



Weak compactness and representation in variable exponent Lebesgue spaces on infinite measure spaces

Francisco L. Hernández¹ · César Ruiz¹ · Mauro Sanchiz¹

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Abstract

Relative weakly compact sets and weak convergence in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ for infinite measure spaces (Ω, μ) are characterized. Criteria recently obtained in [14] for finite measures are here extended to the infinite measure case. In particular, it is showed that the inclusions between variable exponent Lebesgue spaces for infinite measures are never L -weakly compact. A lattice isometric representation of $L^{p(\cdot)}(\Omega)$ as a variable exponent space $L^{q(\cdot)}(0, 1)$ is given.

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1 Introduction

Compactness and weak compactness has always been a relevant matter of study in Banach spaces. Many specific results have been obtained for concrete spaces. Recall, for example, the classical Riesz-Kolmogorof compactness theorem in L_p -spaces ($1 \leq p < \infty$), which has been recently extended to the context of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ (or Nakano spaces) by Górká and Macios [12], Bandaliev and Górká [3] and Dong et al. [9]. These authors give useful criteria according with the underlying measure space (Ω, μ) considered (f.i. Euclidean spaces, metric measure spaces or locally compact groups).

Weak compactness has been also widely studied in many different contexts. Recall, for example, the classical Dunford and Pettis result for $L_1(\Omega)$ spaces characterizing the relative weakly compact subsets as the equi-integrable sets. For Orlicz spaces $L^\varphi(\Omega)$, useful weak

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✉ Mauro Sanchiz
msanchiz@ucm.es

Francisco L. Hernández
pacoh@ucm.es

César Ruiz
cruizb@mat.ucm.es

¹ IMI and Department of Mathematical Analysis and Applied Mathematics, Faculty of Mathematics, Complutense University of Madrid, 28040 Madrid, Spain

compactness criteria were given by Andô in [1] (cf. [21] chapter 4). Later on, many extensions have been given for symmetric function spaces (see f.i. [8] and references within) and also for the vectorial case of Orlicz-Bochner spaces ([20]). In the non-symmetric setting, the weak compactness in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ for finite measure spaces (Ω, μ) has been studied recently in [14], obtaining suitable criteria of weak compactness and weak convergence.

In this paper we broaden the study of weak compactness in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$. One of our goals is to find suitable weak compactness criteria in $L^{p(\cdot)}(\Omega)$ for infinite measure spaces (Ω, μ) extending results obtained in [14] for the case of finite measure. We analyze possible versions of Andô weak compactness characterizations given for Orlicz spaces $L^\varphi(\Omega)$ ([1]). Another goal in this paper is to find lattice isometric representations of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ over infinite measure spaces (Ω, μ) as a suitable variable exponent Lebesgue space $L^{q(\cdot)}(0, 1)$ on the probability measure space $(0, 1)$ with the Lebesgue measure. This kind of representations are a useful tool in the study of the structure of general variable exponent Lebesgue spaces.

Often, to lift properties from function spaces over probability measure spaces to the framework of function spaces on infinite measure spaces requires a careful study and additional conditions are usually needed. For example, while for finite measures the $L^{p(\cdot)}$ -*equi-integrability* is equivalent to the *uniform integrability*, in the case of infinite measures the additional condition of *uniform decay at infinity* (i.e. for every $\varepsilon > 0$ there exists $A \subset \Omega$ with $\mu(A) < \infty$ such that $\sup_{f \in S} \int_{\Omega \setminus A} f \chi_{\Omega \setminus A} < \varepsilon$) is required for the equivalence (see Proposition 3.1). Thus, the Dunford-Pettis Theorem in $L_1(\Omega)$ for σ -finite measure spaces (Ω, μ) is stated as follows (cf. [11] Theorem IV.8.9):

Theorem 1.1 ([10]) *A bounded subset $S \subset L_1(\Omega)$ is relatively weakly compact if and only if it is equi-integrable if and only if it is uniformly integrable and decays uniformly at infinity.*

The paper is organized as follows. After a preliminary section, in Sect. 3 we give a characterization of $L^{p(\cdot)}(\Omega)$ -equi-integrability for infinite measure spaces (Ω, μ) . It is proved that all inclusion operators between $L^{p(\cdot)}(\Omega)$ spaces for infinite measures are not L -weakly compact (Theorem 3.4). Section 4 is devoted to the study of weak compactness in variable exponent Lebesgue spaces $L^{p(\cdot)}(0, \infty)$ on the real interval $(0, \infty)$. We find suitable weak compactness criteria extending those given in [14] for finite measures. To prove it, we make use of an isometric lattice representation of the spaces $L^{p(\cdot)}(0, \infty)$ as a suitable $L^{q(\cdot)}(0, 1)$ space on the interval $(0, 1)$. From this it follows easily that $L^{p(\cdot)}(0, \infty)$ is weakly Banach-Saks if and only if $p^+ < \infty$.

Section 5 is devoted to variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ for abstract σ -finite measure spaces (Ω, μ) . Firstly, we give a lattice isometric representation of $L^{p(\cdot)}(\Omega)$ over non-atomic separable σ -finite spaces as a space $L^{\widehat{p}(\cdot)}(0, \infty)$ for a suitable exponent function $\widehat{p}(\cdot)$ (see Theorem 5.4). This useful representation theorem is obtained using the classical Carathéodory Theorem on algebras of measure (cf. [22] page 399). In particular, it follows that every space $L^{p(\cdot)}(\Omega)$ for a non-atomic separable σ -finite measure space can be lattice represented as a certain variable exponent space $L^{q(\cdot)}(0, 1)$ on the interval $(0, 1)$. This remarkable fact is not true for the class of Orlicz spaces (see Remark 5.9). Finally, as a consequence of the above representation and previous results in Sect. 4, suitable criteria for weak compactness and weak convergence in $L^{p(\cdot)}(\Omega)$ are obtained. Namely, $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if it is bounded,

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{p(t)} d\mu = 0$$

and

$$\lim_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n \cap \Omega_1} |f(t)| d\mu = 0,$$

where $\Omega_1 := p^{-1}(\{1\})$. A sequence (f_n) converges weakly to f in $L^{p(\cdot)}(\Omega)$ if and only if

- (i) $\lim_n \int_A f_n d\mu = \int_A f d\mu$ for each $A \in \Sigma$ with $\mu(A) < \infty$,
- (ii) $\lim_{\lambda \rightarrow 0} \sup_n \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda(f_n - f)|^{p(t)} d\mu = 0$, and
- (iii) $\lim_{A_k \searrow \emptyset} \sup_{n \in \mathbb{N}} \int_{A_k \cap \Omega_1} |f_n(t) - f(t)| d\mu = 0$.

We point out that no extra conditions on the regularity of the exponent functions (like the log-Hölder continuous conditions) will be required along the paper.

2 Preliminaries

Throughout the paper, (Ω, Σ, μ) is a non-atomic σ -finite measurable space (unless specified otherwise) and $L_0(\Omega)$ is the space of all real μ -measurable function classes (where $f = g$ if $f(t) = g(t)$ a.e.- μ). Given a measurable subset $A \subset \Omega$, $\chi_A \in L_0(\Omega)$ denotes the characteristic function of A . A Banach function space $E(\Omega)$ is a Banach lattice formed by measurable function classes. $E(\Omega)$ is said to be order continuous if $\lim_\alpha \|f_\alpha\| = 0$ for every downward directed set (f_α) in $E(\Omega)$ with $\bigwedge_\alpha f_\alpha = 0$.

