Interaction of localized large diffusion and boundary conditions

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1 Introduction

Reaction diffusion models with large diffusion have attracted some attention in the literature. As one expects that large diffusion generates a fast spatial redistribution of spatial heterogeneities, one seeks to prove that the solutions of the PDE tend to become constant in space and thus approach the solutions of an ODE that describes the asymptotic behavior. This was proved for example in [15] for a system of equations possessing an invariant region, with Neumann boundary conditions. The interaction of large diffusion and Robin type boundary conditions was analyzed in [13, 21, 20], while the effect of gradient terms in the equations was considered in [12]. Nonlinear boundary conditions were considered in [37], [34], while linear source terms where studied in [33]. It is important to note that in all these references flux boundary conditions (i.e. linear Robin or nonlinear ones) have a deep impact on the limiting system of ODEs, as these include a suitable component coming from the boundary flux, that couples from the term coming from the interior reaction.

On the other hand, localized large diffusion, that is, the situation in which the region of large diffusion is a proper subdomain of the physical domain, arises in many natural phenomena including cellular biology, where cells regions exhibit prominent differences on their diffusion properties, or in composite materials, where the diffusion of the heat exhibits significant spatial variations. Also, it may appear in population dynamics in which one species diffuses much faster than the others in some determined regions, or one can consider the vibrations of an elastic membrane and assume that some part of the membrane is made of a more rigid material than the rest.

In all these cases, the singular limit of localized large diffusion is described by a limit problem that is not an ODE anymore. Instead, it consists of a PDE, defined in the region where diffusion

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is not large, coupled with some ODE (or system of ODEs) that describes the evolution of the constant asymptotic values of the solutions in the different components of the localized large diffusion region. The coupling includes some nonlocal term describing the flux from one region into the other. This shows that solutions of the original PDE tend to become constant in each of the components of the localized large diffusion regions and the values there couple through the flux on their boundaries, with the values in the rest of the domain. Hence, for example in the case of vibration of an elastic membrane, in the limit, one obtains a multistructure vibrating system composed of a rigid plate bound from its boundary to a membrane. Therefore, passing to the singular limit can be regarded as a process describing a transition phenomena of rigidization or solidification in a composite material. In the case of population dynamics, one obtains in the limit a media where isolas of homogenous distribution of species are surrounded by regions where spatial variations dominate.

The mathematical analysis of these cases was started in [31], studying linear elliptic and eigenvalue problems. Using the results in [4], in [7] the asymptotic behavior of nonlinear parabolic problems was analyzed. It was proved in [7] that the limit problem is well posed and dissipative and that the dynamics is asymptotically upper–semicontinuous: as large diffusion takes place, for large times solutions become closer to solutions on the attractor of the limit problem. Later [9] proved the lower semicontinuity (hence continuity) of the asymptotic dynamics, in the case of Dirichlet boundary conditions, while [10] did it for nonlinear boundary conditions. In [14] an estimate on the rate of convergence of the attractors is obtained for singularly perturbed problems and, in particular, for large diffusion problems. References [23], [24] studied porous media type equations with localized large diffusion, while [35], [11] dealt with p–Laplacian type equations. Notice that in all these references the localized large diffusion region is a proper subdomain not touching the boundary of the physical domain of the PDE. Therefore, the limit problem inherits the same boundary conditions as the approximating problems and the coupling between the PDE and the ODE in the limit problem shows up the nonlocal flux between the regions and reflects the homogenized action of the reaction, interior, term.

In this paper we focus then in the interaction of localized large diffusion and boundary conditions, which can be either of nonlinear or Dirichlet type in different parts of the boundary of the physical domain of the PDE. Hence we show the important role play by the interaction of the large diffusion region with the boundary which causes a non straightforward change in the limit problems. Therefore, we consider a localized large diffusion region, Ω_0 that touches the boundary of the domain and attempt to describe the corresponding limit problem. This would reflect the interaction of large diffusion and boundary conditions and its particular form depends strongly on this interaction. For example, if Ω_0 touches a part of the boundary subject to Dirichlet boundary conditions, then by the asymptotic homogenization, the limit problem imposes $u = 0$ in all Ω_0. Therefore the limit problem can be seen as a standard reaction diffusion problem in a restricted subdomain, $\Omega_1 = \Omega \setminus \overline{\Omega_0}$ with flux and Dirichlet boundary conditions in different parts of its boundary.

On the other hand, if Ω_0 touches only the part of the boundary in which flux conditions are imposed, a new type of limit problem arises which contains a subtle coupling in Ω_0 of the interior reaction and the boundary flux. As solutions tend to become constant in Ω_0 the limit ODE that describes that situation incorporates not only a nonlocal flux from the other part of the domain but also an averaged nonlinear term coming from the boundary flux. These result in a different class of nonstandard problems. Additionally for example, in [7, 9, 10] the nonlinear terms do not depend on $x \in \Omega$ and the region where large diffusion take place does not have contact with the boundary. In such a case, the nonlinear terms in the limit problem are "the same" as in the original problem but acting on functions constant in Ω_0. Here we also allow for spatially dependent nonlinear terms. In such a case a suitable spatial average of nonlinear term enters into the asymptotic equation describing the evolution in Ω_0.

Therefore, in this paper we are able to identify the limit problems for each of the different situations described above and show they are well posed and dissipative. We also study how solutions converge to solutions of the limit problem, that is, we study how solutions become constant in Ω_0 as localized diffusion becomes large. For this, in turn, we show it basically suffices to study convergence of linear elliptic problems and apply several available tools. We also prove upper semicontinuity of the dynamics. Then we prove continuity of extremal equilibria, which are the caps of the global attractors. Finally we discuss the continuity of the asymptotic dynamics under suitable hyperbolicity assumptions.

The paper is organized as follows. In Section 2 we introduce the nonlinear parabolic problems with localized large diffusion to be studied and the resulting limit problems. Section 3 is devoted to analyze the linear elliptic problems. Linear parabolic problems are considered in Section 4. The convergence of solutions of linear elliptic problems are studied in Section 5.1 from which the convergence of the spectrum is developed in Section 5.2. Using these, the convergence of linear parabolic problems is analyzed in Section 5.3. Nonlinear problems are considered in Section 6 where we show the (non standard) limit problem is well posed and dissipative and thus has a global attractor. This attractor is also shown to have upper and lower caps which are extremal equilibria of the limit problem which have some special stability properties. We also give suitable conditions for the extremal equilibria to have constant sign and identify some classes of nonlinearities (including logistic type nonlinearities) for which the maximal equilibria is the unique positive equilibria. Finally, convergence of solutions of nonlinear problems is studied in Section 7. Here we first show that the asymptotic dynamics is upper semicontinuous, proving the same property for the family of global attractors. Then we study the continuity of the extremal equilibria and, under suitable hyperbolicity assumptions, the continuity of all equilibria and the global attractors.

2 Problems

Let Ω be a smooth open connected bounded set of $\mathbb{R}^N$. We consider the nonlinear parabolic problems

$$\left\{\begin{array}{rcll} u^\varepsilon_t - \mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon) + c(x)u^\varepsilon &=& f(x,u^\varepsilon) & \text{in} \quad \Omega \\ d_\varepsilon(x)\dfrac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& g(x,u^\varepsilon) & \text{on} \quad \Gamma_N \\ u^\varepsilon &=& 0 & \text{on} \quad \Gamma_D \\ u^\varepsilon(0) &=& u^\varepsilon_0 & \end{array}\right. \tag{2.1}$$

where Γ_D, Γ_N is a partition of the boundary of Ω, $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$ with $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

Also, let $\Omega_0 \subset \Omega$ such that touches the boundary of Ω, that is

$$\Gamma \cap \overline{\Omega_0} = \Gamma \cap \partial\Omega_0 \neq \emptyset$$

and denote $\Omega_1 = \Omega \setminus \overline{\Omega_0}$ and $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$. Also denote

$$\Gamma^1_D = \Gamma_D \setminus \partial\Omega_0, \quad \Gamma^1_N = \Gamma_N \setminus \partial\Omega_0,$$

hence $\partial\Omega_1 = \Gamma^* \cup \Gamma^1_D \cup \Gamma^1_N$. Notice that either Γ^1_D or Γ^1_N could be empty. For simplicity in the exposition we will assume Ω_0 is connected. We will not make any assumption on the connectedness of Ω_1 though.

We assume that $d_\varepsilon(x)$ are $C^1(\overline{\Omega})$ smooth functions, bounded away from zero, such that

$$0 < m_0 < d_\varepsilon(x) < M_\varepsilon, \quad \text{for all } x \in \Omega; \text{ and} \tag{2.2}$$

$$d_\varepsilon(x) \longrightarrow \left\{\begin{array}{ll} \infty & \text{uniformly in compact sets of } \Omega_0, \\ d_0(x) & \text{uniformly in } \overline{\Omega_1}. \end{array}\right. \tag{2.3}$$

It is expected that, due to large diffusion, u^ε becomes constant in Ω_0 and then in the limit as $\varepsilon \to 0$ we will have to deal with functions of the form

$$u = u_{\Omega_0} \mathcal{X}_{\Omega_0} + u \mathcal{X}_{\Omega_1}$$

where u_{Ω_0} is a constant. Indeed, we are going to show that the solutions of (2.1) approach, as $\varepsilon \to 0$, the solutions of the following limit problems.

First, assuming Ω_0 does not touch the Dirichlet boundary of Ω but touches the flux boundary, that is

$$\Gamma_D \cap \overline{\Omega_0} = \emptyset, \quad \Gamma_N \cap \overline{\Omega_0} \neq \emptyset, \tag{2.4}$$

the limit problem is the parabolic problem in Ω

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}(d_0(x)\nabla u) + c(x)u &=& f(x,u) & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& g(x,u) & \text{on } \Gamma_N^1 \\ u &=& 0 & \text{on } \Gamma_D \\ u &=& u_{\Omega_0} & \text{on } \Gamma^* \\ \dot{u}_{\Omega_0} + \dfrac{1}{|\Omega_0|}\Big[\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0} c + \int_{\Gamma_N \cap \partial\Omega_0} b\Big) u_{\Omega_0}\Big] &=& \dfrac{1}{|\Omega_0|}\Big(\displaystyle\int_{\Omega_0} f(x,u_{\Omega_0}) + \int_{\Gamma_N \cap \partial\Omega_0} g(x,u_{\Omega_0})\Big) & \\ u(0) &=& u_0 & \text{in } \Omega \end{array} \right. \tag{2.5}$$

where $u_{\Omega_0}(t)$ is the constant value of u in Ω_0 and $\vec{n}$ denotes the outward unit normal to the boundary of Ω_1; see Figure 1.

Then, if Ω_0 touches the Dirichlet boundary

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset, \tag{2.6}$$

the limit problem is the parabolic problem in Ω_1

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}(d_0(x)\nabla u) + c(x)u &=& f(x,u) & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& g(x,u) & \text{on } \Gamma_N^1 \\ u &=& 0 & \text{in } \Gamma_D^1 \cup \Gamma^* \\ u(0) &=& u_0 & \end{array} \right. \tag{2.7}$$

extended by zero to Ω_0, that is $u_{\Omega_0} = 0$; see Figure 2.

Observe that (2.7) is mixed Dirichlet–nonlinear boundary value problem as (2.1), but in a reduced domain Ω_1. Also, depending on the configuration of Ω_0, it may happen that $\Gamma_N^1 \cap \Big(\Gamma_D^1 \cup \Gamma^*\Big) \neq \emptyset$.

On the other hand, observe that for (2.5) the coupling between the equation in Ω_0 and Ω_1 is given by the term $\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}}$ that measures the boundary outflow from Ω_1 to Ω_0. Also the two terms with the integral on $\Gamma_N \cap \partial\Omega_0$ reflect the contribution of boundary conditions in Γ_N to the limit problem in the region of large diffusion.

3 Linear elliptic problems

Before continuing we make precise the geometrical and smoothness assumptions we will keep throughout the paper on the domains Ω_0 and Ω_1.

Recall that $\Omega_0 \subset \Omega$ is a connected open set that touches the boundary of Ω

$$\Gamma \cap \overline{\Omega_0} = \Gamma \cap \partial\Omega_0 \neq \emptyset$$

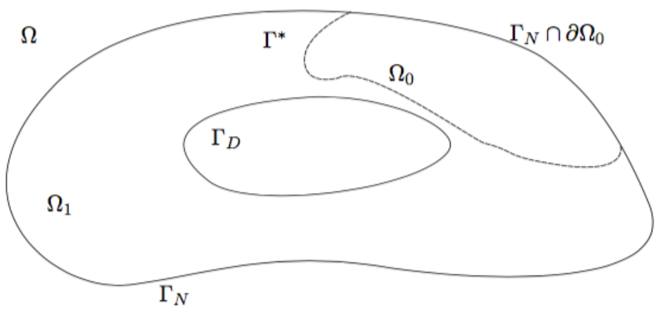

Figure 1: Case $\Gamma_D \cap \overline{\Omega_0} = \emptyset, \quad \Gamma_N \cap \overline{\Omega_0} \neq \emptyset$

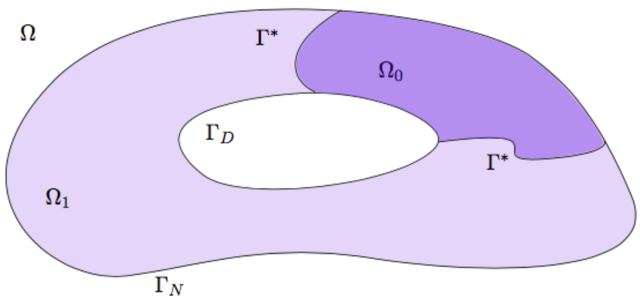

Figure 2: Case $\Gamma_D \cap \overline{\Omega_0} \neq \emptyset$

and denote $\Omega_1 = \Omega \setminus \overline{\Omega_0}$ and

$$\Gamma_D^1 = \Gamma_D \setminus \partial\Omega_0, \quad \Gamma_N^1 = \Gamma_N \setminus \partial\Omega_0, \quad \Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$$

hence $\partial\Omega_0 = \Gamma^* \cup \big(\Gamma_N \cap \partial\Omega_0\big) \cup \big(\Gamma_D \cap \partial\Omega_0\big)$ and $\partial\Omega_1 = \Gamma^* \cup \Gamma_D^1 \cup \Gamma_N^1$.

We will assume that all these boundary sets are smooth enough pieces of $N-1$ dimensional surfaces locally leaving Ω_0 and Ω_1 to one side of themselves and that one can define on them traces of suitable functions as follows.

Let $H^1(\Omega)$ denote the usual Sobolev space. Let

$$H^1_{\Gamma_D}(\Omega) = \big\{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\big\}, \quad H^1_{\Gamma_D}(\Omega_1) = \big\{u \in H^1(\Omega_1) : u = 0 \text{ on } \Gamma_D\big\}.$$

In the case in which $\Gamma_D = \emptyset$ this space is $H^1(\Omega)$. We will keep this notation for the sake of ease in the exposition. We will also consider the set of functions which are constant in Ω_0, that is,

$$L^2_{\Omega_0}(\Omega) = \big\{u \in L^2(\Omega) : u \text{ is constant on } \Omega_0\big\}, \quad H^1_{\Omega_0}(\Omega) = L^2_{\Omega_0}(\Omega) \cap H^1_{\Gamma_D}(\Omega). \tag{3.1}$$

The subspace of function vanishing in Ω_0 will be defined as

$$L^2_{0,\Omega_0}(\Omega) = \big\{u \in L^2(\Omega) : u = 0 \text{ on } \Omega_0\big\}, \quad H^1_{0,\Omega_0}(\Omega) = L^2_{0,\Omega_0}(\Omega) \cap H^1_{\Gamma_D}(\Omega). \tag{3.2}$$

Notice that these spaces are closed subspaces of the ones in (3.1).

With this notation we will assume the following regularity of the boundaries:

(R.1) Functions in $H^1_{\Gamma_D}(\Omega)$, $H^1_{\Gamma_D}(\Omega_1)$, $H^1_{\Omega_0}(\Omega)$ and $H^1_{0,\Omega_0}(\Omega)$ have well defined traces into $H^{1/2}(\Gamma^1_N)$ and $H^{1/2}(\Gamma^*)$ (and null trace into Γ_D). Functions in $H^1_{\Omega_0}(\Omega)$ and $H^1_{0,\Omega_0}(\Omega)$ have constant and null trace, respectively, on Γ^* and $\Gamma_N \cap \partial\Omega_0$.

(R.2) Traces on $\partial\Omega_1$ can be split as traces on Γ^1_N plus traces on Γ^*.

(R.3) Traces of $H^1_{0,\Omega_0}(\Omega)$ are dense in $H^{1/2}(\Gamma^1_N)$.

The dual spaces of the spaces above will be denoted as $H^{-1}(\Omega)$, $H^{-1}_{\Gamma_D}(\Omega)$, $H^{-1}_{\Omega_0}(\Omega)$ and $H^{-1}_{0,\Omega_0}(\Omega)$ respectively.

In order to deal properly with the problems above which include interior and boundary terms we will throughout use the following notation. Whenever O is a regular open set in $\mathbb{R}^N$ and γ is smooth piece of its boundary and f and g are functions defined in O and on γ respectively, we will employ the notation

$$h = f_O + g_\gamma \tag{3.3}$$

to denote the functional defined by

$$\langle h, \varphi \rangle = \int_O f\varphi + \int_\gamma g\varphi$$

for all sufficiently smooth function φ in $\overline{O}$.

3.1 Approximate elliptic problems

With the notation above we consider in this section the linear elliptic problems associated to (2.1), namely

$$\begin{cases} -\mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon) + (\lambda + V(x))u^\varepsilon = f & \text{in} \quad \Omega \\ d_\varepsilon(x)\dfrac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon = g & \text{on} \quad \Gamma_N \\ u^\varepsilon = 0 & \text{on} \quad \Gamma_D \end{cases} \tag{3.4}$$

where $\lambda > 0$ is a positive constant, $V \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_N)$, f and g given in Ω and Γ respectively.

Remark 3.1 *The arguments below are still valid assuming $V \in L^\sigma(\Omega)$, $\sigma > N/2$ and $b \in L^r(\Omega)$, $r > N-1$. We assume further regularity in the sake of clarity in the exposition.*

Let us define

$$Y_\varepsilon = \left\{ z \in H^1_{\Gamma_D}(\Omega) : -\mathrm{div}(d_\varepsilon(x)\nabla z) \in L^2(\Omega) \right\}$$

which is a Hilbert space with the norm $\|z\|^2_{Y_\varepsilon} = \|\mathrm{div}(d_\varepsilon(x)\nabla z)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)}$. Observe that since $d_\varepsilon \in C^1(\overline{\Omega})$ and the lower bound in (2.2) holds, the set Y_ε coincides with $Y = \left\{ z \in H^1_{\Gamma_D}(\Omega),\ \Delta z \in L^2(\Omega) \right\}$ although its norm depends on ε. In Y_ε we can define a normal derivative $d_\varepsilon \frac{\partial u}{\partial \vec{n}} \in H^{-1/2}(\Gamma_N)$ by means of

$$\left\langle d_\varepsilon \frac{\partial u}{\partial \vec{n}}, \gamma(v) \right\rangle_{-1/2,1/2} = \int_\Omega \mathrm{div}(d_\varepsilon(x)\nabla u)v + \int_\Omega d_\varepsilon(x)\nabla u \nabla v \tag{3.5}$$

for any $v \in H^1_{\Gamma_D}(\Omega)$, where γ denotes the trace operator in $H^1_{\Gamma_D}(\Omega)$. The map $u \mapsto d_\varepsilon \frac{\partial u}{\partial \vec{n}}$ is continuous between Y_ε and $H^{-1/2}(\Gamma_N)$, the dual space of $H^{1/2}(\Gamma_N)$.

Then, by integration by parts, weak solutions of (3.4) satisfy

$$\int_\Omega d_\varepsilon(x)\nabla u\nabla\phi + \int_\Omega (\lambda+V(x))u\phi + \int_{\Gamma_N} b(x)u\phi = \int_\Omega f\phi + \int_{\Gamma_N} g\phi, \qquad \forall\phi\in H^1_{\Gamma_D}(\Omega)$$

for any $\phi\in H^1_{\Gamma_D}(\Omega)$. Then Lax–Milgram's theorem gives the following.

