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Algebraic (super-)integrability from commutants of subalgebras in universal enveloping algebras

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Abstract

Starting from a purely algebraic procedure based on the commutant of a subalgebra in the universal enveloping algebra of a given Lie algebra, the notion of algebraic Hamiltonians and the constants of the motion generating a polynomial symmetry algebra is proposed. The case of the special linear Lie algebra $\mathfrak{sl}(n)$ is discussed in detail, where an explicit basis for the commutant with respect to the Cartan subalgebra is obtained, and the order of the polynomial algebra is computed. It is further shown that, with an appropriate realization of $\mathfrak{sl}(n)$, this provides an explicit connection with the generic superintegrable model on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and the related Racah algebra $R(n)$. In particular, we show explicitly how the models on the two-sphere and three-sphere and the associated symmetry algebras can be obtained from the quadratic and cubic polynomial algebras generated by the commutants defined in the enveloping algebra of $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$, respectively. The construction is performed in the classical (or Poisson-Lie) context, where the Berezin bracket replaces the commutator.

Keywords: commutants, Lie algebras, enveloping algebras, Racah algebras, superintegrability

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1. Introduction

A lot of work has been devoted in recent years to a full characterization and structural comprehension of exactly solvable, integrable and superintegrable systems on various classes of spaces, both at the classical and quantum level [1–5]. Several new connections with different areas as nonlinear differential equations, the theory of special functions, the Painlevé transcendents and algebraic structures beyond Lie algebras have emerged from this work, providing alternative techniques and tools complementary to the traditional analytical approach. In particular, certain types of algebras have been shown to be of special interest in this context, from which several criteria for the construction of (super-)integrable models have been deduced (see e.g. [6–9]).

More recently, it was observed that quadratically superintegrable systems can be connected with purely algebraic schemes, where the integrals of the motion are in fact associated to an underlying Lie algebra and the corresponding universal enveloping algebra [10, 11]. Albeit these approaches are mainly based on specific superintegrable systems realized by differential operators, the ansatz can be extrapolated to arbitrary pairs of Lie algebras and subalgebras without reference to a given realization [12], hence allowing us to define generically the notion of algebraic Hamiltonians and their corresponding (algebraic) constants of the motion.

The purpose of this paper is the following. Starting from the conventional notion of a commutant in the enveloping algebra of a Lie algebra $\mathfrak{s}$, we reformulate the problem by means of the Lie-Poisson structure of the corresponding symmetric algebra $S(\mathfrak{s})$, which provides a computationally more adequate frame. We then define the notion of algebraic Hamiltonian with respect to a subalgebra $\mathfrak{a}$, and show that the commutant defines a polynomial algebra that can be identified with the symmetry algebra of the Hamiltonian, with the constants of the motion obtained from the elements belonging to the centralizer $C_{S(\mathfrak{s})}(\mathfrak{a})$. As an illustrating example of the procedure, we analyze the commutant of the Cartan subalgebra $\mathfrak{h}$ in the special linear algebra $\mathfrak{sl}(n)$ and obtain explicit expressions for the linearly independent elements in the enveloping algebra. It is further shown that the polynomial algebra $\mathcal{A}_n$ generated by these linearly independent monomials is of order $n-1$ for $n \geqslant 3$. For values $n \geqslant 3$, the polynomial algebras $\mathcal{A}_n$ are associated, via an appropriate realization of $\mathfrak{sl}(n)$, with the Racah algebra $R(n)$, corresponding to the symmetry algebra of the generic superintegrable models on the sphere $\mathbb{S}^{n-1}$ (see [13–17] and references therein). The reduction of $\mathcal{A}_n$ to $R(n)$ is explicitly computed for $n = 3, 4$, with the general case outlined due to high dimension of the polynomial algebras.

2. The commutant of Lie subalgebras in enveloping algebras

Let $\mathfrak{s}$ be an n-dimensional real or complex Lie algebra and $\mathcal{U}(\mathfrak{s})$ be its universal enveloping algebra. For positive integers p, we define $\mathcal{U}_{(p)}(\mathfrak{s})$ as the subspace generated by monomials $X_1^{a_1} \cdots X_n^{a_n}$ subjected to the constraint $a_1 + a_2 + \cdots + a_n \leqslant p$, where $\{X_1, \cdots, X_n\}$ is an arbitrary basis of $\mathfrak{s}$. In this context, an element $P \in \mathcal{U}(\mathfrak{s})$ has degree d if $d = \inf\left\{k \mid P \in \mathcal{U}_{(k)}(\mathfrak{s})\right\}$. Due to the natural filtration in $\mathcal{U}(\mathfrak{s})$, for any $p, q \geqslant 0$ we have

$$\mathcal{U}_{(0)}(\mathfrak{s}) = \mathbb{C},\ \mathcal{U}_{(p)}(\mathfrak{s})\mathcal{U}_{(q)}(\mathfrak{s}) \subset \mathcal{U}_{(p+q)}(\mathfrak{s}). \tag{2.1}$$

It follows in particular that any $\mathcal{U}_{(p)}(\mathfrak{s})$ is a finite-dimensional representation of $\mathfrak{s}$, a fact that allows us to represent $\mathcal{U}(\mathfrak{s})$ as a sum of finite-dimensional representations of $\mathfrak{s}$ (see [18]).

In terms of a given basis, the adjoint action of $\mathfrak{s}$ on $\mathcal{U}(\mathfrak{s})$ and on the associated symmetric algebra $S(\mathfrak{s})$ is respectively given by

$$\begin{aligned} P \in \mathcal{U}(\mathfrak{s}) \mapsto \quad & P.X_i := [X_i, P] = X_i P - P X_i \in \mathcal{U}(\mathfrak{s}), \\ P(x_1, \cdots, x_n) \in S(\mathfrak{s}) \mapsto \quad & \widehat{X}_i(P) = C_{ij}^k x_k \frac{\partial P}{\partial x_j} \in S(\mathfrak{s}), \end{aligned} \tag{2.2}$$

where in particular the differential operators $\widehat{X}_i = C_{ij}^k x_k \dfrac{\partial}{\partial x_j}$ correspond to infinitesimal generator of the one-parameter subgroup associated to the generators X_i by the coadjoint representation [19]. The symmetric algebra $S(\mathfrak{s})$, that we can identify with $\mathbb{K}[x_1, \cdots, x_n]$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, admits naturally a structure of Poisson algebra through the prescription[3]

$$\{P, Q\} = C_{ij}^k x_k \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_j}, \; P, Q \in S(\mathfrak{s}). \tag{2.3}$$

It follows that $(S(\mathfrak{s}), \{,\})$ is a Lie algebra containing a subalgebra isomorphic to $\mathfrak{s}$. By means of the symmetrization map

$$\Lambda\left(x_{j_1} \cdots x_{j_p}\right) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} X_{j_{\sigma(1)}} \cdots X_{j_{\sigma(p)}}, \tag{2.4}$$

with Σ_p being the symmetric group of p letters, we obtain the canonical linear isomorphism $\Lambda : S(\mathfrak{s}) \to \mathcal{U}(\mathfrak{s})$ that commutes with the adjoint action. If $S^{(p)}(\mathfrak{s})$ denotes the homogeneous polynomials of degree p, the identity $\mathcal{U}^{(p)}(\mathfrak{s}) = \Lambda\left(S^{(p)}(\mathfrak{s})\right)$ induces the decomposition $\mathcal{U}_{(p)}(\mathfrak{s}) = \sum_{k=0}^{p} \mathcal{U}^{(k)}(\mathfrak{s})$, implying that for arbitrary $P \in \mathcal{U}_{(p)}(\mathfrak{s}), Q \in \mathcal{U}_{(q)}(\mathfrak{s})$, the relation

$$[P, Q] \in \mathcal{U}_{(p+q-1)}(\mathfrak{s})$$

holds. A distinguished (Abelian) subalgebra is given by the centre of $\mathcal{U}(\mathfrak{s})$

$$Z(\mathcal{U}(\mathfrak{s})) = \{P \in \mathcal{U}(\mathfrak{s}) \mid [\mathfrak{s}, P] = 0\}, \tag{2.5}$$

consisting of the invariant polynomials of $\mathfrak{s}$. Within the commutative frame with the Lie-Poisson bracket, we have the centre

$$Z(S(\mathfrak{s})) = \{P \in S(\mathfrak{s}) \mid \{P, Q\} = 0, \; Q \in S(\mathfrak{s})\},$$

which is not only linearly but also algebraically isomorphic to $Z(\mathcal{U}(\mathfrak{s}))$, albeit only for the class of nilpotent Lie algebras this algebraic isomorphism coincides with Λ (see e.g. [21]).

Definition 1. The commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ of a subalgebra $\mathfrak{a} \subset \mathfrak{s}$ is defined as the centralizer of $\mathfrak{a}$ in $\mathcal{U}(\mathfrak{s})$:

$$C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = \{Q \in \mathcal{U}(\mathfrak{s}) \mid [P, Q] = 0, \quad \forall P \in \mathfrak{a}\}. \tag{2.6}$$

Commutants of subalgebras can have quite a complicated structure, and no generic characterization exists for arbitrary types of Lie algebras and subalgebras. However, for semisimple or reductive Lie algebras, there are several criteria that allow their study (see e.g. [18]). In this context, it can be ensured that the commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ is Noetherian and finitely generated whenever the subalgebra $\mathfrak{a}$ is reductive in $\mathfrak{s}$[4]. We call $\{P_1, \cdots, P_s\}$ a (linear) basis of the commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ if it is spanned as vector space by the elements

$$P_1^{a_1} P_2^{a_2} \cdots P_s^{a_s}, \quad a_i \in \mathbb{N} \cup 0, \tag{2.7}$$

[3] Usually called the Lie-Poisson or Berezin bracket [20].

[4] This means that for any element X in $\mathfrak{a}$ the adjoint operator is semisimple [18].

and where the coefficients a_i are subjected to some algebraic constraints. It is important to note that these polynomials are linearly but generally not algebraically independent. The requirement on linear independence is necessary if the elements in $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ are supposed to generate a (finite-dimensional) polynomial algebra. If $P, Q \in C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$, the Jacobi identity implies that

$$[\mathfrak{a}, [P, Q]] + [Q, [\mathfrak{a}, P]] + [P, [Q, \mathfrak{a}]] = 0, \tag{2.8}$$

and since PQ annihilates the subalgebra $\mathfrak{a}$, it admits an expression of the type (2.7) and is thus specified by at most s scalars a_i (see (2.7)). In this sense, we say that the linear dimension of the commutant is s, denoting it by $\dim_L C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) = s$. As a general rule, for a maximal set of algebraically independent polynomials that commute with $\mathfrak{a}$, we do not obtain a polynomial algebra with respect to the commutator, as the algebraic dependence relations may be determined by rational non-polynomial functions, hence not belonging to the enveloping algebra, but to its field of fractions [22].

A commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ contains, in particular, the invariant polynomials of the subalgebra $\mathfrak{a}$, in addition to the Casimir operators (whenever these exist) $\mathcal{C}_1, \cdots, \mathcal{C}_\ell$ of $\mathfrak{s}$. With respect to the latter, the commutant $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ has the structure of a free module over $\mathbb{C}[\mathcal{C}_1, \cdots, \mathcal{C}_\ell]$, as follows at once from the Schur lemma (see [18]).

Using the canonical isomorphism Λ allows us to translate the problem of determining the commutant of a subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ in the enveloping algebra $\mathcal{U}(\mathfrak{s})$ to the Lie-Poisson context: for $\mathfrak{a}^* \subset \mathfrak{s}^*$ we define the centralizer

$$C_{S(\mathfrak{s})}(\mathfrak{a}) = \{Q \in S(\mathfrak{s}) \mid \{P, Q\} = 0,\ P \in \mathfrak{a}\}.$$

Elements in the latter space are obtained as (polynomial) solutions of the system of partial differential equations

$$\widehat{X}_i(Q) := \{x_i, Q\} = C_{ij}^k x_k \frac{\partial Q}{\partial x_j} = 0,\ 1 \leqslant i \leqslant m = \dim \mathfrak{a}, \tag{2.9}$$

where $\{x_1, \cdots, x_m\}$ are coordinates in a dual basis of $\mathfrak{a}^*$ (see equation (2.2)). The number of functionally independent solutions of system (2.9) is given by $r_0 = \dim \mathfrak{s} - \operatorname{rank}(A)$, where A is the $m \times n$-matrix with entries $\left(C_{ij}^k x_k\right)$ [23, 24]. As solutions of (2.9) are not necessarily polynomials, the number ξ_0 of independent polynomials commuting with $\mathfrak{a}$ is upper bounded by $\xi_0 \leqslant r_0$, thus providing a lower bound for the linear dimension of the commutant: $\dim_L C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a}) \geqslant \xi_0$.

2.1. Algebraic integrability and superintegrability

The analysis of superintegrable systems from the perspective of algebraic structures (see e.g. [8, 9, 16, 25, 26]) suggest to consider a more general frame, which is suitably placed within enveloping algebras. In this context, various approaches have recently been proposed to define an algebraic notion of integrability and superintegrability [10, 12].

Definition 2. Let $\mathfrak{a} \subset \mathfrak{s}$ be a Lie subalgebra and $C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$ be the commutant. An algebraic Hamiltonian with respect to $\mathfrak{a}$ is defined as

$$\mathcal{H}_a = \sum_{i,j} \alpha_{ij} X_i X_j + \sum_k \beta_k X_k + \sum_\ell \gamma_\ell \mathcal{C}_\ell, \tag{2.10}$$

where $X_i, X_j, X_k \in \mathfrak{a}$, $\mathcal{C}_\ell$ is a Casimir operator of $\mathfrak{s}$ and α_{ij}, β_k, γ_ℓ are parameters.

