

ASSOCIATE SPACES OF LOGARITHMIC INTERPOLATION SPACES AND GENERALIZED LORENTZ-ZYGMUND SPACES

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ABSTRACT. We determine the associate space of the logarithmic interpolation space $(X_0, X_1)_{1,q,\mathbb{A}}$ where X_0 and X_1 are Banach function spaces over a σ -finite measure space (Ω, μ) . Particularizing the results for the case of the couple (L_1, L_∞) over a non-atomic measure space, we recover results of Opic and Pick on associate spaces of generalized Lorentz-Zygmund spaces $L_{(\infty,q;\mathbb{A})}$. We also establish the corresponding results for sequence spaces.

1. INTRODUCTION

Logarithmic spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ are interpolation spaces which are quite close to the space A_1 . This fact is useful in several situations (see, for example, [10, 6, 8, 3]). When $A_0 \cap A_1$ is dense in A_0 and A_1 , the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ has been computed in [8] for $1 \leq q \leq \infty$, and in [3] for the case $0 < q < 1$. Curiously, as it is pointed out in [3, Remark 4.5], although the couple (L_1, L_∞) does not satisfy that $L_1 \cap L_\infty$ is dense in L_∞ , writing down the duality results for (L_1, L_∞) the outcome coincides with the associate spaces of generalized Lorentz-Zygmund spaces $L_{(\infty,q;\mathbb{A})}$, determined by Opic and Pick in [16]. The aim of the present paper is to clarify this coincidence.

We compute the associate space of $(X_0, X_1)_{1,q,\mathbb{A}}$ where X_j are Banach function spaces on a σ -finite measure space (Ω, μ) . This is done in Section 3 with the help of the description of logarithmic spaces in terms of the J-functional. Since there is no J-description in a certain range of the parameters (see [8, Proposition 3.4]), we show first in Section 2 that in such range the space $(A_0, A_1)_{1,q,\mathbb{A}}$ turns out to be equal to the sum of A_1 with a certain J-space modified. This result is of independent interest,

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it complements those of [8, 3] and it is useful in Section 3. Finally, in Section 4, we show some applications of the abstract results. We first consider a non-atomic σ -finite measure space and applying the results to the couple (L_1, L_∞) we recover the results of Opic and Pick [16] on associate spaces of generalized Lorentz-Zygmund spaces $L_{(\infty, q; \mathbb{A})}$. Then we establish the corresponding results for the sequence spaces $\ell_{(\infty, q; \alpha)}$. For this aim, we work with the measure space $(\mathbb{N}, \#)$, where $\#$ is the counting measure, which is completely atomic. The sequence case has not been studied previously.

2. LOGARITHMIC INTERPOLATION METHODS

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. We set $\Sigma(\bar{A}) = A_0 + A_1$ and $\Delta(\bar{A}) = A_0 \cap A_1$. These spaces become Banach spaces when normed by

$$\|a\|_{A_0+A_1} = \|a\|_{\Sigma(\bar{A})} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}$$

and

$$\|a\|_{A_0 \cap A_1} = \|a\|_{\Delta(\bar{A})} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}.$$

For $t > 0$, the *Peetre's K- and J-functionals* are defined by

$$K(t, a) = K(t, a; \bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1,$$

and

$$J(t, a) = J(t, a; \bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, a \in A_0 \cap A_1.$$

Note that $K(1, \cdot) = \|\cdot\|_{A_0+A_1}$ and $J(1, \cdot) = \|\cdot\|_{A_0 \cap A_1}$.

The *Gagliardo completion* A_j^\sim of A_j consists of all those $a \in \Sigma(\bar{A})$ for which there exists a bounded sequence (a_n) in A_j which converges to a in $\Sigma(\bar{A})$. The norm in A_j^\sim is given by

$$\|a\|_{A_j^\sim} = \inf_{(a_n)} \left(\sup_{n \in \mathbb{N}} (\|a_n\|_{A_j}) \right) = \sup_{0 < t < \infty} \frac{K(t, a)}{t^j}, \quad j = 0, 1$$

(see [1, Theorem V.1.4]). We have $A_j \hookrightarrow A_j^\sim \hookrightarrow \Sigma(\bar{A})$, where \hookrightarrow means continuous embedding. Furthermore, for the Banach couple $\overline{A^\sim} = (A_0^\sim, A_1^\sim)$, we have

$$(2.1) \quad K(t, a; \overline{A^\sim}) = K(t, a; \bar{A}), \quad t > 0, \quad a \in \Sigma(\bar{A})$$

(see [1, Theorem V.1.5]). The couple \bar{A} is said to be a *Gagliardo couple* if $A_0 = A_0^\sim$ and $A_1 = A_1^\sim$. Examples of Gagliardo couples are (L_1, L_∞) and (ℓ_1, ℓ_∞) . See Section 4 for details.

For $t > 0$, let $\ell(t) = 1 + |\log t|$ and $\ell\ell(t) = 1 + \log(1 + |\log t|)$. For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ write

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0, \alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty, \end{cases}$$

and define $\ell\ell^{\mathbb{A}}(t)$ similarly.

Let $0 < q \leq \infty$ and $\mathbb{A} \in \mathbb{R}^2$. The *logarithmic interpolation space* $\bar{A}_{1,q,\mathbb{A}} = (A_0, A_1)_{1,q,\mathbb{A}}$ is formed of all $a \in \Sigma(\bar{A})$ which have a finite quasi-norm

$$\|a\|_{1,q,\mathbb{A}} = \left(\int_0^\infty [t^{-1}\ell^{\mathbb{A}}(t)K(t,a)]^q \frac{dt}{t} \right)^{1/q}$$

(as usual, the integral should be replaced by the supremum when $q = \infty$). See [11, 12, 8, 3]. The functional $\|\cdot\|_{1,q,\mathbb{A}}$ is a norm if $1 \leq q \leq \infty$. We shall assume that

$$(2.2) \quad \begin{cases} \alpha_0 + 1/q < 0 & \text{if } 0 < q < \infty, \\ \alpha_0 < 0 & \text{if } q = \infty, \end{cases}$$

in order to avoid that $(A_0, A_1)_{1,q,\mathbb{A}} = \{0\}$ (see [12, Theorem 2.2]). The space $(A_0, A_1)_{1,q,\mathbb{A}}$ also makes sense when $q = \infty$ and $\alpha_0 = 0$. However we do not study this limit case here because the space $(A_0, A_1)_{1,q,(0,\alpha_\infty)}$ has a different structure than the spaces $(A_0, A_1)_{1,q,\mathbb{A}}$ when q and \mathbb{A} satisfy (2.2). We refer to [4] for the properties of the space $(A_0, A_1)_{1,q,(0,\alpha_\infty)}$.

This construction produces *exact interpolation spaces*. More precisely, if $\bar{B} = (B_0, B_1)$ is another Banach couple and $T : \Sigma(\bar{A}) \rightarrow \Sigma(\bar{B})$ is a linear operator whose restrictions to A_j define a bounded linear operator from A_j to B_j for $j = 0, 1$, then

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \rightarrow (B_0, B_1)_{1,q,\mathbb{A}}$$

is also bounded and

$$\|T\|_{\bar{A}_{1,q,\mathbb{A}}, \bar{B}_{1,q,\mathbb{A}}} \leq \max\{\|T\|_{A_0, B_0}, \|T\|_{A_1, B_1}\}.$$

It is not hard to check that the quasi-norm of $(A_0, A_1)_{1,q,\mathbb{A}}$ is equivalent to

$$\|a\|_{\bar{A}_{1,q,\mathbb{A}}} = \left(\sum_{m=-\infty}^{\infty} [2^{-m}\ell^{\mathbb{A}}(2^m)K(2^m, a)]^q \right)^{1/q}.$$

Here the ℓ_q -quasi-norm should be replaced by the ℓ_∞ -norm if $q = \infty$.