Given a μ -measurable function $p : \Omega \rightarrow [1, \infty)$ (called exponent function), the *variable exponent Lebesgue space* (or Nakano space) $L^{p(\cdot)}(\Omega)$ is defined to be the set of all real measurable function classes such that the modular $\rho_{p(\cdot)}(f/r)$ is finite for some $r > 0$, where

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(t)|^{p(t)} d\mu(t) < \infty.$$

The associated Luxemburg norm is defined as

$$\|f\|_{p(\cdot)} := \inf\{r > 0 : \rho_{p(\cdot)}(f/r) \leq 1\}.$$

With the usual pointwise order, $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach function space. The case of $p(\cdot) = p$ constant will be denoted by $L_p(\Omega)$.

Let $p^- := \text{ess inf}\{p(t) : t \in \Omega\}$ and $p^+ := \text{ess sup}\{p(t) : t \in \Omega\}$. By $p^+_{|A}$ and $p^-_{|A}$ we denote the essential supremum and infimum of the function $p(\cdot)$ over a measurable subset $A \subset \Omega$. The conjugate function $p^*(\cdot)$ of $p(\cdot)$ is defined by the equation $\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1$ almost everywhere $t \in \Omega$. Thus, for $p^+ < \infty$ the topological dual of the space $L^{p(\cdot)}(\Omega)$ is the variable exponent space $L^{p^*(\cdot)}(\Omega)$.

A $L^{p(\cdot)}(\Omega)$ space is separable if and only if $p^+ < \infty$ or, equivalently, if and only if $L^{p(\cdot)}(\Omega)$ contains no isomorphic copy of ℓ_∞ . In the sequel, only separable variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ will be considered (note that $p^*(\cdot)$ is bounded if and only if $p^- > 1$, so we can consider non separable dual spaces $L^{p^*(\cdot)}(\Omega)$). An space $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p^- \leq p^+ < \infty$. This is also equivalent to $L^{p(\cdot)}(\Omega)$ being uniformly convex ([17] Theorem 3.3). Notice that, for $p^+ < \infty$, $\|f\|_{p(\cdot)} = 1$ if and only if the modular $\rho_{p(\cdot)}(f) = 1$. Also for $p^+ < \infty$, a sequence $(f_n) \subset L^{p(\cdot)}(\Omega)$ satisfies $\lim_{n \rightarrow \infty} \|f_n\|_{p(\cdot)} = 0$ if and only if $\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(f_n) = 0$ (cf. [7]). By $B_{L^{p(\cdot)}}$ we denote the closed unit ball of $L^{p(\cdot)}(\Omega)$. The essential range of the exponent function $p(\cdot)$ is defined as

$$R_{p(\cdot)} := \{q \in [1, \infty) : \forall \varepsilon > 0 \ \mu(p^{-1}(q - \varepsilon, q + \varepsilon)) > 0\}.$$

It is a closed subset of $[1, \infty)$ and it is compact when $p(\cdot)$ is essentially bounded. Fixed an exponent function $p(\cdot)$, we denote $\Omega_1^{p(\cdot)}$ ($\equiv \Omega_1$) the subset of Ω defined by $\Omega_1 := p^{-1}(\{1\})$.

Given a sequence (p_n) in $[1, \infty)$, the *Nakano sequence space* ℓ_{p_n} is the Banach lattice of sequences (x_n) such that $\sum_{n=1}^\infty \left| \frac{x_n}{r} \right|^{p_n} < \infty$ for some $r > 0$, equipped with the Luxemburg norm

$$\|(x_n)\|_{\ell_{p_n}} = \inf \left\{ r > 0 : \sum_{n=1}^\infty \left| \frac{x_n}{r} \right|^{p_n} \leq 1 \right\}$$

We refer to [4, 6, 7, 16] for other definitions and basic facts regarding variable exponent Lebesgue spaces and Banach function spaces.

3 $L^{p(\cdot)}$ equi-integrability

Given a Banach function space $E(\Omega)$, a subset $S \subset E(\Omega)$ is *equi-integrable* if, for every decreasing sequence of measurable sets (A_n) in Ω with zero measure intersection (denoted shortly by $A_n \searrow \emptyset$),

$$\lim_{A_n \searrow \emptyset} \sup_{f \in S} \|f \chi_{A_n}\| = 0.$$

A subset $S \subset E(\Omega)$ is said to be *uniformly integrable* if

$$\lim_{\mu(A) \rightarrow 0} \sup_{f \in S} \|f \chi_A\| = 0. \tag{*}$$

In Banach function spaces on finite measure spaces, both properties are equivalent (cf. [15] Lemma 2.1) However, for infinite measure spaces, uniform integrability is not enough for getting equi-integrability and an additional condition is required.

Proposition 3.1 *Let $E(\Omega)$ be an order continuous Banach function space on an infinite measure space (Ω, μ) and let $S \subset E(\Omega)$ bounded. Then, S is equi-integrable if and only if it is uniformly integrable and it decays uniformly at infinity, i.e. for every $\varepsilon > 0$ there exists $A \subset \Omega$ with $\mu(A) < \infty$ such that*

$$\sup_{f \in S} \|f \chi_{\Omega \setminus A}\| < \varepsilon. \tag{**}$$

Let us first characterize equi-integrability by a more suitable property:

Lemma 3.2 *Let $E(\Omega)$ be an order continuous Banach function space and $S \subset E(\Omega)$. Then, S is equi-integrable if and only if, for every disjoint sequence of measurable sets (A_n) ,*

$$\lim_{n \rightarrow \infty} \sup_{f \in S} \|f \chi_{A_n}\| = 0.$$

Proof Let S be equi-integrable. For every disjoint sequence (A_n) in Ω , take the decreasing sequence (B_n) defined by $B_n := \bigcup_{m=n}^\infty A_m$ which satisfies $B_n \searrow \emptyset$. Then,

$$\lim_{n \rightarrow \infty} \sup_{f \in S} \|f \chi_{A_n}\| \leq \lim_{n \rightarrow \infty} \sup_{f \in S} \|f \chi_{B_n}\| = \lim_{B_n \searrow \emptyset} \sup_{f \in S} \|f \chi_{B_n}\| = 0.$$

Conversely, if S is not equi-integrable, there exist $\varepsilon > 0$, a decreasing sequence $A_n \searrow \emptyset$ in Ω and a sequence (f_n) in S such that, for all $n \in \mathbb{N}$,

$$\|f_n \chi_{A_n}\| \geq \varepsilon.$$

Now, using that $E(\Omega)$ is order continuous, given $n_1 \in \mathbb{N}$ we can find $n_2 > n_1$ large enough so that, for $B_1 = A_{n_1} \setminus A_{n_2}$,

$$\|f_{n_1} \chi_{B_1}\| \geq \frac{\varepsilon}{2}.$$

Then, we can find n_3 large enough so that, for $B_2 = A_{n_2} \setminus A_{n_3}$,

$$\|f_{n_2} \chi_{B_2}\| \geq \frac{\varepsilon}{2}.$$

Thus, we can define the disjoint sequence (B_m) by $B_m := A_{n_m} \setminus A_{n_{m+1}}$ from the subsequence (A_{n_m}) and, taking the subsequence (f_{n_m}) , for every $m \in \mathbb{N}$ we get

$$\|f_{n_m} \chi_{B_m}\| \geq \frac{\varepsilon}{2},$$

ending the proof. \square

Proof of Proposition 3.1 If $\lim_{\mu(A) \rightarrow 0} \sup_{f \in S} \|f \chi_A\| \neq 0$, we can take a sequence (A_n) in Ω with $\mu(A_{n+1}) \leq \frac{\mu(A_n)}{2}$ and a sequence (f_n) in S such that

$$\|f_n \chi_{A_n}\| \geq \varepsilon.$$

Fixed some n_1 , take n_2 large enough so that

$$\|f_{n_1} \chi_{A_{n_1} \setminus \bigcup_{k=n_2}^{\infty} A_k}\| \geq \frac{\varepsilon}{2}.$$

Then, we can take n_3 large enough so that

$$\|f_{n_2} \chi_{A_{n_2} \setminus \bigcup_{k=n_3}^{\infty} A_k}\| \geq \frac{\varepsilon}{2}.$$

Thus, we construct the disjoint sequence (B_m) , where $B_m := A_{n_m} \setminus \bigcup_{k=n_{m+1}}^{\infty} A_k$, which satisfies, for every natural m ,

$$\|f_m \chi_{B_m}\| \geq \frac{\varepsilon}{2}.$$

Hence, by Lemma 3.2, S is not equi-integrable.