Theorem 3.2 *There exists $\lambda_0=\lambda_0(\|V_-\|_{L^\infty(\Omega)},\|b_-\|_{L^\infty(\Gamma_N)})$ such that for $\lambda\geq\lambda_0$, the operator $T_\varepsilon=T_\varepsilon(\lambda)$ defined by the bilinear form*

$$\tau_\varepsilon(u,v)=\langle T_\varepsilon u,v\rangle=\int_\Omega d_\varepsilon(x)\nabla u\nabla v+\int_\Omega(\lambda+V(x))uv+\int_{\Gamma_N} b(x)uv \tag{3.6}$$

is an isomorphism between $H^1_{\Gamma_D}(\Omega)$ and $H^{-1}_{\Gamma_D}(\Omega)$, and between Y_ε and $L^2(\Omega)+H^{-1/2}(\Gamma_N)$. Furthermore, in the latter case, T_ε is given by

$$T_\varepsilon u=\Big(-\operatorname{div}(d_\varepsilon(x)\nabla u)+(\lambda+V(x))u\Big)_\Omega+\Big(d_\varepsilon(x)\frac{\partial u}{\partial\vec n}+b(x)u\Big)_{\Gamma_N}.$$

By restriction to $L^2(\Omega)$, T_ε induces a positive selfadjoint operator with compact resolvent, $B_\varepsilon=B_\varepsilon(\lambda)$, and domain

$$D(B_\varepsilon)=\Big\{u\in H^1_{\Gamma_D}(\Omega): -\operatorname{div}(d_\varepsilon\nabla u)\in L^2(\Omega),\ d_\varepsilon\frac{\partial u}{\partial\vec n}+bu=0 \text{ on } \Gamma_N\Big\}\subset Y_\varepsilon.$$

Moreover, for $u\in D(B_\varepsilon)$, $B_\varepsilon u=-\operatorname{div}(d_\varepsilon(x)\nabla u)+(\lambda+V(x))u$.

Proof. It is enough to notice that for every $\eta>0$ we can use that $\|u\|^2_{L^2(\Gamma)}\leq\eta\|u\|^2_{H^1(\Omega)}+C_\eta\|u\|^2_{L^2(\Omega)}$ to get

$$\tau_\varepsilon(u,u)\geq m_0\int_\Omega|\nabla u|^2+\int_\Omega(\lambda-V_-)|u|^2-\int_{\Gamma_N} b_-|u|^2. \tag{3.7}$$

The rest is standard. ■

Remark 3.3 *Notice in particular that for $\lambda\in\mathbb{R}$, using the notation in (3.3), if $u\in H^1_{\Gamma_D}(\Omega)$ is a solution of*

$$T_\varepsilon u=h=f_\Omega+g_{\Gamma_N}\in L^2(\Omega)+H^{-1/2}(\Gamma_N)$$

(which exists at least for λ large enough) then $u\in Y_\varepsilon$ and is a weak solution of (3.4).

In the following, we will not make any essential distinction between T_ε and B_ε.

3.2 Elliptic limit problems

In this section, we study the limit problems of those considered in the previous section: Dirichlet-Diffusion and Robin-Diffusion interaction. As we show, the large difussion induces a spatial homogenization in Ω_0 which makes the solution of the limit problem, as $\varepsilon\to 0$, become constant (zero in the Dirichlet-Diffusion case). This process affects in a different manner to each of the cases considered.

Therefore, to set the limit problem we will consider the spaces (3.1), which reflect this homogenization process, and

$$Y_{\Omega_0}=\left\{z\in H^1_{\Omega_0}(\Omega): -\operatorname{div}(d_0(x)\nabla z)\in L^2(\Omega_1)\right\},$$

which is a Hilbert space with the norm $\|z\|^2_{Y_{\Omega_0}} = \|\mathrm{div}(d_0(x)\nabla z)\|^2_{L^2(\Omega_1)} + \|z\|^2_{L^2(\Omega)}$. Also, as in (3.5), the normal derivative on $\partial\Omega_1$ can be defined by means of

$$\left\langle d_0 \frac{\partial u}{\partial \vec{n}}, \gamma(v) \right\rangle_{-1/2,1/2} = \int_{\Omega_1} \mathrm{div}(d_0(x)\nabla u)v + \int_{\Omega_1} d_0(x)\nabla u \nabla v \tag{3.8}$$

for any $v \in H^1_{\Gamma_D}(\Omega_1)$. Thus $d_0 \frac{\partial u}{\partial \vec{n}} \in H^{-1/2}(\Gamma^1_N \cup \Gamma^*)$.

We will denote by u_{Ω_0} the constant value that the elements in the spaces defined above take in Ω_0. Hence a function in $L^2_{\Omega_0}(\Omega)$ can be written as

$$u = u_{\Omega_0}\mathcal{X}_{\Omega_0} + u\mathcal{X}_{\Omega_1}.$$

3.2.1 Flux-Diffusion interaction

In this section we assume (2.4). The following result analogous to Theorem 3.2 states the existence of solution for the limit problem in the case of flux-diffusion interaction

Theorem 3.4 *There exits $\lambda_0 = \lambda_0(\|V_-\|_{L^\infty(\Omega)}, \|b_-\|_{L^\infty(\Gamma_N)})$ such that for $\lambda \geq \lambda_0$, the operator $T_0 = T_0(\lambda)$ defined by the bilinear form*

$$\tau_0(u,v) = \int_{\Omega_1} d_0(x)\nabla u \nabla v + \int_{\Omega_1} (\lambda + V(x))uv + \int_{\Gamma^1_N} b(x)uv + u_{\Omega_0}v_{\Omega_0}\Big(\int_{\Omega_0}(\lambda+V) + \int_{\Gamma_N\cap\partial\Omega_0} b\Big) \tag{3.9}$$

as $\tau_0(u,v) = \langle T_0 u, v\rangle$ for $u, v \in H^1_{\Omega_0}(\Omega)$, is an isomorphism between $H^1_{\Omega_0}(\Omega)$ and its dual space which we denote by $H^{-1}_{\Omega_0}(\Omega)$ and, by restriction, between Y_{Ω_0} and $L^2_{\Omega_0}(\Omega) + H^{-1/2}(\Gamma^1_N) \subset H^{-1}_{\Omega_0}(\Omega)$. In the latter case it is given by

$$\begin{aligned} T_0 u &= \frac{1}{|\Omega_0|}\Big[\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0}(\lambda+V) + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)u_{\Omega_0}\Big]\mathcal{X}_{\Omega_0} \\ &\quad + \Big(-\mathrm{div}(d_0(x)\nabla u) + (\lambda + V(x))u\Big)\mathcal{X}_{\Omega_1} + \Big(d_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u\Big)_{\Gamma^1_N} \end{aligned} \tag{3.10}$$

where u_{Ω_0} is the constant value of u on Ω_0 and $d_0\frac{\partial u}{\partial \vec{n}}$ is the normal derivative of u in the direction of the unit outwards normal vector in Ω_1. Furthermore the restriction of T_0 to $L^2_{\Omega_0}(\Omega)$ induces a positive unbounded self-adjoint operator, $B_0 = B_0(\lambda)$, with compact resolvent and domain

$$D(B_0) = \left\{ u \in H^1_{\Omega_0}(\Omega) : -\mathrm{div}(d_0(x)\nabla u) \in L^2_{\Omega_0}(\Omega) \ \textit{and}\ d_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0 \ \textit{on}\ \Gamma^1_N \right\},$$

and for $u \in D(B_0)$ we have

$$\begin{aligned} B_0 u &= \frac{1}{|\Omega_0|}\Big[\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0}(\lambda+V) + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)u_{\Omega_0}\Big]\mathcal{X}_{\Omega_0} \\ &\quad + \Big(-\mathrm{div}(d_0(x)\nabla u) + (\lambda + V(x))u\Big)\mathcal{X}_{\Omega_1} \end{aligned} \tag{3.11}$$

Proof. Notice that, extending d_0 to Ω_0 with value, say, equal to 1, the bilinear form τ_0 is the restriction to $H^1_{\Omega_0}(\Omega)$ of the continuous bilinear form in $H^1_{\Gamma_D}(\Omega)$

$$\tilde{\tau}_0(u,v) = \int_{\Omega} d_0(x)\nabla u \nabla v + \int_{\Omega} (\lambda + V(x))uv + \int_{\Gamma_N} b(x)uv.$$

This bilinear form is symmetric, continuous and coercive in $H^1_{\Omega_0}(\Omega)$ for λ as in the statement. In that case, we can use Lax–Milgram theorem.

Now take $h = f_\Omega + g_{\Gamma^1_N} \in L^2_{\Omega_0}(\Omega) + H^{-1/2}(\Gamma^1_N) \subset H^{-1}_{\Omega_0}(\Omega)$, as in (3.3), and $u \in H^1_{\Omega_0}(\Omega)$ such that $T_0(u) = h$. We take first a test function $v \in C^\infty_c(\Omega_1)$ which gives $-\text{div}(d_0(x)\nabla u) + (\lambda + V(x))u = f$ in Ω_1 and, in particular, $u \in Y_{\Omega_0}$.

Now we take $v \in H^1_{0,\Omega_0}(\Omega)$ as in (3.2) and using (3.8) and the regularity assumptions **(R.1)** and **(R.2)** gives

$$\left\langle d_0 \frac{\partial u}{\partial \vec{n}}, \gamma(v)_{|\Gamma^1_N} \right\rangle_{-1/2,1/2} + \int_{\Gamma^1_N} buv = \left\langle g, v_{|\Gamma^1_N} \right\rangle.$$

Now **(R.3)** implies $d_0 \frac{\partial u}{\partial \vec{n}} + bu = g$ in $H^{-1/2}(\Gamma^1_N)$.

Finally we take $v \in H^1_{\Omega_0}(\Omega)$ and use again (3.8), **(R.1)** and **(R.2)** and the informations above to obtain

$$\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0} (\lambda + V) + \int_{\Gamma_N \cap \partial\Omega_0} b \Big) u_{\Omega_0} = |\Omega_0| f_{\Omega_0}.$$

In particular, we get (3.10). ■

Remark 3.5 *Observe that we have actually proved above that, using the notation in (3.3), if $\lambda \in \mathbb{R}$ and $u \in H^1_{\Omega_0}(\Omega)$ a solution of*

$$T_0 u = f_\Omega + g_{\Gamma^1_N} \in L^2_{\Omega_0}(\Omega) + H^{-1/2}(\Gamma^1_N)$$

with $f = f_{\Omega_0}\mathcal{X}_{\Omega_0} + f\mathcal{X}_{\Omega_1}$, (which exists at least for large λ), then $u \in Y_{\Omega_0}$ is a weak solution of

$$\left\{ \begin{array}{rcll} -\text{div}(d_0(x)\nabla u) + (\lambda + V(x))u &=& f & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& g & \text{on } \Gamma^1_N \\ u &=& 0 & \text{on } \Gamma_D \\ u &=& u_{\Omega_0} & \text{on } \Gamma^* \\ \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0} (\lambda + V) + \int_{\Gamma_N \cap \partial\Omega_0} b \Big) u_{\Omega_0} \right] &=& f_{\Omega_0} & \end{array} \right. \tag{3.12}$$

We will make no essential distinction between T_0 and B_0.

3.2.2 Dirichlet-Diffusion interaction

In this case, we assume, as in (2.6) that Ω_0 touches the Dirichlet boundary of Ω

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset.$$

Now since the solution vanishes in part of the boundary of Ω_0, the solution of the limit problem must be 0 in Ω_0, that is, $u_{\Omega_0} = 0$. Hence, we will consider the spaces (3.2) and

$$Y_{0,\Omega_0} = \left\{ z \in H^1_{\Omega_0}(\Omega) : -\text{div}(d_0(x)\nabla z) \in L^2(\Omega_1) \right\}.$$

In an analogous way to Theorem 3.4, we get now the following.

Theorem 3.6 *There exits $\lambda_0 = \lambda_0(\|V_-\|_{L^\infty(\Omega_1)}, \|b_-\|_{L^\infty(\Gamma^1_N)})$ such that for $\lambda \geq \lambda_0$, the operator $T_{0,0} = T_{0,0}(\lambda)$ defined by the bilinear form*

$$\tau_{0,0}(u, v) = \langle T_0 u, v \rangle = \int_{\Omega_1} d_0(x)\nabla u \nabla v + \int_{\Omega_1} (\lambda + V(x))uv + \int_{\Gamma^1_N} b(x)uv \tag{3.13}$$

for $u, v \in H^1_{0,\Omega_0}(\Omega)$ is an isomorphism between $H^1_{0,\Omega_0}(\Omega)$ and its dual space which we denote by $H^{-1}_{0,\Omega_0}(\Omega)$ and, by restriction, between Y_{0,Ω_0} and $L^2_{0,\Omega_0}(\Omega)+H^{-1/2}(\Gamma^1_N) \subset H^{-1}_{0,\Omega_0}(\Omega)$. In the latter case it is given by

$$T_{0,0}u = \Big(-\operatorname{div}(d_0(x)\nabla u) + (\lambda + V(x))u \Big)_{\Omega_1} + \Big(d_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u \Big)_{\Gamma^1_N}.$$

Furthermore the restriction of $T_{0,0}$ to $L^2_{0,\Omega_0}(\Omega)$ induces a positive unbounded self-adjoint operator, $B_{0,0} = B_{0,0}(\lambda)$, with compact resolvent and domain

$$D(B_{0,0}) = \Big\{ u \in H^1_{0,\Omega_0}(\Omega) : -\operatorname{div}(d_0(x)\nabla u) \in L^2_{0,\Omega_0}(\Omega) \text{ and } d_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0 \text{ on } \Gamma^1_N \Big\},$$

and for $u \in D(B_{0,0})$, $B_{0,0}(u) = \Big(-\operatorname{div}(d_0(x)\nabla u) + (\lambda + V)u \Big)\mathcal{X}_{\Omega_1}$.

Proof. Note that $\tau_{0,0}$ is the restriction to $H^1_{0,\Omega_0}(\Omega)$ of the bilinear form τ_0 in $H^1_{\Omega_0}(\Omega)$ in Theorem 3.4 and is symmetric continuous and coercive for λ as in the statement.

Also, as in the proof of Theorem 3.4 if $h = f_\Omega + g_{\Gamma^1_N} \in L^2_{0,\Omega_0}(\Omega)+H^{-1/2}(\Gamma^1_N) \subset H^{-1}_{0,\Omega_0}(\Omega)$ and $u \in H^1_{0,\Omega_0}(\Omega)$ is such that $T_{0,0}(u) = h$, taking test functions $v \in C^\infty_c(\Omega_1)$, we get $-\operatorname{div}(d_0(x)\nabla u)+(\lambda + V(x))u = f$ in Ω_1 and, in particular, $u \in Y_{0,\Omega_0}$.

Also, taking test functions $v \in H^1_{0,\Omega_0}(\Omega)$ gives again $d_0 \frac{\partial u}{\partial \vec{n}} + bu = g$ in $H^{-1/2}(\Gamma^1_N)$. ■

Remark 3.7 *If $\lambda \in \mathbb{R}$ and $u \in H^1_{\Omega_0}(\Omega)$ is a solution of*

$$T_{0,0}u = f_\Omega + g_{\Gamma^1_N} \in L^2_{0,\Omega_0}(\Omega) + H^{-1/2}(\Gamma^1_N)$$

(which exists at least for λ large enough), then $u \in Y_{0,\Omega_0}$ and is a weak solution of

$$\begin{cases} -\operatorname{div}(d_0(x)\nabla u) + (\lambda + V(x))u = f & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = g & \text{on } \Gamma^1_N \\ u = 0 & \text{in } \Omega_0 \\ u = 0 & \text{on } \Gamma_D \cup \Gamma^*. \end{cases} \tag{3.14}$$

Also notice that the operators $T_{0,0}$ and $B_{0,0}$ above (and their domains) are nothing but the restrictions of the operators T_0 and B_0 to functions that vanish in Ω_0.

We will make no essential distinction between $T_{0,0}$ and $B_{0,0}$.

3.3 Elliptic regularity and maximum principles

One of the main features of the elliptic problem (3.4) are the well known regularity and comparison properties, see e.g. [1, 18, 27, 25, 17, 26]. Most of these results apply as well to (3.14), despite it may happen that $\Gamma^1_N \cap \Big(\Gamma_D \cup \Gamma^*\Big) \neq \emptyset$.

In this section we show that the limit problem (3.12) also share similar features. For this, with the notation above, fix $\lambda = \lambda_0$, with λ_0 as in either Theorem 3.2, 3.4 or 3.6 and denote $B_\varepsilon = B_\varepsilon(\lambda_0)$, $B_0 = B_0(\lambda_0)$ and $B_{0,0} = B_{0,0}(\lambda_0)$ (which we will not distinguish from T_ε, T_0 or $T_{0,0}$). Note that restricted to $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively, these operators are selfadjoint, positive and have compact resolvent.

Also recall that we say that a form $h \in H^{-1}_{\Gamma_D}(\Omega)$, or $h \in H^{-1}_{\Omega_0}(\Omega)$ or $h \in H^{-1}_{0,\Omega_0}(\Omega)$, is nonnegative, $h \geq 0$, if for all nonnegative test function $0 \leq \varphi \in H^1_{\Gamma_D}(\Omega)$, or $H^1_{\Omega_0}(\Omega)$ or $H^1_{0,\Omega_0}(\Omega)$ respectively we have $\langle h, \varphi \rangle \geq 0$.

Proposition 3.8 *i) If $h \geq 0$ as above then the solution of $B_\varepsilon(u) = h$, $B_0(u) = h$ or $B_{0,0}(u) = h$, satisfies $u \geq 0$.*
ii) The operators $B_\varepsilon, B_0, B_{0,0}$ above have order preserving resolvent in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively.
iii) If $f \in L^p(\Omega)$ with $p > N/2$ and $g \in L^r(\Gamma_N)$ with $r > N-1$ the the solutions of either one of the problems (3.4), (3.12), (3.14) satisfies $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ and

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_N)})$$

for a constant depending on $\Omega, d_\varepsilon, d_0, \lambda, \|b\|_{L^\infty(\Gamma_N)}, \|V\|_{L^\infty(\Omega)}$.
iv) The eigenfunctions of B_ε, B_0 or $B_{0,0}$ are Hölder continuous in $\overline{\Omega}$. The first eigenvalue of (3.4) is positive and simple with a positive associated eigenfunction. The same applies to (3.12) and (3.14) if Ω_1 is connected.

Proof. Recall the associated bilinear forms τ_ε, τ_0 and $\tau_{0,0}$ as in Theorems 3.2, 3.4 or 3.6.
i) Observe that if $v \in H^1_{\Gamma_D}(\Omega)$, or $H^1_{\Omega_0}(\Omega)$ or $H^1_{0,\Omega_0}(\Omega)$ respectively then the same holds for the negative part of v, $v^- = \max\{-v, 0\}$ and then if $B_\varepsilon u = h$ we get

$$\tau_\varepsilon(-u^-, -u^-) = \tau_\varepsilon(u, -u^-) = \langle h, -u^- \rangle \leq 0.$$

Since the bilinear form is coercive we get $u^- = 0$ and then $u \geq 0$. The other problems are treated in a similar way.
ii) Observe that if $v \in H^1_{\Gamma_D}(\Omega)$, or $H^1_{\Omega_0}(\Omega)$ or $H^1_{0,\Omega_0}(\Omega)$ respectively, then the same holds for $|v|$ and $\tau_\varepsilon(|v|, |v|) = \tau_\varepsilon(v, v)$, $\tau_0(|v|, |v|) = \tau_0(v, v)$ or $\tau_{0,0}(|v|, |v|) = \tau_{0,0}(v, v)$ respectively. Then the result follow from Theorem 1.3.2 in [16].
iii) For (3.4) and (3.14) the result follow directly from part vi) in Lemma B.1 in [4]. On the other hand, for (3.12), restricting the problem to Ω_1 notice that $u = u_{\Omega_0}$ in Γ^*, which is a Hölder function and then again part vi) in Lemma B.1 in [4] concludes $u \in C^\alpha(\overline{\Omega_1})$ and hence $u \in C^\alpha(\overline{\Omega})$.
iv) For the Hölder regularity of eigenfunctions, note that they satisfy $B_\varepsilon(u) = f$, $B_0(u) = f$ or $B_{0,0}(u) = f$ with $f = \mu u$. Then bootstrap arguments using part i) in Lemma B.1 in [4] gives that $f = \mu u \in L^p(\Omega)$ for some $p > N/2$ and then part iii) concludes.

Now notice that the first eigenvalues satisfy

$$\Lambda_\varepsilon = \inf_{v \in H^1_{\Gamma_D}(\Omega)} \frac{\tau_\varepsilon(v,v)}{\|v\|_{L^2(\Omega)}}, \quad \Lambda_0 = \inf_{v \in H^1_{\Omega_0}(\Omega)} \frac{\tau_0(v,v)}{\|v\|_{L^2(\Omega)}}, \quad \Lambda_{0,0} = \inf_{v \in H^1_{0,\Omega_0}(\Omega)} \frac{\tau_0(v,v)}{\|v\|_{L^2(\Omega)}}$$

and they are attained on the associated eigenfunctions. From the property of the quadratic forms in part ii) we, have that if φ is an eigenfunction then so is $|\varphi|$.

For (3.4) the strong maximum principle implies that then $|\varphi| > 0$ in Ω and then Λ_ε is simple. For (3.12) and (3.14) the same applies in Ω_1 if Ω_1 is connected, see Proposition 3.9 below, and then $\Lambda_0, \Lambda_{0,0}$ are simple. ■

Now we prove the strong maximum principle used above, in a version that is suitable to our purposes in this paper. Much finer versions can be found in [30, 28], from which we borrow the key techniques. Note that no regularity is assumed on the open set $\mathcal{O}$.

Proposition 3.9 *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded open set and let $d_0 \in L^\infty(\mathcal{O})$ with $\inf_{\mathcal{O}} d_0 > 0$ and $V \in L^p(\mathcal{O})$ with $p > N/2$. Finally assume $0 \leq u \in H^1(\mathcal{O})$ satisfies*

$$\int_{\mathcal{O}} d_0(x) \nabla u \nabla v + \int_{\mathcal{O}} V(x) u v \geq 0 \tag{3.15}$$

for all $0 \leq v \in H^1_0(\mathcal{O})$.