Next we introduce the corresponding J-spaces. We assume that

$$(2.3) \quad \begin{cases} \alpha_\infty \geq 0 & \text{if } 0 < q \leq 1, \\ \alpha_\infty - 1/q' > 0 & \text{if } 1 < q \leq \infty, \end{cases}$$

where for $1 \leq q \leq \infty$ the parameter q' is given by equality $1/q + 1/q' = 1$. The space $\bar{A}_{1,q,\mathbb{A}}^J = (A_0, A_1)_{1,q,\mathbb{A}}^J$ consists of all $a \in \Sigma(\bar{A})$ for which there exists $(u_m)_{m \in \mathbb{Z}} \subseteq \Delta(\bar{A})$ such that

$$a = \sum_{m=-\infty}^{\infty} u_m \text{ (convergence in } \Sigma(\bar{A}) \text{)}$$

and

$$\left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} < \infty.$$

The quasi-norm in $(A_0, A_1)_{1,q,\mathbb{A}}^J$ is

$$\|a\|_{\bar{A}_{1,q,\mathbb{A}}^J} = \inf \left\{ \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

If $1 < q \leq \infty$ and $\alpha_\infty = 1/q'$, the J-space still makes sense if we replace $\ell^{\mathbb{A}}(t)$ by $\ell^{\mathbb{A}}(t)\ell^{\mathbb{B}}(t)$ with $\mathbb{B} = (\beta_0, \beta_\infty)$ and $\beta_\infty - 1/q' > 0$. We denote the corresponding space by $\bar{A}_{1,q,\mathbb{A},\mathbb{B}}^J = (A_0, A_1)_{1,q,\mathbb{A},\mathbb{B}}^J$. The space $\bar{A}_{1,q,\mathbb{A},\mathbb{B}}^J$ is also well-defined for $0 < q \leq 1$, $\alpha_\infty = 0$ and $\beta_\infty \geq 0$. The quasi-norm in $\bar{A}_{1,q,\mathbb{A},\mathbb{B}}^J$ is

$$\|a\|_{\bar{A}_{1,q,\mathbb{A},\mathbb{B}}^J} = \inf \left\{ \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) \ell^{\mathbb{B}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

It is easy to check that the J-spaces are also exact interpolation spaces.

According to [8, Theorems 3.5 and 3.6], if \mathbb{A} and $1 \leq q \leq \infty$ satisfy (2.2) then we have with equivalence of norms

$$(2.4) \quad (A_0, A_1)_{1,q,\mathbb{A}} = \begin{cases} (A_0, A_1)_{1,q,\mathbb{A}+1}^J & \text{if } \alpha_\infty + 1/q > 0, \\ (A_0, A_1)_{1,q,\mathbb{A}+1,(0,1)}^J & \text{if } \alpha_\infty = -1/q \text{ and } q < \infty. \end{cases}$$

Here for $\lambda \in \mathbb{R}$, we put $\mathbb{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$.

If $0 < q < 1$ and (2.2) holds, then by [3, Theorem 3.2] we have with equivalence of quasi-norms

$$(2.5) \quad (A_0, A_1)_{1,q,\mathbb{A}} = \begin{cases} (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}+1/q}^J & \text{if } \alpha_\infty + 1/q > 0, \\ (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}+1/q,(0,1/q)}^J & \text{if } \alpha_\infty = -1/q. \end{cases}$$

Moreover, if $0 < q < 1$ and we assume in addition that $A_0 \cap A_1$ is dense in A_0 and A_1 , then

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^\sim, A_1^\sim)_{1,q,(\alpha_0+1/q,0)}^J \text{ for } \alpha_\infty + 1/q < 0.$$

In general there is no description for $(A_0, A_1)_{1,q,\mathbb{A}}$ as a J-space in the case

$$(2.6) \quad \begin{cases} \alpha_\infty + 1/q < 0 & \text{if } q < \infty, \\ \alpha_\infty \leq 0 & \text{if } q = \infty \end{cases}$$

(see [8, Proposition 3.4]). However, we show next that in this range $(A_0, A_1)_{1,q,\mathbb{A}}$ is the sum of A_1 with a modified J-space.

In what follows, if X and Y are quantities depending on certain parameters some of them being the significant parameters in our reasoning, we write $X \lesssim Y$ if $X \leq cY$ with a constant $c > 0$ independent of the significant parameters. We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Similarly, if $\|\cdot\|$ and $\|\|\cdot\|\|$ are quasi-norms on a space A , we put $\|a\| \lesssim \|\|a\|\|$ if there is a constant $c > 0$ such that $\|a\| \leq c\|\|a\|\|$ for any $a \in A$. We write $\|a\| \sim \|\|a\|\|$ if $\|a\| \lesssim \|\|a\|\|$ and $\|\|a\|\| \lesssim \|a\|$.

Put $\mathbb{Z}^- = \{0, -1, -2, -3, \dots\}$. If $0 < q \leq \infty$ and $\alpha_0 \in \mathbb{R}$, we write $[\bar{A}]_{1,q,\alpha_0}^J = [A_0, A_1]_{1,q,\alpha_0}^J$ for the collection of all $a \in \Sigma(\bar{A})$ such that there exists $(u_n)_{n \in \mathbb{Z}^-} \subseteq \Delta(\bar{A})$ satisfying

$$a = \sum_{n=-\infty}^0 u_n \text{ (convergence in } A_0 + A_1)$$

and

$$\left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0} (2^n) J(2^n, u_n)]^q \right)^{1/q} < \infty.$$

We endow $[A_0, A_1]_{1,q,\alpha_0}^J$ with the quasi-norm

$$\|a\|_{[A_0, A_1]_{1,q,\alpha_0}^J} = \inf \left\{ \left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0} (2^n) J(2^n, u_n)]^q \right)^{1/q} : a = \sum_{n=-\infty}^0 u_n \right\}.$$

We claim that

$$A_0 \cap A_1 \hookrightarrow [A_0, A_1]_{1,q,\alpha_0}^J \hookrightarrow A_0 + A_1.$$

Indeed, take any $a \in A_0 \cap A_1$ and for any $n \in \mathbb{Z}^-$, put $u_n = \delta_n^0 a$ where δ_n^m is the Kronecker delta. So, $a = \sum_{n=-\infty}^0 u_n$ and $\|a\|_{[A_0, A_1]_{1,q,\alpha_0}^J} \leq J(1, a) = \|a\|_{A_0 \cap A_1}$. On the other hand, take any $a \in [A_0, A_1]_{1,q,\alpha_0}^J$ and let $a = \sum_{n=-\infty}^0 u_n$ be a representation with

$$\left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0} (2^n) J(2^n, u_n)]^q \right)^{1/q} \leq 2 \|a\|_{[A_0, A_1]_{1,q,\alpha_0}^J}.$$

Then

$$\begin{aligned} \|a\|_{A_0+A_1} &= K(1, a) \leq \sum_{n=-\infty}^0 K(1, u_n) \\ &\leq \sum_{n=-\infty}^0 \min(1, 2^{-n}) J(2^n, u_n) = \sum_{n=-\infty}^0 J(2^n, u_n). \end{aligned}$$

If $1 \leq q \leq \infty$, applying Hölder's inequality, we obtain that

$$\begin{aligned} \|a\|_{A_0+A_1} &\leq \left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n)]^q \right)^{1/q} \left(\sum_{n=-\infty}^0 [2^n \ell^{-\alpha_0}(2^n)]^{q'} \right)^{1/q'} \\ &\lesssim \left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n)]^q \right)^{1/q} \\ &\lesssim \|a\|_{[A_0, A_1]_{1, q, \alpha_0}^J}. \end{aligned}$$

If $0 < q \leq 1$, we get

$$\begin{aligned} \|a\|_{A_0+A_1} &\leq \left(\sum_{n=-\infty}^0 J(2^n, u_n)^q \right)^{1/q} \\ &\leq \left(\sum_{n=-\infty}^0 [2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n)]^q \right)^{1/q} \sup_{n \in \mathbb{Z}^-} (2^n \ell^{-\alpha_0}(2^n)) \\ &\lesssim \|a\|_{[A_0, A_1]_{1, q, \alpha_0}^J}. \end{aligned}$$

This proves that $[A_0, A_1]_{1, q, \alpha_0}^J \hookrightarrow A_0 + A_1$.

It is also not hard to check that these modified J-spaces are exact interpolation spaces.

Lemma 2.1. *Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfy (2.3). Given any Banach couple $\bar{A} = (A_0, A_1)$, we have with equivalent quasi-norms*

$$A_1 + (A_0, A_1)_{1, q, \mathbb{A}}^J = A_1 + [A_0, A_1]_{1, q, \alpha_0}^J.$$

Proof. Let $v = a_1 + a$ with $a_1 \in A_1$ and $a \in (A_0, A_1)_{1, q, \mathbb{A}}^J$. Find $(u_m)_{m \in \mathbb{Z}} \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ and

$$\left(\sum_{m=-\infty}^{\infty} [2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, u_m)]^q \right)^{1/q} \leq 2 \|a\|_{(A_0, A_1)_{1, q, \mathbb{A}}^J}.$$

Then $w = \sum_{m=1}^{\infty} u_m$ belongs to A_1 . Indeed, if $0 < q \leq 1$, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \|u_m\|_{A_1} &\leq \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) \leq \left(\sum_{m=1}^{\infty} [2^{-m} J(2^m, u_m)]^q \right)^{1/q} \\ &\leq \left(\sum_{m=1}^{\infty} [2^{-m} \ell^{\alpha_{\infty}}(2^m) J(2^m, u_m)]^q \right)^{1/q} \sup_{m \in \mathbb{N}} \ell^{-\alpha_{\infty}}(2^m) \\ &\lesssim \|a\|_{(A_0, A_1)_{1, q, \mathbb{A}}^J} \end{aligned}$$

where we have used that $\alpha_{\infty} \geq 0$ in the last inequality. If $1 < q \leq \infty$, we proceed using Hölder's inequality. We get

$$\begin{aligned} \sum_{m=1}^{\infty} \|u_m\|_{A_1} &\leq \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) \\ &\leq \left(\sum_{m=1}^{\infty} [2^{-m} \ell^{\alpha_{\infty}}(2^m) J(2^m, u_m)]^q \right)^{1/q} \left(\sum_{m=1}^{\infty} \ell^{-\alpha_{\infty} q'}(2^m) \right)^{1/q'} \\ &\lesssim \|a\|_{(A_0, A_1)_{1, q, \mathbb{A}}^J} \end{aligned}$$

because $\alpha_{\infty} - 1/q' > 0$.

