Pullback attractors and extremal complete trajectories for non-autonomous reaction-diffusion problems.

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Abstract

We analyse the dynamics of the non-autonomous nonlinear reaction-diffusion equation $u_t - \Delta u = f(t,x,u)$, subject to appropriate boundary conditions, proving the existence of two bounding complete trajectories, one maximal and one minimal. Our main assumption is that the nonlinear term satisfies a bound of the form $f(t,x,u)u \le C(t,x)|u|^2 + D(t,x)|u|$, where the linear evolution operator associated with $\Delta + C(t,x)$ is exponentially stable. As an important step in our argument we give a detailed analysis of the exponential stability properties of the evolution operator for the non-autonomous linear problem $u_t - \Delta u = C(t,x)u$ between different L^p spaces.

1 Introduction

In this paper we analyse the dynamics of the following non–autonomous nonlinear parabolic model problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in} \quad \Omega \quad t > s \\ u(s) &=& u_s & \\ u &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1.1}$$

where Ω is a bounded domain in $I\!R^N$ and $f(t,x,u) : I\!R \times \Omega \times I\!R \to I\!R$ is a suitable smooth function. We denote the solution of this equation by $u(t,s;u_s)$.

Our goal is to prove that under suitable conditions there exist two *extremal* complete trajectories (defined for all $t \in I\!R$), one maximal and one minimal, which give bounds for the asymptotic behaviour of solutions in an appropriate sense. We will assume that the nonlinear term satisfies

$$f(t,x,u)u \leq C(t,x)|u|^2 + D(t,x)|u| \quad \text{for all } u \in I\!R \tag{1.2}$$

for some $C \in C^{\alpha}(I\!R, L^p(\Omega))$ with $0 < \alpha \leq 1$ and $p > N/2$, and some function D with values in $L^r(\Omega)$, $1 \leq r \leq \infty$. The assumption crucial to our analysis is that the evolution operator associated with $\Delta + C(t,x)$ is exponentially stable.

This paper makes two essentially independent contributions, which are combined in our treatment of the reaction-diffusion equation (1.1).

The first is a well-developed general theory concerning the properties (in particular the exponential stability) of the evolution operators associated with non-autonomous *linear* problems of the form

$$u_t - \Delta u = C(t,x)u \tag{1.3}$$

posed in $L^q(\Omega)$ $(1 < q < \infty)$ or $C(\overline{\Omega})$, with $C \in C^{\alpha}(I\!R, L^p(\Omega))$, where $0 < \alpha \leq 1$ and $p > N/2$. We present a complete study of the norms of the solution operator $U_C(t,s)$ between different L^p spaces.

The second ingredient is a dynamical argument that, under certain natural conditions, guarantees the existence of extremal complete trajectories of a nonlinear problem. The argument makes key use of the order-preserving property of our equations, thereby deducing the existence of extremal trajectories for the nonlinear equation (1.1) from their existence for the associated linear problem (1.3). Although for the sake of simplicity we consider this problem in the phase space $C(\overline{\Omega})$, under suitable growth conditions we could also consider Sobolev spaces of initial data in $L^q(\Omega)$: the dynamical arguments would remain almost identical, but the analysis would become much more technically involved.

Also, note that in our analysis no prescribed time dependence is assumed (e.g. periodic, quasiperiodic or almost periodic).

Since our fundamental tools are comparison techniques, the results are valid for more general operators than the Laplacian and other boundary conditions provided that the problem admits a comparison principle. We also use the smoothing effect of the equations in an essential way.

The dynamical results here are the non-autonomous counterpart of those for autonomous parabolic problems established in Rodríguez-Bernal and Vidal-López [19] and Vidal-López [23].

1.1 Summary of results for the nonlinear equation

Since the initial time plays a central role in non-autonomous problems, in order to analyse the behaviour of solutions of (1.1) it is natural to make use of the notions of pullback attraction and pullback attractors. The basic idea behind these is that the relevant dynamics at the current time t are those that have arisen from initial conditions long ago, i.e. we take $s \to -\infty$ in order to discount the transient behaviour (rather than taking $t \to \infty$ which is more natural in the autonomous case). The *pullback attractor* is then the set of possible current states, $\{\mathcal{A}(t)\}_{t\in I\!R}$, for solutions that started arbitrarily far in the past. Of course, the more familiar concept of forwards attraction (as $t \to +\infty$) is still relevant, although some care is needed with the definition of a 'forwards attractor' in non-autonomous systems.

In this paper we are going to show (under suitable conditions on f, C, and D as described above) that there exist two extremal complete trajectories for (1.1), $\varphi_M(t,x)$ and $\varphi_m(t,x)$, that are maximal and minimal respectively, in the sense that any other complete trajectory $\psi(t,x)$ satisfies

$$\varphi_m(t,x) \le \psi(t,x) \le \varphi_M(t,x) \qquad \text{for all} \qquad t \in I\!R.$$

We also prove that if $f(t,x,u)$ is T–periodic in time then so are φ_m and φ_M.

A relatively simple argument shows that the 'order interval' $[\varphi_m(t), \varphi_M(t)]$, consisting of all functions lying between φ_m and φ_M, is positively invariant, i.e. for any $t > s$

$$\varphi_m(s,x) \le u_s(x) \le \varphi_M(s,x) \quad \Rightarrow \quad \varphi_m(t,x) \le u(t,s;u_s)(x) \le \varphi_M(t,x),$$

and also attracts the dynamics of the system uniformly in the pullback sense, i.e. for every $t \in I\!R$ we have

$$\varphi_m(t,x) \le \liminf_{s\to-\infty} u(t,s,x;u_s) \le \limsup_{s\to-\infty} u(t,s,x;u_s) \le \varphi_M(t,x) \tag{1.4}$$

uniformly in $x \in \overline{\Omega}$ for all $\{u_s\}$ in a bounded set of initial data B.

Moreover, $\varphi_M(t)$ is globally asymptotically stable from above in the pullback sense, i.e. for all $v \in C_b(I\!R, X)$ such that $v \ge \varphi_M$ we have

$$\lim_{s\to-\infty} u(t,s;v_s)(x) = \varphi_M(t,x)$$

uniformly in $x \in \Omega$. In a similar sense, $\varphi_m(t)$ is globally asymptotically stable from below.

As a consequence, there exists a pullback attractor for (7.1), denoted by $\mathcal{A} = \{\mathcal{A}(t)\}_t$, and

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)] \quad \text{for all } t \in I\!R.$$

The two extremal trajectories lie in the pullback attractor: $\varphi_m(t), \varphi_M(t) \in \mathcal{A}(t)$ for all $t \in I\!R$.

A full and exact statement of these results is given in Theorem 7.1.

Observe that it is possible (and it is indeed the case in certain problems) that the extremal solutions are not uniformly bounded for all t. While in such a case the pullback attractor can still exist, there can be no bounded forwards attractor.

1.2 Outline of the paper

In Section 2 we recall some definitions from the theory of attractors and order-preserving dynamical systems. In particular we introduce the notion of pullback attraction in a formal way.

In Section 3 we first sketch the dynamical arguments using the simple example of a scalar ordinary differential equation, and then give a more extended presentation in the context of the reaction-diffusion model, but with no attempt to treat the problem in full generality or to give all the details of the proofs. This avoids the technicalities from the theory of PDEs required to obtain the sharp results that follow later, but enables us to present the underlying ideas in what we hope is a relatively transparent way.

In Section 4 we analyse in detail the evolution operators associated with linear homogeneous non-autonomous parabolic equations. In particular we discuss questions related to regularisation and exponential stability in several function spaces. We give sufficient conditions for exponential stability, and prove its persistence under various classes of perturbation.

In Section 5 we study complete trajectories for inhomogeneous linear non-autonomous parabolic equations, giving suitable conditions for their existence and analysing their asymptotic behaviour both as $t \to +\infty$ and as $t \to -\infty$. Further to this in Section 6 we consider asymptotically autonomous and asymptotically periodic problems: we prove that in such cases the complete trajectories inherit the properties of the underlying equation (asymptotically autonomous/periodic).

In Section 7 we prove our main result concerning extremal complete trajectories and the pullback attractor for (1.1), as outlined above. In Section 8 we analyse the case in which the extremal trajectories are bounded forward in time and give a description of the asymptotic behaviour of (1.1) starting from the pullback attractor.

In Section 9 we show how the general results from previous sections can be applied to some logistic non-autonomous model problems. Finally, in Section 10 we extend the results to some non-autonomous parabolic equations with nonlinear non-autonomous boundary conditions.

2 Some useful concepts for non-autonomous equations.

Throughout the paper we will recast our equations as abstract families of (non-autonomous) evolution operators acting on an appropriate phase space.

Definition 2.1 *Given a metric space (X, d), we say that a family of mappings $\{U(t,s)\}_{t\geq s}$ is a* process *or a* family of evolution operators *if it satisfies*

1. $U(t,t) = I$ *for all* $t \in I\!R$,
2. $U(t,s)U(s,r)u = U(t,r)u$ *for all* $r \leq s \leq t$, $u \in X$, *and*
3. $u \mapsto U(t,r)u$ *is continuous in* X, $t > r$.

2.1 Different notions of attraction in non-autonomous problems

We begin with some useful definitions from the theory of attractors for non-autonomous systems which we will use throughout this paper (see for example Crauel, Debussche, & Flandoli [8], Kloeden & Schmalfuß [14], or Schmalfuß [21]).

We define formally the notions of attraction and absorbtion in both the 'pullback' and 'forwards' senses. In what follows we denote by $\mathcal{B}$ and $\mathcal{K}$ time-dependent families $\{B(s)\}_{s\in\mathbb{R}}$ and $\{K(s)\}_{s\in\mathbb{R}}$ of bounded sets. We begin with attraction.

Definition 2.2 *i) We say that* $\mathcal{K}$ attracts $\mathcal{B}$ in the pullback sense *if for each* $t_0 \in \mathbb{R}$

$$\lim_{s\to-\infty} \mathrm{dist}(U(t_0,s)B(s), K(t_0)) = 0.$$

ii) We say that $\mathcal{K}$ attracts $\mathcal{B}$ (forwards in time) *if for each* $s \in \mathbb{R}$

$$\lim_{t\to\infty} \mathrm{dist}(U(t,s)B(s), K(t)) = 0.$$

We say that $\mathcal{K}$ attracts bounded sets *(in whichever sense) if the above definitions hold for* $B(t) \equiv B$*, where* B *is a fixed bounded set.*

Stronger than this, but key to the existence results for pullback and forwards attractors, is the notion of an absorbing set.

Definition 2.3 *i) A bounded set* K absorbs $\mathcal{B}$ in the pullback sense at time t_0 *if there exists* $T = T(t_0, \mathcal{B}) \le t_0$ *such that*

$$U(t_0,s)B(s) \subset K \quad \textit{for all } s \le T \le t_0;$$

ii) a bounded set $K \subset X$ absorbs $\mathcal{B}$ forwards in time *if for each* $s \in \mathbb{R}$ *there exists* $T = T(s,\mathcal{B}) \ge s$ *such that*

$$U(t,s)B(s) \subset K \quad \textit{for all } t \ge T.$$

A time-dependent set $\mathcal{K}$ is invariant if it preserved under the action of $U(t,s)$:

Definition 2.4 *We say that* $\mathcal{K}$ *is* forwards invariant *(with respect to* U*) if*

$$U(t,s)K(s) \subseteq K(t) \quad \textit{for all } t \ge s,$$

and that $\mathcal{K}$ *is* invariant *(with respect to* U*) if*

$$U(t,s)K(s) = K(t) \quad \textit{for all } t \ge s.$$

In the following we will fix some nonempty class $\mathcal{D}$ of families of bounded sets of X, $\{B(s)\}_{s\in\mathbb{R}}$, as *the basin of attraction.* See Schmalfuß [21] for details of some of the properties required for such a class (a "universe"), but we remark here that in particular the classes that we will consider will include all time-independent bounded sets, i.e. families where $B(t) = B$ for all $t \in \mathbb{R}$ where $B \subset X$ is bounded.

As a general notation used below, if an element in $\mathcal{D}$ is of the form $\{v(s)\}_s$ with $v(s)$ being a single element in X then we denote it by v_s.

We are now in a position to define the pullback attractor.

Definition 2.5 *We say that a family of compact sets $\mathcal{A} = \{\mathcal{A}(t)\}_t$ in X is the* pullback attractor *(for U) with respect to $\mathcal{D}$ if it is invariant with respect to U, pullback attracts all $\mathcal{B} \in \mathcal{D}$, and is minimal in the sense that if $\{K(t)\}_{t\in\mathbb{R}}$ is another pullback attracting family of closed sets then $\mathcal{A}(t) \subseteq K(t)$ for all $t \in \mathbb{R}$.*

To treat the asymptotic behaviour of solutions forwards in time we define the notion of a forwards attractor.

Definition 2.6 *We say that a compact set $\mathcal{F}$ is the* forwards attractor *for U if $\mathcal{F}$ is the minimal compact set such that for any $s \in \mathbb{R}$ and any bounded set $B \subset X$,*

$$\lim_{t\to\infty} \operatorname{dist}(U(t,s)B, \mathcal{F}) = 0.$$

Note that the notion of a pullback attractor, as introduced above, is relative to some chosen basin of attraction $\mathcal{D}$. On the other hand the domain of attraction for the forwards attractor is restricted, as is customary, to the class of time-independent bounded sets.

The next result reproduces the standard conditions guaranteeing the existence of a pullback attractor (see Crauel et al. [8], Langa and Suárez [15], Schmalfuß [20]).

Theorem 2.7 *If there exists a time-dependent compact set that is pullback absorbing for all $\mathcal{B} \in \mathcal{D}$ then there exists a pullback attractor with respect to $\mathcal{D}$.*

For a somewhat similar result for the case of the forwards attractor, see Section 8.

2.2 Order-preserving & exponentially stable evolution operators

One of the main tools we use in our analysis of (1.1) is the monotonicity of solutions, in various senses. To formalise these, suppose that we have an order structure on the phase space X, which we will denote by $\leq$. We will use evolution operators that preserve the order in the following sense:

Definition 2.8 *We say that an evolution operator is* order-preserving *if there exists an order relation in X ($\leq$) such that*

$$u_0 \leq v_0 \;\Rightarrow\; U(t,s)u_0 \leq U(t,s)v_0 \quad \text{for all } t \geq s$$

while both solutions exist.

Definition 2.9 *Given $u \leq v$, the* order interval *defined by u and v is*

$$[u,v] = \{w \in X : \; u \leq w \leq v\}.$$

The next definition gives us the non-autonomous analogues of the concepts of an equilibrium point and of sub- and super- solutions from the theory of autonomous problems (see Amann [1], Arnold and Chueshov [4]). In particular the notion of a complete trajectory is central to all that follows.

Definition 2.10 *We say that a continuous map $v : \mathbb{R} \to X$ is a* complete trajectory *for U if for all $t \geq s$*

$$U(t,s)v(s) = v(t).$$

We say that v is a super-trajectory *for U if for all $t \geq s$*

$$U(t,s)v(s) \leq v(t),$$

and that v is a sub-trajectory *for U if for all $t \geq s$*

$$U(t,s)v(s) \geq v(t).$$

Finally, we give define a concept that will be crucial in the rest of this work, namely, an exponentially stable evolution operator.

Definition 2.11 *If X is a Banach space and $U(t,s) \in \mathcal{L}(X)$, we say that the evolution operator $U(t,s)$ is* exponentially stable *if for some $\beta > 0$ and $M > 0$*

$$\|U(t,s)\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{-\beta(t-s)} \quad \textit{for all } t > s.$$

2.3 Existence, uniqueness, and comparison results for our parabolic problem

We now recall some existence and uniqueness results for the nonlinear parabolic problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in} \quad \Omega, \quad t > s \\ u &=& 0 & \text{on} \quad \partial\Omega \quad t > s \\ u(s) &=& u_0, & \end{array} \right. \tag{2.1}$$

posed in $X = C(\overline{\Omega})$. The following theorem gives the existence of a local solution for (2.1) (e.g. see Amann [2], Danners and Koch–Medina [9], Henry [12] or Lunardi [16] or Mora [18]).

Theorem 2.12 *Suppose that $f(t,x,u)$ is a continuous function, locally Hölder in t and locally Lipschitz in u. Then for every $u_0 \in X = C(\overline{\Omega})$ there exists a unique local solution $u(t,s;u_0)$ of (2.1) given by the variation of constants formula*

$$u(t,s;u_0) = \mathrm{e}^{\Delta(t-s)}u_0 + \int_s^t \mathrm{e}^{\Delta(t-\tau)} f(\tau, u(\tau,s;u_0))\,\mathrm{d}\tau.$$

Moreover, if $u_0 \in C_0(\overline{\Omega})$, the class of continuous functions vanishing at $\partial\Omega$, then u is continuous at $t = s$.

If the solutions of (2.1) are globally defined then $U(t,s)u_0 = u(t,s;u_0)$ defines an evolution operator in $X = C(\overline{\Omega})$ as in Definition 2.1. Due to the smoothing effect of (2.1) we know that for any $t > s$, $U(t,s)$ is a continuous and bounded map from $C(\overline{\Omega})$ to $C^1_0(\overline{\Omega})$ (the class of C^1 functions vanishing in $\partial\Omega$).

If we consider the problem posed in $L^q(\Omega)$ with $1 < q < \infty$ then the smoothing property of the evolution operator guarantees that all the solutions enter $C(\overline{\Omega})$ immediately (for $t > 0$), and so it is sufficient to study the problem in the phase space $C(\overline{\Omega})$. However, notice that we need to impose some growth restrictions on f to ensure the existence of a solution of problem (7.1).

Notice that if $u_0 \in C_0(\overline{\Omega})$ then the mild solution defined in Theorem 2.12 is continuous at $t = s$ and then Theorem 2.12 is obtained directly from the references above.

However, if we deal with initial data $u_0 \in C(\overline{\Omega})$ or even $u_0 \in L^\infty(\Omega)$ then the mild solution defined above is not continuous at $t = s$ and this makes the proof of Theorem 2.12 more subtle. To prove the result in such case, problem (2.1) is considered in a Lebesgue space $L^q(\Omega)$ with $1 < q < \infty$. Then, given any initial data $u_0 \in L^\infty(\Omega) \subset L^q(\Omega)$ and truncating the nonlinear term we obtain the existence and uniqueness of a solution in $L^q(\Omega)$ starting from u_0 for a certain time interval. Notice that by truncating the nonlinearity we avoid the need to impose growth restrictions on f. Now, by the smoothing property of the evolution operator in $L^q(\Omega)$, the solution belongs to $C_0(\overline{\Omega})$ for all $t > s$ while it is defined. Thus, we have proved the existence of a unique solution starting from $u_0 \in L^\infty(\Omega)$.

One of the main tools we will use is the following consequence of the maximum principle that we will refer to as the *comparison principle* (see e.g. Appendix A in Arrieta et al. [5]). In all that follows we use $f \leq g$ to denote the standard ordering, i.e. $f(x) \leq g(x)$ for almost every $x \in \Omega$.

Theorem 2.13 *Let $f, g : I\!\!R \times \Omega \times I\!\!R \to I\!\!R$ be continuous functions, locally Hölder in t and locally Lipschitz in u. Suppose that for all $t \in I\!\!R$, $x \in \Omega$ and $u \in I\!\!R$ we have*

$$f(t, x, u) \leq g(t, x, u).$$

Then, if $u_0 \leq v_0$ are two ordered initial conditions in $X = C(\overline{\Omega})$, we have

$$u_f(t, s, x; u_0) \leq u_g(t, s, x; v_0)$$

while both solutions exist, where we have denoted by $u_f(t, s, x; u_0)$ the solution at time t of problem (2.1) with initial data u_0 at time s and nonlinear term f.

3 A short review of the dynamical plot.

The statements and proofs in Sections 5 and 6 contain dynamical arguments that could be hidden on a first reading because they make use of technical arguments from the theory of PDEs (in particular various smoothing properties between different spaces) in order to treat the problem in wide generality.

