TRANSMISSION PROBLEMS FOR SIMPLY CONNECTED DOMAINS IN THE COMPLEX PLANE

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ABSTRACT. We study existence and uniqueness of a transmission problem in simply connected domains in the plane with data in weighted Lebesgue spaces by first investigating solvability results of a related novel problem associated to a homeomorphism in the real line and domains given by the upper and lower half planes. Our techniques are based on the use of conformal maps and Rellich identities for the Hilbert transform, and are motivated by previous works concerning the Dirichlet, Neumann and Zaremba problems.

1. INTRODUCTION AND MAIN RESULTS

Transmission problems model diffusion processes with discontinuities across interfaces. They arise in areas such as electrodynamics of solid media and solid mechanics as well as in the study of vibrating folded membranes and composite plates, among other settings. The pioneering work [27] related to partial differential equations in classical elasticity along with subsequent investigations in [7, 23, 31, 29] propelled an active field of research. Well-posedness, regularity theory, interfaces and numerous other aspects have been extensively studied ever since; see for instance the articles [12, 13, 14, 22] and the more recent works [6, 21, 24, 30], as well as references therein. We refer the reader to the monograph [5] for a comprehensive treatment of the theory of transmission problems.

In this article, we study existence and uniqueness of solutions of a transmission problem in the plane with data in weighted Lebesgue spaces, which we next describe. Let Λ be an unbounded rectifiable Jordan curve that divides the complex plane in two simply connected domains, Ω^+ (upper graph domain) and Ω^- (lower graph domain), and consider the following transmission problem (see Figure 1):

$$\begin{cases} \Delta v^\pm = 0 & \text{in } \Omega^\pm, \\ v^+ = v^- & \text{on } \Lambda, \\ \partial_{\mathbf{n}} v^+ - \mu\, \partial_{\mathbf{n}} v^- = g & \text{on } \Lambda. \end{cases} \tag{1.1}$$

Here, Δ is the Laplacian in $\mathbb{R}^2$, $\mathbf{n}$ is the inward unit normal to Ω^+, $\partial_{\mathbf{n}} v^\pm$ denotes the normal derivative of $v^\pm$, and $\mu \in \mathbb{R}$ is a fixed parameter. The datum g belongs to some weighted Lebesgue space on Λ and equalities on the boundary are interpreted almost everywhere with respect to arc length. Estimates in weighted Lebesgue spaces for appropriate maximal operators (for instance, the non-tangential maximal operator when the domains have the cone property) of the gradients of the solutions are also required (see Definition 2.10 for details). We note that when $\mu = 0$, (1.1) reduces to a Neumann problem for v^+ and to a Dirichlet problem for v^-.

The solvability of the transmission problem (1.1) with data in unweighted L^p spaces was studied in [13] for the case when Λ is the graph of a Lipschitz function. It holds that if $\mu \neq 1$, then there exists $\varepsilon = \varepsilon(\Lambda, \mu) > 0$ such that the transmission problem has a unique solution

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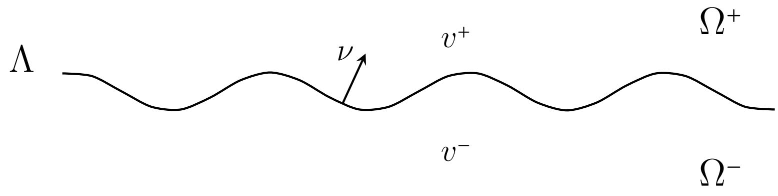


FIGURE 1. The transmission problem (1.1)

(up to constants) for all $g \in L^p(\Lambda)$ provided that $1 < p < 2 + \varepsilon$; moreover, there are integral representation formulas for the solution in terms of harmonic layer potentials. When $\mu = 1$, problem (1.1) is well-posed for any datum in $L^p(\Lambda)$ and $1 < p < \infty$. It was proved in [25] that the range $1 < p \le 2$ is sharp in the sense that for any $p > 2$ there exist Lipschitz domains $\Omega^\pm$ and $0 < \mu < 1$ such that (1.1) is not well-posed in $L^p(\Lambda)$.

In this article, we investigate new aspects associated to the solvability of the transmission problem (1.1) in weighted Lebesgue spaces, and specially in the case $p = 2$, by first studying a related problem corresponding to $\Omega^\pm = \mathbb{R}^2_\pm$, which is interesting in its own right and for which we obtain novel results. Our techniques are based on the use of conformal maps and Rellich identities for the Hilbert Transform, and are motivated by the works [20, 19, 18], as well as the articles [11, 9, 8] dealing with the Dirichlet, Neumann and Zaremba problems.

Since $\Omega^\pm$ are simply connected, they are conformally equivalent to $\mathbb{R}^2_\pm$. Let $\Phi_\pm : \mathbb{R}^2_\pm \to \Omega^\pm$ be conformal maps such that $\Phi_\pm(\infty) = \infty$. Then $\Phi_\pm$ extend as homeomorphisms from $\overline{\mathbb{R}^2_\pm}$ onto $\overline{\Omega^\pm}$ (Carathéodory's theorem) and $\Phi'_\pm(x)$ exist and are not zero for almost every $x \in \mathbb{R}$ (see Section 2.4). For an almost everywhere differentiable function β defined on $\mathbb{R}$ with values in the domain of a complex-valued function g, set

$$T_\beta g = |\beta'|\,(g \circ \beta).$$

Consider solutions $v^\pm$ of the transmission problem (1.1) and define $u^\pm = v^\pm \circ \Phi_\pm$.

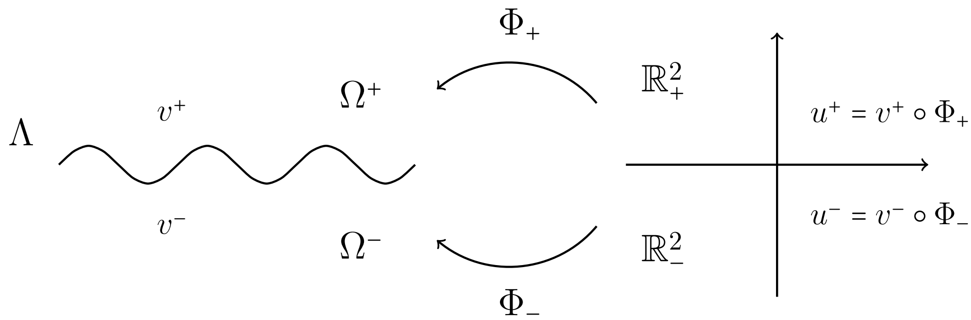


FIGURE 2. The conformal maps $\Phi_\pm$

Then $u^\pm$ are harmonic in $\mathbb{R}^2_\pm$ and $u^+ \circ \Psi = u^-$ on $\mathbb{R}$, where $\Psi = \Phi_+^{-1} \circ \Phi_-$. Using that $\partial_y u^\pm = T_{\Phi_\pm}(\partial_{\mathbf{n}} v^\pm)$ on $\mathbb{R}$ (see Section 5), we have

$$T_{\Phi_+^{-1}}(\partial_y u^+) - \mu\, T_{\Phi_-^{-1}}(\partial_y u^-) = \partial_{\mathbf{n}} v^+ - \mu\, \partial_{\mathbf{n}} v^- = g \quad \text{on } \Lambda.$$

This leads to

$$T_{\Phi_-} T_{\Phi_+^{-1}}(\partial_y u^+) - \mu\, \partial_y u^- = T_{\Phi_-} g \quad \text{on } \mathbb{R},$$

and noting that $T_\Psi = T_{\Phi_-} \circ T_{\Phi_+^{-1}}$, we obtain

$$T_\Psi(\partial_y u^+) - \mu\, \partial_y u^- = T_{\Phi_-} g \quad \text{on } \mathbb{R}.$$

We then have that $u^{\pm}$ satisfy $\Delta u^{\pm} = 0$ in $\mathbb{R}^2_{\pm}$, $u^+ \circ \Psi = u^-$ on $\mathbb{R}$, and $T_\Psi(\partial_y u^+) - \mu\,\partial_y u^- = T_{\Phi_-} g$ on $\mathbb{R}$. More generally, we will consider a homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$ and study the solvability of the following problem with $f \in L^p(\mathbb{R}, w)$, where w is a weight in the Muckenhoupt class $A_p(\mathbb{R})$:

$$\begin{cases} \Delta u^{\pm} = 0 & \text{in } \mathbb{R}^2_{\pm}, \\ u^+ \circ \Psi = u^- & \text{on } \mathbb{R}, \\ T_\Psi(\partial_y u^+) - \mu\,\partial_y u^- = f & \text{on } \mathbb{R}. \end{cases} \tag{1.2}$$

We note that when $\Psi(x) = x$, (1.2) corresponds to the transmission problem (1.1) with $\Omega^{\pm} = \mathbb{R}^2_{\pm}$, which can be solved through the Neumann problem and the reflection principle for harmonic functions. We will denote by $P_\Psi(\mu)$ the transmission problem (1.2). As in the case of (1.1), $P_\Psi(0)$ reduces to a Neumann problem for u^+ and to a Dirichlet problem for u^-, whose solvability with data in weighted Lebesgue and Lorentz spaces was studied in [9, 11]; therefore, we will always assume that $\mu \neq 0$.

By studying existence and uniqueness of solutions of (1.2), we obtain results on existence and uniqueness of solutions of (1.1), and conversely:

Theorem 1.1. *Let $1 < p < \infty$, $\mu \neq 0$, ν be a weight on Λ, and $w = |\Phi_-'|^{1-p}(\nu \circ \Phi_-)$; assume $\Psi = \Phi_+^{-1} \circ \Phi_-$ is locally absolutely continuous. Then the transmission problem* (1.1) *is uniquely solvable in $L^p(\Lambda, \nu)$ if and only if $P_\Psi(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$.*

We refer the reader to Definitions 2.3 and 2.10 regarding the concepts of unique solvability for $P_\Psi(\mu)$ and (1.1), respectively. As corollaries of Theorem 1.1 we obtain unique solvability of (1.1) in $L^2(\Lambda)$, $L^2(\Lambda, |(\Phi_+^{-1})'|^{-1})$ and $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$ (see Section 5). In particular, we recover results in [13, Theorem 1.1] for $p = 2$ and $n = 2$ in the case when Λ is the graph of a Lipschitz function. In the following, if Λ is the graph of a Lipschitz function, we will refer to Ω^+ as an upper Lipschitz graph domain and to Ω^- as a lower Lipschitz graph domain. Our results are true for a general unbounded rectifiable Jordan curve Λ; for instance, we have the following corollary when Λ is the hyperbola $y = 1/x$:

Corollary 1.2. *Let $\Omega^{\pm}$ be the upper and lower graph domains associated to the hyperbola $y = 1/x$, $x > 0$. Then the transmission problem* (1.1) *is uniquely solvable in $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$ for $\mu \neq 0$ such that $|\mu| < \frac{-\sqrt{3}+\sqrt{3+2^{1/3}}}{2} \approx 0.165953$ or $|\mu| > \frac{2}{-\sqrt{3}+\sqrt{3+2^{1/3}}} \approx 6.02579$.*

We start our study of $P_\Psi(\mu)$ by proving necessary and sufficient conditions for the solvability and unique solvability of $P_\Psi(\mu)$ in $L^p(\mathbb{R}, w)$ in terms of the surjectivity and invertibility, respectively, of the operator $HT_\Psi + \mu T_\Psi H$, where H is the Hilbert transform. More precisely, we prove the following result (see Section 3):

Theorem 1.3. *Let $\mu \neq 0$, $1 < p < \infty$, and $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism with $\Psi' > 0$ almost everywhere; assume $w \in A_p(\mathbb{R})$ is such that $\widetilde{w}_p = |(\Psi^{-1})'|^{1-p}(w \circ \Psi^{-1}) \in A_p(\mathbb{R})$. It holds that*

(a) $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R}, w)$ if and only if the operator $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is surjective;

(b) $P_\Psi(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$ if and only if the operator $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is invertible.

Theorem 1.3 is used to prove solvability results for $P_\Psi(\mu)$ in weighted L^2 spaces. For instance, we give sufficient conditions on the homeomorphism Ψ for the solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, \frac{1}{\Psi'})$ (see Section 4 for other related results):

Theorem 1.4. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism such that $\Psi' > 0$ almost everywhere and satisfies*

$$\frac{1}{\Psi'} = \operatorname{Re}\left(\frac{1}{\Phi'}\right) \tag{1.3}$$

for some conformal map Φ from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain. Define

$$k_\Psi := \left\| \Psi' \operatorname{Im}\left(\frac{1}{\Phi'}\right)\right\|_{L^\infty(\mathbb{R})}.$$

Then for every $0 < |\mu| < 1$ such that

$$1 - \mu^2 - 2\, k_\Psi\, |\mu| > 0, \tag{1.4}$$

the transmission problems $P_\Psi(\mu)$ and $P_\Psi(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Psi'})$.

We also present several examples where the conformal map Φ in Theorem 1.4 is associated to domains such as an infinite staircase, symmetric cones and hyperbolas, as well as to the Helson-Szegö representation of A_2 weights.

Returning to the motivation problem described in this section, we consider homeomorphisms $\Psi : \mathbb{R} \to \mathbb{R}$ of the form $\Psi = \Phi_+^{-1} \circ \Phi_-$ and prove the following result:

Theorem 1.5. *Let $\Psi = \Phi_+^{-1} \circ \Phi_-$ with $\Omega^\pm$ upper and lower Lipschitz graph domains. Then $P_\Psi(\mu)$ is solvable in $L^2(\mathbb{R}, |\Phi_-'|^{-1})$ for $\mu > 0$.*

Homeomorphisms of the form $\Psi = \Phi_+^{-1} \circ \Phi_-$ are referred to as conformal weldings and have been extensively studied. More generally, a homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ is a conformal welding if there exist a Jordan curve Γ in $\overline{\mathbb{C}}$ with complementary domains Ω_1 and Ω_2, and conformal mappings $\phi_1 : \mathbb{R}^2_+ \to \Omega_1$ and $\phi_2 : \mathbb{R}^2_- \to \Omega_2$ such that $\psi = \phi_1^{-1} \circ \phi_2$. A sufficient condition for a homeomorphism ψ to be a conformal welding is quasi-symmetry: if ψ is an increasing function so that there is a constant $M > 0$ such that

$$\frac{1}{M} \leq \frac{\psi(x+t) - \psi(x)}{\psi(x) - \psi(x-t)} \leq M, \quad \forall x \in \mathbb{R}, t > 0,$$

then ψ is a conformal welding. This condition is equivalent to saying that the push forward of the Lebesgue measure under ψ is a doubling measure. We refer the reader to [1, 2, 3, 4, 17, 33] for more information regarding conformal weldings.

