PHYSICAL REVIEW D **74,** 105004 (2006)

Noncommutative Einstein-Maxwell pp-waves

S. Marculescu
Fachbereich Physik, Universität Siegen, D-57068 Siegen, Germany

F. Ruiz Ruiz
Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain
(Received 27 July 2006; published 2 November 2006)

The field equations coupling a Seiberg-Witten electromagnetic field to noncommutative gravity, as described by a formal power series in the noncommutativity parameters $\theta^{\alpha\beta}$, is investigated. A large family of solutions, up to order one in $\theta^{\alpha\beta}$, describing Einstein-Maxwell null pp-waves is obtained. The order-one contributions can be viewed as providing noncommutative corrections to pp-waves. In our solutions, noncommutativity enters the spacetime metric through a conformal factor and is responsible for dilating/contracting the separation between points in the same null surface. The noncommutative corrections to the electromagnetic waves, while preserving the wave null character, include constant polarization, higher harmonic generation, and inhomogeneous susceptibility. As compared to pure noncommutative gravity, the novelty is that nonzero corrections to the metric already occur at order one in $\theta^{\alpha\beta}$.

DOI: 10.1103/PhysRevD.74.105004 PACS numbers: 11.10.Nx, 04.40.Nr

I. INTRODUCTION

The idea that physics at the Planck length may be a probe for noncommutativity [1] has made of noncommutative gravity a central issue in the related literature; see Ref. [2] for a review. By now, several noncommutative deformations of general relativity in various dimensions [3] have been proposed with varied luck, many of which involve complexification of the metric and of the local Lorentz group. Lately a more fundamental theory of noncommutative gravity based on a deformation of the group of diffeomorphisms [4] has been proposed, and its connection with the Seiberg-Witten limit [5] of graviton interactions in bosonic string theory has also been studied [6]. This formulation has the property, shared by many phenomenological approaches [3], that an expansion in powers of the noncommutativity parameters $\theta^{\alpha\beta}$ reveals that corrections to Einstein gravity start at order two. However, no explicit solution to the corresponding field equations has been found to this order.

In this paper, inspired by recent results on noncommutatively smeared Schwarzschild black holes [7], we couple general relativity to a Seiberg-Witten electromagnetic (EM) field up to order one in $\theta^{\alpha\beta}$. Our model is based on two assumptions. The first one is that Einstein gravity should remain applicable. This is justifiable since, as already mentioned, in more fundamental approaches [4,6] and in many deformation approaches [3,8] noncommutative corrections start at order two in $\theta^{\alpha\beta}$. The second hypothesis is that noncommutative gravity modifications should already occur at order one in $\theta^{\alpha\beta}$, since matter distributions from field theory classical actions receive contributions to this order. In other words, even though gravity lacks first-order corrections in $\theta^{\alpha\beta}$, the right-hand side of the Einstein equations provides such corrections. Based on these assumptions, we consider a model that couples gravity, described by the Einstein-Hilbert action, to the order-zero and order-one terms of the Seiberg-Witten expansion for the action of an Abelian gauge field. This yields the classical action in (2.1) below, which defines our model. In this paper we will look at the corrections to the metric due to the occurrence of $\theta^{\alpha\beta}$ on the right-hand side of the Einstein equations, and to the $\theta^{\alpha\beta}$-dependence of the EM field.

Because of their relevance in general relativity and in string propagation on gravitational backgrounds, we are interested here in finding pp-wave solutions. We anticipate ourselves and mention that we are able to construct a large variety of noncommutative null pp-wave spacetimes. In all of them, the noncommutativity dependence on the metric is through a conformal factor which is a function of the coordinate labeling the null surface of spacetime. In turn the EM field receives in general three different types of noncommutative corrections: a dynamically generated constant polarization/induction contribution that can be tuned at will, a susceptibility inhomogeneous polarization due to gravity, and a nonlinear dipolar contribution, ultimately caused by the Seiberg-Witten map, which generates higher harmonics. These EM waves do not exhibit modified dispersion relations and are of a different type to those encountered in flat spacetime [9].

The paper is organized as follows. We introduce in Sec. II the classical action defining our model and describe the assumptions behind it. In Sec. III we derive the corresponding Einstein and EM field equations. Since $\theta^{\alpha\beta}$ is treated as a small parameter, the field equations decouple into an order-zero system of equations, describing conventional Einstein-Maxwell theory, and a complicated order-one system of equations. Also in Sec. III we make a conformal ansatz for the noncommutative correction to the metric that simplifies very much the order-one equations. We consider in Sec. IV solutions of the order-zero problem describing null Einstein-Maxwell pp-waves, and ask ourselves whether the order-one equations have solutions preserving the pp-wave nature of such order-zero backgrounds. The answer is in the affirmative and does not rely on the details of the equations but rather on the conformal ansatz made for the noncommutative correction to the metric. In Sec. V we move on to study particular cases of noncommutative pp-wave spacetimes. The physical significance of the noncommutative corrections to the EM field and a comparison with known solutions in the literature for flat spacetime is performed in Sec. VI. We conclude in Sec. VII.