Our aim in this section is to give a more abstract, but thereby we hope more transparent, outline of the key arguments without these technical distractions. We begin by showcasing the argument in the context of a scalar ODE, and then present an abstract version of the PDE results.

3.1 A simplified version of the dynamical argument for scalar ODEs

Since our argument makes fundamental use of the order-preserving properties of equation (1.1), we will use a scalar ODE (which is also order-preserving) as a model problem, and prove the existence of maximal and minimal complete trajectories (in a certain class) for

$$\dot{x} = f(t,x) \qquad x(s) = x_s \in I\!R, \tag{3.1}$$

where we assume that f is sufficiently smooth to guarantee the existence of unique solutions for any x_s and all $t \geq s$. It will be convenient to write the solution of this equation at time t in terms of a solution operator $U(t,s)$, i.e. $x(t,s;x_0) = U(t,s)x_0$. Because we have a scalar ODE solutions maintain their initial ordering, i.e.

$$x_0 \leq y_0 \qquad \Rightarrow \qquad U(t,s)x_0 \leq U(t,s)y_0 \quad \text{for all} \quad t \geq s.$$

Under the assumption that f is bounded above by a linear function of x,

$$xf(t,x) \leq -C(t)x^2 + D(t)|x|, \tag{3.2}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|x| \leq -C(t)|x| + D(t),$$

and so we can expect to obtain some results on the behaviour of the nonlinear equation by understanding that of the inhomogeneous linear equation

$$\dot{y} = -C(t)y + D(t). \tag{3.3}$$

Indeed, we have the 'comparison principle'

$$|U(t,s)x_0| \leq y(t,s;|x_0|). \tag{3.4}$$

In order to control the behaviour of (3.3) we assume that the solution operator $\Phi(t,s)$ of the corresponding homogeneous problem $\dot{z} = -C(t)z$ is exponentially stable, i.e. for some $M, \beta > 0$,

$$0 \leq \Phi(t,s) \leq M\mathrm{e}^{-\beta(t-s)} \qquad \text{for all} \qquad t \geq s.$$

It is then easy to show that if $D \in \mathcal{D}_\beta$, where

$$\mathcal{D}_\beta = \{x(\cdot): \text{ for some } \gamma < \beta, \quad \mathrm{e}^{\gamma t}|x(t)| \to 0 \quad \text{as} \quad t \to -\infty\},$$

then

$$\phi(t) = \int_{-\infty}^{t} \Phi(t,s)D(s)\,\mathrm{d}s$$

is the unique complete trajectory of (3.3) in $\mathcal{D}_\beta$, and that it attracts all $y \in \mathcal{D}_\beta$ 'in the pullback sense', i.e.

$$\lim_{s\to-\infty} y(t,s;y(s)) = \phi(t) \qquad \text{for all} \qquad y \in \mathcal{D}_\beta. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\limsup_{s\to-\infty} |U(t,s)x_0| \le \phi(t), \tag{3.6}$$

and that $\phi(t)$ is a super-trajectory of (3.1), i.e. that

$$U(t,s)\phi(s) \le y(t,s;\phi(s)) = \phi(t) \qquad \text{for all} \qquad t \ge s.$$

We now consider a candidate for the maximal complete trajectory of (3.1), namely

$$\varphi_M(t) = \lim_{s\to-\infty} U(t,s)\phi(s). \tag{3.7}$$

Since $\phi(\cdot)$ is a super-trajectory, for each fixed t we have

$$U(t,s)\phi(s) = U(t,s+\epsilon)U(s+\epsilon,s)\phi(s) \le U(t,s+\epsilon)\phi(s+\epsilon),$$

and so $U(t,s)\phi(s)$ is non-increasing as $s \to -\infty$. Since we also know from (3.6) that $\liminf_{s\to-\infty} U(t,s)\phi(s) \ge -\phi(t)$, we know that the limit in (3.7) exists pointwise. That $\varphi_M(t)$ is a (complete) trajectory of (3.1) follows from the continuity of $U(t,s)$,

$$\begin{aligned} U(t,s)\varphi_M(s) &= U(t,s)\lim_{r\to-\infty} U(s,r)\phi(r) = \lim_{r\to-\infty} U(t,s)U(s,r)\phi(r) \\ &= \lim_{r\to-\infty} U(t,r)\phi(r) = \varphi_M(t), \end{aligned}$$

which also shows that the limit in (3.7) is in fact a continuous function of t.

To prove that φ_M is the *maximal* complete trajectory, suppose that $x(t)$ is another complete trajectory contained in $\mathcal{D}_\beta$. Then

$$U(r,s)x(s) \le y(r,s;|x(s)|),$$

and acting with $U(t,r)$ on both sides we have

$$x(t) \le U(t,r)y(r,s;|x(s)|).$$

Since this holds for all s we use the fact that $|x(\cdot)| \in \mathcal{D}_\beta$ to take limits as $s \to -\infty$ to obtain

$$x(t) \le U(t,r)\limsup_{s\to-\infty} y(t,s;|x(s)|) \le U(t,r)\phi(r);$$

since this holds for all r we now take the limit as $t \to -\infty$ to obtain $x(t) \le \varphi_M(t)$ as required.

Of course, in our more detailed analysis we also consider other variations and refinements of this argument, but it is the use of the comparison principle and the 'pullback' idea that is at its heart.

3.2 Review of the arguments for parabolic PDEs

Here we give a short review of the dynamical arguments in the context of a PDE problem, and point the reader towards the complete proofs in the following sections. The arguments are given in more detail than those for the ODE in the previous section.

The type of linear equations that we will consider in Sections 5 and 6 can be written in an abstract form as

$$\begin{cases} v_t + A(t)v &= f(t), \quad t > s \\ v(s) &= v_s \end{cases}$$

in a certain Banach space X, with $A(t)$ a time–dependent operator in X with a time-independent domain and $f : I\!R \to X$.

We begin by studying the homogeneous case, i.e., $f \equiv 0$,

$$\begin{cases} w_t + A(t)w &= 0, \quad t > s \\ w(s) &= w_0. \end{cases} \tag{3.8}$$

We assume that $A(t)$ defines a linear evolution operator $U(t,s)$ and then (3.8) has a unique solution $w(t,s;w_0) = U(t,s)w_0$.

For $\beta > 0$ we define $\mathcal{D}_\beta = \mathcal{D}_\beta(I\!R, X)$, the "basin of attraction", consisting of families of bounded sets that grow slower than $\mathrm{e}^{-\beta t}$ as $t \to -\infty$, i.e. families of bounded sets of the form $\{B(t)\}_t$ such that for some $\gamma < \beta$ we have

$$\mathrm{e}^{\gamma t}\|B(t)\|_X \to 0 \qquad \text{as } t \to -\infty,$$

where

$$\|B\|_X := \sup_{b \in B} \|b\|_X.$$

In a slight abuse of notation we will also say of a function $f : I\!R \to X$ that '$f \in \mathcal{D}_\beta$' if $\{f(s)\}_{s \in I\!R} \in \mathcal{D}_\beta$.

In this case Theorem 5.1 can be stated as:

Theorem 3.1 *Suppose that the evolution operator $U(t,s)$ for (5.1) in X is exponentially stable, i.e, for some $\beta > 0$*

$$\|U(t,s)\|_{\mathcal{L}(X)} \le M\mathrm{e}^{-\beta(t-s)} \quad \textit{for all } t > s. \tag{3.9}$$

Then the unique complete trajectory for (3.8) in $\mathcal{D}_\beta$ is the trivial solution. Indeed, $\mathcal{A} = \{\mathcal{A}(t)\}_t$ such that $\mathcal{A}(t) = \{0\}$ is the pullback attractor with respect to $\mathcal{D}_\beta$.

Moreover, the trivial solution also attracts bounded sets in X forwards in time.

We then consider the following linear inhomogeneous problem

$$\begin{cases} v_t + A(t)v &= f(t), \quad t > s, \\ v(s) &= v_s \end{cases} \tag{3.10}$$

which, under suitable assumptions on f, has a unique solution given by the variation of constants formula, i.e.

$$v(t,s;v_s) = U(t,s)v_s + \int_s^t U(t,\tau)f(\tau)\,\mathrm{d}\tau. \tag{3.11}$$

Assuming the exponential stability of the evolution operator U associated with $A(t)$ as above (see (3.9)) we can obtain following result (see Theorem 5.3):

Theorem 3.2 *Suppose that the evolution operator $U(t,s)$ is exponentially stable in X, i.e.,*

$$\|U(t,s)\|_{\mathcal{L}(X)} \le M\mathrm{e}^{-\beta(t-s)} \quad \textit{with } \beta > 0 \textit{ and } M \ge 1.$$

i) If $f \in \mathcal{D}_\beta(I\!R, X)$ then (3.10) has a unique complete trajectory $\phi \in \mathcal{D}_\beta$.
ii) If $f \in L^\sigma(I\!R, X)$ then (3.10) has a unique complete trajectory $\phi \in C_b(I\!R, X) \cap L^\sigma(I\!R, X) \subset \mathcal{D}_\beta$, and ϕ is the unique complete trajectory within $\mathcal{D}_\beta$.

Furthermore, $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ is the pullback attractor with respect to $\mathcal{D}_\beta$ for (3.10).

Also, $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ attracts bounded sets of X forwards in time. More precisely, for every bounded set $B \subset X$, we have

$$\|v(t,s;v_0) - \phi(t)\|_X \le K\mathrm{e}^{-\beta(t-s)}, \qquad t > s, \tag{3.12}$$

for all $v_0 \in B$, where $K = K(B)$.

Proof. It can be shown that

$$\phi(t) = \int_{-\infty}^t U(t,\tau)f(\tau)\mathrm{d}\tau \tag{3.13}$$

is a complete trajectory for (3.10).

Let $\mathcal{B} = \{B(s)\}_s \in \mathcal{D}_\beta$ and fix $\{v_s\}_s \in \mathcal{B}$. Let $w(t,s;v_s) = v(t,s;v_s) - \phi(t)$. Then w solves the homogeneous problem

$$\left\{ \begin{array}{rcll} w_t + A(t)w &=& 0, & t > s \\ w(s) &=& v_s - \phi(s). & \end{array} \right. \tag{3.14}$$

So, since $\{B(s) - \phi(s)\}_s \in \mathcal{D}_\beta$, from Theorem 3.1, we have

$$w(t,s;v_s) \to 0 \quad \text{as} \quad s \to -\infty$$

uniformly for $v_s \in B(s)$, where $\{B(s)\}_s \in \mathcal{D}_\beta$.

Therefore, the complete trajectory ϕ is the pullback attractor for (3.10).

Notice that if we fix $s \in I\!R$ and a bounded set $B \subset X$, we have

$$\|v(t,s;v_0) - \phi(t)\|_X \le K\mathrm{e}^{-\beta(t-s)}\|v_0 - \phi(s)\|_X \le K_1\mathrm{e}^{-\beta(t-s)} \to 0 \tag{3.15}$$

as $t \to +\infty$, for all $v_0 \in B$, where K_1 depends on the bounded set B. Hence $\phi(t)$ also attracts bounded sets of X forwards in time.

We now give a brief outline of the proof that $\phi(t)$ is well-defined and belongs to the right space. For further details see the proof of Theorem 5.3. In the first case, when $f \in \mathcal{D}_\beta$ we have $\mathrm{e}^{\gamma t}\|f(t)\|_X \to 0$ as $t \to -\infty$ for some $\gamma < \beta$: with this choice of γ we can write

$$\mathrm{e}^{\gamma t}\phi(t) = \int_{-\infty}^{t} \mathrm{e}^{\gamma(t-\tau)}U(t,\tau)\mathrm{e}^{\gamma\tau}f(\tau)\mathrm{d}\tau,$$

and taking norms yields

$$\begin{aligned}
\mathrm{e}^{\gamma t}\|\phi(t)\|_X &\leq M\left(\int_{-\infty}^{t} \mathrm{e}^{-(\beta-\gamma)(t-\tau)}\,\mathrm{d}\tau\right)\sup_{\tau\leq t}\left(\mathrm{e}^{\gamma\tau}\|f(\tau)\|_X\right) \\
&= \frac{M}{\beta-\gamma}\sup_{\tau\leq t}\left(\mathrm{e}^{\gamma\tau}\|f(\tau)\|_X\right).
\end{aligned}$$

Since $\mathrm{e}^{\gamma t}\|f(t)\|_X \to 0$ as $t \to -\infty$ this implies that $\phi \in \mathcal{D}_\beta$ and, by Theorem 3.1, ϕ is the unique complete trajectory in $\mathcal{D}_\beta$.

If $f \in L^\infty(\mathbb{R}, X)$ then a simple computation gives

$$\|\phi(t)\|_X \leq \limsup_{s\to-\infty}\frac{M}{\beta}\left(1-\mathrm{e}^{-\beta(t-s)}\right)\|f\|_{L^\infty(\mathbb{R},X)} \leq \frac{M}{\beta}\|f\|_{L^\infty(\mathbb{R},X)}.$$

The proof of the fact that $\phi \in L^1(\mathbb{R}, X)$ when $f \in L^1(\mathbb{R}, X)$ follows using Fubini's Theorem (for further details see Theorem 5.3). The result in the case $f \in L^r(\mathbb{R}, X)$ follows from the interpolation theorem for $L^r(\mathbb{R}, X)$ spaces. Finally, using the Hölder inequality one can prove that in any case $\phi \in L^\infty(\mathbb{R}, X)$. ■

When f is integrable or, more generally, when f is small at $\pm\infty$, we then show that ϕ vanishes at $t = \pm\infty$ (see part (iii) of Theorem 5.3). More precisely (see Corollary 5.6):

Corollary 3.3 *Assume that either*

$$f \in L^\sigma(\mathbb{R}, X) \quad \text{with} \quad 1 \leq \sigma < \infty,$$

or that

$$f \in L^\infty(\mathbb{R}, X) \quad \text{and} \quad \|f(t)\|_X \to 0 \quad \text{as} \quad t \to \pm\infty.$$

Then ϕ, the complete trajectory given in the theorem above, also satisfies $\phi \in C_0(\mathbb{R}, X)$.

We next consider the T-periodic problem associated with (3.10), i.e., we suppose that A and f are T-periodic. The following result states that, in the T-periodic case, the unique complete trajectory given by Theorem 3.2 is T-periodic (see Corollary 5.8).

Corollary 3.4 *Assume that the evolution operator associated with $A(t)$ is exponentially stable (see (3.9)) and $f \in L^\infty(\mathbb{R}, X)$. If both A and f are T-periodic functions then the unique complete trajectory $\phi \in \mathcal{D}_\beta$ for (3.10) is T-periodic.*

Proof. Note that the hypotheses of Theorem 3.2 ii) with $\sigma = \infty$ hold. Thus, let $\phi \in \mathcal{D}_\beta$ be the unique complete trajectory given by Theorem 3.2. Then,

$$\phi_t(t) + A(t)\phi(t) = f(t)$$

and, by the periodicity of A and f we have

$$\phi_t(t) + A(t+T)\phi(t) = f(t+T)$$

which, after a change of variables, gives

$$\phi_t(t-T) + A(t)\phi(t-T) = f(t).$$

So, $w(t) = \phi(t-T)$ is a complete trajectory of the problem (3.10). But Theorem 3.2 guarantees the uniqueness of such a complete trajectory, and so we must have $\phi(t) = w(t)$ for all $t \in \mathbb{R}$, that is, $\phi(t) = \phi(t-T)$ for all $t \in \mathbb{R}$. In other words, ϕ is T-periodic. ■

We then consider asymptotically autonomous linear problems. That is, we suppose that in problem (3.10), $A(t) \to A^\pm$ and $f(t) \to f^\pm$ as $t \to \pm\infty$ (in an appropriate sense). As we said before it is not our purpose here to give detailed proofs of the results but only to give an idea of how to proceed, so we delay a formal statement to Theorem 6.1 and content ourselves for now with an outline of the argument.

Assume that the semigroup generated by A^+ in X has exponential decay. Then there exists a unique solution ϕ^+ of the equation

$$A^+\phi^+ = f^+. \tag{3.16}$$

Setting $w(t) = v(t) - \phi^+$ we have,

$$\left\{ \begin{array}{rcl} w_t + A(t)w &=& f(t) - A(t)\phi^+ \\ w(s) &=& v_0 - \phi^+. \end{array} \right.$$

Now,

$$f(t) - A(t)\phi^+ = (f(t) - f^+) + (f^+ - A(t)\phi^+)$$

and, from the definition of ϕ^+

$$f^+ - A(t)\phi^+ = (A(t) - A^+)\phi^+.$$

Thus,

$$f(t) - A(t)\phi^+ = (f(t) - f^+) + (A(t) - A^+)\phi^+ \to 0$$

as $t \to \infty$.

Then, a similar argument to that in the proof of Corollary 3.3 and part iii) of Theorem 5.3 guarantees that $w(t,s;v_0 - \phi^+) \to 0$. Therefore

$$v(t,s;u_0) \to \phi^+.$$

In particular, $\phi(t) \to \phi^+$ as t tends to infinity.

The study of the behaviour as t tends to $-\infty$ follows similar arguments.

The case in which $A(t)$ and $f(t)$ converge (in the appropriate senses) to a T-periodic operator or function, respectively, is obtained using an analogous argument (see Theorem 6.2).

In the following sections we will use the arguments above on the PDE model problem

$$\left\{\begin{array}{rcll} v_t - \Delta v &=& C(t,x)v + D(t,x), & \text{in} \quad \Omega, \quad t > s, \\ v &=& 0, & \text{on} \quad \partial\Omega, \quad t > s \\ v(s) &=& v_s & \end{array}\right. \tag{3.17}$$

For this, we will take $X = L^q(\Omega)$, $1 < q < \infty$ or $X = C(\overline{\Omega})$. Then, we will obtain sharp conditions on the time-dependent potential $C(t,x)$ such that the corresponding evolution operator $U = U_C(t,s)$ is well-defined and exponentially stable, see Section 4.

Afterwards, using the smoothing effect of $U_C(t,s)$ we will give sharp conditions on $D(t,x)$ for the existence of the complete trajectory ϕ for the linear problem. Note that the smoothing effect allows us to reproduce the arguments above without assuming that for all $t \in I\!R$, $f(t) \in X$. Instead, we need only assume that $f(t)$ takes values in a weaker Lebesgue space $L^r(\Omega)$ for some suitable $r < q$. For the case of asymptotically autonomous or periodic problems, we give adequate conditions on the convergence of $C(t,x)$ and $D(t,x)$ as $t \to \pm\infty$ to obtain the results above in a rigorous way. All these arguments require the use of technical results from the theory of PDEs.

4 Evolution operators for linear non-autonomous problems.

We now apply the abstract results about linear evolution operators found in Amann [3] to the following problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& C(t,x)u, & \text{in} \quad \Omega, \quad t > s \\ u &=& 0, & \text{on} \quad \partial\Omega, \quad t > s \\ u(s) &=& u_0 & \end{array}\right. \tag{4.1}$$

posed in $X = L^q(\Omega)$ with $1 < q < \infty$ or in $X = C(\overline{\Omega})$.

If $C \in C^\alpha(I\!R, L^p(\Omega))$, with $0 < \alpha \leq 1$ and some $p > N/2$, then the time-dependent operator $A(t) = \Delta + C(t,x)$ satisfies (4.2.1), p. 55, in Amann [3]. Therefore, by Theorem 4.4.1, p. 63, in Amann [3], $A(t)$ generates an evolution operator with constant domain $D(A(t)) = W^{2,q}_D(\Omega) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$, $1 < q < \infty$, for all $t \in I\!R$. We denote this evolution operator by $U_C(t,s)$, i.e. $u(t,s;u_0) = U_C(t,s)u_0$ is the solution of (4.1).