The advantage of such a definition relies upon the fact that the commutant automatically provides a set of constants of the motion. If $P \in C_{\mathcal{U}(\mathfrak{s})}(\mathfrak{a})$, it commutes with both the subalgebra

and the Casimir operators of $\mathfrak{s}$, hence satisfies the condition $[\mathcal{H}_a, P] = 0$. The notion can be easily translated to the Lie-Poisson frame, considering the Hamiltonian

$$\mathcal{H} = \sum_{i,j} \alpha_{ij} x_i x_j + \sum_k \beta_k x_k + \sum_\ell \gamma_\ell c_\ell, \tag{2.11}$$

with c_ℓ being the symmetric counterpart of the Casimir operators. If $P = \lambda(P_0)$ for some $P_0 \in C_{S(\mathfrak{s})}(\mathfrak{a})$, then clearly $\{\mathcal{H}, P_0\} = 0$. Depending on the number of functionally independent solutions of system (2.9), a sufficient number of constants of the motion that implies the (super-)integrability of the corresponding system can be extracted [10, 12].

The preceding prescription leads to two possible cases that are conveniently separated:

(a) The polynomials in $C_{S(\mathfrak{s})}(\mathfrak{a})$ have all vanishing commutator/Berezin bracket, in which case the algebraic Hamiltonian $\mathcal{H}_a$ is integrable and the symmetry algebra is Abelian.
(b) There are non-commuting elements in $C_{S(\mathfrak{s})}(\mathfrak{a})$. In this case we say that the algebraic Hamiltonian $\mathcal{H}_a$ possesses a non-commutative (super-)integrability property and the symmetry algebra is a non-Abelian polynomial algebra. The dimension of the symmetry algebra will be given by the number of linearly independent elements among the constants of the motion.

Once a specific realization (by differential operators) of the Lie algebra $\mathfrak{s}$ has been chosen, the Casimir operators and the previously computed constants of the motion can either decompose or satisfy additional dependence relations in the functional space associated to the realization, imposing further constraints among the generators that can reduce the dimension of the symmetry algebra. The realized Hamiltonian and invariants eventually still lead to well-defined exactly, quasi-exactly, integrable or even superintegrable systems in the usual sense, i.e. where the variables are in terms of generalized coordinates and their momentum (see e.g. [3, 6, 15, 27, 28] and references therein). Such an approach has been successfully applied to various Lie algebras such as $\mathfrak{su}(3)$ and $\mathfrak{gl}(3)$ in order to recover the Smorodinsky–Winternitz systems, as well as the generic model on the sphere $\mathbb{S}^2$, which connects to the 58 superintegrable systems on conformally flat spaces [29, 30]. The approach chosen here is quite different, and is mainly based on the algebraic setting, where integrals are polynomials in the enveloping algebra of a Lie algebra and the (non-commutative) integrability or superintegrability will be deduced from commutation relations of the polynomials in enveloping algebra of $\mathfrak{s}$, or either using the simpler relations in terms of the Lie-Poisson or Berezin bracket.

In the following, the construction will be performed in the classical (or Poisson-Lie) context, where the Berezin bracket replaces the commutator, for computational reasons. It is understood that all the algebraic structures considered in this paper also have a quantum analog, i.e. when using the commutator and the symmetrization map (2.4).

3. The commutant of $\mathfrak{h}$ in $\mathfrak{sl}(n)$

In this section we compute the commutant of the special linear Lie algebra $\mathfrak{sl}(n)$ with respect to the Cartan subalgebra $\mathfrak{h}$. It is shown that the elements in a basis of polynomials commuting with $\mathfrak{h}$ can be identified with cycles in the symmetric group Σ_n, thus providing the dimension for arbitrary values of n. It is further shown that the resulting polynomial algebras are deeply related, via an appropriate realization, with the description of superintegrable models on the spheres and the Racah algebras [4, 15, 17].

We consider the special linear Lie algebra $\mathfrak{sl}(n)$ in its defining representation. A basis is given by the generators E_{ij} with $1 \leqslant i,j \leqslant n$ subjected to the constraint $\sum_{i=1}^{n} E_{ii} = 0$. The commutator is then given by

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} \qquad (1 \leqslant i,j,k,l \leqslant n). \tag{3.1}$$

It can be easily verified that a minimal set of generators for $\mathfrak{sl}(n)$ is given by the $2(n-1)$ elements $E_{i,i+1}, E_{i+1,i}$ for $1 \leqslant i \leqslant n-1$, with the Cartan subalgebra $\langle$ being determined by

$$[E_{i,i+1}, E_{i+1,i}] = E_{i,i} - E_{i+1,i+1} := H_i \qquad (1 \leqslant i \leqslant n-1). \tag{3.2}$$

As the restriction of the Killing form κ to the Cartan subalgebra $\mathfrak{h}$ is non-degenerate, $\mathfrak{h}$ is reductive in $\mathfrak{sl}(n)$ and thus the commutant $C_{\mathcal{U}(\mathfrak{sl}(n))}(\mathfrak{h})$ is finitely generated, so that it admits a basis of type (2.7).

As already mentioned, for computational purposes, it is more convenient to use the Poisson–Lie setting, i.e. to determine the centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ of $\mathfrak{h}$ in $S(\mathfrak{sl}(n))$. To this extent, we have to solve the system of partial differential equations (PDEs)

$$\{f, h_i\} = 0, \quad 1 \leqslant i \leqslant n-1. \tag{3.3}$$

Using the analytical approach, it can be easily verified that the system (3.3) possesses $\dim \mathfrak{sl}(n) - n + 1 = n(n-1)$ functionally independent solutions, that can always be chosen as polynomials since the Lie algebra is semisimple. In particular, the Cartan subalgebra $\mathfrak{h}$ is contained in $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$, and corresponds to the only linear elements in the centralizer. The symmetric counterpart $c^{[2]}, \cdots, c^{[n]}$ of the Casimir operators of $\mathfrak{sl}(n)$ also belong to $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$. In order to determine the generic shape of polynomials satisfying the system (3.3), suppose that $P = P_1 + \cdots + P_r$ is a polynomial with homogeneous components P_k of degree $k = 1, \cdots, r$. Then $\{h_i, P\} = \{h_i, P_1\} + \cdots + \{h_i, P_r\} = 0$ if and only if

$$\{h_i, P_k\} = 0, \quad 1 \leqslant i \leqslant n-1, \quad 1 \leqslant k \leqslant r. \tag{3.4}$$

This allows to reduce the analysis of the centralizer to homogeneous polynomials. On the other hand, if $P_k = \lambda^{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k}$ is homogeneous of degree k, with x_{i_j} arbitrary elements of $\mathfrak{sl}(n)^*$, we conclude from the properties of the Lie-Poisson bracket that

$$\{h_\ell, P_k\} = \lambda^{i_1 \cdots i_k} \left(\mu^\ell_{i_1} + \cdots + \mu^\ell_{i_k}\right) x_{i_1} \cdots x_{i_k}, \tag{3.5}$$

where $\mu^\ell_{i_j}$ is the eigenvalue of x_{i_j} with respect to h_ℓ. As the monomials $x_{i_1} \cdots x_{i_k}$ are independent, equation (3.4) is satisfied if and only if $\mu^\ell_{i_1} + \cdots + \mu^\ell_{i_k} = 0$ for each ℓ and $i_1, \cdots, i_k$. We conclude that it suffices to determine, for each order $d \geqslant 2$, a maximal set of linearly independent monomials that satisfy the system (3.3). Any other element is obtained as a polynomial in these elements.

For computational purposes, it is convenient to use the following (lexicographically ordered) basis in $\mathfrak{sl}(n)^*$:

$$\begin{aligned} h_i, &\quad 1 \leqslant i \leqslant n-1 \\ e_{i,i+s}, &\quad 1 \leqslant i \leqslant n-1,\ 1 \leqslant s \leqslant n-i \\ e_{i+s,i}, &\quad 1 \leqslant i \leqslant n-1,\ 1 \leqslant s \leqslant n-i. \end{aligned} \tag{3.6}$$

It follows at once from (3.2) that for arbitrary indices i,j,k we have

$$\{h_i, e_{j,k}\} = \delta^j_i e_{i,k} - \delta^j_{1+i} e_{i+1,k} - \delta^k_i e_{j,i} + \delta^k_{1+i} e_{j,1+i}, \tag{3.7}$$

showing that $\mathfrak{h}^*$ acts diagonally on the generators of the basis. Rewriting the latter expression as

$$\{h_i, e_{j,k}\} = \mu^i_{j,k} e_{j,k}, \tag{3.8}$$

we denote by $\mu^i_{j,k}$ the weight of $e_{j,k}$ with respect to the Cartan generator h_i. It can be easily verified that for $1 \leqslant j \leqslant n-1$ these weights are given by

$$\mu^{j-1}_{j,j+1} = -1, \quad \mu^{j}_{j,j+1} = 2, \quad \mu^{j+1}_{j,j+1} = -1, \quad \mu^{\ell}_{j,j+1} = 0,\ \ell \neq j-1, j, j+1. \tag{3.9}$$

If $\mathbf{A}$ denotes the matrix with entries $\mu^i_{j,j+1}$, it follows at once from (3.9) that $\mathbf{A}$ coincides with the Cartan matrix of $\mathfrak{sl}(n)$:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ & & & \ddots & & \\ & & & 2 & -1 & 0 \\ & & & -1 & 2 & -1 \\ & & & 0 & -1 & 2 \end{pmatrix}. \tag{3.10}$$

This shows that the (column) vectors $\mathbf{v}_{j,k} = \left(\mu^1_{j,k}, \cdots, \mu^{n-1}_{j,k}\right)$ with $1 \leqslant j < k \leqslant n$ actually behave like the positive roots of the root system $\mathcal{R}$ associated to $\mathfrak{sl}(n)$, with the $\mathbf{v}_{j,j+1}$ corresponding to the simple roots, implying that the $\mathbf{v}_{j,k}$ are linear combinations of the $\mathbf{v}_{j,j+1}$ with positive integer coefficients [31]. Specifically, for any $e_{j,k}$ with $j < k$ we have the following algebraic relation for its weight vector:

$$\mathbf{v}_{j,k} = \sum_{s=j}^{k-1} \mathbf{v}_{s,s+1}, \quad 1 \leqslant j < k \leqslant n. \tag{3.11}$$

In analogous way, for the generators $e_{k,j}$ with $j < k$, the eigenvalue vector is given by $\mathbf{v}_{k,j} = -\mathbf{v}_{j,k}$. It is worthy to mention two immediate but relevant consequences of (3.11):

(a) Linear combinations of the $\mathbf{v}_{j,k}$ with $j < k$ (respectively $j > k$) and positive integer coefficients cannot be zero, as the vectors $\mathbf{v}_{j,j+1}$ are linearly independent.
(b) For any $\mathbf{v}_{j,k}$, the multiplicity of the weight vector $\mathbf{v}_{s,s+1}$ ($1 \leqslant s \leqslant n-1$) in the sum (3.11) is either one or zero.

These properties enable us to construct recursively a maximal set of linearly independent sets that generate the centralizer. As already mentioned, the only linear solutions to (3.3) are the Cartan generators themselves. As any element in $\mathfrak{h}^*$ has weight zero, it follows that for any monomial P commuting with $\mathfrak{h}^*$, the product $F(h_1, \cdots, h_{n-1})P$ also satisfies (3.3). In order to discard these decomposable solutions, we can suppose without loss of generality that the monomials P satisfying system (3.3) additionally fulfill the constraint

$$\frac{\partial P}{\partial h_\ell} = 0, \quad 1 \leqslant \ell \leqslant n-1. \tag{3.12}$$

We start with $d = 2$ and consider quadratic monomials $e_{j,k}e_{l,m}$. By condition (3.5), the monomial is a solution of system (3.3) if and only if the eigenvalue vectors satisfy the constraint $\mathbf{v}_{j,k} + \mathbf{v}_{l,m} = 0$, hence $\mathbf{v}_{j,k} = -\mathbf{v}_{l,m} = \mathbf{v}_{m,l}$ must hold, implying that $m = j$ and $l = k$. We conclude that the monomial has the form

$$p_{j,k} = e_{j,k}e_{k,j}, \quad 1 \leqslant j < k \leqslant n. \tag{3.13}$$

As can be easily seen, the number of independent monomials of this type is given by the number of transpositions in the symmetric group Σ_n, and equals $\nu_2 = \frac{n!}{2(n-2)!}$. We observe that the monomials $p_{i,j}$ are not only linearly independent, but moreover functionally independent. This can be easily verified considering the Jacobian matrix A_J with respect to the set of variables $e_{j,k}$ with $j < k$, the determinant of which is given by

$$\det(A_J) = \prod_{s=1}^{n-1} e_{s+1,s} \neq 0. \tag{3.14}$$

Let us now consider $d = 3$ and a cubic monomial $e_{i_1,j_1}e_{i_2,j_2}e_{i_3,j_3}$. It is a solution whenever $\mathbf{v}_{i_1,j_1} + \mathbf{v}_{i_2,j_2} + \mathbf{v}_{i_3,j_3} = 0$ holds. As we are assuming that the constraint (3.12) holds, none of the weight vector vanishes. On the other hand, If $i_\alpha < j_\alpha$ for $\alpha = 1,2,3$,[5] the sum of the weight vectors cannot be zero as a consequence of equation (3.11). Up to a reordering of the indices, there are two possibilities: either $i_\alpha < j_\alpha$ for $\alpha = 1,2$ and $i_3 > j_3$ or $i_\alpha > j_\alpha$ for $\alpha = 1,2$ and $i_3 < j_3$. For both cases, the argumentation is the same, thus let us assume that the first possibility is given. We can further assume that $i_1 \leqslant i_2$. Equation (3.11) leads to the identity

$$\mathbf{v}_{i_1,j_1} + \mathbf{v}_{i_2,j_2} = \sum_{s=i_1}^{j_1-1} \mathbf{v}_{s,s+1} + \sum_{s=i_2}^{j_2-1} \mathbf{v}_{s,s+1} = -\mathbf{v}_{i_3,j_3} = \mathbf{v}_{j_3,i_3} = \sum_{s=j_3}^{i_3-1} \mathbf{v}_{s,s+1}. \tag{3.15}$$

On the right-hand side of the identity, each $\mathbf{v}_{s,s+1}$ appears with multiplicity one, with the chain from $s = j_3$ to $s = i_3 - 1$ being uninterrupted. Therefore, the sums of the left-hand side must exhibit the same property. In order to avoid multiplicities, we must have the ordering of indices $i_1 < j_1 \leqslant i_2 < j_2$, while the identities $i_1 = j_3$, $j_1 = i_2$, $j_2 = i_3$ are a consequence of the fact that the sequence from $s = j_3$ to $s = i_3 - 1$ is uninterrupted. We conclude that the cubic monomial is given by

$$e_{i_1,j_1}e_{i_2,j_2}e_{i_3,j_3} = e_{i_1,j_1}e_{j_1,j_2}e_{j_2,i_1}, \tag{3.16}$$

where the indices i_1, j_1 and j_2 are all distinct. As before, identifying the index set with the three-cycle $(i_1 j_1 j_2)$, we obtain that the number of linearly independent cubic monomials Poisson-commuting with $\mathfrak{h}^*$ is given by $\nu_3 = \frac{n!}{3(n-3)!}$.