Then u is either zero or strictly positive a.e. in each connected component of $\mathcal{O}$.

Proof. Since $u \geq 0$ a.e. in $\mathcal{O}$, assume $u \neq 0$ and $u = 0$ a.e. in $A \subset \mathcal{O}$ with $|A| > 0$. Then for any $\varphi \in C_c^\infty(\mathcal{O})$ consider the function $v = \frac{\varphi^2}{1+u}$. Since $0 \leq \frac{1}{1+u} \leq 1$ then $v \in L^2(\mathcal{O})$ and clearly

$$\nabla\Big(\frac{\varphi^2}{1+u}\Big) = -\frac{\varphi^2 \nabla u}{(1+u)^2} + 2\frac{\varphi\nabla\varphi}{1+u}$$

where both terms in the right hand side belong to $L^2(\mathcal{O})$. Hence $v \in H_0^1(\mathcal{O})$ and taking this test function in (3.15) gives

$$\int_{\mathcal{O}} d_0(x)\frac{|\nabla u|^2}{(1+u)^2}\varphi^2 \leq \int_{\mathcal{O}} 2d_0(x)\frac{\varphi}{1+u}\nabla\varphi\nabla u + \int_{\mathcal{O}} V(x)\frac{u}{1+u}\varphi^2.$$

Using the upper and lower bounds on d_0, $0 \leq \frac{u}{1+u} \leq 1$ and Young's inequality, we have

$$\int_{\mathcal{O}} 2d_0(x)\frac{\varphi}{1+u}\nabla\varphi\nabla u \leq \varepsilon \int_{\mathcal{O}} \frac{|\nabla u|^2}{(1+u)^2}\varphi^2 + C_\varepsilon \int_{\mathcal{O}} |\nabla\varphi|^2$$

that leads to

$$\int_{\mathcal{O}} \frac{|\nabla u|^2}{(1+u)^2}\varphi^2 \leq C(\varphi) = c\int_{\mathcal{O}} |\nabla\varphi|^2 + c\int_{\mathcal{O}} V^+(x)\varphi^2,$$

for all $\varphi \in C_c^\infty(\mathcal{O})$.

Observe now that $0 \leq \log(1+u) \leq u$ implies $\log(1+u) \in L^2(\mathcal{O})$ and $\nabla \log(1+u) = \frac{\nabla u}{(1+u)} \in L^2(\mathcal{O})$, hence $\log(1+u) \in H^1(\mathcal{O})$. Thus, the last inequality above can be written as

$$\int_{\mathcal{O}} |\nabla \log(1+u)|^2\varphi^2 \leq C(\varphi), \qquad \varphi \in C_c^\infty(\mathcal{O}).$$

Then we can take any smooth open connected subset $\omega \Subset \mathcal{O}$ such that $|A \cap \omega| > 0$, to obtain

$$\int_\omega |\nabla \log(1+u)|^2 \leq C(\omega).$$

Since $\log(1+u) = 0$ a.e. $A \cap \omega$ then the Poincaré inequality, see Lemma 3.10 below, leads to

$$\int_\omega |\log(1+u)|^2 \leq C(\omega)$$

where $C(\omega)$ does not depend on u.

Setting $F(s) = |\log(1+s)|^2$, which is increasing, and $F(\infty) = \infty$, observe that we can apply the inequality above to $\frac{u}{\varepsilon}$ for any $\varepsilon > 0$ and then we get

$$\int_\omega F(\frac{u}{\varepsilon}) \leq C(\omega).$$

Chebyshev inequality then leads to

$$F(\frac{t}{\varepsilon})|\{u > t\} \cap \omega| \leq C(\omega)$$

for any $t > 0$. Letting $\varepsilon \to 0$ gives $|\{u > t\} \cap \omega| = 0$ for any $t > 0$. That is $u = 0$ a.e. $\omega \Subset \mathcal{O}$ and hence in any connected component of $\mathcal{O}$ that intersects A with positive measure. ■

Now we prove the Poincaré inequality used above.

Lemma 3.10 *Assume ω is a smooth connected open set and $A \subset \omega$ such that $|A| > 0$. Then there exists a positive constant such that for all $u \in H^1(\omega)$ such that $u = 0$ a.e. in A*

$$\int_\omega |\nabla u|^2 \geq c \int_\omega |u|^2.$$

Proof. If the inequality is not true then there exists a bounded sequence $\{u_m\}_m \subset H^1(\omega)$ such that $u_m = 0$ a.e. in A with $\int_\omega |u_m|^2 = 1$ for all $m \in \mathbb{N}$ and $\int_\omega |\nabla u_m|^2 \to 0$. Since ω is regular, the embedding $H^1(\omega) \subset L^2(\omega)$ is compact, and we can assume that u_m converges in $L^2(\omega)$, a.e. in ω and weakly in $H^1(\omega)$ to $u \in H^1(\omega)$ with $\int_\omega |u|^2 = 1$. Then, by weak lower semicontinuity, we get $\int_\omega |\nabla u|^2 = 0$ and then $u = \frac{1}{|\omega|^{1/2}}$ in ω, which contradicts that $u = 0$ a.e. in A. ■

4 Linear parabolic problems

With the notation above, fix $\lambda = \lambda_0$, with λ_0 as in either Theorem 3.2, 3.4 or 3.6. Then the operators $B_\varepsilon = B_\varepsilon(\lambda_0)$, $B_0 = B_0(\lambda_0)$ and $B_{0,0} = B_{0,0}(\lambda_0)$ are selfadjoint, positive in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively. Moreover these operators have compact resolvent. In particular, they are sectorial and define asymptotically decaying analytic semigroups $e^{-B_\varepsilon t}$, $e^{-B_0 t}$, $e^{-B_{0,0} t}$ in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively; see [1, 22].

Also, they have associated the corresponding family of fractional power spaces X^α_ε, X^α_0, $X^\alpha_{0,0}$ for $\alpha \in \mathbb{R}$, which satisfy, for $\alpha \geq 0$, $X^{-\alpha}_\varepsilon = \left(X^\alpha_\varepsilon\right)'$, $X^{-\alpha}_0 = \left(X^\alpha_0\right)'$, $X^{-\alpha}_{0,0} = \left(X^\alpha_{0,0}\right)'$. In particular

$$X^0_\varepsilon = L^2(\Omega),\ X^0_0 = L^2_{\Omega_0}(\Omega),\ X^0_{0,0} = L^2_{0,\Omega_0}(\Omega),$$

$$X^{1/2}_\varepsilon = H^1_{\Gamma_D}(\Omega),\ \text{with norm } \|u\|^2_{1/2,\varepsilon} = \tau_\varepsilon(u,u),$$

$$X^{1/2}_0 = H^1_{\Omega_0}(\Omega),\ \text{with norm } \|u\|^2_{1/2,0} = \tau_0(u,u),$$

$$X^{1/2}_{0,0} = H^1_{0,\Omega_0}(\Omega),\ \text{with norm } \|u\|^2_{1/2,0,0} = \tau_{0,0}(u,u).$$

Even more for $-1 \leq \beta \leq \alpha \leq 1$, we have for $t > 0$,

$$\|e^{-B_\varepsilon t}\|_{\mathcal{L}(X^\beta_\varepsilon, X^\alpha_\varepsilon)} \leq \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t}, \quad \|e^{-B_0 t}\|_{\mathcal{L}(X^\beta_0, X^\alpha_0)} \leq \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t}, \quad \|e^{-B_{0,0} t}\|_{\mathcal{L}(X^\beta_{0,0}, X^\alpha_{0,0})} \leq \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t},$$

for some $\omega > 0$ and $M_\omega = M_{\omega,\beta,\alpha}$ independent of ε.

To see this, note that $-B_\varepsilon$, $-B_0$ and $-B_{0,0}$ are selfadjoint with numerical range in $(-\infty, -\omega)$ for some $\omega > 0$ independent of ε. Then from Theorem 1.3.9 in [29] we can obtain uniform estimates on the resolvent operators on any closed sector not intersecting the numerical range. From this and Theorems 1.3.4 and 1.4.3 in [22] we get the uniform estimate for M_ω.

Now from the proof of Theorem 1.4.4 in [22] we get that the constant in the interpolation inequality

$$\|u\|_{\gamma,\varepsilon} \leq M \|u\|^\theta_{\alpha,\varepsilon} \|u\|^{1-\theta}_{\beta,\varepsilon}$$

for $-1 \leq \beta \leq \gamma \leq \alpha \leq 1$, and $\gamma = \theta\alpha + (1-\theta)\beta$ can be taken independent of ε.

Finally, observe that the uniform coercitivity of the forms in Theorem 3.2 imply that

$$X^0_{0,0} \subset X^0_0 \subset X^0_\varepsilon = L^2(\Omega), \qquad X^{1/2}_{0,0} \subset X^{1/2}_0 \subset X^{1/2}_\varepsilon \subset H^1(\Omega)$$

with embedding constants independent of ε. Then from Theorem 1.15.3 in [36] and Remark 1.15.2 in [2] (characterizing fractional power spaces as interpolation ones) we get that for $0 \leq \alpha \leq 1/2$,

$$X^\alpha_{0,0} \subset X^\alpha_0 \subset X^\alpha_\varepsilon \subset H^{2\alpha}(\Omega)$$

with embedding constants independent of ε. By duality,

$$H^{-2\alpha}(\Omega) := (H^{2\alpha}(\Omega))' \hookrightarrow X_\varepsilon^{-\alpha} \hookrightarrow X_0^{-\alpha} \hookrightarrow X_{0,0}^{-\alpha}$$

for $0 < \alpha \le 1/2$ with embedding constants independent of ε.

Then for $\lambda \in \mathbb{R}$ we define the selfadjoint operators

$$A_\varepsilon = B_\varepsilon + (\lambda - \lambda_0)I, \quad A_0 = B_0 + (\lambda - \lambda_0)I, \quad A_{0,0} = B_{0,0} + (\lambda - \lambda_0)I \tag{4.1}$$

in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively. These operators induce analytic semigroups with the same fractional power spaces as above X_ε^α, X_0^α, $X_{0,0}^\alpha$ for $\alpha \in \mathbb{R}$ and satisfy for $-1 \le \beta \le \alpha \le 1$ and $t > 0$,

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon^\beta, X_\varepsilon^\alpha)} \le \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t}, \quad \|e^{-A_0 t}\|_{\mathcal{L}(X_0^\beta, X_0^\alpha)} \le \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t}, \quad \|e^{-A_{0,0} t}\|_{\mathcal{L}(X_{0,0}^\beta, X_{0,0}^\alpha)} \le \frac{M_\omega}{t^{\alpha-\beta}} e^{-\omega t},$$

for some $\omega(\lambda) \in \mathbb{R}$ and $M_\omega = M_{\omega,\beta,\alpha}(\lambda)$ independent of ε.

Notice that from Theorems 3.2, 3.4 or 3.6 for $u_0^\varepsilon \in X_\varepsilon^\alpha$, $\alpha \in \mathbb{R}$, we have that $u^\varepsilon(t; u_0^\varepsilon) = e^{-A_\varepsilon t} u_0^\varepsilon$ is a solution of

$$\left\{ \begin{array}{rcll} u_t^\varepsilon - \operatorname{div}(d_\varepsilon(x)\nabla u^\varepsilon) + (\lambda + V(x))u^\varepsilon &=& 0 & \text{in } \Omega, \ t > 0 \\ d_\varepsilon(x)\dfrac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon &=& 0 & \text{on } \Gamma_N \\ u^\varepsilon &=& 0 & \text{on } \Gamma_D \\ u^\varepsilon(0) &=& u_0^\varepsilon & \text{in } \Omega. \end{array} \right. \tag{4.2}$$

On the other hand, for $u_0 \in X_0^\alpha$, $\alpha \in \mathbb{R}$, we have that $u(t; u_0) = e^{-A_0 t} u_0$ is a solution of

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}(d_0(x)\nabla u) + (\lambda + V(x))u &=& 0 & \text{in } \Omega_1, \ t > 0 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& 0 & \text{on } \Gamma_N^1 \\ u &=& 0 & \text{on } \Gamma_D \\ u &=& u_{\Omega_0} & \text{on } \Gamma^* \\ \dot{u}_{\Omega_0} + \dfrac{1}{|\Omega_0|}\Big[\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0} (\lambda + V) + \int_{\Gamma_N \cap \partial\Omega_0} b\Big)\Big] u_{\Omega_0} &=& 0 & \\ u(0) &=& u_0 & \text{in } \Omega. \end{array} \right. \tag{4.3}$$

Then, in case (2.6), that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

for $u_0 \in X_{0,0}^\alpha$, $\alpha \in \mathbb{R}$, we have that $u(t; u_0) = e^{-A_{0,0} t} u_0$ is a solution of

$$\left\{ \begin{array}{rcll} u_t - \operatorname{div}(d_0(x)\nabla u) + (\lambda + V(x))u &=& 0 & \text{in } \Omega_1, \ t > 0 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& 0 & \text{on } \Gamma_N^1 \\ u &=& 0 & \text{on } \Gamma_D^1 \cup \Gamma^* \\ u &=& 0 & \text{in } \Omega_0 \\ u(0) &=& u_0 & \text{in } \Omega. \end{array} \right. \tag{4.4}$$

Remark 4.1 *The semigroups above enjoy several comparison properties. First, from part ii) in Proposition 3.8 and Theorem 1.3.2 in [16] we get at once that $e^{-B_\varepsilon t}$, $e^{-B_0 t}$, $e^{-B_{0,0} t}$ are order preserving in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ and $L^2_{0,\Omega_0}(\Omega)$ respectively.*

Also recall that for linear forms in $H^{-1}_{\Gamma_D}(\Omega)$, $H^{-1}_{\Omega_0}(\Omega)$ or $H^{-1}_{0,\Omega_0}(\Omega)$, we say $h_1 \le h_2$ iff $h_2 - h_1 \ge 0$. This defines an order in $H^{-1}_{\Gamma_D}(\Omega)$, $H^{-1}_{\Omega_0}(\Omega)$ or $H^{-1}_{0,\Omega_0}(\Omega)$ which is consistent with the ordering in $L^2(\Omega)$, $L^2_{\Omega_0}(\Omega)$ or $L^2_{0,\Omega_0}(\Omega)$ respectively and for which the cone on nonnegative functions in *the latter spaces are dense in the cone of nonnegative forms in the former ones. To see this, note that if for example* $0 \le h \in H^{-1}_{\Omega_0}(\Omega)$ *then* $0 \le u(t) = e^{-B_0 t}h \in L^2_{\Omega_0}(\Omega)$ *and* $u(t) \to h$ *as* $t \to 0$ *in* $H^{-1}_{\Omega_0}(\Omega)$. *Notice that* $0 \le u(t) = e^{-B_0 t}h$ *follows by using* $-u^-(t)$ *as a test function. The result for* $e^{-B_\varepsilon t}$ *and* $e^{-B_{0,0} t}$ *are well known but can be also be obtained along the same lines.*

From this it is not difficult to see that the family of spaces X^α_ε, X^α_0, $X^\alpha_{0,0}$ *for* $\alpha \in \mathbb{R}$, *are scales of ordered spaces as in Definition A.8 in [4]. Therefore all the abstract comparison results in that reference hold for the problems in this paper.*

These results will be used below for the corresponding nonlinear parabolic problems in Section 6.

5 Convergence of linear problems

5.1 Elliptic problems

Now we show that weak solutions of problems (3.4) converge in certain sense, as $\varepsilon \to 0$ to weak solutions of problems (3.12) or (3.14), depending on the configuration of the region of large diffusion.

First we state the following general result, irrespective of the problems considered. The proofs are analogous to those of Lemma 4.3 and Theorem 4.4 i) in [31], respectively.

Lemma 5.1 *Assume (2.3) and* $\{u^\varepsilon\}_\varepsilon \subset H^1(\Omega)$ *are given such that* $u^\varepsilon \rightharpoonup u$ *weakly in* $H^1(\Omega_1)$. *Then*

$$\int_{\Omega_1} d_0(x)\,|\nabla u|^2 \le \liminf_{\varepsilon} \int_{\Omega_1} d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 .$$

If, in addition, $u^\varepsilon \to u$ *in* $L^2(\Omega_1)$, *then* $u^\varepsilon \to u$ *in* $H^1(\Omega_1)$ *if and only if*

$$\int_{\Omega_1} d_0(x)\,|\nabla u|^2 = \lim_{\varepsilon} \int_{\Omega_1} d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 .$$

Lemma 5.2 *Assume (2.2) and that for* $0 < \varepsilon \le \varepsilon_0$, *the sequence* $\{u^\varepsilon\}_\varepsilon \subset H^1(\Omega)$ *is bounded in* $L^2(\Omega)$ *and* $0 \le \int_\Omega d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 \le M$ *with* M *a constant not depending on* ε. *Then, taking a subsequence if necessary,* u^ε *converges weakly in* $H^1(\Omega)$ *and strongly in* $L^2(\Omega)$ *to a function* $u \in H^1(\Omega)$ *that is constant in* Ω_0. *Moreover,*

$$\lim_{\varepsilon} \int_{\Omega} d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 = \int_{\Omega} d_0(x)\,|\nabla u|^2 = \int_{\Omega_1} d_0(x)\,|\nabla u|^2$$

if and only if u^ε *strongly converges to* u *in* $H^1(\Omega)$ *and* $\lim_\varepsilon \int_{\Omega_0} d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 = 0$.

With this, we can now obtain the convergence of the approximate problems to the limit one.

Theorem 5.3 *Let* $\lambda \in \mathbb{R}$. *Assume that for* $0 < \varepsilon < \varepsilon_0$, *the sequence* $\{h_\varepsilon\}_\varepsilon \subset H^{-1}_{\Gamma_D}(\Omega)$ *is bounded and weakly converges to* $h \in H^{-1}_{\Gamma_D}(\Omega)$. *Assume also that the sequence* $\{u^\varepsilon\}_\varepsilon \subset H^1_{\Gamma_D}(\Omega)$ *is bounded in* $L^2(\Omega)$ *and satisfies*

$$T_\varepsilon u^\varepsilon = h^\varepsilon$$

with T_ε *given by (3.6).*

Then $0 \le \int_\Omega d_\varepsilon(x)\,|\nabla u^\varepsilon|^2 \le M$ *for some constant not depending on* ε *and, taking a subsequence as needed,* $u^\varepsilon \rightharpoonup u \in H^1_{\Omega_0}(\Omega)$ *weakly in* $H^1(\Omega)$; $Vu^\varepsilon \to Vu$ *in* $L^2(\Omega)$; *and* $bu^\varepsilon \to bu$ *in* $L^2(\Gamma_N)$. *Furthermore,* $u \in H^1_{\Omega_0}(\Omega)$ *and satisfies*

$$T_0 u = h|_{H^1_{\Omega_0}(\Omega)} \quad \text{in } H^{-1}_{\Omega_0}(\Omega),$$

with T_0 given by (3.10).

Finally, $u^\varepsilon \to u$ strongly in $H^1_{\Gamma_D}(\Omega)$ if and only if $\langle h_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle h, u\rangle_{-1,1}$. The latter holds if, for instance, $h^\varepsilon \to h$ strongly in $H^{-1}_{\Gamma_D}(\Omega)$. Also, in such a case,

$$\lim_{\varepsilon} \tau_\varepsilon(u^\varepsilon, u^\varepsilon) = \tau_0(u,u), \tag{5.1}$$

and

$$\lim_{\varepsilon\to 0} \tau_\varepsilon(u^\varepsilon - u, u^\varepsilon - u) = 0. \tag{5.2}$$

where τ_0 is given by (3.9). In particular

$$\lim_{\varepsilon} \int_{\Omega_0} d_\varepsilon \left|\nabla u^\varepsilon\right|^2 = 0, \qquad \lim_{\varepsilon} \int_{\Omega} d_\varepsilon(x) \left|\nabla u^\varepsilon\right|^2 = \int_{\Omega} d_0(x) \left|\nabla u\right|^2 = \int_{\Omega_1} d_0(x) \left|\nabla u\right|^2 .$$

In the case in which (2.6) holds, that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

then $u \in H^1_{0,\Omega_0}(\Omega)$ and $T_{0,0}u = h|_{H^1_{0,\Omega_0}(\Omega)}$ in $H^{-1}_{0,\Omega_0}(\Omega)$, with $T_{0,0}$ given (3.13).