The organization of the article is as follows. In Section 2, we present some notation, definitions and preliminary results that will be used throughout the article. These include the definitions of classes of weights and several associated properties, the precise definitions of solvability for (1.1) and $P_\Psi(\mu)$, and statements on the Neumann problem in the upper half plane proved in [9] along with new results regarding its uniqueness of solutions and boundary data (Theorem 2.5 and Lemma 2.8). Section 3 contains the proof of Theorem 1.3 and the statement and proof of Corollary 3.4, which gives symmetric properties associated to the solvability of $P_\Psi(\mu)$. The proofs of Theorem 1.4 and Theorem 1.5 are presented in Section 4 along with the statements and proofs of related results (Theorem 4.3 and Theorem 4.4) and examples. A main tool in the proofs of the results in Section 4 is the use of a type of Rellich identity for the Hilbert Transform proved in [10], which we describe in Section 4.1. Section 5 contains the proof of Theorem 1.1 and the statements of results on the solvability of (1.1) in $L^2(\Lambda)$, $L^2(\Lambda, |(\Phi_+^{-1})'|^{-1})$ and $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$ that follow from Theorem 1.1 and the theorems in Section 4. An application of Theorem 4.3 related to the hyperbola $y = 1/x$, $x > 0$, and the proof of Corollary 1.2 are presented on Section 6

2. Notation and preliminaries

In this section we present some notation, definitions and preliminary results that will be used throughout the article.

The notation $A \lesssim B$ means that there is a constant $c > 0$ such that $A \le cB$; c may depend on some of the parameters used but not on the functions or variables involved. We will use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2.1. **Weights.** Consider a weight w on $\mathbb{R}$ (i.e. a non-negative locally integrable function defined in $\mathbb{R}$). For $1 \le p \le \infty$, the space $L^p(\mathbb{R}, w)$ is the class of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R},w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)\,dx \right)^{\frac{1}{p}} < \infty,$$

with the corresponding changes for $p = \infty$. When $w \equiv 1$, we use the notation $L^p(\mathbb{R})$ instead of $L^p(\mathbb{R}, w)$.

If $1 < p < \infty$, the Muckenhoupt class $A_p(\mathbb{R})$ is given by the weights w such that

$$[w]_p = \sup_{I \subset \mathbb{R}} \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ and p' is the conjugate exponent of p (i.e. $1/p + 1/p' = 1$). We recall that $w \in A_p(\mathbb{R})$ if and only if $w^{1-p'} \in A_{p'}(\mathbb{R})$, $A_p(\mathbb{R}) \subset A_q(\mathbb{R})$ if $p < q$ and, if $w \in A_p(\mathbb{R})$ then $w \in A_{p-\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$.

The Hilbert transform H is defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \frac{f(x-t)}{t}\,dt.$$

The Muckenhoupt classes characterize boundedness properties of H in the sense that H is bounded on $L^p(\mathbb{R}, w)$ if and only if $w \in A_p(\mathbb{R})$ (see Hunt–Muckenhoupt-Wheeden [16]).

2.1.1. *Homeomorphisms and weights.* Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism and w a weight in $\mathbb{R}$; define

$$\widetilde{w}_p = |(\Psi^{-1})'|^{1-p}(w \circ \Psi^{-1}) = T_{\Psi^{-1}}(w|\Psi'|^p). \tag{2.1}$$

Then $T_\Psi : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ and we have

$$\begin{aligned}\|T_\Psi h\|^p_{L^p(\mathbb{R},w)} &= \int_{\mathbb{R}} |h(\Psi(x))|^p |\Psi'(x)|^p w(x)\,dx = \int_{\mathbb{R}} |h(y)|^p |\Psi'(\Psi^{-1}(y))|^{p-1} w(\Psi^{-1}(y))\,dy \\ &= \int_{\mathbb{R}} |h(y)|^p |(\Psi^{-1})'(y)|^{1-p} w(\Psi^{-1}(y))\,dy = \|h\|^p_{L^p(\mathbb{R},\widetilde{w}_p)}.\end{aligned}$$

Remark 2.1. When $p = 2$, we will use the notation $\widetilde{w}$ instead of $\widetilde{w}_2$.

The following remark states necessary and sufficient conditions for $\widetilde{w}_p$ to be in $A_p(\mathbb{R})$ when $w \in A_p(\mathbb{R})$.

Remark 2.2. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism. If $w \in A_p(\mathbb{R})$, then

$$\widetilde{w}_p \in A_p(\mathbb{R}) \iff \left(\frac{\int_I |\Psi'(x)|^p w(x)\,dx}{\int_I w(x)\,dx} \right)^{\frac{1}{p}} \approx \frac{|\Psi(I)|}{|I|}, \qquad \text{for all intervals } I \subset \mathbb{R},$$

where $\Psi(I)$ is the image of I under Ψ. Indeed, using (2.1) and a change of variable we see that for an interval $J \subset \mathbb{R}$ we have

$$\left(\frac{1}{|J|}\int_J \widetilde{w}_p(y)\,dy\right)\left(\frac{1}{|J|}\int_J \widetilde{w}_p(y)^{1-p'}\,dy\right)^{p-1}$$
$$= \left(\frac{1}{|\Psi(I)|}\int_I |\Psi'(x)|^p w(x)\,dx\right)\left(\frac{1}{|\Psi(I)|}\int_I w(x)^{1-p'}\,dx\right)^{p-1},$$

where $I = \Psi^{-1}(J)$. Since $w \in A_p(\mathbb{R})$, it holds that

$$\left(\int_I w(x)^{1-p'}dx\right)^{p-1} \approx \frac{|I|^p}{\int_I w(x)\,dx}.$$

The above leads to

$$\left(\frac{1}{|J|}\int_J \widetilde{w}_p(y)\,dy\right)\left(\frac{1}{|J|}\int_J \widetilde{w}_p(y)^{1-p'}\,dy\right)^{p-1} \approx \left(\frac{|I|}{|\Psi(I)|}\right)^p \frac{\int_I |\Psi'(x)|^p w(x)\,dx}{\int_I w(x)\,dx},$$

from which the desired result follows.

We remark that the above gives in particular that if $w \equiv 1$, then $\widetilde{w}_p \in A_p(\mathbb{R})$ iff $|\Psi'|$ satisfies a reverse Hölder inequality with exponent p; this is,

$$\left(\frac{1}{|I|}\int_I |\Psi'(x)|^p\,dx\right)^{\frac{1}{p}} \approx \frac{1}{|I|}\int_I |\Psi'(x)|\,dx,$$

for all intervals $I \subset \mathbb{R}$.

2.2. **What does solvability of $P_\Psi(\mu)$ mean?** Given $0 < \alpha < \pi/2$, $\mathcal{M}_\alpha^\pm$ will denote the non-tangential maximal operators given by

$$\mathcal{M}_\alpha^\pm(F)(x) = \sup_{z \in \Gamma_\alpha^\pm(x)} |F(z)|, \quad x \in \mathbb{R},$$

where F is a complex-valued function defined in the complex plane and

$$\Gamma_\alpha^+(x) = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0 \text{ and } |\mathrm{Re}(z) - x| < \tan(\alpha)\mathrm{Im}(z)\},$$
$$\Gamma_\alpha^-(x) = \{z \in \mathbb{C} : \overline{z} \in \Gamma_\alpha^+(x)\}.$$

For $F^+ = F\chi_{\mathbb{R}_+^2}$ and $x \in \mathbb{R}$, we say that F^+ converges non-tangentially to a complex number $F(x)$ at x if $\lim_{z\to x, z\in\Gamma_\alpha^+(x)} F^+(z) = F(x)$. A similar definition applies to $F^- = F\chi_{\mathbb{R}_-^2}$ using $\Gamma_\alpha^-(x)$.

We next present the definition of solvability of $P_\Psi(\mu)$ in $L^p(\mathbb{R}, w)$.

Definition 2.3. Given a homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$, $\mu \neq 0$ and a weight w in $\mathbb{R}$, we say that $P_\Psi(\mu)$ *is solvable in* $L^p(\mathbb{R}, w)$ if, for every $f \in L^p(\mathbb{R}, w)$, there are harmonic functions $u^\pm$ in $\mathbb{R}_\pm^2$ such that

(a) $u^\pm$ and $\partial_y u^\pm$ on $\mathbb{R}$ are the traces of $u^\pm$ and $\partial_y u^\pm$, respectively, in the sense of non-tangential convergence,
(b) $u^+ \circ \Psi = u^-$ and $T_\Psi(\partial_y u^+) - \mu\,\partial_y u^- = f$ almost everywhere in $\mathbb{R}$,
(c) if $0 < \alpha < \pi/2$, then

$$\|\mathcal{M}_\alpha^+ \nabla u^+\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|f\|_{L^p(\mathbb{R},w)} \quad \text{and} \quad \|\mathcal{M}_\alpha^- \nabla u^-\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)}.$$

We will say that $P_\Psi(\mu)$ *is uniquely solvable in* $L^p(\mathbb{R}, w)$ if $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R}, w)$ and solutions are unique modulo constants.

2.3. **The Neumann problem in the upper half-plane.** The results presented in this section will be useful for some of the proofs in this article.

Consider the classical Neumann boundary value problem in $\mathbb{R}^2_+$:

$$\Delta u = 0 \text{ on } \mathbb{R}^2_+ \qquad \text{and} \qquad \partial_y u = f \text{ on } \mathbb{R}, \tag{2.2}$$

where the equality $\partial_y u = f$ is interpreted in the sense of non-tangential convergence. For $f : \mathbb{R} \to \mathbb{C}$ define

$$u_f(x,y) := \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{1+|t|}\right) f(t)\,dt, \quad (x,y) \in \mathbb{R}^2_+. \tag{2.3}$$

We note that the integral on the right-hand side of (2.3) is absolutely convergent for all f satisfying $\int_{\mathbb{R}} |f(x)|(1+|x|)^{-1}\,dx < \infty$; in particular, u_f is well defined and absolutely convergent for all $f \in L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$.

The Neumann problem (2.2) is solvable in $L^p(\mathbb{R}, w)$ for $w \in A_p(\mathbb{R})$; more precisely, the following result holds.

Theorem 2.4 (Solvability of the Neumann problem in $L^p(\mathbb{R}, w)$; Theorem 1.3 in [9])**.** *Let $1 < p < \infty$ and $0 < \alpha < \pi/2$. If $w \in A_p(\mathbb{R})$ and $f \in L^p(\mathbb{R}, w)$, then u_f is harmonic in $\mathbb{R}^2_+$, $\partial_y u_f = f$ on $\mathbb{R}$ in the sense of non-tangential convergence and*

$$\|\mathcal{M}^+_\alpha(\nabla u_f)\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)},$$

where the implicit constant is independent of f.

The following result shows that the solution of the Neumann problem with datum in $L^p(\mathbb{R}, w)$ for $w \in A_p(\mathbb{R})$ is unique up to a constant.

Theorem 2.5 (Uniqueness of solutions for the Neumann problem)**.** *Let $f \in L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$. If u is a solution of the Neumann problem* (2.2) *with datum f and $\mathcal{M}^+_\alpha \nabla u \in L^p(\mathbb{R}, w)$, then there exists a constant C such that $u = u_f + C$.*

Proof. Fix $(x,y) \in \mathbb{R}^2_+$ and $\{z_k\}_k \subset (0,\infty)$ such that $\lim_{k\to\infty} z_k = 0$. Since

$$\left|\log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{(1+|t|)}\right)\right| \lesssim \frac{1}{1+|t|}, \qquad \forall t \in \mathbb{R},$$

we have

$$\left|\partial_2 u(t,z_k)\log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{(1+|t|)}\right)\right| \lesssim \frac{\mathcal{M}^+_\alpha \nabla u(t)}{1+|t|}, \quad \forall t \in \mathbb{R}.$$

The right-hand side is an integrable function since $\mathcal{M}^+_\alpha \nabla u \in L^p(\mathbb{R}, w)$ and $w \in A_p(\mathbb{R})$. Hence, by the dominated convergence theorem and the fact that $\lim_{k\to\infty} \partial_2 u(t,z_k) = f(t)$ for almost every $t \in \mathbb{R}$, we obtain

$$\lim_{k\to\infty}\int_{\mathbb{R}} \partial_2 u(t,z_k)\log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{(1+|t|)}\right) dt = \int_{\mathbb{R}} f(t)\log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{(1+|t|)}\right) dt.$$

For each k, define

$$U_k(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\tfrac{\sqrt{(x-t)^2+y^2}}{(1+|t|)}\right)\partial_2 u(t,z_k)\,dt \quad \text{and} \quad V_k(x,y) = u(x,y+z_k).$$

Setting $W_k(x,y) = U_k(x,y) - V_k(x,y)$, we have

$$\partial_2 W_k(x,0) = \partial_2 U_k(x,0) - \partial_2 V_k(x,0) = \partial_2 u(x,z_k) - \partial_2 u(x,z_k) = 0 \quad \text{a.e } x \in \mathbb{R}.$$