Before proceeding with the analysis, we observe that the monomial solutions of equation (3.3) up to degree three subjected to the constraint (3.12) contain a maximal set of functionally independent solutions. Considering the monomials $p_{1,j,k}$ with $1 < j < k$ jointly with the basis of the Cartan subalgebra and the $p_{r,s}$, we get exactly $n^2 - n$ elements, that is, the number of independent solutions of the system (3.3). The Cartan generators are clearly independent, so that it suffices to show that the nonlinear polynomials above are independent. We consider the complementary Φ of the set of variables $\{h_1, \cdots, h_{n-1}, e_{s,s-1},\ 2 \leqslant s \leqslant n\}$ in the basis (3.6), and determine the Jacobian matrix A_J with respect to the variables in Φ. A long but routine computation shows that

[5] The same holds if $i_\alpha > j_\alpha$ for $\alpha = 1,2,3$.

$$\det(A_{\mathrm{J}}) = \prod_{e_{j,k}\notin\Psi} e_{j,k} \times \prod_{k=2}^{n-1} e_{1,k}^{n-k-1} \times \prod_{k=4}^{n} e_{k,1}^{k-3} \neq 0, \tag{3.17}$$

where $\Psi = \{e_{1,n},\, e_{k,s},\, 4 \leqslant k \leqslant n,\, k-2 \leqslant s \leqslant n-2\}$. This shows in particular that the $p_{i,j}$ along with $p_{1,k,l}$ constitute a maximal set of functionally independent solutions of (3.3) satisfying the constraint (3.12).

In order to simplify the analysis of linearly independent monomial solutions $e_{i_1,j_1}\cdots e_{i_d,j_d}$ for orders $d \geqslant 4$, we first extract some general consequences from equation (3.11). First of all, if there exists some reordering of the indices and an integer $q < d$ such that $\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_q,j_q} = 0$, then clearly

$$\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_d,j_d} = \mathbf{v}_{i_{q+1},j_{q+1}} + \cdots + \mathbf{v}_{i_d,j_d} = 0 \tag{3.18}$$

and the monomial is decomposable as a product of a monomial of order q and a monomial of order $d-q$. We can thus restrict the analysis to those sums of weight vectors for which no partial sum of $q < d$ indices vanishes. We next consider a partition of the index set $\mathcal{S}$ as follows:

$$\mathcal{S}_+ = \{i_\alpha < j_\alpha \mid 1 \leq \alpha \leqslant d\}, \quad \mathcal{S}_- = \{i_\alpha > j_\alpha \mid 1 \leq \alpha \leqslant d\}. \tag{3.19}$$

Lemma. *If the cardinal card$(\mathcal{S}_-) \geqslant 2$, then the monomial $e_{i_1,j_1}\cdots e_{i_d,j_d}$ is decomposable.*

Proof. Suppose that card$(\mathcal{S}_-) = 2$. Reordering the indices if necessary, the condition (3.5) can be rewritten as

$$\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_{d-2},j_{d-2}} = -\mathbf{v}_{i_{d-1},j_{d-1}} - \mathbf{v}_{i_d,j_d} = \mathbf{v}_{j_{d-1},i_{d-1}} + \mathbf{v}_{j_d,i_d}. \tag{3.20}$$

Without loss of generality we can suppose that the first indices are ordered lexicographically, i.e. $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{d-2}$ and $j_{d-1} \leqslant j_d$. By the decomposition (3.11)

$$\sum_{s=i_1}^{j_1-1} \mathbf{v}_{s,s+1} + \cdots + \sum_{s=i_{d-2}}^{j_{d-2}-1} \mathbf{v}_{s,s+1} = \sum_{s=j_{d-1}}^{i_{d-1}-1} \mathbf{v}_{s,s+1} + \sum_{s=j_d}^{i_d-1} \mathbf{v}_{s,s+1}. \tag{3.21}$$

It follows from the sum on the right-hand side that the multiplicity of each $\mathbf{v}_{s,s+1}$ is at most two. Further, there must exist a partition of the indices such that

$$\{(i_a,j_a),\ 1 \leqslant a \leqslant d-2\} = \{(\alpha_r,\beta_r),\ 1 \leqslant r \leqslant m_0\} \cup \{(\lambda_s,\nu_s),\ 1 \leqslant s \leqslant d-2-m_0\} \tag{3.22}$$

and such that

$$\sum_{s=j_{d-1}}^{i_{d-1}-1} \mathbf{v}_{s,s+1} = \sum_{r=1}^{m_0} \mathbf{v}_{\alpha_r,\beta_r}, \quad \sum_{s=j_d}^{i_d-1} \mathbf{v}_{s,s+1} = \sum_{s=1}^{d-2-m_0} \mathbf{v}_{\lambda_s,\nu_s}. \tag{3.23}$$

From these identities we conclude that

$$\begin{aligned} j_{d-1} = \alpha_1,\ \beta_r = \alpha_{r+1}\ (1 \leqslant r \leqslant m_0-1),\ \beta_{m_0} = i_{d-1}, \\ j_d = \lambda_1,\ \nu_s = \lambda_{s+1}\ (1 \leqslant s \leqslant d-3-m_0),\ \nu_{d-2-m_0} = i_d, \end{aligned} \tag{3.24}$$

showing that the monomial decomposes as

$$e_{i_1,j_1}\cdots e_{i_d,j_d} = \left(\prod_{r=1}^{m_0-1} e_{\alpha_r,\alpha_{r+1}}\right) e_{\alpha_{m_0},\alpha_1} \left(\prod_{s=1}^{d-2-m_0-1} e_{\lambda_s,\lambda_{s+1}}\right) e_{\lambda_{d-2-m_0},\lambda_1}. \tag{3.25}$$

The same argument holds for any card$(\mathcal{S}_-) > 2$. □

As a consequence of this result, an indecomposable monomial of order d satisfying equations (3.3) and (3.12) satisfies either $\mathrm{card}\,(\mathcal{S}_-) = 1$ or $\mathrm{card}\,(\mathcal{S}_-) = d-1$. In the following, we will assume that the first possibility holds, the second being completely equivalent.

Let the monomial $e_{i_1,j_1}\cdots e_{i_d,j_d}$ satisfy the system (3.3). By the preceding lemma, it suffices to consider the case where $i_a < j_a$ for $1 \leqslant a \leqslant d-1$ and $i_d > j_d$. Writing the condition $\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_d,j_d} = 0$ on the weight vectors as

$$\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_{d-1},j_{d-1}} = -\mathbf{v}_{i_d,j_d} = \mathbf{v}_{j_d,i_d}, \tag{3.26}$$

and using again that multiplicities in the decomposition (3.11) are at most one, a routine computation shows that the monomial necessarily has the form

$$p_{i_1,\cdots,i_d} = e_{i_1,i_2}e_{i_2,i_3}\cdots e_{i_{d-1},i_d}e_{i_d,i_1}. \tag{3.27}$$

The number of linearly independent elements of this type is given by $\nu_d = \frac{n!}{d(n-d)!}$. It is important to observe that, due to the commutativity of the variables $h_\ell, e_{j,k}$, the cyclic symmetry of indices in the monomials $p_{i_1,\cdots,i_d}$ gives rise to the same element, i.e.

$$p_{i_1,\cdots,i_d} = p_{i_2,\cdots,i_d,i_1} = \cdots = p_{i_d,i_1\cdots,i_{d-1}}. \tag{3.28}$$

This property allows us to identify the monomials of degree d satisfying equations (3.3) and (3.12) with the $d-$cycles of Σ_n.

Lemma. *Any monomial P of order $d > n$ satisfying equation (3.3) is decomposable.*

Proof. Let $P = e_{i_1,j_1}\cdots e_{i_{n+1},j_{n+1}}$ be a monomial solution of (3.3). If $\mathrm{card}\,(\mathcal{S}_-) \geqslant 2$, the previous results show that P is decomposable, hence we can suppose that $\mathrm{card}\,(\mathcal{S}_-) = 1$. Without loss of generality, we can assume that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$ and that $i_{n+1} > j_{n+1}$. The condition on the eigenvalues reads

$$\mathbf{v}_{i_1,j_1} + \cdots + \mathbf{v}_{i_n,j_n} = \mathbf{v}_{j_{n+1},i_{n+1}}. \tag{3.29}$$

It is clear that the indices i_k cannot be all distinct, as otherwise, up to a reordering of indices, we would have $i_k = k$ for $1 \leqslant k \leqslant n$ and thus $k < j_k \leqslant n$. However, as the right-hand side of (3.29) has no multiplicities greater than one, necessarily $j_k = i_{k+1} = (k+1)$ for $1 \leqslant k \leqslant n-1$ and $j_n = i_{n+1}$. As $i_n = n < j_n \leqslant n$, this leads to a contradiction. Therefore, there must exist at least two equal indices $i_k = i_{k+1}$ for some $1 \leqslant k \leqslant n-1$. Let $\xi_0 = \min(j_k, j_{k+1})$. It follows from the decomposition that all $\mathbf{v}_{s,s+1}$ with $i_k \leqslant s \leqslant \xi_0$ have at least multiplicity two. Again, as each term on the right-hand side of (3.29) has multiplicity one, and since by assumption $i_k < j_k$ holds for $k \leqslant n$, one of the following constraints must hold:

$$\mathbf{v}_{i_k,j_k} + \cdots + \mathbf{v}_{i_{\xi_0},j_{\xi_0}} = 0 \quad \text{or} \quad \mathbf{v}_{i_k,j_k} + \cdots + \mathbf{v}_{i_{\xi_1},j_{\xi_1}}, \tag{3.30}$$

where $\xi_1 = \max(j_k, j_{k+1})$. But each of these possibilities implies that the monomial P is decomposable, proving the assumption. □

Using the analogy with the symmetric group, the result is intuitively clear, as there do not exist d-cycles in Σ_n for $d > n$. Summarizing, it has been shown that a basis of linearly independent elements in the centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ is given by the polynomials

$$h_\ell, \quad p_{i_1,\cdots,i_d}, \quad 1 \leqslant \ell \leqslant n-1, \quad 2 \leqslant d \leqslant n. \tag{3.31}$$

Its linear dimension is therefore given by

$$\dim_L C_{S(\mathfrak{sl}(n))}(\mathfrak{h}) = (n-1) + \sum_{d=2}^{n} \nu_d = \sum_{d=1}^{n} \frac{n!}{(n-d)!d} - 1. \tag{3.32}$$

As only n^2-n of these elements are functionally independent, the basis elements will satisfy several algebraic relations, the number of which increases with the value of n. In the following, we present some of these relations.

Proposition. *For any $n \geqslant 3$ and $3 \leqslant d \leqslant n$, following dependence relations hold:*

$$\left(\prod_{u=1}^{k-1} p_{i_u,i_{u+1}}\right) p_{i_k,i_1} = p_{i_1,i_2,\cdots,i_k} p_{i_1,i_k,i_{k-1},\cdots,i_2},$$

$$\prod_{1\leqslant i<j\leqslant n} p_{i,j} = p_{1,2,\cdots,n}\, p_{1,n-1,\cdots,2} \prod_{m=1}^{n-2}\prod_{s=m+2}^{n} p_{m,s}, \tag{3.33}$$

$$\prod_{i_1\neq i_2\neq\cdots\neq i_k} p_{i_1,i_2,\cdots,i_k} = \left(\prod_{r<s} p_{r,s}\right)^{\varphi(k)}, \quad \varphi(k) = \prod_{s=2}^{k-1}(n-s).$$

The proof follows by direct verification, taking into account the structure of the monomials (3.31) and the number (3.32) of linearly independent elements. Although these identities do not exhaust all possible dependence relations, they are useful for simplifying computations. As follows from (3.31), the commutant $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ defines a polynomial algebra $\mathcal{A}_n$ with respect to the Lie-Poisson bracket, and possessing an $(n-1)$-dimensional centre generated by the $h_1,\cdots,h_{n-1}$.