Using Theorem 4.4.1, p. 63, and Lemma 5.1.3, p. 69, in Amann [3] plus the sharp Sobolev embeddings of complex interpolation spaces (see Amann [2]), it can be shown that for each q and r with $1 \leq q \leq r \leq \infty$ the evolution operator $U_C(t,s)$ satisfies

$$\|U_C(t,s)u_0\|_{L^r(\Omega)} \leq M \frac{e^{\delta(t-s)}}{(t-s)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}} \|u_0\|_{L^q(\Omega)}, \qquad t > s \tag{4.2}$$

for some $M > 0$ and $\delta \in I\!R$ (potentially depending on r and q).

Finally, Theorem 6.4.2, p. 85, in Amann [3] guarantees that the evolution operator $U_C(t,s)$ is order-preserving.

4.1 Exponential stability in L^q.

The following results show that in fact the exponent δ in (4.2) is independent of q and r and is strongly related to the exponential growth of the evolution operator. In particular they show that the evolution operator is exponentially stable in $L^q(\Omega)$ iff it is so in $L^r(\Omega)$ for any $1 < r, q \leq \infty$. As exponential stability will be a crucial property that we will use repeatedly below, these results will be very useful in what follows.

Lemma 4.1 *Assume that $U = U_C$, as above, is an evolution operator in $L^q(\Omega)$, $1 < q \leq \infty$, such that there exist $M > 0$ and $\beta \in I\!R$ such that*

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \textit{for all} \quad t > s. \tag{4.3}$$

Then, as an operator in $L^r(\Omega)$, with $1 < r \leq \infty$, U satisfies

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq K e^{\beta(t-s)} \quad \textit{for all} \quad t - s > 1$$

for some $K \geq 1$. In particular, the exponential type of the evolution operator, that is, the best exponent β in (4.3), is independent of the $L^q(\Omega)$ space.

Proof. First, note that from (4.2) we have

$$\|U(t+1,t)\|_{\mathcal{L}(L^r(\Omega),L^q(\Omega))} \leq C \quad \text{for all} \quad t \in I\!R, \quad q \geq r \tag{4.4}$$

$$\|U(t+1,t)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \leq C \quad \text{for all} \quad t \in I\!R, \quad q \leq r \tag{4.5}$$

Now, suppose that $r \geq q$, so that $L^r(\Omega) \subseteq L^q(\Omega)$. Then, since $U(t+1,s) = U(t+1,t)U(t,s)$,

$$\|U(t+1,s)u_0\|_{L^r(\Omega)} \leq \|U(t+1,t)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \|U(t,s)u_0\|_{L^q(\Omega)}.$$

Using now (4.3) and (4.5) we have

$$\|U(t+1,s)u_0\|_{L^r(\Omega)} \leq CMe^{-\beta}e^{\beta(t+1-s)}\|u_0\|_{L^q(\Omega)} \leq CMe^{-\beta}e^{\beta(t+1-s)}\|u_0\|_{L^r(\Omega)}.$$

Thus

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq K e^{\beta(t-s)}$$

for all $t - s > 1$.

Suppose now that $1 < r < q$, and therefore $L^q(\Omega) \subset L^r(\Omega)$. Now, we remark that $U(t+1,s) = U(t+1,s+1)U(s+1,s)$. So, using (4.3) and (4.4)

$$\begin{aligned}
\|U(t+1,s)u_0\|_{L^r(\Omega)} &\leq C\|U(t+1,s)u_0\|_{L^q(\Omega)} \\
&\leq C\|U(t+1,s+1)\|_{\mathcal{L}(L^q(\Omega))}\|U(s+1,s)u_0\|_{L^q(\Omega)} \\
&\leq CMe^{\beta(t-s)}\|U(s+1,s)\|_{\mathcal{L}(L^r,L^q(\Omega))}\|u_0\|_{L^r(\Omega)} \\
&\leq CMe^{-\beta}e^{\beta(t+1-s)}\|u_0\|_{L^r(\Omega)}.
\end{aligned}$$

Thus,

$$\|U(t,s)\|_{\mathcal{L}(L^r(\Omega))} \leq Ke^{\beta(t-s)}$$

for all $t - s > 1$. ■

We also have the following estimate between different Lebesgue spaces:

Lemma 4.2 *Suppose that $U(t,s)$ satisfies (4.3). Then, for $1 < q \leq r \leq \infty$*

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \leq \begin{cases} K(t-s)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} & \text{if } t-s \leq 2 \\ Ke^{\beta(t-s)} & \text{if } t-s > 2 \end{cases} \tag{4.6}$$

for some constant K.

Proof. From (4.2) for $t - s \leq 2$, there exists a constant K_1 such that

$$\|U(t,s)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} \leq K_1(t-s)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}$$

and, for $t - s > 2$, from (4.5), there exists a constant K_2 such that

$$\begin{aligned}
\|U(t,s)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))} &\leq \|U(t,t-1)\|_{\mathcal{L}(L^q(\Omega),L^r(\Omega))}\|U(t-1,s)\|_{\mathcal{L}(L^q(\Omega))} \\
&\leq K_2e^{\beta(t-s)}.
\end{aligned}$$

Thus, (4.6) holds for some $K \geq 1$. ■

4.2 Sufficient conditions for exponential stability

Now we give sufficient conditions for the exponential stability of an evolution operator $U(t,s) = U_C(t,s)$, for which we will make use of the Hilbert structure of the space $L^2(\Omega)$ and Lemma 4.1. We therefore consider

$$\begin{cases} u_t - \Delta u = C(t,x)u & \text{in } \Omega, \quad t > s \\ u = 0 & \text{on } \partial\Omega, \quad t > s \\ u(s) = u_0. \end{cases} \tag{4.7}$$

In the simplest case, when C does not depend on t, i.e. $C(t,x) = C(x)$ and the operator $\Delta + C(x)$ does not depend on time, we know that the semigroup associated with $\Delta + C(x)$ is exponentially stable if and only if the first eigenvalue of

$$\begin{cases} -(\Delta + C(x))u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is positive.

To treat the time-dependent case, we therefore take $X = L^2(\Omega)$ and for any fixed $t \in I\!R$, consider the first eigenvalue of

$$\left\{ \begin{array}{rcll} -\Delta u - C(t,x)u & = & \lambda(t)u & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right.$$

which satisfies

$$\int_\Omega \left(|\nabla\varphi|^2 - C(t,x)|\varphi|^2 \right) \, \mathrm{d}x \geq \lambda_1(t)\|\varphi\|^2, \tag{4.8}$$

for all smooth functions φ vanishing on $\partial\Omega$, where we have denoted by $\|\cdot\|$ the norm in $L^2(\Omega)$.

Multiplying the first equation in (4.7) by $u(t)$ and integrating in Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \int_\Omega \left(|\nabla u|^2 - C(t,x)|u|^2 \right) \, \mathrm{d}x = 0.$$

By (4.8) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \lambda_1(t)\|u(t)\|^2 \leq 0$$

and by Gronwall's Lemma

$$\|u(t)\|^2 \leq \mathrm{e}^{-\int_s^t \lambda_1(r)\,\mathrm{d}r}\|u(s)\|^2.$$

Exponential stability is therefore guaranteed provided that, for some $R, \beta > 0$ and $t \geq R$, $s \leq -R$, with R large enough, we have

$$\frac{\int_s^t \lambda_1(r)\,\mathrm{d}r}{t-s} \geq 2\beta$$

which, in turn, is satisfied if

$$\liminf_{t\to\pm\infty} \lambda_1(t) > 0.$$

We have thus proved:

Lemma 4.3 *Let $C \in C^\alpha(I\!R, L^p(\Omega))$ with $0 < \alpha \leq 1$ and $p > N/2$. Suppose that*

$$\liminf_{t\to\pm\infty} \lambda_1(t) > 0$$

where $\lambda_1(t)$ is the first eigenvalue of the problem

$$\left\{ \begin{array}{rcll} -\Delta u - C(t,x)u & = & \lambda(t)u & \textit{in} \quad \Omega \\ u & = & 0 & \textit{on} \quad \partial\Omega. \end{array} \right.$$

Then $\Delta + C(t,x)$ generates an exponentially stable evolution operator in $L^q(\Omega)$ for all $1 < q \leq \infty$.

4.3 Persistence of exponential stability under perturbation

We now turn our attention to perturbations of the evolution operators $U = U_C$ defined by the solutions of (4.1) in $L^q(\Omega)$, $1 < q \leq \infty$. Our goal is to estimate the effects of the perturbation on the exponential type of the resulting evolution operator.

Proposition 4.4 *Assume that $U = U_C$ is the evolution operator defined by the solutions of (4.1) in $L^q(\Omega)$, $1 < q \leq \infty$, as above, and that there exist $M > 0$ and $\beta \in \mathbb{R}$ such that*

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \text{for all} \quad t > s. \tag{4.9}$$

Assume that $P \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, is a given time–dependent perturbation of C, and denote by P^+ the positive part of P.
i) If $P^+ \in L^1(\mathbb{R}, L^\infty(\Omega))$ then

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq K e^{\beta(t-s)} \quad \text{for all} \quad t > s.$$

for some constant K.
ii) If $P^+ \in L^\sigma(\mathbb{R}, L^p(\Omega))$, with $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, then for every $\varepsilon > 0$ there exists a K_ε such that

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq K_\varepsilon e^{(\beta+\varepsilon)(t-s)} \quad \text{for all} \quad t > s.$$

iii) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega))$ then

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq K e^{(\beta+\gamma)(t-s)} \quad \text{for all} \quad t > s$$

for some γ which depends on $\|P^+\|_{L^\infty(\mathbb{R}, L^p(\Omega))}$ and for some constant K.
iv) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega)) \cap L^1(\mathbb{R}, L^p(\Omega))$, $p > N/2$, then for every $\varepsilon > 0$ there exists a K_ε such that

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq K_\varepsilon e^{(\beta+\varepsilon)(t-s)} \quad \text{for all} \quad t > s.$$

Proof. First we prove that non-positive perturbations do not increase the exponential type of the evolution operator. More precisely, we prove that if $0 \geq P \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$ then

$$|U_{C+P}(t,s)u_0| \leq U_C(t,s)|u_0|$$

pointwise in Ω for every $u_0 \in L^q(\Omega)$. To see this note first that if $u_0 \geq 0$ then $U_{C+P}(t,s)u_0 \geq 0$ which implies that $|U_{C+P}(t,s)u_0| \leq U_{C+P}(t,s)|u_0|$. Therefore it is enough to prove the claim for non-negative initial data. In such a case, let $u(t,s;u_0) = U_{C+P}(t,s)u_0 \geq 0$ then, since $P \leq 0$ we have

$$\begin{cases} u_t - \Delta u = C(t,x)u + P(t,x)u \leq C(t,x)u \\ u(s) = u_0. \end{cases}$$

Hence, by the comparison principle, $0 \leq u(t,s;u_0) \leq U_C(t,s)u_0$ and the claim is proved.

Now let P be as in the statement of the proposition, i.e., $P \in C^{\alpha}(I\!R, L^p(\Omega)$ with $0 < \alpha \leq 1$ for some $p > N/2$. Writing $P = P^+ - P^-$ and using the evolution operator $U_{C-P^-}(t,s)$, which still satisfies (4.9), we have, by the variation of constants formula, that for every $u_0 \in L^q(\Omega)$ the solution $u(t,s,u_0) = U_{C+P}(t,s)u_0$ satisfies

$$u(t,s;u_0) = U_{C-P^-}(t,s)u_0 + \int_s^t U_{C-P^-}(t,\tau)P^+(\tau)u(\tau,s;u_0)\,\mathrm{d}\tau.$$

Case A). Assume that $p \geq q'$. Then the term $P^+(\tau)u(\tau,s;u_0)$ can be estimated, using Hölder's inequality, in $L^r(\Omega)$ with $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Hence denoting $z(t) = \mathrm{e}^{-\beta(t-s)}\|u(t,s,u_0)\|_{L^q(\Omega)}$, using (4.2), and (4.9) we get

$$z(t) \leq M\|u_0\|_{L^q(\Omega)} + \int_s^t \frac{M}{(t-\tau)^{\frac{N}{2p}}}\|P^+(\tau)\|_{L^p(\Omega)}z(\tau)\,\mathrm{d}\tau$$

and moreover, for every $s \leq t_0 \leq t$

$$z(t) \leq Mz(t_0) + \int_{t_0}^t \frac{M}{(t-\tau)^{\frac{N}{2p}}}\|P^+(\tau)\|_{L^p(\Omega)}z(\tau)\,\mathrm{d}\tau$$

The argument in this case is concluded using the singular Gronwall Lemma below, with $\beta = \frac{N}{2p} < 1$.

Case B). Assume that $p < q'$. Now the term $P^+(\tau)u(\tau,s;u_0)$ can only be estimated (using Hölder's inequality) in $L^1(\Omega)$, but since the case $q = p'$ is included in Case A), above, we get

$$\begin{aligned}\|u(t,s;u_0)\|_{L^q(\Omega)} \;&\leq\; M\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^q(\Omega)} \\ &\quad + \int_s^t \frac{M\mathrm{e}^{\beta(t-\tau)}}{(t-\tau)^{\frac{N}{2q'}}}\|P^+(\tau)\|_{L^p(\Omega)}\|u(\tau,s;u_0)\|_{L^{p'}(\Omega)}\,\mathrm{d}\tau\end{aligned}$$

and then

$$\begin{aligned}\|u(t,s;u_0)\|_{L^q(\Omega)} \;&\leq\; M\mathrm{e}^{\beta(t-s)}\|u_0\|_{L^q(\Omega)} \\ &\quad + M^2\mathrm{e}^{\mu(t-s)}\|u_0\|_{L^q(\Omega)}\int_s^t \frac{\|P^+(\tau)\|_{L^p(\Omega)}}{(t-\tau)^{\frac{N}{2q'}}(\tau-s)^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p'})}}\,\mathrm{d}\tau\end{aligned}$$

where μ equals $\beta+\varepsilon$ or $\beta+\gamma$ according to cases ii), iii) or iv) of the statement. Now the result follows after using Hölder's inequality and observing that setting $\tau = s+z(t-s)$ we get

$$\int_s^t \frac{\mathrm{d}\tau}{(t-\tau)^{\frac{\sigma' N}{2q'}}(\tau-s)^{\sigma'\frac{N}{2}(\frac{1}{q}-\frac{1}{p'})}} = (t-s)^{1-\frac{\sigma' N}{2p}}\int_0^1 \frac{\mathrm{d}z}{(1-z)^{\frac{\sigma' N}{2q'}}z^{\sigma'\frac{N}{2}(\frac{1}{q}-\frac{1}{p'})}}$$

with $\frac{\sigma' N}{2q'} < 1$ and $\sigma' \frac{N}{2}(\frac{1}{q} - \frac{1}{p'}) < 1$ because $\frac{\sigma' N}{2} < p < q'$. Therefore,

$$\|u(t,s;u_0)\|_{L^q(\Omega)} \leq M^2 e^{\mu(t-s)} \|u_0\|_{L^q(\Omega)} \Big(1 + (t-s)^{\frac{1}{\sigma'} - \frac{N}{2p}} C(\sigma,q,p) \|P^+\|_{L^\sigma(\mathbb{R}, L^p(\Omega))}\Big).$$

■

We now prove the singular Gronwall lemma used above.

Lemma 4.5 A singular Gronwall lemma

Assume that $a \in L^\sigma([0,\infty))$ with $1 \leq \sigma \leq \infty$ and that $z(t) \geq 0$ is a locally bounded function that for every $0 \leq t_0 \leq t$ satisfies

$$z(t) \leq M z(t_0) + \int_{t_0}^{t} \frac{a(\tau)}{(t-\tau)^\beta} z(\tau)\, d\tau \tag{4.10}$$

with $\beta\sigma' < 1$. Then for $t \geq 0$

$$0 \leq z(t) \leq M(\gamma) e^{\gamma t}$$

where $\gamma = 0$ if $\sigma = 1$ (and $\beta = 0$), γ is arbitrarily small if $1 < \sigma < \infty$ and $\beta\sigma' < 1$, or γ is proportional to $\|a\|_{L^\infty(0,\infty)}^{1/(1-\beta)}$ if $\sigma = \infty$ and $0 \leq \beta < 1$.

In particular, if $a \in L^\infty([0,\infty)) \cap L^1([0,\infty))$ and $0 \leq \beta < 1$ then for $t \geq 0$

$$0 \leq z(t) \leq M(\gamma) e^{\gamma t}$$

where γ is arbitrarily small.

Proof. Note that the case $\sigma = 1$, $\beta = 0$ reduces to the usual Gronwall lemma and then $z(t) \leq M z(0) e^{\int_0^t a(s)\, ds}$ and the result is obvious.

On the other hand the case $\sigma = \infty$ and $0 \leq \beta < 1$ is a particular case of the singular Gronwall lemma in Henry [12, Lemma 7.1.1, page 188] which gives $\gamma = (\|a\|_{L^\infty(0,\infty)} \Gamma(1-\beta))^{1/(1-\beta)}$.

Therefore, we will consider now the case $1 < \sigma < \infty$ and $\beta\sigma' < 1$. Note that in this case we can take T_0 large enough such that $\|a\|_{L^\sigma(T_0,\infty)}$ is as small as we want. Also, from (4.10) we get that for $T_0 \leq t_0 \leq t \leq t_0 + T$ we have, denoting $w(t_0, T) = \sup_{t_0 \leq \tau \leq t_0 + T} z(\tau)$ and using Hölder's inequality

$$\begin{aligned} z(t) &\leq M z(t_0) + w(t_0,T) \|a\|_{L^\sigma(t_0, t_0+T)} \Big(\int_{t_0}^{t} \frac{1}{(t-\tau)^{\beta\sigma'}}\, d\tau \Big)^{1/\sigma'} \\ &\leq M z(t_0) + w(t_0,T)\delta(T_0,T) \end{aligned}$$

where we have set $\delta(T_0,T) = \|a\|_{L^\sigma(T_0,\infty)} C(\beta,\sigma') T^{1/\sigma' - \beta}$, for some constant $C(\beta,\sigma')$.

Now, given T_0, choose T such that $\delta(T_0,T) = \|a\|_{L^\sigma(T_0,\infty)} C(\beta,\sigma') T^{1/\sigma'-\beta} = 1/2$. Taking the supremum for $t_0 \leq t \leq t_0 + T$ we get

$$z(t) \leq w(t_0) \leq 2M z(t_0) \quad \text{for all} \quad t_0 \leq t \leq t_0 + T.$$

Writing $t_1 = t_0 + T$ and repeating the process and the estimate above we get a sequence $t_n = t_0 + nT$ such that

$$z(t) \leq (2M)^n z(t_0), \quad \text{for all} \quad t_0 + (n-1)T \leq t \leq t_0 + nT.$$

From here it follows that

$$z(t) \leq (2M)^{\frac{t-t_0}{T}+1} z(t_0) \leq (2M)^{\frac{t}{T}+1} z(t_0), \quad \text{for all} \quad t \geq t_0.$$

Since choosing T_0 large enough, we can make T as large as we want, we obtain the result.

Finally, if $a \in L^\infty([0,\infty)) \cap L^1([0,\infty))$ and $0 \leq \beta < 1$ then we can always choose σ such that $a \in L^\sigma([0,\infty))$ and $\beta\sigma' < 1$ and we are finished. ■

As a consequence of the above results we get the following corollary which will be of great help below.

Corollary 4.6 *Under the assumptions of Proposition 4.4, assume furthermore that the evolution operator $U_C(t,s)$ is exponentially stable in $L^q(\Omega)$, i.e. that (4.9) is satisfied with $\beta < 0$.*
i) If $P^+ \in L^1(\mathbb{R}, L^\infty(\Omega))$, or $P^+ \in L^\sigma(\mathbb{R}, L^p(\Omega))$ with $1 < \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, then the evolution operator $U = U_{C+P}$ is exponentially stable in $L^q(\Omega)$.
ii) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega))$ with $p > \frac{N}{2}$, then the evolution operator $U = U_{C+P}$ is exponentially stable in $L^q(\Omega)$ provided that

$$\beta + \left(M \|P^+\|_{L^\infty(\mathbb{R}, L^p(\Omega))} \Gamma(1-\delta)\right)^{1/(1-\delta)} < 0,$$

where $\delta = \frac{N}{2p} < 1$.
iii) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega)) \cap L^1(\mathbb{R}, L^p(\Omega))$ with $p > \frac{N}{2}$, then the evolution operator $U = U_{C+P}$ is exponentially stable in $L^q(\Omega)$.

