

Power divergence approach for one-shot device testing under competing risks.

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Abstract

Most work on one-shot devices assume that there is only one possible cause of device failure. However, in practice, it is often the case that the products under study can experience any one of various possible causes of failure. Robust estimators and Wald-type tests are developed here for the case of one-shot devices under competing risks. An extensive simulation study illustrates the robustness of these divergence-based estimators and test procedures based on them. A data-driven procedure is proposed for choosing the optimal estimator for any given data set which is then applied to an example in the context of survival analysis.

1 Introduction

In lifetime data analysis, it is often the case that the products under study can experience one of different types of failure. For example, in the context of survival analysis, we can have several different types of failure (death, relapse, opportunistic infection, etc.) that are of interest to us, leading to the so-called “competing risks” scenario. A competing risk is an event whose occurrence precludes the occurrence of the primary event of interest. In a study examining time to death attributable, for instance, to cardiovascular causes, death attributable to noncardiovascular causes would be a competing risk. Crowder (2001) has presented review of this competing risks problem for which one needs to estimate the failure rates for each cause. Balakrishnan et al. (2016a, 2016b) and So (2016) have discussed the problem of one-shot devices under competing risk for the first time.

One-shot device testing data, also known as current status data in survival analysis, come from testing one-shot devices that are tested only once and get destroyed right after the test. It is very common that one-shot devices contain multiple components and failure of any of them will lead to the failure of the device. In Balakrishnan et al (2019a, 2019b, 2019c, 2020), new estimators and Wald-type tests for one-shot devices models have been proposed. The estimators introduced, namely, weighted minimum density power divergence estimators, and the corresponding Wald-type tests, demonstrated good behaviour and performance in terms of robustness without a significant loss of efficiency. However, it was assumed that there is only one survival endpoint of interest, and that censoring is independent of the event in interest. The main purpose of this work is to develop weighted minimum density power divergence estimators as well as Wald-type test statistics under competing risk models for one-shot device testing assuming exponential lifetimes.

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In Section 2, we present the model formulation as well as the notation to be used the rest of the paper. In this section, we also describe the relation between maximum likelihood estimator and minimization of the Kullback-Leibler divergence between appropriate distributions. The weighted minimum density power divergence estimators for one-shot device testing exponential model under competing risks are then developed in Section 3. Their asymptotic distribution and a new family of Wald-type test statistics based on them are also presented in this section. In Section 4, an extensive Monte Carlo simulation study is carried out for demonstrating the robust behaviour of the proposed estimators as well as the testing procedures. An ad hoc procedure for the choice of the optimal estimator is also provided in this section. The developed methods are then applied to a pharmacology data for illustrative purposes. Finally, some concluding remarks are made in Section 6, while the proofs of all the main results are presented in Appendix A.

2 Model formulation

In this section, we shall introduce the notation necessary for the developments in this paper, paying special attention to the maximum likelihood estimator (MLE) of the model, as well as its relation with the minimization of Kullback-Leibler divergence.

The setting for an accelerate life-test for one-shot devices under competing risks considered here is stratified in I testing conditions as follows:

1. The tests are checked at inspection times IT_i , for $i = 1, \dots, I$;
2. The devices are tested under J different stress levels, $\mathbf{x}_i = (x_{i1}, \dots, x_{iJ})^T$, for $i = 1, \dots, I$;
3. K_i devices are tested in the i th test condition, for $i = 1, \dots, I$;
4. The number of devices failed due to the r -th cause under the i -th test condition is denoted by n_{ir} , for $i = 1, \dots, I$, $r = 1, \dots, R$;
5. The number of devices that survive under the i -th test condition is denoted by $n_{i0} = K_i - \sum_{r=1}^R n_{ir}$.

Table 1: One-shot device testing under competing risks.

Condition	Times	Devices	Survivals	Failures			Stress levels		
				Cause 1	...	Cause R	Stress 1	...	Stress J
1	IT_1	K_1	n_{10}	n_{11}	...	n_{1R}	x_{11}	...	x_{1J}
2	IT_2	K_2	n_{20}	n_{21}	...	n_{2R}	x_{21}	...	x_{2J}
\vdots	\vdots	\vdots		\vdots		\vdots	\vdots		\vdots
I	IT_I	K_I	n_{I0}	n_{I1}	...	n_{IR}	x_{I1}	...	x_{IJ}

This setting is summarized in Table 1. For simplicity, and as considered in Balakrishnan et al. (2016a), we will limit in this paper, the number of stress levels to $J = 1$ and the number of competing causes to $R = 2$, even though inference for the general case when $J > 1$ and $R > 2$ can be presented in an analogous manner.

Let us denote the random variable for the failure time due to causes 1 and 2 as T_{irk} , for $r = 1, 2$, $i = 1, \dots, I$, and $k = 1, \dots, K_i$, respectively. We now assume that T_{irk} follows

an exponential distribution with failure rate parameter $\lambda_{ir}(\boldsymbol{\theta})$ and its probability density function

$$\begin{aligned} f_r(t; x_i, \boldsymbol{\theta}) &= \lambda_{ir}(\boldsymbol{\theta}) e^{-\lambda_{ir}(\boldsymbol{\theta})t}, \quad t > 0, \\ \lambda_{ir}(\boldsymbol{\theta}) &= \theta_{r0} \exp(\theta_{r1} x_i), \\ \boldsymbol{\theta} &= (\theta_{10}, \theta_{11}, \theta_{20}, \theta_{21})^T, \quad \theta_{r0}, \theta_{r1} > 0, \quad r = 1, 2, \end{aligned}$$

where x_i is the stress factor of the condition i and $\boldsymbol{\theta}$ is the model parameter vector, with $\boldsymbol{\theta} \in \mathbb{R}^4$.

We shall use $\pi_{i0}(\boldsymbol{\theta})$, $\pi_{i1}(\boldsymbol{\theta})$ and $\pi_{i2}(\boldsymbol{\theta})$ for the survival probability, failure probability due to cause 1 and failure probability due to cause 2, respectively. Their expressions are

$$\begin{aligned} \pi_{i0}(\boldsymbol{\theta}) &= (1 - F_1(IT_i; x_i, \boldsymbol{\theta}))(1 - F_2(IT_i; x_i, \boldsymbol{\theta})) = \exp(-(\lambda_{i1} + \lambda_{i2})IT_i), \\ \pi_{i1}(\boldsymbol{\theta}) &= \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)), \\ \pi_{i2}(\boldsymbol{\theta}) &= \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)), \end{aligned}$$

where $\lambda_{ir} = \lambda_{ir}(\boldsymbol{\theta})$, $r = 1, 2$. Derivations of these expressions can be found in So (2016, p. 151). Now, the likelihood function is given by

$$\mathcal{L}(n_{01}, \dots, n_{I2}; \boldsymbol{\theta}) \propto \prod_{i=1}^I \pi_{i0}(\boldsymbol{\theta})^{n_{i0}} \pi_{i1}(\boldsymbol{\theta})^{n_{i1}} \pi_{i2}(\boldsymbol{\theta})^{n_{i2}}, \quad (1)$$

where $n_{0i} + n_{1i} + n_{2i} = K_i$, $i = 1, \dots, I$.

Definition 1 (MLE, classical definition) *The maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$, denoted by $\hat{\boldsymbol{\theta}}$, is obtained by maximizing the likelihood function in (1) or, equivalently, its logarithm.*

We will present an alternative definition of the MLE later on (see Definition 3). Let us introduce the following probability vectors:

$$\hat{\mathbf{p}}_i = (\hat{p}_{i0}, \hat{p}_{i1}, \hat{p}_{i2})^T = \frac{1}{K_i} (n_{i0}, n_{i1}, n_{i2})^T, \quad i = 1, \dots, I, \quad (2)$$

$$\boldsymbol{\pi}_i(\boldsymbol{\theta}) = (\pi_{i0}(\boldsymbol{\theta}), \pi_{i1}(\boldsymbol{\theta}), \pi_{i2}(\boldsymbol{\theta}))^T, \quad i = 1, \dots, I. \quad (3)$$

The Kullback-Leibler divergence measure (see, for instance, Pardo (2006)), between $\hat{\mathbf{p}}_i$ and $\boldsymbol{\pi}_i(\boldsymbol{\theta})$, is given by

$$\begin{aligned} d_{KL}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) &= \sum_{r=0}^2 \hat{p}_{ir} \log \left(\frac{\hat{p}_{ir}}{\pi_{ir}(\boldsymbol{\theta})} \right) \\ &= \hat{p}_{i0} \log \left(\frac{\hat{p}_{i0}}{\pi_{i0}(\boldsymbol{\theta})} \right) + \hat{p}_{i1} \log \left(\frac{\hat{p}_{i1}}{\pi_{i1}(\boldsymbol{\theta})} \right) + \hat{p}_{i2} \log \left(\frac{\hat{p}_{i2}}{\pi_{i2}(\boldsymbol{\theta})} \right) \\ &= \frac{n_{i0}}{K_i} \log \left(\frac{n_{i0}/K_i}{\pi_{i0}(\boldsymbol{\theta})} \right) + \frac{n_{i1}}{K_i} \log \left(\frac{n_{i1}/K_i}{\pi_{i1}(\boldsymbol{\theta})} \right) + \frac{n_{i2}}{K_i} \log \left(\frac{n_{i2}/K_i}{\pi_{i2}(\boldsymbol{\theta})} \right) \\ &= \frac{1}{K_i} \left\{ n_{i0} \log \left(\frac{n_{i0}/K_i}{\pi_{i0}(\boldsymbol{\theta})} \right) + n_{i1} \log \left(\frac{n_{i1}/K_i}{\pi_{i1}(\boldsymbol{\theta})} \right) + n_{i2} \log \left(\frac{n_{i2}/K_i}{\pi_{i2}(\boldsymbol{\theta})} \right) \right\}, \end{aligned}$$

and the weighted Kullback-Leibler divergence measure is given by

$$\begin{aligned} d_{KL}^W(\boldsymbol{\theta}) &= \sum_{i=1}^I \frac{K_i}{K} d_{KL}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) \\ &= \frac{1}{K} \sum_{i=1}^I \left\{ n_{i0} \log \left(\frac{n_{i0}/K_i}{\pi_{i0}(\boldsymbol{\theta})} \right) + n_{i1} \log \left(\frac{n_{i1}/K_i}{\pi_{i1}(\boldsymbol{\theta})} \right) + n_{i2} \log \left(\frac{n_{i2}/K_i}{\pi_{i2}(\boldsymbol{\theta})} \right) \right\}, \end{aligned}$$

with $K = K_1 + \dots + K_I$.