We remark that $n=2$ is the only value where the number (3.32) of linearly independent monomials coincides with the cardinal of a maximal set of functionally independent solutions of the system (3.3). In this case, a basis of the centralizer $C_{\mathfrak{sl}(2)^*}(\mathfrak{h}^*)$ is given by $h_1, p_{1,2}$, and the resulting algebra is two-dimensional and Abelian. In particular, the symmetric counterpart $c^{[2]}$ of the Casimir operator $\mathcal{C}$ of $\mathfrak{sl}(2)$ is given by $c^{[2]} = \frac{1}{4}h_1^2 + p_{1,2}$.

For computational purposes, it may be convenient to consider some variant of the basis (3.31) that takes into account additional symmetry properties, in order to simplify the expression of the Lie-Poisson brackets in the centralizer. For instance, the generic bracket of two quadratic monomials is given by

$$\begin{aligned}\{p_{i_1,j_1}, p_{i_2,j_2}\} = {}& \delta_{i_1}^{i_2}\left(p_{i_1,j_1,j_2} - p_{i_1,j_2,j_1}\right) + \delta_{j_1}^{j_2}\left(p_{i_1,i_2,j_2} - p_{i_1,j_2,i_2}\right)\\ &+ \delta_{i_1}^{j_2}\left(p_{i_1,j_1,i_2} - p_{i_1,i_2,j_1}\right) + \delta_{j_1}^{i_2}\left(p_{i_1,j_2,i_2} - p_{i_1,i_2,j_2}\right),\end{aligned} \tag{3.34}$$

an identity that suggests to consider an alternative choice of cubic monomials, namely replacing $p_{i,j,k}$ by their symmetric and skew-symmetric counterparts

$$g_{i,j,k} = \frac{1}{2}\left(p_{i,j,k} + p_{i,k,j}\right), \quad f_{i,j,k} = \frac{1}{2}\left(p_{i,j,k} - p_{i,k,j}\right), \tag{3.35}$$

and where the coefficient $\frac{1}{2}$ is a consequence of the index cyclic symmetry (3.28). The same transformation can be considered for higher orders, where it may be appropriate to consider monomials possessing additional symmetry properties to simplify the expression of the Lie-Poisson bracket.

The polynomial algebra $\mathcal{A}_n$ is of order at most $n-1$ in the generators satisfying the condition (3.12). This can be easily verified considering for example the monomials of maximal degree $P = p_{1,2,\cdots,n}$ and $Q = p_{1,2,n,n-1,\cdots,4,3}$, where the brackets gives, among other terms

$$\{P,Q\} = p_{1,2,3}p_{3,4}\cdots p_{n-1,n}p_{2,n} + \cdots, \quad n \geqslant 4 \tag{3.36}$$

and where the first term cannot be written in terms of a lower number of basis elements. This actually corresponds to the maximal possible order of brackets. As the bracket of a monomial

of degree d (i.e. $2d$ indices) with one of degree m (i.e. $2m$ indices) has degree $d+m-1$, the total number of indices in each terms is given by $2d+2m-2$. Each quadratic monomial $p_{i,j}$ involves four indices, thus the maximal number ℓ_0 of quadratic terms in a Lie-Poisson bracket is upper bounded by $\ell_0 < \frac{1}{4}(2d+2m-2)$. Longest possible chains can result for $d=m=n$, with $\ell_0 < \frac{1}{4}(4n-2)$. This shows that $\ell_0 = n-2$ is the maximal value, with six remaining indices that must correspond to a cubic monomial, hence leading to a maximal length of $n-1$.

Using the canonical chain of embeddings $\mathfrak{sl}(2) \subset \mathfrak{sl}(3) \subset \cdots \subset \mathfrak{sl}(n)$, it further follows from the basis (3.31) that for the polynomial algebra $\mathcal{A}_n$ associated to centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ we have the ascending filtration $\mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \subset \mathcal{A}_n$, meaning that each $\mathcal{A}_k$ can be seen as a non-central extension of $\mathcal{A}_{k-1}$.

4. $C_{S(\mathfrak{sl}(3))}(\mathfrak{h})$ and Racah-type algebras

For $n=3$, the system (3.3) possesses six functionally independent solutions, while the dimension of the centralizer $C_{S(\mathfrak{sl}(3))}(\mathfrak{h})$, according to formula (3.32), is seven. Following (3.31), a basis may be taken as $\mathcal{B} = \{h_1, h_2, p_{1,2}, p_{1,3}, p_{2,3}, p_{1,2,3}, p_{1,3,2}\}$. The algebraic dependence of these elements is given by

$$p_{1,2}p_{1,3}p_{2,3} - p_{1,2,3}p_{1,3,2} = 0. \tag{4.1}$$

We remark in particular that $p_{1,3,2}$ cannot be expressed as a polynomial in the remaining monomials $h_1, h_2, p_{1,2}, p_{1,3}, p_{2,3}, p_{1,2,3}$, showing that a maximal set of functionally independent polynomial solutions of (3.3) does not generate in general a polynomial algebra. The elements in $\mathcal{B}$ generate a quadratic algebra with nontrivial brackets

$$\begin{aligned}
\{p_{1,2}, p_{1,3}\} &= -\{p_{1,2}, p_{2,3}\} = \{p_{1,3}, p_{2,3}\} = p_{1,2,3} - p_{1,3,2},\\
\{p_{1,2}, p_{1,2,3}\} &= p_{1,2}(p_{1,3} - p_{2,3}) - h_1 p_{1,2,3},\\
\{p_{1,3}, p_{1,2,3}\} &= p_{1,3}(p_{2,3} - p_{1,2}) + (h_1 + h_2) p_{1,2,3},\\
\{p_{2,3}, p_{1,2,3}\} &= p_{2,3}(p_{1,2} - p_{1,3}) - h_2 p_{1,2,3},\\
\{p_{1,2}, p_{1,3,2}\} &= -p_{1,2}(p_{1,3} - p_{2,3}) + h_1 p_{1,3,2},\\
\{p_{1,3}, p_{1,2,3}\} &= -p_{1,3}(p_{2,3} - p_{1,2}) - (h_1 + h_2) p_{1,3,2},\\
\{p_{2,3}, p_{1,2,3}\} &= -p_{2,3}(p_{1,2} - p_{1,3}) + h_2 p_{1,3,2},\\
\{p_{1,2,3}, p_{1,3,2}\} &= h_1 p_{1,3} p_{2,3} + h_2 p_{1,2} p_{1,3} - (h_1 + h_2) p_{1,2} p_{2,3}.
\end{aligned} \tag{4.2}$$

The classical counterpart of the Casimir operators of $\mathfrak{sl}(3)$ turn out to be:

$$\begin{aligned}
c^{[2]} &= p_{1,2} + p_{1,3} + p_{2,3} + \frac{1}{3}(h_1^2 + h_1 h_2 + h_2^2),\\
c^{[3]} &= p_{1,2,3} + p_{1,3,2} + \frac{1}{3}\left((h_1 + 2h_2)p_{1,2} + (h_1 - h_2)p_{1,3} - (2h_1 + h_2)p_{2,3}\right)\\
&\quad + \frac{1}{9} h_1 h_2 (h_1 - h_2) + \frac{2}{27}(h_1^3 - h_2^3).
\end{aligned} \tag{4.3}$$

4.1. Connection with Racah-type algebras: change of basis

We show now that there exists an explicit connection of the previous quadratic algebra associated to $C_{S(\mathfrak{sl}(3))}(\mathfrak{h})$ with a Racah-type quadratic algebra. To this extent, instead of h_1, h_2 we consider the three elements

$$c_1 := \frac{1}{3}(2h_1 + h_2), \quad c_2 := \frac{1}{3}(h_2 - h_1), \quad c_3 := -\frac{1}{3}(h_1 + 2h_2), \tag{4.4}$$

subjected to the linear relation:

$$c_1 + c_2 + c_3 = 0, \tag{4.5}$$

as well as $c_{ij} := p_{i,j}$. We also introduce skew-symmetric and symmetric elements (see (3.35))

$$f_{123} := \frac{1}{2}(p_{1,3,2} - p_{1,2,3}), \quad g_{123} := \frac{1}{2}(p_{1,3,2} + p_{1,2,3}). \tag{4.6}$$

In terms of these generators and central elements, the relations of the quadratic algebra read:

$$\begin{aligned}
\{c_{12}, c_{23}\} &= \{c_{23}, c_{13}\} = \{c_{13}, c_{12}\} = 2f_{123}, \\
\{c_{12}, f_{123}\} &= (c_{23} - c_{13})c_{12} + (c_1 - c_2)g_{123} \\
\{c_{13}, f_{123}\} &= (c_{12} - c_{23})c_{13} + (c_3 - c_1)g_{123} \\
\{c_{23}, f_{123}\} &= (c_{13} - c_{12})c_{23} + (c_2 - c_3)g_{123} \\
\{c_{12}, g_{123}\} &= (c_1 - c_2)f_{123} \\
\{c_{13}, g_{123}\} &= (c_3 - c_1)f_{123} \\
\{c_{23}, g_{123}\} &= (c_2 - c_3)f_{123} \\
\{f_{123}, g_{123}\} &= \frac{1}{2}\big((c_1 - c_3)c_{12}c_{23} + (c_3 - c_2)c_{12}c_{13} + (c_2 - c_1)c_{13}c_{23}\big),
\end{aligned} \tag{4.7}$$

to which the algebraic dependency relation

$$g_{123}^2 - f^2{}_{123} - c_{12}c_{23}c_{13} = 0 \tag{4.8}$$

must be added. The Casimir of the quadratic algebra reads

$$K = f_{123}^2 + c_{12}c_{23}c_{13} - (c_1c_{23} + c_2c_{13} + c_3c_{12})\left(g_{123} - \frac{1}{4}(c_1c_{23} + c_2c_{13} + c_3c_{12})\right). \tag{4.9}$$

This can be re-expressed in terms of the third order Casimir of $\mathfrak{sl}(3)^*$ and the central elements as:

$$K = \frac{1}{4}\big(c^{[3]} - c_1c_2c_3\big)^2. \tag{4.10}$$

At this point, to make the connection with Racah-type algebras more explicit, we consider the second and third-order Casimir elements $\{c^{[2]}, c^{[3]}\}$ in (4.3). The generators are related to the Casimirs through the following functional relations:

$$c_{12} + c_{13} + c_{23} = c^{[2]} - \frac{1}{2}(c_1^2 + c_2^2 + c_3^2) \tag{4.11}$$

$$2g_{123} - c_3c_{12} - c_2c_{13} - c_1c_{23} = c^{[3]} - \frac{1}{3}(c_1^3 + c_2^3 + c_3^3). \tag{4.12}$$

Taking into account these relations, and considering the following redefinitions:

$$\begin{aligned}
&\bar{c}_i : \ = c_i/2, \quad \bar{c}_{12} := c_{12} + \frac{1}{4}(c_1 - c_2)^2, \quad \bar{c}_{13} := c_{13} + \frac{1}{4}(c_1 - c_3)^2, \\
&\bar{c}_{23} : \ = c_{23} + \frac{1}{4}(c_2 - c_3)^2,
\end{aligned} \tag{4.13}$$

we get the quadratic algebra:

$$f_{123} := \frac{1}{2}\{\bar{c}_{12}, \bar{c}_{23}\} = \frac{1}{2}\{\bar{c}_{23}, \bar{c}_{13}\} = \frac{1}{2}\{\bar{c}_{13}, \bar{c}_{12}\} \tag{4.14}$$

$$
\begin{aligned}
\{\bar{c}_{12},f_{123}\} &= (\bar{c}_{23}-\bar{c}_{13})\bar{c}_{12}+(\bar{c}_1-\bar{c}_2)\big(c^{[3]}+(\bar{c}_1+\bar{c}_2)c^{[2]}-(\bar{c}_1+\bar{c}_2)^3\big)\\
\{\bar{c}_{13},f_{123}\} &= (\bar{c}_{12}-\bar{c}_{23})\bar{c}_{13}+(\bar{c}_3-\bar{c}_1)\big(c^{[3]}+(\bar{c}_1+\bar{c}_3)c^{[2]}-(\bar{c}_1+\bar{c}_3)^3\big)\\
\{\bar{c}_{23},f_{123}\} &= (\bar{c}_{13}-\bar{c}_{12})\bar{c}_{23}+(\bar{c}_2-\bar{c}_3)\big(c^{[3]}+(\bar{c}_2+\bar{c}_3)c^{[2]}-(\bar{c}_2+\bar{c}_3)^3\big)\\
\{\bar{c}_{12},g_{123}\} &= 2(\bar{c}_1-\bar{c}_2)f_{123}\\
\{\bar{c}_{13},g_{123}\} &= 2(\bar{c}_3-\bar{c}_1)f_{123}\\
\{\bar{c}_{23},g_{123}\} &= 2(\bar{c}_2-\bar{c}_3)f_{123}\\
\{f_{123},g_{123}\} &= (\bar{c}_1-\bar{c}_3)\bar{c}_{12}\bar{c}_{23}+(\bar{c}_3-\bar{c}_2)\bar{c}_{12}\bar{c}_{13}+(\bar{c}_2-\bar{c}_1)\bar{c}_{13}\bar{c}_{23}\\
&\quad -(\bar{c}_1-\bar{c}_2)\times(\bar{c}_2-\bar{c}_3)(\bar{c}_3-\bar{c}_1)\Bigg(c^{[2]}-\frac{1}{3}\big((\bar{c}_1-\bar{c}_2)^2\\
&\quad +(\bar{c}_1-\bar{c}_2)(\bar{c}_2-\bar{c}_3)+(\bar{c}_2-\bar{c}_3)^2\big)\Bigg).
\end{aligned}
\tag{4.15}
$$