A close look at the proof above prompts the following remark which will be used below:

Remark 4.7 *Notice that Proposition 4.4 and Corollary 4.6 remain true if we only assume that*

$$P^+ \in L^\sigma([s_0,\infty), L^p(\Omega))$$

for some $s_0 \in \mathbb{R}$ and σ and p as in the statements of these results. In this case we obtain the estimate

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_{s_0} \mathrm{e}^{(\beta+\gamma)(t-s)} \quad \textit{for all} \quad t > s > s_0$$

where γ is arbitrarily small or depends on $\|P^+\|_{L^\sigma([s_0,\infty),L^p(\Omega))}$ according to the cases above.

In order to obtain a constant M_{s_0} independent of s_0 we will then need to have a uniform bound on $\|P^+\|_{L^\sigma([s_0,\infty),L^p(\Omega))}$, which requires $P^+ \in L^\sigma(\mathbb{R}, L^p(\Omega))$.

The next corollary gives a result that will be useful for the study of asymptotically autonomous problems.

Corollary 4.8 *Let $C \in C^{\alpha}(I\!R, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, such that the evolution operator generated by $\Delta + C(t,x)$ is exponentially stable.*
i) If there exist $C^+ \in L^p(\Omega)$ and $T_0 \in I\!R$ such that $C - C^+ \in L^{\sigma}([T_0,\infty), L^p(\Omega))$ with either $1 \leq \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, or $\sigma = \infty$ and $p > \frac{N}{2}$ and

$$\lim_{t\to\infty} \|C(t) - C^+\|_{L^p(\Omega)} = 0$$

then the semigroup generated by $\Delta + C^+$ has exponential decay.
ii) If there exist $C^- \in L^p(\Omega)$ and $T_0 \in I\!R$ such that $C - C^- \in L^{\sigma}((-\infty,T_0], L^p(\Omega))$ with either $1 \leq \sigma < \infty$ and $p > \frac{N\sigma'}{2}$, or $\sigma = \infty$ and $p > \frac{N}{2}$ and

$$\lim_{t\to-\infty} \|C(t) - C^-\|_{L^p(\Omega)} = 0$$

then the semigroup generated by $\Delta + C^-$ has exponential decay.

Proof. Since the evolution operator U_C is exponentially stable we have, for some $\beta < 0$,

$$\|U_C(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M e^{\beta(t-s)} \quad \text{for all} \quad t > s.$$

i) Set $P(t,x) = C^+(x) - C(t,x)$. Our assumptions imply that for s_0 large enough the norm $\|P\|_{L^{\sigma}([s_0,\infty),L^p(\Omega))}$ is as small as we want. Therefore from Proposition 4.4 and Remark 4.7 we know that

$$\|U_{C+P}(t,s)\|_{\mathcal{L}(L^q(\Omega))} \leq M_{s_0} e^{(\beta+\varepsilon)(t-s)} \quad \text{for all} \quad t > s > s_0 \tag{4.11}$$

for arbitrarily small ε.

Since $C(t,x) + P(t,x) = C^+(x)$ we know that $T_{C^+}(t) = U_{C^+}(t + s_0, s_0)$, $t > 0$ is an autonomous evolution operator, i.e. a semigroup which is actually the semigroup generated by $\Delta + C^+$. Hence, from (4.11), $T_{C^+}(t)$ has exponential decay.

ii) Set $P(t,x) = C^-(x) - C(t,x)$. Our assumptions now imply that for t_0 sufficiently negative the norm $\|P\|_{L^{\sigma}([s_0,t_0],L^p(\Omega))}$ is as small as we want. Therefore from Proposition 4.4 and Remark 4.7 it follows that (4.11) holds for $s_0 \leq s < t \leq t_0$ with arbitrarily small ε.

As before, since $C(t,x) + P(t,x) = C^-(x)$ the semigroup generated by $\Delta + C^-$ satisfies $T_{C^-}(t) = U_{C+P}(t_0, t_0 - t)$. Hence, from (4.11), we can find t such that

$$\|T_{C^-}(t)\|_{\mathcal{L}(L^q(\Omega))} < 1$$

and once more obtain exponential decay. ∎

5 Complete trajectories for the linear problem

5.1 The homogeneous case

We begin by studying the homogeneous case. For this, we will consider the following problem

$$\left\{ \begin{array}{rcll} w_t - \Delta w & = & C(t,x)w, & \text{in } \;\Omega, \quad t > s \\ w & = & 0 & \text{on } \;\partial\Omega, \quad t > s \\ w(s) & = & w_0 & \end{array} \right. \tag{5.1}$$

where Ω is a bounded domain in $I\!R^N$ and $C \in C^\alpha(I\!R, L^p(\Omega))$ with $0 < \alpha \le 1$ and some $p > N/2$.

Let $U_C(t,s)$ be the evolution operator associated with the above problem in $X = C(\overline{\Omega})$ or in $X = L^q(\Omega)$ with $1 < q < \infty$, i.e. $U_C(t,s)w_0 = w(t,s;w_0)$.

As in Section 3.2, given $\beta > 0$ we define $\mathcal{D}_\beta = \mathcal{D}_\beta(I\!R, X)$, the "basin of attraction", consisting of families of bounded sets that grow slower than $\mathrm{e}^{-\beta t}$ as $t \to -\infty$, i.e. families of bounded sets of the form $\{B(t)\}_t$ such that for some $\gamma < \beta$ we have

$$\mathrm{e}^{\gamma t}\|B(t)\|_X \to 0 \qquad \text{as } t \to -\infty,$$

where

$$\|B\|_X := \sup_{b\in B} \|b\|_X.$$

As remarked before, in a slight abuse of notation we can also include single-valued functions in $\mathcal{D}_\beta$ via the identity $\psi \leftrightarrow \{\psi(t)\}_{t\in I\!R}$. Similarly, we can regard $\mathcal{D}_\beta$ as containing all fixed bounded sets B, via the identity $B \leftrightarrow \{B(t)\}_{t\in I\!R}$, where $B(t) = B$ for all $t \in I\!R$.

Outstandingly we note that although the class $\mathcal{D}_\beta$ imposes some bound on the growth as $t \to -\infty$, it imposes no restrictions at all as $t \to \infty$.

Theorem 5.1 *Let $X = C(\overline{\Omega})$ or $L^q(\Omega)$ with $1 < q < \infty$. Suppose that the evolution operator $U_C(t,s)$ for (5.1) is exponentially stable in X, i.e. for some $\beta > 0$*

$$\|U_C(t,s)\|_{\mathcal{L}(X)} \le M\mathrm{e}^{-\beta(t-s)} \quad \textit{for all } t > s. \tag{5.2}$$

Then the unique complete trajectory for (5.1) in $\mathcal{D}_\beta$ is the trivial solution. Indeed, $\mathcal{A} = \{\mathcal{A}(t)\}_t$ with $\mathcal{A}(t) = 0$ is the pullback attractor with respect to $\mathcal{D}_\beta$.

Moreover, the trivial solution also attracts bounded sets in X forwards in time.

Proof. It is clear that $\{0\}_t \in \mathcal{D}_\beta$ is a complete trajectory for (5.1) so we only have to prove the uniqueness. Let $\psi \in \mathcal{D}_\beta$ be a complete trajectory. Then

$$\psi(t) = U_C(t,s)\psi(s) \quad \text{for all } t \ge s$$

and if we take norms in the above expression, for some $\gamma < \beta$ we have

$$\|\psi(t)\|_X \le \|U_C(t,s)\|_{\mathcal{L}(X)}\ \|\psi(s)\|_X \le M\mathrm{e}^{-\beta(t-s)}M_1(t)\mathrm{e}^{\gamma s}.$$

Letting s tend to $-\infty$ shows that

$$\|\psi(t)\|_X = 0 \quad \text{for all } t \in I\!R$$

since $\gamma < \beta$. So the unique bounded complete trajectory in $\mathcal{D}_\beta$ is 0.

Now let $\{B(s)\}_s \in \mathcal{D}_\beta$. Then, for all $w_s \in B(s)$,

$$\begin{aligned}\|U_C(t,s)w_s\|_X &\le M\mathrm{e}^{-\beta(t-s)}\|B(s)\|_X \\ &\le M\mathrm{e}^{-\beta(t-s)}M_1(t)\mathrm{e}^{\gamma s} = M_2(t)\mathrm{e}^{(\beta-\gamma)s}\end{aligned} \tag{5.3}$$

for some $\gamma < \beta$. Thus, taking limits as $s \to -\infty$ we have

$$U_C(t,s)w_s \to 0 \quad \text{as } s \to -\infty$$

for all $t \in I\!R$.

The attraction forwards in time follows immediately from the asymptotic stability condition (5.2). ■

Remark 5.2 *To prove the previous theorem in the case of attraction of bounded sets (i.e. with the basin of attraction consisting of families of bounded sets not depending on time) it is not necessary to assume the exponential stability of U. It is enough to suppose that the evolution semigroup decays to zero as s tends to $-\infty$, i.e. that*

$$\|U(t,s)\|_{\mathcal{L}(X)} \to 0 \quad \text{as } s \to -\infty.$$

5.2 The inhomogeneous problem

We now consider the following linear inhomogeneous problem

$$\left\{ \begin{array}{rcll} v_t - \Delta v &=& C(t,x)v + D(t,x), & \text{in } \Omega, \quad t > s, \\ v &=& 0 & \text{on } \partial\Omega, \quad t > s, \\ v(s) &=& v_s & \end{array} \right. \tag{5.4}$$

posed in either $X = C(\overline{\Omega})$ or $X = L^q(\Omega)$ with $1 < q < \infty$.

Assume that $C \in C^\alpha(I\!R, L^p(\Omega))$ with $0 < \alpha \le 1$ and $p > N/2$, and $D \in L^1_{\text{loc}}(I\!R, L^r(\Omega))$, for some $\frac{qN}{N+2q} < r \le \infty$ if $X = L^q(\Omega)$ or $r > N/2$ if $X = C(\overline{\Omega})$ respectively. Then there exists a unique solution of (5.4) given by the variation of constants formula, i.e.

$$v(t,s,v_s) = U_C(t,s)v_s + \int_s^t U_C(t,\tau)D(\tau)\,\mathrm{d}\tau \tag{5.5}$$

(e.g. see Theorem 1.2.1, p. 43, in Amann [3] or Henry [12] or Lunardi [16]).

We will also assume the exponential stability of the evolution operator U_C associated with $\Delta + C(t,x)$ as in Section 5.1.

The following result establishes the existence of a unique complete trajectory for (5.4) under two different types of conditions on the behaviour of D as $t \to -\infty$.

Theorem 5.3 *Let $X = C(\overline{\Omega})$ or $X = L^q(\Omega)$ with $1 < q < \infty$. Suppose that the evolution operator $U_C(t,s)$ is exponentially stable in X, i.e.*

$$\|U_C(t,s)\|_{\mathcal{L}(X)} \le M\mathrm{e}^{-\beta(t-s)} \quad \text{with } \beta > 0 \text{ and } M \ge 1.$$

i) Assume that

$$D \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$$

with $\frac{Nq}{N+2q} < r \le \infty$ if $X = L^q(\Omega)$, $1 < q < \infty$, or $N/2 < r \le \infty$ if $X = C(\overline{\Omega})$.

Then there exists a unique complete trajectory $\phi \in \mathcal{D}_\beta = \mathcal{D}_\beta(I\!R, X)$ for (5.4).
ii) Assume now that for some σ with $1 \le \sigma \le \infty$

$$D \in L^\sigma((-\infty, T), L^r(\Omega)) \quad \textit{for each} \quad T < \infty$$

or that

$$D \in L^\sigma(I\!R, L^r(\Omega))$$

(which corresponds to $T = \infty$ above), for some r with $\frac{Nq}{N+2q} < r \le \infty$ if $X = L^q(\Omega)$ or with $N/2 < r \le \infty$ if $X = C(\overline{\Omega})$.

Then there exists a complete trajectory for (5.4), $\phi \in L^\sigma((-\infty, T), X) \cap C(I\!R, X)$, for each $T < \infty$ (or $\phi \in L^\sigma(I\!R, X) \cap C(I\!R, X)$ if $T = \infty$).

Assume in addition either that $1 < \sigma \le \infty$ and $\frac{N\sigma' q}{N\sigma' + 2q} < r \le \infty$, if $X = L^q(\Omega)$, or $N\sigma'/2 < r \le \infty$, if $X = C(\overline{\Omega})$; or that $\sigma = 1$ and $q \le r \le \infty$, if $X = L^q(\Omega)$, or $r = \infty$, if $X = C(\overline{\Omega})$: then $\phi \in C_b((-\infty, T), X) \subset \mathcal{D}_\beta$ (or $\phi \in C_b(I\!R, X) \subset \mathcal{D}_\beta$ if $T = \infty$) and is the unique complete trajectory within this class.

In either one of the cases above in which the complete trajectory $\phi \in \mathcal{D}_\beta$, the family $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ is the pullback attractor for (5.4) with respect to $\mathcal{D}_\beta$.

$\{\phi(t)\}_t$ also attracts bounded sets of X forwards in time. More precisely, for every bounded set $B \subset X$ we have

$$\|v(t, s; v_0) - \phi(t)\|_X \le K\mathrm{e}^{-\beta(t-s)}, \qquad t > s \tag{5.6}$$

for all $v_0 \in B$, where $K = K(B)$.

Proof. First, we prove the existence of a complete trajectory for (5.4). We set

$$\phi(t) = \int_{-\infty}^{t} U_C(t, \tau) D(\tau) \mathrm{d}\tau. \tag{5.7}$$

If $\phi(t)$ is well-defined then it is a complete trajectory for (5.4) since, given $t \ge s$,

$$\begin{aligned} \phi(t) - U_C(t, s)\phi(s) &= \int_{-\infty}^{t} U_C(t, \tau) D(\tau) \mathrm{d}\tau - U_C(t, s) \int_{-\infty}^{s} U_C(s, \tau) D(\tau) \mathrm{d}\tau \\ &= \int_{s}^{t} U_C(t, \tau) D(\tau) \mathrm{d}\tau \end{aligned} \tag{5.8}$$

and moreover in such a case we will automatically have $\phi \in C(I\!R, X)$. We will show below that $\phi(t)$ is well-defined and belongs to $\mathcal{D}_\beta = \mathcal{D}_\beta(I\!R, X)$. For now we assume that this has been proved.

Let $\mathcal{B} = \{B(s)\}_s \in \mathcal{D}_\beta$ and fix $\{v_s\}_s \in \mathcal{B}$. Then, the solution of (5.4) is given by the variation of constants formula (5.5). Let $w(t, s; v_s) = v(t, s; v_s) - \phi(t)$. Then w solves the homogeneous problem

$$\left\{ \begin{array}{rcll} w_t - \Delta w &=& C(t, x) w, & \text{in } \Omega, \quad t > s \\ w(s) &=& v_s - \phi(s) & \\ w &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{5.9}$$

So, since $\{B(s)-\phi(s)\}_s \in \mathcal{D}_\beta$, from Theorem 5.1 we have

$$w(t,s;v_s) \to 0 \quad \text{as} \quad s \to -\infty$$

uniformly for $v_s \in B(s)$, where $\{B(s)\}_s \in \mathcal{D}_\beta$. Thus, for all $t \in I\!R$

$$v(t,s;v_s) \to \phi(t) \quad \text{as } s \to -\infty.$$

So we have proved that $\mathcal{A} = \{\phi(t)\}$ is the pullback attractor.

Notice that if we fix $s \in I\!R$ and a bounded set $B \subset X$ we have

$$\|v(t,s;v_0) - \phi(t)\|_X \le K\mathrm{e}^{-\beta(t-s)}\|v_0 - \phi(s)\|_X \le K_1 \mathrm{e}^{-\beta(t-s)} \to 0 \tag{5.10}$$

as $t \to +\infty$, for all $v_0 \in B$, where K_1 depends on the bounded set B. Hence $\phi(t)$ also attracts bounded sets of X forwards in time.

We now prove that $\phi(t)$ is well-defined.

i) Since $\frac{Nq}{N+2q} < r \le \infty$ if $X = L^q(\Omega)$, $1 < q < \infty$, or $N/2 < r \le \infty$, if $X = C(\overline{\Omega})$ and $D \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$, then, in (5.7) we get, for each $t \le t_0$, and for some $\gamma < \beta$,

$$\mathrm{e}^{\gamma t}\phi(t) = \int_{-\infty}^t \mathrm{e}^{\gamma(t-\tau)} U_C(t,\tau)\mathrm{e}^{\gamma\tau} D(\tau)\,\mathrm{d}\tau.$$

Using (4.2) we get

$$\mathrm{e}^{\gamma t}\|\phi(t)\|_X \le M \sup_{\tau \le t} \mathrm{e}^{\gamma\tau}\|D(\tau)\|_{L^r(\Omega)} \int_{-\infty}^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \mathrm{e}^{-(\beta-\gamma)(t-\tau)}\,\mathrm{d}\tau$$

with $1 < q \le \infty$.

Now, since $r > \frac{qN}{N+2q}$, if $1 < q < \infty$, or $N/2 < r \le \infty$ if $q = \infty$, we have that $\frac{N}{2}\left(\frac{1}{r} - \frac{1}{q}\right) < 1$ and, therefore, the integral term above is bounded independently of t.

Hence $\phi \in \mathcal{D}_\beta$ and, by Theorem 5.1, ϕ is the unique complete trajectory in $\mathcal{D}_\beta$.

ii) We assume now $D \in L^\sigma((-\infty,T), L^r(\Omega))$ for each $T < \infty$ (or even $T = \infty$ if $D \in L^\sigma(I\!R, L^r(\Omega))$) and distinguish below several cases.

Case a) Suppose that $D \in L^\sigma((-\infty,T), L^r(\Omega))$, $1 \le \sigma \le \infty$ and $q \le r \le \infty$, if $X = L^q(\Omega)$, $1 < q < \infty$, or $1 \le \sigma \le \infty$ and $r = \infty$ if $X = C(\overline{\Omega})$, respectively.

We start with the case $\sigma = \infty$. Then, from (5.7), for $t < T$,

$$\begin{aligned} \|\phi(t)\|_X &\le \limsup_{s\to-\infty} \int_s^t \|U_C(t,\tau)D(\tau)\|_X\,\mathrm{d}\tau \\ &\le \limsup_{s\to-\infty} \frac{M}{\beta}\left(1 - \mathrm{e}^{-\beta(t-s)}\right) \sup_{\tau\le t}\|D(\tau)\|_X \\ &\le \frac{M}{\beta}\|D\|_{L^\infty((-\infty,T),X)}. \end{aligned} \tag{5.11}$$

Thus $\phi \in L^\infty((-\infty,T),X)$.

Now we prove the result in the case $\sigma = 1$, i.e. $D \in L^1((-\infty,T),X)$. From (5.7), we get

$$\begin{aligned}
\|\phi\|_{L^1((-\infty,T),X)} &= \int_{-\infty}^{T} \|\phi(t)\|_X \,\mathrm{d}t \le \int_{-\infty}^{T}\int_{-\infty}^{t} \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau\,\mathrm{d}t \\
&\le \int_{-\infty}^{T}\int_{-\infty}^{t} M\mathrm{e}^{-\beta(t-\tau)}\|D(\tau)\|_X \,\mathrm{d}\tau\,\mathrm{d}t \\
&\le \int_{-\infty}^{T}\int_{\tau}^{T} M\mathrm{e}^{-\beta(t-\tau)}\|D(\tau)\|_X \,\mathrm{d}t\,\mathrm{d}\tau \\
&\le C\int_{-\infty}^{T} \|D(\tau)\|_X \,\mathrm{d}\tau = C\|D\|_{L^1((-\infty,T),X)}
\end{aligned}$$

where we have used Fubini's Theorem and the boundedness of $\int_\tau^T M\mathrm{e}^{-\beta(t-\tau)}\mathrm{d}t$ independent of τ and $T \le \infty$. Thus, $\phi \in L^1((-\infty,T),X)$.