Assume now that there exists some $\varepsilon > 0$ such that, for every A with $\mu(A) < \infty$, it holds $\sup_{f \in S} \|f \chi_{\Omega \setminus A}\| \geq \varepsilon$. Then we can take an increasing sequence of sets $(A_n) \nearrow \Omega$ of finite measure and (f_n) in S such that, for all $n \in \mathbb{N}$,

$$\|f_n \chi_{\Omega \setminus A_n}\| \geq \varepsilon.$$

Now, as before, fixed n_1 take n_2 large enough so that

$$\|f_{n_1} \chi_{A_{n_2} \setminus A_{n_1}}\| \geq \frac{\varepsilon}{2}.$$

In the same way, we define the disjoint sequence (B_m) by $B_m = A_{n_{m+1}} \setminus A_{n_m}$ satisfying

$$\|f_m \chi_{B_m}\| \geq \frac{\varepsilon}{2},$$

so S is not equi-integrable.

Conversely, assume that, given $\varepsilon > 0$, by $(*)$ there exists $\delta > 0$ so that, for every A with $\mu(A) < \delta$,

$$\sup_{f \in S} \|f \chi_A\| < \frac{\varepsilon}{2}.$$

And by (**), take $B \subset \Omega$ with $\mu(B) < \infty$ such that

$$\sup_{f \in S} \|f \chi_{\Omega \setminus B}\| < \frac{\varepsilon}{2}.$$

Then, for every disjoint sequence (A_n) , as $(A_n \cap B) \subset B$, for n large enough we have $\mu(A_n \cap B) < \delta$, hence

$$\sup_{f \in S} \|f \chi_{A_n}\| \leq \sup_{f \in S} \|f \chi_{A_n \cap B}\| + \|f \chi_{\Omega \setminus B}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, by Lemma 3.2, S is equi-integrable. □

Conditions (*) and (**) separately do not imply $L^{p(\cdot)}$ -equi-integrability. The set $S_1 = \{\chi_{[n, n+1)} : n \in \mathbb{N}\}$ in a separable $L^{p(\cdot)}(0, \infty)$ space satisfies (*), but it is not equi-integrable. Neither the set $S_2 = \{\frac{\chi_{A_n}}{\mu(A_n)^{p(\cdot)}} : n \in \mathbb{N}\}$, where (A_n) is a disjoint sequence contained in a set $A \subset \Omega$ of finite measure, is equi-integrable in spite of satisfying (**).

Recall that an inclusion operator $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is said to be *L-weakly compact* when the unit ball $B_{L^{q(\cdot)}}$ is an equi-integrable set in $L^{p(\cdot)}(\Omega)$. In [14] it is studied the *L-weak compactness* of inclusions $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ for $\mu(\Omega) < \infty$. In this case of finite measures, the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ holds if and only if $p(\cdot) \leq q(\cdot)$. A sufficient condition for this inclusion to be *L-weakly compact* is that $\text{ess inf}_{t \in \Omega} (q(t) - p(t)) > 0$ (but it is not necessary) (see [14] Proposition 3.4).

Inclusions between variable exponent Lebesgue spaces for infinite measure spaces are more restricted but nonetheless they exist (in contrast with the behavior of classical Lebesgue spaces L_p). The following is known (cf. [7] Theorem 3.3.1. See also [6] and [2]):

Proposition 3.3 *The inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ holds if and only if $p(t) \leq q(t)$ a.e- $t \in \Omega$ and there exists $\lambda > 1$ such that*

$$\int_{\Omega_d} \lambda^{-r(t)} d\mu < \infty,$$

where $\Omega_d = \{t \in \Omega : p(t) < q(t)\}$ and $r(\cdot)$ is the defect exponent defined by $\frac{1}{r(\cdot)} := \frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}$.

Let us remark that if the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ holds and $\mu(\Omega_d) = \infty$, then for every $\varepsilon > 0$ the set $D_\varepsilon := \{t \in \Omega_d : q(t) < p(t) + \varepsilon\}$ has infinite measure.

Indeed, assume that there exists $\varepsilon > 0$ such that $\mu(D_\varepsilon) < \infty$. Then, $\mu(\Omega_d \setminus D_\varepsilon) = \infty$ with $r_{|\Omega_d \setminus D_\varepsilon}^+ \leq \frac{p^+ q^+}{\varepsilon} < \infty$. Hence, for every $\lambda > 1$,

$$\int_{\Omega_d} \lambda^{-r(t)} d\mu \geq \int_{\Omega_d \setminus D_\varepsilon} \lambda^{-r(t)} d\mu \geq \lambda^{-\frac{p^+ q^+}{\varepsilon}} \mu(\Omega_d \setminus D_\varepsilon) = \infty,$$

which, using the above Proposition, gives a contradiction.

Theorem 3.4 *Let $L^{p(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ with $\mu(\Omega) = \infty$. If the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ holds, then it is not *L-weakly compact*.*

Proof To prove this theorem, it will be enough to find a disjoint sequence (f_n) which be seminormalized in $L^{p(\cdot)}(\Omega)$ as well as in $L^{q(\cdot)}(\Omega)$. To do so, we distinguish two cases. If $\mu(\Omega_d) < \infty$, then $\mu(\Omega \setminus \Omega_d) > 0$. Hence, as $p(t) \leq q(t)$ for almost every $t \in \Omega$ for the

inclusion to hold, $L^{q(\cdot)}(\Omega \setminus \Omega_d) = L^{p(\cdot)}(\Omega \setminus \Omega_d)$ and every disjoint sequence of functions (f_n) supported in $\Omega \setminus \Omega_d$ will have equal norm in $L^{p(\cdot)}(\Omega)$ and in $L^{q(\cdot)}(\Omega)$.

If, on the contrary, $\mu(\Omega_d) = \infty$, we take $(f_n := \chi_{A_n})$, where (A_n) is a disjoint sequence satisfying that $\mu(A_n) = 1$ and, for every natural n ,

$$q_{|A_n}^+ - p_{|A_n}^- < \frac{1}{n}.$$

Indeed, in that case we have

$$\ell_{p_{|A_n}^-} \hookrightarrow [f_n]_{p(\cdot)} \hookrightarrow \ell_{p_{|A_n}^+}$$

and

$$\ell_{q_{|A_n}^-} \hookrightarrow [f_n]_{q(\cdot)} \hookrightarrow \ell_{q_{|A_n}^+},$$

and, using ([19], cf. [13] Lemma 3.2), we have $\ell_{p_{|A_n}^-} \simeq \ell_{p_{|A_n}^+} \simeq \ell_{q_{|A_n}^-} \simeq \ell_{q_{|A_n}^+}$. Hence, we deduce $[f_n]_{p(\cdot)} \simeq [f_n]_{q(\cdot)}$.