Remark 5.4 *Notice that the condition that $\{u^\varepsilon\}_\varepsilon$ is bounded holds for $\lambda > 0$ large enough.*

Proof. Let $\lambda \in \mathbb{R}$ and $\{h_\varepsilon\}_\varepsilon \subset H^{-1}_{\Gamma_D}(\Omega)$ be a bounded sequence, weakly converging to $h \in H^{-1}_{\Gamma_D}(\Omega)$. Then,

$$\tau_\varepsilon(u^\varepsilon, u^\varepsilon) = \int_{\Omega} d_\varepsilon(x) \left|\nabla u^\varepsilon\right|^2 + \int_{\Omega} (\lambda + V(x)) \left|u^\varepsilon\right|^2 + \int_{\Gamma_N} b(x) \left|u^\varepsilon\right|^2 = \langle h_\varepsilon, u^\varepsilon\rangle_{-1,1} . \tag{5.3}$$

From (3.7) and the boundedness of h_ε and u^ε we get $\int_\Omega d_\varepsilon(x) \left|\nabla u^\varepsilon\right|^2 \leq M$. Therefore, we can apply Lemma 5.2 and obtain that, for a suitable subsequence, $u^\varepsilon \to u \in H^1_{\Omega_0}(\Omega)$ weakly in $H^1_{\Gamma_D}(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\Gamma_N)$. Furthermore, since $V \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma_N)$, we have that $Vu^\varepsilon \to Vu$ in $L^2(\Omega)$ and $bu^\varepsilon \to bu$ in $L^2(\Gamma_N)$.

Now, given $\phi \in H^1_{\Omega_0}(\Omega)$, we have

$$\tau_\varepsilon(u^\varepsilon, \phi) = \int_{\Omega_1} d_\varepsilon(x) \nabla u^\varepsilon \nabla \phi + \int_{\Omega} (\lambda + V(x)) u^\varepsilon \phi + \int_{\Gamma_N} b(x) u^\varepsilon \phi = \langle h_\varepsilon, \phi\rangle_{-1,1}.$$

We can then pass to the limit in ε to obtain,

$$\tau_0(u,\phi) = \int_{\Omega_1} d_0(x) \nabla u \nabla \phi + \int_{\Omega_1} (\lambda + V(x)) u \phi + \int_{\Gamma^1_N} b(x) u \phi + u_{\Omega_0} \phi_{\Omega_0} \Big(\int_{\Omega_0} (\lambda + V) + \int_{\Gamma_N \cap \partial\Omega_0} b \Big) = \langle h, \phi\rangle_{-1,1},$$

i.e., $T_0 u = h|_{H^1_{\Omega_0}(\Omega)}$. Taking $\phi = u$, we have

$$\int_{\Omega_1} d_0(x) \left|\nabla u\right|^2 + \int_{\Omega_1} (\lambda + V(x)) \left|u\right|^2 + \int_{\Gamma^1_N} b(x) \left|u\right|^2 + \left|u_{\Omega_0}\right|^2 \Big(\int_{\Omega_0} (\lambda + V) + \int_{\Gamma_N \cap \partial\Omega_0} b \Big) = \langle h, u\rangle_{-1,1}. \tag{5.4}$$

In particular, from (5.3) and (5.4), we have $\lim \int_\Omega d_\varepsilon(x) \left|\nabla u^\varepsilon\right|^2 = \int_{\Omega_1} d_0(x) \left|\nabla u\right|^2$ if and only if $\langle h_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle h, u\rangle_{-1,1}$. The latter holds if either h_ε or u^ε converge strongly. In such a case, we get (5.1). Since we already proved $\lim_{\varepsilon\to 0} \tau_\varepsilon(u^\varepsilon, \phi) = \tau_0(u, \phi)$ then we also get (5.2). The rest is immediate.

In the case (2.6), from **(R.1)** we obtain that $u \in H^1_{0,\Omega_0}(\Omega)$ and taking $\phi \in H^1_{0,\Omega_0}(\Omega)$, and passing to the limit in $\tau_\varepsilon(u^\varepsilon, \phi) = \langle h_\varepsilon, \phi\rangle_{-1,1}$ we obtain $\tau_{0,0}(u, \phi) = \langle h, \phi\rangle_{-1,1}$. ■

We now consider the particular case when $h_\varepsilon = f^\varepsilon_\Omega + g^\varepsilon_{\Gamma_N}$ as in (3.3), and show how the solution and its derivatives behave in the different regions of the domain.

Corollary 5.5 *Under the hypothesis of the Theorem 5.3, assume $h^\varepsilon = f^\varepsilon_\Omega + g^\varepsilon_{\Gamma_N}$ and $f^\varepsilon \rightharpoonup f$ weakly in $L^2(\Omega)$ and $g^\varepsilon \rightharpoonup g$ weakly in $H^{-1/2}(\Gamma_N)$. Then u^ε is a weak solution of (3.4) and*

$$-\mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon) \rightharpoonup f - (\lambda+V)u_{\Omega_0} \quad \textit{weakly in } L^2(\Omega_0)$$

$$-\mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon) \rightharpoonup -\mathrm{div}(d_0(x)\nabla u) \quad \textit{weakly in } L^2(\Omega_1)$$

$$d_\varepsilon \frac{\partial u^\varepsilon}{\partial \vec{n}} \rightharpoonup d_0 \frac{\partial u}{\partial \vec{n}} \quad \textit{weakly in } H^{-1/2}(\Gamma^* \cup \Gamma^1_N).$$

Also, $u^\varepsilon \to u$ strongly in $H^1_{\Gamma_D}(\Omega)$ if and only if $\langle g_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle g, u\rangle_{-1,1}$. In such a case,

$$\lim_\varepsilon \int_{\Omega_0} d_\varepsilon |\nabla u^\varepsilon|^2 = 0, \qquad \lim_\varepsilon \int_\Omega d_\varepsilon(x) |\nabla u^\varepsilon|^2 = \int_\Omega d_0(x) |\nabla u|^2 = \int_{\Omega_1} d_0(x) |\nabla u|^2 .$$

That happens if, for example, $g^\varepsilon \rightharpoonup g$ weakly in $H^{-s}(\Gamma_N)$ for $0 \le s < 1/2$.

If, in particular, $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\Gamma_N)$, then u is a weak solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}(d_0(x)\nabla u) + (\lambda + V(x))u &=& f & \textit{in } \ \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& g & \textit{on } \ \Gamma^1_N \\ u &=& 0 & \textit{on } \ \Gamma_D \\ u &=& u_{\Omega_0} & \textit{on } \ \Gamma^* \\ \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0} (\lambda+V) + \int_{\Gamma_N \cap \partial\Omega_0} b\Big) u_{\Omega_0}\right] &=& \dfrac{1}{|\Omega_0|}\Big(\displaystyle\int_{\Omega_0} f + \int_{\Gamma_N\cap\partial\Omega_0} g\Big). & \end{array} \right. \tag{5.5}$$

In the case in which (2.6) holds, that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

then $u \in H^1_{0,\Omega_0}(\Omega)$ and it is a weak solution of (3.14).

Proof. It is clear that u^ε is a weak solution of (3.4). Also since we can assume $u^\varepsilon \to u$ strongly in $L^2(\Omega)$ then $\langle h_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle h, u\rangle_{-1,1}$ if and only if $\langle g_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle g, u\rangle_{-1,1}$ and in such a case we get the strong convergence in $H^1_{\Gamma_D}(\Omega)$. Notice that since $u^\varepsilon \to u$ weakly in $H^1_{\Gamma_D}(\Omega)$, we can assume the traces converge strongly in $H^s(\Gamma_N)$ for $0 \le s < 1/2$. Hence if $g^\varepsilon \rightharpoonup g$ weakly in $H^{-s}(\Gamma_N)$ we get $\langle g_\varepsilon, u^\varepsilon\rangle_{-1,1} \to \langle g, u\rangle_{-1,1}$.

Since u^ε is a weak solution of (3.4) then, $-\mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon) = f^\varepsilon - (\lambda+V)u^\varepsilon$ converges to $f - (\lambda+V)u = -\mathrm{div}(d_0(x)\nabla u)$ weakly in $L^2(\Omega_1)$ and to $f - (\lambda+V)u_{\Omega_0}$ weakly in $L^2(\Omega_0)$.

On the other hand, by (3.8), for all $\phi \in H^1_{\Omega_0}(\Omega)$ we have

$$\left\langle d_\varepsilon \frac{\partial u^\varepsilon}{\partial \vec{n}}, \gamma(\phi) \right\rangle_{\partial\Omega_1} = \int_{\Omega_1} \mathrm{div}(d_\varepsilon(x)\nabla u^\varepsilon)\phi + \int_{\Omega_1} d_\varepsilon(x)\nabla u^\varepsilon \nabla\phi$$

where the right hand side converges to

$$\int_{\Omega_1} \mathrm{div}(d_0(x)\nabla u)\phi + \int_{\Omega_1} d_0(x)\nabla u\nabla\phi = \left\langle d_0 \frac{\partial u}{\partial \vec{n}}, \gamma(\phi) \right\rangle_{\partial\Omega_1}.$$

Since $\phi = 0$ in Γ^1_D and on $\partial\Omega_1 = \Gamma^* \cup \Gamma^1_D \cup \Gamma^1_N$, we get

$$d_\varepsilon \frac{\partial u^\varepsilon}{\partial \vec{n}} \rightharpoonup d_0 \frac{\partial u}{\partial \vec{n}} \quad \text{weakly in } H^{-1/2}(\Gamma^* \cup \Gamma^1_N).$$

If, in particular, $g^\varepsilon \rightharpoonup g$ weakly in $L^2(\Gamma_N)$, it is easy to see that $h|_{H^1_{\Omega_0}(\Omega)} \in H^{-1}_{\Omega_0}(\Omega)$ is given by

$$h|_{H^1_{\Omega_0}(\Omega)} = \Big(\frac{1}{|\Omega_0|}\big(\int_{\Omega_0} f + \int_{\Gamma_N\cap\partial\Omega_0} g\big)\mathcal{X}_{\Omega_0} + f\mathcal{X}_{\Omega_1}\Big)_\Omega + g_{\Gamma^1_N}. \tag{5.6}$$

In particular, by (3.12), u is a weak solution of (5.5).

In case (2.6) holds it is easy to check that

$$h^{*,0} := h|_{H^1_{0,\Omega_0}(\Omega)} = P(h) = \Big(f\mathcal{X}_{\Omega_1}\Big)_\Omega + g_{\Gamma^1_N}.$$

and then $u \in H^1_{0,\Omega_0}(\Omega)$ and it is a weak solution of (3.14). ■

In view of (5.6) we define the following operator.

Definition 5.6 *For a linear form $h = f_\Omega + g_{\Gamma_N}$ as in (3.3), with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ we define*

$$P(h) = \Big(\frac{1}{|\Omega_0|}\big(\int_{\Omega_0} f + \int_{\Gamma_N\cap\partial\Omega_0} g\big)\mathcal{X}_{\Omega_0} + f\mathcal{X}_{\Omega_1}\Big)_\Omega + g_{\Gamma^1_N} \tag{5.7}$$

and, in case (2.6) holds,

$$P(h) = \Big(f\mathcal{X}_{\Omega_1}\Big)_\Omega + g_{\Gamma^1_N}. \tag{5.8}$$

Notice that

$$P : L^2(\Omega) + L^2(\Gamma_N) \to L^2_{\Omega_0}(\Omega) + L^2(\Gamma^1_N), \tag{5.9}$$

or $P : L^2(\Omega) + L^2(\Gamma_N) \to L^2_{0,\Omega_0}(\Omega_1) + L^2(\Gamma^1_N)$ in the case in which (2.6) holds, is linear, continuous and monotone in the sense that if $f_1, f_2 \in L^2(\Omega)$ and $g_1, g_2 \in L^2(\Gamma_N)$ with

$$f_1 \le f_2,\ g_1 \le g_2 \quad \text{implies} \quad P(h_1) \le P(h_2)$$

in the sense that for every $0 \le \phi \in H^1_{\Omega_0}(\Omega)$, or $0 \le \phi \in H^1_{0,\Omega_0}(\Omega)$ respectively, we have

$$\langle P(h_1), \phi\rangle \le \langle P(h_2), \phi\rangle,$$

where $h_i = (f_i)_\Omega + (g_i)_{\Gamma_N}$.

Remark 5.7 *All this allows to understand the limit process in Theorem 5.3 and Corollary 5.5 as follows.*

Denote by L_ε, L_0 and $L_{0,0}$ the operators as in Theorem 3.2, 3.4 and 3.6 when $V = b = \lambda = 0$. Then with the notation in those theorems and in (4.1) we have

$$A_\varepsilon = L_\varepsilon + (\lambda + V)_\Omega + b_{\Gamma_N}$$

where we are adding multiplication operators in Ω and Γ_N to L_ε. Then

$$A_0 = L_0 + F + P\big((\lambda + V)_\Omega + b_{\Gamma_N}\big)$$

where $F(u)$ accounts for the flux term $\frac{1}{|\Omega_0|}\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}}$ in (3.10), while in case (2.6) holds

$$A_{0,0} = L_{0,0} + P\big((\lambda + V)_\Omega + b_{\Gamma_N}\big).$$

Then, using (4.1), Theorem 5.3 and Corollary 5.5 state that solutions of

$$A_\varepsilon u^\varepsilon = L_\varepsilon u^\varepsilon + ((\lambda + V)u^\varepsilon)_\Omega + (bu^\varepsilon)_{\Gamma_N} = f^\varepsilon_\Omega + g^\varepsilon_{\Gamma_N} \quad \text{in } H^{-1}_{\Gamma_D}(\Omega),$$

converge to solutions of

$$A_0 u = L_0 u + F(u) + P\big((\lambda+V)_\Omega + b_{\Gamma_N}\big)u = P\big(f_\Omega + g_{\Gamma_N}\big) \quad \text{in } H^{-1}_{\Omega_0}(\Omega),$$

or

$$A_{0,0} u = L_{0,0} u + P\big((\lambda+V)_\Omega + b_{\Gamma_N}\big)u = P\big(f_\Omega + g_{\Gamma_N}\big) \quad \text{in } H^{-1}_{0,\Omega_0}(\Omega)$$

respectively.

5.2 Spectral convergence

As mentioned above the operators $A_\varepsilon, A_0, A_{0,0}$ in (4.1) have compact resolvents. Thus, they have a discrete spectrum formed by an increasing sequence of eigenvalues of finite multiplicity, $\sigma(A_\varepsilon) = \{\mu_n^\varepsilon\}_n$ and $\sigma(A_0) = \{\mu_n^0\}_n$, $\sigma(A_{0,0}) = \left\{\mu_n^{0,0}\right\}_n$. From Theorems 3.2, 3.4 or 3.6 observe that the eigenfunctions of the approximate problems satisfy

$$\left\{\begin{array}{rcll} -\mathrm{div}(d_\varepsilon(x)\nabla u) + (\lambda + V(x))u &=& \mu u & \text{in } \Omega \\ d_\varepsilon(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& 0 & \text{on } \Gamma_N \\ u &=& 0 & \text{on } \Gamma_D \end{array}\right. \tag{5.10}$$

and those of the limit problem satisfy

$$\left\{\begin{array}{rcll} -\mathrm{div}(d_0(x)\nabla u) + (\lambda + V(x))u &=& \mu u & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u &=& 0 & \text{on } \Gamma_N^1 \\ u &=& 0 & \text{on } \Gamma_D \\ u &=& u_{\Omega_0} & \text{on } \Gamma^* \\ \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0 \frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0}(\lambda+V) + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)u_{\Omega_0}\right] &=& \mu u_{\Omega_0}. & \end{array}\right. \tag{5.11}$$

In the case in which (2.6) holds, the eigenvalue problem reduces to

$$\left\{\begin{array}{rll} -\mathrm{div}(d_0(x)\nabla u) + (\lambda + V(x))u = \mu u & \text{in} & \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = 0 & \text{on} & \Gamma_N^1 \\ u = 0 & \text{in} & \Omega_0 \\ u = 0 & \text{on} & \Gamma_D \cup \Gamma^*. \end{array}\right. \tag{5.12}$$

The next result states the spectral convergence of the approximate problems (3.4) to the limit problem (3.12) or (3.14) in case (2.6) holds.

Theorem 5.8 *The following statements hold:*
i) If for some $\varepsilon_n \to 0$ and $\mu_n \in \sigma(A_{\varepsilon_n})$ we have $\mu_n \to \mu_0$ then $\mu_0 \in \sigma(A_0)$.

If, in addition, we have eigenfunctions $\phi_n \in E(\mu_n)$ such that $\|\phi_n\|_{L^2(\Omega)} = 1$ then any subsequence that converges weakly in $L^2(\Omega)$ homogenizes to some $\phi \in E(\mu_0)$, that is, as in (5.2), $\lim_{n\to\infty} \tau_{\varepsilon_n}(\phi_n - \phi, \phi_n - \phi) = 0$.
ii) For any $\mu_0 \in \sigma(A_0)$ there exists sequences $\varepsilon_n \to 0$ and $\mu_n \in \sigma(A_{\varepsilon_n})$ such that $\mu_n \to \mu_0$.
iii) For any $\mu_0 \in \sigma(A_0)$ with multiplicity M, there exists $r_0 > 0$ such that for any $0 < r < r_0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the operator A_ε has exactly M eigenvalues in the ball $B(\mu_0, r) \subset \mathbb{C}$, counting multiplicities.

In case (2.6) holds, that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

then the statements above hold replacing A_0 with $A_{0,0}$.

Proof. Note that by (4.1) it is enough to prove the results for the positive operators $B_\varepsilon, B_0, B_{0,0}$ in (4.1). To achieve the result we will prove first that the inverses of these operators are close in some sense below.

Step 1. Consider any family $\{f_\varepsilon\}_\varepsilon$ with $f_\varepsilon \in L^2(\Omega)$ and $\|f_\varepsilon\|_{L^2(\Omega)} = 1$. Then consider any sequence of positive numbers $\varepsilon_n \to 0$ and consider $u^{\varepsilon_n} = B_{\varepsilon_n}^{-1} f_{\varepsilon_n}$.

Then, from Theorem 5.3 there exists a subsequence, that we denote the same, such that $f_{\varepsilon_n} \rightharpoonup f$ weakly in $L^2(\Omega)$ and $u^{\varepsilon_n} \to u \in H^1_{\Omega_0}(\Omega)$ weakly in $H^1_{\Gamma_D}(\Omega)$ and strongly in $L^2(\Omega)$ where $u = B_0^{-1} f_0$ with

$$f_0 = \left(\fint_{\Omega_0} f \right) \mathcal{X}_{\Omega_0} + f \mathcal{X}_{\Omega_1}.$$

by Corollary 5.5 (with $g^\varepsilon = 0$).

Step 2. Assume moreover that $f_\varepsilon \to f \in L^2_{\Omega_0}(\Omega)$ in $L^2(\Omega)$. Since $\langle f_{\varepsilon_n}, u^{\varepsilon_n} \rangle \to \langle f, u \rangle$ again Theorem 5.3 implies that $u^{\varepsilon_n} \to u \in H^1_{\Omega_0}(\Omega)$ strongly in $H^1_{\Gamma_D}(\Omega)$. Even more we have that (5.2) holds for the sequence u^{ε_n}.

Step 3. All the above implies that $B_\varepsilon^{-1} \to B_0^{-1}$ in the sense of (4.5) in [5]. Then arguing as in the proof of Lemma 4.9 in [5] we get that for any compact set $K \subset \rho(B_0)$ there exists $\varepsilon_K > 0$ such that for any $0 < \varepsilon < \varepsilon_K$ we have $K \subset \rho(B_\varepsilon)$. This proves the first part of i).

Now we end the proof of i). For this assume $\phi_n \in E(\mu_n)$ such that $\|\phi_n\|_{L^2(\Omega)} = 1$ and $B_{\varepsilon_n}(\phi_n) = \mu_n \phi_n$. Then for any subsequence that converges weakly in $L^2(\Omega)$, which we denote the same, by Theorem 5.3, arguing as in Steps 1 and 2 above, we have that ϕ_n homogenizes, as in (5.2), to some ϕ in $H^1_{\Omega_0}(\Omega)$ such that $B_0(\phi) = \mu_0 \phi$. Since $\|\phi\|_{L^2(\Omega)} = 1$ we get $\mu_0 \in \sigma(B_0)$.

Step 4. Parts ii) and iii) follow as in Theorem 4.10 in [5] from the convergence of the spectral projections as we now sketch. Recall that given a sectorial operator S in a Banach space such that its spectrum satisfies $\sigma(S) = \sigma_1 \cup \sigma_2$ with σ_i open and closed in the spectrum and σ_1 bounded, then the spectral projection associated to σ_1 is

$$Q(\sigma_1, S) = \frac{1}{2\pi i} \int_\Gamma (zI - S)^{-1} \, dz$$

where Γ is a simple, positively oriented curve in the resolvent of S surrounding σ_1 and leaving σ_2 in its exterior. Then $W(\sigma_1, S) = Q(\sigma_1, S)X$ is the (generalized) eigenspace associated to σ_1, see [22].

In our case, we take $\sigma_1 = \{\mu_0\}$ with $\mu_0 \in \sigma(B_0)$ and $S = B_0$. Then, we take Γ a small circle enclosing μ_0 and no other point of the spectrum of B_0. Hence from the first part in Step 3, the trace of Γ is in the resolvent of B_ε for small enough ε. Also the convergence above $B_\varepsilon^{-1} \to B_0^{-1}$ translates into a similar convergence $Q(\mu_0, B_\varepsilon) \to Q(\mu_0, B_0)$. This, in particular, implies that for small enough ε, $W(\mu_0, B_\varepsilon)$ and $W(\mu_0, B_0)$ have the same dimension. This proves iii).