Theorem 2 *The likelihood function $\mathcal{L}(n_{01}, \dots, n_{I2}; \boldsymbol{\theta})$, given in (1), is related to the weighted Kullback-Leibler divergence measure through*

$$d_{KL}^W(\boldsymbol{\theta}) = \sum_{i=1}^I \frac{K_i}{K} d_{KL}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) = c - \frac{1}{K} \log \mathcal{L}(n_{01}, \dots, n_{I2}; \boldsymbol{\theta}), \quad (4)$$

with c being a constant, not dependent on $\boldsymbol{\theta}$.

Definition 3 (MLE, alternative definition) *The MLE of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}$, can be obtained by the minimization of the weighted Kullback-Leibler divergence measure given in (4).*

Example 4 (The BDC experiment) *The benzidine dihydrochloride (BDC) experiment, studied in Lindsey and Ryan (1993) and conducted at the National Center for Toxicological Research, examines the incidence in mice of liver tumors induced by the drug. Two different doses of drug are induced in the mice: 60 parts per million ($w=1$) and 400 parts per million ($w=2$) and two causes of death are recorded: died without tumor ($\delta_{ijk} = 1$) and died with tumor ($\delta_{ijk} = 2$). These data are presented in Table 2. With these data, we*

Table 2: BDC experiment

		$\delta_{ijk} = 0$	$\delta_{ijk} = 1$	$\delta_{ijk} = 2$
$IT_1 = 9.37$	$w = 1$	70	2	0
	$w = 2$	22	3	0
$IT_2 = 14.07$	$w = 1$	48	1	0
	$w = 2$	14	4	17
$IT_3 = 18.7$	$w = 1$	35	4	7
	$w = 2$	1	1	9

obtained the MLE of the vector of parameters and also measured the discrepancy of the corresponding estimated rates and the observed ones, given by

$$\frac{1}{3I} \sum_{i=1}^I \sum_{r=0}^2 \left| \frac{n_{ir} - K_i \pi_{ir}(\hat{\boldsymbol{\theta}})}{K_i} \right|. \quad (5)$$

These results are presented in Table 3. In the ensuing work, we will present alternative estimators to the MLE, which are seen to provide better performance in terms of robustness.

Table 3: MLEs of parameters based on the BDC experiment

	θ_{10}	θ_{11}	θ_{20}	θ_{21}	estimated error
MLE	0.00089	1.3191	0.00028	2.493	0.1051

3 Weighted minimum density power divergence estimator

In this section, we shall introduce the weighted minimum density power divergence estimator as a natural extension of the MLE. For this purpose, we shall introduce the ordinary density power divergence (DPD). Given these two probability vectors $\hat{\mathbf{p}}_i$ and $\boldsymbol{\pi}_i(\boldsymbol{\theta})$, defined in (2) and (3), respectively, the DPD between both probability vectors is given by

$$\begin{aligned} d_\beta(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) &= \left(\pi_{i0}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i1}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i2}^{\beta+1}(\boldsymbol{\theta}) \right) \\ &\quad - \frac{\beta+1}{\beta} \left(\hat{p}_{i0} \pi_{i0}^\beta(\boldsymbol{\theta}) + \hat{p}_{i1} \pi_{i1}^\beta(\boldsymbol{\theta}) + \hat{p}_{i2} \pi_{i2}^\beta(\boldsymbol{\theta}) \right) \\ &\quad + \frac{1}{\beta} \left(\hat{p}_{i0}^{\beta+1} + \hat{p}_{i1}^{\beta+1} + \hat{p}_{i2}^{\beta+1} \right), \quad \text{if } \beta > 0, \end{aligned}$$

and $d_{\beta=0}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) = \lim_{\beta \rightarrow 0^+} d_\beta(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) = d_{KL}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta}))$, for $\beta = 0$.

The weighted DPD is given by

$$\begin{aligned} d_\beta^W(\boldsymbol{\theta}) &= \sum_{i=1}^I \frac{K_i}{K} \left[\left(\pi_{i0}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i1}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i2}^{\beta+1}(\boldsymbol{\theta}) \right) \right. \\ &\quad \left. - \frac{\beta+1}{\beta} \left(\hat{p}_{i0} \pi_{i0}^\beta(\boldsymbol{\theta}) + \hat{p}_{i1} \pi_{i1}^\beta(\boldsymbol{\theta}) + \hat{p}_{i2} \pi_{i2}^\beta(\boldsymbol{\theta}) \right) + \frac{1}{\beta} \left(\hat{p}_{i0}^{\beta+1} + \hat{p}_{i1}^{\beta+1} + \hat{p}_{i2}^{\beta+1} \right) \right] \end{aligned}$$

but the term $\frac{1}{\beta} \left(\hat{p}_{i0}^{\beta+1} + \hat{p}_{i1}^{\beta+1} + \hat{p}_{i2}^{\beta+1} \right)$, $i = 1, \dots, I$, does not have any role in its minimization with respect to $\boldsymbol{\theta}$. Therefore, in order to minimize $d_\beta^W(\boldsymbol{\theta})$, we can consider the equivalent measure

$$\begin{aligned} {}^*d_\beta^W(\boldsymbol{\theta}) &= \sum_{i=1}^I \frac{K_i}{K} \left[\left(\pi_{i0}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i1}^{\beta+1}(\boldsymbol{\theta}) + \pi_{i2}^{\beta+1}(\boldsymbol{\theta}) \right) \right. \\ &\quad \left. - \frac{\beta+1}{\beta} \left(\hat{p}_{i0} \pi_{i0}^\beta(\boldsymbol{\theta}) + \hat{p}_{i1} \pi_{i1}^\beta(\boldsymbol{\theta}) + \hat{p}_{i2} \pi_{i2}^\beta(\boldsymbol{\theta}) \right) \right]. \end{aligned} \quad (6)$$

Definition 5 We can define the weighted minimum density power divergence estimator of $\boldsymbol{\theta}$ as

$$\hat{\boldsymbol{\theta}}_\beta = \arg \min_{\boldsymbol{\theta} \in \Theta} {}^*d_\beta^W(\boldsymbol{\theta}), \quad \text{for } \beta > 0$$

and for $\beta = 0$ we get the weighted maximum likelihood estimator.

Theorem 6 The weighted minimum density power estimator of $\boldsymbol{\theta}$, with tuning parameter $\beta \geq 0$, $\hat{\boldsymbol{\theta}}_\beta$, can be obtained as the solution of the following equation:

$$\sum_{i=1}^I K_i \left\{ -\pi_{i0}(\boldsymbol{\theta}) IT_i \left[\pi_{i0}(\boldsymbol{\theta})^{\beta-1} (\pi_{i0}(\boldsymbol{\theta}) - p_{i0}) - (1 - \pi_{i0}(\boldsymbol{\theta}))^{\beta-1} \Gamma_{i,\beta} \right] \mathbf{l}_i + (1 - \pi_{i0}(\boldsymbol{\theta}))^\beta \Gamma_{i,\beta}^* \right\} = \mathbf{0}_4,$$

where

$$\Gamma_{i,\beta} = \frac{\lambda_{i1}^\beta \left[\frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}(1 - \pi_{i0}(\boldsymbol{\theta})) - p_{i1} \right] + \lambda_{i2}^\beta \left[\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}(1 - \pi_{i0}(\boldsymbol{\theta})) - p_{i2} \right]}{(\lambda_{i1} + \lambda_{i2})^\beta},$$

$$\Gamma_{i,\beta}^* = \frac{\lambda_{i1}^{\beta-1} \left[\frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}(1 - \pi_{i0}(\boldsymbol{\theta})) - p_{i1} \right] - \lambda_{i2}^{\beta-1} \left[\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}(1 - \pi_{i0}(\boldsymbol{\theta})) - p_{i2} \right]}{(\lambda_{i1} + \lambda_{i2})^{\beta-1}},$$

$$\mathbf{l}_i = (\lambda_{i1}/\theta_{10}, \lambda_{i1}x_i, \lambda_{i2}/\theta_{20}, \lambda_{i2}x_i)^T \text{ and } \mathbf{r}_i = \frac{\lambda_{i1}\lambda_{i2}}{(\lambda_{i1}+\lambda_{i2})^2}(1/\theta_{10}, x_i, -1/\theta_{20}, -x_i)^T.$$

Now, by using the Theorem 3.1 in Ghosh and Basu (2013), we can obtain the asymptotic distribution of the above weighted minimum density power divergence estimator.