By the reflection principle, W_k admits a harmonic extension $\widetilde{W_k}$ to $\mathbb{R}^2$ which satisfies

$$\mathcal{M}^+_\alpha \nabla \widetilde{W}^+_k \in L^p(\mathbb{R}, w) \quad \text{and} \quad \mathcal{M}^-_\alpha \nabla \widetilde{W}^-_k \in L^p(\mathbb{R}, w),$$

where $\widetilde{W}_k^{\pm} = \widetilde{W}_k \chi_{\mathbb{R}^2_{\pm}}$. Let $z = (x,y) \in \mathbb{R}^2_+$; by the mean value property applied to $\partial_1 \widetilde{W}_k$, we have

$$\begin{aligned}
|\partial_1 W_k(z)| &\leq \frac{1}{|B(z,r)|} \int_{B(z,r)} \left|\partial_1 \widetilde{W}_k(s)\right| ds \\
&= \frac{1}{|B(z,r)|} \int_{B(z,r)\cap\mathbb{R}^2_+} \left|\partial_1 \widetilde{W}_k(s)\right| ds + \frac{1}{|B(z,r)|} \int_{B(z,r)\cap\mathbb{R}^2_-} \left|\partial_1 \widetilde{W}_k(s)\right| ds \\
&\lesssim \frac{1}{r} \int_{x-r}^{x+r} \mathcal{M}_\alpha^+ \nabla \widetilde{W}_k^+(t) dt + \frac{1}{r} \int_{x-r}^{x+r} \mathcal{M}_\alpha^- \nabla \widetilde{W}_k^-(t) dt.
\end{aligned}$$

Since $w \in A_p(\mathbb{R})$, it follows that

$$\begin{aligned}
\frac{1}{r} \int_{x-r}^{x+r} \mathcal{M}_\alpha^+ \nabla \widetilde{W}_k^+(t) dt &\lesssim \left(\frac{1}{r} \int_{x-r}^{x+r} \left|\mathcal{M}_\alpha^+ \nabla \widetilde{W}_k^+(t)\right|^p w(t) dt \right)^{1/p} \left(\frac{1}{r} \int_{x-r}^{x+r} w^{1-p'}(t)\, dt \right)^{1/p'} \\
&\lesssim \frac{1}{r} \|\mathcal{M}_\alpha^+ \nabla \widetilde{W}_k^+\|_{L^p(\mathbb{R},w)} \left(\int_{x-r}^{x+r} w^{1-p'}(t)\, dt \right)^{1/p'} \lesssim \frac{\|\mathcal{M}_\alpha^+ \nabla \widetilde{W}_k^+\|_{L^p(\mathbb{R},w)}}{(\int_{x-r}^{x+r} w(x)\, dx)^{1/p}},
\end{aligned}$$

and similarly for the term with $\mathcal{M}_\alpha^- \nabla \widetilde{W}_k^-$. Letting $r \to \infty$, we obtain that $\partial_1 W_k(z) = 0$; the same reasoning gives that $\partial_2 W_k(z) = 0$. Hence W_k is constant; that is, there exists $C_k \in \mathbb{R}$ so that

$$U_k(x,y) = V_k(x,y) + C_k, \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Letting $k \to \infty$ we obtain the desired result. □

The next two lemmas deal with the boundary values of u_f.

Lemma 2.6 (Lemma 3.3 in [8]). *Let $1 < p < \infty$, $w \in A_p(\mathbb{R})$ and $f \in L^p(\mathbb{R}, w)$.*

(a) The function $x \to \int_{\mathbb{R}} \left|\log\left(\frac{|x-t|}{1+|t|}\right) f(t)\right| dt$ is locally integrable in $\mathbb{R}$. Moreover, the function given by

$$\mathcal{B}f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \log\left(\frac{|x-t|}{1+|t|}\right) f(t)\, dt \tag{2.4}$$

satisfies $(\mathcal{B}f)' = Hf$ in the sense of distributions.

(b) $u_f = \mathcal{B}f$ almost everywhere on $\mathbb{R}$ in the sense of non-tangential convergence.

Remark 2.7. Let $1 < p < \infty$, $w \in A_p(\mathbb{R})$ and $f \in L^p(\mathbb{R}, w)$. We note that $u(x,y) = -u_f(x,-y)$ with $x \in \mathbb{R}$ and $y < 0$ is a solution of the following Neumann problem in $\mathbb{R}^2_-$:

$$\Delta u = 0 \text{ on } \mathbb{R}^2_- \qquad \text{and} \qquad \partial_y u = f \text{ on } \mathbb{R},$$

Also, Lemma 2.6 gives $u = -\mathcal{B}f$ almost everywhere on $\mathbb{R}$ in the sense of non-tangential convergence and $\partial_x u = -Hf$ in the sense of distributions on $\mathbb{R}$.

Lemma 2.8. *If $\Psi : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous homeomorphism, $w \in A_p(\mathbb{R})$ and $f \in L^p(\mathbb{R}, w)$, then $\mathcal{B}f \circ \Psi$ is locally integrable in $\mathbb{R}$.*

Proof. Given $M > 0$, we will prove that

$$\int_{|x|\leq M} \int_{\mathbb{R}} \left|\log\left(\frac{|\Psi(x) - t|}{1 + |t|}\right)\right| |f(t)|\, dt\, dx < \infty.$$

Let $\bar{M}$ be such that $|\Psi(x)| \leq \bar{M}$ for $|x| \leq M$.

We have

$$\begin{aligned}
\left|1 - \frac{|\Psi(x) - t|}{1 + |t|}\right| &= \left|\frac{1 + |t| - |\Psi(x) - t|}{1 + |t|}\right| \\
&\leq \frac{1 + ||t| - |t - \Psi(x)||}{1 + |t|} \leq \frac{1 + |\Psi(x)|}{1 + |t|} \leq \frac{1 + \bar{M}}{1 + |t|} \qquad \text{for } |x| \leq M.
\end{aligned}$$

Choose $\delta > 0$ such that

$$y > 0,\ |1-y| \le \delta \implies |\log y| \lesssim |1-y|\,;$$

then let K be so that

$$|t| \ge K \implies \frac{1+\bar{M}}{1+|t|} \le \delta.$$

All of the above implies that

$$|t| \ge K \implies \left|\log\left(\frac{|\Psi(x)-t|}{1+|t|}\right)\right| \lesssim \frac{1+\bar{M}}{1+|t|} \qquad \text{for } |x| \le M;$$

therefore

$$\int_{|x|\le M}\int_{|t|>K}\left|\log\left(\frac{|\Psi(x)-t|}{1+|t|}\right)\right||f(t)|\,dt\,dx \lesssim \int_{|x|\le M}\int_{|t|>K}\frac{|f(t)|}{1+|t|}\,dt\,dx < \infty$$

since $f \in L^p(\mathbb{R}, w)$ and $w \in A_p(\mathbb{R})$.

We next prove that

$$\int_{|x|\le M}\int_{|t|\le K}\left|\log\left(\frac{|\Psi(x)-t|}{1+|t|}\right)\right||f(t)|\,dt\,dx < \infty.$$

We have

$$\begin{aligned}
&\int_{|x|\le M}\int_{|t|\le K}\left|\log\left(\frac{|\Psi(x)-t|}{1+|t|}\right)\right||f(t)|\,dt\,dx\\
&\qquad\lesssim 1 + \int_{|x|\le M}\int_{|t|\le K}|\log|\Psi(x)-t||\,|f(t)|\,dt\,dx\\
&\qquad= 1 + \int_{|y|\le \bar{M}}\left|(\Psi^{-1})'(y)\right|\int_{|t|\le K}|\log|y-t||\,|f(t)|\,dt\,dy
\end{aligned}$$

It is then enough to show that

$$\int_{|t|\le K}|\log|y-t||\,|f(t)|\,dt \lesssim 1 \quad \text{for } |y| \le \bar{M}, \tag{2.5}$$

$$\int_{|y|\le\bar{M}}\left|(\Psi^{-1})'(y)\right|\,dy < \infty. \tag{2.6}$$

Regarding (2.5), we have

$$\int_{|t|\le K}|\log|y-t||\,|f(t)|\,dt \lesssim \|f\|_{L^p(\mathbb{R},w)}\left(\int_{|t|\le K}|\log|y-t||^{p'}\,w(t)^{1-p'}\,dt\right)^{\frac{1}{p'}}.$$

Since $w^{1-p'} \in A_{p'}(\mathbb{R})$, by the reverse Hölder inequality for Muckenhoupt weights, there exists $r > 1$ such that $\int_{|t|\le K} w(t)^{(1-p')r}\,dt < \infty$. Then, for $|y| \le \bar{M}$, we obtain

$$\begin{aligned}
\int_{|t|\le K}|\log|y-t||^{p'}\,w(t)^{1-p'}\,dt &\le \left(\int_{|t|\le K} w(t)^{(1-p')r}\,dt\right)^{\frac{1}{r}}\left(\int_{|t|\le K}|\log|y-t||^{p'r'}\,dt\right)^{\frac{1}{r'}}\\
&\le \left(\int_{|t|<K} w(t)^{(1-p')r}\,dt\right)^{\frac{1}{r}}\left(\int_{|s|\le\bar{M}+K}|\log|s||^{p'r'}\,ds\right)^{\frac{1}{r'}}\\
&< \infty.
\end{aligned}$$

As for (2.6), let $\bar{\bar{M}}$ be such that $|\Psi^{-1}(y)| \le \bar{\bar{M}}$ for $|y| \le \bar{M}$; we then get

$$\int_{|y|\le\bar{M}}|(\Psi^{-1})'(y)|\,dy = \int_{|x|\le\bar{\bar{M}}}|(\Psi^{-1})'(\Psi(x))|\Psi'(x)|\,dx = \int_{|x|\le\bar{\bar{M}}} 1\,dx < \infty.$$

$\square$

2.4. **Conformal maps, domains and definition of solvability for** (1.1)**.** We formalize in this section the definitions of the conformal maps $\Phi_\pm$ and the domains $\Omega^\pm$ given in Section 1 as well as the definition of solvability of (1.1).

Let Λ be a rectifiable Jordan curve in the complex plane given parametrically by $x+i\gamma(x)$ for $x\in\mathbb{R}$, where γ is a real-valued, and consider the domains

$$\Omega^+ = \{z\in\mathbb{C} : \mathrm{Im}(z) > \gamma(\mathrm{Re}(z))\}, \qquad \Omega^- = \mathrm{int}((\Omega^+)^c); \tag{2.7}$$

note that $\Lambda = \partial\Omega^+ = \partial\Omega^-$. The set Ω^+ will be called an *upper graph domain* and Ω^- will be referred to as a *lower graph domain*.

Since $\Omega^\pm$ are simply connected, they are conformally equivalent to $\mathbb{R}^2_\pm$. Let $\Phi_\pm : \mathbb{R}^2_\pm \to \Omega^\pm$ be conformal maps such that $\Phi_\pm(\infty) = \infty$. Then $\Phi_\pm$ extend as homeomorphisms from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega^\pm}$ and $\Phi_\pm(x)$, $x\in\mathbb{R}$, are absolutely continuous when restricted to any finite interval; in particular, $\Phi'_\pm(x)$ exist for almost every $x\in\mathbb{R}$ and are locally integrable. Moreover, $\Phi'_\pm(x)\neq 0$ for almost every $x\in\mathbb{R}$, $\lim_{z\to x}\Phi'_\pm(z) = \Phi'_\pm(x)$ for almost every $x\in\mathbb{R}$ in the sense of non-tangential convergence. If $\Phi'_\pm(x)$ exist and are not zero, then they are vectors tangent to Λ at $\Phi_\pm(x)$ and $\mathrm{Re}(\Phi'_\pm) > 0$ almost everywhere on $\mathbb{R}$. We refer the reader to [19, proof of Theorem 1.1] for the proof of those properties.

The arc length measure in Λ will be denoted by ds. Given a weight ν in Λ (i.e. a non-negative locally integrable (with respect to ds) function defined on Λ), we will denote by $L^p(\Lambda,\nu)$ the space of p-integrable functions on Λ with respect to $\nu\, ds$. For future use, we note that

$$\|g\|_{L^p(\Lambda,\nu)} = \|T_{\Phi_\pm} g\|_{L^p(\mathbb{R},|\Phi'_\pm|^{1-p}(\nu\circ\Phi_\pm))}. \tag{2.8}$$

When γ is a Lipschitz function we will call Ω^+ *an upper Lipschitz graph domain* and Ω^- a *lower Lipschitz graph domain*. In this case, it holds that $|\Phi'_\pm|^{-1}\in A_2(\mathbb{R})$ ([19, Theorem 1.10]). Also, setting $\Phi_\pm = \Phi^1_\pm + i\,\Phi^2_\pm$, we have $\mathrm{Re}\left(\frac{1}{\Phi'_\pm}\right) \approx \frac{1}{\Phi^{1\,\prime}_\pm} \approx \frac{1}{|\Phi'_\pm|}$ almost everywhere. Indeed, since $\Phi^2_\pm(x) = \gamma(\Phi^1_\pm(x))$ for $x\in\mathbb{R}$, we obtain

$$\Phi^{2\,\prime}_\pm(x) = \gamma'(\Phi^1_\pm(x))\Phi^{1\,\prime}_\pm(x), \qquad \text{a.e. } x\in\mathbb{R}, \tag{2.9}$$

and therefore

$$\mathrm{Re}\left(\frac{1}{\Phi'_\pm}\right) = \frac{\Phi^{1\,\prime}_\pm}{|\Phi'_\pm|^2} = \frac{\Phi^{1\,\prime}_\pm}{|\Phi^{1\,\prime}_\pm|^2(1+\gamma'(\Phi^1_\pm)^2)}. \tag{2.10}$$

The equality (2.9) gives $|\Phi^{2\,\prime}_\pm| \lesssim \Phi^{1\,\prime}_\pm$ almost everywhere; this and (2.10) lead to the desired result.

2.4.1. *What does solvability of* (1.1) *mean?* We start with a definition that extends the idea of non-tangential convergence to domains whose boundary do not posses the cone property (i.e. there does not exist β such that the cones of aperture β and vertex at points of the boundary are contained in the domain for all points in the boundary).