In turn, by contradiction, part iii) gives part ii). ■

Observe that a somewhat weaker result on the convergence of eigenvalues and eigenfunctions, in the spirit of Theorem 5.1 in [31], can be proved here only using the Rayleigh characterization of eigenvalues.

Also, we get the following result.

Corollary 5.9 *With the notation in Theorem 5.8, the following are equivalent*
i) $\Lambda_0 = \inf\{\mu,\ \mu \in \sigma(A_0)\} > 0$.
ii) There exists $\delta > 0$ *such that for all sufficiently small* ε *we have*

$$\Lambda_\varepsilon = \inf\{\mu,\ \mu \in \sigma(A_\varepsilon)\} > \delta > 0.$$

In case (2.6), that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

then the statement above holds replacing A_0 with $A_{0,0}$.

In the spirit of Remark 5.7 we get now the following.

Remark 5.10 *Observe that Theorem 5.8 state that eigenvalues and eigenfunctions of*

$$A_\varepsilon u^\varepsilon = L_\varepsilon u^\varepsilon + ((\lambda+V)u^\varepsilon)_\Omega + (bu^\varepsilon)_{\Gamma_N} = \mu^\varepsilon u^\varepsilon_\Omega \quad \text{in } H^{-1}_{\Gamma_D}(\Omega),$$

converge to eigenvalues and eigenfunctions of

$$A_0 u = L_0 u + F(u) + P\big((\lambda+V)_\Omega + b_{\Gamma_N}\big)u = \mu u_\Omega \quad \text{in } H^{-1}_{\Omega_0}(\Omega),$$

or

$$A_{0,0} u = L_{0,0} u + P\big((\lambda+V)_\Omega + b_{\Gamma_N}\big)u = \mu u_\Omega \quad \text{in } H^{-1}_{0,\Omega_0}(\Omega),$$

respectively.

5.3 Linear parabolic problems

In this section we prove the convergence of the approximated linear parabolic problems (4.2) towards the linear parabolic limit problem (4.3). In case (2.6) holds, that is

$$\Gamma_D \cap \overline{\Omega_0} \neq \emptyset,$$

the limit problem will be (4.4).

We have the following continuity result for the approximate parabolic homogeneous problems, whose proof is analogous to that of Proposition 4.6, p. 48, in [7], based on the spectral decomposition provided by Theorem 5.8. A similar result can be found in Theorem 3.4 in [10].

Proposition 5.11 *With the notation in Corollary 5.9, assume $\Lambda_0 > 0$. Then, given $\gamma \in [0,1)$ there exist $\alpha \in ((1+\gamma)/2, 1)$ and a function $\tilde{c}(\cdot) \geq 0$ with $\tilde{c}(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for all $h \in H^{-\gamma}(\Omega) \equiv (H^\gamma(\Omega))'$*

$$\|e^{-A_\varepsilon t}h - e^{-A_0 t}h^*\|_{H^1(\Omega)} \leq \tilde{c}(\varepsilon)t^{-\alpha}\|h\|_{H^{-\gamma}(\Omega)}, \qquad t > 0$$

where we have denoted by h^* the restriction of h to $H^\gamma_{\Omega_0}(\Omega)$, so $h^* \in (H^\gamma_{\Omega_0}(\Omega))'$.

In case (2.6) holds the result remain true with $A_{0,0}$ replacing A_0 and h^* is the restriction of h to $H^\gamma_{0,\Omega_0}(\Omega)$, so $h^* \in (H^\gamma_{0,\Omega_0}(\Omega))'$.

In particular, using Definition 5.6 and arguing as in Corollary 4.8 and Corollary 4.9 in [7], taking $\gamma = 1/2$, we have the following.

Corollary 5.12 *With the notation in Corollary 5.9, assume $\Lambda_0 > 0$ and $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$.*
i) There exist $\alpha \in (3/4, 1)$ and a function $\tilde{c}(\cdot) > 0$ with $\tilde{c}(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\|e^{-A_\varepsilon t}(f_\Omega + g_{\Gamma_N}) - e^{-A_0 t}P(f_\Omega + g_{\Gamma_N})\|_{H^1(\Omega)} \leq \tilde{c}(\varepsilon)t^{-\alpha}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}), \qquad t > 0$$

where P is given in (5.7).
ii) There exist $k > 0$ and $M > 0$ such that

$$\|e^{-A_0 t}P(f_\Omega + g_\Gamma)\|_{L^\infty(\Omega)} \leq Mt^{-k}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}).$$

In case (2.6) holds the result remain true with $A_{0,0}$ replacing A_0.

Proof. Part i) is immediate from Proposition 5.11 with $\gamma = 1/2$. For part ii), from Corollary 5.9 we have that for $\varepsilon > 0$ small enough, $\Lambda_\varepsilon > \delta > 0$ for some $\delta > 0$. We know from [4], Lemma 4.4, applied to A_ε that there exists $\tilde{M}$ and $\tilde{N}$, independent of ε, such that $\|e^{-A_\varepsilon t}\psi\|_{L^\infty(\Omega)} \leq \tilde{M}t^{-\tilde{N}}\|\psi\|_{L^2(\Omega)}$. Passing to the limit as $\varepsilon \to 0$ and using part i) we have that $\|e^{-A_0 t}P(\psi)\|_{L^\infty(\Omega)} \leq \tilde{M}t^{-\tilde{N}}\|\psi\|_{L^2(\Omega)}$.

Using this, the semigroup property for $e^{-A_0 t}$, the embedding $L^2_{\Omega_0}(\Omega) + L^2(\Gamma^1_N) \subset H^{-1}_{\Omega_0}(\Omega) = X_0^{-1/2}$ and the continuity of P in (5.9), we obtain

$$\begin{aligned}
\|e^{-A_0 t}P(f_\Omega + g_{\Gamma_N})\|_{L^\infty(\Omega)} &\leq \tilde{M}(t/2)^{-\tilde{N}}\|e^{-A_0 t/2}P(f_\Omega + g_{\Gamma_N})\|_{L^2(\Omega)} \\
&\leq \tilde{M}(t/2)^{-\tilde{N}}Ct^{-1/2}\|P(f_\Omega + g_{\Gamma_N})\|_{X_0^{-1/2}} \\
&\leq Mt^{-k}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}).
\end{aligned}$$

In case (2.6) holds the proof follows the same lines with obvious changes. ■

6 Nonlinear parabolic problems

The next step in our study is to understand the behaviour of the corresponding nonlinear evolution problems. In particular, we obtain conditions ensuring the existence of an attractor for the approximated and limit problems as well as their upper semicontinuity.

Note that an essential tool are the comparison principles that can be obtained for all the problems in this paper from Apendix A in [4], see Remark 4.1.

6.1 Approximate nonlinear parabolic problems

We now consider the nonlinear parabolic problems (2.1) with initial data $u_0 \in X$, where X is a space in the class $\mathcal{E}$ defined by

$$\mathcal{E} = \{L^q(\Omega),\ W^{1,q}_{\Gamma_D}(\Omega),\ 1 < q < \infty\}.$$

We also assume $c \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma_N)$ for the sake of clarity in the exposition (see Remark 3.1). Notice that from Theorem 3.2, (4.1) and the notation in (3.3), problem (2.1) can be written as

$$u_t + A_\varepsilon u = h(x, u) := f(x, u)_\Omega + g(x, u)_{\Gamma_N} \tag{6.1}$$

and the solutions are given by the variations of constants formula

$$u^\varepsilon(t, u_0) = \mathrm{e}^{-A_\varepsilon t}u_0 + \int_0^t \mathrm{e}^{-A_\varepsilon(t-s)}h(u^\varepsilon(s, u_0))\mathrm{d}s. \tag{6.2}$$

Suppose that f and g satisfy the following growth condition

$(\mathbf{G})_\mathbf{X}$: $f(x,\cdot), g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz uniformly in $x \in \Omega$ and $x \in \Gamma_N$, respectively. In addition,

1. If $X = L^q(\Omega)$, assume that f and g satisfy a relation of the form

$$|j(x, u) - j(x, v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1) \tag{6.3}$$

with exponents ρ_f and ρ_g respectively, such that, for $N \geq 2$ (resp. $N = 1$)

$$\rho_f \leq \rho_\Omega := 1 + \frac{2q}{N}, \quad \rho_g \leq \rho_\Gamma := 1 + \frac{1}{N} \quad (\text{resp. } \rho_g \leq \rho_\Gamma := 1 + q).$$

2. If $X = W^{1,q}_{\Gamma_D}(\Omega)$, assume that one of the following conditions holds
   (a) $q > N$ (hence, no growth assumption is assumed on f or g).
   (b) $q = N$ and f, g satisfy that for all $\eta > 0$ there exists $c_\eta > 0$ such that

$$|j(x,u) - j(x,v)| \leq c_\eta(e^{\eta|u|\frac{N}{N-1}} + e^{\eta|v|\frac{N}{N-1}})|u - v|. \tag{6.4}$$

   (c) $1 < q < N$ and f, g satisfy (6.3) with exponents

$$\rho_f \leq \rho_\Omega := 1 + \frac{2q}{N-q} \quad \rho_g \leq \rho_\Gamma := 1 + \frac{q}{N-q}.$$

Also assume that there exist $C_0 \in L^p(\Omega)$, $0 \leq C_1 \in L^p(\Omega)$, $p > N/2$, $B_0 \in L^r(\Gamma_N)$ and $0 \leq B_1 \in L^r(\Gamma_N)$, $r > N-1$, such that

$$sf(x,s) \leq C_0(x)s^2 + C_1(x)|s| \quad x \in \Omega \tag{6.5}$$
$$sg(x,s) \leq B_0(x)s^2 + B_1(x)|s| \quad x \in \Gamma_N \tag{6.6}$$

for all $s \in \mathbb{R}$. Finally, assume that the first eigenvalue, Λ_ε, of the following problem is positive

$$\begin{cases} -\mathrm{div}(d_\varepsilon(x)\nabla u) + (c(x) - C_0(x))u = \mu u & \text{in } \Omega \\ d_\varepsilon(x)\frac{\partial u}{\partial \vec{n}} + (b(x) - B_0(x))u = 0 & \text{on } \Gamma_N \\ u = 0 & \text{on } \Gamma_D. \end{cases} \tag{6.7}$$

From Theorem 2.4 and 3.4 in [4] we have the existence and uniqueness of solution as well as existence of a global attractor for the problem (2.1). Namely, we have

Theorem 6.1 *Let $0 < \varepsilon < \varepsilon_0$. Let X any of the spaces in the class $\mathcal{E}$. Suppose that f and g are locally Lipschitz functions satisfying $(G)_X$ and (6.5), (6.6). Then, given $u_0 \in X$ there exists a unique solution $u^\varepsilon(\cdot; u_0)$ of problem (2.1) which is defined for all time $t \geq 0$. Also, this solution continuously depend on the data and is classical for $t > 0$. Finally, the following smoothing effect holds: if $u_0 \in X$ then $u^\varepsilon(t; u_0) \in Y$ for any other space Y in the class $\mathcal{E}$, $t > 0$.*

Therefore, we can define the semigroup $S_\varepsilon(t)$ in X given by the solutions of (2.1) with initial dara $u_0 \in X$.

If, in addition, (6.7) holds, this semigroup has a global attractor $\mathcal{A}^\varepsilon{}_X$ in X. Also, for any Y in the class $\mathcal{E}$ such that $Y \subset X$ there exists the global attractor $\mathcal{A}^\varepsilon{}_Y$. In fact, $\mathcal{A}^\varepsilon{}_X = \mathcal{A}^\varepsilon{}_Y$ and attracts bounded sets of X in the topology of Y.

Then we have the following uniform estimates for the solution of (2.1). Notice that although the problem can be set in an L^p–setting, due to the variations in the diffusion coefficients, the uniform bounds for the spatial derivatives of the solutions are obtained only in $H^1(\Omega)$; see Theorem 4.5 and Theorem 4.6 in [4].

Theorem 6.2 *If $\Lambda_\varepsilon > \delta > 0$ for some δ not depending on ε then for all $0 < \varepsilon \leq \varepsilon_0$,*

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{u_0\in\mathcal{A}^\varepsilon} (\|u_0\|_{H^1(\Omega)} + \|u_0\|_{L^\infty(\Omega)}) \leq K$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \sup_{u_0\in\mathcal{A}^\varepsilon} \sup_{t\in\mathbb{R}} \|u^\varepsilon_t(t, u_0)\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq K$$

with K not depending on ε. Also, for functions in the attractors, $\mathcal{A}^\varepsilon$, there are uniform C^α bounds on each compact set of Ω_1 with some $\alpha = \alpha(K) \in (0,1)$.

Observe that from Corollary 5.9 the assumption in Theorem 6.2 is equivalent to the positivity of the first eigenvalue, Λ_0, of the problem

$$\left\{\begin{array}{rcll} -\mathrm{div}(d_0(x)\nabla u) + (c(x) - C_0(x))u & = & \mu u & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial u}{\partial \vec{n}} + (b(x) - B_0(x))u & = & 0 & \text{on } \Gamma_N^1 \\ u & = & 0 & \text{on } \Gamma_D \\ u & = & u_{\Omega_0} & \text{on } \Gamma^* \\ \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0\frac{\partial u}{\partial \vec{n}} + \Big(\int_{\Omega_0}(c - C_0) + \int_{\Gamma_N \cap \partial\Omega_0}(b - B_0)\Big)u_{\Omega_0}\right] & = & \mu u_{\Omega_0}, & \end{array}\right. \tag{6.8}$$

or the positivity of the first eigenvalue, $\Lambda_{0,0}$, of the simplified problem analogous to (5.12) in case (2.6) holds.

We can now apply Theorems 3.15 and 3.16 in [32] to obtain

Theorem 6.3 *In the assumptions of Theorem 6.1, there exist two extremal equilibria* $\varphi_m^\varepsilon, \varphi_M^\varepsilon \in \mathcal{A}^\varepsilon$ *such that* $\varphi_m^\varepsilon \le \varphi_M^\varepsilon$, $\mathcal{A}^\varepsilon \subset [\varphi_m^\varepsilon, \varphi_M^\varepsilon]$ *and*

$$\varphi_m^\varepsilon \le \liminf_{t\to\infty} u^\varepsilon(t, x; u_0) \le \limsup_{t\to\infty} u^\varepsilon(t, x; u_0) \le \varphi_M^\varepsilon$$

uniformly in $x \in \Omega$ *for* u_0 *in bounded sets of* X.

From the uniform estimates above, we can assume henceforth that the nonlinear terms $f(x,u)$, $g(x,u)$ have been truncated for large values of $|u|$ in such a way that they become globally bounded and Lipschitz and satisfy (6.5), (6.6) and (6.8).

For the dynamics of (2.1) equilibria play a relevant role. Not only extremal equilibria as in Theorem 6.3 but all equilibria have a decissive influence in local and global dynamics. Indeed, after the truncation above, (2.1) is a gradient system in $H^1(\Omega)$ with energy

$$V_\varepsilon(u) = \frac{1}{2}\int_\Omega d_\varepsilon |\nabla u|^2 + \int_\Omega (c|u|^2 - F_0(x,u)) + \int_{\Gamma_N} (b|u|^2 - G_0(x,u))$$

with $F_0(x,s) = \int_0^s f(x,s)\,ds$ and $G_0(x,s) = \int_0^s g(x,s)\,ds$. Therefore

$$\mathcal{A}_\varepsilon = W^u(E_\varepsilon) \tag{6.9}$$

where E_ε denote the set of equilibria; see [19]. Therefore stability of equilibria plays an important role in the complexity of the dynamics of (2.1). This, in turn, depends on the properties of the linearization around an equilibrium φ^ε which amounts to solving the linearized problems

$$\left\{\begin{array}{rcll} v_t^\varepsilon - \mathrm{div}(d_\varepsilon(x)\nabla v^\varepsilon) + c(x)v^\varepsilon & = & \partial_u f(x, \varphi^\varepsilon)v^\varepsilon & \text{in } \Omega \\ d_\varepsilon(x)\dfrac{\partial v^\varepsilon}{\partial \vec{n}} + b(x)v^\varepsilon & = & \partial_u g(x, \varphi^\varepsilon)v^\varepsilon & \text{on } \Gamma_N \\ v^\varepsilon & = & 0 & \text{on } \Gamma_D. \end{array}\right.$$

The solution of the linearized equations determine the structure of the local invariant manifolds near the equilibrium; see [22, 19] and Lemma 6.7 below.

6.2 The nonlinear parabolic limit problems

In the following we will concentrate in the case of the limit problem (2.5). In case (2.6) holds, that is for (2.7), analogous result hold with obvious changes.

As mentioned earlier, from the uniform estimates above, we will assume henceforth that the nonlinear terms $f(x,u)$, $g(x,u)$ have been truncated for large values of $|u|$ in such a way that they become globally bounded and Lipschitz and satisfy (6.5), (6.6) and (6.8). Notice that from (3.11) and (4.1), problem (2.5) can be written as

$$u_t + A_0 u = H(x,u) = f_*(x,u)_\Omega + g(x,u)_{\Gamma_N^1} \quad \text{in } H^{-1}_{\Omega_0}(\Omega), \tag{6.10}$$

where $u = u_{\Omega_0}\mathcal{X}_{\Omega_0} + u\mathcal{X}_{\Omega_1}$ and

$$f_*(x,u) = \frac{1}{|\Omega_0|}\left[\int_{\Omega_0} f(x,u_{\Omega_0})\,\mathrm{d}x + \int_{\Gamma_N\cap\partial\Omega_0} g(x,u_{\Omega_0})\,\mathrm{d}x\right]\mathcal{X}_{\Omega_0} + f(x,u)\mathcal{X}_{\Omega_1}.$$

Observe that using the notation in Remark 5.7 we have

$$A_0 = L_0 + F + P(c_\Omega + b_{\Gamma_N}). \tag{6.11}$$

As for nonlinear terms, for a function $u = u_{\Omega_0}\mathcal{X}_{\Omega_0} + u\mathcal{X}_{\Omega_1}$ defined up to the boundary, using the notation in (3.3) and (5.7) (resp. (5.8)), the nonlinear term above $H(u) = f_*(u)_\Omega + g(u)_{\Gamma_N^1}$ can be written as

$$H(u) = Ph(u) = P(f(u)_\Omega + g(u)_{\Gamma_N}) \tag{6.12}$$

where P is defined in (5.7) (resp. (5.8)). Hence the solutions of (2.5) are given by the corresponding variation of constants formula

$$u(t,u_0) = \mathrm{e}^{-A_0 t}u_0 + \int_0^t \mathrm{e}^{-A_0(t-s)}H(u(s,u_0))\mathrm{d}s. \tag{6.13}$$

Since f and g are bounded and globally Lipschitz, the nonlinear term satisfies

$$H = (f_*)_\Omega + g_{\Gamma_N^1} : H^1_{\Omega_0}(\Omega) \to L^2_{\Omega_0}(\Omega) + L^2(\Gamma_N^1) \subset H^{-s}_{\Omega_0}(\Omega) \tag{6.14}$$

for any $s > 1/2$, and is globally bounded and Lipschitz. Thus, as in Proposition 3.1 in [7], using Remark 4.1 and the abstract comparison results in Appendix A in [4], we can prove the following result.

Proposition 6.4 *For any $u_0 \in L^2_{\Omega_0}(\Omega)$ there exists a unique globally defined mild solution of (2.5), i.e. $u_t + A_0 u = H(u)$, with initial data u_0, $u(\cdot,u_0) \in C([0,\infty), L^2_{\Omega_0}(\Omega)) \cap C((0,\infty), H^1_{\Omega_0}(\Omega))$. Moreover this solution depends continuously on the initial data and satisfies $u, u_t \in C((0,\infty), X_0^\gamma)$, for any $\gamma < 3/4$. Moreover if u_0 lies in a bounded set of $L^2_{\Omega_0}(\Omega)$ then, for $t > 0$, fixed $u(t,u_0)$ lies in a bounded subset of X_0^γ for any $\gamma < 3/4$. If the initial data is in $H^1_{\Omega_0}(\Omega) = X_0^{1/2}$ then the solution is also in $C([0,\infty), H^1_{\Omega_0}(\Omega))$.*

Also, the comparison principle holds for problem (2.5).

Now observe that for $x \in \Omega_1$ we have

$$sf_*(x,s) \le C_0(x)|s|^2 + C_1(x)|s|, \quad s \in \mathbb{R}$$

and for $x \in \Omega_0$ we have

$$\begin{aligned} sf_*(x,s) &\le \frac{1}{|\Omega_0|}\Big(\int_{\Omega_0} f(x,s)s\,\mathrm{d}x + \int_{\Gamma_N\cap\partial\Omega_0} g(x,s)s\,\mathrm{d}x\Big) \\ &\le \frac{1}{|\Omega_0|}\Big(\int_{\Omega_0}[C_0(x)|s|^2 + C_1(x)|s|]\,\mathrm{d}x + \int_{\Gamma_N\cap\partial\Omega_0}[B_0(x)|s|^2 + B_1(x)|s|]\,\mathrm{d}x\Big) \\ &\le \tilde{c}_0|s|^2 + \tilde{c}_1|s|, \quad s \in \mathbb{R} \end{aligned}$$

where

$$\tilde{c}_0 = \frac{1}{|\Omega_0|}\Big(\int_{\Omega_0} C_0 + \int_{\Gamma_N\cap\partial\Omega_0} B_0\Big) \quad \text{and} \quad \tilde{c}_1 = \frac{1}{|\Omega_0|}\Big(\int_{\Omega_0} C_1 + \int_{\Gamma_N\cap\partial\Omega_0} B_1\Big).$$

Hence, as in (6.5),

$$sf_*(x,s) \le \tilde{C}_0(x)s^2 + \tilde{C}_1(x)|s| \quad x \in \Omega \tag{6.15}$$

with

$$\tilde{C}_0(x) = \begin{cases} C_0(x) & x \in \Omega_1 \\ \tilde{c}_0 & x \in \Omega_0 \end{cases}, \quad \tilde{C}_1(x) = \begin{cases} C_1(x) & x \in \Omega_1 \\ \tilde{c}_1 & x \in \Omega_0. \end{cases}$$

Using this we obtain the following result that states the existence of the attractor for (2.5) and some of its properties.