Now from the interpolation theorem for L^p-spaces (see Theorem 5.2.3, p. 111, in Bergh and Löfström [6]) we have that if $D \in L^\sigma((-\infty,T),X)$, $1 \le \sigma \le \infty$, then $\phi \in L^\sigma((-\infty,T),X)$ and

$$\|\phi\|_{L^\sigma((-\infty,T),X)} \le C\|D\|_{L^\sigma((-\infty,T),X)}.$$

Finally, we prove that if we take $D \in L^\sigma((-\infty,T),X)$, $1 \le \sigma < \infty$ then $\phi \in L^\infty((-\infty,T),X)$. Indeed, let $1 < \sigma < \infty$ then, from expression (5.7), using Hölder's inequality, we get for $t < T$,

$$\begin{aligned}
\|\phi(t)\|_X &\le \limsup_{s\to-\infty}\int_s^t \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau \\
&\le M\limsup_{s\to-\infty}\left(\frac{1}{\beta\sigma'}\Big(1-\mathrm{e}^{-\beta\sigma'(t-s)}\Big)\right)^{1/\sigma'}\left(\int_{-\infty}^{t}\|D(\tau)\|_X^\sigma\,\mathrm{d}\tau\right)^{1/\sigma} \qquad (5.12)\\
&\le \frac{M}{(\beta\sigma')^{1/\sigma'}}\|D\|_{L^\sigma((-\infty,T),X)}.
\end{aligned}$$

Hence $\phi \in L^\infty((-\infty,T),X)$. The case $\sigma = 1$ is proved in an analogous way.

Case b) Let $X = L^q(\Omega)$, $1 < q < \infty$. Suppose that $D \in L^\sigma((-\infty,T),L^r(\Omega))$, $1 \le \sigma \le \infty$, $\frac{Nq}{N+2q} < r < q$. In this case, we need to use L^p-L^q smoothing estimates for the evolution operator, see (4.2).

We start with the case $\sigma = \infty$. Suppose that $D \in L^\infty((-\infty,T),L^r(\Omega))$ with $\frac{qN}{N+2q} < r < q$. Then, for $t < T$,

$$\begin{aligned}
\|\phi(t)\|_{L^q(\Omega)} &\le \int_{-\infty}^{t}\|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\
&\le M\sup_{\tau\le t}\|D(\tau)\|_{L^r(\Omega)}\int_{-\infty}^{t}(t-\tau)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\mathrm{e}^{-\beta(t-\tau)}\,\mathrm{d}\tau \qquad (5.13)
\end{aligned}$$

where we have used (4.2). Now, since $r > \frac{qN}{N+2q}$ we have $\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right) < 1$ and, therefore, the integral term above is bounded independently of t. Thus, from (5.13),

$$\|\phi(t)\|_{L^\infty((-\infty,T),L^q(\Omega))} \le C\|D\|_{L^\infty((-\infty,T),L^r(\Omega))}.$$

Now assume that $\sigma = 1$, i.e. that $D \in L^1((-\infty,T),L^r(\Omega))$. Then, using (4.2) as in (5.13), we have

$$\begin{aligned}
\|\phi\|_{L^1((-\infty,T),L^q(\Omega))} &\le \int_{-\infty}^{T}\int_{-\infty}^{t}(t-\tau)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}M\mathrm{e}^{-\beta(t-\tau)}\|D(\tau)\|_{L^r(\Omega)}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{T}\left[\int_{0}^{T-\tau}Ms^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\mathrm{e}^{-\beta s}\,\mathrm{d}t\right]\|D(\tau)\|_{L^r(\Omega)}\,\mathrm{d}\tau \\
&\le C\int_{-\infty}^{T}\|D(\tau)\|_{L^r(\Omega)}\,\mathrm{d}\tau = C\|D\|_{L^1((-\infty,T),L^r(\Omega))}
\end{aligned}$$

where we have used Fubini's Theorem and the boundedness of $\int_0^\infty s^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\mathrm{e}^{-\beta s}\mathrm{d}s$, since $r > \frac{Nq}{N+2q}$.

Again, the result in the case $D \in L^\sigma((-\infty,T),L^r(\Omega))$, $1 < \sigma < \infty$, follows from the interpolation theorem for L^p spaces as in Case a) above.

Finally, we show that if $D \in L^\sigma((-\infty,T),L^r(\Omega))$, $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma'+2q} < r < q$, then $\phi \in L^\infty((-\infty,T),L^q(\Omega))$. Indeed, let $1 < \sigma < \infty$ then, as in (5.13), using the Hölder inequality, we get for $t < T$,

$$\begin{aligned}
\|\phi(t)\|_{L^q(\Omega)} &\le \limsup_{s\to-\infty}\int_s^t \|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\
&\le \left(\int_0^\infty s^{-\frac{\sigma' N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\mathrm{e}^{-\beta\sigma' s}\right)^{1/\sigma'}\left(\int_{-\infty}^{t}\|D(\tau)\|^\sigma_{L^r(\Omega)}\,\mathrm{d}\tau\right)^{1/\sigma} \qquad (5.14)\\
&\le C\|D\|_{L^\sigma((-\infty,T),L^r(\Omega))}
\end{aligned}$$

where we have used that $\frac{\sigma' N}{2}\left(\frac{1}{r}-\frac{1}{q}\right) < 1$ since $r > \frac{N\sigma' q}{N\sigma'+2q}$. Thus, $\phi \in L^\infty((-\infty,T),L^q(\Omega))$.

Case c) Let $X = C(\overline{\Omega})$. Assume $D \in L^\sigma((-\infty,T),L^r(\Omega))$ with $1 \le \sigma \le \infty$, $r > N/2$. This case follows as in Case b) with $q = \infty$ and, of course, $r = \infty$ if $\sigma = 1$ or $r > N\sigma'/2$ if $1 < \sigma \le \infty$. ■

Remark 5.4 *In the first part of case ii) of the theorem we prove the existence of a complete trajectory $\phi \in L^\sigma((-\infty,T),X)\cap C(\mathbb{R},X)$. In particular, this complete trajectory can be unbounded in time. Furthermore, $\phi(t)$ can grow very fast as $t \to -\infty$ and may be even not belong to $\mathcal{D}_\beta$. For this reason we cannot prove the uniqueness of such a $\phi(t)$. Nevertheless, the complete trajectory $\phi(t)$ is unique in the class $\phi + \mathcal{D}_\beta$.*

Remark 5.5 *Observe that with similar arguments as in the proof of the theorem above, if we define* $E_\alpha = \{f \in C(I\!R, X) : \ \mathrm{e}^{-\alpha|t|} f \in C_b(I\!R, X)\}$, *with* $\beta > \alpha > 0$, *one can show that there exists a unique complete trajectory in* E_α *provided that* $D \in E_\alpha$. *In such a case we have to restrict the basin of attraction* $\mathcal{D}$ *to families of bounded sets in* $\mathcal{D}_{\beta-\alpha}$.

Considering only pullback attraction it is enough to work in

$$E_\alpha^- = \{f \in C(I\!R, X) : \ \mathrm{e}^{\alpha t} f \in C_b((-\infty, \tau), X) \text{ for some } \tau \in I\!R\},$$

and analogously if we consider forward attraction we can use

$$E_\alpha^+ = \{f \in C(I\!R, X) : \ \mathrm{e}^{-\alpha t} f \in C_b((\tau, \infty), X) \text{ for some } \tau \in I\!R\}.$$

5.3 Asymptotic behaviour as $t \to \pm\infty$

Given the above theorem it is natural to consider the asymptotic behaviour of the complete trajectory as $t \to \pm\infty$. In fact with a closer look at the above proof we can show that, in the cases that $\phi(t)$ remains bounded and integrable as $t \to \pm\infty$, it actually converges to zero:

Corollary 5.6 *Let* $X = C(\overline{\Omega})$ *or* $L^q(\Omega)$ *with* $1 < q < \infty$. *Suppose that the evolution operator associated with* $\Delta + C(t, x)$ *is exponentially stable.*
i) Assume that $D \in L^\sigma((-\infty, T), L^r(\Omega))$, *for* $T < \infty$, *with either* $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r \leq \infty$, *or* $\sigma = 1$ *and* $q \leq r \leq \infty$, *or* $\sigma = \infty$ *and* $\frac{Nq}{N+2q} < r \leq \infty$. *In the case* $\sigma = \infty$ *assume in addition that*

$$\lim_{t \to -\infty} \|D(t)\|_{L^r(\Omega)} = 0.$$

Then $\phi(t) \to 0$ *in* X *as* $t \to -\infty$.
ii) Assume that D *satisfies the assumptions in Theorem 5.3 and also that* $D \in L^\sigma((T, \infty), L^r(\Omega))$, *for* $T > -\infty$, *with* σ *and* r *as in case i) above. Assume in addition that, if* $\sigma = \infty$,

$$\lim_{t \to \infty} \|D(t)\|_{L^r(\Omega)} = 0.$$

Then $\phi(t) \to 0$ *in* X *as* $t \to \infty$. *Hence, for every bounded set* $B \subset X$, *we have*

$$v(t, s; v_0) \to 0 \quad \text{in} \quad X \quad \text{as} \quad t \to \infty$$

uniformly for $v_0 \in B$.

Proof. Case i) is a direct consequence of inequalities (5.11), (5.12), (5.13) and (5.14).

Therefore it remains to prove that $\phi(t) \to 0$ in X as $t \to \infty$. In such a case, the rest of the result is a consequence of (5.6). Hence, note that for any solution of (5.4) we have (see (5.5))

$$v(t, s; v_s) = U_C(t, s) v_s + \int_s^t U_C(t, \tau) D(\tau) \, \mathrm{d}\tau$$

and since the linear evolution operator is exponentially stable, the first term tends to zero as $t \to \infty$. For the integral term, let $t > T > s$ to be fixed later. Then

$$\left\| \int_s^t U_C(t,\tau) D(\tau)\, \mathrm{d}\tau \right\|_X \le \int_s^T \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau + \int_T^t \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau$$

Case a) Suppose that either $X = L^q(\Omega)$, $1 < q < \infty$, and $1 \le \sigma \le \infty$, $r \ge q$; or $X = C(\overline{\Omega})$, $1 \le \sigma \le \infty$ and $r = \infty$.

Suppose $\sigma = \infty$. Given $\epsilon > 0$, on the one hand we have

$$\begin{aligned} \int_T^t \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau &\le \int_T^t M\mathrm{e}^{-\beta(t-\tau)} \|D(\tau)\|_X \,\mathrm{d}\tau \\ &\le \frac{M}{\beta}\left(1 - \mathrm{e}^{-\beta(t-T)}\right) \operatorname*{ess\,sup}_{\tau \ge T} \|D(\tau)\|_X < \frac{\epsilon}{2} \end{aligned} \tag{5.15}$$

choosing T large enough. On the other hand,

$$\begin{aligned} \int_s^T \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau &\le \int_s^T M\mathrm{e}^{-\beta(t-\tau)} \|D(\tau)\|_X \,\mathrm{d}\tau \\ &= \frac{M}{\beta} \mathrm{e}^{-\beta(t-T)} \left(1 - \mathrm{e}^{-\beta(T-s)}\right) \|D\|_{L^\infty((s,\infty),X)} \\ &< \frac{\epsilon}{2} \end{aligned} \tag{5.16}$$

choosing t large enough. Thus, for all t large enough we have

$$\|\phi(t)\|_X < \epsilon,$$

i.e. $\phi(t) \to 0$ in X as $t \to \infty$.

In the case $D \in L^\sigma((T,\infty),X)$, $1 < \sigma < \infty$, arguing as in (5.12) in the proof of Theorem 5.3, we have

$$\int_T^t \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau \le M \left(\frac{1 - \mathrm{e}^{-\beta\sigma'(t-T)}}{\beta\sigma'} \right)^{1/\sigma'} \|D\|_{L^\sigma((T,t),X)} < \frac{\epsilon}{2}$$

and

$$\int_s^T \|U_C(t,\tau)D(\tau)\|_X \,\mathrm{d}\tau \le M\mathrm{e}^{-\beta(t-T)} \left(\frac{1 - \mathrm{e}^{-\beta\sigma'(T-s)}}{\beta\sigma'} \right)^{1/\sigma'} \|D\|_{L^\sigma((s,\infty),X)} < \frac{\epsilon}{2}$$

for T and t large enough.

The case $\sigma = 1$ follows in an analogous way.

Case b) Suppose that $X = L^q(\Omega)$, $1 < q < \infty$, and $1 \le \sigma \le \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r < q$.

Suppose that $\sigma = \infty$. In such a case, $r > \frac{Nq}{N+2q}$. Arguing as in (5.13) in the proof of Theorem 5.3, we have that for T large and $t \to \infty$

$$\begin{aligned}\int_T^t &\|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\ &\le M \operatorname*{ess\,sup}_{\tau\ge T} \|D(\tau)\|_{L^r(\Omega)} \int_T^t \frac{\mathrm{e}^{-\beta(t-\tau)}}{(t-\tau)^{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}}\,\mathrm{d}\tau < \frac{\epsilon}{2}\end{aligned}$$

and

$$\begin{aligned}\int_s^T &\|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\ &\le M(t-T)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \int_s^T \mathrm{e}^{-\beta(t-\tau)}\|D(\tau)\|_{L^r(\Omega)}\,\mathrm{d}\tau \\ &\le \frac{K}{\beta}\,\mathrm{e}^{-\beta(t-T)}\left(1-\mathrm{e}^{-\beta(T-s)}\right)\|D\|_{L^\infty((s,\infty),L^r(\Omega))} < \frac{\epsilon}{2}\end{aligned}$$

for some constant $K > 0$.

Suppose now that $1 < \sigma < \infty$. Again, arguing as in (5.14) in the proof of Theorem 5.3, we have that for T and t large enough

$$\begin{aligned}\int_T^t &\|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\ &\le M\left(\int_T^t \frac{\mathrm{e}^{-\beta\sigma'(t-\tau)}}{(t-\tau)^{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)\sigma'}}\,\mathrm{d}\tau\right)^{1/\sigma'} \|D\|_{L^\sigma((T,t),L^r(\Omega))} < \frac{\epsilon}{2}\end{aligned}$$

and

$$\begin{aligned}\int_s^T &\|U_C(t,\tau)D(\tau)\|_{L^q(\Omega)}\,\mathrm{d}\tau \\ &\le M(t-T)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\left(\int_s^T \mathrm{e}^{-\beta\sigma'(t-\tau)}\,\mathrm{d}\tau\right)^{1/\sigma'} \|D\|_{L^\sigma((s,\infty),L^r(\Omega))} \\ &\le K\mathrm{e}^{-\beta(t-T)}\left(\frac{1-\mathrm{e}^{-\beta\sigma'(T-s)}}{\beta\sigma'}\right)^{1/\sigma'} \|D\|_{L^\infty((s,\infty),L^r(\Omega))} < \frac{\epsilon}{2}\end{aligned}$$

for some constant $K > 0$.

Case c) Let $X = C(\overline{\Omega})$ and assume that $D \in L^\sigma((T,\infty),L^r(\Omega))$ with $1 \le \sigma \le \infty$, $r > N/2$. This case follows as in Case b) with $q = \infty$ and, of course, $r = \infty$. ■

Now, by combining the arguments from the proofs of Theorem 5.3 and Corollary 5.6, we obtain the following result.

Corollary 5.7 *Let $X = C(\overline{\Omega})$ or $X = L^q(\Omega)$ with $1 < q < \infty$. Suppose that the evolution operator associated with $\Delta + C(t,x)$ is exponentially stable.*

Assume D_0 is such that Theorem 5.3 applies and denote by ϕ_0 the corresponding complete trajectory. Assume in addition that $D - D_0$ is such that Corollary 5.6 also applies.

Then (5.4) has a complete trajectory, ϕ, and $\phi - \phi_0 \to 0$ as t tends either to $+\infty$ or to $-\infty$, according to the cases in Corollary 5.6.

Note that this corollary can be applied for example if $D_0 \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$ while $D - D_0 \in L^\sigma((-\infty, T), L^r(\Omega))$. In such a case D might not satisfy the assumptions in Theorem 5.3.

5.4 The periodic problem

Finally we consider the T-periodic problem associated with (5.4), i.e. we suppose that $C(t,x)$ and $D(t,x)$ are T-periodic functions. In this case the unique complete trajectory given by Theorem 5.3 is T-periodic.

Corollary 5.8 *Let $X = L^q(\Omega)$, $1 < q < \infty$, or $X = C(\overline{\Omega})$. Assume $C \in C^\alpha(I\!R, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, the evolution operator associated with $\Delta + C(t,x)$ is exponentially stable and*

$$D \in L^\infty(I\!R, L^r(\Omega))$$

for some $\frac{Nq}{N+2q} < r \leq \infty$ if $X = L^q(\Omega)$ or $N/2 < r \leq \infty$ if $X = C(\overline{\Omega})$.

If $C(t,x)$ and $D(t,x)$ are T-periodic functions then the unique complete trajectory $\phi \in \mathcal{D}_\beta$ for (5.4) is T-periodic.

Proof. Note that the hypotheses in Theorem 5.3 hold since $D \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$. Let $\phi \in \mathcal{D}_\beta$ be the unique complete trajectory given by Theorem 5.3. Then,

$$\phi_t(t) - \Delta\phi(t) = C(t,x)\phi(t) + D(t,x)$$

and, by the periodicity of C and D we have

$$\phi_t(t) - \Delta\phi(t) = C(t+T,x)\phi(t) + D(t+T,x)$$

which after a change of variables gives

$$\phi_t(t-T) - \Delta\phi(t-T) = C(t,x)\phi(t-T) + D(t,x)$$

So, $w(t) = \phi(t-T)$ is a complete trajectory of the problem (5.4). But, from the Theorem 5.3 this complete trajectory is unique, and so we have $\phi(t) = w(t)$ for all $t \in I\!R$, that is, $\phi(t) = \phi(t-T)$ for all $t \in I\!R$. In other words, ϕ is T-periodic. ∎

6 Asymptotically autonomous and asymptotically periodic linear problems

In this section we study the linear evolution problem

$$\left\{\begin{array}{rcll} v_t - \Delta v &=& C(t,x)v + D(t,x), & \text{in} \quad \Omega, \qquad t > s \\ v(s) &=& v_0 & \\ v_{|\partial\Omega} &=& 0, & \text{on} \quad \partial\Omega \end{array}\right. \tag{6.1}$$

where $C(t,x)$ and $D(t,x)$ converge in some sense as $t \to \pm\infty$.

When C and D converge to time–independent functions we will show below that under suitable conditions the pullback and forwards asymptotic behaviour of the solutions of (6.1) are described in terms of suitable functions $\phi^{\pm}(x)$ which can be characterised as solutions of some elliptic problems.

An analogous result will be proved for the case where C and D converge to periodic functions.

Theorem 6.1 *Let $X = L^q(\Omega)$, $1 < q < \infty$ or $X = C(\overline{\Omega})$. Suppose also that the evolution operator associated with $\Delta + C(t,x)$ is exponentially stable.*
i) Assume that there exists a $C^- \in L^p(\Omega)$ for some $p > N/2$, such that, for every $T < \infty$,

$$C - C^- \in L^\sigma((-\infty, T), L^p(\Omega))$$

with $p > \frac{N\sigma'}{2}$, and

$$\lim_{t\to-\infty} \|C(t) - C^-\|_{L^p(\Omega)} = 0$$

if $\sigma = \infty$.