Therefore, $v = (a_1 + w) + \sum_{m=-\infty}^0 u_m$ belongs to $A_1 + [A_0, A_1]_{1, q, \alpha_0}^J$ with

$$\begin{aligned} \|v\|_{A_1 + [A_0, A_1]_{1, q, \alpha_0}^J} &\leq \|a_1 + w\|_{A_1} + \left(\sum_{m=-\infty}^0 [2^{-m} \ell^{\alpha_0}(2^m) J(2^m, u_m)]^q \right)^{1/q} \\ &\lesssim \|a_1\|_{A_1} + \|a\|_{(A_0, A_1)_{1, q, \mathbb{A}}^J}. \end{aligned}$$

This yields that

$$A_1 + (A_0, A_1)_{1, q, \mathbb{A}}^J \hookrightarrow A_1 + [A_0, A_1]_{1, q, \alpha_0}^J.$$

The converse inclusion is clear. \square

Now we are ready to show the announced result for the modified J-spaces.

Theorem 2.2. *Let $\mathbb{A} = (\alpha_0, \alpha_{\infty}) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfy (2.2) and (2.6). Given any Banach couple $\bar{A} = (A_0, A_1)$ we have with equivalence of quasi-norms*

$$(A_0, A_1)_{1, q, \mathbb{A}} = \begin{cases} A_1 + [A_0, A_1]_{1, q, \alpha_0+1}^J & \text{if } 1 \leq q \leq \infty, \\ A_1^{\sim} + [A_0^{\sim}, A_1^{\sim}]_{1, q, \alpha_0+1/q}^J & \text{if } 0 < q < 1. \end{cases}$$

Proof. The argument in the proof of [8, Lemma 2.3] for $1 \leq q \leq \infty$ is still valid for $0 < q \leq \infty$ showing that in the assumption (2.6) we have

$$(2.7) \quad \|a\|_{\bar{A}_{1, q, \mathbb{A}}} \sim \left(\int_0^1 [t^{-1} K(t, a) \ell^{\alpha_0}(t)]^q \frac{dt}{t} \right)^{1/q}.$$

In particular, if $a \in A_1$, we obtain

$$\|a\|_{\bar{A}_{1,q,\mathbb{A}}} \lesssim \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t} \right)^{1/q} \|a\|_{A_1} \lesssim \|a\|_{A_1}.$$

This yields that

$$(2.8) \quad A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

Take any $\alpha \in \mathbb{R}$ with $\alpha + 1/q > 0$. We claim that

$$(2.9) \quad (A_0, A_1)_{1,q,\mathbb{A}} = A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)}$$

with equivalent quasi-norms. Indeed, by (2.7) and (2.8) we have that

$$(A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \quad \text{and} \quad A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

So,

$$A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}}.$$

Conversely, if $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, we can write $a = a_0 + a_1$ with $a_j \in A_j$ and

$$(2.10) \quad \|a_0\|_{A_0} + \|a_1\|_{A_1} \leq 2 \|a\|_{A_0+A_1}.$$

Now we check that a_0 belongs to $(A_0, A_1)_{1,q,(\alpha_0,\alpha)}$. We have

$$\begin{aligned} \|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} &\lesssim \left(\int_1^\infty [t^{-1}K(t, a_0)\ell^\alpha(t)]^q \frac{dt}{t} \right)^{1/q} \\ &\quad + \left(\int_0^1 [t^{-1}K(t, a_1)\ell^{\alpha_0}(t)]^q \frac{dt}{t} \right)^{1/q} + \left(\int_0^1 [t^{-1}K(t, a)\ell^{\alpha_0}(t)]^q \frac{dt}{t} \right)^{1/q} \\ &\lesssim \left(\int_1^\infty [t^{-1}\ell^\alpha(t)]^q \frac{dt}{t} \right)^{1/q} \|a_0\|_{A_0} + \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t} \right)^{1/q} \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbb{A}}} \\ &\lesssim \|a\|_{A_0+A_1} + \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbb{A}}} \end{aligned}$$

where we have used (2.10) in the last inequality. Hence,

$$\|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} \lesssim \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbb{A}}}.$$

This shows that $(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)}$ with

$$\begin{aligned} \|a\|_{A_1+(A_0,A_1)_{1,q,(\alpha_0,\alpha)}} &\leq \|a_1\|_{A_1} + \|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} \\ &\lesssim \|a\|_{A_0+A_1} + \|a\|_{\bar{A}_{1,q,\mathbb{A}}} \\ &\lesssim \|a\|_{\bar{A}_{1,q,\mathbb{A}}} \end{aligned}$$

which establishes (2.9).

Combining (2.9) with (2.4) and Lemma 2.1, we conclude for $1 \leq q \leq \infty$ that

$$\begin{aligned} (A_0, A_1)_{1,q,\mathbb{A}} &= A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \\ &= A_1 + (A_0, A_1)_{1,q,(\alpha_0+1,\alpha+1)}^J \\ &= A_1 + [A_0, A_1]_{1,q,\alpha_0+1}^J. \end{aligned}$$

If $0 < q < 1$, we use (2.1), (2.9), (2.5) and Lemma 2.1 to derive

$$\begin{aligned} (A_0, A_1)_{1,q,\mathbb{A}} &= (A_0^\sim, A_1^\sim)_{1,q,\mathbb{A}} \\ &= A_1^\sim + (A_0^\sim, A_1^\sim)_{1,q,(\alpha_0,\alpha)} \\ &= A_1^\sim + (A_0^\sim, A_1^\sim)_{1,q,(\alpha_0+1/q,\alpha+1/q)}^J \\ &= A_1^\sim + [A_0^\sim, A_1^\sim]_{1,q,\alpha_0+1/q}^J. \end{aligned}$$

The proof is complete. \square

3. ASSOCIATE SPACES

In what follows, (Ω, μ) is a σ -finite measure space and \mathcal{M} is the collection of all (equivalence classes of) scalar valued μ -measurable functions on Ω which are finite μ -almost everywhere. We endow \mathcal{M} with the topology of convergence in measure on sets of finite measure.

The notion of Banach function space as described in [1] and [9] includes the Fatou property. However, in other books one can find a similar concept but leaving out the Fatou property. See [18], [14] and [15]. In this paper we follow this last point of view.

By a *Banach function space* we mean a Banach space $(X, \|\cdot\|_X)$ of functions in \mathcal{M} satisfying the following three properties:

- (i) Whenever $g \in \mathcal{M}$, $f \in X$ and $|g(x)| \leq |f(x)|$ μ -a.e., then $g \in X$ and $\|g\|_X \leq \|f\|_X$ (lattice property).
- (ii) $\chi_E \in X$ for every $E \subseteq \Omega$ with $\mu(E) < \infty$.
- (iii) For every $E \subseteq \Omega$ with $\mu(E) < \infty$ there is $c_E > 0$ such that $\int_E |f| d\mu \leq c_E \|f\|_X$ for every $f \in X$.

Clearly, simple functions are contained in X and $\|\chi_E\|_X = \mu(E)$ for $E \subseteq \Omega$.

The argument in [1, p. 4] based on (iii) can still be applied with the result that

$$(3.1) \quad X \hookrightarrow \mathcal{M}.$$

Lebesgue spaces L_p , Lorentz spaces $L_{p,q}$, Orlicz spaces L^Φ are examples of Banach function spaces. Other examples are the generalized Lorentz-Zygmund spaces $L_{(p,q;\mathbb{A})}$

formed by all those $f \in \mathcal{M}$ satisfying that

$$(3.2) \quad \|f\|_{L_{(p,q;\mathbb{A})}} = \left(\int_0^\infty \left[t^{1/p-1} \ell^{\mathbb{A}}(t) \int_0^t f^*(s) ds \right]^q \frac{dt}{t} \right)^{1/q} < \infty$$

(the outer integral should be replaced by the supremum if $q = \infty$). Here $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and f^* stands for the decreasing rearrangement of f , defined by

$$f^*(t) = \inf\{\delta > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > \delta\}) \leq t\}, \quad t > 0.$$

Spaces $L_{(p,q;\mathbb{A})}$ make also sense if $0 < q < 1$ but then (3.2) is no longer a norm but a quasi-norm. We refer to [16] and [9] for properties of generalized Lorentz-Zygmund spaces. In order to avoid that $L_{(p,q;\mathbb{A})} = \{0\}$ one should assume that any of the following conditions holds

$$\begin{cases} 1 < p < \infty; \\ p = \infty, & q < \infty, \quad \alpha_0 + 1/q < 0; \\ p = \infty, & q = \infty, \quad \alpha_0 \leq 0; \\ p = 1, & q < \infty, \quad \alpha_\infty + 1/q < 0; \\ p = 1, & q = \infty, \quad \alpha_\infty \leq 0; \end{cases}$$

(see [16, Lemma 3.5/(ii)]).