Theorem 6.5 *Assume (6.5), (6.6) and that the first eigenvalue, Λ_0, of (6.8) is positive. Then, problem (2.5) has a global attractor $\mathcal{A}^0$, which is a compact set of $H^1_{\Omega_0}(\Omega)$ and bounded in $L^\infty(\Omega)$. The attractor $\mathcal{A}^0$ attracts in $H^1_{\Omega_0}(\Omega)$ bounded sets of $L^2_{\Omega_0}(\Omega)$ and is a bounded set in $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Moreover, there exist extremal equilibria $\varphi_m, \varphi_M \in \mathcal{A}^0$ such that $\varphi_m \le \varphi_M$, $\mathcal{A}^0 \subset [\varphi_m, \varphi_M]$ and

$$\varphi_m \le \liminf_{t\to\infty} u(t,x;u_0) \le \limsup_{t\to\infty} u(t,x;u_0) \le \varphi_M \tag{6.16}$$

uniformly in $x \in \Omega$ and for u_0 in bounded sets of $L^2_{\Omega_0}(\Omega)$.

Proof. By Proposition 6.4 the semigroup of solutions associated to (2.5), $S_0(t)u_0 = u(t,u_0)$, is bounded from $L^2_{\Omega_0}(\Omega)$ into X_0^γ for any $\gamma < 3/4$, which is compactly embedded into $X_0^{1/2} = H^1_{\Omega_0}(\Omega)$. Hence the semigroup is compact.

By comparison, using (6.15) and (6.6), for $u_0 \in L^2_{\Omega_0}(\Omega)$ the solution $u(t,u_0)$ in Proposition 6.4 satisfies

$$|u(t,x,u_0)| \le U(t,x,|u_0|), \quad t>0, \ x \in \Omega,$$

where U satisfies

$$U_t + A_0 U = \big(\tilde{C}_0 U + \tilde{C}_1\big)_\Omega + \big(B_0 U + B_1\big)_{\Gamma_N^1} \tag{6.17}$$

that is,

$$\begin{cases} U_t & - & \operatorname{div}(d_0(x)\nabla U) + (c(x) - C_0(x))U = C_1(x) & \text{in} & \Omega_1 \\ & & d_0(x)\dfrac{\partial U}{\partial \vec{n}} + (b(x) - B_0(x))U = B_1(x) & \text{on} & \Gamma_N^1 \\ U & = & 0 & \text{on} & \Gamma_D \\ U & = & U_{\Omega_0} & \text{on} & \Gamma^* \\ \dot{U}_{\Omega_0} & + & \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0 \dfrac{\partial U}{\partial \vec{n}} + \tilde{c}_2 U_{\Omega_0}\right] = \tilde{c}_1 & & \\ U(0) & = & |u_0| & \text{in} & \Omega, \end{cases}$$

with $\tilde{c}_2 = \Big(\displaystyle\int_{\Omega_0}(c - C_0) + \int_{\Gamma_N \cap \partial\Omega_0}(b - B_0)\Big)$. Notice that with Definition 5.6 the right hand side in (6.17) coincides with

$$P\big(((\tilde{C}_0)_\Omega + (B_0)_{\Gamma_N})U\big) + P\big((\tilde{C}_1)_\Omega + (B_1)_{\Gamma_N}\big).$$

Also, note that (6.17) can be written as $U_t + L_1 U = (\tilde{C}_1)_\Omega + (B_1)_{\Gamma_N^1}$, with $L_1 U = A_0 U - P\big(((\tilde{C}_0)_\Omega + (B_0)_{\Gamma_N})U\big)$ hence

$$U(t,|u_0|) = e^{-L_1 t}|u_0| + \int_0^t e^{-L_1(t-s)}\Big((\tilde{C}_1)_\Omega + (B_1)_{\Gamma_N^1}\Big)\, ds.$$

Since the first eigenvalue of (6.8) is positive then the first eigenvalue of L_1 is $\Lambda_0 > 0$ and then U converges in, say $L^2_{\Omega_0}(\Omega)$, to

$$\phi^0 = \int_0^\infty e^{-L_1 s}\Big((\tilde{C}_1)_\Omega + (B_1)_{\Gamma_N^1}\Big)\, ds,$$

which is the unique solution of

$$\left\{\begin{array}{rcll} -\mathrm{div}(d_0(x)\nabla\phi^0) + (c(x) - C_0(x))\phi^0 &=& C_1(x) & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial\phi^0}{\partial\vec{n}} + (b(x) - B_0(x))\phi^0 &=& B_1(x) & \text{on } \Gamma_N^1 \\ \phi^0 &=& 0 & \text{on } \Gamma_D \\ \phi^0 &=& \phi^0_{\Omega_0} & \text{on } \Gamma^* \\ \dfrac{1}{|\Omega_0|}\left[\displaystyle\int_{\Gamma^*} d_0\frac{\partial\phi^0}{\partial\vec{n}} + \tilde{c}_2\phi^0_{\Omega_0}\right] &=& \tilde{c}_1. & \end{array}\right. \tag{6.18}$$

Also we can write

$$U(t, |u_0|) = e^{-L_1 t}\big(|u_0| - \phi^0\big) + \phi^0$$

and then

$$\limsup_{t\to\infty} |u(t, x; u_0)| \le \phi^0(x), \quad x \in \Omega.$$

From Proposition 3.8 we get $0 \le \phi_0 \in C^\alpha(\overline{\Omega})$. Now Corollary 5.12 gives

$$\|U(t, |u_0|)\|_{L^\infty(\Omega)} \le Mt^{-k}\||u_0| - \phi^0\|_{L^2(\Omega)} + \|\phi^0\|_{L^\infty(\Omega)} \to \|\phi^0\|_{L^\infty(\Omega)}$$

as $t \to \infty$. From here the semigroup S_0 has an absorbing ball in $L^\infty(\Omega)$ and, from Proposition 6.4, an absorbing ball in X_0^γ for any $\gamma < 3/4$. Thus, from [19] we get the existence of a global attractor as in the statement.

The Hölder estimate on the attractor are obtained as follows. If $u(\cdot)$ is an orbit in the attractor $\mathcal{A}_0$, then we know that it is defined for all $t \in \mathbb{R}$, $u, u_t \in C(\mathbb{R}, H^1_{\Omega_0}(\Omega))$, that is $u \in C^1(\mathbb{R}, H^1_{\Omega_0}(\Omega))$. Multiplying (2.5) by u_t, integrating by parts and in time for $t \in (0, 1)$, and using that the attractor is a bounded set of $H^1(\Omega)$ and $L^\infty(\Omega)$, we can show that for any orbit $u(t)$ in the attractor $\mathcal{A}_0$, we have that $\int_0^1 \|u_t(s)\|^2_{L^2(\Omega)} ds \le C$ for some constant C independent of the orbit on the attractor.

Now, if we denote by $v(t) = u_t(t)$ then, v satisfies the linearized equation along $u(t)$, given by

$$\left\{\begin{array}{ll} v_t - \mathrm{div}(d_0(x)\nabla v) + c(x)v = \partial_u f(x, u)v & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial v}{\partial\vec{n}} + b(x)v = \partial_u g(x, u)v & \text{on } \Gamma_N^1 \\ v = 0 & \text{on } \Gamma_D \\ v = v_{\Omega_0} & \text{on } \Gamma^* \\ \dot{v}_{\Omega_0} + \dfrac{1}{|\Omega_0|}\Big[\displaystyle\int_{\Gamma^*} d_0\frac{\partial v}{\partial\vec{n}} + \Big(\int_{\Omega_0} c + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)v_{\Omega_0}\Big] = & \\ \qquad = \dfrac{1}{|\Omega_0|}\Big(\displaystyle\int_{\Omega_0} \partial_u f(x, u_{\Omega_0}) + \int_{\Gamma_N\cap\partial\Omega_0} \partial_u g(x, u_{\Omega_0})\Big)v_{\Omega_0} & \\ v(0) = u_t(0) & \text{in } \Omega. \end{array}\right.$$

This is justified by Lemma 6.7 and Corollary 6.9 below.

By comparison, denoting by $D_0 = \sup\{\|\partial_u f(\phi)\|_{L^\infty(\Omega)} : \phi \in \mathcal{A}_0\}$ and $E_0 = \sup\{\|\partial_u g(\phi)\|_{L^\infty(\Omega)} : \phi \in \mathcal{A}_0\}$ then for $s \in (0, 1)$ we have $|v(t + s)| \le w(t)$ for $t > 0$ where w is the solution of the linear problem

$$\begin{cases} w_t - \operatorname{div}(d_0(x)\nabla w) + c(x)w = D_0 w & \text{in } \Omega_1 \\ d_0(x)\frac{\partial w}{\partial \vec{n}} + b(x)w = E_0 w & \text{on } \Gamma_N^1 \\ w = 0 & \text{on } \Gamma_D \\ w = w_{\Omega_0} & \text{on } \Gamma^* \\ \dot{w}_{\Omega_0} + \frac{1}{|\Omega_0|}\Big[\int_{\Gamma_0} d_0(x)\frac{\partial w}{\partial \vec{n}} + \Big(\int_{\Omega_0} c + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)\Big]w_{\Omega_0} = \big(D_0 + \dfrac{|\Gamma_N\cap\partial\Omega_0|}{|\Omega_0|}\big)w_{\Omega_0}, & \\ w(0) = |v(s)|. & \end{cases}$$

By Corollary 5.12 we have that $w(2) \in L^\infty(\Omega)$ and $\|w(2)\|_{L^\infty(\Omega)} \le M_0\|v(s)\|_{L^2(\Omega)}$ for any $s \in (0,1)$, which implies that $\|v(2)\|^2_{L^\infty(\Omega)} \le M_0^2 \int_0^1 \|v(s)\|^2_{L^2(\Omega)} ds \le M_0^2 C$. By the invariance of the attractor we obtain that there exists a constant M_2 such that $\|u_t(t)\|_{L^\infty(\Omega)} \le M_2$ for any orbit $u(\cdot)$ of the attractor.

Now, for t fixed, we can rewrite equation (2.5) as an elliptic equation in Ω_1, as

$$\begin{cases} -\operatorname{div}(d_0(x)\nabla u) + c(x)u = f(x,u) - u_t & \text{in } \Omega_1 \\ d_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = g(x,u) & \text{on } \Gamma_N^1 \\ u = 0 & \text{on } \Gamma_D, \\ u = u_{\Omega_0} & \text{on } \Gamma^*. \end{cases}$$

Since $f(x,u) - u_t \in L^\infty(\Omega)$, $g(x,u) \in L^\infty(\Gamma)$, and $u_{\Omega_0} \in L^\infty(\Gamma^*)$, with uniform bounds for $t \in \mathbb{R}$ and $u(\cdot)$ on the attractor, by part iii) in Proposition 3.8, we obtain that u lies in a bounded set of $C^\alpha(\bar{\Omega}_1)$. Since u is constant in Ω_0 we obtain that u lies in a bounded set of $C^\alpha(\bar{\Omega})$.

The existence of the extremal equilibria and the asymptotic estimate (6.16) follows from Theorem 3.2 in [32]; see also Corollary 3.11 in that reference. Observe that the uniformity in (6.16) for $x \in \Omega$ follows because for any initial data $0 \le u_0 \in L^2_{\Omega_0}(\Omega)$ the arguments above imply that for a given $\delta > 0$, there exists $T > 0$ such that $u(t; u_0) \le \phi^0 + \delta$ for $t \ge T$. Note we can assume that $u(T; \phi^0 + \delta) \le \phi^0 + \delta$ which implies that as $n \to \infty$, $u(nT; \phi^0 + \delta) \to \varphi_M$ monotonically. Since φ_M is continuous, see Remark 6.6, Dini's criterium (c.f. [8, p. 194]) implies the convergence is uniform. From this, as in the proof of Theorem 3.2 in [32] we also get $u(t+T; \phi^0+\delta) \le u(t; \phi^0+\delta) \to \varphi_M$ uniformly in Ω. The argument involving φ_m is analogous. ■

Remark 6.6 *i) Observe that the extremal equilibria satisfy*

$$|\varphi_m|, |\varphi_M| \le \phi^0$$

with ϕ^0 as in (6.18). Then from Propositions 3.8 and 3.9 we get ϕ^0 is Hölder continuous and either zero or strictly positive a.e. in each connected component of Ω_1.
ii) Also, any equilibrium of (2.5) satisfies

$$A_0\varphi = H(x,\varphi) \quad \text{in } H^{-1}_{\Omega_0}(\Omega), \tag{6.19}$$

that is

$$\begin{cases} -\operatorname{div}(d_0(x)\nabla\varphi) + c(x)\varphi = f(x,\varphi) & \text{in } \Omega_1 \\ d_0(x)\dfrac{\partial\varphi}{\partial\vec{n}} + b(x)\varphi = g(x,\varphi) & \text{on } \Gamma_N^1 \\ \varphi = 0 & \text{on } \Gamma_D \\ \varphi = \varphi_{\Omega_0} & \text{on } \Gamma^* \\ \dfrac{1}{|\Omega_0|}\Big[\int_{\Gamma^*} d_0\dfrac{\partial\varphi}{\partial\vec{n}} + \Big(\int_{\Omega_0} c + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)\varphi_{\Omega_0}\Big] = & \dfrac{1}{|\Omega_0|}\Big(\int_{\Omega_0} f(x,\varphi_{\Omega_0}) + \int_{\Gamma_N\cap\partial\Omega_0} g(x,\varphi_{\Omega_0})\Big). \end{cases} \tag{6.20}$$

Since f and g are bounded and globally Lipschitz, then Proposition 3.8 imples that φ is Hölder continuous in $\overline{\Omega}$.

Now we prove the following result on the differentiability of the nonlinear terms in problems (2.1) and (2.5) used in the proof of Theorem 6.5 above (see also comment after (6.9)). Recall that after the results in Section 6.1, we assumed in Section 6.2 that f and g have been truncated for large values of u in such a way that they are bounded and globally Lipschitz. Below we assume furthermore that they are $f, g \in C^2_{\rm b}(\overline{\Omega}\times\mathbb{R})$.

Lemma 6.7 *Assume $f, g \in C^2_{\rm b}(\overline{\Omega}\times\mathbb{R})$. We consider the operator $h : H^1_{\Gamma_D}(\Omega) \to H^{-\gamma}_{\Gamma_D}(\Omega)$ for $\gamma > 1/2$ (see (6.14)) given by*

$$h(u) = f(\cdot, u)_\Omega + g(\cdot, u)_{\Gamma_N}, \quad u \in H^1_{\Gamma_D}(\Omega)$$

as in (6.1).

Then, $h \in C^{1,\sigma}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))$ with $\sigma = \min\left\{1, \frac{2\gamma}{N-2}\right\}$ and for all $u, v \in H^1_{\Gamma_D}(\Omega)$

$$Dh(u)(v) = \big(\partial_u f(\cdot, u)v\big)_\Omega + \big(\partial_u g(\cdot, u)v\big)_{\Gamma_N}$$

in the sense of (3.3).

Proof. Let $u, v \in H^1_{\Gamma_D}(\Omega)$. Then observe that for $x \in \Omega$

$$|f(x, u(x)) - f(x, v(x)) - \partial_u f(x, u(x))(u(x) - v(x))| = |\big(\partial_u f(x, \xi_x) - \partial_u f(x, u(x))\big)(u(x) - v(x))|$$

for some $\xi_x = \alpha_x u(x) + (1-\alpha_x)v(x)$, $0 \le \alpha_x \le 1$. Using this and that $\partial_u f$ and $\partial^2_u f$ are bounded, we get for $0 < \sigma \le 1$

$$|\partial_u f(x, \xi_x) - \partial_u f(x, u(x))| \le |\partial_u f(x, \xi_x) - \partial_u f(x, u(x))|^\sigma |\partial_u f(x, \xi_x) - \partial_u f(x, u(x))|^{1-\sigma} \le C|u(x) - v(x)|^\sigma. \tag{6.21}$$

Hence

$$|f(x, u(x)) - f(x, v(x)) - \partial_u f(x, u(x))(u(x) - v(x))| \le C|u(x) - v(x)|^{1+\sigma}.$$

Now consider $\varphi \in H^\gamma_{\Gamma_D}(\Omega)$. Then the estimate above, Hölder's inequality and Sobolev embeddings $H^\gamma_{\Gamma_D}(\Omega) \subset L^{\frac{2N}{N-2\gamma}}(\Omega)$ give

$$\left|\int_\Omega [f(\cdot, u) - f(\cdot, v) - \partial_u f(\cdot, u)(u - v)]\varphi\right| \le C\int_\Omega |u-v|^{1+\sigma}|\varphi| \le C\|u-v\|^{1+\sigma}_{L^{(1+\sigma)\frac{2N}{N+2\gamma}}(\Omega)} \|\varphi\|_{L^{\frac{2N}{N-2\gamma}}(\Omega)}.$$

Using now $H^1_{\Gamma_D}(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ we get

$$\left|\int_\Omega [f(\cdot, u) - f(\cdot, v) - \partial_u f(\cdot, u)(u - v)]\varphi\right| \le C\|u - v\|^{1+\sigma}_{H^1_{\Gamma_D}(\Omega)} \|\varphi\|_{H^\gamma_{\Gamma_D}(\Omega)}$$

as long as $\sigma \le \min\{1, \frac{2+2\gamma}{N-2}\}$. Thus

$$\|f(\cdot, u) - f(\cdot, v) - \partial_u f(\cdot, u)(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)} \le C\|u - v\|^{1+\sigma}_{H^1_{\Gamma_D}(\Omega)}.$$

Since $\partial_u f(\cdot, u) \in L^\infty(\Omega)$ then, as a multiplication operator, it defines a linear operator in $\mathcal{L}(H^1_{\Gamma_D}(\Omega), L^2(\Omega))$ and thus in $\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))$. Therefore for all $u, z \in H^1_{\Gamma_D}(\Omega)$

$$Df_\Omega(u)(z) = \big(\partial_u f(\cdot, u)z\big)_\Omega \in H^{-\gamma}_{\Gamma_D}(\Omega)$$

as in (3.3).

Analogously, for $x \in \Gamma_N$ and $0 < \sigma \leq 1$

$$|g(x, u(x)) - g(x, v(x)) - \partial_u g(x, u(x))(u(x) - v(x))| \leq C|u(x) - v(x)|^{1+\sigma}.$$

Now for $\varphi \in H^{\gamma}_{\Gamma_D}(\Omega)$, Hölder's inequality and Sobolev embeddings $H^{\gamma}_{\Gamma_D}(\Omega) \subset L^{\frac{2(N-1)}{N-2\gamma}}(\Gamma_N)$ give

$$\left| \int_{\Gamma_N} [g(\cdot, u) - g(\cdot, v) - \partial_u g(\cdot, u)(u - v)]\varphi \right| \leq C \int_{\Gamma_N} |u - v|^{1+\sigma}|\varphi| \leq$$

$$\leq C\|u - v\|^{1+\sigma}_{L^{(1+\sigma)\frac{2(N-1)}{N-2+2\gamma}}(\Gamma_N)} \|\varphi\|_{L^{\frac{2(N-1)}{N-2\gamma}}(\Gamma_N)}.$$

Using now $H^1_{\Gamma_D}(\Omega) \subset L^{\frac{2(N-1)}{N-2}}(\Gamma_N)$ we get

$$\left| \int_{\Gamma_N} [g(\cdot, u) - g(\cdot, v) - \partial_u g(\cdot, u)(u - v)]\varphi \right| \leq C\|u - v\|^{1+\sigma}_{H^1_{\Gamma_D}(\Omega)} \|\varphi\|_{H^{\gamma}_{\Gamma_D}(\Omega)}$$

as long as $\sigma \leq \min\{1, \frac{4\gamma-2}{N-2\gamma}\}$. Thus

$$\|g(\cdot, u) - g(\cdot, v) - \partial_u g(\cdot, u)(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)} \leq C\|u - v\|^{1+\sigma}_{H^1_{\Gamma_D}(\Omega)}.$$

Since $\partial_u g(\cdot, u) \in L^{\infty}(\Gamma_N)$ then, as a multiplication operator on the boundary, it defines a linear operator in $\mathcal{L}(H^1_{\Gamma_D}(\Omega), L^2(\Gamma_N))$ and thus in $\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))$. Therefore for all $u, z \in H^1_{\Gamma_D}(\Omega)$

$$Dg_{\Gamma_N}(u)(z) = \big(\partial_u g(\cdot, u)z\big)_{\Gamma_N} \in H^{-\gamma}_{\Gamma_D}(\Omega)$$

as in (3.3).