II. CLASSICAL ACTION

Our interest is to study the noncommutative coupling of an Abelian gauge field A_α to Einstein gravity to order one in the noncommutativity parameters $\theta^{\alpha\beta}$. Let us first write the classical action that defines our model and then discuss how it arises. The action is given by

$$S = \int d^4x\sqrt{-g}\bigg[\frac{R}{2\kappa^2} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\theta^{\mu\nu}\Big(F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}F_{\mu\nu}F_{\alpha\beta}\Big)F^{\alpha\beta}\bigg] + O(\theta^2), \quad (2.1)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength. To explain this action, we situate ourselves in a reference frame in which $\theta^{\alpha\beta}$ is constant and invoke the two hypotheses mentioned in the introduction. The first one was that general relativity should be applicable up to order in $\theta^{\alpha\beta}$, since existing formulations [4] of noncommutative gravity based on a deformation of the diffeomorphism group introduced noncommutative corrections to general relativity at order two and higher. Hence, the gravity part of the classical action up to order one should be the Einstein-Hilbert action, which is the first term in (2.1). The second and third terms are the most straightforward generalization to curved spacetime of the action provided in flat spacetime by the Seiberg-Witten map for a U(1) gauge field. These terms are in agreement with the second assumption in the Introduction. Next we adopt the observer point of view [10] and regard $\theta^{\alpha\beta}$ as a contravariant two-tensor. This implies that the action (2.1) is invariant under conventional gauge transformations of A_α and diffeomorphisms. Note that, since we are not dealing with Moyal products, we are not running into the problems associated with nonconstant noncommutativity [11]. This defines our model and is the starting point for our analysis.

This action can be viewed as collecting the order-zero and order-one terms of a formal power series in $\theta^{\alpha\beta}$ for a classical action describing the Seiberg-Witten coupling of gravity and electromagnetism. Our solutions must be understood in this way, as providing the first nontrivial terms of a formal power series in $\theta^{\alpha\beta}$ describing noncommutative deformations of both gravitational and EM fields. The lack of first-order contributions to the gravity sector of the action [3,4,6,8], together with the well-known form of the first term in the Seiberg-Witten expansion of the classical action for an Abelian gauge field, speaks in favor of the generality of the action (2.1).

III. FIELD EQUATIONS AND CONFORMAL ANSATZ

Varying the action (2.1) with respect to the metric and the gauge field we obtain the field equations. Substituting in them the expansions

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + g^{(1)}_{\alpha\beta} + \cdots \qquad F_{\alpha\beta} = F^{(0)}_{\alpha\beta} + F^{(1)}_{\alpha\beta} + \cdots \quad (3.1)$$

for $g_{\alpha\beta}$ and $F_{\alpha\beta}$ in powers of $\theta^{\alpha\beta}$, retaining contributions up to order one and identifying coefficients of the same order, we obtain

$$\bar{R}_{\alpha\beta} = \kappa^2\bar{T}_{\alpha\beta} \quad (3.2)$$

$$\bar{\nabla}_\alpha\bar{F}^{\alpha\beta} = 0 \quad (3.3)$$

for the Einstein and field equations at order zero, and

$$\hat{R}_{\alpha\beta} = \kappa^2\hat{T}_{\alpha\beta} \quad (3.4)$$

$$\bar{\nabla}_\alpha(\hat{F}^{\alpha\beta} + \hat{H}^{\alpha\beta}) = L^\beta \quad (3.5)$$

for the Einstein and field equations at order one. Here the notation is as follows. Quantities of order zero in $\theta^{\alpha\beta}$ are denoted with a bar and quantities of order one with a hat, so that $\bar{g}_{\alpha\beta} = g^{(0)}_{\alpha\beta}$, $\hat{g}_{\alpha\beta} = g^{(1)}_{\alpha\beta}$, etc. The order-zero contribution to the Ricci tensor is constructed from the metric $\bar{g}_{\alpha\beta}$, while the order-zero contribution to the energy-momentum tensor reads

$$\bar{T}_{\alpha\beta} = \bar{F}_\alpha{}^\mu\bar{F}_{\beta\mu} - \tfrac{1}{4}\bar{g}_{\alpha\beta}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}.$$