Definition 2.9. Let Ω be a simply connected domain in the complex plane and Φ a conformal map from $\mathbb{R}^2_+$ or $\mathbb{R}^2_-$ onto Ω. Given $\mathcal{R}:\Omega\to\mathbb{C}$, $r:\partial\Omega\to\mathbb{C}$ and $\xi\in\partial\Omega$, we say that $\mathcal{R}(z)$ converges to $r(\xi)$ in the sense of Φ *non-tangential convergence* if, for some $0<\alpha<\pi/2$, $\lim\mathcal{R}(z) = r(\xi)$ as $z\to\xi$ with $z\in\Phi(\Gamma_\alpha(\Phi^{-1}(\xi)))$, where $\Gamma_\alpha = \Gamma^+_\alpha$ if the domain of Φ is $\mathbb{R}^2_+$ and $\Gamma_\alpha = \Gamma^-_\alpha$ if the domain of Φ is $\mathbb{R}^2_-$.

We note that by [19, Lemma 1.13], Φ non-tangential convergence implies non-tangential convergence when Ω is an upper or lower Lipschitz graph domain, and the definitions are equivalent in this setting if the Lipschitz constant of $\partial\Omega$ is less than 1.

We next present the definition of solvability of (1.1) in $L^p(\Lambda,\nu)$.

Definition 2.10. Consider domains $\Omega^\pm$ as in (2.7) and corresponding conformal maps $\Phi_\pm : \mathbb{R}^2_\pm \to \Omega^\pm$. Given $\mu \neq 0$ and a weight ν in Λ, we say that (1.1) *is solvable in* $L^p(\Lambda, \nu)$ if, for every $g \in L^p(\Lambda, \nu)$, there are harmonic functions $v^\pm$ in $\Omega^\pm$ such that

(a) $v^\pm$ and $\partial_{\mathbf{n}} v^\pm$ on Λ are the traces of $v^\pm$ and $\partial_{\mathbf{n}} v^\pm$, respectively, in the sense of $\Phi_\pm$ non-tangential convergence and $\partial_{\mathbf{n}} v^\pm \in L^p(\Lambda, \nu)$,

(b) $v^+ = v^-$ and $\partial_{\mathbf{n}} v^+ - \mu\, \partial_{\mathbf{n}} v^- = g$ almost everywhere on Λ with respect to arc length,

(c) if $0 < \alpha < \pi/2$, then

$$\|\mathcal{M}^+_\alpha \nabla(v^+ \circ \Phi_+)\|_{L^p(\mathbb{R}, \widetilde{w}_p)} \lesssim \|g\|_{L^p(\Lambda,\nu)} \quad \text{and} \quad \|\mathcal{M}^-_\alpha \nabla(v^- \circ \Phi_-)\|_{L^p(\mathbb{R},w)} \lesssim \|g\|_{L^p(\Lambda,\nu)},$$

where $w = |(\Phi_-)'|^{1-p}\, (\nu \circ \Phi_-)$ and $\widetilde{w}_p$ is as in (2.1) with $\Psi = \Phi_+^{-1} \circ \Phi_-$.

We will say that (1.1) *is uniquely solvable in* $L^p(\Lambda, \nu)$ if (1.1) is solvable in $L^p(\Lambda, \nu)$ and solutions are unique modulo constants.

3. General results for the solvability of $P_\Psi(\mu)$ in $L^p(\mathbb{R}, w)$

In this section, we prove Theorem 1.3, which states general necessary and sufficient conditions for the solvability and unique solvability of $P_\Psi(\mu)$ in $L^p(\mathbb{R}, w)$ in terms of the surjectivity and invertibility, respectively, of the operator $HT_\Psi + \mu T_\Psi H$. This will be useful for the proofs of the statements in Section 4 regarding other conditions that imply solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, w)$. We also present in this section Corollary 3.4, which gives symmetric properties associated to the solvability of $P_\Psi(\mu)$.

The following lemma will be used in the proof of Theorem 1.3:

Lemma 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ be a surjective bounded linear operator. Then for every $y \in \mathcal{Y}$, there exists $x_y \in \mathcal{X}$ such that $\mathcal{T}(x_y) = y$ and $\|x_y\|_{\mathcal{X}} \lesssim \|y\|_{\mathcal{Y}}$.*

Proof. Consider the Banach space $\bar{\mathcal{X}} = \mathcal{X}/\ker(\mathcal{T})$ with the norm

$$\|[x]\|_{\bar{\mathcal{X}}} = \inf\{\|x'\|_{\mathcal{X}} : \mathcal{T}(x) = \mathcal{T}(x')\},$$

where $[x]$ denotes the equivalence class of x. Define $\bar{\mathcal{T}} : \bar{\mathcal{X}} \to \mathcal{Y}$ such that $\bar{\mathcal{T}}([x]) = \mathcal{T}(x)$.

It easily follows that $\bar{\mathcal{T}}$ is a bijective bounded linear operator and therefore

$$\|[x]\|_{\bar{\mathcal{X}}} \approx \|\bar{\mathcal{T}}([x])\|_{\mathcal{Y}} \quad \forall [x] \in \bar{\mathcal{X}}.$$

Given $y \in \mathcal{Y}$, $y \neq 0$, let $x \in \mathcal{X}$ be such that $\mathcal{T}(x) = y$ and choose $x_y \in [x]$ satisfying $\|x_y\|_{\mathcal{X}} < 2\|[x]\|_{\bar{\mathcal{X}}}$; then

$$\|x_y\|_{\mathcal{X}} \lesssim \|\bar{\mathcal{T}}([x])\|_{\mathcal{Y}} = \|\mathcal{T}(x_y)\|_{\mathcal{Y}} = \|y\|_{\mathcal{Y}}.$$

If $y = 0$, the result follows by choosing $x_y = 0$.

□

We next prove Theorem 1.3.

Proof of Theorem 1.3. We first prove (a) and then (b).

Proof of (a). We first show that if $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is surjective then $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R}, w)$. Let $f \in L^p(\mathbb{R}, w)$; then Lemma 3.1 gives that there exists $h \in L^p(\mathbb{R}, \widetilde{w}_p)$ such that $(HT_\Psi + \mu T_\Psi H)(h) = Hf$ and

$$\|h\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|Hf\|_{L^p(\mathbb{R},w)}, \tag{3.1}$$

with the implicit constant independent of h and f.

Consider the following Neumann problem in the upper-half plane:

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathbb{R}^2_+, \\ \partial_y u^+ = h & \text{on } \mathbb{R}. \end{cases} \tag{3.2}$$

Let u^+ be as given by Theorem 2.4. By Lemma 2.6, u^+ is well defined in $\mathbb{R}$ and $\partial_x u^+ = Hh$ almost everywhere in $\mathbb{R}$.

Let U^- be the solution of the following Neumann problem as given by Remark 2.7:

$$\begin{cases} \Delta U^- = 0 & \text{in } \mathbb{R}^2_-, \\ \partial_y U^- = \frac{T_\Psi h - f}{\mu} & \text{on } \mathbb{R}. \end{cases} \tag{3.3}$$

Again, U^- is well-defined in $\mathbb{R}$ and $\partial_x U^- = -H(\frac{T_\Psi h-f}{\mu})$ almost everywhere in $\mathbb{R}$. Since $(HT_\Psi + \mu T_\Psi H)(h) = Hf$, we get

$$\partial_x U^- = -H(\tfrac{T_\Psi h-f}{\mu}) = T_\Psi Hh \quad \text{a.e in } \mathbb{R}.$$

Moreover, recalling that $\Psi' > 0$, we obtain

$$\partial_x(u^+ \circ \Psi) = (\partial_x u^+ \circ \Psi)\,\Psi' = T_\Psi \partial_x u^+ = T_\Psi Hh \quad \text{a.e in } \mathbb{R}.$$

Noting that $T_\Psi Hh$ is locally integrable, we then have

$$\partial_x(u^+ \circ \Psi) = \partial_x U^-$$

in the sense of distributions. Both, $u^+ \circ \Psi$ and U^- are locally integrable functions on $\mathbb{R}$ (in particular, they are distributions) by Lemmas 2.6 and 2.8, respectively. We then conclude that $u^+ \circ \Psi = U^- + C$ almost everywhere on $\mathbb{R}$ for some constant C. Defining $u^- = U^- + C$, we have that $u^\pm$ satisfy $P_\Psi(\mu)$.

We next prove the estimates for the non-tangential maximal operator. Since u^+ is a solution of the Neumann problem with datum h in $L^p(\mathbb{R}, \widetilde{w}_p)$, we have

$$\|\mathcal{M}_\alpha^+ \nabla u^+\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|h\|_{L^p(\mathbb{R},\widetilde{w}_p)};$$

moreover, (3.1) and the fact that $w \in A_p(\mathbb{R})$ lead to

$$\|h\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|Hf\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)}. \tag{3.4}$$

As a consequence, we obtain

$$\|\mathcal{M}_\alpha^+ \nabla u^+\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|f\|_{L^p(\mathbb{R},w)}.$$

Using that u^- is a solution of the Neumann problem with datum $\frac{T_\Psi h-f}{\mu}$ in $L^p(\mathbb{R}, w)$, the fact that $\|T_\Psi h\|_{L^p(\mathbb{R},w)} = \|h\|_{L^p(\mathbb{R},\widetilde{w}_p)}$, and (3.4), we see that

$$\|\mathcal{M}_\alpha^- \nabla u^-\|_{L^p(\mathbb{R},w)} \lesssim \|\tfrac{T_\Psi h-f}{\mu}\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)}. \tag{3.5}$$

We next show that if $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R}, w)$, then the operator $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is surjective. Let $g, f \in L^p(\mathbb{R}, w)$ be such that $g = Hf$ and consider the solutions $u^\pm$ of $P_\Psi(\mu)$ with datum f; set $h = \partial_y u^+$ and note that $h \in L^p(\mathbb{R}, \widetilde{w}_p)$ in view of the estimates satisfied by $\mathcal{M}_\alpha^+ \nabla u^+$. We have

$$\partial_x(u^+ \circ \Psi) = \partial_x u^- \quad \text{a.e. in } \mathbb{R};$$

also, since u^+ is a solution of (3.2) and u^- is a solution of (3.3), we have

$$\partial_x u^+ = Hh \quad \text{and} \quad \partial_x u^- = -H\left(\frac{T_\Psi h - f}{\mu}\right).$$

This leads to $H(\frac{T_\Psi h-f}{\mu}) = -T_\Psi Hh$, from which we get $(HT_\Psi + \mu T_\Psi H)(h) = Hf = g$.

Proof of (b). Assume first that $P_\Psi(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$. By Part(a), $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is surjective.

If $HT_\Psi + \mu T_\Psi H$ is not injective, then there exists $\eta \in L^p(\mathbb{R}, \widetilde{w}_p)$ such that $\eta \neq 0$ and $(HT_\Psi + \mu T_\Psi H)(\eta) = 0$. Let $f \in L^p(\mathbb{R}, w)$; then $Hf \in L^p(\mathbb{R}, w)$ and by Lemma 3.1, there exists $h_1 \in L^p(\mathbb{R}, \widetilde{w}_p)$ such that $(HT_\Psi + T_\Psi H)(h_1) = Hf$ and

$$\|h_1\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|Hf\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)}.$$

Define
$$h_2 = h_1 + \frac{\|f\|_{L^p(\mathbb{R},w)}}{\|\eta\|_{L^p(\mathbb{R},\widetilde{w}_p)}}\eta \in L^p(\mathbb{R},\widetilde{w}_p),$$
which also satisfies $(HT_\Psi + T_\Psi H)(h_2) = Hf$ and
$$\|h_2\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|f\|_{L^p(\mathbb{R},w)}.$$
The argument in the proof of Part (a) gives solutions $u_1^\pm$ and $u_2^\pm$ of $P_\Psi(\mu)$ with datum f which are associated to h_1 and h_2, respectively, through (3.2) and (3.3). Since $h_1 \neq h_2$, we have that $u_1^\pm \neq u_2^\pm$, which contradicts the fact that $P_\Psi(\mu)$ is uniquely solvable. We then conclude that $HT_\Psi + \mu T_\Psi H$ is injective.

Conversely, assume that $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R},\widetilde{w}_p) \to L^p(\mathbb{R},w)$ is invertible. Then $P_\Psi(\mu)$ is solvable by Part (a). We next show uniqueness of solutions modulo constants.

If $u_1^\pm$ and $u_2^\pm$ are solutions of $P_\Psi(\mu)$ with datum $f \in L^p(\mathbb{R},w)$, then $u^\pm = u_1^\pm - u_2^\pm$ are harmonic functions in $\mathbb{R}^2_\pm$ that satisfy Items (a) and (b) (with datum zero) of Definition 2.3; moreover $\mathcal{M}_\alpha^+\nabla u^+ \in L^p(\mathbb{R},\widetilde{w}_p)$ and $\mathcal{M}_\alpha^-\nabla u^- \in L^p(\mathbb{R},w)$ by Item (c) of Definition 2.3 for $u_1^\pm$ and $u_2^\pm$, which imply that $\partial_y u^+ \in L^p(\mathbb{R},\widetilde{w}_p)$ and $\partial_y u^- \in L^p(\mathbb{R},w)$. Since u^+ is a solution of the Neumann problem with datum $\partial_y u^+$, we have
$$\partial_x u^+ = H(\partial_y u^+).$$
Since u^- is a solution of the Neumann problem with datum $\partial_y u^- = \frac{T_\Psi(\partial_y u^+)}{\mu}$ and $u^+\circ\Psi = u^-$, we have
$$T_\Psi H(\partial_y u^+) = (\partial_x u^+ \circ \Psi)\,\Psi' = \partial_x(u^+\circ\Psi) = \partial_x u^- = -\frac{H(T_\Psi(\partial_y u^+))}{\mu}.$$
We then obtain
$$(HT_\Psi + \mu T_\Psi H)(\partial_y u^+) = 0$$
and, since $HT_\Psi + \mu T_\Psi H$ is injective, it follows that $\partial_y u^+ = 0$. Also, $\partial_y u^- = \frac{T_\Psi(\partial_y u^+)}{\mu} = 0$. By Theorem 2.5, we have that $u^\pm$ are constants in $\mathbb{R}^2_\pm$. We then conclude that $P_\Psi(\mu)$ is uniquely solvable. □

Remark 3.2. Given a homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$, set
$$S = HT_{\Psi^{-1}}HT_\Psi : L^p(\mathbb{R},\widetilde{w}_p) \to L^p(\mathbb{R},\widetilde{w}_p);$$
then S is a bounded invertible operator. Note that
$$HT_\Psi + \mu T_\Psi H = HT_\Psi(I - \mu T_{\Psi^{-1}}HT_\Psi H) = HT_\Psi(I - \mu S^{-1}) = HT_\Psi(S - \mu I)S^{-1}.$$
We then have that $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R},\widetilde{w}_p) \to L^p(\mathbb{R},w)$ is surjective (injective) if and only if $S - \mu I : L^p(\mathbb{R},\widetilde{w}_p) \to L^p(\mathbb{R},\widetilde{w}_p)$ is surjective (injective).