A quick check shows that $\{\bar{c}_{12}+\bar{c}_{13}+\bar{c}_{23},P\}=0$ for $P=f_{123}$ and $P=g_{123}$, as expected. In this rescaled basis, the previous relations can be cast into the form:

$$
\begin{aligned}
&\bar{c}_1+\bar{c}_2+\bar{c}_3=0,\bar{c}_{12}+\bar{c}_{13}+\bar{c}_{23}=c^{[2]}+\bar{c}_1^2+\bar{c}_2^2+\bar{c}_3^2,\\
&g_{123}-\bar{c}_3\bar{c}_{12}-\bar{c}_2\bar{c}_{13}-\bar{c}_1\bar{c}_{23}=\frac{1}{2}c^{[3]}+\frac{5}{3}(\bar{c}_1^3+\bar{c}_2^3+\bar{c}_3^3)
\end{aligned}
\tag{4.16}
$$

together with:

$$
g_{123}^2-f^2{}_{123}-\big(\bar{c}_{12}-(\bar{c}_1-\bar{c}_2)^2\big)\big(\bar{c}_{13}-(\bar{c}_1-\bar{c}_3)^2\big)\big(\bar{c}_{23}-(\bar{c}_2-\bar{c}_3)^2\big)=0. \tag{4.17}
$$

At this point, two comments are in order. First, we notice that from the functional relations it is possible to extrapolate the Casimir associated to the above quadratic algebra. In particular, from (4.16) we can obtain g_{123} expressed in terms of the other generators and central elements. Then, we can replace its expression into (4.17). The resulting equation can be separated into two parts, with the Casimir written in terms of generators and central elements on the left-hand side, whereas on the right-hand side we obtain an expression depending only on the central elements. Explicitly, on the left-hand side we have

$$
\begin{aligned}
K=&f_{123}^2+\bar{c}_{12}\bar{c}_{13}\bar{c}_{23}-\sum_{i\neq j\neq k}\bar{c}_k^2\bar{c}_{ij}^2-\sum_{i\neq j\neq k}(\bar{c}_i^2+\bar{c}_j^2)\bar{c}_{ki}\bar{c}_{kj}\\
&+\sum_{i\neq j\neq k}\left((\bar{c}_i-\bar{c}_j)^2(\bar{c}_i-\bar{c}_k)^2-\bar{c}_i\left(c^{[3]}+\frac{10}{3}(\bar{c}_1^3+\bar{c}_2^3+\bar{c}_3^3)\right)\right)\bar{c}_{jk},
\end{aligned}
\tag{4.18}
$$

while the right-hand side of the equation reads:

$$
K=\frac{1}{4}\big(c^{[3]}\big)^2+\frac{5}{3}(\bar{c}_1^3+\bar{c}_2^3+\bar{c}_3^3)c^{[3]}+2(\bar{c}_1^2+\bar{c}_2^2)(\bar{c}_1^2+\bar{c}_3^2)(\bar{c}_2^2+\bar{c}_3^2). \tag{4.19}
$$

We further observe that, taking into account the functional relation (4.17), we can restrict to consider just a subset of the relations (4.14) and (4.15). These share some similarities with the defining relations of the Racah algebra $R(3)$ [14, 26]. In this setting, however, we see that the second-order and third-order Casimir of the Lie algebra appear on the right hand side of the relations. To the best of our knowledge, this represents a new Racah-type quadratic algebra, here arising from suitable combinations of the elements of the commutant of the Cartan subalgebra. We prove in the following that the well-known relations of Racah algebra $R(3)$ can be recovered after considering explicit canonical realizations for the generators, the latter being associated to the generic superintegrable system on the two-sphere after reduction.

4.2. Connection with the superintegrable system on the two-sphere $\mathbb{S}^2$

In order to provide an actual physical application to superintegrable systems we provide here a connection with the generic superintegrable system on the two-sphere $\mathbb{S}^2$, determined by the Hamiltonian model (see [4]):

$$\begin{aligned} H &= \frac{p_1^2}{2}+\frac{p_2^2}{2}+\frac{p_3^2}{2}+\frac{\alpha_1^2}{2s_1^2}+\frac{\alpha_2^2}{2s_2^2}+\frac{\alpha_3^2}{2s_3^2} \\ &= \frac{1}{2}\sum_{1\leqslant i<j}^{3}(s_ip_j-s_jp_i)^2+\frac{1}{2}\sum_{i=1}^{3}\frac{\alpha_i^2}{s_i^2} \quad \cup \quad \begin{cases} s_1p_1+s_2p_2+s_3p_3=0 \\ s_1^2+s_2^2+s_3^2=1 \end{cases} \end{aligned} \tag{4.20}$$

This problem has been faced in the paper [16], in relation to the Lie algebra $\mathfrak{su}(3)$ (see e.g. [32]). In this context, the canonical realization for the generators of the underlying Lie algebra $\mathfrak{sl}(3)$ that allows us to connect with the generic superintegrable system on the sphere (4.20) reads:

$$\begin{aligned} h_1 &= \mathrm{i}(\alpha_1-\alpha_2), \quad h_2=\mathrm{i}(\alpha_2-\alpha_3), \\ &\begin{cases} e_{ij}=-\frac{1}{2}\left((s_ip_j-s_jp_i)-\mathrm{i}\left(\alpha_i\frac{s_j}{s_i}+\alpha_j\frac{s_i}{s_j}\right)\right) \\ e_{ji}=-\frac{1}{2}\left((s_jp_i-s_ip_j)-\mathrm{i}\left(\alpha_i\frac{s_j}{s_i}+\alpha_j\frac{s_i}{s_j}\right)\right) \end{cases} \qquad (1\leqslant i<j\leqslant 3). \end{aligned} \tag{4.21}$$

As a result of (4.21), we get the following identifications for the second and third-order Casimir invariants:

$$c^{[2]}=-\frac{1}{2}\left(H+\frac{1}{6}(\alpha_1+\alpha_2+\alpha_3)^2\right) \tag{4.22}$$

$$c^{[3]}=\frac{\mathrm{i}}{3}(\alpha_1+\alpha_2+\alpha_3)c^{[2]}+\frac{\mathrm{i}}{27}(\alpha_1+\alpha_2+\alpha_3)^3. \tag{4.23}$$

Notice that the third-order Casimir collapses to a combination of the second-order Casimir and the constants α_i appearing in the Hamiltonian, the latter being:

$$H=-2c^{[2]}-\frac{1}{6}(\alpha_1+\alpha_2+\alpha_3)^2. \tag{4.24}$$

In terms of these elements, exactly the same relations given in (4.7) are obtained. In particular, the element g_{123} collapses as a consequence of the reduction:

$$g_{123}=\mathrm{i}\,(\alpha_3c_{12}+\alpha_2c_{13}+\alpha_1c_{23}+\alpha_1\alpha_2\alpha_3). \tag{4.25}$$

This leads to the usual Racah algebra $R(3)$. In fact, we see that the elements c_{ij} of the commutant of the Cartan subalgebra adopt the well-known expression:

$$c_{ij}=-\frac{1}{4}\left((s_ip_j-s_jp_i)^2+\alpha_i^2\frac{s_j^2}{s_i^2}+\alpha_j^2\frac{s_i^2}{s_j^2}+2\alpha_i\alpha_j\right) \qquad 1\leqslant i<j\leqslant 3 \tag{4.26}$$

namely, they collapse to the constants of the motion associated to the model (4.24). At this point, by performing the change of basis as in (4.13), re-expressing all in terms of the Hamiltonian H and the rescaled constants of motion, that now read:

$$\bar{c}_{ij}=-\frac{1}{4}\left((s_ip_j-s_jp_i)^2+(s_i^2+s_j^2)\left(\frac{\alpha_i^2}{s_i^2}+\frac{\alpha_j^2}{s_j^2}\right)\right) \qquad (1\leqslant i<j\leqslant 3) \tag{4.27}$$

we obtain the symmetry algebra associated to the generic superintegrable system on the sphere $\mathbb{S}^2$:

$$\begin{aligned} f_{123} &= \frac{1}{2}\{\bar{c}_{12},\bar{c}_{23}\} = \frac{1}{2}\{\bar{c}_{23},\bar{c}_{13}\} = \frac{1}{2}\{\bar{c}_{13},\bar{c}_{12}\}, \\ \{\bar{c}_{12},f_{123}\} &= \bar{c}_{12}(\bar{c}_{23}-\bar{c}_{13}) + \frac{1}{16}(\alpha_1^2-\alpha_2^2)(2H-\alpha_3^2) \\ \{\bar{c}_{13},f_{123}\} &= \bar{c}_{13}(\bar{c}_{12}-\bar{c}_{23}) + \frac{1}{16}(\alpha_3^2-\alpha_1^2)(2H-\alpha_2^2) \\ \{\bar{c}_{23},f_{123}\} &= \bar{c}_{23}(\bar{c}_{13}-\bar{c}_{12}) + \frac{1}{16}(\alpha_2^2-\alpha_3^2)(2H-\alpha_1^2). \end{aligned} \tag{4.28}$$

along with the functional (actually linear) relation among the four constants of the motion:

$$\frac{H}{2} + \bar{c}_{12} + \bar{c}_{13} + \bar{c}_{23} + \frac{1}{4}(\alpha_1^2+\alpha_2^2+\alpha_3^2) = 0. \tag{4.29}$$

Notice that the Casimir (4.19) collapses to the following quadratic function of the Hamiltonian:

$$K = \Omega_1(\alpha_1,\alpha_2,\alpha_3)H^2 + \Omega_2(\alpha_1,\alpha_2,\alpha_3)H + \Omega_3(\alpha_1,\alpha_2,\alpha_3), \tag{4.30}$$

where:

$$\Omega_1(\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{144}(\alpha_1+\alpha_2+\alpha_3)^2 \tag{4.31}$$

$$\begin{aligned} \Omega_2(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{144}(\alpha_1+\alpha_2+\alpha_3)\Bigg(&(\alpha_1+\alpha_2+\alpha_3)^3 \\ &- 5\sum_{i\neq j\neq k}\alpha_i(\alpha_j+\alpha_k)^2 + 30\alpha_1\alpha_2\alpha_3\Bigg) \end{aligned} \tag{4.32}$$

$$\begin{aligned} \Omega_3(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{576}\big(&\alpha_1^6 - 4(\alpha_2+\alpha_3)\alpha_1^5 + \left(9\alpha_2^2+2\alpha_3\alpha_2+9\alpha_3^2\right)\alpha_1^4 \\ &- 2(\alpha_2+\alpha_3)\alpha_1^3\left(4\alpha_2^2-3\alpha_3\alpha_2+4\alpha_3^2\right) + \left(9\alpha_2^4-2\alpha_3\alpha_2^3\right. \\ &\left.+6\alpha_3^2\alpha_2^2-2\alpha_3^3\alpha_2+9\alpha_3^4\right)\alpha_1^2 - 2(\alpha_2+\alpha_3)\left(\alpha_2^2+\alpha_3^2\right) \\ &\times\left(2\alpha_2^2-3\alpha_3\alpha_2+2\alpha_3^2\right)\alpha_1 + \left(\alpha_2^2+\alpha_3^2\right) \\ &\times\left(\alpha_2^4-4\alpha_3\alpha_2^3+8\alpha_3^2\alpha_2^2-4\alpha_3^3\alpha_2+\alpha_3^4\right)\big). \end{aligned} \tag{4.33}$$

It is not surprising that in terms of the following quantities:

$$C_i := -\alpha_i^2/4, \quad C_{ij} := \bar{c}_{ij}, \quad C_{123} := -H/2, \quad F_{123} := f_{123} \tag{4.34}$$

the relations of the rank-one Racah algebra $R(3)$ are satisfied [13, 14]:

$$\begin{aligned} F_{123} &= \frac{1}{2}\{C_{12},C_{23}\} = \frac{1}{2}\{C_{23},C_{13}\} = \frac{1}{2}\{C_{13},C_{12}\}, \\ \{C_{12},F_{123}\} &= (C_{23}-C_{13})C_{12} + (C_2-C_1)(C_3-C_{123}) \\ \{C_{13},F_{123}\} &= (C_{12}-C_{23})C_{13} + (C_1-C_3)(C_2-C_{123}) \\ \{C_{23},F_{123}\} &= (C_{13}-C_{12})C_{23} + (C_3-C_2)(C_1-C_{123}), \end{aligned} \tag{4.35}$$

together with the linear relation

$$C_{123} = C_{12}+C_{13}+C_{23}-C_1-C_2-C_3. \tag{4.36}$$

The latter is obtained after considering the explicit realisation in terms of canonical coordinates (s_i, p_i). An alternative presentation, that will turn out to be the most suitable for higher-rank cases, is the one involving the generators [33]:

$$P_{ii} := 2C_i \qquad P_{ij} := C_{ij} - C_i - C_j, \tag{4.37}$$

explicitly, in the given canonical realisation:

$$P_{ii} = -\alpha_i^2/2, \qquad P_{ij} = -\frac{1}{4}\left((s_i p_j - s_j p_i)^2 + \left(\alpha_i^2 \frac{s_j^2}{s_i^2} + \alpha_j^2 \frac{s_i^2}{s_j^2}\right)\right). \tag{4.38}$$

In terms of these generators, the rank-one Racah algebra $R(3)$ can be rewritten as:

$$\begin{aligned} F_{123} &= \frac{1}{2}\{P_{12}, P_{23}\} = \frac{1}{2}\{P_{23}, P_{13}\} = \frac{1}{2}\{P_{13}, P_{12}\}, \\ \{P_{12}, F_{123}\} &= (P_{12} + P_{11})P_{23} - (P_{12} + P_{22})P_{13} \\ \{P_{13}, F_{123}\} &= (P_{13} + P_{33})P_{12} - (P_{13} + P_{11})P_{23} \\ \{P_{23}, F_{123}\} &= (P_{23} + P_{22})P_{13} - (P_{23} + P_{33})P_{12}, \end{aligned} \tag{4.39}$$

where the Hamiltonian of the model is given by

$$H = -2\sum_{1 \leqslant i < j}^{3} P_{ij} - \sum_{i=1}^{3} P_{ii}. \tag{4.40}$$

At this stage, it is worthy to be mentioned that the algebra can be rewritten in a more compact form if one takes into account the symmetry properties of the generators with respect to the indices. In fact, considering that $P_{ij} = P_{ji}$ is symmetric on exchange of two indices, whereas the three-indices generator $F_{ijk} = -F_{jik} = F_{jki}$ is antisymmetric, for $i \neq j \neq k \in \{1,2,3\}$, the algebra can be rewritten as [33]:

$$\{P_{ij}, P_{jk}\} = 2F_{ijk}, \qquad \{P_{jk}, F_{ijk}\} = (P_{jk} + P_{jj})P_{ik} - (P_{jk} + P_{kk})P_{ij}. \tag{4.41}$$

4.2.1. The general form of the quadratic algebra. In analogy with the Racah case, we can use the symmetry properties on the indices of generators in $C_{\mathfrak{sl}(3)^*}(\mathfrak{h}^*)$ to present the associated quadratic algebra in a more general form.