In turn, the order-one contributions $\hat{R}_{\alpha\beta}$ and $\hat{T}_{\alpha\beta}$ to the Ricci and the energy-momentum tensors have the form

$$\hat{R}_{\alpha\beta} = \tfrac{1}{2}\bar{g}^{\gamma\delta}(\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{g}_{\gamma\delta} - \bar{\nabla}_\gamma\bar{\nabla}_\beta\hat{g}_{\alpha\delta} - \bar{\nabla}_\gamma\bar{\nabla}_\alpha\hat{g}_{\beta\delta} + \bar{\nabla}_\gamma\bar{\nabla}_\delta\hat{g}_{\alpha\beta})$$

$$\hat{T}_{\alpha\beta} = [\hat{K}_{(\alpha}{}^\mu + \hat{F}_{(\alpha}{}^\mu - \tfrac{1}{2}\hat{g}_\rho{}^\mu\bar{F}_{(\alpha}{}^\rho]\bar{F}_{\beta)\mu} - \tfrac{1}{2}\bar{g}_{\alpha\beta}(\hat{K}_{\mu\nu} + \hat{F}_{\mu\nu} - \bar{F}_\mu{}^\rho\hat{g}_{\rho\nu})\bar{F}^{\mu\nu} - \tfrac{1}{4}\hat{g}_{\alpha\beta}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu},$$

where we have defined

$$\hat{K}_{\alpha\beta} = \theta^{\mu\nu}(\bar{F}_{\mu\alpha}\bar{F}_{\nu\beta} - \tfrac{1}{4}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}),$$

and introduced the notation $X_{(\alpha}Y_{\beta)} = X_\alpha Y_\beta + X_\beta Y_\alpha$. In Eq. (3.5) $\hat{H}^{\alpha\beta}$ and $\hat{L}^\beta$ stand for

$$\hat{H}^{\alpha\beta} = \theta^{\mu\nu}(\bar{F}_\mu{}^\alpha \bar{F}_\nu{}^\beta - \tfrac{1}{2}\bar{F}_{\mu\nu}\bar{F}^{\alpha\beta}) - (\theta^{\alpha\mu}\bar{F}^{\beta\nu} - \theta^{\beta\mu}\bar{F}^{\alpha\nu})\bar{F}_{\mu\nu} - \tfrac{1}{4}\theta^{\alpha\beta}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}$$
$$\hat{L}^\beta = \bar{g}^{\mu\beta}\bar{\nabla}_\alpha(\hat{g}_{\mu\nu}\bar{F}^{\alpha\nu}) - \bar{g}^{\mu\alpha}\bar{\nabla}_\alpha(\hat{g}_{\mu\nu}\bar{F}^{\beta\nu}) - \hat{\Gamma}^\alpha_{\alpha\mu}\bar{F}^{\mu\beta},$$

with the order-one contribution $\hat{\Gamma}^\alpha_{\beta\gamma}$ to the Christoffel symbol given by

$$\hat{\Gamma}^{\ \alpha}_{\beta\gamma} = \tfrac{1}{2}\bar{g}^{\alpha\delta}(\bar{\nabla}_\beta\hat{g}_{\delta\gamma} + \bar{\nabla}_\gamma\hat{g}_{\beta\delta} - \bar{\nabla}_\delta\hat{g}_{\beta\gamma}). \tag{3.6}$$

In these expressions indices are lowered, raised and contracted with the zeroth order metric $\bar{g}_{\alpha\beta}$, so that $\bar{F}^{\alpha\beta} = \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\bar{F}_{\mu\nu}$, $\hat{T}^\alpha{}_\alpha = \bar{g}^{\alpha\beta}\hat{T}_{\alpha\beta}$, etc.

It may look at first sight that expanding the field strength $F_{\alpha\beta}$ in Eq. (3.1) in a power series of $\theta^{\alpha\beta}$ clashes with the idea that the first-order noncommutative correction to $\bar{F}_{\alpha\beta}$ is determined by the Seiberg-Witten map. Note, however, that $\hat{T}_{\alpha\beta}$ will in general produce an order-one correction to the metric, which in turn may react back and modify the field strength. Solutions with $\hat{F}_{\alpha\beta} = 0$ will correspond to the case in which the Seiberg-Witten order-one noncommutative correction remains unchanged by gravity and all noncommutative corrections to $\bar{F}_{\alpha\beta}$ are provided by the map itself.