We next note that, since $S - \mu I = \mu(\frac{1}{\mu}S - I)$, then $S - \mu I$ is invertible if $\|S\| < |\mu|$. Also, since $S - \mu I = S(I - \mu S^{-1})$, then $S - \mu I$ is invertible if $|\mu| < \|S^{-1}\|^{-1}$. As a consequence $P_\Psi(\mu)$ is solvable for $|\mu|$ sufficiently small and for $|\mu|$ sufficiently large.

Remark 3.3. By [13, Section 2], the transmission problem (1.1) is uniquely solvable in $L^p(\Lambda)$ when $\mu = 1$, $1 < p < \infty$ and $\Omega^\pm$ are upper and lower Lipschitz graph domains. We can then apply Theorem 1.1 to conclude that $P_\Psi(1)$ with $\Psi = \Phi_+^{-1}\circ\Phi_-$ (assuming Ψ locally absolutely continuous) is uniquely solvable in $L^p(\mathbb{R}, |\Phi_-'|^{1-p})$. Noting that for $w = |\Phi_-'|^{1-p}$, we have $\widetilde{w}_p = |\Phi_+'|^{1-p}$, assuming both weights are in $A_p(\mathbb{R})$, Theorem 1.3 and Remark 3.2 give that the operator $S - I : L^p(\mathbb{R}, |\Phi_+'|^{1-p}) \to L^p(\mathbb{R}, |\Phi_+'|^{1-p})$ is invertible. This is the case for $p = 2$ since $|\Phi_-'|^{-1}$ and $|\Phi_+'|^{-1}$ are in $A_2(\mathbb{R})$.

Theorem 1.3 leads to the following symmetry properties for the solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R},w)$.

Corollary 3.4. *Let $\mu \neq 0$, $1 < p < \infty$, and $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism with $\Psi' > 0$ almost everywhere; assume $w \in A_p(\mathbb{R})$ is such that $\widetilde{w}_p \in A_p(\mathbb{R})$. It then holds that*

(a) $P_\Psi(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$ if and only if $P_\Psi(1/\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$.

(b) $P_\Psi(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, w)$ if and only if $P_{\Psi^{-1}}(\mu)$ is uniquely solvable in $L^p(\mathbb{R}, \widetilde{w}_p)$.

Proof. The result will follow by doing simple manipulations of the operator $HT_\Psi + \mu T_\Psi H$ and using the characterization of solvability of $P_\Psi(\mu)$ given in Theorem 1.3.

Proof of (a). We have:

$$HT_\Psi + \mu T_\Psi H = -\mu H\big(HT_\Psi + \tfrac{1}{\mu} T_\Psi H\big)H.$$

Since $H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, \widetilde{w}_p)$ is invertible, it follows that $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is invertible if and only if $HT_\Psi + \frac{1}{\mu} T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is invertible.

Proof of (b). Similarly:

$$HT_\Psi + \mu T_\Psi H = \mu T_\Psi\big(HT_{\Psi^{-1}} + \tfrac{1}{\mu} T_{\Psi^{-1}} H\big)T_\Psi.$$

Since $T_\Psi : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is invertible, it follows that $HT_\Psi + \mu T_\Psi H : L^p(\mathbb{R}, \widetilde{w}_p) \to L^p(\mathbb{R}, w)$ is invertible if and only if $HT_{\Psi^{-1}} + \frac{1}{\mu} T_{\Psi^{-1}} H : L^p(\mathbb{R}, w) \to L^p(\mathbb{R}, \widetilde{w}_p)$ is invertible. Finally, by Part (a), we get the result. □

We end this section with a lemma that gives sufficient conditions for a family of linear operators to be invertible; this result will be used in Section 4. See [26], we include the proof for the sake of completeness.

Lemma 3.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $I \subset \mathbb{R}$ be an open interval. Consider a family $\{\mathcal{T}_\mu\}_{\mu \in I} \subset \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{T}_{\mu_0}$ is invertible for some $\mu_0 \in I$, $\mathcal{T}_\mu(\mathcal{X})$ is closed in $\mathcal{Y}$, $\dim(\ker(\mathcal{T}_\mu)) = 0$ for all $\mu \in I$, and $\mu \to \mathcal{T}_\mu$ is continuous in I. Then $\mathcal{T}_\mu$ is invertible for all $\mu \in I$.*

Proof. In view of the fact that $\dim(\ker(\mathcal{T}_\mu)) = 0$, we only need to show that $\mathcal{T}_\mu$ is surjective. Define

$$\mathcal{U} = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : T(\mathcal{X}) \text{ is closed in } \mathcal{Y} \text{ and } \dim(\ker(T)) < \infty\}.$$

Note that $\{\mathcal{T}_\mu\}_{\mu \in I} \subset \mathcal{U}$. Let $\text{Ind} : \mathcal{U} \to \mathbb{Z} \cup \{-\infty\}$ be the index function given by

$$\text{Ind}(T) = \dim(\ker(T)) - \dim(\mathcal{Y}/T(\mathcal{X})), \quad T \in \mathcal{U}. \tag{3.6}$$

By [26, Theorem 2.2], $\mathcal{U}$ is open and Ind is continuous in $\mathcal{U}$. This and the continuity of $\mu \to \mathcal{T}_\mu$ in I imply that the function $f : I \to \mathbb{Z} \cup \{-\infty\}$ defined by $f(\mu) = \text{Ind}(\mathcal{T}_\mu)$ is continuous in I. Since $f(\mu_0) = 0$ we obtain that $f(\mu) = 0$ for $\mu \in I$.

We then conclude that $\dim(\mathcal{Y}/\mathcal{T}_\mu(\mathcal{X})) = \dim(\ker(\mathcal{T}_\mu)) - \text{Ind}(\mathcal{T}_\mu) = 0$; thus $\mathcal{T}_\mu(\mathcal{X})$ is dense in $\mathcal{Y}$, which along with the assumption that $\mathcal{T}_\mu(\mathcal{X})$ is closed, implies that $\mathcal{T}_\mu$ is surjective.

□

4. Solvability results for $P_\Psi(\mu)$ in $L^2(\mathbb{R}, w)$

In this section, we present different settings that lead to solvability results for $P_\Psi(\mu)$ in weighted L^2 spaces, which are used in Section 5 to study solvability of the transmission problem (1.1) and are interesting in their own right.

In Section 4.2, we prove Theorem 1.4 and state Theorems 4.3 and 4.4; all of these results give sufficient conditions on the homeomorphism Ψ for the solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, \frac{1}{\Psi'})$; we also present examples associated to domains that include an infinite staircase and symmetric cones, as well as an example related to the Helson-Szegö representation of A_2 weights. In Section 4.2, we prove Theorem 1.5 , which deals with the solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, |\Phi_-'|^{-1})$ when $\Psi = \Phi_+^{-1} \circ \Phi_-$. A main tool in the proofs of all these results is the use of a Rellich identity for the Hilbert transform, which we present in Section 4.1.

4.1. **Rellich identity.** For a real-valued Schwarz function f, the following formula known in the literature as "the magic formula" holds true (see [15, (5.1.23), p. 320]):

$$(Hf)^2 - f^2 = 2H(fHf).$$

From here, it can be easily deduced that if v is a weight for which Hv is well defined and f is a real-valued Schwarz function, then

$$\int_{\mathbb{R}} (Hf)^2 v dx = \int_{\mathbb{R}} f^2 v dx - 2\int_{\mathbb{R}} f\, Hf Hv dx. \tag{4.1}$$

A second formula of this type was proved in [10] using Rellich identity and connections with the Neumann problem on Lipschitz graph domains.

Theorem 4.1 (Theorem 1.2 in [10])**.** *Let Φ be a conformal map from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain and $f \in L^2(\mathbb{R}, |\Phi'|^{-1})$ be real-valued. Then*

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx - 2\int_{\mathbb{R}} f\, Hf \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx. \tag{4.2}$$

Remark 4.2. If Φ is a conformal map from $\mathbb{R}^2_-$ onto a lower Lipschitz graph domain, the Rellich formula (4.2) becomes,

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx + 2\int_{\mathbb{R}} f\, Hf \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx. \tag{4.3}$$

This follows by applying (4.2) to the conformal map $-\Phi(-x,-y)$, which satisfies the hypothesis of Theorem 4.1.

4.2. **Solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, \frac{1}{\Psi'})$.** Our results in this section use the Rellich identity (4.2) to obtain solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, \frac{1}{\Psi'})$ under different assumptions on Ψ. We start with the proof of Theorem 1.4.

Proof of Theorem 1.4. Note that if $w = \frac{1}{\Psi'}$, then $\widetilde{w} = 1$; also, $w \in A_2(\mathbb{R})$ since $|\Phi'|^{-1} \in A_2(\mathbb{R})$ and $|\Phi'|^{-1} \sim \operatorname{Re}\left(\frac{1}{\Phi'}\right)$. By Theorem 1.3, Remark 3.2 and Part (a) of Corollary 3.4, it is enough to see that the operator $S - \mu I : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible for $0 < |\mu| < 1$ satisfying (1.4).

Set $h = \Psi' \operatorname{Im}\left(\frac{1}{\Phi'}\right)$ and let $f \in L^2(\mathbb{R})$; without loss of generality we may assume that f is real-valued. Noting that $T_\Psi f \in L^2(\mathbb{R}, \frac{1}{\Psi'}) = L^2(\mathbb{R}, |\Phi'|^{-1})$, we apply (4.2) to the function $T_\Psi f$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}} (HT_\Psi f)^2 \frac{1}{\Psi'} dx &= \int_{\mathbb{R}} (HT_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx \\
&= \int_{\mathbb{R}} (T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx - 2\int_{\mathbb{R}} T_\Psi f\, HT_\Psi f \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx \\
&= \int_{\mathbb{R}} (T_\Psi f)^2 \frac{1}{\Psi'} dx - 2\int_{\mathbb{R}} T_\Psi f\, HT_\Psi f \frac{h}{\Psi'} dx.
\end{aligned}$$

A change of variables gives $\int_{\mathbb{R}} FG\frac{1}{\Psi'}\, dx = \int_{\mathbb{R}} T_{\Psi^{-1}}F\, T_{\Psi^{-1}}G\, dx$, and therefore it follows that

$$\int_{\mathbb{R}} (T_{\Psi^{-1}} HT_\Psi f)^2 dx = \int_{\mathbb{R}} f^2 dx - 2\int_{\mathbb{R}} f\, T_{\Psi^{-1}}\, HT_\Psi f\, (h \circ \Psi^{-1}) dx;$$

equivalently,

$$\int_{\mathbb{R}} (HSf)^2 dx = \int_{\mathbb{R}} f^2 dx + 2\int_{\mathbb{R}} f\, HSf\, (h \circ \Psi^{-1}) dx.$$

This leads to

$$\int_{\mathbb{R}} H(S-\mu I)f\, H(S+\mu I)f dx = \int_{\mathbb{R}} (HSf)^2 dx - \mu^2 \int_{\mathbb{R}} (Hf)^2 dx$$

$$= \int_{\mathbb{R}} f^2 dx + 2\int_{\mathbb{R}} f\, HSf\, (h\circ\Psi^{-1}) dx - \mu^2 \int_{\mathbb{R}} (Hf)^2 dx$$

$$= \left(1-\mu^2\right)\int_{\mathbb{R}} f^2 dx + 2\int_{\mathbb{R}} f\, H[S-\mu I]f\, (h\circ\Psi^{-1}) dx + 2\mu \int_{\mathbb{R}} fHf\, (h\circ\Psi^{-1}) dx, \tag{4.4}$$

that is

$$\left(1-\mu^2\right)\int_{\mathbb{R}} f^2 dx + 2\mu\int_{\mathbb{R}} fHf\, (h\circ\Psi^{-1}) dx \tag{4.5}$$

$$= \int_{\mathbb{R}} H(S-\mu I)f\, H(S+\mu I)f dx - 2\int_{\mathbb{R}} f\, H(S-\mu I)f\, (h\circ\Psi^{-1}) dx \tag{4.6}$$

Observe that (4.6) is controlled up to a constant by

$$\|(S-\mu I)f\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})},$$

and, since $\|h\circ\Psi^{-1}\|_\infty \leq k_\Psi$, the second term of (4.5) satisfies

$$\left|2\mu\int_{\mathbb{R}} fHf(h\circ\Psi^{-1})dx\right| \leq 2|\mu|k_\Psi\|f\|^2_{L^2(\mathbb{R})},$$

giving that

$$2\mu\int_{\mathbb{R}} fHf\,(h\circ\Psi^{-1})dx \geq -2|\mu|k_\Psi\|f\|^2_{L^2(\mathbb{R})}.$$

As a consequence,

$$(1-\mu^2 - 2\,k_\Psi\,|\mu|)\|f\|_{L^2(\mathbb{R})} \lesssim \|(S-\mu I)f\|_{L^2(\mathbb{R})},$$

and hence, given (1.4), we conclude that $S-\mu I$ is injective and has closed range. An application of Lemma 3.5 with $\mu_0 = 0$ gives that $S-\mu I$ is invertible.

□

4.2.1. *Applications of Theorem 1.4.* We next present examples of homeomorphisms Ψ that satisfy the hypothesis of Theorem 1.4.