The defining elements of the quadratic algebra are defined for $i \neq j \neq k \in \{1,2,3\}$ as:

$$\begin{aligned} c_i &:= \frac{1}{3}\sum_{j=1}^{2}(3-j)h_j - \sum_{j=1}^{i-1} h_j \qquad c_{ij} := p_{i,j}, \\ f_{ijk} &:= \frac{1}{2}(p_{i,k,j} - p_{i,j,k}) \qquad g_{ijk} := \frac{1}{2}(p_{i,k,j} + p_{i,j,k}). \end{aligned} \tag{4.42}$$

The elements c_i play the role of central elements of the algebra. In particular, they depend on the Cartan elements $\{h_1, h_2\}$ in such a way $c_1 + c_2 + c_3 = 0$. The two and three indices elements satisfy the following symmetry properties:

$$c_{ij} = c_{ji} \qquad \text{(symmetric)} \tag{4.43}$$

$$f_{ijk} = -f_{jik} = f_{jki} \qquad \text{(skew-symmetric)} \tag{4.44}$$

$$g_{ijk} = g_{jik} = g_{jki} \qquad \text{(symmetric)} \tag{4.45}$$

The previous elements, for $i \neq j \neq k \in \{1,2,3\}$ satisfy the following quadratic algebra:

$$\begin{aligned}
\{c_i,\cdot\} &= 0\\
\{c_{ij},c_{jk}\} &= 2f_{ijk}\\
\{c_{jk},f_{ijk}\} &= (c_{ik}-c_{ij})c_{jk}+(c_j-c_k)g_{ijk}\\
\{c_{jk},g_{ijk}\} &= (c_j-c_k)f_{ijk}\\
\{f_{ijk},g_{ijk}\} &= \frac{1}{2}\big((c_i-c_k)c_{ij}c_{jk}+(c_k-c_j)c_{ki}c_{ij}+(c_j-c_i)c_{jk}c_{ki}\big)
\end{aligned} \tag{4.46}$$

together with the additional functional relation:

$$f_{ijk}f_{kji}+g_{ijk}g_{kji}=c_{ij}c_{jk}c_{ki}. \tag{4.47}$$

With this notation, the Casimir invariant of the quadratic algebra can be expressed as:

$$K_{ijk}=f_{ijk}f_{jki}+c_{ij}c_{jk}c_{ki}-(c_ic_{jk}+c_jc_{ik}+c_kc_{ij})\left(g_{ijk}-\frac{1}{4}(c_ic_{jk}+c_jc_{ik}+c_kc_{ij})\right), \tag{4.48}$$

where the symmetry properties $K_{ijk}=K_{jik}=K_{jki}$ hold. In particular,

$$K_{ijk}=\frac{1}{4}\big(c^{[3]}-c_ic_jc_k\big)^2. \tag{4.49}$$

4.3. Systems on the three-sphere and $\mathfrak{sl}(4)$

We next consider the rank three Lie algebra $\mathfrak{sl}(4)$. In this case, the system (3.3) has 12 functionally independent solutions, while the centralizer $C_{S(\mathfrak{sl}(4))}(\mathfrak{h})$ is of dimension 23. Following (3.32), a basis of $C_{\mathfrak{sl}(4)^*}(\mathfrak{h}^*)$ is given by

$$\begin{aligned}
&h_1,\quad h_2,\quad h_3,\quad p_{1,2},\quad p_{1,3},\quad p_{1,4},\quad p_{2,3},\quad p_{2,4},\quad p_{3,4},\quad p_{1,2,3},\\
&p_{1,2,4},\quad p_{1,3,4},\quad p_{2,3,4},\quad p_{1,3,2},\quad p_{1,4,2},\quad p_{1,4,3},\quad p_{2,4,3},\\
&p_{1,2,3,4},\quad p_{1,2,4,3},\quad p_{1,3,2,4},\quad p_{1,4,2,3},\quad p_{1,4,3,2},\quad p_{1,3,4,2},
\end{aligned} \tag{4.50}$$

with $\{h_1,h_2,h_3,p_{1,2},p_{1,3},p_{1,4},p_{2,3},p_{2,4},p_{3,4},p_{1,2,3},p_{1,2,4},p_{1,3,4}\}$ being functionally independent.

In analogy to the previous case, we can use symmetry properties on the indices of the generators to determine a more suitable basis that simplifies the expression of the Lie-Poisson brackets and results in a more transparent description of the polynomial algebra generated by the monomials in (4.50). We redefine the basis $i \neq j \neq k \neq l \in \{1,2,3,4\}$ as follows:

$$\begin{aligned}
c_i &:= \frac{1}{4}\sum_{j=1}^{3}(4-j)h_j-\sum_{j=1}^{i-1}h_j, \quad c_{ij}:=p_{i,j},\\
f_{ijk} &:= \frac{1}{2}(p_{i,k,j}-p_{i,j,k}), \quad g_{ijk}:=\frac{1}{2}(p_{i,k,j}+p_{i,j,k}).
\end{aligned} \tag{4.51}$$

Again, the one index elements c_i play the role of central elements for the algebra, and fulfill the linear relation $c_1+c_2+c_3+c_4=0$. The c_{ij} and g_{ijk} are symmetric, while the f_{ijk} are skew-symmetric. For elements depending on four indices, instead of using the symmetric and skew-symmetric counterparts, it is computationally convenient to consider the elements

$$f_{ijkl}:=\frac{1}{2}(p_{i,l,k,j}-p_{i,j,k,l}), \qquad g_{ijkl}:=\frac{1}{2}(p_{i,l,k,j}+p_{i,j,k,l}). \tag{4.52}$$

These four-indices elements f_{ijkl} and g_{ijkl} satisfy the following symmetry properties:

$$f_{ijkl}=-f_{jilk}, \quad g_{ijkl}=g_{jilk}, \tag{4.53}$$

meaning that they are antisymmetric and symmetric with respect to the simultaneous exchange of the first and second pair of indices, and of the central pair with the external one, respectively. Besides, they both satisfy the cyclic property. The composition of these two properties provides several equalities for the four indices elements. Those equalities will be taken into account when presenting the algebra in general form.

For $i \neq j \neq k \neq l \in \{1,2,3,4\}$, in terms of the previously defined elements, the following cubic algebra relations are obtained. The elements with one index are central, hence $\{c_i, \cdot\} = 0$. For the case of two indices elements we get

$$\{c_{ij}, c_{kl}\} = 0, \quad \{c_{ij}, c_{jk}\} = 2f_{ijk}, \tag{4.54}$$

whereas for the bracket of quadratic and cubic elements we obtain

$$\begin{aligned}
\{c_{jk}, f_{ijk}\} &= (c_{ik} - c_{ij})c_{jk} + (c_j - c_k)g_{ijk}, \\
\{c_{jk}, g_{ijk}\} &= (c_j - c_k)f_{ijk}, \\
\{c_{kl}, f_{ijk}\} &= g_{ijlk} - g_{ijkl}, \\
\{c_{kl}, g_{ijk}\} &= f_{ijlk} - f_{ijkl}, \\
\{c_{jk}, f_{ijk}\} &= (c_{ik} - c_{ij})c_{jk} + (c_j - c_k)g_{ijk} \\
\{c_{jk}, g_{ijk}\} &= (c_j - c_k)f_{ijk} \\
\{c_{kl}, f_{ijk}\} &= g_{ijlk} - g_{ijkl} \\
\{c_{kl}, g_{ijk}\} &= f_{ijlk} - f_{ijkl}.
\end{aligned} \tag{4.55}$$

We next consider the brackets of elements with three indices:

$$\begin{aligned}
\{f_{ijk}, g_{ijk}\} &= \frac{1}{2}\big((c_i - c_k)c_{ij}c_{jk} + (c_k - c_j)c_{ki}c_{ij} + (c_j - c_i)c_{jk}c_{ki}\big), \\
\{f_{ijk}, f_{jkl}\} &= \frac{1}{2}((c_{ij} - c_{ki})f_{jkl} + (c_{kl} - c_{jl})f_{ijk} + (f_{ilj} + f_{ilk})c_{jk} + (c_j - c_k)f_{ijlk}), \\
\{g_{ijk}, g_{jkl}\} &= \frac{1}{2}((c_{ij} - c_{ki})f_{jkl} + (c_{kl} - c_{jl})f_{ijk} + (f_{ijl} + f_{ikl})c_{jk} + (c_j - c_k)f_{iklj}), \\
\{f_{ijk}, g_{jkl}\} &= \frac{1}{2}(c_{ij} - c_{ki})g_{jkl} + (c_{kl} - c_{jl})g_{ijk} + (g_{ijl} - g_{ikl})c_{jk} + (c_k - c_j)g_{iklj}).
\end{aligned} \tag{4.56}$$

Elements with two indices with those having four indices close as

$$\begin{aligned}
\{c_{kl}, f_{ijkl}\} &= (g_{ilj} - g_{ikj})c_{kl} + (c_k - c_l)g_{ijkl}, \\
\{c_{kl}, f_{ijlk}\} &= (g_{ikj} - g_{ilj})c_{kl} + (c_l - c_k)g_{ijlk}, \\
\{c_{kl}, f_{iljk}\} &= (c_{jl} - c_{jk})g_{ikl} + (c_{ik} - c_{il})g_{jkl}, \\
\{c_{kl}, g_{ijkl}\} &= (f_{ijl} - f_{ijk})c_{kl} + (c_k - c_l)f_{ijkl}, \\
\{c_{kl}, g_{ijlk}\} &= (f_{ijk} - f_{ijl})c_{kl} + (c_l - c_k)f_{ijlk}, \\
\{c_{kl}, g_{iljk}\} &= (c_{jk} - c_{jl})f_{ikl} + (c_{ik} - c_{il})f_{jkl},
\end{aligned} \tag{4.57}$$

while for elements with three and four indices, respectively, we get the relations