From the above remark it follows that $\mu(D_{\frac{1}{2n}}) = \infty$ for every natural n . We take a finite partition $1 = x_1 < \dots < x_i < \dots < x_m = q^+$, $1 \leq i \leq m$, of the interval $[1, q^+]$ where $x_{i+1} - x_i < \frac{1}{2n}$ for every i . We can ensure that, for some j ,

$$\mu\left(D_{\frac{1}{2n}} \cap q^{-1}([x_j, x_{j+1}])\right) = \infty.$$

Now, due to the definition of $D_{\frac{1}{2n}}$, it is also true that

$$\mu\left(D_{\frac{1}{2n}} \cap q^{-1}([x_j, x_{j+1}]) \cap p^{-1}\left([x_j - \frac{1}{2n}, x_{j+1}]\right)\right) = \infty.$$

Thus, we deduce that there exists an infinite measure set E_n (the set above) such that

$$q_{|E_n}^+ - p_{|E_n}^- \leq x_{j+1} - \left(x_j - \frac{1}{2n}\right) \leq \frac{1}{n}.$$

Finally, since $\mu(E_n) = \infty$ for each natural n , we take the subsets which satisfy the required properties, namely $A_n \subset \left(E_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)\right)$ having $\mu(A_n) = 1$. □

For example, the inclusion $L^{\frac{2r+3}{r+1}}(0, \infty) \subset L^{\frac{2r+3}{r+2}}(0, \infty)$ is not L -weakly compact.

4 Weak compactness in $L^{p(\cdot)}(0, \infty)$

In this section we study the weak compactness of subsets of $L^{p(\cdot)}(0, \infty)$ spaces defined on the infinite interval $(0, \infty)$ with the Lebesgue measure μ . Weak compactness is derived from the equi-integrability (cf. [18] Proposition 3.6.5), but the converse is not true in general. We will use an isometric representation of $L^{p(\cdot)}(0, \infty)$ as a space $L^{q(\cdot)}(0, 1)$ on the finite interval for a suitable exponent $q(\cdot)$ to obtain weak compactness criteria in $L^{p(\cdot)}(0, \infty)$ from the ones given in [14] for finite measures.

Fixed an exponent function $p(\cdot)$ on $(0, \infty)$, let $\tau : (0, 1) \rightarrow (0, \infty)$ be an diphomeomorphism (for example the tangent function $\tau(s) = \tan(\frac{\pi}{2}s)$). If we consider the exponent function $q(\cdot) : (0, 1) \rightarrow [1, \infty)$ defined by $q(s) := p(\tau(s))$, we have, for $f \in L^{p(\cdot)}(0, \infty)$,

$$\int_0^\infty |f(t)|^{p(t)} dt = \int_0^1 |f(\tau(s))|^{q(s)} |\tau'(s)| ds = \int_0^1 \left(|f(\tau(s))\tau'(s)^{\frac{1}{q(s)}} \right)^{q(s)} ds.$$

Thus, the operator $T_\tau : L^{p(\cdot)}(0, \infty) \rightarrow L^{q(\cdot)}(0, 1)$ defined by

$$(T_\tau f)(s) = \left| \tau'(s)^{\frac{1}{q(s)}} \right| f(\tau(s))$$

is an isomodular lattice isomorphism. The inverse operator T_τ^{-1} is given by

$$(T_\tau^{-1}g)(t) = \left| (\tau')^{-1}(t)^{\frac{1}{p(t)}} \right| g(\tau^{-1}(t)) = (T_{\tau^{-1}}g)(t).$$

This representation has been already used in ([23] Lemma 3.2) for other purposes. Given $S \subset L^{p(\cdot)}(0, \infty)$, we study now the weak compactness of $T_\tau(S)$.

Recall that in the case of finite measures we have ([14] Theorem 4.2) that a subset $S \subset L^{p(\cdot)}(\Omega)$, for $\mu(\Omega) < \infty$ and $p^+ < \infty$, is relatively weakly compact if and only if S is norm bounded and, for each $g \in L^{p^*(\cdot)}(\Omega)$,

$$\lim_{\mu(E) \rightarrow 0} \sup_{f \in S} \int_E |fg| d\mu = 0.$$

Thus, applying this and the isometry T_τ (shortly denoted by T), we get that $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if and only if, for every $g \in L^{q^*(\cdot)}(0, 1)$,

$$\lim_{\mu(E) \rightarrow 0} \sup_{f \in S} \int_E |Tf \cdot g| ds = 0,$$

with $q(s) = p(\tau(s))$. Consider now the isometry $T_* : L^{p^*(\cdot)}(0, \infty) \rightarrow L^{q^*(\cdot)}(0, 1)$

$$(T_*g)(s) = \left| \tau'(s)^{\frac{1}{q^*(s)}} \right| g(\tau(s)).$$

As $T_*(L^{p^*(\cdot)}(0, \infty)) = L^{q^*(\cdot)}(0, 1)$ coincides with the dual of $T(L^{p(\cdot)}(0, \infty)) = L^{q(\cdot)}(0, 1)$, we can rewrite the above expression as

$$\lim_{\mu(E) \rightarrow 0} \sup_{f \in S} \int_E |Tf \cdot T_*g| ds = 0$$

for $g \in L^{p^*(\cdot)}(0, 1)$. Equivalently, $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if, for each $g \in L^{p^*(\cdot)}(0, \infty)$, the set $S'_g = \{Tf \cdot T_*g : f \in S\}$ is uniformly integrable in $L_1(0, 1)$. As this is equivalent to being equi-integrable (for finite measure spaces), we rewrite the expression as

$$\lim_{E_n \searrow \emptyset} \sup_{f \in S} \int_{E_n} |Tf \cdot T_*g| ds = 0.$$

So, we can give a weak compactness criteria in $L^{p(\cdot)}(0, \infty)$ in terms of equi-integrability:

Theorem 4.1 *Let $L^{p(\cdot)}(0, \infty)$ with $p^+ < \infty$. A subset $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if and only if it is norm bounded and, for each $g \in L^{p^*(\cdot)}(0, \infty)$ and each sequence of measurable sets $A_n \searrow \emptyset$,*

$$\lim_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n} |fg| dt = 0.$$

Proof Through the previous reasoning, we need only to show that, for $A_n \searrow \emptyset$ in $(0, \infty)$,

$$\limsup_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n} |fg| dt = 0$$

if and only if, for $E_n \searrow \emptyset$ in $(0, 1)$,

$$\limsup_{E_n \searrow \emptyset} \sup_{f \in S} \int_{E_n} |Tf \cdot T_*g| ds = 0.$$

This is easily proved using Lemma 3.2 and that the dipheomorphisms τ and τ^{-1} preserve disjointness. Indeed, if there exist a disjoint sequence (A_n) in $(0, \infty)$ and (f_n) in S such that

$$\int_{A_n} |f_n g| dt \geq \varepsilon,$$

we obtain, using the change of variable $t = \tau(s)$, another disjoint sequence $(\tau^{-1}(A_n))$ in $(0, 1)$ such that

$$\int_{\tau^{-1}(A_n)} |Tf_n \cdot T_*g| ds \geq \varepsilon,$$

(using that $\frac{dt}{ds} = \tau'(s) = \tau'(s)^{\frac{1}{q(s)} + \frac{1}{q^*(s)}}$).