Let $\Phi = \Phi^1 + i\,\Phi^2$ be a conformal map from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain associated to a Lipschitz curve $t+i\gamma(t)$ for $t\in\mathbb{R}$. By (2.10), if Ψ is given as in (1.3), we have

$$\frac{1}{\Psi'} = \operatorname{Re}\left(\frac{1}{\Phi'}\right) = \frac{\Phi^{1\prime}}{|\Phi'|^2} = \frac{\Phi^{1\prime}}{|\Phi^{1\prime}|^2(1+\gamma'(\Phi^1(x))^2)},$$

and since $\Phi^{1\prime} > 0$ almost everywhere, we obtain that

$$\Psi'(x) = \Phi^{1\prime}(x)(1+\gamma'(\Phi^1(x))^2). \tag{4.7}$$

We present specific examples of the above:

(a) Example related to the infinite staircase: Let Ω to be the "infinite rotated staircase" with interior angles alternately equal to $\pi/2$ and $3\pi/2$ as shown in Figure 3.

In this case, a conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$ is given by

$$\Phi(z) = \frac{\sqrt{2}\pi}{4}e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\int_0^z \sqrt{\tan y}\,dy,$$

which satisfies

$$\Phi'(x) = e^{-i\frac{\pi}{4}}\sqrt{|\tan x|}, \quad x\in\mathbb{R}.$$

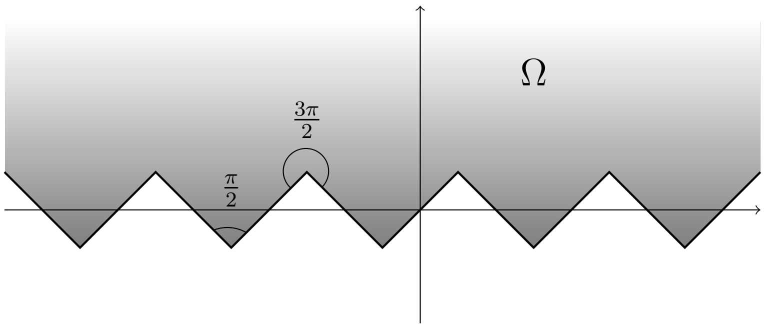


FIGURE 3. Infinite staircase

Then, if $\Psi : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous homeomorphism such that $\frac{1}{\Psi'} = \operatorname{Re}\left(\frac{1}{\Phi'}\right)$, using (4.7) and the fact that $|\gamma'(t)| = 1$ almost everywhere give

$$\Psi'(x) = \sqrt{2|\tan x|}.$$

Also, since $\operatorname{Re}(\Phi')(x) = -\operatorname{Im}(\Phi')(x)$ for $x \in \mathbb{R}$, it follows that $k_\Psi = 1$. Then $P_\Psi(\mu)$ is uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Psi'})$ for $\mu \neq 0$ such that $|\mu| < \sqrt{2} - 1$ or $|\mu| > \frac{1}{\sqrt{2}-1}$.

(b) Example related to a symmetric infinite sector of aperture α: Let Ω be a cone with aperture $\alpha\pi$, with $\alpha \in (0, 2)$, which is symmetric about the imaginary axis (see Figure 4). Consider the conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$ such that

$$\Phi(z) = e^{i\frac{(1-\alpha)}{2}\pi} z^\alpha = ie^{-i\frac{\alpha}{2}\pi} e^{\alpha(\log|z| + i\operatorname{Arg}(z))}, \tag{4.8}$$

where we chose the branch cut $\{iy : y \leq 0\}$, so that Φ is analytic on $\mathbb{R}^2_+$.

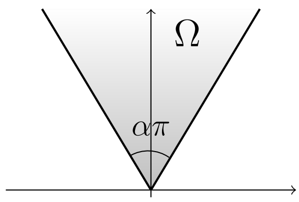


FIGURE 4. Symmetric cone with aperture $\alpha\pi$.

It follows that $\operatorname{Re}\left(\frac{1}{\Phi'}\right) = \alpha^{-1}\sin(\frac{\alpha\pi}{2})|x|^{1-\alpha}$ and therefore, if $\Psi : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous homeomorphism such that $\frac{1}{\Psi'} = \operatorname{Re}\left(\frac{1}{\Phi'}\right)$, we have $\Psi'(x) = \frac{\alpha}{\sin(\frac{\alpha\pi}{2})}|x|^{\alpha-1}$. Note that $\Psi' \in A_2(\mathbb{R})$ if and only if $0 < \alpha < 2$, as expected. We have $\operatorname{Im}\left(\frac{1}{\Phi'}\right) = -\alpha^{-1}\cos(\frac{\alpha\pi}{2})\operatorname{sgn}(x)|x|^{1-\alpha}$ which together with the expression for Ψ' gives that $k_\Psi = |\cot(\frac{\alpha\pi}{2})|$. In particular, if $\alpha = \frac{1}{2}$, then $k_\Psi = 1$ and $P_\Psi(\mu)$ is uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Psi'})$ for $\mu \neq 0$ such that $|\mu| < \sqrt{2} - 1$ or $|\mu| > \frac{1}{\sqrt{2}-1}$.

(c) Examples of homeomorphisms using the Helson-Szegö representation of A_2 weights: If f is such that $\|f\|_\infty < \frac{\pi}{2}$, by the Helson-Szegö representation of A_2 weights, there is a conformal map Φ from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain such that $\Phi' = e^{Hf}e^{-if}$ on $\mathbb{R}$ (see proof of [19, Lemma 1.11]). We have that

$$\Phi_1'(x) = e^{Hf}\cos f, \qquad \Phi_2'(x) = -e^{Hf}\sin f$$

and, recalling (2.9), we obtain

$$\gamma'(\Phi_1(x)) = -\tan f(x).$$

Thus, if $\Psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\frac{1}{\Psi'} = \mathrm{Re}\left(\frac{1}{\Phi'}\right)$, then (4.7) gives

$$\Psi'(x) = e^{Hf} \cos f \, (1 + \tan^2 f) = \frac{e^{Hf}}{\cos f}.$$

In this case, $k_\Psi = \| \tan f \|_\infty$.

4.2.2. *Extensions of Theorem 1.4.* In general, given a homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$, it is not easy to see whether condition (1.3) holds. However, if $\Psi' \in A_2(\mathbb{R})$, using the Helson-Szegö representation of A_2 weights, it follows that there exists a conformal map Φ from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain and a constant $A_\Psi > 0$ such that

$$\frac{A_\Psi}{\Psi'} \leq \mathrm{Re}\left(\frac{1}{\Phi'}\right) \leq \frac{1}{\Psi'}.$$

With obvious modification in the proof of Theorem 1.4, we get the following result.

Theorem 4.3. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism such that $\Psi' > 0$ almost everywhere and $\Psi' \in A_2(\mathbb{R})$. Consider a conformal map Φ from $\mathbb{R}^2_+$ onto an upper Lipschitz graph domain such that*

$$\frac{A_\Psi}{\Psi'} \leq \mathrm{Re}\left(\frac{1}{\Phi'}\right) \leq \frac{1}{\Psi'} \tag{4.9}$$

for some positive constant A_Ψ. Define

$$k_\Psi := \left\| \Psi' \, \mathrm{Im}\left(\frac{1}{\Phi'}\right) \right\|_{L^\infty(\mathbb{R})}.$$

Then for every $0 < |\mu| < 1$ such that

$$A_\Psi - \mu^2 - 2\, k_\Psi \, |\mu| > 0,$$

the transmission problems $P_\Psi(\mu)$ and $P_\Psi(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Psi'})$.

We note that, if L is the Lipschitz constant associated to the boundary of $\Phi(\mathbb{R}^2_+)$ with Φ as in Theorem 4.3, then

$$\begin{aligned} k_\Psi = \left\| \Psi' \, \mathrm{Re}\left(\frac{1}{\Phi'}\right) \frac{\mathrm{Im}\left(\frac{1}{\Phi'}\right)}{\mathrm{Re}\left(\frac{1}{\Phi'}\right)} \right\|_{L^\infty(\mathbb{R})} &\leq \left\| \Psi' \, \mathrm{Re}\left(\frac{1}{\Phi'}\right) \right\|_{L^\infty(\mathbb{R})} \left\| \frac{\mathrm{Im}\left(\frac{1}{\Phi'}\right)}{\mathrm{Re}\left(\frac{1}{\Phi'}\right)} \right\|_{L^\infty(\mathbb{R})} \\ &= \left\| \Psi' \, \mathrm{Re}\left(\frac{1}{\Phi'}\right) \right\|_{L^\infty(\mathbb{R})} \left\| \frac{\mathrm{Im}(\Phi')}{\mathrm{Re}(\Phi')} \right\|_{L^\infty(\mathbb{R})} \leq L. \end{aligned}$$

An application of Theorem 4.3, associated to the hyperbola $y = 1/x$, $x > 0$, is presented in Section 6.

Also, with the same proof as in Theorem 1.4 and using formula (4.1), we obtain the following result.

Theorem 4.4. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous homeomorphism such that $\Psi' > 0$ almost everywhere, $\Psi' \in A_2(\mathbb{R})$ and*

$$\left| H\left(\frac{1}{\Psi'}\right) \right| \leq \frac{C_\Psi}{\Psi'}$$

for some positive constant C_Ψ. Then for every $0 < |\mu| < 1$ satisfying

$$1 - \mu^2 - 2\, C_\Psi \, |\mu| > 0,$$

the transmission problems $P_\Psi(\mu)$ and $P_\Psi(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Psi'})$.

4.2.3. *An application of Theorem 4.4: perturbation of the identity.* Given $\varepsilon > 0$ consider

$$\Theta_\varepsilon(x) = x + \varepsilon \arctan(x);$$

then

$$\Theta'_\varepsilon(x) = 1 + \frac{\varepsilon}{1+x^2}, \quad \frac{1}{\Theta'_\varepsilon(x)} = 1 - \frac{\varepsilon}{1+\varepsilon+x^2} \quad \text{and} \quad H\left(\frac{1}{\Theta'_\varepsilon}\right)(x) = -\frac{\varepsilon}{\sqrt{1+\varepsilon}}\,\frac{x}{1+\varepsilon+x^2}.$$

We have

$$\left|H\left(\frac{1}{\Theta'_\varepsilon}\right)(x)\,\Theta'_\varepsilon(x)\right| = \left|\frac{\varepsilon}{\sqrt{1+\varepsilon}}\,\frac{x}{1+x^2}\right| \le \frac{\varepsilon}{2\sqrt{1+\varepsilon}}.$$

Theorem 4.4 implies that $P_{\Theta_\varepsilon}(\mu)$ and $P_{\Theta_\varepsilon}(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R}) = L^2(\mathbb{R}, \frac{1}{\Theta'_\varepsilon})$ (note that $\frac{1}{\Theta'_\varepsilon}$ behaves like a constant) for μ such that $1 - \mu^2 - \frac{\varepsilon}{\sqrt{1+\varepsilon}}|\mu| > 0$, this is, $|\mu| \le \frac{1}{\sqrt{1+\varepsilon}}$.

4.3. **Solvability of $P_\Psi(\mu)$ in $L^2(\mathbb{R}, |\Phi'_-|^{-1})$ with $\Psi = \Phi_+^{-1} \circ \Phi_-$.** In this section, we prove Theorem 1.5. The next lemma will be used in its proof.

Lemma 4.5. *If Ψ is as above, we have*

$$\int_{\mathbb{R}} T_{\Psi^{-1}} f\, T_{\Psi^{-1}} g \,\operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx = \int_{\mathbb{R}} f\, g\, \operatorname{Re}\left(\frac{1}{\Phi'_-}\right) dx,$$
$$\int_{\mathbb{R}} T_{\Psi^{-1}} f\, T_{\Psi^{-1}} g \,\operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx = \int_{\mathbb{R}} f\, g\, \operatorname{Im}\left(\frac{1}{\Phi'_-}\right) dx.$$

Proof. We will only prove the first equality.

$$\begin{aligned}
&\int_{\mathbb{R}} T_{\Psi^{-1}} f\, T_{\Psi^{-1}} g \,\operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&= \int_{\mathbb{R}} |(\Psi^{-1})'(x)|^2 f(\Psi^{-1}(x)) g(\Psi^{-1}(x)) \operatorname{Re}\left(\frac{\Phi'_+(x)}{|\Phi'_+(x)|^2}\right) dx \\
&= \int_{\mathbb{R}} |(\Phi_-^{-1})'(\Phi_+(x))\Phi'_+(x)|^2\, f(\Phi_-^{-1}(\Phi_+(x)))\, g(\Phi_-^{-1}(\Phi_+(x)))\, \operatorname{Re}\left(\frac{\Phi'_+(x)}{|\Phi'_+(x)|^2}\right) dx \\
&= \int_{\Lambda} |(\Phi_-^{-1})'(y)|^2\, f(\Phi_-^{-1}(y))\, g(\Phi_-^{-1}(y))\, \operatorname{Re}\left(\frac{\Phi'_+(\Phi_+^{-1}(y))}{|\Phi'_+(\Phi_+^{-1}(y))|}\right) dy \\
&= \int_{\Lambda} |(\Phi_-^{-1})'(y)|^2\, f(\Phi_-^{-1}(y))\, g(\Phi_-^{-1}(y))\, \operatorname{Re}\left(\frac{\Phi'_-(\Phi_-^{-1}(y))}{|\Phi'_-(\Phi_-^{-1}(y))|}\right) dy \\
&= \int_{\mathbb{R}} f\, g\, \operatorname{Re}\left(\frac{1}{\Phi'_-}\right) dx.
\end{aligned}$$

□

Proof of Theorem 1.5. By Theorem 1.3, Remarks 3.2 and 3.3 and Part (a) of Corollary 3.4, it is enough to show that $S - \mu I$, where $S = HT_{\Psi^{-1}}HT_\Psi$, is invertible in $L^2(\mathbb{R}, |\Phi'_+|^{-1})$ for $0 < \mu < 1$, noting that $\widetilde{w} = |\Phi'_+|^{-1}$ for $w = |\Phi'_-|^{-1}$ and both belong to $A_2(\mathbb{R})$.