Theorem 7 *Let $\boldsymbol{\theta}^0$ be the true value of the parameter $\boldsymbol{\theta}$. The asymptotic distribution of the weighted minimum density power divergence estimator of $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}_\beta$, is given by*

$$\sqrt{K} \left(\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^0 \right) \xrightarrow{K \rightarrow \infty} \mathcal{N} \left(\mathbf{0}_4, \mathbf{J}_\beta^{-1}(\boldsymbol{\theta}^0) \mathbf{K}_\beta(\boldsymbol{\theta}^0) \mathbf{J}_\beta^{-1}(\boldsymbol{\theta}^0) \right),$$

where

$$\mathbf{J}_\beta(\boldsymbol{\theta}) = \sum_{i=1}^I \sum_{r=0}^2 \frac{K_i}{K} \mathbf{u}_{ir}^*(\boldsymbol{\theta}) \mathbf{u}_{ir}^{*T}(\boldsymbol{\theta}) \pi_{ir}^{\beta-1}(\boldsymbol{\theta}), \quad (7)$$

$$\mathbf{K}_\beta(\boldsymbol{\theta}) = \sum_{i=1}^I \sum_{r=0}^2 \frac{K_i}{K} \mathbf{u}_{ir}^*(\boldsymbol{\theta}) \mathbf{u}_{ir}^{*T}(\boldsymbol{\theta}) \pi_{ir}^{2\beta-1}(\boldsymbol{\theta}) - \sum_{i=1}^I \frac{K_i}{K} \boldsymbol{\xi}_{i,\beta}(\boldsymbol{\theta}) \boldsymbol{\xi}_{i,\beta}^T(\boldsymbol{\theta}), \quad (8)$$

with $\boldsymbol{\xi}_{i,\beta}(\boldsymbol{\theta}) = \sum_{r=0}^2 \mathbf{u}_{ir}^*(\boldsymbol{\theta}) \pi_{ir}^\beta(\boldsymbol{\theta})$ and $\mathbf{u}_{ir}^*(\boldsymbol{\theta}) = \frac{\partial \pi_{ir}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T}$, where

$$\begin{aligned} \frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= -IT_i \pi_{i0}(\boldsymbol{\theta}) \mathbf{l}_i, \\ \frac{\partial \pi_{i1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} IT_i \pi_{i0}(\boldsymbol{\theta}) \mathbf{l}_i + (1 - \pi_{i0}(\boldsymbol{\theta})) \mathbf{r}_i, \\ \frac{\partial \pi_{i2}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} IT_i \pi_{i0}(\boldsymbol{\theta}) \mathbf{l}_i - (1 - \pi_{i0}(\boldsymbol{\theta})) \mathbf{r}_i, \end{aligned}$$

$$\mathbf{l}_i = (\lambda_{i1}/\theta_{10}, \lambda_{i1}x_i, \lambda_{i2}/\theta_{20}, \lambda_{i2}x_i)^T \text{ and } \mathbf{r}_i = \frac{\lambda_{i1}\lambda_{i2}}{(\lambda_{i1}+\lambda_{i2})^2}(1/\theta_{10}, x_i, -1/\theta_{20}, -x_i)^T.$$

3.1 Wald-type test statistics

Let us consider the function $\mathbf{m} : \mathbb{R}^{J+1} \rightarrow \mathbb{R}^r$, where $r \leq 4$. Then

$$\mathbf{m}(\boldsymbol{\theta}) = \mathbf{0}_r, \quad (9)$$

with $\mathbf{0}_r$ being the null column vector of dimension r , which represents the null hypothesis. We assume that the $4 \times r$ matrix

$$\mathbf{M}(\boldsymbol{\theta}) = \frac{\partial \mathbf{m}^T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

exists and is continuous in " $\boldsymbol{\theta}$ " and that $\text{rank}(\mathbf{M}(\boldsymbol{\theta})) = r$. For testing

$$H_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0 \text{ against } H_1 : \boldsymbol{\theta} \notin \boldsymbol{\Theta}_0, \quad (10)$$

where

$$\Theta_0 = \{\theta \in \Theta_0 : m(\theta) = \mathbf{0}_r\},$$

we can consider the following Wald-type test statistics:

$$W_K(\hat{\theta}_\beta) = K m^T(\hat{\theta}_\beta) \left(M^T(\hat{\theta}_\beta) \Sigma(\hat{\theta}_\beta) M(\hat{\theta}_\beta) \right)^{-1} m(\hat{\theta}_\beta),$$

where

$$\Sigma_\beta(\hat{\theta}_\beta) = J_\beta^{-1}(\hat{\theta}_\beta) K_\beta(\hat{\theta}_\beta) J_\beta^{-1}(\hat{\theta}_\beta),$$

and $J_\beta(\theta)$ and $K_\beta(\theta)$ are as given in (7) and (8), respectively. Wald-type test statistics based on weighted minimum density power divergence estimator have been considered previously by Basu et al. (2015) and Ghosh et al. (2016).

Theorem 8 *Under the null hypothesis, we have*

$$W_K(\hat{\theta}_\beta) \xrightarrow[K \rightarrow \infty]{\mathcal{L}} \chi_r^2.$$

Based on Theorem 8, we can reject the null hypothesis, in (10), if

$$W_K(\hat{\theta}_\beta) > \chi_{r,\alpha}^2, \quad (11)$$

where $\chi_{r,\alpha}^2$ is the upper α percentage point of χ_r^2 distribution.

Some results about the power function of the proposed Wald-type tests are presented in Appendix A.2.

Remark 9 (Robustness properties) *The influence function (IF) is a classical tool to measure the local robustness of an estimator (Hampel et al., 1968). In Balakrishnan et al. (2019a, 2020), the robustness of the weighted minimum density estimators and tests, for $\beta > 0$, was theoretically derived through local dependence under the exponential assumption but in a non-competing risk framework, for large leverages \mathbf{x}_i s. Analogous computations would result in the same conclusion for the competing risks scenario. However, we could not directly infer about the robustness against outliers in the response variable which are, in fact, the misspecification errors. In the next section, a simulation study is carried out in order to empirically illustrate the robustness of the proposed statistics with $\beta > 0$, and the non-robustness when $\beta = 0$, also against such misspecification errors.*

4 Simulation Study

In this section, a Monte Carlo simulation study that examines the accuracy of the proposed weighted minimum density power divergence estimators is presented. Section 4.1 focuses on the efficiency, measured in terms of root of mean square error (RMSE), mean bias error (MBE) and mean absolute error (MAE), of the estimators of model parameters, while Section 4.2 examines the behavior of Wald-type tests developed in preceding sections. Finally, in Section 4.3, an ad hoc procedure for the choice of the tuning parameter is proposed. Every step of simulation was tested under $S = 5,000$ replications with R statistical software.

Paying special attention to the robustness issue, we will consider in this context, “outlying cells” rather than “outlying observations” (see Balakrishnan et al., 2019a, 2019b).

This means that devices under a specific testing condition (cell) will not follow the general one-shot device model considered, contributing to an increase in the values of the divergence measure between the data and the corresponding fitted values. In this cell, the number of devices failed will be lower or higher than expected. This is similar to the principle of inflated models in distribution theory (see Lambert (1992) and Heilbron (1994)). The main purpose of this study is to show that within the family of weighted minimum density power divergence estimators, developed in the preceding sections, there are estimators with better robustness properties than the MLE, and the Wald-type tests constructed based on them are at the same time more robust than the classical Wald test constructed based on the MLE.

4.1 Weighted minimum density power divergence estimators

The lifetimes of devices are simulated for different levels of reliability and different sample sizes, under 4 different stress conditions with 1 stress factor at 4 levels. Then, all devices under each stress condition are inspected at 3 different inspection times, depending on the level of reliability. The corresponding data will then be collected under $I = 12$ test conditions.

4.1.1 Balanced data: Effect of the sample size

Firstly, a balanced data with equal sample size for each group was considered. K_i was taken to range from small to large sample sizes, two causes of failure were considered, and the model parameters were set to be $\theta = (\theta_{10}, 0.05, \theta_{20}, 0.08)^T$ with $\theta_{10} \in \{0.008, 0.004, 0.001\}$ and $\theta_{20} \in \{0.0008, 0.0004, 0.0001\}$ for devices with low, moderate and high reliability, respectively. To prevent many zero-observations in test groups, the inspection times were set as $IT \in \{5, 10, 20\}$ for the case of low reliability, $IT \in \{7, 15, 25\}$ for the case of moderate reliability, and $IT \in \{10, 20, 30\}$ for the case of high reliability. To evaluate the robustness of the weighted minimum density power divergence estimators, we studied their behavior in the presence of an outlying cell for the first testing condition in our table. This cell was generated under the parameters $\tilde{\theta} = (\theta_{10}, 0.05, \theta_{20}, 0.15)^T$. See Table 4 for a summary of these scenarios. RMSEs, MAEs and MBEs of model parameters were then computed for the cases of both pure and contaminated data and are plotted in Figures 4, 5 and 6, respectively, with similar conclusions for the three error measures.