We write $L_{(p,q;\alpha_0)}(0, 1)$ if the integral in (3.2) is taken only on the interval $(0, 1)$ instead of $(0, \infty)$.

The *associate space* X' of the Banach function space X consists of all $g \in \mathcal{M}$ such that

$$\int_{\Omega} |fg| d\mu < \infty \text{ for every } f \in X.$$

It is also a Banach function space over Ω endowed with the norm

$$\|g\|_{X'} = \sup \left\{ \int_{\Omega} |fg| d\mu : \|f\|_X \leq 1 \right\}.$$

Indeed, the arguments in the proof of [1, Theorem I.2.2] can be applied to show that $(X', \|\cdot\|_{X'})$ is a norm space of functions in \mathcal{M} which satisfies the corresponding versions of (i), (ii) and (iii). Moreover, using the definition of $\|\cdot\|_{X'}$, it is not hard to check that if $(g_n) \subseteq X'$ and $\sum_{n=1}^{\infty} \|g_n\|_{X'} < \infty$ then the function $g = \sum_{n=1}^{\infty} g_n$ belongs to X' and $\|g - \sum_{j=1}^n g_j\|_{X'} \rightarrow 0$ as $n \rightarrow \infty$.

We also have that $\int_{\Omega} |fg| d\mu \leq \|f\|_X \|g\|_{X'}$.

If $(Y, \|\cdot\|_Y)$ is a quasi-Banach space of functions in \mathcal{M} such that the corresponding versions of (i), (ii) and (iii) hold, then we define Y' as above.

We let ℓ_q be the usual space of scalar q -summable sequences with indices on \mathbb{Z} . It is known that $\ell'_q = \ell_{q'}$ for $1 \leq q \leq \infty$ where $1/q + 1/q' = 1$. For later use we compute now the associate space of the quasi-Banach space ℓ_q when $0 < q < 1$.

Lemma 3.1. *Let $0 < q < 1$. Then $\ell'_q = \ell_\infty$ with equality of norms.*

Proof. Take any $\eta = (\eta_m) \in \ell_\infty$ and $\xi = (\xi_m) \in \ell_q$. It follows from

$$\sum_{m=-\infty}^{\infty} |\xi_m \eta_m| \leq \left(\sum_{m=-\infty}^{\infty} |\xi_m|^q |\eta_m|^q \right)^{1/q} \leq \|\xi\|_{\ell_q} \|\eta\|_{\ell_\infty}$$

that $\ell_\infty \hookrightarrow \ell'_q$ and that the embedding has norm less than or equal to 1. Conversely, take any $\eta = (\eta_m) \in \ell'_q$ and for $n \in \mathbb{Z}$ let $e_n = (\delta_m^n)_{m \in \mathbb{Z}}$. We have

$$|\eta_n| = \sum_{m=-\infty}^{\infty} |\eta_m \delta_m^n| \leq \|\eta\|_{\ell'_q} \|e_n\|_{\ell_q} = \|\eta\|_{\ell'_q}.$$

Hence, η belongs to ℓ_∞ and $\|\eta\|_{\ell_\infty} \leq \|\eta\|_{\ell'_q}$. This completes the proof. \square

Let X_0, X_1 be Banach function spaces over Ω . According to (3.1), we have that $X_j \hookrightarrow \mathcal{M}$. Hence $\bar{X} = (X_0, X_1)$ is a Banach couple. Subsequently, we are going to determine the associate space of $\bar{X}_{1,q,\mathbb{A}}$ and $\bar{X}_{1,q,\mathbb{A}}^J$. We work under different assumptions on \bar{X} than in [13] and [5], but ideas of those papers will be useful for our considerations.

Let $g \in \mathcal{M}$ and $f \in X_0 \cap X_1$ with $|g(x)| \leq |f(x)|$ μ -a.e., then it is clear that $g \in X_0 \cap X_1$ with $J(t, g; \bar{X}) \leq J(t, f; \bar{X})$, $t > 0$. So, $X_0 \cap X_1$ is a Banach function space with the norm $J(t, \cdot; \bar{X})$. As for the K-functional, using that

$$K(t, f; \bar{X}) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| \leq f_0 + f_1, f_j \geq 0, f_j \in X_j\}$$

(see, for example, [7, Lemma 3.1]), it follows that $K(t, g; \bar{X}) \leq K(t, f; \bar{X})$ provided that $|g| \leq |f|$, $f \in X_0 + X_1$. Now it is not hard to check that $X_0 + X_1$ is also a Banach function space.

The properties above of the J- and K-functionals also yield that for $1 \leq q \leq \infty$ the spaces $\bar{X}_{1,q,\mathbb{A}}$, $\bar{X}_{1,q,\mathbb{A}}^J$ and $\bar{X}_{1,q,\mathbb{A},\mathbb{B}}^J$ are Banach function spaces. To check (iii) one can rely on the exact interpolation property of the logarithmic interpolation methods applied to the operator $f \rightsquigarrow \int_E f d\mu$. If $0 < q < 1$, these spaces have also properties (i), (ii), (iii) but $\|\cdot\|_{\bar{X}_{1,q,\mathbb{A}}}$, $\|\cdot\|_{\bar{X}_{1,q,\mathbb{A}}^J}$ and $\|\cdot\|_{\bar{X}_{1,q,\mathbb{A},\mathbb{B}}^J}$ are only quasi-norms.

Let $\overline{X'} = (X'_0, X'_1)$ be the Banach couple formed by the associate spaces. If $f \in X_0 \cap X_1$, $g \in X'_0 + X'_1$ with $g = g_0 + g_1$, $g_j \in X'_j$ and $t > 0$, we have

$$\begin{aligned} \int_{\Omega} |fg| d\mu &\leq \int_{\Omega} |fg_0| d\mu + \int_{\Omega} |fg_1| d\mu \\ &\leq \|f\|_{X_0} \|g_0\|_{X'_0} + \|f\|_{X_1} \|g_1\|_{X'_1} \\ &\leq J(t, f; \overline{X}) \left(\|g_0\|_{X'_0} + t^{-1} \|g_1\|_{X'_1} \right). \end{aligned}$$

This yields

$$(3.3) \quad \int_{\Omega} |fg| d\mu \leq J(t, f; \overline{X}) K(t^{-1}, g; \overline{X'}), \quad t > 0.$$

Furthermore, we have that

$$(3.4) \quad J(t, g; \overline{X'}) = \sup_{f \in X_0 + X_1} \frac{\int_{\Omega} |fg| d\mu}{K(t^{-1}, f; \overline{X})}, \quad g \in X'_0 \cap X'_1, \quad t > 0.$$

Indeed, let $g \in X'_0 \cap X'_1$. Proceeding as to establish (3.3) we get

$$\sup_{f \in X_0 + X_1} \frac{\int_{\Omega} |fg| d\mu}{K(t^{-1}, f; \overline{X})} \leq J(t, g; \overline{X'}).$$

To check the converse inequality we write λX for the space X normed by $\lambda \|\cdot\|_X$. Since the embeddings $X_0 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot))$ and $t^{-1}X_1 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot))$ have norm less than or equal to 1, we have $(X_0 + X_1, K(t^{-1}, \cdot))' \hookrightarrow X'_0$ and $(X_0 + X_1, K(t^{-1}, \cdot))' \hookrightarrow tX'_1$ with norm less than or equal to 1. Hence

$$(X_0 + X_1, K(t^{-1}, \cdot))' \hookrightarrow (X'_0 \cap X'_1, J(t, \cdot))$$

with

$$J(t, g; \overline{X'}) \leq \sup_{f \in X_0 + X_1} \frac{\int_{\Omega} |fg| d\mu}{K(t^{-1}, f; \overline{X})}.$$

This establishes (3.4).

Recall that a Banach function space X over Ω is said to have *absolutely continuous norm* if for any $f \in X$ and any decreasing sequence (E_n) of μ -measurable sets with empty intersection we have that $\|f\chi_{E_n}\|_X \downarrow 0$ as $n \rightarrow \infty$. If X has absolutely continuous norm then X' coincides with the dual space X^* of X (see [18, Theorem 15.72.5, p. 480]).