Also, assume that there exists a $D^- \in L^r(\Omega)$ such that, for every $T < \infty$,

$$D - D^- \in L^\sigma((-\infty, T), L^r(\Omega))$$

with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, and

$$\lim_{t\to-\infty} \|D(t) - D^-\|_{L^r(\Omega)} = 0.$$

Then there exists a unique solution ϕ^- of

$$\left\{\begin{array}{rcl} -\Delta\phi^- &=& C^-(x)\phi^- + D^-(x) \\ \phi^-_{|\partial\Omega} &=& 0 \end{array}\right. \tag{6.2}$$

and (6.1) has a pullback attractor given by the unique complete trajectory for (6.1) $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ which satisfies $\phi(t) \to \phi^-$ in X as $t \to -\infty$.

ii) Assume that C and D satisfy the assumptions in Theorem 5.3. In addition assume that there exists a $C^+ \in L^p(\Omega)$ for some $p > N/2$, such that for $-\infty < T$,

$$C \in \mathcal{D}_\beta(I\!R, L^p(\Omega)) \quad \textit{and} \quad C - C^+ \in L^\sigma((T,\infty), L^p(\Omega))$$

with $p > \frac{N\sigma'}{2}$, *and*

$$\lim_{t\to\infty} \|C(t) - C^+\|_{L^p(\Omega)} = 0$$

if $\sigma = \infty$.

Also, assume that there exists a $D^+ \in L^r(\Omega)$ *such that, for* $-\infty < T$,

$$D \in \mathcal{D}_\beta(I\!R, L^r(\Omega)) \quad \text{and} \quad D - D^+ \in L^\sigma((T,\infty), L^r(\Omega))$$

with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma'+2q} < r \leq \infty$, $\sigma = 1$ *and* $q \leq r \leq \infty$ *or* $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, *and*

$$\lim_{t\to\infty} \|D(t) - D^+\|_{L^r(\Omega)} = 0.$$

Then there exists a unique solution ϕ^+ *of*

$$\left\{\begin{array}{rcl} -\Delta\phi^+ &=& C^+(x)\phi^+ + D^+(x) \\ \phi^+_{|\partial\Omega} &=& 0 \end{array}\right. \tag{6.3}$$

and for every bounded set $B \subset X$, *we have* $v(t,s,u_0) \to \phi^+$ *in* X *as* $t \to \infty$, *uniformly for* $u_0 \in B$. *Moreover, there exists a pullback attractor* $\mathcal{A}$ *given by* $\mathcal{A}(t) = \phi(t)$ *for all* $t \in I\!R$ *where* $\phi(t)$ *is the unique complete trajectory for (6.1) which satisfies* $\phi(t) \to \phi^+$ *as* $t \to \infty$.

Proof. From Corollary 4.8 we have that $C^\pm$ are such that the semigroups generated by $\Delta + C^\pm(x)$ have exponential decay. Thus, problems (6.2) and (6.3) have unique solutions $\phi^\pm$. Take then any $v_0 \in X$ and let $v(t,s,v_0)$ be the unique solution of (6.1). Then $w = v(t,s,v_0) - \phi^\pm$ satisfies

$$\left\{\begin{array}{rcl} w_t - (\Delta + C(t,x))w &=& D(t,x) + (\Delta + C(t,x))\phi^\pm = \tilde{D}^\pm(t,x) \\ w(s) &=& v_0 - \phi^\pm \\ w_{|\partial\Omega} &=& 0 \end{array}\right. \tag{6.4}$$

where

$$\begin{array}{rcl} \tilde{D}^\pm(t,x) &=& D(t,x) + [\Delta + C(t,x)]\phi^\pm \\ &=& (D(t,x) - D^\pm(x)) + (C(t,x) - C^\pm(x))\phi^\pm. \end{array}$$

Note that, by elliptic regularity, $D^\pm \in L^r(\Omega)$ implies that $\phi^\pm \in L^s(\Omega)$ for all s such that $\frac{1}{r} - \frac{2}{N} < \frac{1}{s}$ and then, for each t, $(C(t) - C^\pm)\phi^\pm \in L^m(\Omega)$ with $\frac{1}{m} = \frac{1}{s} + \frac{1}{p} > \frac{1}{r} - \frac{2}{N} + \frac{1}{p}$ and since $p > N/2$ we can take $m > r$. Therefore, for each t, $\tilde{D}^\pm(t) \in L^r(\Omega)$.

Hence in case i), note that we have $\tilde{D}^-(t,x) = (D(t,x) - D^-(x)) + (C(t,x) - C^-(x))\phi^-$ and for all $T \in I\!R$, $\tilde{D}^- \in L^\sigma((-\infty,T], L^r(\Omega))$ with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma'+2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, and

$$\lim_{t\to-\infty} \|\tilde{D}^-(t)\|_{L^r(\Omega)} = 0.$$

Then by part i) of Corollary 5.6 we get the result.

For case ii) we have $\tilde{D}^+(t,x) = (D(t,x) - D^+(x)) + (C(t,x) - C^+(x))\phi^+$ which satisfies $\tilde{D}^+ \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$ and for all $T \in I\!R$, $\tilde{D}^+ \in L^\sigma([T,\infty), L^r(\Omega))$ with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r \le \infty$, $\sigma = 1$ and $q \le r \le \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \le \infty$, and

$$\lim_{t\to\infty} \|\tilde{D}^+(t)\|_{L^r(\Omega)} = 0.$$

Hence part ii) of Corollary 5.6 gives the result. ■

Analogously, for the case of asymptotically periodic problems, we have the following result.

Theorem 6.2 *Let $X = C(\overline{\Omega})$ or $X = L^q(\Omega)$ with $1 < q < \infty$.*

Suppose that the evolution operators associated with $\Delta + C(t,x)$ and $\Delta + C^\pm(t,x)$ are exponentially stable, where $C^\pm \in C^\alpha(I\!R, L^p(\Omega))$, with $0 < \alpha \le 1$ and some $p > N/2$, are T-periodic functions.

In addition assume that $D^\pm \in L^\infty(I\!R, L^r(\Omega))$, for some $\frac{Nq}{N+2q} < r \le \infty$ if $X = L^q(\Omega)$ or $N/2 < r \le \infty$ if $X = C(\overline{\Omega})$, are T-periodic functions.

Define $\phi^\pm(t)$ as the unique complete trajectories of the periodic problems

$$\left\{ \begin{array}{rcl} z_t^\pm - \Delta z^\pm & = & C^\pm(t,x) z^\pm + D^\pm(t,x) \\ z^\pm_{|\partial\Omega} & = & 0 \end{array} \right. \tag{6.5}$$

which are T–periodic by Corollary 5.8.
i) Assume that for every $T_0 < \infty$,

$$C - C^- \in L^\sigma((-\infty, T_0), L^p(\Omega))$$

with $p > \frac{N\sigma'}{2}$, and

$$\lim_{t\to-\infty} \|C(t) - C^-(t)\|_{L^p(\Omega)} = 0$$

if $\sigma = \infty$.

Also, assume that, for every $T_0 < \infty$,

$$D - D^- \in L^\sigma((-\infty, T_0), L^r(\Omega))$$

with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r \le \infty$, $\sigma = 1$ and $q \le r \le \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \le \infty$, and

$$\lim_{t\to-\infty} \|D(t) - D^-(t)\|_{L^r(\Omega)} = 0.$$

Then, (6.1) has a pullback attractor given by a complete trajectory for (6.1) $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ which satisfies $\phi(t) - \phi^-(t) \to 0$ in X, as $t \to -\infty$.

ii) Assume that for $-\infty < T_0$,

$$C \in \mathcal{D}_\beta(I\!R, L^p(\Omega)) \quad \text{and} \quad C - C^+ \in L^\sigma((T_0, \infty), L^p(\Omega))$$

with $p > \frac{N\sigma'}{2}$, *and*

$$\lim_{t\to\infty} \|C(t) - C^+(t)\|_{L^p(\Omega)} = 0$$

if $\sigma = \infty$.

Also, assume that, for $-\infty < T_0$,

$$D \in \mathcal{D}_\beta(\mathbb{R}, L^r(\Omega)) \quad \text{and} \quad D - D^+ \in L^\sigma((T_0,\infty), L^r(\Omega))$$

with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma'+2q} < r \leq \infty$, $\sigma = 1$ *and* $q \leq r \leq \infty$ *or* $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, *and*

$$\lim_{t\to\infty} \|D(t) - D^+(t)\|_{L^r(\Omega)} = 0.$$

Then, for every bounded set $B \subset X$, *we have* $v(t,s,u_0) - \phi^+(t) \to 0$ *in* X, *as* $t \to \infty$, *uniformly for* $u_0 \in B$. *Moreover, there exists a pullback attractor* $\mathcal{A}$ *given by* $\mathcal{A}(t) = \{\phi(t)\}$ *for all* $t \in \mathbb{R}$ *where* $\phi(t)$ *is the unique complete trajectory for (6.1) which satisfies* $\phi(t) - \phi^+(t) \to 0$ *as* $t \to \infty$.

Proof. Since the evolution operators associated with $\Delta + C^\pm(t,x)$ are exponentially stable we know, from Corollary 5.8, that problems ($6.5^\pm$) have unique complete trajectories $\phi^\pm$ which are T–periodic. Take any $v_0 \in X$ and let $v(t,s,v_0)$ be the unique solution of (6.1). Then $w = v(t,s,v_0) - \phi^\pm(t)$ satisfies

$$\left\{ \begin{array}{rcl} w_t - (\Delta + C(t,x))w &=& \tilde{D}^\pm(t,x) \\ w(s) &=& v_0 - \phi^\pm(s) \\ w_{|\partial\Omega} &=& 0 \end{array} \right. \tag{6.6}$$

where

$$\tilde{D}^\pm(t,x) = (D(t,x) - D^\pm(t,x)) + (C(t,x) - C^\pm(t,x))\phi^\pm(t).$$

Note that, by parabolic regularity, for each t, $D^\pm(t) \in L^r(\Omega)$ implies that $\phi^\pm(t) \in L^s(\Omega)$ for all s such that $\frac{1}{r} - \frac{2}{N} < \frac{1}{s}$ and then, for each t, $(C(t) - C^\pm(t))\phi^\pm(t) \in L^m(\Omega)$ with $\frac{1}{m} = \frac{1}{s} + \frac{1}{p} > \frac{1}{r} - \frac{2}{N} + \frac{1}{p}$ and since $p > N/2$ we can take $m > r$. Therefore, for each t, $\tilde{D}^\pm(t) \in L^r(\Omega)$.

Hence in case i), note that we have

$$\tilde{D}^-(t,x) = (D(t,x) - D^-(t,x)) + (C(t,x) - C^-(t,x))\phi^-(t)$$

and for all $T_0 \in \mathbb{R}$, $\tilde{D}^- \in L^\sigma((-\infty,T_0], L^r(\Omega))$ with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma'+2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, and

$$\lim_{t\to-\infty} \|\tilde{D}^-(t)\|_{L^r(\Omega)} = 0.$$

Then from part i) of Corollary 5.6 we get the result.

For the case ii) we have

$$\tilde{D}^+(t,x) = (D(t,x) - D^+(t,x)) + (C(t,x) - C^+(t,x))\phi^+(t)$$

which satisfies $\tilde{D}^+ \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$ and for $T_0 \in I\!R$, $\tilde{D}^+ \in L^\sigma([T_0, \infty), L^r(\Omega))$ with either $1 < \sigma < \infty$, $\frac{N\sigma' q}{N\sigma' + 2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N+2q} < r \leq \infty$, and

$$\lim_{t \to \infty} \|\tilde{D}^+(t)\|_{L^r(\Omega)} = 0.$$

Hence part ii) of Corollary 5.6 gives the result. ∎

7 The nonlinear problem

We will now consider the nonlinear non-autonomous problem

$$\begin{cases} u_t - \Delta u &= f(t,x,u), \quad \text{in} \quad \Omega, \qquad t > s \\ u(s) &= u_0 \in X \\ u_{|\partial\Omega} &= 0, \qquad\qquad \text{on} \quad \partial\Omega \end{cases} \tag{7.1}$$

where $X = C(\overline{\Omega})$ and $f(t,x,u)$ is continuous, locally Hölder in t and locally Lipschitz in u. Hence, from the results quoted in Section 2, (7.1) has a unique locally defined smooth solution for every $u_0 \in X$.

Suppose that f satisfies the dissipativity condition

$$uf(t,x,u) \leq C(t,x)u^2 + D(t,x)|u| \tag{7.2}$$

with $C \in C^\alpha(I\!R, L^p(\Omega))$, for some α with $0 < \alpha \leq 1$ and some $p > N/2$, and that $D \geq 0$ with values in $L^r(\Omega)$. Our key assumption is that the evolution operator associated with $\Delta + C(t,x)$, which we continue to denote by $U_C(t,s)$, is exponentially stable.

To ensure that the solutions of (7.1) are globally defined forward in time we only need to prove that the solutions of (7.1) are bounded for all $t \geq s$, which will follow from the dissipativity property of f (7.2) (see (7.6) in the proof of Lemma 7.2 below).

Then the solutions of the problem (7.1) define evolution operator given by

$$U(t,s)u_0 = u(t,s,x;u_0) \quad t \geq s$$

and this operator is order-preserving by Theorem 2.13 (see also [5]).

The next result guarantees the existence of two extremal complete trajectories for (7.1) which are 'attracting' in a certain sense. A related result can be found in Langa and Suárez [15] for abstract evolution operators given the assumption either of the existence of a pair of sub- and super-trajectories, or the existence of a pullback attractor for the system embedded in an order interval. In the first case, the authors prove the existence of extremal complete trajectories between the sub and the super-trajectory (see Remark 7.5 below for more details).

Theorem 7.1 *Suppose that $X = C(\overline{\Omega})$ and that f is continuous, locally Hölder in t, locally Lipschitz in u, and satisfies (7.2) with $C \in C^\alpha(I\!R, L^p(\Omega))$ for some α with $0 < \alpha \leq 1$ and some $p > N/2$.*

Assume in addition that the evolution operator associated with $\Delta + C(t,x)$ is exponentially stable with exponent β and that $D(t,x)$ is such that the linear problem (5.1) has a pullback attractor in the class $\mathcal{D}_\beta$, given by a complete trajectory $\{\phi(t)\}_t$, e.g. as in Theorems 5.3, 6.1 or 6.2.

Then the solutions of (7.1) are global and we can define $U(t,s)$, the evolution operator defined by the solutions of (7.1), for all $t \geq s$.

Moreover, there exist two extremal complete trajectories that are elements of $\mathcal{D}_\beta$, φ_M and φ_m, maximal and minimal respectively, in the sense that any other complete trajectory for U in $\mathcal{D}_\beta$, ψ, satisfies $\varphi_m(t) \leq \psi(t) \leq \varphi_M(t)$ for all $t \in I\!R$.

The order interval $I(t) = [\varphi_m(t), \varphi_M(t)]$ is forward invariant and attracts the dynamics of the system uniformly in the pullback sense, i.e. for all $t \in I\!R$ we have

$$\varphi_m(t,x) \leq \liminf_{s\to-\infty} u(t,s,x;v_s) \leq \limsup_{s\to-\infty} u(t,s,x;v_s) \leq \varphi_M(t,x) \tag{7.3}$$

uniformly in $x \in \overline{\Omega}$ for all v_s with $v_s \in B(s)$, where $\{B(s)\}_s \in \mathcal{D}_\beta$. Moreover, $\varphi_M(t)$ is globally asymptotically stable from above in the pullback sense, i.e. for all $v \in \mathcal{D}_\beta(I\!R, X)$, $v \geq \varphi_M$ we have

$$\lim_{s\to-\infty} u(t,s;v_s) = \varphi_M(t).$$

Similarly, $\varphi_m(t)$ is globally asymptotically stable from below in the pullback sense.

As a consequence, there exists a pullback attractor for U with respect to $\mathcal{D}_\beta$, denoted by $\mathcal{A} = \{\mathcal{A}(t)\}_t$, and

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)] \quad \textit{for all } t \in I\!R.$$

Moreover, $\varphi_m(t), \varphi_M(t) \in \mathcal{A}(t)$ for all $t \in I\!R$.

We prove the theorem in two steps. First, we prove that the solutions of (7.1) are asymptotically bounded by the unique complete trajectory of the linear problem (5.4) (with C and D from (7.2)) and then we prove Theorem 7.1 proper.

Lemma 7.2 *Under the assumptions of Theorem 7.1, the solutions of (7.1) are global and satisfy*

$$\limsup_{s\to-\infty} |u(t,s,x;v_s)| \leq \phi(t,x) \quad \textit{for all } t \in I\!R \tag{7.4}$$

uniformly in $x \in \overline{\Omega}$ for every v_s with $v_s \in B(s)$ where $\{B(s)\}_s \in \mathcal{D}_\beta$, where $\phi(t)$ is the pullback attractor in the class $\mathcal{D}$, given by a complete trajectory $\{\phi(t)\}_t$ for the problem

$$\left\{ \begin{array}{rcll} v_t - \Delta v &=& C(t,x)v + D(t,x), & \textit{in} \quad \Omega, \quad t > s \\ v_{|\partial\Omega} &=& 0, & \textit{on} \quad \partial\Omega \end{array} \right. \tag{7.5}$$

Moreover, the order interval $[-\phi(t), \phi(t)]$ is forward invariant for (7.1).

Remark 7.3 *In particular, since the limit in (7.4) is uniform in $x \in \overline{\Omega}$, the order interval $[-\phi(t) - \delta, \phi(t) + \delta]$ is pullback absorbing at time t for the solutions of (7.1). In fact, for any fixed $t \in I\!R$ and $\delta > 0$, there exists a time s_0 such that*

$$-\phi(t) - \delta \leq u(t,s;v_s) \leq \phi(t) + \delta$$

for all $s < s_0$.

Proof of Lemma 7.2. We know that there exists a unique bounded complete trajectory for (7.5) which we denote by $\phi(t)$. Furthermore, $\mathcal{A} = \{\mathcal{A}(t)\}_t = \{\phi(t)\}_t$ is the pullback attractor for this problem. Given $u_0 \in X$, let $v(t, s, x; u_0)$ be the solution at time t of the problem (7.5) starting from u_0 and $u(t, s, x; u_0)$ the solution at time t of (7.1) with initial data u_0. We fix $\{B(s)\}_s \in \mathcal{D}$ and $v_s \in B(s)$. By (7.2) and the comparison principle, see Theorem 2.13,

$$|u(t, s, x; v_s)| \leq v(t, s, x; |v_s|) \tag{7.6}$$

while both solutions exist. In particular, from here we get bounds on the solution of (7.1) on finite time intervals and hence the solution is defined for all $t > s$.

Now, we have

$$\lim_{s \to -\infty} v(t, s, x; |v_s|) = \phi(t, x)$$

in $C(\overline{\Omega})$. Thus

$$\limsup_{s \to -\infty} u(t, s, x; v_s) \leq \phi(t, x)$$

uniformly in $x \in \overline{\Omega}$ and $v_s \in B(s)$. Arguing with $-v(t, s, x; -|v_s|)$ instead of $v(t, s, x; |v_s|)$, we have

$$\limsup_{s \to -\infty} |u(t, s, x; v_s)| \leq \phi(t, x)$$

for all $\{v_s\}$ in $\{B(s)\}_s \in \mathcal{D}$.

Finally, notice that if $\{u_s\}$ is such that $u_s \leq \phi(s)$ then, by the comparison principle,

$$u(t, s; u_s) \leq u(t, s; \phi(t)) \leq v(t, s; \phi(t)) = \phi(t) \quad \text{for all} \quad t > s.$$

Taking now $\{u_s\}$ such that $u_s \geq -\phi(s)$ we have

$$u(t, s; u_s) \geq u(t, s; -\phi(s)) \geq -v(t, s; -\phi(s)) = -\phi(t) \quad \text{for all} \quad t > s.$$

Thus,

$$U(t, s)[-\phi(s), \phi(s)] \subset [-\phi(t), \phi(t)],$$

i.e., $\{[-\phi(t), \phi(t)]\}_t$ is forward invariant for U. ∎

Using this lemma we can now prove Theorem 7.1.