In Eqs. (3.2) and (3.3) we recognize the field equations for the coupling of a conventional Abelian gauge field to gravity. This problem has been extensively studied in the literature, see e.g. [12]. The order-one noncommutative corrections $\hat{g}_{\alpha\beta}$ and $\hat{F}_{\alpha\beta}$ are in turn governed by Eqs. (3.4) and (3.5). We will look for solutions for $\hat{g}_{\alpha\beta}$ of the form

$$\hat{g}_{\alpha\beta} = \hat{\omega}\bar{g}_{\alpha\beta}, \tag{3.7}$$

with $\hat{\omega}$ a function of x^α of order one in $\theta^{\alpha\beta}$. For this ansatz, the first-order contribution to the energy-momentum tensor takes the form

$$\hat{T}_{\alpha\beta} = \left[\hat{K}_{(\alpha}{}^\mu + \hat{F}_{(\alpha}{}^\mu - \frac{\hat{\omega}}{2}\bar{F}_{(\alpha}{}^\mu\right]\bar{F}_{\beta)\mu} - \frac{1}{2}\bar{g}_{\alpha\beta}\left(\hat{K}_{\mu\nu} + \hat{F}_{\mu\nu} - \frac{\hat{\omega}}{2}\bar{F}_{\mu\nu}\right)\bar{F}^{\mu\nu}, \tag{3.8}$$

while, very importantly, $\hat{L}^\beta$ vanishes. The first-order equations (3.4) and (3.5) then read

$$\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{\omega} = \kappa^2\hat{T}_{\alpha\beta} \tag{3.9}$$

$$\bar{\nabla}_\alpha(\hat{F}^{\alpha\beta} + \hat{H}^{\alpha\beta}) = 0. \tag{3.10}$$

The function $\hat{\omega}$ satisfies the constraint $\bar{\nabla}^2\hat{\omega} = 0$. This is so since the order-one contribution to the Ricci tensor is $\hat{R}_{\alpha\beta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{\omega} + \frac{1}{2}\bar{g}_{\alpha\beta}\bar{\nabla}^2\hat{\omega}$ and $\hat{T}_{\alpha\beta}$ is traceless, so that $\hat{R}^\alpha{}_\alpha$ vanishes. The way to proceed is now clear. Take a solution of the order-zero equations (3.2) and (3.3), substitute it in Eqs. (3.9) and (3.10), and solve the resulting equations for $\hat{\omega}$ and $\hat{F}_{\alpha\beta}$. In the next sections we will find solutions to these equations describing noncommutative modifications of pp-waves.

IV. NONCOMMUTATIVE pp-WAVES: GENERAL CONSIDERATIONS

As is known [12], all homogeneous null Einstein-Maxwell fields are represented by the pp-wave spacetime metric

$$ds^2 = 2dudv + 2H(u,x,y)du^2 - dx^2 - dy^2, \tag{4.1}$$

where the coefficient H has one of the following two forms:

$$\text{case 0: } H = \frac{b^2\sigma^2}{2}(x^2+y^2) + a\sigma^2[(x^2-y^2)\cos(2c\sigma u) + 2xy\sin(2c\sigma u)] \tag{4.2}$$

$$\text{case 1: } H = \frac{b^2}{2u^2}(x^2+y^2) + \frac{a}{u^2}[(x^2-y^2) \times \cos(2c\ln(\sigma u)) + 2xy\sin(2c\ln(\sigma u))]. \tag{4.3}$$

The EM field, in turn, has only nonzero components

$$\text{case 0: } \bar{F}_{ux} = \bar{E} = \frac{\sqrt{2}b\sigma}{\kappa}\cos\phi(\sigma u) \qquad \bar{F}_{uy} = \bar{B} = -\frac{\sqrt{2}b\sigma}{\kappa}\sin\phi(\sigma u) \tag{4.4}$$

$$\text{case 1: } \bar{F}_{ux} = \bar{E} = \frac{\sqrt{2}b}{\kappa u}\cos\phi(\ln(\sigma u)) \qquad \bar{F}_{uy} = \bar{B} = -\frac{\sqrt{2}b}{\kappa u}\sin\phi(\ln(\sigma u)). \tag{4.5}$$

In these equations, $u = t - z$, $v = t + z$, x, and y are spacetime coordinates, σ is an arbitrary mass scale, a, b, c are arbitrary real constants, and ϕ is an arbitrary function of its argument. In case 0, the coordinate u may take on arbitrary values, whereas in case 1, one has $u > 0$. The only nonvanishing Christoffel symbols are

$$\bar{\Gamma}^{\ v}_{uu} = \partial_u H \quad \bar{\Gamma}^v_{ui} = \bar{\Gamma}^i_{uu} = \partial_i H \quad (i = x, y)$$

It is worth noting that the change $\sigma u' = \ln(\sigma u)$ in case 1 yields for $\bar{F}_{\alpha\beta}$ and H the same expressions as in case 0, but changes the metric coefficient $\bar{g}_{uv}$ from 1 to $e^{\sigma u'}$. Here we will not make this change and stick to the notation presented above. The reason for this is that it is customary in the literature to write pp-waves in coordinate systems for which the uv-metric coefficient is 1. We finally note that the vector $\bar{\xi}_\alpha = \delta_{\alpha u}$ is null ($\bar{\xi}^2 = 0$) and covariantly constant ($\bar{\nabla}_\alpha\bar{\xi}_\beta = 0$) and that the null surfaces of the metric are $u =$ const. As regards the EM field $\bar{F}_{\alpha\beta}$, it is obvious that it is null ($\bar{F}^{\alpha\beta}\bar{F}_{\alpha\beta} = 0$) and that it needs not be a plane wave, since the function ϕ in (4.4) and (4.5) is arbitrary. Plane EM waves correspond to $\phi(w) = w$.