Subsequently, \mathbb{K} stands for the scalar field, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Lemma 3.2. *Let $\overline{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω . Suppose that X_0 or X_1 has absolutely continuous norm. Then*

$$(3.5) \quad K(t^{-1}, g; \overline{X'}) = \sup_{f \in X_0 \cap X_1} \frac{\int_{\Omega} |fg| d\mu}{J(t, f; \overline{X})}, \quad g \in X'_0 + X'_1, \quad t > 0.$$

Proof. Inequality

$$\sup_{f \in X_0 \cap X_1} \frac{\int_{\Omega} |fg| d\mu}{J(t, f; \bar{X})} \leq K(t^{-1}, g; \bar{X}'), \quad g \in X'_0 + X'_1, \quad t > 0,$$

follows from (3.3). To check the reverse inequality, consider $X_0 \cap X_1$ with the norm $J(t, \cdot; \bar{X})$, endow $X'_0 + X'_1$ with the norm $K(t^{-1}, \cdot; \bar{X}')$ and take any $g \in (X_0 \cap X_1)'$. Then the functional T assigning to any $f \in X_0 \cap X_1$ the scalar $Tf = \int_{\Omega} fgd\mu$ belongs to $(X_0 \cap X_1)^*$ with

$$\|T\|_{(X_0 \cap X_1)^*} = \sup \left\{ \left| \int_{\Omega} fgd\mu \right| : J(t, f; \bar{X}) \leq 1 \right\} = \|g\|_{(X_0 \cap X_1)'}$$

Consider the space $X_0 \times X_1$ normed by

$$\|(f_0, f_1)\|_{X_0 \times X_1} = \max\{\|f_0\|_{X_0}, t\|f_1\|_{X_1}\}$$

and put $A = \{(f_0, f_1) \in X_0 \times X_1 : f_0 = f_1\}$. The linear functional $F : A \rightarrow \mathbb{K}$ defined by

$$F(f_0, f_1) = T\left(\frac{f_0 + f_1}{2}\right) = \int_{\Omega} f_0gd\mu, \quad (f_0, f_1) \in A,$$

is bounded with

$$\|F\|_{A^*} = \sup \left\{ \left| \int_{\Omega} fgd\mu \right| : f \in X_0 \cap X_1 \text{ with } J(t, f; \bar{X}) \leq 1 \right\} = \|g\|_{(X_0 \cap X_1)'}$$

According to the Hahn-Banach theorem, we can extend F to a bounded linear functional $\hat{F} \in (X_0 \times X_1)^*$ with $\|\hat{F}\|_{(X_0 \times X_1)^*} = \|g\|_{(X_0 \cap X_1)'}$. Hence, there are $L_j \in X_j^*$, $j = 0, 1$, such that

$$(3.6) \quad L_0f_0 = \hat{F}(f_0, 0), \quad L_1f_1 = \hat{F}(0, f_1) \quad \text{and} \quad \hat{F}(f_0, f_1) = L_0f_0 + L_1f_1.$$

Assume that X_0 has absolutely continuous norm. Then $X_0^* = X'_0$ and so there is $g_0 \in X'_0$ such that $L_0f_0 = \int_{\Omega} f_0g_0d\mu$ and $\|L_0\|_{X_0^*} = \|g_0\|_{X'_0}$. For any $f \in X_0 \cap X_1$, we have

$$\int_{\Omega} fgd\mu = \hat{F}(f, f) = L_0f + L_1f = \int_{\Omega} fg_0d\mu + L_1f.$$

Whence

$$L_1f = \int_{\Omega} f(g - g_0)d\mu, \quad f \in X_0 \cap X_1.$$

We claim that $g - g_0 \in X'_1$. Indeed, take any $f \in X_1$. We can find an increasing sequence of simple functions (f_n) such that $0 \leq f_n \uparrow |f|$. Since $(f_n) \subseteq X_0 \cap X_1$, we get

$$\begin{aligned} \int_{\Omega} f_n |g - g_0| d\mu &= \left| \int_{\Omega} (\operatorname{sgn}(g - g_0) f_n) (g - g_0) d\mu \right| \\ &= |L_1(\operatorname{sgn}(g - g_0) f_n)| \\ &\leq \|L_1\|_{X_1^*} \|f_n\|_{X_1} \\ &\leq \|L_1\|_{X_1^*} \|f\|_{X_1}. \end{aligned}$$

Whence, using the monotone convergence theorem, we derive that

$$\int_{\Omega} |f| |g - g_0| d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n |g - g_0| d\mu < \infty.$$

This shows that $g_1 = g - g_0$ belongs to X'_1 with $\|g_1\|_{X'} \leq \|L_1\|_{X_1^*}$. So, $g = g_0 + g_1 \in X'_0 + X'_1$. Moreover, given any $\varepsilon > 0$, we have

$$\begin{aligned} \|g\|_{X'_0 + X'_1} - \varepsilon(1 + t^{-1}) &\leq \|g_0\|_{X'_0} - \varepsilon + t^{-1}(\|g_1\|_{X'_1} - \varepsilon) \\ &\leq \|L_0\|_{X_0^*} - \varepsilon + t^{-1}(\|L_1\|_{X_1^*} - \varepsilon) \\ &\leq |L_0 f_0| + t^{-1} |L_1 f_1| \end{aligned}$$

for some $f_j \in X_j$ with $\|f_j\|_{X_j} \leq 1$. Therefore, using (3.6), we get

$$\begin{aligned} \|g\|_{X'_0 + X'_1} - \varepsilon(1 + t^{-1}) &\leq L_0 \left(\frac{|L_0 f_0|}{L_0 f_0} f_0 \right) + L_1 \left(t^{-1} \frac{|L_1 f_1|}{L_1 f_1} f_1 \right) \\ &= \hat{F} \left(\frac{|L_0 f_0|}{L_0 f_0} f_0, t^{-1} \frac{|L_1 f_1|}{L_1 f_1} f_1 \right) \\ &\leq \|\hat{F}\|_{(X_0 \times X_1)^*} \max \left(\left\| \frac{|L_0 f_0|}{L_0 f_0} f_0 \right\|_{X_0}, t t^{-1} \left\| \frac{|L_1 f_1|}{L_1 f_1} f_1 \right\|_{X_1} \right) \\ &\leq \|\hat{F}\|_{(X_0 \times X_1)^*} = \|g\|_{(X_0 \cap X_1)'}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, this yields that

$$K(t^{-1}, g; \bar{X}') = \|g\|_{X'_0 + X'_1} \leq \|g\|_{(X_0 \cap X_1)'} = \sup_{f \in X_0 \cap X_1} \frac{\int_{\Omega} |f g| d\mu}{J(t, f; \bar{X})}$$

and completes the proof.

The case when X_1 has absolutely continuous norm can be treated analogously. \square

Next we determine the associate space of $\bar{X}_{1, q, \mathbb{A}}^J$. If $\mathbb{A} = (\alpha_0, \alpha_{\infty}) \in \mathbb{R}^2$, we put $-\mathbb{A} = (-\alpha_0, -\alpha_{\infty})$ and we write $\tilde{\mathbb{A}} = (\alpha_{\infty}, \alpha_0)$ for the reverse pair.

Theorem 3.3. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω . Suppose that X_0 or X_1 has absolutely continuous norm. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $0 < q \leq \infty$ satisfying (2.3). Put*

$$q^* = \begin{cases} q' & \text{if } 1 \leq q \leq \infty, \\ \infty & \text{if } 0 < q < 1. \end{cases}$$

Then $(\bar{X}_{1,q,\mathbb{A}}^J)' = \bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}$.

Proof. We proceed following the lines of [13, Theorem 3.4]. Take any $g \in (\bar{X}_{1,q,\mathbb{A}}^J)'$ and any $\varepsilon > 0$. According to (3.5), for any $m \in \mathbb{Z}$, there is $f_m \in X_0 \cap X_1$ such that

$$(1 - \varepsilon)K(2^{-m}, |g|; \bar{X}') \leq J(2^m, |f_m|; \bar{X})^{-1} \int_{\Omega} |f_m g| d\mu.$$

Take any sequence (δ_m) of non-negative scalars such that $(\sum_{m=-\infty}^{\infty} (2^{-m} \ell^{\mathbb{A}}(2^m) \delta_m)^q)^{1/q} \leq 1$. Put $u_m = J(2^m, |f_m|; \bar{X})^{-1} \delta_m |f_m|$. Then the function $f = \sum_{m=-\infty}^{\infty} u_m$ belongs to $\bar{X}_{1,q,\mathbb{A}}^J$ and

$$\begin{aligned} \|f\|_{\bar{X}_{1,q,\mathbb{A}}^J} &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) \delta_m \right]^q \right)^{1/q} \leq 1. \end{aligned}$$

Moreover,

$$\begin{aligned} (1 - \varepsilon) \sum_{m=-\infty}^{\infty} 2^m \ell^{-\mathbb{A}}(2^m) K(2^{-m}, |g|; \bar{X}') 2^{-m} \ell^{\mathbb{A}}(2^m) \delta_m \\ \leq \sum_{m=-\infty}^{\infty} \delta_m J(2^m, |f_m|; \bar{X})^{-1} \int_{\Omega} |f_m g| d\mu \\ = \int_{\Omega} |f g| d\mu \leq \|g\|_{(\bar{X}_{1,q,\mathbb{A}}^J)'}. \end{aligned}$$

Using that $\ell'_q = \ell_{q^*}$, we derive that $g \in \bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}$ and that $\|g\|_{\bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}} \leq \|g\|_{(\bar{X}_{1,q,\mathbb{A}}^J)'}$.