Analogously, if there exist a disjoint sequence (E_n) in $(0, 1)$ and (f_n) in S such that

$$\int_{E_n} |Tf_n \cdot T_*g| ds \geq \varepsilon$$

we obtain, using the change of variable $s = \tau^{-1}(t)$, another disjoint sequence $(\tau(E_n))$ in $(0, \infty)$ such that

$$\int_{\tau(E_n)} |f_n g| dt \geq \varepsilon.$$

□

Now, by Proposition 3.1, we deduce that weakly compact sets $S \subset L^{p(\cdot)}(0, \infty)$ are characterized as those sets which satisfy, for every $g \in L^{p^*(\cdot)}(0, \infty)$, that

$$\lim_{\mu(A) \rightarrow 0} \sup_{f \in S} \int_{A_n} |fg| dt = 0 \tag{1}$$

and that, for every $\varepsilon > 0$, there exists some $A \subset (0, \infty)$ with $\mu(A) < \infty$ such that

$$\sup_{f \in S} \int_{(0, \infty) \setminus A} |fg| dt < \varepsilon. \tag{2}$$

Another criterion for finite measure spaces (given in [14] Proposition 4.3) states that in $L^{p(\cdot)}(\Omega)$ with $\mu(\Omega_1) = 0$, a subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if

$$\lim_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} |\lambda f(t)|^{p(t)} dt = 0.$$

As the zero measure sets are preserved under τ , we can apply the isometry T again obtaining that $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if and only if

$$\lim_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_0^1 |\lambda Tf(s)|^{q(s)} ds = 0.$$

And we can enunciate it avoiding to leave the space $L^{p(\cdot)}(0, \infty)$:

Theorem 4.2 *Let $L^{p(\cdot)}(0, \infty)$ with $p_+ < \infty$ and $\mu(\Omega_1) = 0$. A subset $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if and only if it is norm bounded and*

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_0^\infty |\lambda f(t)|^{p(t)} dt = 0.$$

Proof We just have to prove that this condition is equivalent to:

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_0^1 |\lambda T f(s)|^{q(s)} ds = 0.$$

But, as the isometry T is also isomodular, it is clear that

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \rho_{q(\cdot)}(\lambda T(f)) = \limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \rho_{q(\cdot)}(T(\lambda f)) = \limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \rho_{p(\cdot)}(\lambda f).$$

□

If $\mu(\Omega_1) > 0$, combining the above result and Dunford-Pettis Theorem 1.1 we get:

Theorem 4.3 *Let $L^{p(\cdot)}(0, \infty)$ with $p^+ < \infty$. A subset $S \subset L^{p(\cdot)}(0, \infty)$ is relatively weakly compact if and only if it is norm bounded,*

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{(0, \infty) \setminus \Omega_1} |\lambda f(t)|^{p(t)} d\mu = 0$$

and

$$\limsup_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n \cap \Omega_1} |f(t)| d\mu = 0.$$

We pass to characterize weakly convergent sequences in $L^{p(\cdot)}(0, \infty)$:

Proposition 4.4 *Let $L^{p(\cdot)}(0, \infty)$ with $p^+ < \infty$ and a sequence (f_n) and f in $L^{p(\cdot)}(0, \infty)$. Then, $f_n \rightarrow f$ weakly if and only if*

- (i) $\lim_n \int_A (f_n - f) d\mu = 0$ for each $A \in \Sigma$ with $\mu(A) < \infty$,
- (ii) $\lim_{\mu(A) \rightarrow 0} \sup_n \int_A |(f_n - f)g| d\mu = 0$ for each function $g \in L^{p^*(\cdot)}(0, \infty)$, and
- (iii) For each function $g \in L^{p^*(\cdot)}(0, \infty)$ and for each $\varepsilon > 0$, there exists some $A \in \Sigma$ with $\mu(A) < \infty$ such that $\sup_{n \in \mathbb{N}} \int_{(0, \infty) \setminus A} |(f_n - f)g| d\mu \leq \varepsilon$.

Proof (\Rightarrow): Clearly, (i) holds since $\chi_A \in L^{p^*(\cdot)}$, and conditions (ii) and (iii) follow from above Theorem 4.1 and Proposition 3.1.

(\Leftarrow): We can assume w.l.o.g. $f = 0$. If $g \in L^{p^*(\cdot)}(0, \infty)$ is a simple function with finite measure support, then it follows directly from (i) that $\lim_n \int_0^\infty f_n g d\mu = 0$. Assume now that g is a bounded function. Given $\varepsilon > 0$, by condition (iii) there exists $A \subset (0, \infty)$ with $\mu(A) < \infty$ such that

$$\sup_{n \in \mathbb{N}} \int_{(0, \infty) \setminus A} |f_n g| d\mu \leq \frac{\varepsilon}{3}$$

and we can take a simple function g_s such that $\|(g - g_s)\chi_A\|_\infty < \frac{\varepsilon}{3}$. Thus,

$$\begin{aligned} \int_0^\infty |f_n g| d\mu &\leq \int_A |f_n(g - g_s)| d\mu + \int_A |f_n g_s| d\mu + \int_{(0,\infty)\setminus A} |f_n g| d\mu \\ &\leq \frac{\varepsilon}{3} \int_A |f_n| d\mu + \int_A |f_n g_s| d\mu + \frac{\varepsilon}{3} \end{aligned}$$

and hence $\int_0^\infty |f_n g| d\mu \leq \varepsilon$ for $n \in \mathbb{N}$ large enough.

Now, for an arbitrary $g \in L^{p^{*(\cdot)}}(0, \infty)$, by condition (ii), there exists $\delta > 0$ such that $\int_A |f_n g| d\mu < \frac{\varepsilon}{2}$ if $\mu(A) < \delta$. Consider $G_m = \{t \in (0, \infty) : |g(t)| \leq m\}$ with m large enough so that $\mu(G_m^c) < \delta$. Then,

$$\int_0^\infty |f_n g| d\mu = \int_{G_m} |f_n g| d\mu + \int_{G_m^c} |f_n g| d\mu \leq \int_0^\infty |f_n g \chi_{G_m}| d\mu + \frac{\varepsilon}{2}.$$

Since $g \chi_{G_m}$ is bounded, we have $\int_0^\infty |f_n g| d\mu \leq \varepsilon$ from $n \in \mathbb{N}$ large enough. □

Proposition 4.5 *Let $L^{p^{(\cdot)}}(0, \infty)$ with $p^+ < \infty$ and $\mu(\Omega_1) = 0$. A sequence (f_n) in $L^{p^{(\cdot)}}(0, \infty)$ converges weakly to $f \in L^{p^{(\cdot)}}(0, \infty)$ if and only if*

- (i) $\lim_n \int_A f_n d\mu = \int_A f d\mu$ for each $A \in \Sigma$ with $\mu(A) < \infty$, and
- (ii) $\lim_{\lambda \rightarrow 0} \sup_{n \in \mathbb{N}} \frac{1}{\lambda} \int_0^\infty |\lambda(f_n - f)|^{p^{(t)}} d\mu = 0$.

Proof Clearly, (i) is equivalent to condition (i) in the above Proposition 4.4 and, using Theorem 4.2, we get also condition (ii).

Conversely, let $g \in L^{p^{*(\cdot)}}(0, \infty)$ and $r > 0$ such that $\int_0^\infty |r g(t)|^{p^{*(t)}} d\mu < \infty$. By hypothesis (ii), given $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that

$$\sup_{n \in \mathbb{N}} \frac{1}{\lambda_0} \int_0^\infty |\lambda_0(f_n(t) - f(t))|^{p^{(t)}} d\mu < \frac{\varepsilon r}{2}.$$

Take $A \subset (0, \infty)$ with $\mu(A) < \infty$ such that

$$\int_{(0,\infty)\setminus A} |r g(t)|^{p^{*(t)}} d\mu < \frac{\varepsilon}{2}.$$

Take $\delta > 0$ such that, for every measurable set E with $\mu(E) < \delta$,

$$\int_E |r g(t)|^{p^{*(t)}} d\mu < \frac{\varepsilon \lambda_0 r}{2}.$$

Thus, using Young inequality ([7] Lemma 3.2.20), we have:

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \int_{(0,\infty)\setminus A} |(f_n(t) - f(t))g(t)| d\mu \\ &\leq \frac{1}{\lambda_0 r} \left[\sup_{n \in \mathbb{N}} \int_{(0,\infty)\setminus A} |\lambda_0(f_n(t) - f(t))|^{p^{(t)}} d\mu + \int_{(0,\infty)\setminus A} |r g(t)|^{p^{*(t)}} d\mu \right] \\ &< \frac{1}{r} \left(\frac{\varepsilon r}{2} \right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

as well as

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \int_E |(f_n(t) - f(t))g(t)| d\mu \\ &\leq \frac{1}{\lambda_0 r} \left[\sup_{n \in \mathbb{N}} \int_E |\lambda_0(f_n(t) - f(t))|^{p^{(t)}} d\mu + \int_E |r g(t)|^{p^{*(t)}} d\mu \right] \end{aligned}$$

$$< \frac{1}{r} \left(\frac{\varepsilon r}{2}\right) + \frac{1}{\lambda_0 r} \left(\frac{\varepsilon \lambda_0 r}{2}\right) = \varepsilon,$$

and so the conditions (iii) and (ii) of the above Proposition 4.4 are respectively satisfied. Thus, we conclude that (f_n) is weakly convergent to f . □

Another consequence of the isometric representation of $L^{p(\cdot)}(0, \infty)$ as $L^{q(\cdot)}(0, 1)$ for $q(s) = p(\tau(s))$ is (using [14] Theorem 5.1):

Proposition 4.6 *A $L^{p(\cdot)}(0, \infty)$ space is weakly Banach-Saks if and only if $p^+ < \infty$.*

5 A representation of $L^{p(\cdot)}(\Omega)$

In this section (Ω, Σ, μ) will denote an abstract non-atomic separable σ -finite measure space (i.e. Σ endowed with the usual metric $d([A], [B]) = \mu((A \setminus B) \cup (B \setminus A))$, for $A, B \in \Sigma$, is separable). The real interval $(0, \infty)$ with the Lebesgue measure will be denoted by $((0, \infty), \mathcal{B}, \lambda)$. In the previous Sect. 4, we have obtained weak compactness criteria in $L^{p(\cdot)}(0, \infty)$ by using the lattice isometry T_τ between $L^{p(\cdot)}(0, \infty)$ and a certain $L^{q(\cdot)}(0, 1)$ space. We would like now to formulate this result for general separable σ -finite measure spaces (Ω, μ) . In order to do it, we are going to give a lattice isomorphic representation of the space $L^{p(\cdot)}(\Omega)$ as a certain variable exponent space $L^{\widehat{p}(\cdot)}(0, \infty)$ in the interval $(0, \infty)$. For that, we will use a version of the classical Carathéodory Theorem on algebras of measure for σ -finite measure spaces. Recall that \mathcal{N}_μ and \mathcal{N}_λ are the classes of subsets of measure zero in Ω and $(0, \infty)$ respectively.

Theorem 5.1 (Carathéodory, [22] page 399) *Let (Ω, Σ, μ) be a non-atomic separable finite measure space and $((0, \mu(\Omega)), \mathcal{B}, \lambda)$ be the interval $(0, \mu(\Omega))$ with the σ -algebra of Borel sets \mathcal{B} and the Lebesgue measure λ . Then, there exists an isomorphism between the algebras of measure $\phi : \Sigma/\mathcal{N}_\mu \rightarrow \mathcal{B}/\mathcal{N}_\lambda$ that preserves the measures i.e. $\lambda(\phi(A)) = \mu(A)$.*

For σ -finite measure spaces, we have the following extension:

Proposition 5.2 *Let (Ω, Σ, μ) be a non-atomic separable σ -finite measure space and $((0, \infty), \mathcal{B}, \lambda)$. Then, there exists an isomorphism $\phi : \Sigma/\mathcal{N}_\mu \rightarrow \mathcal{B}/\mathcal{N}_\lambda$ that preserves the measures i.e. $\lambda(\phi(A)) = \mu(A)$.*

Proof Since Ω is σ -finite, consider a sequence (A_n) of sets with $\mu(A_n) < \infty$ such that $\Omega = \bigcup_{n=1}^\infty A_n$. Taking $B_n := A_n \setminus \bigcup_{i < n} A_i$, we have $\Omega = \bigcup_{n=1}^\infty B_n$ with (B_n) mutually disjoint and $\mu(B_n) < \infty$. Thus, if we take an increasing sequence $(x_n) \nearrow \infty$ in $(0, \infty)$ with $x_1 = 0$ and, for all $n \in \mathbb{N}$, $x_{n+1} - x_n = \mu(B_n)$, then the above Carathéodory theorem says that, for every natural n , the measure spaces $(B_n, \Sigma|_{B_n}, \mu|_{B_n})$ and $((x_n, x_{n+1}], \mathcal{B}|_{(x_n, x_{n+1}]}, \lambda|_{(x_n, x_{n+1}]})$ are isomorphic (modulo sets of zero measure). Also, $(\bigcup_{n=1}^\infty \Sigma|_{B_n})/\mathcal{N}_\mu = \bigcup_{n=1}^\infty (\Sigma|_{B_n}/\mathcal{N}_\mu)$ and $(\bigcup_{n=1}^\infty \mathcal{B}|_{(x_n, x_{n+1}]})/\mathcal{N}_\lambda = \bigcup_{n=1}^\infty (\mathcal{B}|_{(x_n, x_{n+1]}}/\mathcal{N}_\lambda)$. In consequence, Σ/\mathcal{N}_μ and $\mathcal{B}/\mathcal{N}_\lambda$ are isomorphic too. □

We can now pursue a representation Theorem. To begin with, given the measure preserving isomorphism ϕ from Σ/\mathcal{N}_μ to $\mathcal{B}/\mathcal{N}_\lambda$, take any simple function $S = \sum_{n \in \Delta} a_n \chi_{A_n}$ in $L_0(\Omega)$ (we assume simple functions in this section with the canonical decomposition, i.e. (A_n) is a disjoint finite sequence such that $\Omega = \bigcup_{n \in \Delta} A_n$). Define the transformation

$$\widehat{S} := \sum_{n \in \Delta} a_n \chi_{\phi(A_n)}$$

into a simple function in $L_0(0, \infty)$. Recall that given a positive measurable function f , there exists an increasing sequence of simple functions (S_n) such that $S_n \nearrow f$ pointwise (cf. [5]). Thus, for a positive function $f \in L_0(\Omega)$, we define the transformation

$$\widehat{f} := \lim_{n \rightarrow \infty} \widehat{S}_n$$

into $L_0(0, \infty)$, where (S_n) is an increasing sequence of positive simple functions such that $S_n \nearrow f$, since the sequence (\widehat{S}_n) is also increasing a.e- λ . The next proposition shows that \widehat{f} is well defined.

Proposition 5.3 *Let (S_n) and (R_n) be increasing sequences of positive simple functions in $L_0(\Omega)$ such that both $S_n \nearrow f$, $R_n \nearrow f$ pointwise. Then, both $\widehat{S}_n \nearrow \widehat{f}$ and $\widehat{R}_n \nearrow \widehat{f}$.*

Proof As (\widehat{S}_n) and (\widehat{R}_n) are increasing sequences, they have a λ -pointwise limit, noted $\widehat{S}_n \nearrow \widehat{f}$ and $\widehat{R}_n \nearrow \widehat{g}$. We have to prove that $\widehat{f}(s) = \widehat{g}(s)$ a.e- λ . Assume there is $\varepsilon > 0$ such that,

$$\lambda(D_\varepsilon) := \lambda(\{x : |\widehat{f}(x) - \widehat{g}(x)| > \varepsilon\}) > 0.$$

Then, we can take $C \subset D_\varepsilon$ with $0 < \mu(C) < \infty$. Let $A = \phi^{-1}(C) \in \Sigma$ so $\mu(A) = \lambda(C)$. By Egorov's theorem, we can suppose w.l.o.g. that the convergence on C (and on $\phi^{-1}(C) = A$) is uniform, i.e, $S_n|_A \nearrow f|_A$, $R_n|_A \nearrow f|_A$, $\widehat{S}_n|_C \nearrow \widehat{f}|_C$ and $\widehat{R}_n|_C \nearrow \widehat{g}|_C$ uniformly. So,

$$|S_n - R_n| \leq |S_n - f| + |f - R_n| \rightarrow 0, \text{ uniformly on } A,$$

and then $|\widehat{S}_n - \widehat{R}_n| \rightarrow 0$ uniformly on C . Finally,

$$|\widehat{f}|_C - \widehat{g}|_C| \leq |\widehat{f}|_C - \widehat{S}_n|_C| + |\widehat{S}_n|_C - \widehat{R}_n|_C| + |\widehat{R}_n|_C - \widehat{g}|_C| \rightarrow 0$$

uniformly in C , which is a contradiction. □

Similarly, it holds that if (S_n) is a decreasing sequence of positive simple functions in $L_0(\Omega)$ such that $S_n \searrow f$ pointwise, then, $\widehat{S}_n \searrow \widehat{f}$ a.e- λ .