$$
\begin{aligned}
\{f_{jkl},f_{ijkl}\} =&\frac{1}{2}\big((c_{jk}-c_{kl})f_{ijkl}+(g_{ikl}-g_{ijk})f_{jkl}+(f_{ikl}-f_{ijk})g_{jkl}+\big((c_l-c_k)c_{jk}\\
&+(c_k-c_j)c_{kl}\big)f_{ijl}\big),\\
\{f_{jkl},f_{ijlk}\} =&\frac{1}{2}\big((c_{kl}-c_{jl})f_{ijlk}+(g_{ikl}-g_{ijl})f_{jkl}+(f_{ijl}+f_{ikl})g_{jkl}+((c_j-c_l)c_{kl}\\
&+(c_l-c_k)c_{jl})f_{ijk}\big),\\
\{f_{jkl},f_{iljk}\} =&\frac{1}{2}\big((c_{jl}-c_{jk})f_{iljk}+(g_{ijk}-g_{ijl})f_{jkl}+(f_{ijk}+f_{ijl})g_{jkl}+((c_j-c_k)c_{jl}\\
&+(c_l-c_j)c_{jk})f_{ikl}),\\
\{g_{jkl},g_{ijkl}\} =&\frac{1}{2}\big((c_{jk}-c_{kl})f_{ijkl}+(g_{ikl}-g_{ijk})f_{jkl}+(f_{ikl}-f_{ijk})g_{jkl}+\big((c_k-c_l)c_{jk}\\
&+(c_j-c_k)c_{kl}\big)f_{ijl}\big),\\
\{g_{jkl},g_{ijlk}\} =&\frac{1}{2}\big((c_{jl}-c_{kl})f_{ijlk}+(g_{ijl}-g_{ikl})f_{jkl}-(f_{ijl}+f_{ikl})g_{jkl}+((c_j-c_l)c_{kl}\\
&+(c_l-c_k)c_{jl})f_{ijk}\big),\\
\{g_{jkl},g_{iljk}\} =&\frac{1}{2}\big((c_{jl}-c_{jk})f_{iljk}+(g_{ijk}-g_{ijl})f_{jkl}+(f_{ijk}+f_{ijl})g_{jkl}+((c_k-c_j)c_{jl}\\
&+(c_j-c_l)c_{jk})f_{ikl}),\\
\{f_{jkl},g_{ijkl}\} =&\frac{1}{2}\left((c_{jk}-c_{kl})g_{ijkl}+(c_{ij}-c_{il})c_{jk}c_{kl}+(f_{ikl}-f_{ijk})f_{jkl}+(g_{ikl}-g_{ijk})g_{jkl}\right.\\
&\left.+\left((c_k-c_j)c_{kl}+(c_l-c_k)c_{jk}\right)g_{ijl}\right),\\
\{f_{jkl},g_{ijkl}\} =&\frac{1}{2}\left((c_{jk}-c_{kl})g_{ijkl}+(c_{ij}-c_{il})c_{jk}c_{kl}+(f_{ikl}-f_{ijk})f_{jkl}+(g_{ikl}-g_{ijk})g_{jkl}\right.\\
&\left.+\left((c_k-c_j)c_{kl}+(c_l-c_k)c_{jk}\right)g_{ijl}\right),\\
\{f_{jkl},g_{ijlk}\} =&\frac{1}{2}\left((c_{kl}-c_{jl})g_{ijlk}+(c_{ik}-c_{ij})c_{jl}c_{kl}-(f_{ijl}+f_{ikl})f_{jkl}+(g_{ijl}-g_{ikl})g_{jkl}\right.\\
&\left.+\left((c_j-c_l)c_{kl}+(c_l-c_k)c_{jl}\right)g_{ijk}\right),\\
\{f_{jkl},g_{iljk}\} =&\frac{1}{2}\left((c_{jl}-c_{jk})g_{iljk}+(c_{il}-c_{ik})c_{jk}c_{jl}+(f_{ijk}+f_{ijl})f_{jkl}+(g_{ijk}-g_{ijl})g_{jkl}\right.\\
&\left.+\left((c_j-c_l)c_{jk}+(c_k-c_l)c_{jl}\right)g_{ikl}\right),\\
\{g_{jkl},f_{ijkl}\} =&\frac{1}{2}\left((c_{jk}-c_{kl})g_{ijkl}+(c_{il}-c_{ij})c_{jk}c_{kl}+(f_{ikl}-f_{ijk})f_{jkl}+(g_{ikl}-g_{ijk})g_{jkl}\right.\\
&\left.+\left((c_j-c_k)c_{kl}+(c_k-c_l)c_{jk}\right)g_{ijl}\right),\\
\{g_{jkl},f_{ijlk}\} =&\frac{1}{2}\left((c_{jl}-c_{kl})g_{ijlk}+(c_{ik}-c_{ij})c_{jl}c_{kl}+(f_{ijl}+f_{ikl})f_{jkl}+(g_{ikl}-g_{ijl})g_{jkl}\right.\\
&\left.+\left((c_j-c_l)c_{kl}+(c_l-c_k)c_{jl}\right)g_{ijk}\right),\\
\{g_{jkl},f_{iljk}\} =&\frac{1}{2}\left((c_{jl}-c_{jk})g_{iljk}+(c_{il}-c_{ik})c_{jk}c_{jl}+(f_{ijk}+f_{ijl})f_{jkl}+(g_{ijk}-g_{ijl})g_{jkl}\right.\\
&\left.+\left((c_j-c_l)c_{jk}+(c_k-c_l)c_{jl}\right)g_{ikl}\right).
\end{aligned}
$$

Finally, the brackets of elements with four indices elements lead to the following identities

$$
\begin{aligned}
\{f_{ijkl},f_{ijlk}\} =&\frac{1}{2}\big(\big((c_{jk}+c_{jl}-c_{kl})f_{ikl}+(c_{kl}-c_{ik}-c_{il})f_{jkl}\big)c_{ij}+\big((c_{ij}-c_{il}-c_{jl})f_{ijk}\\
&+(c_{ik}+c_{jk}-c_{ij})f_{ijl}\big)c_{kl}+(c_i-c_j)(f_{ikl}g_{jkl}+f_{jkl}g_{ikl})+(c_l-c_k)(f_{ijk}g_{ijl}+f_{ijl}g_{ijk})\big),\\
\{f_{ijkl},f_{iljk}\} =&\frac{1}{2}\big(\big((c_{ik}+c_{kl}-c_{il})f_{ijl}+(c_{il}-c_{ij}-c_{jl})f_{ikl}\big)c_{jk}+\big((c_{jl}+c_{kl}-c_{jk})f_{ijk}\\
&+(c_{jk}-c_{ij}-c_{ik})f_{jkl}\big)c_{il}+(c_i-c_l)(f_{ijk}g_{jkl}+f_{jkl}g_{ijk})+(c_j-c_k)(f_{ikl}g_{ijl}+f_{ijl}g_{ikl})\big),\\
\{f_{ijlk},f_{iljk}\} =&\frac{1}{2}\big(\big((c_{ik}-c_{il}-c_{kl})f_{ijk}+(c_{ik}-c_{ij}-c_{jk})f_{ikl}\big)c_{jl}+\big((c_{jl}-c_{jk}-c_{kl})f_{ijl}\\
&+(c_{jl}-c_{ij}-c_{il})f_{jkl}\big)c_{ik}+(c_i-c_k)(f_{jkl}g_{ijl}-f_{ijl}g_{jkl})+(c_l-c_j)(f_{ijk}g_{ikl}-f_{ikl}g_{ijk})\big),\\
\{g_{ijkl},g_{ijlk}\} =&\frac{1}{2}\big(\big((c_{kl}-c_{jk}-c_{jl})f_{ikl}+(c_{ik}+c_{il}-c_{kl})f_{jkl}\big)c_{ij}+\big((c_{ij}-c_{il}-c_{jl})f_{ijk}\\
&+(c_{ik}+c_{jk}-c_{ij})f_{ijl}\big)c_{kl}+(c_j-c_i)(f_{ikl}g_{jkl}+f_{jkl}g_{ikl})+(c_l-c_k)(f_{ijl}g_{ijk}+f_{ijk}g_{ijl})\big),\\
\{g_{ijkl},g_{iljk}\} =&\frac{1}{2}\big(\big((c_{il}-c_{ik}-c_{kl})f_{ijl}+(c_{ij}+c_{jl}-c_{il})f_{ikl}\big)c_{jk}+\big((c_{jl}+c_{kl}-c_{jk})f_{ijk}\\
&+(c_{jk}-c_{ij}-c_{ik})f_{jkl}\big)c_{il}+(c_i-c_l)(f_{ijk}g_{jkl}+f_{jkl}g_{ijk})+(c_k-c_j)(f_{ikl}g_{ijl}+f_{ijl}g_{ikl})\big),\\
\{g_{ijlk},g_{iljk}\} =&\big(\big((c_{ik}-c_{il}-c_{kl})f_{ijk}+(c_{ik}-c_{ij}-c_{jk})f_{ikl}\big)c_{jl}+\big((c_{jk}+c_{kl}-c_{jl})f_{ijl}\\
&+(c_{ij}+c_{il}-c_{jl})f_{jkl}\big)c_{ik}+(c_k-c_i)(f_{jkl}g_{ijl}-f_{ijl}g_{jkl})+(c_l-c_j)(f_{ijk}g_{ikl}-f_{ikl}g_{ijk})\big),\\
\{f_{ijkl},g_{ijkl}\} =&\frac{1}{2}\big(\big((c_i-c_l)c_{ij}-(c_i-c_j)c_{il}\big)c_{jk}c_{kl}+\big((c_l-c_k)c_{jk}+(c_k-c_j)c_{kl}\big)c_{ij}c_{il}\big),\\
\{f_{ijkl},g_{ijlk}\} =&\frac{1}{2}\big(\big((c_{kl}-c_{jk}-c_{jl})g_{ikl}+(c_{ik}+c_{il}-c_{kl})g_{jkl}\big)c_{ij}+\big((c_{ij}-c_{il}-c_{jl})g_{ijk}\\
&+(c_{il}+c_{jk}-c_{ij})g_{ijl}\big)c_{kl}+(c_l-c_k)(f_{ijk}f_{ijl}+g_{ijk}g_{ijl})+(c_j-c_i)(f_{ikl}f_{jkl}+g_{ikl}g_{jkl})\big),\\
\{f_{ijkl},g_{iljk}\} =&\frac{1}{2}\big(\big((c_{il}-c_{ik}-c_{kl})g_{ijl}+(c_{ij}-c_{il}+c_{jl})g_{ikl}\big)c_{jk}+\big((c_{jl}+c_{kl}-c_{jk})g_{ijk}\\
&+(c_{jk}-c_{ij}-c_{ik})g_{jkl}\big)c_{il}+(c_k-c_j)(f_{ijl}f_{ikl}+g_{ijl}g_{ikl})+(c_i-c_l)(f_{ijk}f_{jkl}+g_{ijk}g_{jkl})\big),\\
\{f_{ijlk},g_{ijkl}\} =&\frac{1}{2}\big(\big((c_{kl}-c_{jk}-c_{jl})g_{ikl}+(c_{ik}+c_{il}-c_{kl})g_{jkl}\big)c_{ij}+\big((c_{il}+c_{jl}-c_{ij})g_{ijk}\\
&+(c_{ij}-c_{ik}-c_{jk})g_{ijl}\big)c_{kl}+(c_k-c_l)(f_{ijk}f_{ijl}+g_{ijk}g_{ijl})+(c_j-c_i)(f_{ikl}f_{jkl}+g_{ikl}g_{jkl})\big),\\
\{f_{ijlk},g_{iljk}\} =&\frac{1}{2}\big(\big((c_{ik}-c_{il}-c_{kl})g_{ijk}+(c_{ij}-c_{ik}+c_{jk})g_{ikl}\big)c_{jl}+\big((c_{jk}-c_{jl}+c_{kl})g_{ijl}\\
&+(c_{jl}-c_{ij}-c_{il})g_{jkl}\big)c_{ik}+(c_j-c_l)(f_{ijk}f_{ikl}-g_{ijk}g_{ikl})+(c_k-c_i)(f_{ijl}f_{jkl}-g_{ijl}g_{jkl})\big),\\
\{f_{iljk},g_{ijkl}\} =&\frac{1}{2}\big(\big((c_{jk}-c_{jl}-c_{kl})g_{ijk}+(c_{ij}+c_{ik}-c_{jk})g_{jkl}\big)c_{il}+\big((c_{il}-c_{ik}-c_{kl})g_{ijl}\\
&+(c_{ij}-c_{il}+c_{jl})g_{ikl}\big)c_{jk}+(c_l-c_i)(f_{ijk}f_{jkl}+g_{ijk}g_{jkl})+(c_k-c_j)(f_{ijl}f_{ikl}+g_{ijl}g_{ikl})\big),\\
\{f_{iljk},g_{ijlk}\} =&\frac{1}{2}\big(\big((c_{jk}-c_{jl}+c_{kl})g_{ijl}+(c_{jl}-c_{ij}-c_{il})g_{jkl}\big)c_{ik}+\big((c_{il}+c_{kl}-c_{ik})g_{ijk}\\
&+(c_{ik}-c_{ij}-c_{jk})g_{ikl}\big)c_{jl}+(c_l-c_j)(f_{ijk}f_{ikl}-g_{ijk}g_{ikl})+(c_k-c_i)(f_{jkl}f_{ijl}-g_{jkl}g_{ijl})\big).
\end{aligned}
$$

Besides the dependence relations enumerated in (3.33), there are some additional functional identities like

$$\begin{aligned} f_{ijk}f_{kji} + g_{ijk}g_{kji} &= c_{ij}c_{jk}c_{ki}, \\ f_{ijkl}f_{lkji} + g_{ijkl}g_{lkji} &= c_{ij}c_{jk}c_{kl}c_{li}. \end{aligned} \tag{4.58}$$

Once we have described the general structure of the cubic algebra $\mathcal{A}_4$, we next show that, once a realization associated to the generic superintegrable systems on the three-sphere $\mathbb{S}^3$ is considered, the cubic structure reduces to that of the quadratic algebra $R(4)$.