Conversely, take any $g \in \bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}$, let $f \in \bar{X}_{1,q,\mathbb{A}}^J$ and take any J-representation $f = \sum_{m=-\infty}^{\infty} f_m$ of f . By (3.3), we have

$$\int_{\Omega} |f_m g| d\mu \leq J(2^m, f_m; \bar{X}) K(2^{-m}, g; \bar{X}'), \quad m \in \mathbb{Z}.$$

Hence, if $1 \leq q \leq \infty$, it follows by using Hölder's inequality that

$$\begin{aligned} \int_{\Omega} |fg| d\mu &\leq \sum_{m=-\infty}^{\infty} J(2^m, f_m; \bar{X})K(2^{-m}, g; \bar{X}') \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, f_m; \bar{X}) \right]^q \right)^{1/q} \\ &\quad \times \left(\sum_{m=-\infty}^{\infty} \left[2^m \ell^{-\mathbb{A}}(2^m) K(2^{-m}, g; \bar{X}') \right]^{q'} \right)^{1/q'}. \end{aligned}$$

If $0 < q < 1$, we obtain

$$\begin{aligned} \int_{\Omega} |fg| d\mu &\leq \sum_{m=-\infty}^{\infty} J(2^m, f_m; \bar{X})K(2^{-m}, g; \bar{X}') \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[J(2^m, f_m; \bar{X})K(2^{-m}, g; \bar{X}') \right]^q \right)^{1/q} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) J(2^m, f_m; \bar{X}) \right]^q \right)^{1/q} \\ &\quad \times \sup_{m \in \mathbb{Z}} \{ 2^m \ell^{-\mathbb{A}}(2^m) K(2^{-m}, g; \bar{X}') \}. \end{aligned}$$

Therefore, for any $0 < q \leq \infty$, we get that

$$\int_{\Omega} |fg| d\mu \leq \|f\|_{\bar{X}_{1,q,\mathbb{A}}^J} \|g\|_{\bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}}.$$

This shows that g belongs to $(\bar{X}_{1,q,\mathbb{A}}^J)'$ with $\|g\|_{(\bar{X}_{1,q,\mathbb{A}}^J)'} \leq \|g\|_{\bar{X}'_{1,q^*,-\tilde{\mathbb{A}}}}$. The proof is complete. \square

Remark 3.4. For $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ and $1 < q \leq \infty$ with $\alpha_\infty = 1/q'$ and $\beta_\infty > 1/q'$, or $0 < q \leq 1$, $\alpha_\infty = 0$ and $\beta_\infty \geq 0$, one can determine the associate space of $(X_0, X_1)_{1,q,\mathbb{A},\mathbb{B}}^J$ proceeding in a similar way. If X_0 or X_1 has absolutely continuous norm, the outcome is

$$(3.7) \quad \left((X_0, X_1)_{1,q,\mathbb{A},\mathbb{B}}^J \right)' = (X'_0, X'_1)_{1,q^*,-\tilde{\mathbb{A}},-\tilde{\mathbb{B}}}$$

where the quasi-norm in the K -space is given by

$$\|g\|_{\bar{X}'_{1,q^*,-\tilde{\mathbb{A}},-\tilde{\mathbb{B}}}} = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{-\tilde{\mathbb{A}}}(2^m) \ell^{-\tilde{\mathbb{B}}}(2^m) K(2^m, g; \bar{X}') \right]^{q^*} \right)^{1/q^*}.$$

Remark 3.5. *The same techniques are useful to determine the associate space of the modified J -space $[X_0, X_1]_{1,q,\alpha_0}^J$. The relevant K -spaces are now*

$$[X_0, X_1]_{1,q,\alpha}^K = \{f \in X_0 + X_1 : \|f\|_{[X_0, X_1]_{1,q,\alpha}^K} = \left(\sum_{m=0}^{\infty} [2^{-m} \ell^\alpha(2^m) K(2^m, f; \bar{X})]^q \right)^{1/q} < \infty\}.$$

We have

$$(3.8) \quad \left([X_0, X_1]_{1,q,\alpha_0}^J \right)' = [X_0', X_1']_{1,q^*, -\alpha_0}^K$$

provided that X_0 or X_1 has absolutely continuous norm.

Remark 3.6. *On the contrary to the duality formulae of [8] and [3] where it is essential that $A_0 \cap A_1$ is dense in A_0 and A_1 , such assumption is not needed in Theorem 3.3. It suffices that X_0 or X_1 has absolutely continuous norm. Moreover, the parameter q can also take the value ∞ in the Theorem 3.3.*

Next we determine the associate spaces of K -spaces. We start with the case $1 \leq q \leq \infty$. We put $1/q + 1/q' = 1$.

Theorem 3.7. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over the space Ω . Suppose that X_0 or X_1 has absolutely continuous norm. Let $1 \leq q \leq \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (2.2). Then we have with equivalence of norms:*

- (i) *If $\alpha_\infty + 1/q > 0$, then $(\bar{X}_{1,q,\mathbb{A}})' = \overline{X'}_{1,q', -\tilde{\mathbb{A}}-1}$.*
- (ii) *If $q < \infty$ and $\alpha_\infty = -1/q$, then $(\bar{X}_{1,q,\mathbb{A}})' = \overline{X'}_{1,q', -\tilde{\mathbb{A}}-1, (-1,0)}$.*
- (iii) *If $\alpha_\infty + 1/q < 0$ and $q < \infty$, or $\alpha_\infty \leq 0$ and $q = \infty$, then*

$$(\bar{X}_{1,q,\mathbb{A}})' = X_1' \cap [X_0', X_1']_{1,q', -\alpha_0-1}^K.$$

Proof. Statements (i) and (ii) follow from (2.4), Theorem 3.3 and Remark 3.4. To prove (iii) we use Theorem 2.2, (3.4) and Remark 3.5. \square

Next we deal with the case $0 < q < 1$.

Theorem 3.8. *Let $\bar{X} = (X_0, X_1)$ be a Gagliardo couple of Banach function spaces over Ω . Suppose that X_0 or X_1 has absolutely continuous norm. Let $0 < q < 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (2.2). Then we have with equivalence of norms:*

- (i) *If $\alpha_\infty + 1/q > 0$, $(\bar{X}_{1,q,\mathbb{A}})' = \overline{X'}_{1,\infty, -\tilde{\mathbb{A}}-1/q}$.*
- (ii) *If $\alpha_\infty + 1/q = 0$, $(\bar{X}_{1,q,\mathbb{A}})' = \overline{X'}_{1,\infty, -\tilde{\mathbb{A}}-1/q, (-1/q,0)}$.*
- (iii) *If $\alpha_\infty + 1/q < 0$, $(\bar{X}_{1,q,\mathbb{A}})' = X_1' \cap [X_0', X_1']_{1,\infty, -\alpha_0-1/q}^K$.*

Proof. Since \bar{X} is a Gagliardo couple, we have that $X_j^\sim = X_j$, $j = 0, 1$. Statements follow from (2.5), Theorems 2.2 and 3.3, and Remarks 3.4 and 3.5. \square

4. GENERALIZED LORENTZ-ZYGMUND SPACES

First we assume that (Ω, μ) is a non-atomic σ -finite measure space and we deal with the spaces $L_{(\infty, q; \mathbb{A})}$ (see (3.2)). Their associate spaces have been determined by Opic and Pick [16, Theorem 6.2/(ii),(v) and Theorem 6.6/(ii),(iv)] by means of direct calculations. In this section we derive them from the abstract results obtained in Section 3 as an specific example.

Consider the Banach couple (L_1, L_∞) . It is well-known that

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s) ds, \quad t > 0,$$

(see, for example, [2, Theorem 5.2.1]). From this equality it is not hard to check that (L_1, L_∞) is a Gagliardo couple (see the comment after [1, Theorem V.1.6]). Besides, the norm of L_1 is absolutely continuous.