For the case of pure data, MLE presents the best behaviour (overall in the model with high reliability) and an increment in the tuning parameter β leads to a gradual loss in terms of efficiency. However, in the case of contaminated data, MLE turns to be the worst estimator, and weighted minimum density power divergence estimators with $\beta > 0$ present much more robust behaviour. Note that, as expected, an increase in the sample size improves the efficiency of the estimators, both for pure and contaminated data.

4.1.2 Unbalanced data: Effect of the degree of contamination

Now, we consider an unbalanced data with unequal sample sizes for the test conditions. This data set, which consists a total of $K = 300$ devices, is presented in Table 5. A competing risks model, with two different causes of failure, was generated with parameters $\theta = (0.001, 0.05, 0.0001, 0.08)^T$. To examine the robustness in this accelerated life test (ALT) plan (in which the devices are tested under high stress levels, so that more failures can be observed), we increased each of the parameters of the outlying first cell (Figure 1). The contaminated parameters are expressed by $\tilde{\theta}_{10}, \tilde{\theta}_{11}, \tilde{\theta}_{20}$ and $\tilde{\alpha}_{21}$, respectively.

Table 4: Parameter values used in the simulation. Study of efficiency.

Reliability	Parameters	Symbols	Values
<i>Low reliability</i>	Risk 1	θ_{10}, θ_{11}	0.008, 0.05
	Risk 2	θ_{20}, θ_{21}	0.0008, 0.08
	Contamination	$\tilde{\theta}_{21}$	0.15
	Temperature ($^{\circ}\text{C}$)	$\mathbf{x}^T = (x_1, x_2, x_3, x_4)$	(35, 45, 55, 65)
	Inspection Time (days)	$IT = \{IT_1, IT_2, IT_3\}$	{5, 10, 20}
<i>Moderate reliability</i>	Risk 1	θ_{10}, θ_{11}	0.004, 0.05
	Risk 2	θ_{20}, θ_{21}	0.0004, 0.08
	Contamination	$\tilde{\theta}_{21}$	0.15
	Temperature ($^{\circ}\text{C}$)	$\mathbf{x}^T = (x_1, x_2, x_3, x_4)$	(35, 45, 55, 65)
	Inspection Time (days)	$IT = (IT_1, IT_2, IT_3)$	{7, 15, 25}
<i>High reliability</i>	Risk 1	θ_{10}, θ_{11}	0.001, 0.05
	Risk 2	θ_{20}, θ_{21}	0.0001, 0.08
	Contamination	$\tilde{\theta}_{21}$	0.15
	Temperature ($^{\circ}\text{C}$)	$\mathbf{x}^T = (x_1, x_2, x_3, x_4)$	(35, 45, 55, 65)
	Inspection Time (days)	$IT = (IT_1, IT_2, IT_3)$	{10, 20, 30}

Table 5: ALT plan, unbalanced data.

i	x_i	IT_i	K_i
1	35	10	50
2	45	10	40
3	55	10	20
4	65	10	40
5	35	20	20
6	45	20	20
7	55	20	30
8	65	20	20
9	35	30	20
10	45	30	20
11	55	30	10
12	65	30	10

When there is no contamination in the cell or the degree of contamination is very low, and in concordance with results obtained in the previous scenario, MLE is observed to be the most efficient estimator. However, when the degree of contamination increases, there is an increase in the error for all the estimators, but weighted minimum density power divergence estimators are shown to be much more robust. This is also the case for whatever choice of the contamination parameters we considered.

4.2 Wald-type tests

Let us consider the balanced data under moderate reliability defined in the previous section. To compute the accuracy in terms of contrast, we consider the testing problem

$$H_0 : \theta_{21} = 0.08 \quad \text{vs.} \quad H_1 : \theta_{21} \neq 0.08. \quad (12)$$

For computing the empirical test level, we measured the proportion of test statistics exceeding the corresponding chi-square critical value. The simulated test powers were also obtained under H_1 in (12) in a similar manner. We used a nominal level of 0.05. Table 6

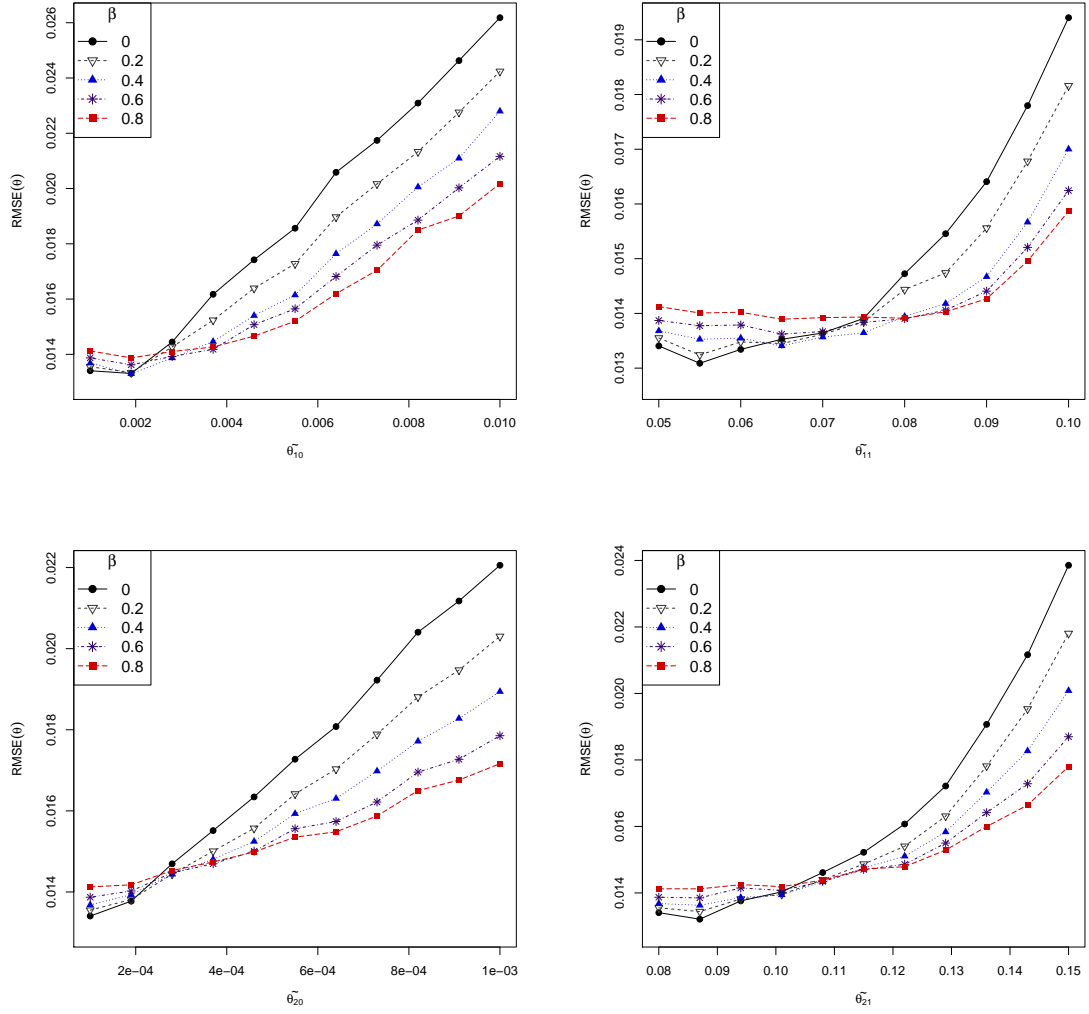


Figure 1: RMSEs of the weighted minimum density power divergence estimators of θ for different contamination parameter values. Unbalanced data.

summarizes the model considered for this purpose. As in the previous section, an outlying cell with $\tilde{\theta}_{21} = 0.15$ is considered to illustrate the robustness of the proposed Wald-type tests (Figure 2).

Table 6: Parameter values used in the simulation study of Wald-type tests.

Study	Parameters	Symbols	Values
<i>Levels</i>	Model True Parameters	$\theta^T = (\theta_{10}, \theta_{11}, \theta_{20}, \theta_{21})$	(0.004, 0.05, 0.0004, 0.08)
<i>Powers</i>	Model True Parameters	$\theta^T = (\theta_{10}, \theta_{11}, \theta_{20}, \theta_{21})$	(0.004, 0.05, 0.0004, 0.09)

In the case of pure data, we see how a big sample size is needed to obtain empirical tests close to the nominal level. In the case of contaminated data, empirical test levels are far away from the nominal level, with the MLE again presenting the least robust behaviour.