4.4. Connection with the superintegrable system on the three-sphere $\mathbb{S}^3$

The superintegrable system on the three sphere $\mathbb{S}^3$ is defined by the Hamiltonian:

$$\begin{aligned} H &= \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{p_3^2}{2} + \frac{p_4^2}{2} + \frac{\alpha_1^2}{2s_1^2} + \frac{\alpha_2^2}{2s_2^2} + \frac{\alpha_3^2}{2s_3^2} + \frac{\alpha_4^2}{2s_4^2} \\ &= \frac{1}{2}\sum_{1\leqslant i<j}^{4}(s_ip_j - s_jp_i)^2 + \frac{1}{2}\sum_{i=1}^{4}\frac{\alpha_i^2}{s_i^2} \end{aligned} \tag{4.59}$$

subjected to the constraints

$$s_1p_1 + s_2p_2 + s_3p_3 + s_4p_4 = 0, \quad s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1. \tag{4.60}$$

In analogy with the rank-two case, we consider the canonical realization for the generators of $\mathfrak{sl}(4)$:

$$\begin{aligned} &h_1 = \mathrm{i}(\alpha_1 - \alpha_2), \quad h_2 = \mathrm{i}(\alpha_2 - \alpha_3), \quad h_3 = \mathrm{i}(\alpha_3 - \alpha_4), \\ &e_{ij} = -\frac{1}{2}\left((s_ip_j - s_jp_i) - \mathrm{i}\left(\alpha_i\frac{s_j}{s_i} + \alpha_j\frac{s_i}{s_j}\right)\right), \\ &e_{ji} = -\frac{1}{2}\left((s_jp_i - s_ip_j) - \mathrm{i}\left(\alpha_i\frac{s_j}{s_i} + \alpha_j\frac{s_i}{s_j}\right)\right). \end{aligned} \tag{4.61}$$

From this realization, it is straightforward to derive the following identifications with the quadratic, cubic and quartic Casimir invariants of $\mathfrak{sl}(4)^*$:

$$\begin{aligned} c^{[2]} &= -\frac{1}{2}\left(H + \frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2\right) \\ c^{[3]} &= \frac{\mathrm{i}}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)c^{[2]} + \frac{\mathrm{i}}{16}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^3 \\ c^{[4]} &= -\frac{1}{16}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2c^{[2]} + \frac{1}{8}(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^3 \\ &\quad -\frac{1}{8}\left((\alpha_1+\alpha_2+\alpha_3)^2 + 3(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2 \\ &\quad +\frac{1}{2}(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\ &\quad -\frac{9}{256}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^4 + 2\alpha_1\alpha_2\alpha_3\alpha_4. \end{aligned} \tag{4.62}$$

We observe that both the third-order and fourth-order Casimir invariants collapse to a combination of the quadratic one and the constants α_i appearing in the Hamiltonian, the latter being given by:

$$H = -2c^{[2]} - \frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2. \tag{4.63}$$

Again, the two indices elements c_{ij} belonging to the commutant of the Cartan subalgebra assume the well-known expression:

$$c_{ij} = -\frac{1}{4}\left((s_ip_j - s_jp_i)^2 + \alpha_i^2\frac{s_j^2}{s_i^2} + \alpha_j^2\frac{s_i^2}{s_j^2} + 2\alpha_i\alpha_j\right) \qquad (1 \leqslant i < j \leqslant 4). \tag{4.64}$$

These elements, together with the elements $\{f_{ijk}, g_{ijk}\}$ and $\{f_{ijkl}, g_{ijkl}\}$, reproduce exactly the relations of the cubic algebra $\mathcal{A}_4$ presented in the previous paragraph (equations (4.54) onwards) and the related functional relations. The crucial point in this construction is that the g_{ijk}, as well as the four-indices elements f_{ijkl}, g_{ijkl} are all expressible in terms of elements with a lower number of indices after the reduction to canonical coordinates. More explicitly, the following identities are satisfied for any $i \neq j \neq k \neq l \in \{1,2,3,4\}$:

$$\begin{aligned} g_{ijk} =& \frac{\mathrm{i}}{2}(\alpha_k c_{ij} + \alpha_j c_{ik} + \alpha_i c_{jk} + \alpha_i\alpha_j\alpha_k), \\ g_{ijkl} =& \frac{1}{2}(c_{ij}c_{kl} + c_{il}c_{jk} - c_{ik}c_{jl} - \alpha_i\alpha_k c_{jl} - \alpha_j\alpha_l c_{ik} - \alpha_i\alpha_j\alpha_k\alpha_l), \\ f_{ijkl} =& \frac{\mathrm{i}}{2}(\alpha_l f_{ijk} + \alpha_k f_{ijl} + \alpha_j f_{ikl} + \alpha_i f_{jkl}). \end{aligned} \tag{4.65}$$

This means specifically that, as a result of the canonical realization, the cubic algebra collapses to a quadratic one with basis elements:

$$\alpha_i,\ c_{ij},\ f_{ijk},\ H. \tag{4.66}$$

However remarkable this result may appear, it is albeit expected, as the algebra associated to the generic superintegrable system on the three-sphere $\mathbb{S}^3$, the rank-two Racah algebra $R(4)$, is quadratic. To make this connection with $R(4)$ more explicit, we introduce the constants of the motion

$$\bar{c}_{ij} := -\frac{1}{4}\left((s_ip_j - s_jp_i)^2 + (s_i^2 + s_j^2)\left(\frac{\alpha_i^2}{s_i^2} + \frac{\alpha_j^2}{s_j^2}\right)\right), \qquad (1 \leqslant i < j \leqslant 4), \tag{4.67}$$

such that the following constraint holds:

$$\frac{H}{2} + \bar{c}_{12} + \bar{c}_{13} + \bar{c}_{14} + \bar{c}_{23} + \bar{c}_{24} + \bar{c}_{34} + \frac{1}{2}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) = 0. \tag{4.68}$$

Besides the Abelian relations involving the central elements and $\{\bar{c}_{ij}, \bar{c}_{kl}\} = 0$ for disjoint pairs of indices (ij) and (kl), we have the defining relations:

$$f_{ijk} := \frac{1}{2}\{\bar{c}_{ij}, \bar{c}_{jk}\} = \frac{1}{2}\{\bar{c}_{jk}, \bar{c}_{ik}\} = \frac{1}{2}\{\bar{c}_{ik}, \bar{c}_{ij}\} \qquad (1 \leqslant i < j < k \leqslant 4). \tag{4.69}$$

Thus the connection with the Racah algebra $R(4)$, considering that several different bases are conceivable to give a presentation (see e.g. [34]), is obtained through the following identifications:

$$\begin{aligned} C_i :=& -\alpha_i^2/4, \quad 1 \leqslant i \leqslant 4, \\ C_{ij} :=& \bar{c}_{ij}, \quad 1 \leqslant i < j \leqslant 4, \\ F_{ijk} :=& f_{ijk}, \quad 1 \leqslant i < j < k \leqslant 4. \end{aligned} \tag{4.70}$$

In analogy with the lower-rank case, we introduce the generators P_{ij} and P_{ii} (with the obvious new range for the indices) as defined in (4.38). Taking into account the symmetry properties

of the generators P_{ij}, F_{ijk}, for $i \neq j \neq k \neq l \in \{1,2,3,4\}$, it is easily checked that the following relations of $R(4)$ [33] are satisfied by the elements:

$$\begin{aligned}
&\{P_{ij},P_{kl}\} = 0, \quad \{P_{ij},P_{jk}\} = 2F_{ijk},\\
&\{P_{jk},F_{ijk}\} = (P_{jk}+P_{jj})P_{ik} - (P_{jk}+P_{kk})P_{ij},\\
&\{P_{kl},F_{ijk}\} = P_{ik}P_{jl} - P_{il}P_{jk},\\
&\{F_{ijk},F_{jkl}\} = -(F_{ijl}+F_{ikl})P_{jk}.
\end{aligned} \tag{4.71}$$

The Hamiltonian of the model can itself be expressed as:

$$H = -2\sum_{1\leqslant i<j}^{4} P_{ij} - \sum_{i=1}^{4} P_{ii}. \tag{4.72}$$

4.5. Extrapolation to the general case $n \geqslant 5$

The pattern observed for $n = 4$ remains valid for values $n \geqslant 5$, in spite of the fact that the polynomial algebra $\mathcal{A}_n$ is of order $n-1$. From the basis (3.31) we can again consider alternative elements that reflect some symmetry or skew-symmetry properties with respect to the indices, generalizing the previous cases f_{ijkl}, g_{ijkl} etc to a higher number of indices. Concerning the superintegrable model on the sphere $\mathbb{S}^{n-1}$, the corresponding Hamiltonian H is given by

$$H = \sum_{k=1}^{n}\frac{p_k^2}{2} + \frac{1}{2}\sum_{k=1}^{n}\frac{\alpha_k^2}{s_k^2} = \frac{1}{2}\sum_{1\leqslant i<j}(s_ip_j - s_jp_i)^2 + \frac{1}{2}\sum_{k=1}^{n}\frac{\alpha_k^2}{s_k^2},$$

where the constraints

$$\sum_{k=1}^{n} s_kp_k = 0, \ \sum_{k=1}^{n} s_k^2 = 1$$

are satisfied. The appropriate realization of $\mathfrak{sl}(n)$ is given by

$$\begin{aligned}
&h_k = \mathrm{i}\,(\alpha_k - \alpha_{k+1}),\ 1 \leqslant k \leq n-1\\
&e_{ij} = -\frac{1}{2}\left((s_ip_j - s_jp_i) - \mathrm{i}\left(\alpha_i\frac{s_j}{s_i} + \alpha_j\frac{s_i}{s_j}\right)\right).
\end{aligned}$$

Considering the constants of the motion $\bar{c}_{ij}$ as defined in (4.67), a long but routine computation leads that H, $\bar{c}_{ij}$ and $\sum_{k=1}^{n}\alpha_k^2$ are linearly dependent, as expected.

On the other hand, the Casimir invariants $c^{[k]}$ of $\mathfrak{sl}(n)$ are be easily obtained as trace operators according to the formula

$$c^{[k]} = \frac{1}{k}\mathrm{Tr}\begin{pmatrix} \Delta_1 & e_{12} & \cdots & e_{1,n-1} & e_{1n}\\ e_{21} & \Delta_2 & \cdots & e_{2,n-1} & e_{2n}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ e_{n-1,1} & e_{n-1,2} & \cdots & \Delta_{n-1} & e_{n-1,n}\\ e_{n1} & e_{n2} & \cdots & e_{n,n-1} & \Delta_n \end{pmatrix}^k, \tag{4.73}$$

where

$$\Delta_k = \sum_{s=k}^{n-1}\frac{n-s}{n}h_s - \sum_{s=1}^{k-1}\frac{s}{n}h_s, \quad 1 \leqslant k \leqslant n. \tag{4.74}$$

The quadratic Casimir invariant $c^{[2]}$ is related to the Hamiltonian through the dependence relation

$$2c^{[2]} + H + \frac{n-2}{2n}\left(\sum_{k=1}^{n}\alpha_k\right)^2 = 0, \quad n \geqslant 5, \tag{4.75}$$

hence leading to the identity

$$H = -2c^{[2]} - \frac{n-2}{2n}\left(\sum_{k=1}^{n}\alpha_k\right)^2. \tag{4.76}$$

Clearly, the general formula above also holds for the lower-rank cases discussed in the previous sections. A rather cumbersome but straightforward computation shows that for any $n \geqslant 5$, analogously to the lower-rank cases, the Casimir invariants $c^{[k]}$ of order $k \geqslant 3$ collapse to a combination of $c^{[2]}$ and the constants α_i appearing in the Hamiltonian, hence providing the connection with the Racah algebra $R(n)$, ultimately allowing us to write the Hamiltonian as

$$H = -2\sum_{1\leqslant i<j}^{n} P_{ij} - \sum_{i=1}^{n} P_{ii}, \tag{4.77}$$

where the generators P_{ij} for $1 \leqslant i,j \leqslant n$ are defined as in equation (4.37).

5. Concluding remarks

Using the Lie-Poisson reformulation of commutants in enveloping algebras of Lie algebras $\mathfrak{s}$, a notion of algebraic (super)integrability based on algebraic Hamiltonians and constants of the motion has been proposed. These first integrals are obtained as elements of the centralizer $C_{S(\mathfrak{s})}(\mathfrak{a})$ of a given subalgebra $\mathfrak{a}$ in the symmetric algebra of $\mathfrak{s}$. As $C_{S(\mathfrak{s})}(\mathfrak{a})$ determines a finitely-generated polynomial algebra $\mathcal{A}$, the latter can be identified with the symmetry algebra of the system defined by the algebraic Hamiltonian. For a given realization of the Lie algebra $\mathfrak{s}$, the symmetry algebra $\mathcal{A}$ can eventually reduce its polynomial order, due to additional constraints that are determined by the realization but that are not relations that hold generically in the enveloping algebra of $\mathfrak{s}$.

In order to illustrate the procedure, we have determined a basis for the centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n)$ for any values $n \geqslant 2$. The resulting polynomial algebra $\mathcal{A}_n$ has been shown to be quadratic for $n = 3$ and of order $n-1$ for $n \geqslant 4$, further defining a chain $\mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \subset \mathcal{A}_n$ adapted to the canonical embedding $\mathfrak{sl}(2) \subset \mathfrak{sl}(3) \subset \cdots \subset \mathfrak{sl}(n)$. It has been shown explicitly for $n = 3,4$ that for suitable realizations of $\mathfrak{sl}(n)$, the polynomial algebra reduces to the $(n-2)$-rank Racah algebra $R(n)$ obtained with other techniques (see [13–15] and references therein), providing an alternative derivation of the symmetry algebras of superintegrable models on the sphere $\mathbb{S}^{n-1}$ by means of an algebraic Hamiltonian associated to $\mathcal{A}_n$, showing in particular how the cubic algebra reduces to a quadratic one as a consequence of the realization. The construction can formally be extended to any value $n \geqslant 5$. This is however computationally cumbersome, as the linear dimension of the centralizer increases exponentially. So, for the values $n = 5,6,7$, the dimension is given respectively by (see equation (3.32))

$$\dim_L C_{S(\mathfrak{sl}(5))}(\mathfrak{h}) = 88, \quad \dim_L C_{S(\mathfrak{sl}(6))}(\mathfrak{h}) = 414, \quad \dim_L C_{S(\mathfrak{sl}(7))}(\mathfrak{h}) = 2371, \tag{5.1}$$

making an explicit presentation of the Poisson brackets quite complicated. In any case, the construction is feasible for any $n \geqslant 5$, with the centralizer $C_{S(\mathfrak{sl}(n))}(\mathfrak{h})$ determining a polynomial algebra. The approach by commutants in higher-rank Lie algebras provides an alternative description of these superintegrable models, eventually leading to new integrable models for other appropriate realizations of the Lie algebra.

In this context, it is natural to ask whether for the remaining classical algebras (and their real forms) the commutant of the Cartan subalgebra leads to polynomial algebras that are naturally associated to superintegrable hierarchies that have been studied by other methods. As an extension of this approach, it is also conceivable to combine the approach presented here with the so-called missing label problem [35, 36], where the subalgebras used have a definite physical meaning as internal symmetry algebras. Although it is expected that the computational obstructions are considerable, due to the generally complicated structure of subgroup scalars, it is not excluded that new systems with interesting properties can emerge from this ansatz. Work in this direction is currently in progress.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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