Theorem 4.1. *Let $1 \leq q \leq \infty$, $1/q + 1/q' = 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfying (2.2). We have with equivalence of norms:*

- (i) *If $\alpha_\infty + 1/q > 0$, then $(L_{(\infty, q; \mathbb{A})})' = L_{(1, q'; -\mathbb{A}-1)}$.*
- (ii) *If $\alpha_\infty + 1/q < 0$ and $q < \infty$, or $\alpha_\infty \leq 0$ and $q = \infty$, then*

$$(L_{(\infty, q; \mathbb{A})})' = \left\{ g \in \mathcal{M} : \|g\| = \int_0^\infty g^*(t) dt + \|g\|_{L_{(1, q'; -\alpha_0-1)}(0,1)} < \infty \right\}.$$

- (iii) *If $\alpha_\infty + 1/q = 0$ and $q < \infty$, then $(L_{(\infty, q; \mathbb{A})})' =$*

$$\left\{ g \in \mathcal{M} : \|g\| = \left(\int_0^\infty \left[\ell^{(-\alpha_0-1, -1/q')}(t) \ell^{\ell(0, -1)}(t) \int_0^t g^*(s) ds \right]^{q'} \frac{dt}{t} \right)^{1/q'} < \infty \right\}.$$

Proof. We have

$$(4.1) \quad (L_1, L_\infty)_{1, q, \mathbb{A}} = L_{(\infty, q; \mathbb{A})}$$

because

$$\begin{aligned} \|f\|_{(L_1, L_\infty)_{1, q, \mathbb{A}}} &= \left(\int_0^\infty \left[t^{-1} \ell^{\mathbb{A}}(t) K(t, f) \right]^q \frac{dt}{t} \right)^{1/q} \\ &= \left(\int_0^\infty \left[t^{-1} \ell^{\mathbb{A}}(t) \int_0^t f^*(s) ds \right]^q \frac{dt}{t} \right)^{1/q} = \|f\|_{L_{(\infty, q; \mathbb{A})}}. \end{aligned}$$

Besides, $L'_1 = L_\infty$ and $L'_\infty = L_1$. Hence, it follows from Theorem 3.7/(i) that

$$\begin{aligned} \|g\|_{(L_{(\infty,q;\mathbb{A})})'} &\sim \left(\int_0^\infty \left[t^{-1} \ell^{-\tilde{\mathbb{A}}-1}(t) K(t, g; L_\infty, L_1) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{-\tilde{\mathbb{A}}-1}(t) K(t^{-1}, g; L_1, L_\infty) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{-\mathbb{A}-1}(t) \int_0^t g^*(s) ds \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \|g\|_{L_{(1,q';-\mathbb{A}-1)}}. \end{aligned}$$

This establishes (i). As for (ii), using Theorem 3.7/(iii), we get

$$\|g\|_{(L_{(\infty,q;\mathbb{A})})'} \sim \|g\|_{L_1} + \|g\|_{[L_\infty, L_1]_{1,q',-\alpha_0-1}^K}.$$

On the other hand, reversing the couple, we derive

$$\begin{aligned} \|g\|_{[L_\infty, L_1]_{1,q',-\alpha_0-1}^K} &= \left(\sum_{m=0}^\infty \left[2^{-m} \ell^{-\alpha_0-1}(2^m) K(2^m, g; L_\infty, L_1) \right]^{q'} \right)^{1/q'} \\ &= \left(\sum_{m=0}^\infty \left[\ell^{-\alpha_0-1}(2^m) K(2^{-m}, g; L_1, L_\infty) \right]^{q'} \right)^{1/q'} \\ &\sim \left(\int_0^1 \left[\ell^{-\alpha_0-1}(t) K(t, g; L_1, L_\infty) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^1 \left[\ell^{-\alpha_0-1}(t) \int_0^t g^*(s) ds \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \|g\|_{L_{(1,q';-\alpha_0-1)}}(0, 1). \end{aligned}$$

This completes the proof of (ii). Finally, for (iii), according to Theorem 3.7/(ii), we obtain

$$\begin{aligned} \|g\|_{(L_{(\infty,q;\mathbb{A})})'} &= \left(\int_0^\infty \left[t^{-1} \ell^{-\tilde{\mathbb{A}}-1}(t) \ell \ell^{(-1,0)}(t) K(t, g; L_\infty, L_1) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{-\tilde{\mathbb{A}}-1}(t) \ell \ell^{(-1,0)}(t) K(t^{-1}, g; L_1, L_\infty) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{(-\alpha_0-1, -1/q')}(t) \ell \ell^{(0,-1)}(t) \int_0^t g^*(s) ds \right]^{q'} \frac{dt}{t} \right)^{1/q'}. \end{aligned}$$

□

Theorem 4.2. *Let $0 < q < 1$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ satisfy (2.2). Then we have with equivalence of norms:*

(i) If $\alpha_\infty + 1/q > 0$, $(L_{(\infty, q; \mathbb{A})})' = L_{(1, \infty; -\mathbb{A}-1/q)}$.

(ii) If $\alpha_\infty + 1/q < 0$,

$$(L_{(\infty, q; \mathbb{A})})' = \left\{ g \in \mathcal{M} : \|g\| = \int_0^\infty g^*(t) dt + \|g\|_{L_{(1, \infty; -\alpha_0-1/q)}(0,1)} < \infty \right\}.$$

(iii) If $\alpha_\infty + 1/q = 0$,

$$(L_{(\infty, q; \mathbb{A})})' = \left\{ g \in \mathcal{M} : \|g\| = \sup_{0 < t < \infty} \left(\ell^{(-\alpha_0-1/q, 0)}(t) \ell^{\ell^{(0, -1/q)}(t)} \int_0^t g^*(s) ds \right) < \infty \right\}.$$

Proof. One can proceed from (4.1) as in Theorem 4.1 but replacing Theorem 3.7 by Theorem 3.8. \square

We close the paper by establishing the corresponding results to Theorems 4.1 and 4.2 for sequence spaces. This question has not been discussed in [16], but it can be treated as another specific application of the abstract results of Section 3.

Let $\Omega = \mathbb{N}$ and $\mu = \#$ the counting measure. Given any bounded sequence of scalars $\xi = (\xi_n)$, we put

$$\xi_n^* = \inf \{ \tau > 0 : \# \{ j \in \mathbb{N} : |\xi_j| \geq \tau \} < n \}.$$

The sequence (ξ_n^*) is the decreasing rearrangement of (ξ_n) by magnitude of modulus. If $\xi = (\xi_n)$ converges to zero, then

$$\xi_1^* = \max \{ |\xi_n| : n \in \mathbb{N} \} = |\xi_{n_1}|, \quad \xi_2^* = \max \{ |\xi_n| : n \in \mathbb{N} \setminus \{n_1\} \} \text{ and so on.}$$

For $\alpha \in \mathbb{R}$, let $\ell_{(\infty, q; \alpha)}$ be the collection of all bounded sequences $\xi = (\xi_n)$ such that

$$\|\xi\|_{\ell_{(\infty, q; \alpha)}} = \left(\sum_{n=1}^{\infty} \left[\ell^\alpha(n) n^{-1} \sum_{j=1}^n \xi_j^* \right]^q n^{-1} \right)^{1/q} < \infty.$$

We put $\ell_{(1, q; \alpha)}$ for the set of all bounded sequences $\xi = (\xi_n)$ such that

$$\|\xi\|_{\ell_{(1, q; \alpha)}} = \left(\sum_{n=1}^{\infty} \left[\ell^\alpha(n) \sum_{j=1}^n \xi_j^* \right]^q n^{-1} \right)^{1/q} < \infty.$$

Replacing the weight $\ell^\alpha(n)$ by $\ell^\alpha(n) \ell^{\beta}(n)$, where $\beta \in \mathbb{R}$, we obtain the spaces $\ell_{(1, q; \alpha, \beta)}$.

To determine the associate space of $\ell_{(\infty, q; \alpha)}$, we work with the Banach couple (ℓ_1, ℓ_∞) . The K-functional for this couple is

$$(4.2) \quad K(n, \xi; \ell_1, \ell_\infty) = \sum_{j=1}^n \xi_j^*, \quad n \in \mathbb{N}$$

(see [17, p. 126]). Since

$$\|\xi\|_{\ell_1^\sim} = \sup_{0 < t < \infty} K(t, \xi) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \xi_j^* = \|\xi\|_{\ell_1}$$

we have that (ℓ_1, ℓ_∞) is a Gagliardo couple. Moreover, ℓ_1 has absolutely continuous norm.