This simulation study has illustrated well the robust properties of the weighted minimum density power divergence estimators for $\beta > 0$, which is inevitably accompanied with a loss of efficiency in the case of pure data. It seems that a moderate low value of the

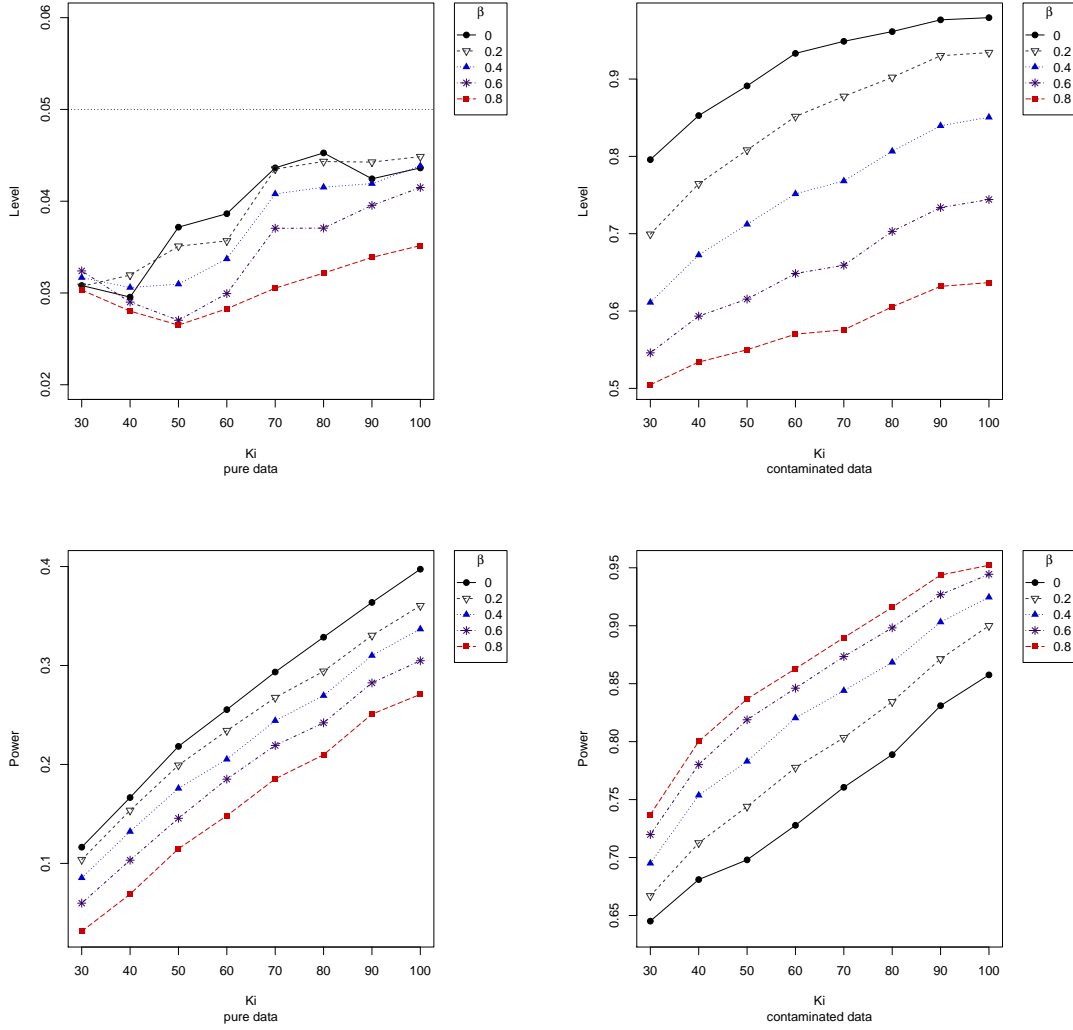


Figure 2: Levels and Powers of the weighted minimum density power divergence estimators-based Wald-type tests for different values of K_i with pure (left) and contaminated data (right), for the case of moderate reliability.

tuning parameter can be a good choice when applying the estimators to a real data set. However, when dealing with specific data sets, especially when we have small data sets, a data driven procedure for the choice of tuning parameter will become necessary.

4.3 Choice of tuning parameter

The problem of choosing the optimal tuning parameter in a DPD-based family of estimators has been extensively discussed in the literature; see, for example, Hong and Kim (2001), Warwick (2001), Warwick and Jones (2005), and Ghosh and Basu (2015). We now adopt the procedure proposed by Warwick and Jones (2005), which consists minimizing the estimated mean square error of the estimators, computed as the sum of estimated squared bias and variance; that is,

$$\widehat{MSE}_\beta = (\widehat{\theta}_\beta - \theta_P)^T (\widehat{\theta}_\beta - \theta_P) + \frac{1}{K} \text{trace} \left[\mathbf{J}_\beta^{-1}(\widehat{\theta}_\beta) \mathbf{K}_\beta(\widehat{\theta}_\beta) \mathbf{J}_\beta^{-1}(\widehat{\theta}_\beta) \right],$$

where θ_P is a pilot estimator, whose choice will be empirically discussed, since the overall procedure depends on this choice. If we take $\theta_P = \hat{\theta}_\beta$, the approach coincides with that of Hong and Kim (2001), but it does not take into account the model misspecification.

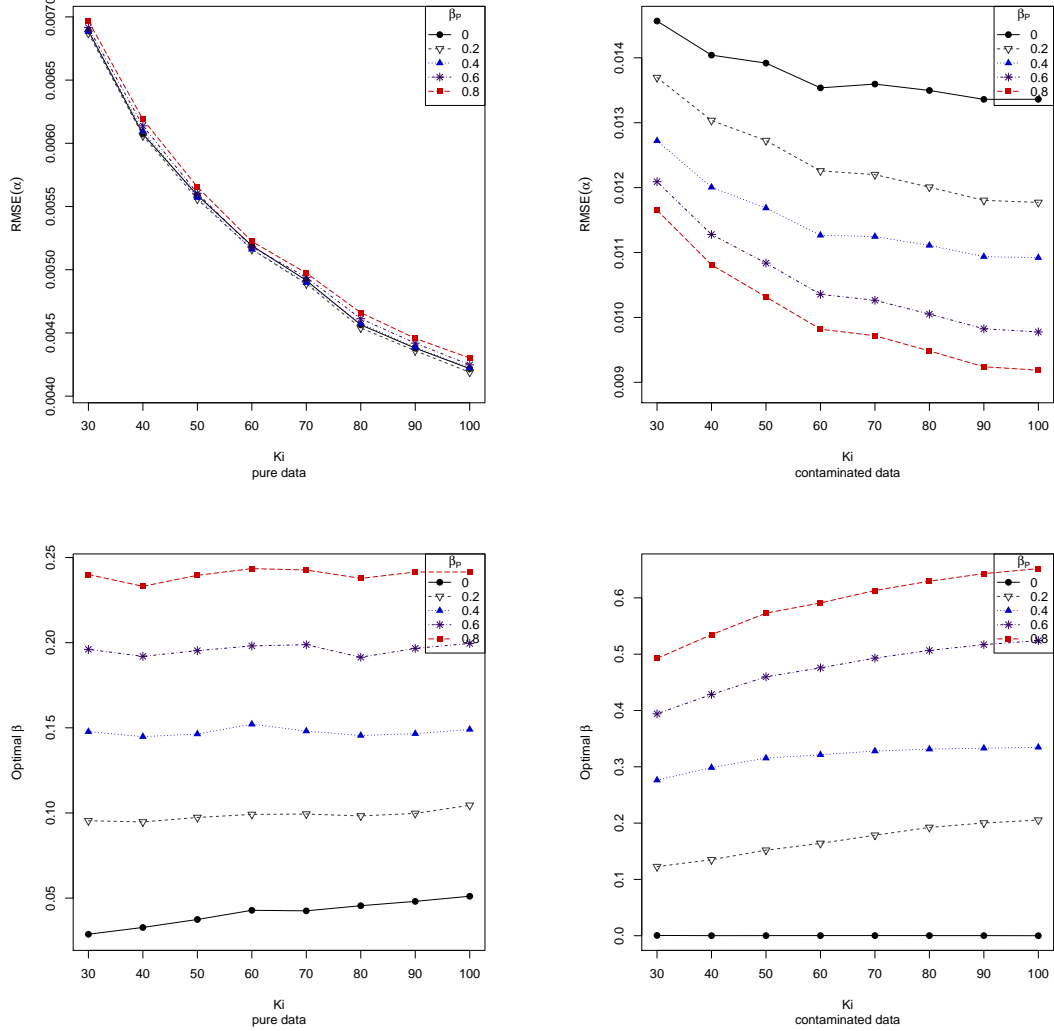


Figure 3: Estimated optimal β and the corresponding RMSEs for different pilot estimators in the proposed ad-hoc approach for the case of moderate reliability.

We consider again the balanced scenario under moderate reliability discussed earlier. For different pilot estimators and a grid of 100 points, optimal tuning parameters and their corresponding RMSEs are computed. The optimal tuning parameter increases when the contamination level increases in the data, and it seems that a moderate value of β is the best choice for the pilot estimator, as suggested in the work of Warwick and Jones (2005).

