mathcal{S}_k := \mathcal{S}_{k-1} \cup \{a + a_k \mid a \in \mathcal{S}_{k-1}\} \cup \{a - a_k \mid a \in \mathcal{S}_{k-1}\},$$

for some $a_k \in U_{\mathbb{K},n}(2^{-k}\epsilon)$ such that Q_{k-1} is not 0. Similarly, we obtain $\mathbb{K}_k := \mathbb{K}(\phi_{u+a,v+a} \mid a \in \mathcal{S}_k)$ whose elements are convergent on $U_{2n}(2^{-k}\epsilon)$. Since in the k repetition the degree of P_k is smaller than that of P_{k-1}, this process eventually stops, say at step s. Letting $\epsilon' := 2^{-s-1}\epsilon$, $\mathcal{C} := \mathcal{S}_s$ and $A_j := A_{s,j}$, for each $0 \leq j \leq \ell_s$, we are done for this i. The elements $A_0, \dots, A_{\ell_s}$ are convergent on $U_{2n}(2\epsilon')$ because they are elements of $\mathbb{K}_s$.

For each i, $1 \leq i \leq n$, denote by ϵ'_i, $\mathcal{C}_i$ and $A^i_0, \dots, A^i_{N_i}$ the elements ϵ', $\mathcal{C}$ and $A_1, \dots, A_{\ell_s}$ previously obtained for that choice of i. To complete the proof, take $\mathcal{C} := \bigcup_i \mathcal{C}_i$, $\epsilon' := \min_i\{\epsilon'_i\}$, and let $\{A_0, \dots, A_N\}$ be the union of the sets $\{A^i_0, \dots, A^i_{N_i}\}$. □

We need two additional lemmas before proving the Extension Theorem.

Lemma 6. *Let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ admit an AAT. Then, $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$.*

Proof. Take $\epsilon > 0$ such that $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ is convergent on $U_n(\epsilon)$. Since ϕ admits an AAT, we know that $\phi(u+v)$ is algebraic over $\mathbb{K}(\phi(u), \phi(v))$. Taking into account transcendence degrees, it follows that $\phi(v)$ is algebraic over $\mathbb{K}(\phi(u+v), \phi(u))$. For some $a \in U_{\mathbb{K},n}(\epsilon)$, we may substitute v by $-u+a$, so $\phi(-u+a)$ is algebraic over $\mathbb{K}(\phi(u))$. By Lemma 3, $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(-u+a))$ and hence over $\mathbb{K}(\phi(u))$. □

Lemma 7. *Let $\epsilon > 0$. Let $\phi \in \mathscr{M}^n_{\mathbb{K},n}$ be convergent on $U_n(\epsilon)$ such that it admits an AAT. Then there exist $\epsilon_1 \in (0, \epsilon]$ and $\Psi := (\psi_0, \dots, \psi_n) \in \mathscr{M}^{n+1}_{\mathbb{K},n}$ convergent on $U_n(\epsilon_1)$ and algebraic over $\mathbb{K}(\phi)$ satisfying $\psi := (\psi_1, \dots, \psi_n)$ admits an AAT, ψ_0 is algebraic over $\mathbb{K}(\psi)$ and, for each $f \in \mathbb{K}(\Psi)$, $f(-u) \in \mathbb{K}(\Psi(u))$, there exists $\delta \in (0, \epsilon_1]$ such that for each $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{K}(\Psi)$ and f_{u+a} is convergent on $U_n(\epsilon_1)$.*

Proof. We will define a field $\mathbb{L}$ generated over $\mathbb{K}$ by certain elements of $\mathscr{M}_{\mathbb{K},n}$, next we will prove that each $f \in \mathbb{L}$ satisfy the conclusion of the lemma and finally we find a primitive element Ψ such that $\mathbb{L} = \mathbb{K}(\Psi)$.

Let $\epsilon' \in (0, \epsilon]$, $\mathcal{C} \subset U_{\mathbb{K},n}(\epsilon)$ and $A_0, \dots, A_N \in \mathbb{K}(\phi_{(u+c,v+c)} \mid c \in \mathcal{C})$ be the ones provided by Lemma 5 for ϕ. Let U be an open dense subset of $U_{\mathbb{K},n}(\epsilon')$ such that

$$U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid \phi(a+c) \in \mathbb{K}^n \text{ for all } c \in \mathcal{C}\}$$

and

$$U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid A_0(u,a), \dots, A_N(u,a) \in \mathscr{M}_{\mathbb{K},n}\}.$$