Theorem 4.3. *Let $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$. Then we have*

$$\ell'_{(\infty, q; \alpha)} = \begin{cases} \ell_{(1, q'; -\alpha-1)} & \text{if } 1 \leq q \leq \infty \quad \text{and} \quad \alpha + 1/q > 0, \\ \ell_{(1, \infty; -\alpha-1/q)} & \text{if } 0 < q < 1 \quad \text{and} \quad \alpha + 1/q > 0, \\ \ell_{(1, q'; -\alpha-1, -1)} & \text{if } 1 \leq q < \infty \quad \text{and} \quad \alpha + 1/q = 0, \\ \ell_{(1, \infty; -\alpha-1/q, -1/q)} & \text{if } 0 < q < 1 \quad \text{and} \quad \alpha + 1/q = 0. \end{cases}$$

Proof. Choose $\alpha_0 \in \mathbb{R}$ with $\alpha_0 + 1/q < 0$. We have $K(t, \xi; \ell_1, \ell_\infty) \leq t \|\xi\|_{\ell_\infty}$. Hence

$$\left(\int_0^1 [t^{-1} \ell^{\alpha_0}(t) K(t, \xi; \ell_1, \ell_\infty)]^q \frac{dt}{t} \right)^{1/q} \leq \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t} \right)^{1/q} \|\xi\|_{\ell_\infty} \lesssim \|\xi\|_{\ell_\infty}.$$

Therefore, using (4.2), we obtain

$$\begin{aligned} \|\xi\|_{(\ell_1, \ell_\infty)_{1, q, (\alpha_0, \alpha)}} &\sim \left(\int_1^\infty [t^{-1} \ell^\alpha(t) K(t, \xi; \ell_1, \ell_\infty)]^q \frac{dt}{t} \right)^{1/q} \\ &\sim \left(\sum_{n=1}^\infty [n^{-1} \ell^\alpha(n) \sum_{j=1}^n \xi_j^*]^q n^{-1} \right)^{1/q} \\ &= \|\xi\|_{\ell_{(\infty, q; \alpha)}}. \end{aligned}$$

Whence, if $1 \leq q \leq \infty$ and $\alpha + 1/q > 0$, according to Theorem 3.7/(i), we derive

$$\ell'_{(\infty, q; \alpha)} = (\ell_\infty, \ell_1)_{1, q', (-\alpha-1, -\alpha_0-1)}.$$

Furthermore,

$$\begin{aligned} &\left(\int_1^\infty [t^{-1} \ell^{-\alpha_0-1}(t) K(t, \xi; \ell_\infty, \ell_1)]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &\lesssim \left(\int_1^\infty [t^{-1} \ell^{-\alpha_0-1}(t)]^{q'} \frac{dt}{t} \right)^{1/q'} \|\xi\|_{\ell_\infty} \\ &\lesssim \|\xi\|_{\ell_\infty}. \end{aligned}$$

This yields that

$$\begin{aligned}
\|\xi\|_{(\ell_\infty, \ell_1)_{1, q', (-\alpha-1, -\alpha_0-1)}} &\sim \left(\int_0^1 [t^{-1} \ell^{-\alpha-1}(t) K(t, \xi; \ell_\infty, \ell_1)]^{q'} \frac{dt}{t} \right)^{1/q'} \\
&= \left(\int_0^1 [\ell^{-\alpha-1}(t) K(t^{-1}, \xi; \ell_1, \ell_\infty)]^{q'} \frac{dt}{t} \right)^{1/q'} \\
&\sim \left(\int_1^\infty [\ell^{-\alpha-1}(t) K(t, \xi; \ell_1, \ell_\infty)]^{q'} \frac{dt}{t} \right)^{1/q'} \\
&\sim \left(\sum_{n=1}^\infty [\ell^{-\alpha-1}(n) \sum_{j=1}^n \xi_j^*]^{q'} n^{-1} \right)^{1/q'} \\
&= \|\xi\|_{\ell_{(1, q'; -\alpha-1)}}.
\end{aligned}$$

Consequently, $\ell'_{(\infty, q; \alpha)} = \ell_{(1, q'; -\alpha-1)}$.

The case $\alpha + 1/q = 0$ with $1 \leq q < \infty$ follows similarly but using now Theorem 3.7/(ii).

The remaining cases where $0 < q < 1$ can be derived analogously but replacing Theorem 3.7 by Theorem 3.8. \square

In Theorem 4.3 we have not considered the case $\alpha + 1/q < 0$. The reason is that in this situation we have that $\ell_{(\infty, q; \alpha)} = \ell_\infty$. Indeed,

$$\begin{aligned}
\|\xi\|_{\ell_\infty} &\leq \|\xi\|_{\ell_{(\infty, q; \alpha)}} = \left(\sum_{n=1}^\infty [\ell^\alpha(n) n^{-1} \sum_{j=1}^n \xi_j^*]^q n^{-1} \right)^{1/q} \\
&\leq \left(\sum_{n=1}^\infty \ell^{\alpha q}(n) n^{-1} \right)^{1/q} \|\xi\|_{\ell_\infty} \lesssim \|\xi\|_{\ell_\infty}.
\end{aligned}$$

Therefore,

$$(4.3) \quad \ell'_{(\infty, q; \alpha)} = \ell'_\infty = \ell_1 \text{ for } \alpha + 1/q < 0 \text{ and } 0 < q < \infty, \text{ or } \alpha \leq 0 \text{ and } q = \infty.$$

Still, this last result can be derived from Theorems 3.7/(iii) and 3.8/(iii) noting that in this case $X'_1 = \ell_1$ and that $[\ell_\infty, \ell_1]_{1, q', -\alpha_0-1}^K = \ell_\infty = [\ell_\infty, \ell_1]_{1, \infty, -\alpha_0-1/q}^K$ because

$$\begin{aligned}
\left(\int_1^\infty [t^{-1} \ell^{-\alpha_0-1}(t) K(t, \xi; \ell_\infty, \ell_1)]^{q'} \frac{dt}{t} \right)^{1/q'} &\leq \left(\int_1^\infty [t^{-1} \ell^{-\alpha_0-1}(t)]^{q'} \frac{dt}{t} \right)^{1/q'} \|\xi\|_{\ell_\infty} \\
&\lesssim \|\xi\|_{\ell_\infty}
\end{aligned}$$

and similarly

$$\sup_{1 < t < \infty} [t^{-1} \ell^{-\alpha_0-1/q}(t) K(t, \xi; \ell_\infty, \ell_1)] \lesssim \|\xi\|_{\ell_\infty}.$$

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REFERENCES

- [1] C. Bennett and R. Sharpley, “*Interpolation of Operators*”, Academic Press, Boston, 1988.
- [2] J. Bergh and J. Löfström, “*Interpolation Spaces. An Introduction*”, Springer, Berlin, 1976.
- [3] B.F. Besoy and F. Cobos, *Duality for logarithmic interpolation spaces when $0 < q < 1$ and applications*, J. Math. Anal. Appl. **466** (2018), 373-399.
- [4] B.F. Besoy, F. Cobos and L.M. Fernández-Cabrera, *On the structure of a limit class of logarithmic interpolation spaces*, preprint, Madrid (2019).
- [5] F. Cobos and L.M. Fernández-Cabrera, *The fundamental function of certain interpolation spaces generated by N -tuples of rearrangement-invariant spaces*, in “*Function Spaces and Inequalities*”. Edited by P. Jain and H.-J. Schmeisser, Springer, Singapore, 2017, pp. 1-14.
- [6] F. Cobos, L.M. Fernández-Cabrera and A. Martínez, *On a paper of Edmunds and Opic on limiting interpolation of compact operators between L_p spaces*, Math. Nachr. **288** (2015) 167-175.
- [7] F. Cobos and J. Martín, *On interpolation of function spaces by methods defined by means of polygons*, J. Approx. Theory **132** (2005) 182-203.
- [8] F. Cobos and A. Segurado, *Description of logarithmic interpolation spaces by means of the J -functional and applications*, J. Funct. Anal. **268** (2015) 2906-2945.
- [9] D.E. Edmunds and W.D. Evans, “*Hardy Operators, Function Spaces and Embeddings*”, Springer, Berlin, 2004.
- [10] D.E. Edmunds and B. Opic, *Limiting variants of Krasnosel’skiĭ’s compact interpolation theorem*, J. Funct. Anal. **266** (2014) 3265-3285.
- [11] W.D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. **52** (2000) 920-960.
- [12] W.D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7** (2002) 187-269.
- [13] L. M. Fernández-Cabrera, *The fundamental function of spaces generated by interpolation methods associated to polygons*, Mediterr. J. Math. (2017) **14**:17.
- [14] S.G. Kreĭn, Ju.I. Petunin and E.M. Semenov, “*Interpolation of Linear Operators*”, Amer. Math. Soc., Providence, 1982.
- [15] P. Meyer-Nieberg, “*Banach Lattices*”, Springer, Berlin, 1991.
- [16] B. Opic and L. Pick, *On generalized Lorentz-Zygmund spaces*, Math. Inequal. Appl. **2** (1999) 391-467.
- [17] H. Triebel, “*Interpolation Theory, Function Spaces, Differential Operators*”, North-Holland, Amsterdam, 1978.
- [18] A.C. Zaanen, “*Integration*”, North-Holland, Amsterdam, 1967.

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