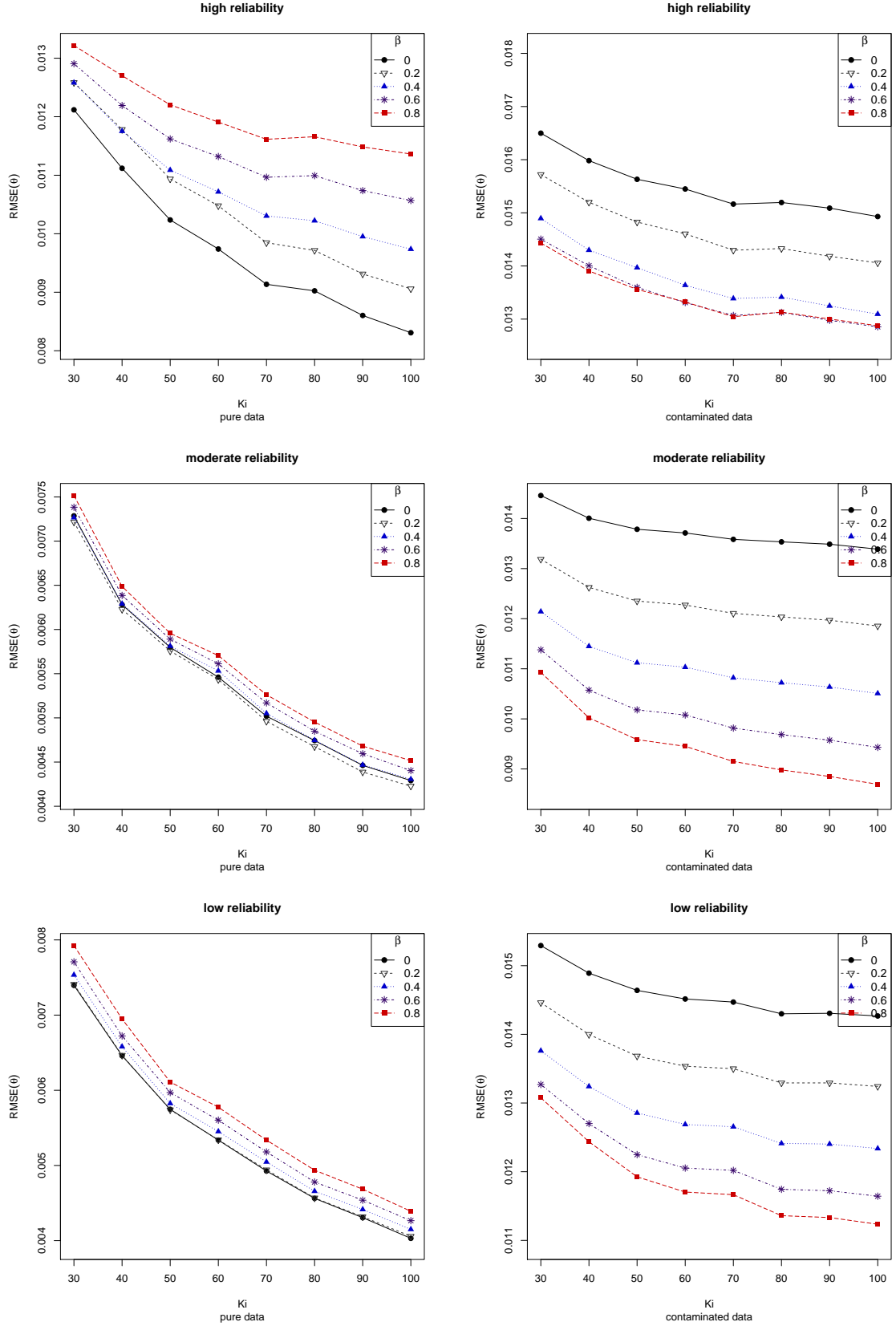


Figure 4: RMSEs of the weighted minimum density power divergence estimators of θ for different values of reliability with pure (left) and contaminated data (right)

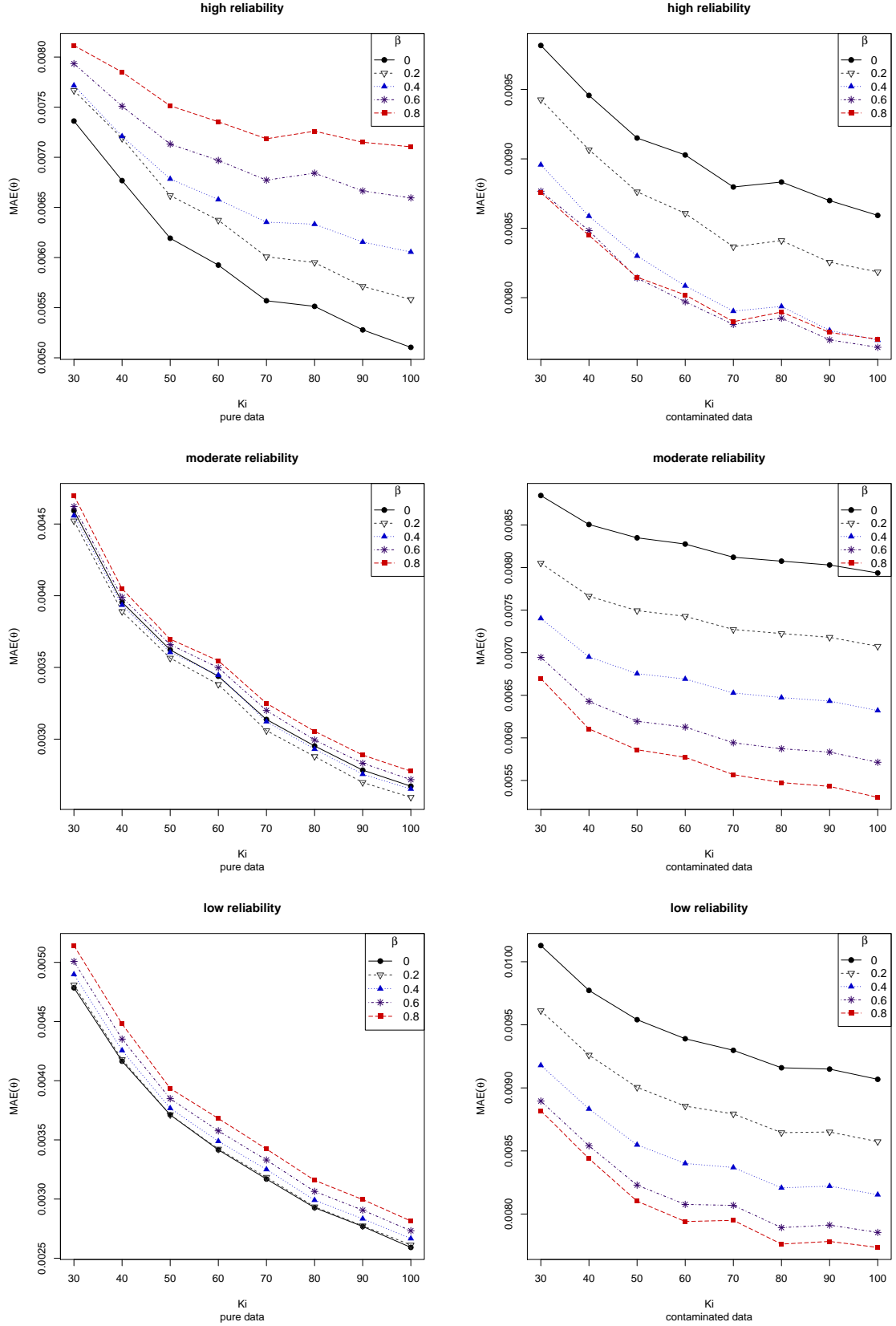


Figure 5: MAEs of the weighted minimum density power divergence estimators of θ for different values of reliability with pure (left) and contaminated data (right).

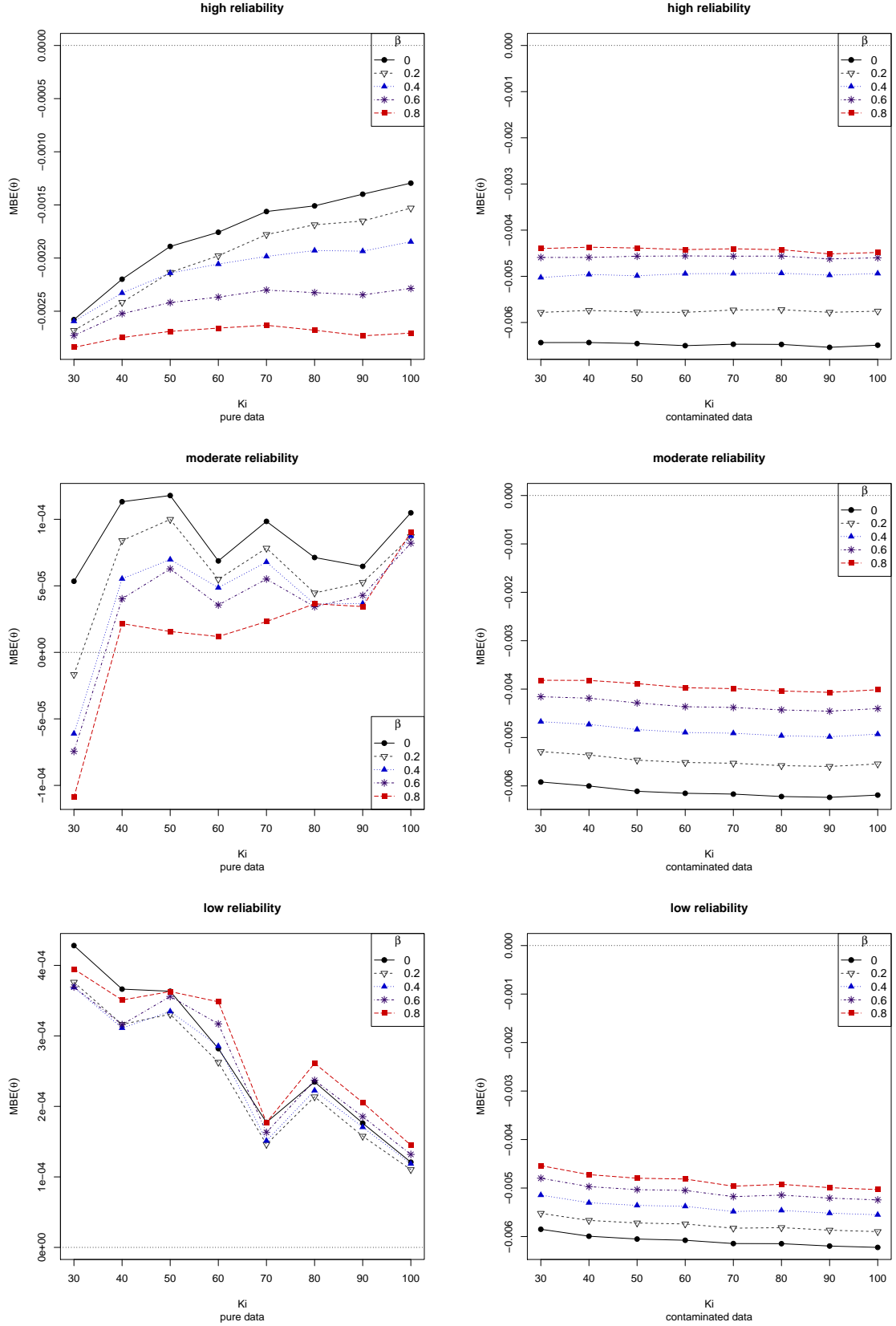


Figure 6: MBEs of the weighted minimum density power divergence estimators of θ for different values of reliability with pure (left) and contaminated data (right).