Proof of Theorem 7.1. Let $U(t, s)$ be the nonlinear evolution operator associated with (7.1). We know that this operator is order-preserving by Theorem 2.13. Moreover, $\phi(t)$ is a super-trajectory since the solution of (7.1) starting from $\phi(s)$ satisfies, by (7.6),

$$u(t, s; \phi(s)) \leq v(t, s; \phi(s)) = \phi(t)$$

where v is the solution of the linear problem (7.5). For the last equality we have used the fact that $\phi(t)$ is a complete trajectory for the linear problem.

Next we prove that, since $\phi(t)$ is a super-trajectory of the nonlinear problem, $U(t,s)\phi(s)$ is monotonic as $s \to -\infty$ and $U(t,s)\phi(s) \to \varphi_M(t)$ as $s \to -\infty$ uniformly in x for all $t \in I\!R$. Indeed, for a fixed $t \in I\!R$ we have, from the definition of a super-trajectory

$$U(t,s)\phi(s) \leq \phi(t) \quad \text{for all } s \leq t$$

in particular,

$$U(s+\epsilon,s)\phi(s) \leq \phi(s+\epsilon) \quad \text{for all } \epsilon > 0.$$

Thus, by monotonicity,

$$U(t,s)\phi(s) = U(t,s+\epsilon)U(s+\epsilon,s)\phi(s) \leq U(t,s+\epsilon)\phi(s+\epsilon).$$

Therefore, $\{U(t,s)\phi(s)\}_s$ is non-increasing as $s \to -\infty$. Moreover, it is bounded from below (by $-\phi(t)-\delta$ for some $\delta > 0$, see Remark 7.3). Thus, it converges pointwise to a certain bounded function that we denote by $\varphi_M(t) \in L^\infty(\Omega)$.

Notice that we can write $U(t,s)\phi(s) = U(t,t-1)U(t-1,s)\phi(s)$, where

$$\{U(t-1,s)\phi(s)\}_{s\leq s_0}$$

is bounded (for some s_0). Thus, by the smoothing property of the evolution operator (see Theorem 2.12 and subsequent remarks) we know that $\{U(t,s)\phi(s)\}_{s\leq s_0} = U(t,t-1)\{U(t-1,s)\phi(s)\}_{s\leq s_0}$ is pre-compact. So, $U(t,s)\phi(s) \to \varphi_M(t) \in C_0(\overline{\Omega})$ uniformly in Ω as $s \to -\infty$.

The continuity of $U(t,s)$ implies that $\varphi_M(t)$ is a complete trajectory for (7.1). Indeed,

$$\begin{aligned} U(t,s)\varphi_M(s) &= U(t,s)\lim_{r\to-\infty} U(s,r)\phi(r) \\ &= \lim_{r\to-\infty} U(t,s)U(s,r)\phi(r) = \lim_{r\to-\infty} U(t,r)\phi(r) \\ &= \varphi_M(t). \end{aligned} \tag{7.7}$$

We now prove that, asymptotically in the pullback sense, all trajectories of equation (7.1) lie below φ_M, uniformly in x. Fix $\{B(s)\}_s \in \mathcal{D}$ and $v_s \in B(s)$. From (7.6) we have

$$u(t,s;v_s) \leq v(t,s;|v_s|) \quad \text{for all } t \geq s.$$

Letting the evolution operator act on both sides, we have by monotonicity

$$U(r,t)u(t,s;v_s) = u(r,s;v_s) \leq U(r,t)v(t,s;|v_s|) \quad \text{for all } r \geq t \geq s.$$

Taking limits as s goes to $-\infty$ we have, for each $x \in \Omega$,

$$\limsup_{s\to-\infty} u(r,s,x;v_s) \leq U(r,t)\phi(t,x) \tag{7.8}$$

for all $t \leq r$, where we have used the continuity of $U(t,s)$. Letting t tend to $-\infty$ we have

$$\limsup_{s\to-\infty} u(r,s,x;v_s) \leq \varphi_M(r,x),$$

as claimed. The maximality of $\varphi_M(t)$ follows from this inequality.

From inequality (7.8) we obtain the global asymptotic stability from above in the pullback sense for the maximal complete trajectory. Indeed, let $r \in I\!R$ be fixed and assume $v_s \geq \varphi_M(s)$ for all s. Then, by monotonicity, for $x \in \Omega$,

$$\varphi_M(r,x) = u(r,s;\varphi_M(s)) \leq u(r,s,x;v_s)$$

for all $s \leq r$. Now, taking limits as $s \to -\infty$ and using (7.8) we have

$$\varphi_M(r,x) \leq \liminf_{s\to-\infty} u(r,s,x;v_s) \leq \limsup_{s\to-\infty} u(r,s,x;v_s) \leq U(r,t)\phi(t,x)$$

for all $t \leq r$. Taking now limits as $t \to -\infty$ we obtain

$$\varphi_M(r,x) \leq \liminf_{s\to-\infty} u(r,s,x;v_s) \leq \limsup_{s\to-\infty} u(r,s,x;v_s) \leq \varphi_M(r,x)$$

Therefore, $u(r,s;v_s) \to \varphi_M(r)$ as $s \to -\infty$ which proves the asymptotic stability from above.

The result for the minimal complete trajectory is proved in an analogous way.

To prove the forward invariance of $I(t)$, take $\{u_s\}_s$ such that

$$\varphi_m(s) \leq u_s \leq \varphi_M(s)$$

for all $s \in I\!R$. Then, letting the evolution operator act, we have, by the comparison principle,

$$\varphi_m(t) = U(t,s)\varphi_m(s) \leq u(t,s;u_s) \leq U(t,s)\varphi_M(s) = \varphi_M(t)$$

So, $U(t,s)I(s) \subset I(t)$, i.e., $I(t)$ is forward invariant.

We now show the existence of the pullback attractor $\mathcal{A}$. As we pointed out in Remark 7.3 the time-dependent order interval $[-\phi(t)-\delta, \phi(t)+\delta]$ in $C(\overline{\Omega})$ is an absorbing set at time t for $U(t,s)$ in the pullback sense. Let

$$J(t) = \overline{U(t,t-1)[-\phi(t-1)-\delta, \phi(t-1)+\delta]}.$$

From the smoothing effect of $U(t,s)$ we know that $J(t)$ is compact in $C(\overline{\Omega})$. Moreover, $J(t)$ is a pullback absorbing set. Thus, from Theorem 2.7 there exists a pullback attractor $\mathcal{A}$ for $U(t,s)$.

Finally, it is clear that $\mathcal{A}(t) \subset I(t)$ and $\varphi_m(t), \varphi_M(t) \in \mathcal{A}(t)$ for all $t \in I\!R$. ■

Remark 7.4 *Observe that if C and D satisfy the assumptions in Corollary 5.6 then $\phi(t)$ converges to 0 in $X = C(\overline{\Omega})$ as $t \to \infty$ or $t \to -\infty$. In particular the same holds true for the solutions of the nonlinear problem (7.1).*

On the other hand, if C and D satisfy the assumptions in Theorem 6.1 or Corollary 5.8 or Theorem or 6.2 then $\phi(t)$ is asymptotically constant or periodic.

Remark 7.5 *Notice that we obtain global information about the dynamics of problem (7.1) as well as uniform properties for the asymptotic behaviour of their solutions. Namely, we obtain information about the dynamics of the problem in the whole phase space (in fact, in the basin of attraction $\mathcal{D}_\beta$) and the uniform convergence of solutions to the order interval defined by the two extremal complete trajectories.*

Moreover, from the proof above it is easy to extend the results obtained for the particular case of problem (1.1) to the general framework of order-preserving evolution operators as considered in Langa and Suárez [15]. Their paper gives a related result (their Theorem 3.4) that guarantees the existence of extremal complete trajectories between an ordered couple of sub- and super-trajectories.

We now consider the T-periodic problem associated with (1.1), i.e., we suppose that $f(t,x,u)$ is a T-periodic function. This kind of problem has been widely studied (see e. g. Danners and Koch–Medina [9] or Hess [13]). Moreover, we suppose that f satisfies (7.2) with T-periodic functions $C(t,x)$ and $D(t,x)$.

A simple application of our main result, Theorem 7.1, gives the existence of extremal T-periodic solutions for the nonlinear problem.

Corollary 7.6 *In the T-periodic equation case, the extremal solutions of (1.1) given in Theorem 7.1 are T-periodic. In particular, there exist two T-periodic extremal solutions of (1.1).*

Proof. From Corollary 5.8 we know that the unique complete trajectory of (7.5) in $\mathcal{D}_\beta$ is T-periodic. We only have to check that the maximal complete trajectory from Theorem 7.1 is T-periodic. But, we know that

$$U(t,s)\phi(s) \to \varphi_M(t) \quad \text{as} \quad s \to -\infty.$$

We can use now that $\phi(s)$ and $f(t,x,u)$ are T-periodic functions and then

$$U(T+t,T+s)\phi(T+s) = U(t,s)\phi(s)$$

where the left-hand side of the equality tends to $\varphi_M(T+t)$ as $s \to -\infty$ and the right-hand side tends to $\varphi_M(t)$ as $s \to -\infty$. So, $\varphi_M(t) = \varphi_M(T+t)$ and φ_M is T-periodic as we wanted to prove. The same argument applies for $\varphi_m(t)$. ■

Remark 7.7 *To study this type of equation it is usual to consider the Poincaré map associated with (1.1): $S = U(T,0)$, where $U(t,s)$ is the evolution operator given by the solutions of (1.1) (see e.g. Hess [13]).*

In this case the evolution operator generated by $\Delta + C(t,x)$ is exponentially stable if and only if the Poincaré map S_C associated with this operator has spectral radius less than one (see Hess [13]).

In such a case, this implies that $1 \in \rho(S_C)$ (the resolvent of S_C) and by Proposition 6.9 in Danners and Koch–Medina [9] we obtain the existence of a unique periodic solution for the linear problem as stated in Corollary 5.8. However, we have given another proof that follows straightforwardly from the fact that we are dealing with equations with periodic coefficients.

8 Asymptotic behaviour forwards in time

In order to study the asymptotic behaviour forwards in time of non-autonomous equation, natural concepts are those of asymptotically compact evolution operators and uniform attractors as defined by Haraux [11] and by Chepyzhov and Vishik [7]:

Definition 8.1
i) We say that $U(t,s)$ is asymptotically compact *at $\sigma \in I\!R$ if there exists a compact set $K_\sigma \subset X$, which may depend on σ, that attracts bounded sets of X forwards in time for the one parameter family $U_\sigma(t,0) = U(t+\sigma,\sigma)$, $t \geq 0$.*

We say that $U(t,s)$ is asymptotically compact *if it is asymptotically compact for all $s \in I\!R$.*
ii) We say that $U(t,s)$ is uniformly asymptotically compact *if there exist a compact subset $K \subset X$ such that for any bounded set $B \subset X$*

$$\lim_{t\to\infty} \sup_{s\in I\!R} \mathrm{dist}(U(t+s,s)B, K) = 0.$$

iii) We say that a compact set $\mathcal{F}_U$ is the uniform attractor *for $U(t,s)$ if it is the minimal compact set satisfying ii) above.*

As we will show below, in some cases, this notion of attractor could be too strong when studying the asymptotic behaviour forwards in time of a non-autonomous equation. For this reason we now construct another kind of attractor giving information about the forward dynamics.

Consider an asymptotically compact evolution operator $U(t,s)$. Then for a fixed $s \in I\!R$ by definition there exists a compact set K_s that attracts bounded sets of initial data forwards in time from the initial time s.

Then, in a standard way, given a bounded set $B \subset X$ we define the ω-limit set from time s as

$$\omega_s(B) = \{u \in X \ : \ \exists\, t_n \uparrow \infty,\ v_n \in B,\ \text{s.t.} \quad U(t_n,s)v_n \to u \text{ as } n \to \infty\}.$$

With this, it is clear that $\mathcal{F}_s = \omega_s(K_s) \subset K_s$ is the minimal compact set that attracts bounded sets of X forwards in time for the one parameter family $U_s(t,0) = U(t+s,s)$, $t \geq 0$.

It is not difficult to show, see Haraux [11], that there exists a monotone relationship between this family of compact sets. Namely,

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \text{for all} \quad s < t.$$

An interesting situation occurs therefore when the compact set K_σ in Definition 8.1 is independent of σ, that is, there exists a compact set $K \subset X$ such that for all $s \in I\!R$ and any bounded set $B \subset X$,

$$\lim_{t\to\infty} \mathrm{dist}_X(U(t,s)B, K) = 0.$$

In such a case we get the existence of a forward attractor in the sense of Definition 2.6 for the problem (1.1) that can be characterised as

$$\mathcal{F} = \overline{\bigcup_{s\in I\!R} \mathcal{F}_s} \subset K$$

where $\mathcal{F}_s = \omega_s(K) \subset K$.

To see how these ideas apply to (7.1) we take $X = C(\overline{\Omega})$. Suppose that f is continuous, locally Hölder in t, locally Lipschitz in u, and satisfies (7.2) with $C \in C^\alpha(I\!R, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, and D is such that there exists a unique complete trajectory $\phi \in L^\infty(I\!R, X)$ for the linear problem (5.4) satisfying

$$\|v(t,s;u_s) - \phi(t)\|_X \leq M\mathrm{e}^{-\beta(t-s)}$$

for all $u_s \in L^\infty(I\!R, X)$ and $M = M(\{u_s\}_s)$ (see Theorems 5.3, 6.1 and 6.2 for such conditions on C and D). Then, given a bounded set B in X and $\epsilon > 0$ there exists a time $T = T(\epsilon)$ such that for every $u_0 \in B$,

$$\|v(t,s;u_0) - \phi(t)\|_X < \epsilon \quad \text{for all} \quad t - s \geq T. \tag{8.1}$$

In particular, for $R = \|\phi\|_{L^\infty(X)} + 1$,

$$\|v(t,x;u_0)\|_X \leq R \quad \text{for all} \quad t - s \geq T.$$

Moreover, we know that $|u(t,s;u_0)| \leq v(t,s;|u_0|)$ for all $t > s$. Hence, for all $t - s \geq T$,

$$\|u(t,s;u_0)\|_X \leq \|v(t,x;u_0)\|_X \leq R, \tag{8.2}$$

i.e, for all $s \in I\!R$, $B_X(0,R)$ is an absorbing set forward in time for $U(t,s)$. Furthermore, by the smoothing property of the evolution operator, the solutions of the nonlinear problem enter some ball in a space Y compactly embedded in X, $B_Y(0,R_Y) \subset B_X(0,R)$. Thus, $K = \overline{B_Y(0,R_Y)} \subset \overline{B_X(0,R)}$ (where the closure is taken in X) is a forward absorbing compact set not depending on s.

It is now clear that in this case the sets $\mathcal{F}_s$ and $\mathcal{F}$ as defined above can also be described as

$$\mathcal{F}_s = \overline{\bigcup_{B\subset\mathcal{B}(X)} \omega_s(B)}$$

where $\mathcal{B}(X)$ denotes the set of all bounded sets of X and

$$\mathcal{F} = \overline{\bigcup_{s\in I\!R} \omega_s(B_X(0,R))}.$$

Remark 8.2 *Notice that the construction above can be carried out for (7.1) without the boundedness assumption on $\phi(t)$. Namely, everything above remains true if we allow $\phi \in \mathcal{D}_\gamma$ for some $0 < \gamma < \beta$ since, in that case,*

$$\begin{aligned}\|v(t,s;u_s) - \phi(t)\|_X &\leq M\mathrm{e}^{-\beta(t-s)}\|u_s - \phi(s)\| \leq M_1\mathrm{e}^{-\beta(t-s)}\mathrm{e}^{-\gamma \mathrm{s}} \\ &\leq M_1\mathrm{e}^{-\gamma \mathrm{t}}\mathrm{e}^{-(\beta-\gamma)(t-s)} = M_1\mathrm{e}^{-\gamma \mathrm{t}}\mathrm{e}^{-(\beta-\gamma)(t-s)}.\end{aligned}$$

And for $t > 0$ we have

$$\|v(t,s;u_s) - \phi(t)\|_X \leq M_1 \mathrm{e}^{-(\beta-\gamma)(t-s)}.$$

Thus, since $\beta - \gamma > 0$, given $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such a that for all $t - s > T$, $t > 0$,

$$\|v(t,s;u_s) - \phi(t)\|_X < \epsilon$$

and now the argument follows as above.

We now state a result about the structure of the forward attractor $\mathcal{F}$ for (7.1). For this, let $\{B(t)\}_t$ be a family that is invariant under $U(t,s)$, i.e. $U(t,s)B(s) = B(t)$ for all $t > s$. We denote by $\omega(B)$ the set

$$\omega(B) = \{u \in X \ : \ \exists\, t_n \uparrow \infty,\ v_n \in B(t_n),\ \text{s.t.} \quad v_n \to u \text{ as } n \to \infty\}.$$

Proposition 8.3 *If $\{B(t)\}_t$ is an invariant set for $U(t,s)$ then*

$$\omega(B) \subset \mathcal{F},$$

where $\mathcal{F}$ is the forwards attractor of $U(t,s)$ as defined above.

In particular, if $\mathcal{A}$ is a pullback attractor for $U(t,s)$ then $\omega(\mathcal{A}) \subset \mathcal{F}$. Moreover, if $\psi(t)$ is a complete trajectory for $U(t,s)$ then $\omega(\psi) \subset \mathcal{F}$.

Proof. From (8.2) and the smoothing effect, if we take $r - s > T$ as above, we have

$$B(r) = U(r,s)B(s) \subset K$$

where $K = \overline{B_Y(0,R_Y)} \subset \overline{B_X(0,R)}$. Thus, for all $t > r$,

$$B(t) = U(t,r)B(r) \subset U(t,r)K.$$

Taking limits as t goes to $+\infty$ we have that

$$\omega(B) \subset \omega_r(K) \subset \mathcal{F}_r \subset \mathcal{F}$$

where $\omega_r(K)$ is the ω-limit set from time r defined above. ∎

Remark 8.4 *As a consequence, the attractor $\mathcal{F}$ can be defined in cases where the uniform attractor cannot, since boundedness of ϕ is needed in the definition of the uniform attractor. Indeed, it can be shown (see Chepyzhov and Vishik [7]) that*

$$\mathcal{F}_U = \overline{\bigcup_{t \in \boldsymbol{R}} \mathcal{A}(t)}$$

where $\mathcal{A}$ is the pullback attractor attracting bounded sets.

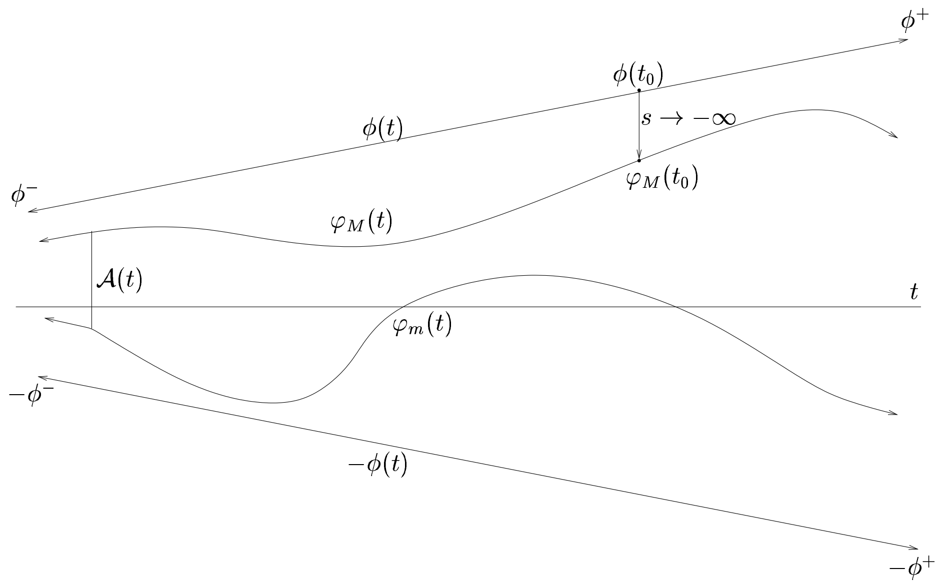

Figure 1: Behaviour of a nonlinear non-autonomous evolution operator bounded by asymptotically autonomous linear ones.