On the other hand

$$\|Df_\Omega(u) - Df_\Omega(v)\|_{\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))} = \sup_{\substack{z \in H^1_{\Gamma_D}(\Omega) \\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \|Df_\Omega(u)z - Df_\Omega(v)z\|_{H^{-\gamma}_{\Gamma_D}(\Omega)}$$

$$= \sup_{\substack{z \in H^1_{\Gamma_D}(\Omega) \\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \sup_{\substack{\varphi \in H^{\gamma}_{\Gamma_D}(\Omega) \\ \|\varphi\|_{H^{\gamma}_{\Gamma_D}(\Omega)}=1}} \left| \int_\Omega \big(\partial_u f(\cdot, u) - \partial_u f(\cdot, v)\big) z\varphi \right| \leq$$

$$\leq \sup_{\substack{z \in H^1_{\Gamma_D}(\Omega) \\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \sup_{\substack{\varphi \in H^{\gamma}_{\Gamma_D}(\Omega) \\ \|\varphi\|_{H^{\gamma}_{\Gamma_D}(\Omega)}=1}} C \int_\Omega |u - v|^{\sigma}|z||\varphi|$$

where in the last step we argued as in (6.21). Now Sobolev embeddings $H^{\gamma}_{\Gamma_D}(\Omega) \subset L^{\frac{2N}{N-2\gamma}}(\Omega)$, $H^1_{\Gamma_D}(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ and Hölder's inequality gives

$$\|Df_\Omega(u) - Df_\Omega(v)\|_{\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))} \leq C\|u - v\|^{\sigma}_{L^{\sigma\frac{2N}{2+2\gamma}}(\Omega)} \leq C\|u - v\|^{\sigma}_{H^1_{\Gamma_D}(\Omega)}$$

as long as $\sigma \leq \min\{1, \frac{2+2\gamma}{N-2}\}$.

Analogously

$$\|Dg_{\Gamma_N}(u) - Dg_{\Gamma_N}(v)\|_{\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))} = \sup_{\substack{z \in H^1_{\Gamma_D}(\Omega) \\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \|Dg_{\Gamma_N}(u)z - Dg_{\Gamma_N}(v)z\|_{H^{-\gamma}_{\Gamma_D}(\Omega)}$$

$$= \sup_{\substack{z\in H^1_{\Gamma_D}(\Omega)\\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \sup_{\substack{\varphi\in H^{\gamma}_{\Gamma_D}(\Omega)\\ \|\varphi\|_{H^{\gamma}_{\Gamma_D}(\Omega)}=1}} \left| \int_{\Gamma_N} \big(\partial_u g(\cdot,u) - \partial_u g(\cdot,v)\big) z\varphi \right| \leq$$

$$\leq \sup_{\substack{z\in H^1_{\Gamma_D}(\Omega)\\ \|z\|_{H^1_{\Gamma_D}(\Omega)}=1}} \sup_{\substack{\varphi\in H^{\gamma}_{\Gamma_D}(\Omega)\\ \|\varphi\|_{H^{\gamma}_{\Gamma_D}(\Omega)}=1}} C \int_{\Gamma_N} |u-v|^{\sigma}|z||\varphi|.$$

Now Sobolev embeddings $H^{\gamma}_{\Gamma_D}(\Omega) \subset L^{\frac{2(N-1)}{N-2\gamma}}(\Gamma_N)$, $H^1_{\Gamma_D}(\Omega) \subset L^{\frac{2(N-1)}{N-2}}(\Gamma_N)$ and Hölder's inequality gives

$$\|Dg_{\Gamma_N}(u) - Dg_{\Gamma_N}(v)\|_{\mathcal{L}(H^1_{\Gamma_D}(\Omega), H^{-\gamma}_{\Gamma_D}(\Omega))} \leq C\|u-v\|^{\sigma}_{L^{\sigma\frac{N-1}{\gamma}}(\Gamma_N)} \leq C\|u-v\|^{\sigma}_{H^1_{\Gamma_D}(\Omega)}$$

as long as $\sigma \leq \min\{1, \frac{2\gamma}{N-2}\}$. ■

Remark 6.8 *A similar result can be found in Lemma 4.3 in [10] with $\sigma = 1$. Due to this, the Sobolev embeddings used there only hold for $N \leq 4$ in Ω and $N \leq 3$ in Γ.*

As a consequence we obtain the following result for the nonlinear terms of the limit problem (2.5) as in (6.10) or (6.12).

Corollary 6.9 *Assume $f, g \in C^2_b(\overline{\Omega} \times \mathbb{R})$. We consider the operator $H : H^1_{\Omega_0}(\Omega) \to H^{-\gamma}_{\Omega_0}(\Omega)$ for $\gamma > 1/2$ given by*

$$H(u) = Ph(u) = P(f(u)_{\Omega} + g(u)_{\Gamma_N}), \quad u \in H^1_{\Omega_0}(\Omega)$$

as in (6.12), where P is given in (5.7).

Then, $H \in C^{1,\sigma}(H^1_{\Omega_0}(\Omega), H^{-\gamma}_{\Omega_0}(\Omega))$ with $\sigma = \min\left\{1, \frac{2\gamma}{N-2}\right\}$ and for all $u, v \in H^1_{\Omega_0}(\Omega)$

$$DH(u)(v) = P\Big(\big(\partial_u f(\cdot,u)v\big)_{\Omega} + \big(\partial_u g(\cdot,u)v\big)_{\Gamma_N}\Big).$$

Proof. Just use P is linear and continuous and Lemma 6.7. ■

As mentioned at the end of Section 6.1 for (2.1), also for problem (2.5) equilibria play an important role in the dynamics. As for problem (2.1), after truncation of nonlinear terms, problem (2.5) is a gradient system in $H^1_{\Omega_0}(\Omega)$ with energy

$$V_0(u) = \frac{1}{2}\int_{\Omega_1} d_0|\nabla u|^2 + \int_{\Omega_1} c|u|^2 + \int_{\Gamma^1_N} b|u|^2 +$$

$$\Big(\int_{\Omega_0} c + \int_{\Gamma_N\cap\partial\Omega_0} b\Big)|u_{\Omega_0}|^2 - \int_{\Omega_0} F_0(x, u_{\Omega_0}) - \int_{\Omega_1} F_0(x,u) - \int_{\Gamma_N\cap\partial\Omega_0} G_0(x, u_{\Omega_0}) - \int_{\Gamma^1_N} G_0(x,u) \tag{6.22}$$

with $F_0(x,s) = \int_0^s f(x,s)\,ds$ and $G_0(x,s) = \int_0^s g(x,s)\,ds$. Therefore

$$\mathcal{A}_0 = W^u(E_0) \tag{6.23}$$

where E_0 denotes the set of equilibria

Now observe that comparison arguments, see Remark 4.1, and Theorem 6.5 imply that φ_M is stable from above. This in turn, implies the following result which, in fact, applies to any equilibrium that is stable from above or from below.

Lemma 6.10 *The first eigenvalue of the linearization at the maximal equilibria φ_M satisfies*

$$\Lambda_0(A_0 - DH(\varphi_M)) \geq 0.$$

Proof. Indeed, if $\Lambda_0(A_0 - DH(\varphi_M)) < 0$ since this eignevalue always has nonnegative associated eigenfunctions, see Proposition 3.8, there would exists initial data above φ_M for which the solution of the nonlinear problem would not converge to φ_M. In fact those solution, following the unstable manifold of φ_M would get away of φ_M and since (6.10) is a gradient system, see (6.22), (6.23), they would have to converge to another equilibrium. Since the initial data are above φ_M they need to converge to φ_M, which is a contradiction. ∎

Now we focus on non-negative solutions of (2.5). First, from comparison we get the following results

Proposition 6.11 *i) Assume $H(x,0) \geq 0$. Then $u_0 \geq 0$ implies $u(t,u_0) \geq 0$. Also $\varphi_M \geq 0$ in Ω.*
ii) Assume $H(x,0) = 0$ then $\varphi_m \leq 0 \leq \varphi_M$.
iii) Assume Ω_1 is connected. Assume also that either $f(x,0), g(x,0) \geq 0$ and not both trivial.
Then, any nonnegative equilibrium of (2.5) is a.e. positive in Ω_1.
iv) Assume Ω_1 is connected, $f(x,0) \equiv g(x,0) \equiv 0$, and there exist $s_0 > 0$, $M \in L^\infty(\Omega)$ and $N \in L^\infty(\Gamma_N)$, such that for all $0 \leq s \leq s_0$,

$$f(x,s) \geq M(x)s \quad \textit{for all} \quad x \in \Omega, \quad g(x,s) \geq N(x)s \quad \textit{for all} \quad x \in \Gamma_N$$

and the first eigenvalue of the operator $A_0 - P(M_\Omega + N_{\Gamma_N})$, see (5.11), is $\lambda_1 < 0$.
Then, the maximal equilibrium of (2.5) φ_M is a.e. positive in Ω_1.

Proof. i) Since $H(x,0) \geq 0$ then 0 is a subsolution of the associated stationary problem, and then $0 \leq u(t,x;0)$ and is increasing in time, see Theorem A.12 in [4] and Lemma 2.9 in [32].

By comparison, if $u_0 \geq 0$ implies $u(t,x;u_0) \geq u(t,x;0) \geq 0$ for all $t \geq 0$. From (6.16) we get $\varphi_M \geq 0$.
ii) If $H(x,0) = 0$ then 0 is an equilibrium of (2.5) and (6.16) gives the result.
iii) Note that $H(x,0) \geq 0$, $H(x,0) \not\equiv 0$, see (6.10). Then, from part i), $0 \leq u(t,x;0)$, is nontrivial, increasing and bounded. Hence the pointwise limit

$$0 \leq \lim_{t\to\infty} u(t,x;0) = \varphi_m^+(x) \leq \varphi_M(x)$$

is also the limit in $L^2_{\Omega_0}(\Omega)$. Thus, $0 \leq \varphi_m^+ \in L^2_{\Omega_0}(\Omega)$ is a nonnegative equilibrium for (2.5); see Lemma 3.1 in [32]. In particular, since φ_m^+ is continuous in $\overline{\Omega}$, see Remark 6.6, then Dini's criterium (c.f. [8, p. 194]) implies the monotonic limit $u(t,x;0) \to \varphi_m^+$ is uniform in $\overline{\Omega}$.

Now φ_m^+ is minimal since any other nonnegative equilibrium ψ, satisfies $0 \leq \psi$, and by the comparison principle, we must have $u(t,x;0) \leq \psi(x)$. So, taking limits as $t \to \infty$ we have $\varphi_m^+ \leq \psi$.

Now we prove φ_m^+ is a.e. strictly positive in Ω_1, which concludes the result. For this, note that from (6.19), (6.20) we can write for $\varphi = \varphi_m^+ \geq 0$

$$-\mathrm{div}(d_0(x)\nabla\varphi) + c(x)\varphi + \mu\varphi = f(x,\varphi) + \mu\varphi \quad \text{in } \Omega_1.$$

Then we claim that we can chose $\mu > 0$ such that $f(x,\varphi) + \mu\varphi \geq 0$ in Ω_1. This, (6.19) and the maximum principle in Proposition 3.9 implies φ_m^+ is a.e. strictly positive in Ω_1. Finally, since $0 \leq \varphi$ is bounded, note that the claim on f holds as long as μ is smaller than the Lipschitz constant of f in a set of the form $[0, s_0] \times \overline{\Omega_1}$. Then $f(x,s) + \mu s$ is increasing in that set and then

$$f(x,s) + \mu s \geq f(x,0) \geq 0, \quad 0 \leq s \leq s_0, \ x \in \Omega_1.$$

iv) Suppose that $f(x,0) \equiv g(x,0) \equiv 0$ as in the statement. Since Ω_1 is connected, by part iv) in Proposition 3.8, there exists $\phi \in H^1_{\Omega_0}(\Omega) \cap C^\alpha(\overline{\Omega})$ a positive eigenfunction for the first eigenvalue $\lambda_1 < 0$ of the operator $A_0 - P(M_\Omega + N_{\Gamma_N})$ normalized as $||\phi||_{L^\infty(\Omega)} = 1$.

Given $s_0 \geq \gamma > 0$ we have that $\gamma\phi$ is a subsolution of the stationary problem since by the assumption on f and g we have

$$A_0 \gamma\phi = P((M\gamma\phi)_\Omega + (N\gamma\phi)_{\Gamma_N}) + \lambda_1 \gamma\phi \leq P(f(\gamma\phi)_\Omega + g(\gamma\phi)_{\Gamma_N}) = H(\gamma\phi).$$

Therefore, $u(t,x;\gamma\phi)$ is monotonic increasing in time (see Lemma 2.9 in [32]). From Theorem 6.5 and (6.16) $u(t,x;\gamma\phi)$ is bounded in $H^1_{\Omega_0}(\Omega) \cap L^\infty(\Omega)$ and then the pointwise limit $u(t,x;\gamma\phi) \to \varphi_\gamma \leq \varphi_M$ is also the limit in $L^2_{\Omega_0}(\Omega)$. From Lemma 3.1 in [32], $0 \leq \varphi_\gamma$ is an equilibrium, that is

$$A_0 \varphi_\gamma = H(x, \varphi_\gamma).$$

In particular, since φ_γ is continuous, see Remark 6.6, in $\overline{\Omega}$ then Dini's criterium (c.f. [8, p. 194]) implies the monotonic limit $u(t,x;\gamma\phi) \to \varphi_\gamma$ is uniform in $\overline{\Omega}$.

Now, given $\gamma < \tilde{\gamma} \leq s_0$ we have that $\gamma\phi \leq \tilde{\gamma}\phi$. Thus $u(t,x;\gamma\phi) \leq u(t,x;\tilde{\gamma}\phi)$ and taking $t \to \infty$ we get $\varphi_\gamma \leq \varphi_{\tilde{\gamma}}$. Thus, $\{\varphi_\gamma\}_{0<\gamma\leq s_0}$ is an ordered set, bounded in $H^1_{\Omega_0}(\Omega) \cap L^\infty(\Omega)$.

In particular for $0 < \gamma \leq s_0$

$$\gamma\phi \leq \varphi_\gamma \leq \varphi_M$$

In particular $s_0\phi \leq \varphi_M$ and φ_M is a.e. positive in Ω_1. ■

From Proposition 6.11 we have $\varphi_M \geq 0$ and a.e. strictly in Ω_1. In the following results, we give conditions for the extremal solution φ_M to be the unique positive equilibrium and analyze its stability. They will apply in particular for logistic type nonlinearities.

Definition 6.12 *Assume A is a measure space and $I \subset \mathbb{R}$. Then for functions defined in $A \times I$ with values in $\mathbb{R}$ we say that $h_1 \leq h_2$ strict in measure if for all $s \in I$, $h_1(x,s) \leq h_2(x,s)$, $x \in A$, strictly in a set of positive measure.*

We say that $h(x,s)$ is strictly decreasing in measure if given $s_1 < s_2$ we have $h(x,s_1) \geq h(x,s_2)$, $x \in A$, with strict inequality in a set of positive measure.

Remark 6.13 *i) If h is smooth in s then $\frac{h(x,s)}{s}$ being strictly decreasing in measure is equivalent to*

$$\partial_s h(x,s) s \leq h(x,s) \quad \textit{strict in measure.} \tag{6.24}$$

ii) Assume h is smooth and $h(x,0) \geq 0$ and strictly concave in measure, i.e. $\partial_s h(x,s)$ is strictly decreasing in measure. By the mean value theorem, for $s > 0$ and certain $0 < \xi_x < s$

$$\frac{h(x,s)}{s} \geq \frac{h(x,s) - h(x,0)}{s} = \partial_s h(x,\xi_x) \geq \partial_s h(x,s) \quad \textit{strict in measure.}$$

Thus, $h(x,s)/s$ is strictly decreasing in measure for $s \geq 0$.
iii) Assume in particular that h has the specific logistic *form*

$$h(x,s) = h_0(x) + m(x)s - n(x)s^\rho$$

with $s > 0$, $\rho > 1$ and $h_0(x), n(x) \geq 0$. Then $h(x,0) = h_0(x) \geq 0$,

$$\partial_s h(x,s) = m(x) - \rho n(x) s^{\rho-1},$$

and

$$\frac{h(x,s)}{s} = \frac{h_0(x)}{s} + m(x) - n(x)s^{\rho-1}$$

are stricltly decreasing in measure for $s > 0$.

Theorem 6.14 *Let* $\varphi_M \geq 0$ *be the maximal positive solution for (2.5). Assume in addition that*

$$\frac{f(x,s)}{s} \quad \text{is strictly decreasing in measure for } s>0 \text{ and } x \in \Omega_1$$

and

$$\frac{g(x,s)}{s} \quad \text{is decreasing in measure for } s>0.$$

Then the first eigenvalue of the linearized problem satisfies $\Lambda_0(A_0 - DH(\varphi_M)) > 0$. *Moreover, there is no other nonnegative solution of (2.5) that is a.e. positive in* Ω_1.

Proof. To shorten the notation in the following we write φ instead of φ_M. First, from Lemma 6.10 we have $\Lambda_0(A_0 - DH(\varphi)) \geq 0$. Suppose $\Lambda_0(A_0 - DH(\varphi)) = 0$ and consider $\phi \geq 0$ an associated eigenfunction, which satisfies $A_0\phi = DH(\varphi)\phi$, see Proposition 3.8.

Using (6.12) and (6.24) we claim

$$\langle H(\varphi), \phi\rangle = \langle P(f(\varphi)_\Omega + g(\varphi)_{\Gamma_N}), \phi\rangle > \langle P((\partial_u f(\varphi)\varphi)_\Omega + (\partial_u g(\varphi)\varphi)_{\Gamma_N}), \phi\rangle = \langle DH(\varphi), \phi\rangle$$

where P is defined in (5.7). Indeed in the left hand side we have

$$\left(\int_{\Omega_0} f(x,\varphi_{\Omega_0}) + \int_{\Gamma_N\cap\partial\Omega_0} g(x,\varphi_{\Omega_0})\right)\phi_{\Omega_0} + \int_{\Omega_1} f(x,\varphi)\phi + \int_{\Gamma_N^1} g(x,\varphi)\phi$$

which by (6.24) is larger than

$$\left(\int_{\Omega_0} \partial_u f(x,\varphi_{\Omega_0}) + \int_{\Gamma_N\cap\partial\Omega_0} \partial_u g(x,\varphi_{\Omega_0})\right)\varphi_{\Omega_0}\phi_{\Omega_0} + \int_{\Omega_1} \partial_u f(x,\varphi)\varphi\phi + \int_{\Gamma_N^1} \partial_u g(x,\varphi)\varphi\phi$$

and this equals $\langle P((\partial_u f(\varphi)\varphi)_\Omega + (\partial_u g(\varphi)\varphi)_{\Gamma_N}), \phi\rangle$. Since $\varphi, \phi > 0$ a.e. in Ω_1 and the assumption on f, then $\int_{\Omega_1} f(x,\varphi)\phi > \int_{\Omega_1} \partial_u f(x,\varphi)\varphi\phi$ and the claim is proved.

Then since $A_0\varphi = H(\varphi)$ and $A_0\phi = DH(\varphi)\phi$,

$$\langle A_0\varphi, \phi\rangle = \langle H(\varphi), \phi\rangle > \langle DH(\varphi), \phi\rangle = \langle A_0\phi, \varphi\rangle$$

which is a contradiction. Thus, $\Lambda_0(A_0 - DH(\varphi)) > 0$.

Now we prove uniqueness of nonnegative solutions which are a.e. positive in Ω_1. Since φ is the maximal solution of (2.5), if there is any other such nonegative solution we then have $0 \leq \psi \leq \varphi$. Then

$$0 = \langle A_0\varphi, \psi\rangle - \langle \varphi, A_0\psi\rangle = \langle H(\varphi), \psi\rangle - \langle \varphi, H(\psi)\rangle.$$

Since $0 \leq \psi \leq \varphi$, from the assumptions on the nonlinear terms, (6.12) and (5.7) the different terms in $\langle H(\varphi), \psi\rangle - \langle \varphi, H(\psi)\rangle$ can be written as follows. In Ω_0, assuming both φ_{Ω_0}, ψ_{Ω_0} are nonzero, we have

$$\int_{\Omega_0}\left(\frac{f(x,\varphi_{\Omega_0})}{\varphi_{\Omega_0}} - \frac{f(x,\psi_{\Omega_0})}{\psi_{\Omega_0}}\right)\varphi_{\Omega_0}\psi_{\Omega_0} + \int_{\Gamma_N\cap\partial\Omega_0}\left(\frac{g(x,\varphi_{\Omega_0})}{\varphi_{\Omega_0}} - \frac{g(x,\psi_{\Omega_0})}{\psi_{\Omega_0}}\right)\varphi_{\Omega_0}\psi_{\Omega_0} \leq 0$$

while this term is zero if $\varphi_{\Omega_0} = \psi_{\Omega_0} = 0$. On the other hand in the regions where both φ, ψ are nonzero, the integrands of $\langle H(\varphi), \psi\rangle - \langle \varphi, H(\psi)\rangle$ in Ω_1 and Γ_N^1 respectively can be written as

$$\left(\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi}\right)\varphi\psi \leq 0, \quad \left(\frac{g(x,\varphi)}{\varphi} - \frac{g(x,\psi)}{\psi}\right)\varphi\psi \leq 0$$

and of course the integrand is zero whenever φ or ψ are zero.