5 Benzidine dihydrochloride (BDC) experiment

Let us reconsider our motivating example, the BDC experiment, to study the performance of the proposed procedures. As noted in Section 2, this experiment considers two different doses of drug induced in the mice: 60 parts per million ($\omega = 1$) and 400 parts per million ($\omega = 2$) and two causes of death are recorded: died without tumor ($\delta_{ijk} = 1$) and died with tumor ($\delta_{ijk} = 2$). The data are presented in Table 2.

Table 7: Estimations for the BDC experiment for different choices of tuning parameters

β	θ_{10}	θ_{11}	θ_{20}	θ_{21}	$E_{\omega=1}^1$	$E_{\omega=2}^1$	$E_{\omega=1}$	$E_{\omega=2}$	$P_{\omega=1}^1$	$P_{\omega=2}^1$
0	0.00089	1.3191	0.00028	2.493	300.545	80.355	150.203	18.952	0.4997	0.2358
0.1	0.00091	1.3072	0.00029	2.465	297.876	80.593	146.984	18.872	0.4934	0.2341
0.2	0.00094	1.2844	0.00031	2.441	295.010	81.658	144.138	18.869	0.4885	0.2310
0.3	0.00097	1.2627	0.00033	2.408	291.902	82.572	140.528	18.818	0.4814	0.2279
0.4	0.00281	0.5329	0.00027	2.531	208.917	122.608	122.893	19.891	0.5882	0.1622
0.5	0.00104	1.2150	0.00036	2.367	285.233	84.626	135.755	18.859	0.4759	0.2228
0.6	0.00285	0.5253	0.00028	2.511	207.847	122.908	121.491	19.884	0.5845	0.1617
0.7	0.00282	0.5277	0.00028	2.503	209.051	123.322	121.277	19.824	0.5801	0.1607
0.8	0.00112	1.1412	0.00041	2.313	284.037	90.723	130.889	18.988	0.4608	0.2093
0.9	0.00271	0.5458	0.00029	2.496	213.458	123.669	122.077	19.741	0.5719	0.1596
1	0.00263	0.5514	0.00030	2.488	219.303	126.339	123.241	19.715	0.5619	0.1560
0.37	0.00279	0.5378	0.00026	2.537	209.275	122.221	123.529	19.946	0.5902	0.1632

Estimators of parameters were obtained for different choices of tuning parameters. We then computed the expected mean lifetime of the devices under the two doses of drug, both for the whole population ($E_{\omega=1}$ and $E_{\omega=2}$) and particularly for the mice that died without tumor ($E_{\omega=1}^1$ and $E_{\omega=2}^1$). We have also computed the probability of failure due to cause 1 (die without tumor) given failure, for both doses of drug ($P_{\omega=1}^1$ and $P_{\omega=2}^1$).

We applied the procedure described in Section 4.3 to determine the optimal tuning parameter for this data set, over a grid of 100 points. The resulting optimal tuning parameter, 0.37, and its corresponding estimators are presented in Table 7.

Finally, we estimate the errors, as given in (5), for different tuning parameters β , and the corresponding results in Table 8. The minimum is obtained for $\beta = 0.8$, while $\beta = 0.37$ also presents a lower estimated error, which is in concordance with the estimate obtained earlier.

Table 8: Estimated errors for the BDC experiment

β	0	0.1	0.2	0.3	0.37	0.4	0.6	0.7	0.8	0.9
est. error	0.1051	0.1049	0.1047	0.1044	0.1043	0.1051	0.1052	0.1050	0.1040	0.1048

6 Concluding Remarks and Future Work

In this paper, a robust divergence-based approach has been developed for the evaluation of one-shot devices with competing causes of failure, under the exponential distribution. The performance of the estimators as well as tests procedures based on them have been compared, through a simulation study and a numerical example, with these based on the classical maximum likelihood estimator.

For further study, we can consider developing results for other lifetime distributions, such as Weibull and gamma. While the exponential distribution has constant hazard rate,

Weibull and gamma lifetime distributions presents a non-constant hazard and practically useful aging properties and may therefore provide a more practical model, even though it will result in a much more complicated analysis. We are currently working on this problem and hope to report the findings in a future paper.

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A Appendix

A.1 Proofs of Results

A.1.1 Proof of Theorem 2

Proof. We have

$$\begin{aligned}
\sum_{i=1}^I \frac{K_i}{K} d_{KL}(\hat{\mathbf{p}}_i, \boldsymbol{\pi}_i(\boldsymbol{\theta})) &= \frac{1}{K} \sum_{i=1}^I \left\{ n_{i0} \log \left(\frac{n_{i0}/K_i}{\pi_{i0}(\boldsymbol{\theta})} \right) + n_{i1} \log \left(\frac{n_{i1}/K_i}{\pi_{i1}(\boldsymbol{\theta})} \right) + n_{i2} \log \left(\frac{n_{i2}/K_i}{\pi_{i2}(\boldsymbol{\theta})} \right) \right\} \\
&= \frac{1}{K} \sum_{i=1}^I \left\{ n_{i0} \log \left(\frac{n_{i0}}{K_i} \right) + n_{i1} \log \left(\frac{n_{i1}}{K_i} \right) + n_{i2} \log \left(\frac{n_{i2}}{K_i} \right) \right\} \\
&\quad - \frac{1}{K} \sum_{i=1}^I \{ n_{i0} \log(\pi_{i0}(\boldsymbol{\theta})) + n_{i1} \log(\pi_{i1}(\boldsymbol{\theta})) + n_{i2} \log(\pi_{i2}(\boldsymbol{\theta})) \} \\
&= c - \frac{1}{K} \log \left(\prod_{i=1}^I \pi_{i0}(\boldsymbol{\theta})^{n_{i0}} \pi_{i1}(\boldsymbol{\theta})^{n_{i1}} \pi_{i2}(\boldsymbol{\theta})^{n_{i2}} \right) \\
&= c - \frac{1}{K} \log(\mathcal{L}(\boldsymbol{\theta} | \boldsymbol{\delta}, \mathbf{IT}, \mathbf{x})),
\end{aligned}$$

where $c = \frac{1}{K} \sum_{i=1}^I \sum_{r=0}^2 \left\{ n_{ir} \log \left(\frac{n_{ir}}{K_i} \right) \right\}$ and it does not depend on the parameter vector $\boldsymbol{\theta}$. ■

A.1.2 Proof of Theorem 6

Proof. The estimating equations are given by

$$\frac{\partial}{\partial \boldsymbol{\theta}} {}^*d_{\beta}^W(\boldsymbol{\theta}) = \mathbf{0}_4, \quad (13)$$

where ${}^*d_{\beta}^{Weighted}(\boldsymbol{\theta})$ is as given in (6). Equation (13) is equivalent to

$$\frac{1}{\beta+1} \frac{\partial}{\partial \boldsymbol{\theta}} \sum_{i=1}^I \sum_{r=0}^2 K_i \pi_{ir}^{\beta+1}(\boldsymbol{\theta}) - \frac{1}{\beta} \frac{\partial}{\partial \boldsymbol{\theta}} \sum_{i=1}^I \sum_{r=0}^2 K_i p_{ir} \pi_{ir}^{\beta}(\boldsymbol{\theta}) = \mathbf{0}_4; \quad (14)$$

that is,

$$\frac{1}{\beta+1} \sum_{i=1}^I \sum_{r=0}^2 K_i (\beta+1) \pi_{ir}^{\beta}(\boldsymbol{\theta}) \frac{\partial \pi_{ir}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - \frac{1}{\beta} \sum_{i=1}^I \sum_{r=0}^2 K_i p_{ir} \beta \pi_{ir}^{\beta-1}(\boldsymbol{\theta}) \frac{\partial \pi_{ir}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0}_4,$$

or, equivalently

$$\sum_{i=1}^I \sum_{r=0}^2 K_i \pi_{ir}^{\beta-1}(\boldsymbol{\theta}) \frac{\partial \pi_{ir}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} [\pi_{ir}(\boldsymbol{\theta}) - p_{ir}] = \mathbf{0}_4.$$