Defining

$$A = 2\int_{\mathbb{R}} (S - \mu I) f\, (T_{\Psi^{-1}} H T_\Psi f)\, \operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx,$$

we have

$$\begin{aligned}
A &= 2\int_{\mathbb{R}} Sf (T_{\Psi^{-1}} H T_\Psi f) \operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx - 2\mu \int_{\mathbb{R}} f (T_{\Psi^{-1}} H T_\Psi f) \operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx \\
&= -2\int_{\mathbb{R}} Sf\, HSf \operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx - 2\mu \int_{\mathbb{R}} T_\Psi f (H T_\Psi f) \operatorname{Im}\left(\frac{1}{\Phi'_-}\right) dx,
\end{aligned}$$

where in the last equality we used Lemma 4.5. Using the Rellich identities (4.2) and (4.3), and Lemma 4.5, it follows that

$$\begin{aligned}
A &= \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx - \int_{\mathbb{R}} (Sf)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&\quad -\mu \int_{\mathbb{R}} (H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_-}\right) dx + \mu \int_{\mathbb{R}} (T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_-}\right) dx \\
&= \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx - \int_{\mathbb{R}} (Sf)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&\quad -\mu \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + \mu \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&= (1-\mu) \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&\quad - \int_{\mathbb{R}} (Sf)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + \mu \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&= (1-\mu) \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + A_1 + A_2.
\end{aligned}$$

We have

$$\begin{aligned}
A_1 &= -\frac{1}{(1-\mu)^2} \int_{\mathbb{R}} ((S-\mu I)f - \mu((S-I)f))^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&= -\frac{1}{(1-\mu)^2} \left[\int_{\mathbb{R}} ((S-\mu I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + \mu^2 \int_{\mathbb{R}} ((S-I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \right] \\
&\quad + \frac{2\mu}{(1-\mu)^2} \int_{\mathbb{R}} ((S-\mu I)f)((S-I)f) \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx,
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \frac{\mu}{(1-\mu)^2} \int_{\mathbb{R}} ((S-\mu I)f - (S-I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&= \frac{\mu}{(1-\mu)^2} \left[\int_{\mathbb{R}} ((S-\mu I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + \int_{\mathbb{R}} ((S-I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \right] \\
&\quad - \frac{2\mu}{(1-\mu)^2} \int_{\mathbb{R}} ((S-\mu I)f)((S-I)f) \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx.
\end{aligned}$$

Therefore,

$$A_1 + A_2 = -\frac{1}{1-\mu} \int_{\mathbb{R}} ((S-\mu I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx + \frac{\mu}{1-\mu} \int_{\mathbb{R}} ((S-I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx.$$

Putting all together, we obtain

$$\begin{aligned}
2 \int_{\mathbb{R}} (S-\mu I)f \, (T_{\Psi^{-1}} H T_\Psi f) \operatorname{Im}\left(\frac{1}{\Phi'_+}\right) dx =& (1-\mu) \int_{\mathbb{R}} (T_{\Psi^{-1}} H T_\Psi f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&- \frac{1}{1-\mu} \int_{\mathbb{R}} ((S-\mu I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx \\
&+ \frac{\mu}{1-\mu} \int_{\mathbb{R}} ((S-I)f)^2 \operatorname{Re}\left(\frac{1}{\Phi'_+}\right) dx.
\end{aligned}$$

Equivalently,

$$(1-\mu)\|T_{\Psi^{-1}}HT_\Psi f\|^2_{L^2(\mathbb{R},\mathrm{Re}(1/\Phi'_+))} + \frac{\mu}{1-\mu}\|(S-I)f\|^2_{L^2(\mathbb{R},\mathrm{Re}(1/\Phi'_+))}$$

(4.10)

$$= 2\int_{\mathbb{R}}(S-\mu I)f\,(T_{\Psi^{-1}}HT_\Psi f)\,\mathrm{Im}\left(\frac{1}{\Phi'_+}\right)dx + \frac{1}{1-\mu}\|(S-\mu I)f\|^2_{L^2(\mathbb{R},\mathrm{Re}(1/\Phi'_+))}.$$

If $\varepsilon > 0$, we have

$$2\int_{\mathbb{R}}(S-\mu I)f\,(T_{\Psi^{-1}}HT_\Psi f)\,\mathrm{Im}\left(\frac{1}{\Phi'_+}\right)dx$$

(4.11)

$$\lesssim \frac{1}{\varepsilon}\|(S-\mu I)f\|^2_{L^2(\mathbb{R},|\Phi'_+|^{-1})} + \varepsilon\|T_{\Psi^{-1}}HT_\Psi f\|^2_{L^2(\mathbb{R},|\Phi'_+|^{-1})}.$$

Recalling that $\mathrm{Re}(\frac{1}{\Phi'_+}) \approx |\Phi'_+|^{-1}$ since Λ is a Lipschitz curve, (4.10) and (4.11) with ε sufficiently small give that

$$\|f\|_{L^2(\mathbb{R},|\Phi'_+|^{-1})} \approx \|(S-I)f\|_{L^2(\mathbb{R},|\Phi'_+|^{-1})} \lesssim \|(S-\mu I)f\|_{L^2(\mathbb{R},|\Phi'_+|^{-1})},$$

where we have used the $S-I$ is invertible and bounded in $L^2(\mathbb{R},|\Phi'_+|^{-1})$ (see Remark 3.3). Therefore $S-\mu I$ is injective and has closed range. An application of Lemma 3.5 with $\mu_0 = 0$ implies that $S-\mu I$ is invertible. □

5. Solvability of (1.1) in weighted L^2 spaces

As explained in Section 1, the transmission problem (1.1) is associated to $P_\Psi(\mu)$ with $\Psi = \Phi_+^{-1}\circ\Phi_-$ and datum $f = T_{\Phi_-}(g)$, where $\Phi_\pm : \mathbb{R}^2_\pm \to \Omega^\pm$ are conformal maps onto upper and lower graph domains as described in Section 2.4 with $\Lambda = \partial\Omega^+ = \partial\Omega^-$. We note that $\Psi' > 0$ almost everywhere and we assume that Ψ is locally absolutely continuous.

In this section we prove Theorem 1.1, which gives necessary and sufficient conditions for the solvability of (1.1) in $L^p(\Lambda,\nu)$ in terms of solvability results for $P_\Psi(\mu)$. We also present particular cases of Theorem 1.1 using the results in Section 4 to obtain solvability of (1.1) in $L^2(\Lambda)$, $L^2(\Lambda,|(\Phi_+^{-1})'|^{-1})$ and $L^2(\Lambda,|(\Phi_-^{-1})'|^{-1})$.

Proof of Theorem 1.1. We first show that $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R},w)$ if and only if the transmission problem (1.1) is solvable in $L^p(\Lambda,\nu)$.

Assume first that $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R},w)$ and let $g\in L^p(\Lambda,\nu)$. Then (2.8) gives that $T_{\Phi_-}g \in L^p(\mathbb{R},w)$ and $\|g\|_{L^p(\Lambda,\nu)} = \|T_{\Phi_-}g\|_{L^p(\mathbb{R},w)}$. Since $P_\Psi(\mu)$ is solvable in $L^p(\mathbb{R},w)$, there exist solutions $u^\pm$ of $P_\Psi(\mu)$ with datum $f = T_{\Phi_-}g$. Define $v^\pm = u^\pm\circ\Phi_\pm^{-1}$, which are harmonic in $\Omega^\pm$. By Item (c) in Definition 2.3, we have

(5.1) $$\|\mathcal{M}^+_\alpha\nabla u^+\|_{L^p(\mathbb{R},\widetilde{w}_p)} \lesssim \|g\|_{L^p(\Lambda,\nu)} \quad\text{and}\quad \|\mathcal{M}^-_\alpha\nabla u^-\|_{L^p(\mathbb{R},w)} \lesssim \|g\|_{L^p(\Lambda,\nu)},$$

as required in Item (c) of Definition 2.10.

Regarding Item (a) of Definition 2.10, we have that $v^\pm$ on Λ are the traces of $v^\pm$ in the sense of $\Phi_\pm$ non-tangential convergence since $u^\pm$ on $\mathbb{R}$ are the traces of $u^\pm$ in the sense of non-tangential convergence by Item (a) of Definition 2.3. Also, the proof of [9, Theorem 1.4] gives that for $\xi\in\Lambda$ such that $\Phi'_+(\Phi_+^{-1}(\xi))$ exists and is non-zero and $z = x+iy$ with $y>0$,

(5.2)

$$\nabla v^+(z)\cdot\mathbf{n}(\xi) = |\Phi'_+(\Phi_+^{-1}(z))|^{-1}\mathrm{Re}\left(\left(\tfrac{\partial u^+}{\partial x}(\Phi_+^{-1}(z)) - i\tfrac{\partial u^+}{\partial y}(\Phi_+^{-1}(z))\right)\tfrac{|\Phi'_+(\Phi_+^{-1}(z))|}{\Phi'_+(\Phi_+^{-1}(z))}\,\mathbf{n}(\xi)\right),$$

where on the left hand side we have the dot product of $\nabla v^+(z)$ with $\mathbf{n}(\xi)$ and, on the right hand side, $\mathbf{n}(\xi)$ is being multiplied as a complex number. A similar formula holds for v^- with u^-, Φ_- and $z = x+iy$ with $y<0$. From here, noting that $\partial_y u^+ \in L^p(\mathbb{R},\widetilde{w}_p)$ and $\partial_y u^- \in L^p(\mathbb{R},w)$ by (5.1), it follows that $\partial_{\mathbf{n}}v^\pm$ on Λ are the traces of $\partial_{\mathbf{n}}v^\pm$ in the sense of

$\Phi_\pm$ non-tangential convergence and $\partial_y u^\pm = T_{\Phi_\pm}(\partial_{\mathbf{n}} v^\pm)$. The later formula also shows that $\partial_{\mathbf{n}} v^\pm \in L^p(\Lambda, \nu)$ by (2.8) (note that $\widetilde{w}_p = |\Phi'_+|^{1-p}\,\nu \circ \Phi_+$).

Finally, since $u^+ \circ \Psi = u^-$ and $T_\Psi(\partial_y u^+) - \mu\, \partial_y u^- = f$ almost everywhere in $\mathbb{R}$ (by Item (b) of Definition 2.3), we also have that $v^+ = v^-$ and $\partial_{\mathbf{n}} v^+ - \mu\, \partial_{\mathbf{n}} v^- = g$ almost everywhere on Λ with respect to arc length, as seen in Section 1; therefore Item (b) of Definition 2.10 is satisfied.

Conversely, assume that the transmission problem (1.1) is solvable in $L^2(\Lambda, \nu)$ and let $f \in L^2(\mathbb{R}, w)$. Setting $g = T_{\Phi_-^{-1}} f$, which satisfies $\|g\|_{L^2(\Lambda,\nu)} = \|f\|_{L^2(\mathbb{R},w)}$ by (2.8), let $v^\pm$ be the solutions of (1.1) with datum g. It then follows that $u^\pm = v^\pm \circ \Phi_\pm$, which are harmonic in $\mathbb{R}^2_\pm$, solve $P_\Psi(\mu)$ with datum f :

Item (c) in Definition 2.3 follows from Item (c) of Definition 2.10.

Regarding Item (a) in Definition 2.3, we have that $u^\pm$ on $\mathbb{R}$ are the traces of $u^\pm$ in the sense of non-tangential convergence since $v^\pm$ on Λ are the traces of $v^\pm$ in the sense of $\Phi_\pm$ non-tangential convergence. Also, for $(x, y) \in \mathbb{R}^2_+$, we have

$$\partial_y u^+(x,y) = (\partial_1 v^+ \circ \Phi_+)(x,y)\, \partial_y(\mathrm{Re}(\Phi_+))(x,y) + (\partial_2 v^+ \circ \Phi_+)(x,y)\, \partial_y(\mathrm{Im}(\Phi_+))(x,y)$$

and similarly for u^- with v^-, Φ_- and $(x, y) \in \mathbb{R}^2_-$. Then, since $\partial_{\mathbf{n}} v^\pm$ converge $\Phi_\pm$ non-tangentially and $\Phi'_\pm$ converge non-tangentially, we obtain that $\partial_y u^\pm$ exists on $\mathbb{R}$ as the trace of $\partial_y u^\pm$ in the sense of non-tangential convergence.

As for Item (b) of Definition 2.3, note first that an analogous formula to the above for $\partial_x u^+$ allows to conclude that $\partial_x u^\pm$ exists on $\mathbb{R}$ as the trace of $\partial_x u^\pm$ in the sense of non-tangential convergence. We can then use (5.2) and its counterpart for v^- and u^- to conclude that $\partial_y u^\pm = T_{\Phi_\pm}(\partial_{\mathbf{n}} v^\pm)$ almost everywhere and proceed as in Section 1 to obtain that $T_\Psi(\partial_y u^+) - \mu\, \partial_y u^- = f$ almost everywhere in $\mathbb{R}$. Finally, $u^+ \circ \Psi = u^-$ almost everywhere in $\mathbb{R}$ since $v^+ = v^-$ almost everywhere in Λ.

The equivalence for unique solvability follows from the relationship $u^\pm = v^\pm \circ \Phi_\pm$ between the solutions of $P_\Psi(\mu)$ and (1.1). □

5.0.1. *Particular cases of Theorem 1.1.* In this section, we state results on the solvability of (1.1) in $L^2(\Lambda)$, $L^2(\Lambda, |(\Phi_+^{-1})'|^{-1})$ and $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$.