We take $\bar{g}_{\alpha\beta}$ and $\bar{F}_{\alpha\beta}$ as above in the reference frame in which $\theta^{\alpha\beta}$ has constant components. It is straightforward to show that

$$\theta^{\mu\nu}\bar{F}_{\mu\alpha}\bar{F}_{\nu\beta} = \tfrac{1}{2}\theta^{\mu\nu}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}, \tag{4.6}$$

which in turn implies

$$\hat{K}_{\alpha\beta} = \tfrac{1}{4}\theta^{\mu\nu}\bar{F}_{\mu\nu}\bar{F}_{\alpha\beta}$$
$$\hat{H}_{\alpha\beta} = (\theta_{\alpha}{}^{\mu}\bar{F}^{\nu}{}_{\beta} - \theta_{\beta}{}^{\mu}\bar{F}^{\nu}{}_{\alpha})\bar{F}_{\mu\nu}.$$

Furthermore, using the notation $\theta \cdot L = \theta^{\alpha\beta}L_{\alpha\beta}$ for any tensor $L_{\alpha\beta}$, and defining

$$e_x = \theta^{ux} \qquad e_y = \theta^{uy},$$

so that $\theta \cdot \bar{F} = 2(e_x\bar{E} + e_y\bar{B})$, we have that the only nonvanishing components of $\hat{K}_{\alpha\beta}$ and $\hat{H}_{\alpha\beta}$ are

$$\hat{K}_{ux} = \tfrac{1}{4}(\theta \cdot \bar{F})\bar{E} \qquad \hat{K}_{uy} = \tfrac{1}{4}(\theta \cdot \bar{F})\bar{B}$$
$$\hat{H}_{ux} = (\bar{E}^2 + \bar{B}^2)e_x \qquad \hat{H}_{uy} = (\bar{E}^2 + \bar{B}^2)e_y.$$

We will look for solutions $\hat{\omega}$ that only depend on u. The metric, including first-order noncommutative corrections, is given by

$$g_{\alpha\beta} = (1 + \hat{\omega})\bar{g}_{\alpha\beta}. \tag{4.7}$$

This yields the following order-one nonzero contributions $\hat{\Gamma}^{\alpha}_{\beta\gamma}$ to the Christoffel symbols:

$$\hat{\Gamma}^{u}_{uu} = \partial_u\hat{\omega} \qquad \hat{\Gamma}^{v}_{uu} = -H\partial_u\hat{\omega}$$
$$\hat{\Gamma}^{v}_{uv} = \tfrac{1}{2}\partial_u\hat{\omega} \qquad \hat{\Gamma}^{i}_{uj} = \tfrac{1}{2}\delta^{i}_{j}\partial_u\hat{\omega}.$$

Let us see that the metric (4.7) is a pp-wave. Since, by definition [12], a spacetime metric is a pp-wave if it admits a null and covariantly constant vector, we must find one such vector for the metric (4.7). It is clear that $\xi_\alpha = (1 + \hat{\omega})\delta_{\alpha u}$ does the job. Indeed, nullity $\xi^2 = 0$ is obvious. As regards covariant constancy, recalling that $\bar{\xi}_\alpha = \delta_{\alpha u}$ was covariantly constant with respect to $\bar{g}_{\alpha\beta}$ and noting the expressions for $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ and $\hat{\Gamma}^{\alpha}_{\beta\gamma}$, we have up to order one

$$\nabla_\alpha\xi_\beta = \bar{\nabla}_\alpha\bar{\xi}_\beta + \partial_\alpha(\hat{\omega}\delta_{\beta u}) - \bar{\Gamma}^{\gamma}_{\alpha\beta}\hat{\omega}\delta_{\gamma u} - \hat{\Gamma}^{\gamma}_{\alpha\beta}\delta_{\gamma u} = 0.$$

To write the metric (4.7) in standard pp-wave coordinates, we perform the change $u \to \tilde{u}$, with $\tilde{u}$ defined by the differential equation

$$(1 + \hat{\omega})du = d\tilde{u}. \tag{4.8}$$

The metric then reads

$$ds^2 = 2d\tilde{u}dv + \frac{2}{1 + \hat{\omega}(\tilde{u})}H(\tilde{u}, x, y)d\tilde{u}^2 - [1 + \hat{\omega}(\tilde{u})](dx^2 + dy^2)$$

and has $\tilde{u} = \text{const}$ as null surfaces. In this new coordinate system, noncommutativity does not explicitly enter in the characterization of the null spacetime surfaces. However, the separation between points in different null surfaces ($d\tilde{u} \neq 0$) and between points within the same null surface ($dx, dy \neq 0$) does in general depend on the noncommutative parameters $\theta^{\alpha\beta}$. We will see below explicit realizations of this. Under the coordinate change (4.8), $\hat{F}_{\alpha\beta}$ and $\theta^{\alpha\beta}$ transform as tensors, so that for example $\tilde{\theta}^{ux} = [1 + \hat{\omega}(\tilde{u})]\theta^{ux}$ and $\theta \cdot \bar{F}$ remains invariant.