But,

$$\begin{aligned} \pi_{i0}(\boldsymbol{\theta}) &= \exp(-(\lambda_{i1} + \lambda_{i2})IT_i), \\ \pi_{i1}(\boldsymbol{\theta}) &= \frac{\lambda_{i1}}{\lambda_{i2} + \lambda_{2i}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)) = \frac{\lambda_{i1}}{\lambda_{i2} + \lambda_{2i}} (1 - \pi_{i0}(\boldsymbol{\theta})), \\ \pi_{i2}(\boldsymbol{\theta}) &= \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)) = \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \pi_{i0}(\boldsymbol{\theta})), \end{aligned}$$

and so

$$\frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -IT_i \pi_{i0}(\boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}} [\lambda_{i1} + \lambda_{i2}] = -IT_i \pi_{i0}(\boldsymbol{\theta}) (\lambda_{i1}/\theta_{10}, \lambda_{i1}x_i, \lambda_{i2}/\theta_{20}, \lambda_{i2}x_i)^T = -IT_i \pi_{i0}(\boldsymbol{\theta}) \mathbf{l}_i,$$

$$\begin{aligned} \frac{\partial \pi_{i1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= (1 - \pi_{i0}(\boldsymbol{\theta})) \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \right] - \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \\ \frac{\partial \pi_{i2}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= (1 - \pi_{i0}(\boldsymbol{\theta})) \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \right] - \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \end{aligned}$$

where

$$\left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \right] = - \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \right] = \frac{\lambda_{i1} \lambda_{i2}}{(\lambda_{i1} + \lambda_{i2})^2} (1/\alpha_{10}, x_i, -1/\alpha_{20}, -x_i)^T.$$

We then obtain the desired result. ■

A.1.3 Proof of Theorem 7

Proof. We have

$$\begin{aligned} \pi_{i0}(\boldsymbol{\theta}) &= \exp(-(\lambda_{i1} + \lambda_{i2})IT_i), \\ \pi_{i1}(\boldsymbol{\theta}) &= \frac{\lambda_{i1}}{\lambda_{i2} + \lambda_{2i}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)) = \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} (1 - \pi_{i0}(\boldsymbol{\theta})), \\ \pi_{i2}(\boldsymbol{\theta}) &= \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \exp(-(\lambda_{i1} + \lambda_{i2})IT_i)) = \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \pi_{i0}(\boldsymbol{\theta})). \end{aligned}$$

It is clear that

$$\frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -IT_i \pi_{i0}(\boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}} [\lambda_{i1} + \lambda_{i2}] = -IT_i \pi_{i0}(\boldsymbol{\theta}) (\lambda_{i1}/\alpha_{10}, \lambda_{i1}x_i, \lambda_{i2}/\alpha_{20}, \lambda_{i2}x_i)^T.$$

On the other hand $\pi_{i1}(\boldsymbol{\theta}) = \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} (1 - \pi_{i0}(\boldsymbol{\theta}))$ and $\pi_{i2}(\boldsymbol{\theta}) = \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} (1 - \pi_{i0}(\boldsymbol{\theta}))$, and so

$$\begin{aligned} \frac{\partial \pi_{i1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= (1 - \pi_{i0}(\boldsymbol{\theta})) \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \right] - \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \\ \frac{\partial \pi_{i2}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= (1 - \pi_{i0}(\boldsymbol{\theta})) \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \right] - \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \frac{\partial \pi_{i0}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}. \end{aligned}$$

Here, we have

$$\left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}} \right] = - \left[\frac{\partial}{\partial \boldsymbol{\theta}} \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \right] = \frac{\lambda_{i1} \lambda_{i2}}{(\lambda_{i1} + \lambda_{i2})^2} (1/\alpha_{10}, x_i, -1/\alpha_{20}, -x_i)^T.$$

■

A.2 Power function of Wald-type tests

In many cases, the power function of the test procedure cannot be derived explicitly in small-sample situation. In the following result, we present a useful asymptotic result for approximating the power function of the Wald-type test statistic given in equation (9).

Theorem 10 *Let $\theta^* \notin \Theta_0$ be the true value of the parameter such that $\hat{\theta}_\beta \xrightarrow{P}_{K \rightarrow \infty} \theta^*$, and let us denote*

$$\ell_\beta(\theta_1, \theta_2) = \mathbf{m}^T(\theta_1) \left(\mathbf{M}^T(\theta_2) \Sigma_\beta(\theta_2) \mathbf{M}(\theta_2) \right)^{-1} \mathbf{m}(\theta_1).$$

We then have

$$\sqrt{K} \left(\ell_\beta(\hat{\theta}_1, \hat{\theta}_2) - \ell_\beta(\theta^*, \theta^*) \right) \xrightarrow{\mathcal{L}}_{K \rightarrow \infty} \mathcal{N}(0, \sigma_{W_{K,\beta}}^2(\theta^*)),$$

where

$$\sigma_{W_{K,\beta}}^2(\theta^*) = \left. \frac{\partial \ell_\beta(\theta, \theta^*)}{\partial \theta^T} \right|_{\theta=\theta^*} \Sigma_\beta(\theta^*) \left. \frac{\partial \ell_\beta(\theta, \theta^*)}{\partial \theta} \right|_{\theta=\theta^*}.$$

Proof. Under the assumption that

$$\hat{\theta}_\beta \xrightarrow{P}_{K \rightarrow \infty} \theta^*,$$

the asymptotic distribution of $\ell_\beta(\hat{\theta}_1, \hat{\theta}_2)$ coincides with the asymptotic distribution of $\ell_\beta(\hat{\theta}_1, \theta^*)$. A first-order Taylor expansion of $\ell_\beta(\hat{\theta}_\beta, \theta)$ at $\hat{\theta}_\beta$ around, θ^* , yields

$$\left(\ell_\beta(\hat{\theta}_\beta, \theta^*) - \ell_\beta(\theta^*, \theta^*) \right) = \left. \frac{\partial \ell_\beta(\theta, \theta^*)}{\partial \theta^T} \right|_{\theta=\theta^*} (\hat{\theta}_\beta - \theta^*) + o_p(K^{-1/2}).$$

Now, the result follows readily since

$$\sqrt{K} (\hat{\theta}_\beta - \theta^*) \xrightarrow{\mathcal{L}}_{K \rightarrow \infty} \mathcal{N}(\mathbf{0}_4, \Sigma_\beta(\theta^*)).$$

■

Remark 11 *Based on Theorem 10, an approximation of the power function of the Wald-type test statistic in (11) at θ^* can be provided as follows:*

$$\begin{aligned} \pi_{W,K}(\theta^*) &= \Pr \left(W_K(\hat{\theta}_\beta) > \chi_{r,\alpha}^2 \right) = \Pr \left(K \left(\ell_\beta(\hat{\theta}_\beta, \theta^*) - \ell_\beta(\theta^*, \theta^*) \right) > \chi_{r,\alpha}^2 - K \ell_\beta(\theta^*, \theta^*) \right) \\ &= \Pr \left(\frac{\sqrt{K} \left(\ell_\beta(\hat{\theta}_\beta, \theta^*) - \ell_\beta(\theta^*, \theta^*) \right)}{\sigma_{W_{K,\beta}}(\theta^*)} > \frac{1}{\sigma_{W_{K,\beta}}(\theta^*)} \left(\frac{\chi_{r,\alpha}^2}{\sqrt{K}} - \sqrt{K} \ell_\beta(\theta^*, \theta^*) \right) \right) \\ &= 1 - \Phi_K \left(\frac{1}{\sigma_{W_{K,\beta}}(\theta^*)} \left(\frac{\chi_{r,\alpha}^2}{\sqrt{K}} - \sqrt{K} \ell_\beta(\theta^*, \theta^*) \right) \right) \end{aligned}$$

for a sequence of distributions functions $\Phi_K(x)$ tending uniformly to the standard normal distribution $\Phi(x)$. It is clear that

$$\lim_{K \rightarrow \infty} \pi_{W,K}(\theta^*) = 1,$$

i.e., the Wald-type test statistic is consistent in the sense of Fraser.

The above approximation of the power function of the Wald-type test statistic can be used to obtain the sample size K necessary in order to attain a prefixed power $\pi_{W,K}(\boldsymbol{\theta}^*) = \pi_0$. To do so, it is necessary to solve the equation

$$\pi_0 = 1 - \Phi_K \left(\frac{1}{\sigma_{W_K, \beta}(\boldsymbol{a}^*)} \left(\frac{\chi_{r, \alpha}^2}{\sqrt{K}} - \sqrt{K} \ell_{\beta}(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) \right).$$

The solution, in K , of the above equation yields $\widehat{K}_{\beta} = \left\lceil \widehat{K}_{\beta}^* \right\rceil + 1$, where

$$\widehat{K}_{\beta}^* = \frac{\widehat{A}_{\beta} + \widehat{B}_{\beta} + \sqrt{\widehat{A}_{\beta}(\widehat{A}_{\beta} + 2\widehat{B}_{\beta})}}{2\ell_{\beta}^2(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*)},$$

with $\widehat{A}_{\beta} = \sigma_{W_K, \beta}^2(\boldsymbol{\theta}^*) (\Phi^{-1}(1 - \pi_0))^2$ and $\widehat{B}_{\beta} = 2\ell_{\beta}(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \chi_{r, \alpha}^2$.

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