Let us consider a linear problem whose unique complete trajectory is unbounded backward in time. Then, the only set that satisfies Definition 8.1 is

$$\mathcal{F}_U = \overline{\bigcup_{t \in \boldsymbol{R}} \phi(t)}$$

which is an unbounded set. Therefore $\mathcal{F}_U$ is not compact. However, the (non-uniform) forward attractor

$$\mathcal{F} = \omega(\phi) = \{u \in X \ : \ \exists t_n \uparrow \infty, \ u_n = \phi(t_n) \ s.t. \ u_n \to u \ as \ n \to \infty\}$$

still exists.

As a particular case we consider now the case of asymptotically autonomous problems. In fact, suppose that f satisfies the dissipativity condition (7.2) with $C(t,x) \to C^+(x)$, $D(t,x) \to D^+(x)$ as $t \to \infty$ as in Theorem 6.1. From the previous results we have, for every $t \in I\!R$,

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)] \subset [-\phi(t), \phi(t)] \subset \overline{B_X(0,R)}$$

and all these sets are forwards invariant. Hence denoting $I = \{I(t)\}_t = [\varphi_m(t), \varphi_M(t)]$, we have from Theorem 6.1 and Proposition 8.3,

$$\omega(\mathcal{A}) \subset \omega(I) \subset \mathcal{F} \subset [-\phi^+, \phi^+].$$

Now observe that since $\{\varphi_m(t)\}_t$, $\{\varphi_M(t)\}_t$ are relatively compact complete trajectories we can consider $\omega(\varphi_M)$ and $\omega(\varphi_m)$, which are compact connected sets of X. Thus,

$$\omega(\mathcal{A}) \subset \omega(I) \subset [\psi_m, \psi_M] \cap \mathcal{F} \subset [-\phi^+, \phi^+]$$

where

$$\psi_m(x) = \inf_{v\in\omega(\varphi_m)} v(x) \quad \text{and} \quad \psi_M(x) = \sup_{v\in\omega(\varphi_M)} v(x).$$

Note that a completely analogous analysis can be carried out backwards in time when $C(t,x) \to C^-(x)$ and $D(t,x) \to D^-(x)$ as $t \to -\infty$ by considering the α–limit set of an invariant set,

$$\alpha(B) = \{u \in X \ : \ \exists\ t_n \uparrow -\infty,\ v_n \in B(t_n),\ \text{s.t.} \quad v_n \to u \text{ as } n \to \infty\}.$$

Continuing with the forward behaviour, assume in addition that $f(t,x,u) \to g(x,u)$ as t goes to ∞, uniformly in $x \in \overline{\Omega}$, for u in bounded sets of X. Then, it is shown in Mischaikow et al. [17] that the evolution operator associated with equation (7.1) is asymptotically autonomous in the sense of Thieme (see Thieme [22]). Thus, the ω-limit set of any point $\omega_s(u_0)$ is invariant under the semiflow $S(t)$ defined by the solutions of the limit equation

$$\left\{\begin{array}{rcl} v_t - \Delta v &=& g(x,v), \quad t>0 \\ v(0) &=& v_0 \\ v_{|\partial\Omega} &=& 0 \end{array}\right. \tag{8.3}$$

(see Theorem 2.5, p. 760 in Thieme [22]). Moreover, if the equilibria of the limit problem are isolated, the existence of a Lyapunov function for the limit problem (8.3) (see Hale [10] or Henry [12]) implies that they are not chained in a cyclic way in the sense of Definition 1.3 in Mischaikow et al. [17].

Then, from Theorem 4.2 and Corollary 4.3, p. 762, in Thieme [22] the ω-limit set of each solution of the non-autonomous problem is an equilibrium point for $S(t)$. So, it follows that

$$\omega(\varphi_m) = \{\varphi_m^\infty\},\ \omega(\varphi_M) = \{\varphi_M^\infty\} \subset \mathcal{F},$$

for some equilibria $\varphi_m^\infty \leq \varphi_M^\infty$ of the limit autonomous problem. Moreover, we have

$$\omega(\mathcal{A}) \subset \omega(I) \subset [\varphi_m^\infty, \varphi_M^\infty] \subset [-\varphi_m^+, \varphi_M^+] \subset [-\phi^+, \phi^+].$$

where φ_m^+, φ_M^+ are the extremal equilibria of the limit problem, see Rodríguez-Bernal and Vidal-López [19] and Vidal-López [23].

Even more, $\omega(\mathcal{A})$ is contained in the attractor of the limit problem (see Theorem 3.7.2, p. 45, in Hale [10] or Theorem 4.3.6, p. 96, in Henry [12]).

On the other hand, from the arguments above it is then clear that we also have

$$\mathcal{F} \subset [\varphi_m^+, \varphi_M^+]$$

since $\mathcal{F}$ can be obtained as a union of ω-limit sets of fixed bounded sets and, from Theorem 3.7.2, p. 45, in Hale [10] or Theorem 4.3.6, p. 96, in [12], it must be an invariant set for the limit problem.

9 An example: the non-autonomous logistic equation

We now consider the non-autonomous logistic equation

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& f(t,x,u), & \text{in } \Omega, \quad t > s \\ u &=& 0 & \text{on } \partial\Omega \\ u(s) &=& v_s & \end{array}\right. \tag{9.1}$$

with the model nonlinearity

$$f(t,x,u) = m(t,x)u - n(t,x)u^3 \tag{9.2}$$

where m and $n \geq 0$ are continuous and locally Hölder in t.

We will show how our techniques can be applied to this problem, although it will be clear from the analysis that much more general classes of nonlinear terms could be considered.

We start with the case in which the asymptotic dynamics of (9.1) is trivial.

Theorem 9.1 *Suppose that $n(t,x) \geq 0$ and that $m(t,x)$ is such that the evolution operator associated with $\Delta + m(t,x)$ is exponentially stable. Then $\|u(t,s;u_0)\|_{L^\infty(\Omega)} \to 0$ as $t \to +\infty$ or $s \to -\infty$ uniformly for u_0 in bounded sets of $X = C(\overline{\Omega})$.*

In particular, the pullback attractor of (9.1) is $\mathcal{A}(t) = \{0\}$ for all $t \in I\!R$ and the forward attractor is $\{0\}$.

Proof. Notice that using $n(t,x) \geq 0$ we have that $f(t,x,u)$ satisfies

$$f(t,x,u)u = m(t,x)u^2 - n(t,x)u^4 \leq m(t,x)u^2 \tag{9.3}$$

for all $t \in I\!R$.

Now, since $m(t,x)$ is such that the evolution operator associated with $\Delta + m(t,x)$ is exponentially stable, it follows from Theorem 7.1, with $C(t,x) = m(t,x)$ and $D(t,x) = 0$ and (9.3) that there exist two extremal bounded complete trajectories for (9.1). But, in this case, both are the same and equal to the trivial one (see Theorem 5.1). So the pullback and forward attractors are 0. In fact, $\phi(t) \equiv 0$ for all $t \in I\!R$. ■

Suppose now that $m(t,x)$ is such that the evolution operator associated with $\Delta + m(t,x)$ is not exponentially stable. In the following result we give conditions to have the existence of a pullback attractor.

Theorem 9.2 *Let $X = C(\overline{\Omega})$. Suppose that the evolution operator generated by $\Delta + m(t,x)$ is not exponentially stable but there exists a decomposition $m(t,x) = m_1(t,x) + m_2(t,x)$ with $m_2(t,x) \geq 0$ and*

$$m_1 \in C^\alpha(I\!R, L^p(\Omega)) \quad \textit{with} \quad 0 < \alpha \leq 1 \quad \textit{and some} \quad p > N/2.$$

such that the evolution operator associated with $\Delta + m_1(t,x)$ is exponentially stable with exponent β. Let

$$D = \left(\frac{m_2^3}{n}\right)^{1/2}$$

and suppose that either

i) $D \in \mathcal{D}_\beta(I\!R, L^r(\Omega))$ *with* $N/2 < r \le \infty$*; or*
ii) for $T < \infty$, $D \in L^\sigma((-\infty, T), L^r(\Omega))$ *with* $1 < \sigma \le \infty$ *and* $N\sigma'/2 < r \le \infty$ *if* $1 < \sigma < \infty$*, or* $N/2 < r \le \infty$ *if* $\sigma = \infty$*.*

Then Theorem 7.1 applies and

1. *There exists a pullback attractor with respect to* $\mathcal{D}_\beta$, $\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)]$ *where* $\varphi_m(t)$ *and* $\varphi_M(t)$ *are the extremal complete trajectories from Theorem 7.1. In particular, the set* $[\varphi_m(t), \varphi_M(t)]$ *is a forward invariant set that is pullback attracting at time* t*.*

2. *For non-negative solutions there also exists a pullback attractor* $\mathcal{A}_+(t) \subset [0, \varphi_M(t)]$*. In particular the set* $[0, \varphi_M(t)]$ *is a pullback attracting invariant set for non-negative solutions.*

Remark 9.3 *Assumption i) implies that* $\phi \in \mathcal{D}_\beta(I\!R, C(\overline{\Omega})$*, while assumption ii) implies that, for each* $T < \infty$, $\phi \in L^\infty((-\infty, T), C(\overline{\Omega})) \subset \mathcal{D}_\beta$*.*

Proof. From Young's inequality applied to f we have

$$f(t, x, u) \le m_1(t, x)u + \left(\frac{8m_2^3(t, x)}{27n(t, x)}\right)^{1/2}.$$

for all $u \ge 0$. A similar expression holds for $u < 0$. Thus,

$$f(t, x, u)u \le m_1(t, x)u^2 + \left(\frac{8m_2^3(t, x)}{27n(t, x)}\right)^{1/2} |u|.$$

Hence, if either i) or ii) hold, we can apply Theorem 7.1 with $C(t, x) = m_1(t, x)$ and $D(t, x) = \left(\frac{8m_2^3(t,x)}{27n(t,x)}\right)^{1/2}$ to deduce the existence of two extremal complete trajectories (φ_m and φ_M) and a pullback attractor $\mathcal{A}(t)$ such that

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)].$$

Moreover, since 0 is a solution of (9.1) and the comparison principle holds, the maximal complete trajectory is non-negative and the minimal one non-positive. So, provided we consider only non-negative solutions, the pullback attractor $\mathcal{A}_+(t)$ satisfies

$$\mathcal{A}_+(t) \subset [0, \varphi_M(t)] \quad \text{for all} \quad t \in I\!R,$$

where $\varphi_M(t)$ is the maximal complete trajectory. ■

Remark 9.4 *Notice that Theorem 5.3, part ii), gives sufficient conditions on* D *to conclude that* ϕ*, and therefore* φ_m, φ_M *are in* $C_b(I\!R, C(\overline{\Omega}))$*. In such a case the arguments in Section 8, regarding the asymptotic behaviour forward in time, apply.*

Also, note that Corollary 5.6 gives conditions on D to conclude that

$$\varphi_m(t), \varphi_M(t) \to 0$$

as t goes to $-\infty$ or ∞, in cases not covered by Theorem 9.1.

However, in general solutions of (9.1) may not be bounded as $t \to +\infty$. For example if $n = n(t)$ tends to zero as $t \to +\infty$ and $m(t,x) = \lambda$, a positive constant larger than the first eigenvalue of the Laplace operator in Ω with Dirichlet boundary conditions (see Langa and Suárez [15], Lemma 4.5). Indeed, assume that on a suitable smooth subdomain $\Omega_0 \subset \Omega$ we have $0 < N(t) = \max_{x\in\Omega_0} n(t,x) \to 0$ as $t \to \infty$ and $m = m(x)$. Then clearly solutions of

$$\begin{cases} w_t - \Delta w &= m(x)w - N(t)w^3 \quad \text{in} \quad \Omega_0, \quad t > s \\ w &= 0 \quad\quad\quad\quad\quad\quad\quad\quad\;\; \text{on} \quad \partial\Omega_0 \\ w(0) &= w_0 \geq 0 \end{cases}$$

give lower bounds for the non-negative solutions of (9.1) restricted to Ω_0. Therefore, if the first eigenvalue of $-\Delta - m(x)I$ with Dirichlet boundary conditions in Ω_0 is negative, using the arguments from the proof of Lemma 4.5 in Langa and Suárez [15], we can show that $w(t,x)$ becomes unbounded in Ω_0 and so do the solutions of (9.1).

We now give four examples which show that sometimes the linear bounds appearing in Lemma 7.2 may have desirable properties even though no special behaviour is prescribed for the nonlinear term. For example, $\phi(t)$ can be independent of t or T-periodic, while the reaction term f is not. We will assume in (9.2) that $n \geq 0$ and that $m(t,x)$ admits a decomposition of the form

$$m(t,x) = m_1(t,x) + m_2(t,x)$$

such that the evolution operator generated by $\Delta + m_1(t,x)$ is exponentially stable.

Example 1. Suppose that $m_1(t,x)$ is T-periodic and that $m_2(t,x) = a(t)g^2(t,x)$, where $g(t,x) \geq 0$ is also T-periodic and $a(t) \geq 0$ is arbitrary. Set

$$n(t,x) = a^3(t)h^2(t,x)$$

for some T-periodic function $h(t,x) \geq 0$. Then f satisfies (7.2) with

$$C(t,x) = m_1(t,x) \quad \text{and} \quad D(t,x) = \frac{8g^3(t,x)}{27h(t,x)}$$

which are T-periodic functions. Hence, $\phi(t)$ is a T-periodic solution of the linear problem, that is, we obtain a T-periodic bound for the pullback attractor of the nonlinear problem.

Example 2. Assume now that

$$m_1(t,x) = m_0(x), \quad m_2(t,x) = a(t,x)g_0^2(x), \text{ and } \quad n(t,x) = a^3(t,x)h_0(x)$$

where $h_0 \geq 0$, m_0, $g_0 \geq 0$ do not depend on t and $a(t,x) \geq 0$ is arbitrary. Then, f satisfies (7.2) with

$$C(t,x) = m_0(x) \quad \text{and} \quad D(t,x) = \frac{8g_0^3(x)}{27h_0(x)}.$$

and the linear problem given by (7.5) is an autonomous parabolic equation. So, its (unique) equilibrium gives bounds for the nonlinear problem, that is, we have a time-independent bound for the pullback attractor of the nonlinear problem.

Example 3. Suppose now that $m_1(t,x)$ is T-periodic,

$$m_2(t,x) = a(t,x)b(t,x)g^2(t,x), \quad \text{and} \quad n(t,x) = a^3(t,x)b(t,x)h^2(t,x).,$$

where$g(t,x) \geq 0$ and $h(t,x) \geq 0$ are both T-periodic and $a(t,x) \geq 0$ is arbitrary. Suppose that $0 \leq b(t,x) \to b_0(x)$ uniformly in x as $t \to \infty$. In this case

$$C(t,x) = m_1(t,x) \quad \text{and} \quad D(t,x) = \frac{8b(t,x)g^3(t,x)}{27h(t,x)}.$$

Notice that if we denote $D^+(t,x) = 4b_0(x)g^3(t,x)/27h(t,x)$ then $D(t,x) - D^+(t,x) \to 0$ uniformly in Ω, as t goes to infinity. Thus $\phi(t)$ is the unique complete trajectory of the asymptotically T-periodic problem (7.5) and therefore $\phi(t)$ is asymptotically T-periodic.

Example 4. In the previous example, suppose that $m_1 \geq 0$, $g \geq 0$ and $h \geq 0$ do not depend on t, i.e. $m_1(t,x) = m_0(x)$,

$$m_2(t,x) = a(t,x)b(t,x)g^2(x), \quad \text{and} \quad n(t,x) = a^3(t,x)b(t,x)h(x),$$

with $a(t,x) \geq 0$ arbitrary and $0 \leq b(t,x) \to b_0(x)$ uniformly in x as $t \to \infty$. Therefore

$$C(t,x) = m_1(x)$$

and

$$D(t,x) = \frac{8b^2(t,x)g_0^3(x)}{27h(x)} \to D^+(x) = \frac{8b_0^2(x)g_0^3(x)}{27h(x)} \quad \text{uniformly as} \quad t \to \infty.$$

Thus $\phi(t)$ satisfies an asymptotically autonomous linear problem. In particular, from the result in Section 8 we have bounds for the asymptotic behaviour of the solutions of the nonlinear problem forward in time.

Note that in all these examples Theorem 9.2 gives conditions for the existence of the pullback attractor.

10 Some other problems

With minor modifications the results of previous sections can be translated to problems other than our model example (1.1).

For example, the results about linear equations in Section 4 remain true for more general equations than (4.1) involving time–dependent operators and boundary conditions.

As a first example, using the results in [9], we can consider operators of the form

$$A(t,D)u = -\sum_{i,j=1}^{N} a_{ij}(x,t)\partial_i\partial_j u + \sum_{i=1}^{N} a_i(x,t)\partial_i u + a(x,t)u$$

with suitable smooth coefficients and either Dirichlet boundary conditions or time–independent boundary conditions of Robin type

$$\mathcal{B}u = \frac{\partial u}{\partial \vec{n}} + b(x)u$$

with no sign conditions on the smooth coefficient $b(x)$. All these operators satisfy the maximum principle [9, page 120] and the estimates in (4.2).

Existence results for the corresponding nonlinear problems, along the lines of those given in Theorem 2.12, can be obtained from the results in [9] and [16].

The analysis of complete trajectories in Section 5 can therefore be carried out without major changes. Of course, the asymptotically autonomous or periodic cases in Section 6 would require a specific although similar treatment.

All the results for the nonlinear equations in Section 7 then follow for this example.

We could also consider the following problem, with non-autonomous nonlinear boundary conditions:

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(t,x,u) & \text{in} \quad \Omega, \quad t>s \\ \frac{\partial u}{\partial n} + b(t,x)u &=& g(t,x,u) & \text{on} \quad \partial\Omega \\ u(s) &=& u_s & \end{array} \right. \tag{10.1}$$

where Ω is a bounded domain in $I\!R^N$, $b(t,x)$ is smooth and $f(t,x,u): I\!R \times \Omega \times I\!R \to I\!R$ and $g(t,x,u): I\!R \times \partial\Omega \times I\!R \to I\!R$ are continuous, locally Hölder in t, locally Lipschitz in u and satisfy

$$f(t,x,u)u \leq C(t,x)|u|^2 + D(t,x)|u| \quad \text{for all } u \in I\!R \tag{10.2}$$

$$g(t,x,u)u \leq B(t,x)|u|^2 + E(t,x)|u| \quad \text{for all } u \in I\!R \tag{10.3}$$

for some suitable smooth functions C, D, B and E. Note that we make no sign assumptions on $b(t,x)$.

In this case the main assumption would be that the evolution operator defined by

$$\left\{ \begin{array}{rcll} v_t - \Delta v &=& C(t,x)v & \text{in} \quad \Omega, \quad t>s \\ \frac{\partial v}{\partial n} + b(t,x)v &=& B(t,x)v & \text{on} \quad \partial\Omega \\ v(s) &=& v_s & \end{array} \right. \tag{10.4}$$

is exponentially stable.

The technical details will be presented elsewhere.

11 Conclusions

We have provided suitable general conditions that imply that certain non-autonomous reaction-diffusion equations have two extremal complete trajectories bounding the pullback attractor. Namely, we need the nonlinear problem to be bounded by two linear problems for which there exists a unique complete trajectory which is globally asymptotically stable. In our analysis no prescribed time dependence is assumed (e.g. periodic, quasiperiodic or almost periodic).

We have also given a result on periodic problems: the existence of two extremal periodic orbits bounding the attractor. In addition we have obtained information about the asymptotic behaviour both forwards and backwards in time of some asymptotically autonomous problems.

In the course of this analysis we have proved some sharp results on the exponential stability of the evolution operator associated with linear evolution equations in Lebesgue spaces.

Finally, we have applied our techniques in the case of a non-autonomous logistic equation.

We hope that with the techniques developed in this paper we will be able to analyse in more detail the dynamical behaviour of important non-autonomous nonlinear models appearing in applications.

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