In particular, $U \subset \{a \in U_{\mathbb{K},n}(\epsilon') \mid \phi(a) \in \mathbb{K}^n\}$ because $0 \in \mathcal{C}$. Since U is open there exist $b \in U$ and $\epsilon'' \in (0, \epsilon' - \|b\|]$ such that

$$V := \{a \in U_{\mathbb{K},n}(\epsilon') \mid \|a - b\| < \epsilon''\} \subset U.$$

Fix such b. Then, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, each $A_j(u, a+b)$, $j = 1, \dots, N$ is an element of $\mathscr{M}_{\mathbb{K},n}$. We note that since each $A_j(u,v)$ is convergent on $U_{2n}(2\epsilon')$ and by definition of b and ϵ'', each $A_j(u, a+b)$ is convergent on $U_n(\epsilon')$, for each $a \in U_{\mathbb{K},n}(\epsilon'')$. Also, since each A_j satisfies the equation (2.2) of Lemma 5,

$$A_j(u, a+b) = A_j(u+a, b) \text{ for all } a \in U_{\mathbb{K},n}(\epsilon''). \tag{2.3}$$

For each $j \in \{0, \dots, N\}$, we define $B_j(u) := A_j(u, b)$. Let

$$\mathbb{L}_1 := \mathbb{K}((B_j)_{u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N).$$

Since, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, each $A_j(u, a+b)$ is convergent on $U_n(\epsilon')$, by equation (2.3) all the elements of $\mathbb{L}_1$ are convergent on $U_n(\epsilon')$ and in particular in $U_n(\epsilon'')$. Let

$$\mathbb{L}_2 := \mathbb{K}((B_j)_{-u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N).$$

Note that all the elements of $\mathbb{L}_2$ are also convergent on $U_n(\epsilon'')$. Hence, if we define

$$\mathbb{L} := \mathbb{K}((B_j)_{u+a}, (B_j)_{-u+a} \mid a \in U_{\mathbb{K},n}(\epsilon''), 0 \le j \le N),$$

all the elements of $\mathbb{L}$ are also convergent on $U_n(\epsilon'')$.

Let us show that

$$\mathbb{L} \subset \mathbb{K}(\phi_{u+c}, \phi_{-u+c} \mid c \in \mathcal{C})$$

and that each element of $\mathbb{L}$ is algebraic over $\mathbb{K}(\phi)$.

We begin proving that

$$\mathbb{L}_1 \subset \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$$

and that each element of $\mathbb{L}_1$ is algebraic over $\mathbb{K}(\phi)$. Fix $j \in \{0, \dots, N\}$ and $a \in U_{\mathbb{K},n}(\epsilon'')$. We recall from Lemma 5 that $A_j(u, v)$ is convergent on $U_{2n}(2\epsilon')$ and $A(u, v) \in \mathbb{K}(\phi_{(u+c,v+c)} \mid c \in \mathcal{C})$. Hence we can evaluate $A_j(u, v)$ at $v = a+b$ to deduce that $A_j(u, a+b) \in \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$. Thus, by equation (2.3), $A_j(u+a, b) \in \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$. Hence, $\mathbb{L}_1 \subset \mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$ and therefore, by Lemma 3, each element of $\mathbb{L}_1$ is algebraic over $\mathbb{K}(\phi)$. By symmetry of $\mathcal{C}$, $\mathbb{L}_2 \subset \mathbb{K}(\phi_{-u+c} \mid c \in \mathcal{C})$ and each element of $\mathbb{L}_2$ is algebraic over $\mathbb{K}(\phi(-u))$. Therefore $\mathbb{L} \subset \mathbb{K}(\phi_{u+c,-u+c} \mid c \in \mathcal{C})$ and, since by Lemma 6 we have that $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$, we deduce that each element of $\mathbb{L}$ is algebraic over $\mathbb{K}(\phi(u))$, as required.

Next, we show that $\phi_1(u+b), \dots, \phi_n(u+b)$ are algebraically independent over $\mathbb{K}$. Let $P \in \mathbb{K}[X_1, \dots, X_n]$ be such that $P(\phi_{u+b}) = 0$. By notation, for each $a \in U_{\mathbb{K},n}(\epsilon'')$, we have that $P(\phi_{u+b}(a)) = 0$ if and only if $P(\phi(a+b)) = 0$. Hence,

$$V \subset \{a \in U_{\mathbb{K},n}(\epsilon) \mid \phi(a) \in \mathbb{K} \text{ and } P(\phi(a)) = 0\}.$$

Since V is open in $U_{\mathbb{K},n}(\epsilon)$, $P(\phi) = 0$ by the identity principle. Since $\phi_1, \dots, \phi_n$ are algebraically independent over $\mathbb{K}$, $P = 0$ and we are done.