Therefore, if $\psi \leq \varphi$ strict in measure in Ω_1 we have

$$\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi} \leq 0 \quad \text{strict in measure in } \Omega_1$$

and then

$$\langle H(\varphi), \psi\rangle - \langle \varphi, H(\psi)\rangle \leq \int_{\Omega_1} f(x,\phi)\psi - f(x,\psi)\phi < 0$$

which is a contradiction. Hence $\psi = \varphi$ in Ω_1. Therefore taking traces in Γ^* we get $\psi_{\Omega_0} = \varphi_{\Omega_0}$ in Ω_0 and hence $\psi = \varphi$ in Ω. ■

7 Convergence for nonlinear problems

7.1 Upper semicontinuity of the attractors

The upper continuity of the attractors of problems (2.1) and (2.5) at $\varepsilon = 0$ follows now from the uniform boundedness of the attractors of the approximate and limit problems, provided by Theorem 6.2 and Theorem 6.5, arguing as in Theorem 5.2. in [7]. In particular, we can compare the asymptotic dynamics of problems (2.1) and (2.5) in the $H^1_{\Gamma_D}(\Omega)$-metric.

Theorem 7.1 *The global attractors $\{\mathcal{A}_\varepsilon\}_\varepsilon$ are upper semicontinuous at $\varepsilon = 0$ in $H^1_{\Gamma_D}(\Omega)$, that is,*

$$\mathrm{dist}_{H^1}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where dist_{H^1} *denotes the Hausdorff semidistance in* $H^1_{\Gamma_D}(\Omega)$.

Proof. Using (5.7), take $\eta_\varepsilon \in \mathcal{A}_\varepsilon$ as initial data for (2.1) and $P\eta_\varepsilon \in L^2_{\Omega_0}(\Omega)$ as initial data for (2.5). Then taking into account Proposition 5.11, Corollary 5.12 and the uniform estimates for the solutions of the approximate problems given in Theorem 6.2 we have, from (6.2) and (6.13), for $t \in (0,\tau)$,

$$\|u^\varepsilon(t,\eta_\varepsilon) - u(t,P\eta_\varepsilon)\|_{H^1(\Omega)} \leq \|\mathrm{e}^{-A_\varepsilon t}\eta_\varepsilon - \mathrm{e}^{-A_0 t}P\eta_\varepsilon\|_{H^1(\Omega)}$$

$$+ \int_0^t \|\mathrm{e}^{-A_\varepsilon(t-s)}h(u^\varepsilon(s,\eta_\varepsilon)) - \mathrm{e}^{-A_0(t-s)}Ph(u^\varepsilon(s,\eta_\varepsilon))\|_{H^1(\Omega)}\mathrm{d}s$$

$$+ \int_0^t \|\mathrm{e}^{-A_0(t-s)}P\Big(h(u^\varepsilon(s,\eta_\varepsilon)) - h(u(s,P\eta_\varepsilon)\Big)\|_{H^1(\Omega)}\mathrm{d}s$$

$$\leq K\tilde{c}(\varepsilon)t^{-\alpha} + K\int_0^t \tilde{c}(\varepsilon)(t-s)^{-\alpha}\mathrm{d}s + \int_0^t (t-s)^{-\beta}L\|u^\varepsilon(t,\eta_\varepsilon) - u(t,P\eta_\varepsilon)\|_{H^1(\Omega)}\mathrm{d}s$$

$$\leq K\tilde{c}(\varepsilon)\tfrac{\tau}{1-\alpha}t^{-\alpha} + L\int_0^t (t-s)^{-\beta}\|u^\varepsilon(s,\eta_\varepsilon) - u(s,P\eta_\varepsilon)\|_{H^1(\Omega)}\mathrm{d}s,$$

where $\alpha \in (3/4,1)$ is as in Corollary 5.12 and $\beta \in (0,1)$. By the singular Gronwall Lemma (see [22]) we obtain the existence of a constant $M = M(\alpha,\beta,L,\tau)$ and a positive function $\tilde{c}(\cdot)$, with $\tilde{c}(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|u^\varepsilon(t,\eta_\varepsilon) - u(t,P\eta_\varepsilon)\|_{H^1(\Omega)} \leq MK\tilde{c}(\varepsilon)t^{-\alpha}, \qquad t \in (0,\tau), \quad \eta_\varepsilon \in \mathcal{A}_\varepsilon. \tag{7.1}$$

Now, since $\cup_\varepsilon \mathcal{A}_\varepsilon$ are bounded in $H^1(\Omega)$ we have that $\cup_\varepsilon P\mathcal{A}_\varepsilon$ is bounded in $L^2_{\Omega_0}(\Omega)$. From Theorem 6.5, $\mathcal{A}_0$ attracts bounded sets of $L^2_{\Omega_0}(\Omega)$. Therefore, given $\delta > 0$, there exists $\tau = \tau(\delta)$ such that $\mathrm{dist}_{H^1}(u(\tau, P\eta_\varepsilon), \mathcal{A}_0) \leq \delta/2$, for all $\eta_\varepsilon \in \mathcal{A}_\varepsilon$ and all $\varepsilon \in (0, \varepsilon_0)$. Furthermore, since the attractors are invariant, given $v_\varepsilon \in \mathcal{A}_\varepsilon$ there exists $\eta_\varepsilon \in \mathcal{A}_\varepsilon$ with $u^\varepsilon(\tau, \eta_\varepsilon) = v_\varepsilon$. Thus, choosing $\varepsilon_1 \in (0, \varepsilon_0)$ such that $MK\tilde{c}(\varepsilon)\tau^{-\alpha} \leq \delta/2$, for all $\varepsilon \in (0, \varepsilon_1)$, we have

$$\mathrm{dist}_{H^1}(v_\varepsilon, \mathcal{A}_0) \leq \|v_\varepsilon - u(\tau, P\eta_\varepsilon)\|_{H^1(\Omega)} + \mathrm{dist}(u(\tau, P\eta_\varepsilon), \mathcal{A}_0) \leq \delta, \quad v_\varepsilon \in \mathcal{A}_\varepsilon, \quad \varepsilon \in (0, \varepsilon_1),$$

from which the upper semicontinuity of the attractors in $H^1(\Omega)$ follows. ■

With this, arguing as in Proposition 5.3 and Corollary 5.4 in [7] we get the following.

Proposition 7.2 *i) Let ε_k be a sequence such that $\varepsilon_k \in (0, \varepsilon_0)$ and $\varepsilon_k \to 0$ as $k \to \infty$. Let $\eta_{\varepsilon_k} \in \mathcal{A}_{\varepsilon_k}$ such that $\eta_{\varepsilon_k} \to \eta_0$ as $k \to \infty$ in $H^1_{\Gamma_D}(\Omega)$. Then, $\eta_0 \in \mathcal{A}_0$ and*

$$u^{\varepsilon_k}(\cdot, \eta_{\varepsilon_k}) \to u(\cdot, \eta_0) \quad \text{in } C([0,T]; H^1_{\Gamma_D}(\Omega)), \text{ as } k \to \infty, \text{ for all } T > 0.$$

ii) For any $\{\varepsilon_k\}_k$ with $\varepsilon_k \to 0$ as $k \to \infty$, and any sequence or complete solutions in the attractors, $u^{\varepsilon_k}(\cdot) \subset \mathcal{A}_{\varepsilon_k}$, there exist a subsequence ε_{k_j} and a complete solution $u^0(\cdot)$ in the attractor $\mathcal{A}_0$ such that

$$u^{\varepsilon_{k_j}}(\cdot) \to u^0(\cdot) \quad \text{in } C([-T,T]; H^1_{\Gamma_D}(\Omega)), \text{ as } j \to \infty \text{ for all } T > 0.$$

iii) For any $\{\varepsilon_k\}_k$ with $\varepsilon_k \to 0$ as $k \to \infty$, and any sequence of equilibria $\varphi^{\varepsilon_k} \in \mathcal{A}_{\varepsilon_k}$, there exists a subsequence ε_{k_j} and an equilibrium of the limit problem $\varphi^0 \in \mathcal{A}_0$ such that

$$\varphi^{\varepsilon_{k_j}} \to \varphi^0 \quad \text{in } H^1_{\Gamma_D}(\Omega), \text{ as } j \to \infty.$$

7.2 Continuity of the extremal equilibria

Notice that, from Proposition 7.2 any accumulation point in $H^1_{\Gamma_D}(\Omega)$, φ^0, of the set of maximal equilibria for the approximate problems as in Theorem 6.3, $\{\varphi^\varepsilon_M\}$, we have $\varphi^0 \in H^1_{\Omega_0}(\Omega)$ and

$$\varphi^0 \leq \varphi^0_M \tag{7.2}$$

where φ^0_M is the extremal equilibrium of problem (2.5) as in Theorem 6.5.

Then we have the following remark regarding the continuity of the maximum equilibria. Notice that the argument for the minimal equilibria is analogous.

Lemma 7.3 *Assume there exist equilibria of the approximate problems φ^ε such that $\varphi^\varepsilon \to \varphi^0_M$ as $\varepsilon \to \infty$. Then the extremal equilibria are continuous in ε, i.e.*

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon_M = \varphi^0_M. \tag{7.3}$$

Proof. Let $\varphi^0 = \lim_{n\to\infty} \varphi^{\varepsilon_n}_M$ any accumulation point in $H^1_{\Gamma_D}(\Omega)$ for the set of maximal equilibria of the approximate problems. Then,

$$\varphi^0_M = \lim_{n\to\infty} \varphi^{\varepsilon_n} \leq \lim_{n\to\infty} \varphi^{\varepsilon_n}_M = \varphi^0 \leq \varphi^0_M$$

and (7.3) holds. ■

Hence, we discuss below the continuity of the extremal equilibria and show how this is related to their stability.

In fact, if $\varphi_M^0 \in H^1_{\Omega_0}(\Omega)$ is the extremal equilibrium for (2.5), or (6.10), then by Lemma 6.10 we have $\Lambda_0(A_0 - DH(\varphi_M^0)) \geq 0$. We show below that if

$$\Lambda_0(A_0 - DH(\varphi_M^0)) > 0 \tag{7.4}$$

then the maximal equilibria of the problems (4.2) are continuous at $\varepsilon = 0$, that is (7.3) holds. This holds in particular under the assumptions in Theorem 6.14. For this we prove that we can use Lemma 7.3.

In order to prove this, we use some ideas developed in [5] and [9, 10] conveniently adapted to the situation in this paper. First note that equilibria of (2.1) and (2.5) satisfy, with the notation in (6.1) and (6.10), respectively,

$$A_\varepsilon \varphi^\varepsilon = h(\varphi^\varepsilon), \quad \text{in } H^{-1}_{\Gamma_D}(\Omega), \qquad A_0\varphi^0 = H(\varphi^0) \quad \text{in } H^{-1}_{\Omega_0}(\Omega). \tag{7.5}$$

Also, their linear operators associated to their linearizations are given by $A_\varepsilon - Dh(\varphi^\varepsilon)$ and $A_0 - DH(\varphi^0)$.

Hence, if φ^0 is hyperbolic, that is, $0 \notin \sigma(A_0 - DH(\varphi^0))$ then we can rewrite the second equation (7.5) as $A_0\varphi^0 - DH(\varphi^0)\varphi^0 = H(\varphi^0) - DH(\varphi^0)\varphi^0$, whence

$$\varphi^0 = (A_0 - DH(\varphi^0))^{-1}(H(\varphi^0) - DH(\varphi^0)\varphi^0).$$

Hence φ^0 is a fixed point of the mapping in $H^1_{\Omega_0}(\Omega)$

$$\Phi_0(v) = (A_0 - DH(\varphi^0))^{-1}(H(v) - DH(\varphi^0)v), \quad v \in H^1_{\Omega_0}(\Omega). \tag{7.6}$$

From the results on linear elliptic problems in Section 3, to search for a nearby equilibria of (2.1) we will search for a nearby fixed point of

$$\Phi_\varepsilon(v) = (A_\varepsilon - Dh(\varphi^0))^{-1}(h(v) - Dh(\varphi^0)v), \quad v \in H^1_{\Gamma_D}(\Omega). \tag{7.7}$$

As in the references above the result we obtain applies to any hyperbolic equilibrium of the limit problem. Namely,

Proposition 7.4 *Let $\varphi^0 \in H^1_{\Omega_0}(\Omega)$ be an equilibrium of (2.5) such that $0 \notin \sigma(A_0 - DH(\varphi^0))$. Then, for certain $\varepsilon_0 > 0$ there exists $\delta > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a unique equilibrium φ^ε of (2.1) in the ball $B(\varphi_M^0; \delta)$ in $H^1_{\Gamma_D}(\Omega)$.*

In particular, there exist a sequence of equilibria of (2.1), φ^ε, with $\varphi^\varepsilon \to \varphi^0$ in $H^1_{\Gamma_D}(\Omega)$. Moreover, if (7.4) holds then (7.3) holds true.

Before proving Proposition 7.4 we prove some preparatory results.

Lemma 7.5 *Let $\varphi^0 \in H^1_{\Omega_0}(\Omega)$ be an equilibrium of (2.5) such that $0 \notin \sigma(A_0 - DH(\varphi^0))$. Then, there exists $\varepsilon_0 > 0$ such that the operator $(A_\varepsilon - Dh(\varphi^0))^{-1} : H^{-\gamma}_{\Gamma_D}(\Omega) \to H^1_{\Gamma_D}(\Omega)$ is uniformly bounded in ε for $0 < \varepsilon < \varepsilon_0$. Namely,*

$$\|(A_\varepsilon - Dh(\varphi^0))^{-1}\|_{\mathcal{L}(H^{-\gamma}_{\Gamma_D}(\Omega), H^1_{\Gamma_D}(\Omega))} \leq C_r$$

where r is such that the spectrum of $A_\varepsilon - Dh(\varphi^0)$ is in $\mathbb{C} \setminus B_r(0)$.

Proof. For this, notice that with the notation in Remark 5.7 we have

$$A_\varepsilon - Dh(\varphi^0) = L_\varepsilon + c_\Omega + b_{\Gamma_N} - Dh(\varphi^0).$$

Since $Dh(\varphi^0) \in L^2(\Omega) + L^2(\Gamma)$ by Theorem 5.8, the spectrum of $A_\varepsilon - Dh(\varphi^0)$ converges to the spectrum of

$$L_0 + F + P\big(c_\Omega + b_{\Gamma_N}\big)u - DH(\varphi^0) = A_0 - DH(\varphi^0),$$

see Remark 5.10.

Hence, there exists $\varepsilon_0, r > 0$ such that $\sigma(A_\varepsilon - Dh(\varphi^0)) \subset \mathbb{C} \setminus B_r(0)$ for all $0 < \varepsilon < \varepsilon_0$. Also given $\varepsilon > 0$ we have a Hilbert basis of $L^2(\Omega)$, $\{\phi_n^\varepsilon\}_n$, of eigenfunctions of $A_\varepsilon - Dh(\varphi^0)$, with corresponding eigenvalues $\{\mu_n^\varepsilon\}_n$. Hence, given $h \in H^{-\gamma}_{\Gamma_D}(\Omega)$ the solution $u \in H^1_{\Gamma_D}(\Omega)$ of

$$(A_\varepsilon - Dh(\varphi^0))u = h$$

can be computed as $u = \sum_k u_k \phi_k^\varepsilon$ with $\mu_k^\varepsilon u_k = h_k :=< h, \phi_k^\varepsilon >$. Hence, using $|\mu_k^\varepsilon| \geq r > 0$,

$$\|(A_\varepsilon - Dh(\varphi^0))^{-1}h\|^2_{H^1(\Omega)} = \sum_k u_k^2 = \sum_k \frac{h_k^2}{|\mu_k^\varepsilon|^2} \leq r^{-2(1-\gamma)} \sum_k \frac{h_k^2}{|\mu_k^\varepsilon|^{2\gamma}} = c_r \|h\|^2_{H^{-\gamma}_{\Gamma_D}(\Omega)}.$$

■

Proof of Proposition 7.4. We prove that Φ_ε in (7.7) is a contraction in the ball $B(\varphi^0; \delta)$ in $H^1_{\Gamma_D}(\Omega)$, for some $\delta > 0$ and all $\varepsilon > 0$ small enough. For this, let $u, v \in B(\varphi^0; \delta)$ and observe

$$\|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)\|_{H^1(\Omega)} \leq \|(A_\varepsilon - Dh(\varphi^0))^{-1}\|_{\mathcal{L}(H^{-\gamma}_{\Gamma_D}(\Omega), H^1_{\Gamma_D}(\Omega))} \|h(u) - h(v) - Dh(\varphi^0)(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)}.$$

Adding and subtracting $Dh(u)(u - v)$ and using Lemma 6.7 we get

$$\begin{aligned}\|h(u) - h(v) - Dh(\varphi^0)(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)} &\leq \|h(u) - h(v) - Dh(u)(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)} \\ + \|(Dh(\varphi^0) - Dh(u))(u - v)\|_{H^{-\gamma}_{\Gamma_D}(\Omega)} &\leq C\|u - v\|^{1+\sigma}_{H^1_{\Gamma_D}(\Omega)} \leq C\delta^\sigma \|u - v\|_{H^1_{\Gamma_D}(\Omega)}\end{aligned}$$

with $\sigma = \min\left\{1, \frac{2\gamma}{N-2}\right\}$. Hence, using Lemma 7.5 we get

$$\|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)\|_{H^1(\Omega)} \leq C\delta^\sigma \|u - v\|_{H^1_{\Gamma_D}(\Omega)}.$$

for every $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$. Taking $\delta > 0$ small enough we have $\|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)\|_{H^1(\Omega)} \leq \frac{1}{2}\|u - v\|_{H^1(\Omega)}$.

Let us show that there exists $\delta > 0$ such that for all $\varepsilon > 0$ small, Φ_ε maps $B(\varphi^0; \delta)$ into itself. Let $v \in B(\varphi^0; \delta)$. Then

$$\begin{aligned}\|\Phi_\varepsilon(v) - \varphi^0\|_{H^1(\Omega)} &\leq \|\Phi_\varepsilon(v) - \Phi_\varepsilon(\varphi^0)\|_{H^1(\Omega)} + \|\Phi_\varepsilon(\varphi^0) - \varphi^0\|_{H^1(\Omega)} \\ &\leq C\delta^\sigma \|v - \varphi^0\|_{H^1(\Omega)} + \|\Phi_\varepsilon(\varphi^0) - \varphi^0\|_{H^1(\Omega)}.\end{aligned}$$

For the last term notice that since $h^* = h(\varphi^0) - Dh(\varphi^0)\varphi^0 \in L^2(\Omega) + L^2(\Gamma)$ then from Theorem 5.3 and Corollary 5.5, we have, as $\varepsilon \to 0$

$$\Phi_\varepsilon(\varphi^0) = (A_\varepsilon - Dh(\varphi^0))^{-1}h^* \to (A_0 - DH(\varphi^0))^{-1}Ph^* = \Phi_0(\varphi^0) = \varphi^0$$

in $H^1(\Omega)$, see Remark 5.7. So, taking $\delta > 0$ small enough we have that for all $0 < \varepsilon < \varepsilon_0$ and $v \in B(\varphi^0; \delta)$, $\|\Phi_\varepsilon(v) - \varphi^0\|_{H^1(\Omega)} < \delta$. ■

Remark 7.6 *Observe that (7.4) and hence Proposition 7.4 hold true for the case of logistic type nonlinearities as in Remark 6.13, or more generaly to nonlinearities like in Theorem 6.14.*

7.3 Lower semicontinuity

Using similar arguments than for extremal equilibria in the section above, one can actually prove lower semicontinuity of the attractors. For this, one has to assume that all equilibria of the limit problems are hyperbolic.

From this it is obtained that just a finite number of them exist and that all of them are isolated. Then, Proposition 7.4 implies that each approximate problem has an equilibrium (and only one) close to each limit equilibrium.

Then, one needs to obtain, in the spirit of Theorem 5.8, the spectral convergence of the linearizations along these convergence sequences of equilibria. The difference with Theorem 5.8 above is that now the linear potentials will depend on ε as well. However the convergence of the equilibria and the differentiability properties of the nonlinear terms in Lemma 6.7 allow to handle this step.

From this, a close revision of the results in [22], allows to prove the convergence of the local unstable manifolds of the equilibria.

Finally, the problems involved are gradient–like which implies that each of the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ are the union of the global unstable manifolds of the equilibria, see (6.23). This allows to conclude the lower semicontinuity of the attractors.

This program has been successfully carried out in different singular perturbation problems, e.g. [6, 9, 5, 3, 10] and references therein. Details are lengthy but can be fulfilled in the case of the present paper with the results in previous sections and the techniques in the papers above.

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