We pass to extend this map to general measurable functions, getting an isometry between the spaces $L^{p(\cdot)}(\Omega)$ and $L^{\widehat{p}(\cdot)}(0, \infty)$ for non-atomic separable σ -finite measure spaces (Ω, μ) .

Theorem 5.4 *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ on a non-atomic separable σ -finite measure space (Ω, μ) . The operator $T_\phi : L^{p(\cdot)}(\Omega) \rightarrow L^{\widehat{p}(\cdot)}(0, \infty)$ defined by*

$$T_\phi(f) := \widehat{f}^+ - \widehat{f}^-,$$

where $f = f^+ - f^-$, is an isomodular lattice isomorphism, i.e. for every $f \in L^{p(\cdot)}(\Omega)$ and denoting $T_\phi(f) = \widehat{f}$,

$$\int_\Omega |f(t)|^{p(t)} d\mu(t) = \int_0^\infty |\widehat{f}(s)|^{\widehat{p}(s)} d\lambda(s).$$

Before we prove this theorem we state several easy preliminary lemmas.

Lemma 5.5 *Let $f, g \in L_0(\Omega)$ with $f, g \geq 0$. Then, $\widehat{f + g} = \widehat{f} + \widehat{g}$.*

Proof Let $S_n \nearrow f$ and $R_n \nearrow g$ pointwise with (S_n) and (R_n) increasing simple functions. We can write

$$S_n + R_n = \sum_{k \in \Delta_n} \sum_{j \in \Gamma_n} (a_{nk} + b_{nj}) \chi_{A_{nk} \cap B_{nj}}.$$

As $S_n + R_n \nearrow f + g$, we have

$$\begin{aligned} \widehat{f + g} &= \lim_{n \rightarrow \infty} \widehat{S_n + R_n} = \lim_{n \rightarrow \infty} \sum_{k \in \Delta_n} \sum_{j \in \Gamma_n} (a_{nk} + b_{nj}) \chi_{\phi(A_{nk} \cap B_{nj})} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k \in \Delta_n} a_{nk} \chi_{\phi(A_{nk})} + \sum_{j \in \Gamma_n} b_{nj} \chi_{\phi(B_{nj})} \right) = \lim_{n \rightarrow \infty} (\widehat{S_n} + \widehat{R_n}) = \widehat{f} + \widehat{g}. \end{aligned}$$

□

Lemma 5.6 *Let $f, g \in L_0(\Omega)$ with $g < 0$ and $f \geq |g|$. Then, $\widehat{f + g} = \widehat{f} + \widehat{g}$.*

Proof Firstly, suppose that g has a finite measure support. Then, we can take increasing sequences of simple functions (S_n) and (R_n) such that $S_n \nearrow f$ and $R_n \nearrow g$ (which exists as g has a finite measure support). Hence, $S_n + R_n \nearrow f + g$, so (for $\widehat{g} = \lim_{n \rightarrow \infty} -(-R_n)$)

$$\widehat{f + g} = \lim_{n \rightarrow \infty} \widehat{S_n + R_n} = \lim_{n \rightarrow \infty} (\widehat{S_n} + \widehat{R_n}) = \lim_{n \rightarrow \infty} (\widehat{S_n} - (-\widehat{R_n})) = \widehat{f} + \widehat{g}.$$

Now, if g has an infinite measure support, let us take $f_n := f \chi_{\Omega_n}$ and $g_n := g \chi_{\Omega_n}$ for a sequence $(\Omega_n) \nearrow \Omega$ with $\mu(\Omega_n) < \infty$. As, $f_n + g_n \nearrow f + g$, then $\widehat{f_n + g_n} \nearrow \widehat{f + g}$. But, for each $n \in \mathbb{N}$, we have $\widehat{f_n + g_n} = \widehat{f_n} + \widehat{g_n}$ and $\widehat{f_n} + \widehat{g_n} \nearrow \widehat{f} + \widehat{g}$, so we conclude $\widehat{f + g} = \widehat{f} + \widehat{g}$. □

Lemma 5.7 *Let $p(\cdot)$ be an exponent function and $f \in L_0(\Omega)$ with $f \geq 0$ and finite measure support. Then, $\widehat{f^p} = \widehat{f}^{\widehat{p}}$.*

Proof As $f = f \chi_{\{f \geq 1\}} + f \chi_{\{0 \leq f < 1\}}$, by Lemma 5.5 we just need to show $\widehat{f^p} = \widehat{f}^{\widehat{p}}$ for $f \geq 1$ and for $0 \leq f < 1$. Both proofs are similar, so we just do the $0 \leq f < 1$ case. Let $S_n = \sum_{k \in \Delta_n} a_{nk} \chi_{A_{nk}} \geq 1$ be an increasing sequence of simple functions such that $S_n \nearrow p$ and let $R_n = \sum_{j \in \Gamma_n} b_{nj} \chi_{B_{nj}} \leq 1$ be a decreasing sequence of simple functions with $R_n \searrow f$ (which exists since f has a finite measure support). Then, the sequence of simple functions $(R_n^{S_n})$ is decreasing and $R_n^{S_n} \searrow f^p$ pointwise. Hence,

$$\begin{aligned} \widehat{f^p} &= \lim_{n \rightarrow \infty} \sum_{j \in \Gamma_n} \sum_{k \in \Delta_n} b_{nj}^{a_{nk}} \chi_{\phi(B_{nj} \cap A_{nk})} \\ &= \lim_{n \rightarrow \infty} \sum_{j \in \Gamma_n} \sum_{k \in \Delta_n} b_{nj}^{a_{nk}} \chi_{\phi(B_{nj}) \cap \phi(A_{nk})} = \lim_{n \rightarrow \infty} \widehat{R_n^{S_n}} = \widehat{f}^{\widehat{p}}. \end{aligned}$$

□

Note that it is also easily deduced that $\widehat{f \chi_A} = \widehat{f} \chi_{\phi(A)}$. We are now ready to

Proof of Theorem 5.4 First, let us show that T_ϕ is linear. Given $f, g \in L_0(\Omega)$, we have

$$\begin{aligned} f + g &= (f + g)^+ - (f + g)^- \\ &= (f^+ + g^+) + (f^+ - g^-) \chi_{\{|f| \geq |g|\}} + (g^+ - f^-) \chi_{\{|f| \leq |g|\}} \\ &\quad - (f^- + g^-) - (f^- - g^+) \chi_{\{|f| > |g|\}} - (g^- - f^+) \chi_{\{|f| < |g|\}}, \end{aligned}$$

so we just have to apply Lemmas 5.5 and 5.6 to get $\widehat{f + g} = \widehat{f} + \widehat{g}$. Also, trivially $\widehat{cf} = c\widehat{f}$ for any constant c .