(a) **Solvability in** $L^2(\Lambda)$: This corresponds to $\nu = 1$, $w = |\Phi'_-|^{-1}$ and $\widetilde{w} = |\Phi'_+|^{-1}$. The estimates for $v^\pm$ then become

$$\|\mathcal{M}^+_\alpha \nabla(v^+ \circ \Phi_+)\|_{L^2(\mathbb{R},|\Phi'_+|^{-1})} \lesssim \|g\|_{L^2(\Lambda)} \quad \text{and} \quad \|\mathcal{M}^-_\alpha \nabla(v^- \circ \Phi_-)\|_{L^2(\mathbb{R},|\Phi'_-|^{-1})} \lesssim \|g\|_{L^2(\Lambda)}.$$

We note that in the case that $\Omega^\pm$ are upper and lower Lipschitz graph domains, these estimates can be rewritten in the following form:

$$\|\widetilde{\mathcal{M}}^+_\alpha \nabla v^+\|_{L^2(\Lambda)} \lesssim \|g\|_{L^2(\Lambda)} \quad \text{and} \quad \|\widetilde{\mathcal{M}}^-_\alpha \nabla v^-\|_{L^2(\Lambda)} \lesssim \|g\|_{L^2(\Lambda)},$$

where $\widetilde{\mathcal{M}}^\pm_\alpha$ are the non-tangential maximal operators associated to the upper and lower Lipschitz graph domains, respectively. This follows from the facts

$$\|\mathcal{M}^+_\alpha \nabla(v^+ \circ \Phi_+)\|_{L^2(\mathbb{R},|\Phi'_+|^{-1})} \approx \|\widetilde{\mathcal{M}}^+_\alpha \nabla v^+\|_{L^2(\Lambda)}, \tag{5.3}$$

$$\|\mathcal{M}^-_\alpha \nabla(v^- \circ \Phi_-)\|_{L^2(\mathbb{R},|\Phi'_-|^{-1})} \approx \|\widetilde{\mathcal{M}}^-_\alpha \nabla v^-\|_{L^2(\Lambda)}. \tag{5.4}$$

For (5.3) see the proof of [9, Theorem 1.4]. The equivalence (5.4) is a consequence of the latter as follows: Let $\widetilde{\Omega} = -\Omega^-$, $v(z) = v^-(-z)$ for $z \in \widetilde{\Omega}$ and $\Phi : \mathbb{R}^2_+ :\to \widetilde{\Omega}$ be defined by $\Phi(x, y) = -\Phi_-(-x, -y)$. It follows that

$$\mathcal{M}^-_\alpha \nabla(v^- \circ \Phi_-)(-x) = \mathcal{M}^+_\alpha \nabla(v \circ \Phi)(x), \quad x \in \mathbb{R},$$

which leads to

$$\|\mathcal{M}^-_\alpha \nabla(v^- \circ \Phi_-)\|_{L^2(\mathbb{R},|\Phi'_-|^{-1})} = \|\mathcal{M}^+_\alpha \nabla(v \circ \Phi)\|_{L^2(\mathbb{R},|\Phi'|^{-1})}. \tag{5.5}$$

The equivalence (5.3) applied with v, Φ and $\widetilde{\Lambda} = -\Lambda$ and the fact that $\widetilde{\mathcal{M}}_\alpha^+ \nabla v(z) = \widetilde{\mathcal{M}}_\alpha^- \nabla v^-(-z)$ give

$$\|\mathcal{M}_\alpha^+ \nabla(v \circ \Phi)\|_{L^2(\mathbb{R},|\Phi'|^{-1})} \approx \|\widetilde{\mathcal{M}}_\alpha^+ \nabla v\|_{L^2(\widetilde{\Lambda})} = \|\widetilde{\mathcal{M}}_\alpha^- \nabla v^-\|_{L^2(\Lambda)}. \tag{5.6}$$

Using (5.5) and (5.6), we obtain (5.4).

The following corollary of Theorems 1.5 and 1.1 recovers results in [13, Theorem 1.1] for $p = 2$ and $n = 2$ for upper and lower Lipschitz graph domains.

Corollary 5.1. *If $\Omega^\pm$ are upper and lower Lipschitz graph domains, the transmission problem* (1.1) *is uniquely solvable in $L^2(\Lambda)$ for every $\mu > 0$.*

We note that the transmission problem (1.1) studied in [13] requires the seemingly more general condition $u^+ - u^- = h$ (rather than $u^+ - u^- = 0$) where h and its derivative are in $L^2(\Lambda)$. However, using [32, Theorem 5.1], Corollary 5.1 also holds for this inhomogeneous version of (1.1).

(b) **Solvability in** $L^2(\Lambda, |(\Phi_+^{-1})'|^{-1})$: This corresponds to $\nu = |(\Phi_+^{-1})'|^{-1}$, $w = \frac{1}{\Psi'}$, and $\widetilde{w} = 1$. The estimates for $v^\pm$ are given by

$$\|\mathcal{M}_\alpha^+ \nabla(v^+ \circ \Phi_+)\|_{L^2(\mathbb{R})} \lesssim \|g\|_{L^2(\Lambda,|(\Phi_+^{-1})'|^{-1})}$$
$$\|\mathcal{M}_\alpha^- \nabla(v^- \circ \Phi_-)\|_{L^2(\mathbb{R},\frac{1}{\Psi'})} \lesssim \|g\|_{L^2(\Lambda,|(\Phi_+^{-1})'|^{-1})}.$$

The following result follows from Theorems 4.3 and 1.1:

Corollary 5.2. *Assume $\Psi' \in A_2(\mathbb{R})$, A_Ψ and k_Ψ are as given in Theorem 4.3 and μ_0 is the positive root of $A_\Psi - \mu^2 - 2\, k_\Psi\, \mu = 0$. Then the transmission problem* (1.1) *is uniquely solvable in $L^2(\Lambda, |(\Phi_+^{-1})'|^{-1})$ for $\mu \neq 0$ such that $|\mu| < \mu_0$ or $|\mu| > 1/\mu_0$.*

Corresponding statements analogous to Corollary 5.2 follow from Theorems 1.4 and 4.4.

(c) **Solvability in** $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$: This case corresponds to $\nu = |(\Phi_-^{-1})'|^{-1}$, $w = 1$ and $\widetilde{w} = |(\Psi^{-1})'|^{-1}$. The estimates for $v^\pm$ are then

$$\|\mathcal{M}_\alpha^+ \nabla(v^+ \circ \Phi_+)\|_{L^2(\mathbb{R},|(\Psi^{-1})'|^{-1})} \lesssim \|g\|_{L^2(\Lambda,|(\Phi_-^{-1})'|^{-1})}$$
$$\|\mathcal{M}_\alpha^- \nabla(v^- \circ \Phi_-)\|_{L^2(\mathbb{R})} \lesssim \|g\|_{L^2(\Lambda,|(\Phi_-^{-1})'|^{-1})}.$$

Corollary 1.2, whose proof is presented in Section 6, is a particular case of this setting.

6. THE HYPERBOLA

In this section, we consider the Jordan curve given by the hyperbola $y = 1/x$, $x > 0$: we present an application of Theorem 4.3 and we prove Corollary 1.2.

6.1. **An application of Theorem 4.3.** Let $\Omega^\pm$ be the upper and lower graph domains associated to $y = 1/x$, $x > 0$. A conformal map from Ω^+ onto $\mathbb{R}^2_+$ is given by $z^2 - 2i$; consider the inverse of this map:

$$\Phi_+(z) = (z + 2i)^{1/2}, \quad z \in \mathbb{R}^2_+.$$

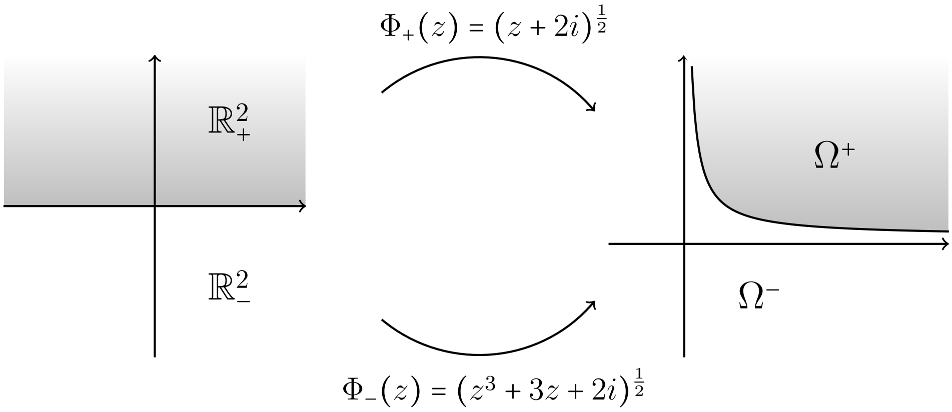
Also, if $h(z) = z^3 - 3z$, then $-ih^{-1}(-iz^2 - 2)$ is a conformal map from Ω^- onto $\mathbb{R}^2_-$; consider the inverse of this map:

$$\Phi_-(z) = (z^3 + 3z + 2i)^{1/2}, \quad z \in \mathbb{R}^2_-.$$

We refer the reader to [28] for the maps Φ_+ and Φ_-, where we have checked that all statements and computations claimed are correct.

It follows that

$$\Psi(x) = \Phi_+^{-1}(\Phi_-(x)) = x^3 + 3x, \quad x \in \mathbb{R},$$



and $\Psi'(x) = 3(1+x^2)$, which does not belong to $A_2(\mathbb{R})$. Hence Ψ does not satisfy the hypothesis of neither Theorem 4.3 nor Theorem 4.4. However, setting $\Theta = \Psi^{-1}$, we have

$$\Theta(x) = \left(\frac{x+\sqrt{4+x^2}}{2}\right)^{1/3} - \left(\frac{2}{x+\sqrt{4+x^2}}\right)^{1/3}, \tag{6.1}$$

and

$$\frac{1}{\Theta'(x)} = 3\cdot 2^{2/3}\frac{\sqrt{4+x^2}(x+\sqrt{4+x^2})^{1/3}}{2+2^{1/3}(x+\sqrt{4+x^2})^{2/3}} \approx (4+x^2)^{\frac{1}{3}} \in A_2(\mathbb{R}).$$

We can then apply Theorem 4.3 with Θ and obtain that if $0 < |\mu| < 1$ satisfies $A_\Theta - \mu^2 - 2\,k_\Theta\,|\mu| > 0$, then $P_\Theta(\mu)$ and $P_\Theta(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R}, \frac{1}{\Theta'})$. Also, since $\widetilde{w} = 1$ for $w = \frac{1}{\Theta'}$, Part (b) of Corollary 3.4 gives that $P_\Psi(\mu)$ and $P_\Psi(1/\mu)$ are uniquely solvable in $L^2(\mathbb{R})$ for the same range of μ.

We next give an explicit value of μ_0 such that $P_\Theta(\mu)$ is uniquely solvable for $|\mu| < \mu_0$ and $|\mu| > 1/\mu_0$ by presenting an example of a conformal map Φ associated to Θ in (6.1) according to the statement of Theorem 4.3. Consider the conformal map

$$\Phi(z) = i(8-4zi)^{\frac{1}{3}}, \quad z \in \mathbb{R}^2_+.$$

We first need to compute A_Θ and k_Θ. We have

$$\Phi'(x) = \frac{4}{3}(8-4xi)^{-\frac{2}{3}}, \quad x \in \mathbb{R};$$

noting that $8-4xi = 4\sqrt{4+x^2}e^{i\arctan(-\frac{x}{2})}$ for $x \in \mathbb{R}$, we obtain

$$\frac{1}{\Phi'(x)} = 3\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}} e^{i\frac{2}{3}\arctan(-\frac{x}{2})}, \quad x \in \mathbb{R}.$$

Then

$$\mathrm{Re}\left(\frac{1}{\Phi'(x)}\right) = 3\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}} \cos\left(\frac{2}{3}\arctan\left(-\frac{x}{2}\right)\right)$$

and, since $-\frac{\pi}{3} \le \frac{2}{3}\arctan(-\frac{x}{2}) \le \frac{\pi}{3}$, we get $\frac{1}{2} \le \cos(\frac{2}{3}\arctan(-\frac{x}{2})) \le 1$. Therefore

$$\frac{3}{2}\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}} \le \mathrm{Re}\left(\frac{1}{\Phi'(x)}\right) \le 3\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}}$$

and

$$\frac{3}{2}\Theta'(x)\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}} \le \Theta'(x)\,\mathrm{Re}\left(\frac{1}{\Phi'(x)}\right) \le 3\Theta'(x)\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}}.$$

It can be checked that

$$4^{-\frac{1}{3}} \le 3\Theta'(x)\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}} \le 1; \tag{6.2}$$

then (4.9) holds with $A_\Theta = 2^{-\frac{5}{3}}$. Regarding k_Θ, we have

$$k_\Theta = \left\|\Theta' \operatorname{Im}\left(\frac{1}{\Phi'}\right)\right\|_{L^\infty(\mathbb{R})} = \left\|3\left(\frac{4+x^2}{4}\right)^{\frac{1}{3}}\Theta'(x)\sin\left(\frac{2}{3}\arctan\left(-\frac{x}{2}\right)\right)\right\|_{L^\infty(\mathbb{R})}$$
$$\le \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

where in the last inequality we have used (6.2) and the fact that $|\frac{2}{3}\arctan\left(-\frac{x}{2}\right)| \le \pi/3$. We next find μ_0; we have

$$A_\Theta - \mu^2 - 2\,k_\Theta\,|\mu| \ge 2^{-\frac{5}{3}} - \mu^2 - \sqrt{3}\,|\mu| > 0,$$

and solving $2^{-\frac{5}{3}} - \mu^2 - \sqrt{3}\,|\mu| = 0$ for a positive root we obtain

$$\mu_0 = \frac{-\sqrt{3}+\sqrt{3+2^{1/3}}}{2} \approx 0.165953;$$

note that $\frac{1}{\mu_0} = \frac{2}{-\sqrt{3}+\sqrt{3+2^{1/3}}} \approx 6.02579$.

6.2. **Proof of Corollary 1.2.** Let $\Phi_\pm$ be as in Section 6.1 and $\Psi = \Phi_+^{-1} \circ \Phi_-$. According to the computations in Section 6.1, $P_\Psi(\mu)$ is uniquely solvable in $L^2(\mathbb{R})$ for $\mu \neq 0$ such that $|\mu| < \mu_0$ or $|\mu| > 1/\mu_0$. By Theorem 1.1, we then conclude that (1.1) is uniquely solvable in $L^2(\Lambda, |(\Phi_-^{-1})'|^{-1})$ for the same range of μ. □

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