The explicit form of $\hat{\omega}$ is to be found from the Einstein equations (3.9), which now take the form

$$\frac{d^2\hat{\omega}}{du^2} = \kappa^2\hat{T}_{\alpha\beta} \quad \text{if } \alpha = \beta = u \tag{4.9}$$

$$0 = \hat{T}_{\alpha\beta} \qquad \text{otherwise.} \tag{4.10}$$

Concerning the noncommutative corrections $\hat{F}_{\alpha\beta}$ to the EM field, it is straightforward to see that the expression (3.8) for $\hat{T}_{\alpha\beta}$ and Eq. (4.10) imply that, except for $\hat{F}_{ux}$ and $\hat{F}_{uy}$, all other components of $\hat{F}_{\alpha\beta}$ vanish. Hence, $F_{\alpha\beta} = \bar{F}_{\alpha\beta} + \hat{F}_{\alpha\beta}$ has F_{ux} and F_{uy} as only nonzero components, and is null for (4.7).

V. NONCOMMUTATIVE pp-WAVES: EXPLICIT SOLUTIONS

In this section we consider some solutions for $\hat{F}_{\alpha\beta}$ of physical interest. We will see that, while the solution for $\hat{\omega}$ is unique, there is a large arbitrariness in the solution for $\hat{F}_{\alpha\beta}$.

A. Vacuum noncommutative metric corrections

Let us first consider $\hat{T}_{\alpha\beta} = 0$, corresponding to vacuum noncommutative metric corrections. For $\hat{T}_{\alpha\beta} = 0$, we demand

$$\hat{F}_{\alpha\beta} = \frac{\hat{\omega}}{2}\bar{F}_{\alpha\beta} - \hat{K}_{\alpha\beta}. \tag{5.1}$$

Now, $2\hat{K}_{\alpha\beta}$ and $\hat{H}_{\alpha\beta}$ are not energetically distinguishable from each other, since they both give the same contribution to the energy-momentum tensor. We therefore replace $2\hat{K}_{\alpha\beta}$ in Eq. (5.1) with a linear combination of $2\hat{K}_{\alpha\beta}$ and $\hat{H}_{\alpha\beta}$ that gives the same contribution to $\hat{T}_{\alpha\beta}$ as $2\hat{K}_{\alpha\beta}$. That is,

$$\hat{F}_{\alpha\beta} = \frac{\hat{\omega}}{2}\bar{F}_{\alpha\beta} - \frac{1}{2}[2a_1\hat{K}_{\alpha\beta} + (1 - a_1)\hat{H}_{\alpha\beta}]. \tag{5.2}$$

Here a_1 may even be regarded, not as a constant, but as a function of u, for it does not enter $\hat{T}_{\alpha\beta} = 0$. Following this line of argumentation, one could think of including in $\hat{F}_{\alpha\beta}$ other antisymmetric two-tensors with the same contribution to the energy-momentum tensor, or with different contributions but such that they cancel among themselves. Using Eq. (4.6) and

$$\bar{F}_{\alpha\mu}\bar{F}^{\mu}{}_{\beta} = (\bar{E}^2 + \bar{B}^2)\delta_{\alpha u}\delta_{\beta u} \qquad \bar{F}_{\alpha\mu}\bar{F}^{\mu\nu}\bar{F}_{\nu\beta} = 0$$
$$\theta^{\mu\nu}(\bar{F}_{\mu\alpha}\bar{F}_{\beta\rho} - \bar{F}_{\mu\beta}\bar{F}_{\alpha\rho})\bar{F}_{\nu}{}^{\rho} = 0,$$

it is not difficult to see, however, that the only nonzero antisymmetric two-tensors that can be formed with one $\theta^{\alpha\beta}$ and two or more $\bar{F}_{\alpha\beta}$ are $\hat{K}_{\alpha\beta}$ and $\hat{H}_{\alpha\beta}$. Hence we stop at (5.2), which can be recast as

$$\hat{F}_{\alpha\beta} = \frac{1}{2}\left(\hat{\omega} - \frac{a_1}{2}\theta\cdot\bar{F}\right)\bar{F}_{\alpha\beta} - \frac{1}{2}(1-a_1)\hat{H}_{\alpha\beta}. \tag{5.3}$$