Next, we show that $\mathbb{L}$ is finitely generated over $\mathbb{K}$ and its transcendence degree is n. Firstly, we note that ϕ is algebraic over $\mathbb{K}(\phi_{u+b})$ because the coordinate functions of ϕ_{u+b} are algebraically independent over $\mathbb{K}$ and ϕ_{u+b} is algebraic over $\mathbb{K}(\phi)$ by Lemma 3. Since ϕ_{u+v} is algebraic over $\mathbb{K}(A_0, \dots, A_N)$, evaluating each $A_j(u, v)$ at $v = b$ we deduce that ϕ_{u+b} is algebraic over $\mathbb{K}(B_0, \dots, B_N)$. Therefore, ϕ is algebraic over $\mathbb{K}(B_0, \dots, B_N)$. On the other hand, $\mathbb{K}(B_0, \dots, B_N)$ is a subset of $\mathbb{K}(\phi_{u+c} \mid c \in \mathcal{C})$ and the latter field is algebraic over $\mathbb{K}(\phi)$ by Lemma 3. Hence the three fields have transcendence degree n over $\mathbb{K}$. Recall that $\mathcal{C} = -\mathcal{C}$, so $\mathbb{K}(\phi_{-u+c} \mid c \in \mathcal{C}) =$

$\mathbb{K}(\phi_{-u-c} \mid c \in \mathcal{C})$. We also note that $\phi(-u)$ is algebraic over $\mathbb{K}(\phi(u))$, so $\mathbb{K}(\phi_{u+c}, \phi_{-u-c} \mid c \in \mathcal{C})$ has transcendence degree n over $\mathbb{K}$. Now, $\mathcal{C}$ is finite and

$$\mathbb{K}(B_0(u), \dots, B_N(u)) \subset \mathbb{L} \subset \mathbb{K}(\phi_{u+c}, \phi_{-u-c} \mid c \in \mathcal{C}),$$

therefore, $\mathbb{L}$ is finitely generated over $\mathbb{K}$ and its transcendence degree is n.

Fix $f \in \mathbb{L}$ and let us check that $f(-u) \in \mathbb{L}$ and that there exists $\delta > 0$ such that for every $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{L}$ and f_{u+a} is convergent on $U_n(\epsilon'')$.

Since $f \in \mathbb{L}$, there exist $m, m' \in \mathbb{N}$, $j(1), \dots, j(m+m') \in \{0, \dots, N\}$ and $a_1, \dots, a_{m+m'} \in U_{\mathbb{K},n}(\epsilon'')$ such that f is a rational function of

$$(B_{j(1)})_{u+a_1}, \dots, (B_{j(m)})_{u+a_m}, (B_{j(m+1)})_{-u+a_{m+1}}, \dots, (B_{j(m+m')})_{-u+a_{m+m'}}.$$

In particular, $f(-u)$ is a rational function of

$$(B_{j(1)})_{-u+a_1}, \dots, (B_{j(m)})_{-u+a_m}, (B_{j(m+1)})_{u+a_{m+1}}, \dots, (B_{j(m+m')})_{u+a_{m+m'}},$$

so $f(-u) \in \mathbb{L}$. Take $\delta > 0$ such that $\delta < \epsilon'' - \max\{\|a_1\|, \dots, \|a_{m+m'}\|\}$. Then, for all $a \in U_{\mathbb{K},n}(\delta)$, $f_{u+a} \in \mathbb{L}$ and f_{u+a} is convergent on $U_n(\epsilon'')$.

Finally, take $\psi_1, \dots, \psi_n \in \mathbb{L}$ algebraically independent over $\mathbb{K}$ and ψ_0 algebraic over $\mathbb{K}(\psi_1, \dots, \psi_n)$ such that $\mathbb{L} = \mathbb{K}(\psi_0, \psi_1, \dots, \psi_n)$. Now, since all the elements of $\mathbb{L}$ are algebraic over $\mathbb{K}(\phi)$, $\psi := (\psi_1, \dots, \psi_n)$ admits an AAT by Lemma 4. $\square$

We now have all the ingredients to prove our main result.

Proof of the Extension Theorem. Let $\phi := (\phi_1, \dots, \phi_n) \in \mathscr{M}_{\mathbb{K},n}^n$ admit an AAT. Take $\epsilon > 0$ such that ϕ is convergent on $U_n(\epsilon)$. Applying Lemma 7 we obtain $\epsilon_1 \in (0, \epsilon]$ and $\Psi := (\psi_0, \dots, \psi_n) \in \mathscr{M}_{\mathbb{K},n}^{n+1}$ as in the lemma. We next check that this Ψ satisfies the conditions of the theorem.