Now we have to show that, for every $f \in L^{p(\cdot)}(\Omega)$,

$$\int_{\Omega} |f(t)|^{p(t)} d\mu(t) = \int_0^{\infty} |\widehat{f}(s)|^{\widehat{p}(s)} d\lambda(s).$$

We only need to do it for positive functions, since

$$\int_{\Omega} |f(t)|^{p(t)} d\mu(t) = \int_{\Omega \cap \{f \geq 0\}} |f^+(t)|^{p(t)} d\mu(t) + \int_{\Omega \cap \{f \leq 0\}} |f^-(t)|^{p(t)} d\mu(t).$$

Suppose first that $f \geq 0$ has a finite measure support. Then, we can take sequences of simple functions (p_n) , (S_n) and (R_n) such that $(p_n) \nearrow p(\cdot)$, $(S_n) \nearrow f\chi_{\{f \geq 1\}}$ and $(R_n) \searrow f\chi_{\{f < 1\}}$. Hence, by the Lebesgue Theorem and Lemma 5.7,

$$\begin{aligned} \int_{\Omega} f(t)^{p(t)} d\mu &= \int_{\Omega \cap \{f \geq 1\}} f(t)^{p(t)} d\mu + \int_{\Omega \cap \{f < 1\}} f(t)^{p(t)} d\mu \\ &= \lim_{n \rightarrow \infty} \int_{\Omega \cap \{f \geq 1\}} S_n(t)^{p_n(t)} d\mu + \lim_{n \rightarrow \infty} \int_{\Omega \cap \{f < 1\}} R_n(t)^{p_n(t)} d\mu \\ &= \lim_{n \rightarrow \infty} \int_{(0, \infty) \cap \{\widehat{f} \geq 1\}} \widehat{S}_n(s)^{\widehat{p}_n(s)} d\lambda + \lim_{n \rightarrow \infty} \int_{(0, \infty) \cap \{\widehat{f} < 1\}} \widehat{R}_n(s)^{\widehat{p}_n(s)} d\lambda \\ &= \int_0^{\infty} \widehat{f}(s)^{\widehat{p}(s)} \chi_{\{\widehat{f} \geq 1\}} d\lambda + \int_0^{\infty} \widehat{f}(s)^{\widehat{p}(s)} \chi_{\{\widehat{f} < 1\}} d\lambda = \int_0^{\infty} \widehat{f}(s)^{\widehat{p}(s)} d\lambda. \end{aligned}$$

Suppose now that $f \geq 0$ has an infinite measure support. Then, we can take $(\Omega_n) \nearrow \Omega$ such that $\mu(\Omega_n) < \infty$ for every natural n . As $\int_{\{f \geq 1\}} f^p(\cdot) < \infty$, then $f\chi_{\{f \geq 1\}}$ has a finite measure support and $f\chi_{\{f < 1\}}$ has an infinite measure support. Let us define $f_n := f\chi_{\Omega_n}$. Hence, every f_n has a finite measure support and, by the dominated convergence theorem, we have

$$\int_{\{f < 1\}} f^{p(t)} d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n^{p(t)} d\mu = \lim_{n \rightarrow \infty} \int_0^{\infty} \widehat{f}_n^{\widehat{p}(s)} d\lambda = \int_{\{\widehat{f} < 1\}} \widehat{f}^{\widehat{p}(s)} d\lambda.$$

Finally, it is clear that T_ϕ is onto using that it is isomodular and that ϕ is bijective. □

It is worth noting that if we compose the above isometry operator T_ϕ with the isometry operator T_τ considered in Sect. 4, we get the following representation theorem:

Corollary 5.8 *Every $L^{p(\cdot)}(\Omega)$ space with $p^+ < \infty$ on a non-atomic separable σ -finite measure space (Ω, μ) is isomodular lattice isomorphic to $L^{q(\cdot)}(0, 1)$ for a suitable exponent function $q(\cdot)$ with the same essential ranges $R_{p(\cdot)} = R_{q(\cdot)}$.*

Remark 5.9 The above is not true in the class of Orlicz spaces: there exist separable Orlicz spaces $L^\varphi(\mu)$ for an infinite measure μ that are not isomorphic to any Orlicz space $L^\psi(0, 1)$ (for example the Orlicz spaces $L^\varphi(\mu) = L_p + L_q(0, \infty)$ for $1 < p < q < 2$, see [16] page 210).

We pass now to give some applications of the above representation theorem. Thus, several results given in Sect. 4 for spaces $L^{p(\cdot)}(0, \infty)$ are now easily extended to the framework of $L^{p(\cdot)}(\Omega)$ spaces for non-atomic separable σ -finite measure spaces (Ω, μ) .

Proposition 5.10 *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if S is norm bounded and, for each $g \in L^{p^*(\cdot)}(\Omega)$ and each sequence of measurable sets $A_n \searrow \emptyset$,*

$$\limsup_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n} |fg| d\mu = 0.$$

Proof Given $S \subset L^{p(\cdot)}(\Omega)$, consider $\widehat{S} \subset L^{\widehat{p}(\cdot)}(0, \infty)$ as in above Theorem 5.4. Since both spaces are isomorphic, S is relatively weakly compact if and only if \widehat{S} is relatively weakly compact. Thus, by Theorem 4.1 we have just to show that

$$\limsup_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n} |fg| d\mu = 0, \quad A_n \subset \Omega,$$

if and only if

$$\limsup_{E_n \searrow \emptyset} \sup_{\widehat{f} \in \widehat{S}} \int_{E_n} |\widehat{f}\widehat{g}| d\lambda = 0, \quad E_n \subset (0, \infty).$$

It is clear by Proposition 5.2 that any decreasing sequence $E_n \searrow \emptyset$ in $(0, \infty)$ can be expressed (up to a zero measure set) as $E_n = \phi(A_n)$ for some decreasing sequence $A_n \searrow \emptyset$ in Ω with $\mu(A_n) = \lambda(E_n)$, and vice versa. Furthermore, for all $x \in \mathbb{R}$, we have $\mu(\{t \in \Omega : (fg)(t) \leq x\}) = \lambda(\{s \in (0, \infty) : (\widehat{f}\widehat{g})(s) \leq x\})$, so the integrals coincide and the equivalence is proved. \square

The following results are deduced similarly. Thus, from Theorem 4.3 and Dunford-Pettis Theorem 1.1 we get:

Proposition 5.11 *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if it is norm bounded,*

$$\limsup_{\lambda \rightarrow 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_\lambda} |\lambda f(t)|^{p(t)} d\mu = 0$$

and

$$\limsup_{A_n \searrow \emptyset} \sup_{f \in S} \int_{A_n \cap \Omega_1} |f(t)| d\mu = 0.$$

Also, from Proposition 4.6 we deduce that $L^{p(\cdot)}(\Omega)$ is weakly Banach-Saks if and only if $p^+ < \infty$.

Finally, criterion for the weak convergence follows from Proposition 4.5:

Proposition 5.12 *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$. A sequence (f_n) in $L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega)$ if and only if the following three conditions hold:*

- (i) $\lim_n \int_A f_n d\mu = \int_A f d\mu$ for each $A \in \Sigma$ with $\mu(A) < \infty$,
- (ii) $\lim_{\lambda \rightarrow 0} \sup_n \frac{1}{\lambda} \int_{\Omega \setminus \Omega_\lambda} |\lambda(f_n - f)|^{p(t)} d\mu = 0$, and
- (iii) $\lim_{A_k \searrow \emptyset} \sup_{n \in \mathbb{N}} \int_{A_k \cap \Omega_1} |f_n(t) - f(t)| d\mu = 0$.

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