For this ansatz, the field equation (3.10) is trivially satisfied for all a_1 and the Einstein equation (4.9) takes the form $\hat{\omega}'' = 0$, where the prime denotes differentiation with respect to u. In what follows we show that its solution is

$$\hat{\omega} = (\theta\cdot\bar{f})(k_1 + k_2\sigma u), \tag{5.4}$$

where k_1 and k_2 are arbitrary real constants and $\theta\cdot\bar{f}$ is the contraction of $\theta^{\alpha\beta}$ with an antisymmetric tensor $\bar{f}_{\alpha\beta}$ whose only nontrivial components are

$$\bar{f}_{ux} = \frac{b\sigma}{\kappa}\cos\eta \qquad \bar{f}_{uy} = \frac{b\sigma}{\kappa}\sin\eta, \tag{5.5}$$

with η a real arbitrary constant. To prove Eq. (5.4), we proceed in three steps.

Step 1.—We first note that the solution to $\hat{\omega}''=0$ is $\hat{\omega} = \hat{k}_1 + \hat{k}_2 u$, with $\hat{k}_1$ and $\hat{k}_2$ integration constants. Since $\hat{k}_1$ and $\hat{k}_2$ are of order one in $\theta^{\alpha\beta}$, they must be contractions of $\theta^{\alpha\beta}$ with antisymmetric tensors of mass dimension 2 taking constant values in the reference frame that we are considering.

Step 2.—Next we show that all tensors of that type for the background metric (4.1) have the form (5.5). To this end, we recall [13] that covariantly constant null antisymmetric two-tensors exist if and only if spacetime is a pp-wave. In addition, metrics that cannot be decomposed into the product of two two-dimensional metrics do not admit covariantly constant non-null antisymmetric two-tensors [14]. Since these two conditions concur for the metric (4.1), all covariantly constant antisymmetric two-tensors in our case are null. Let us take one such tensor, which we denote by $\bar{f}_{\alpha\beta}$, and demand it to be constant, so that $\partial_\gamma\bar{f}_{\alpha\beta} = 0$. Covariant constancy $\bar{\nabla}\gamma\bar{f}_{\alpha\beta} = 0$ then reduces to

$$\bar{\Gamma}^{\mu}_{\gamma\alpha}\bar{f}_{\mu\beta} + \bar{\Gamma}^{\mu}_{\gamma\beta}\bar{f}_{\alpha\mu} = 0.$$

We remind ourselves at this point that a constant and at the same time covariantly constant null antisymmetric two-tensor $\bar{f}_{\alpha\beta}$ can be written as [12]

$$\bar{f}_{\alpha\beta} = \bar{q}_\alpha\bar{p}_\beta - \bar{p}_\alpha\bar{q}_\beta$$

with $\bar{q}_\alpha$ and $\bar{p}_\alpha$ null and spacelike constant vectors satisfying

$$\bar{q}^2 = 0 \qquad \bar{\Gamma}^{\gamma}_{\alpha\beta}\bar{q}_\gamma = 0 \qquad \bar{p}^2 = -1 \qquad \bar{q}\cdot\bar{p} = 0.$$

It is now simple algebra to prove that the only solution to these equations is $\bar{q}_\alpha = (q,0,0,0)$ and $\bar{p}_\alpha = (0,0,\cos\eta,\sin\eta)$, with q and η arbitrary real numbers.

Step 3.—We observe that the constants $\hat{k}_1$ and $\hat{k}_2$ must be zero if the EM background $\bar{F}_{\alpha\beta}$ vanishes, for then one is left with the usual Einstein equations in vacuum, in which no $\theta^{\alpha\beta}$ is involved. Based on this and the observation that $\bar{F}_{\alpha\beta}$ has the form of $\bar{f}_{\alpha\beta}$ constructed in step 2, we take $q = b\sigma/\kappa$ and obtain (5.5).

Let us now go back to $\hat{\omega}$ in Eq. (5.4). Performing the change $u\to\tilde{u}$ in Eq. (4.8), we obtain

$$1 + \hat{\omega} = \sqrt{[1 + k_1(\theta\cdot\bar{f})]^2 + 2k_2(\theta\cdot\bar{f})\sigma\tilde{u}}.$$

For $k_2 = 0$ noncommutativity is felt in the same way in all null surfaces $\tilde{u} = \text{const}$. On the other hand, for $k_2 \neq 0$, null surfaces sort of dilate, since up to order one, the distance between two points on the same null surface $\tilde{u} = \tilde{u}_0$ is

$$ds^2_{\rm null} = [1 + (\theta\cdot\bar{f})(k_1 + k_2\sigma\tilde{u}_0)](dx^2 + dy^2).$$

We postpone to Sec. VI a detailed discussion of the noncommutative correction $\hat{F}_{\alpha\beta}$ to the EM background field given by Eq. (5.3).