(1) By Lemma 7, if $f \in \mathbb{K}(\Psi)$ then $f(-u) \in \mathbb{K}(\Psi)$, so we only have to check $f(u+v) \in \mathbb{K}(\Psi_{(u,v)})$. Fix a non-constant $f \in \mathbb{K}(\Psi)$ and $\delta \in (0, \epsilon_1]$ such that $f_{u+a} \in \mathbb{K}(\Psi)$, for each $a \in U_n(\delta)$, as in Lemma 7. Let $0 < \varepsilon < \delta$ be such that f_{u+v} is convergent on $U_{2n}(\varepsilon)$. Let U be an open connected subset of $U_n(\varepsilon)$ such that $\Psi(u)$ is analytic on U. In particular, $\Psi_{(u,v)}$ is analytic on $U \times U$. On the other hand, if for each $a \in U$ we have that $g(u,v) := f(u+v)$ is not analytic in (a,a) then we would deduce that $f(u)$ is not analytic on an open subset of $U_n(\varepsilon)$, a contradiction. Therefore, shrinking U we can assume that $g(u,v) := f(u+v)$ is also analytic on $U \times U$. By Lemma 7, we have that $g(u,a) \in \mathbb{K}(\Psi(u))$ and $g(a,v) \in \mathbb{K}(\Psi(v))$, for each $a \in U$. Hence, by Bochner [6, Theorem 3], $g(u,v) \in \mathbb{C}(\Psi_{(u,v)})$ on $U \times U$. Since $U \times U$ is an open subset of $U_{2n}(\varepsilon)$, it follows that $g(u,v) \in \mathbb{C}(\Psi_{(u,v)})$ on $U_{2n}(\varepsilon)$. Moreover, clearly $g(u,v) \in \mathbb{K}(\Psi_{(u,v)})$ on $U_{2n}(\varepsilon)$ since both $\Psi \in \mathscr{M}_{\mathbb{K},n}^{n+1}$ and $f \in \mathbb{K}(\Psi)$. This concludes the proof of (1).

(2) We may assume that $\psi_0 \neq 0$. Fix $i \in \{0, \dots, n\}$. We have already shown that $\psi_i(u+v) \in \mathbb{K}(\Psi_{(u,v)})$. Let $A(u,v) := \psi_i(u+v)$. By Lemma 7 and taking a smaller $\epsilon > 0$ if necessary, we may assume that Ψ is convergent on $U_n(\epsilon)$ and $\mathbb{K}(\Psi_{u+a}) \subset \mathbb{K}(\Psi)$, for each $a \in U_{\mathbb{K},n}(\epsilon)$. Let us show that there exists $c \in U_{\mathbb{K},n}(\epsilon)$ such that

$$A(u+c, u-c) \in \mathscr{M}_{\mathbb{K},n}.$$

Take $\alpha, \beta \in \mathcal{O}_{\mathbb{K},2n}$, $\beta \neq 0$ such that $A(u,v) = \frac{\alpha(u,v)}{\beta(u,v)}$. Suppose by contradiction that $\beta(u+c, u-c) = 0$ for all $c \in U_{\mathbb{K},n}(\epsilon)$. Then

$$\beta\left(\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}\right) = 0,$$

for all $a, b \in U_{\mathbb{K},n}(\epsilon/2)$. So $\beta(a,b) = 0$, for all $(a,b) \in U_{\mathbb{K},n}(\epsilon/2)$, that is, $\beta = 0$, which is a contradiction. Consequently,

$$\psi_i(2u) = A(u+c, u-c) \in \mathbb{K}(\Psi_{u+c}(u), \Psi_{u-c}(u)) \subset \mathbb{K}(\Psi(u)).$$

By induction we deduce that

$$\psi_0(u), \ldots, \psi_n(u) \in \mathbb{K}(\Psi(2^{-N}u)),$$

for each $N \in \mathbb{N}$. Hence since $\Psi(2^{-N}u)$ is convergent on $U_n(2^N\epsilon)$, Ψ is also convergent on $U_n(2^N\epsilon)$. Thus each ψ_i is the quotient of two power series convergent in all $\mathbb{C}^n$ (by Poincaré's problem [8, Ch. VIII, §B, Corollary 10]). □

Proof of Corollary 2. Let $\phi \in \mathcal{M}^n_{\mathbb{K},n}$ admit an AAT. By Theorem 1, there exists $\psi \in \mathcal{M}^n_{\mathbb{K},n}$ admitting an AAT whose coordinate functions are the quotient of two convergent whose complex domain of convergence is $\mathbb{C}^n$ and such that ψ is algebraic over $\mathbb{K}(\phi)$. Since the coordinate functions of ψ are algebraically independent, ϕ is algebraic over $\mathbb{K}(\psi)$. □

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