B. Nonvacuum noncommutative metric corrections

We next consider for $\hat{F}_{\alpha\beta}$ the ansatz

$$\hat{F}_{\alpha\beta} = \tfrac{1}{2}(\hat{\omega} - \hat{\varphi})\bar{F}_{\alpha\beta} - \tfrac{1}{2}(1-a_1)\hat{H}_{\alpha\beta}, \tag{5.6}$$

where $\hat{\varphi}$ is a function of u or order one in $\theta^{\alpha\beta}$ to be determined. This $\hat{F}_{\alpha\beta}$ is a nonvacuum generalization of that considered in Eq. (5.3). With respect to the latter, we have replaced the prefactor $\theta\cdot\bar{F}$ in front of $\bar{F}_{\alpha\beta}$ with a function $\hat{\varphi}$. The motivation for doing this is that $\hat{\omega}$ may contribute, through the Einstein equation (4.9), to $\hat{F}_{\alpha\beta}$ with a term proportional to $\bar{F}_{\alpha\beta}$ whose prefactor may not be of the form $\theta\cdot\bar{F}$, and the idea is that $\hat{\varphi}$ accounts for such a contribution. It is straightforward to see that the field equation (3.10) holds trivially for all $\hat{\varphi}$ and a_1. By suitably choosing $\hat{\varphi}$ and a_2, we may construct a large variety of solutions. Let us consider some of them.

Example 1.—Assume $\hat{\varphi} = 0$. If $a_1 = 0$ we fall into the case studied in the previous subsection, so we will consider here $a_1 \neq 0$. The Einstein equation (4.9) then reduces to

$$\frac{d^2\hat{\omega}}{du^2} = -a_1\kappa^2(\bar{E}^2 + \bar{B}^2)(\theta\cdot\bar{F}).$$

Its solution is given by

$$\hat{\omega} = (\theta \cdot \bar{f})(k_1 + k_2 \sigma u) + \hat{\omega}_{\rm p},$$

with $\hat{\omega}_{\rm p}$ a particular solution of the complete equation. Different choices for ϕ in Eqs. (4.4) and (4.5) will yield different particular solutions. Of special relevance are plane EM waves with wave fronts $u = $ const, for which $\phi(w) = w$. In this case, it is straightforward to see then that $\hat{\omega}_{\rm p}$ is given by

$$\text{case 0: } \hat{\omega}_{\rm p} = 2a_1 b^2 (\theta \cdot \bar{F})$$

$$\text{case 1: } \hat{\omega}_{\rm p} = -\frac{a_1 b^2}{5}(\theta \cdot \bar{F} + 3\theta \cdot \tilde{\bar{F}}),$$

where $\theta \cdot \tilde{\bar{F}}$ is the contraction of $\theta^{\alpha\beta}$ with the Hodge dual of $\bar{F}_{\alpha\beta}$. If we now substitute the solution for $\hat{\omega}$ in the expression (5.6) for $\hat{F}_{\alpha\beta}$, we again obtain an expression with qualitatively the same terms as in Eq. (5.3).

Example 2.—Take now $\hat{\varphi} = \hat{\omega} + (\frac{1}{2}a_1 + a_2)\theta \cdot \bar{F}$, with a_2 an arbitrary real constant. The Einstein equation (4.9) then takes the form

$$\frac{d^2 \hat{\omega}}{du^2} - \kappa^2 (\bar{E}^2 + \bar{B}^2)\hat{\omega} = a_2 \kappa^2 (\bar{E}^2 + \bar{B}^2)(\theta \cdot \bar{F}).$$

This is again a second order differential equation that can be solved without difficulty. For a plane wave EM background, the solution is given by

$$\text{case 0: } \hat{\omega} = (\theta \cdot \bar{f})(k_1 e^{\sqrt{2}b\sigma u} + k_2 e^{-\sqrt{2}b\sigma u}) - \frac{2a_2 b^2}{(2b^2 + 1)}\theta \cdot \bar{F}$$

$$\text{case 1: } \hat{\omega} = (\theta \cdot \bar{f})[k_1 (\sigma u)^{\lambda_+} + k_2 (\sigma u)^{\lambda_-}] - \frac{a_2 b^2}{5 - 2b^2 + 2b^4}[(2b^2 - 1)\theta \cdot \bar{F} - 3\theta \cdot \tilde{\bar{F}}],$$

with $2\lambda_\pm = 1 \pm \sqrt{1 + 8b^2}$. In case 0, the range of variation for the coordinate u goes from $-\infty$ to $+\infty$. The homogeneous part of $\hat{\omega}$ then dominates and as $u \to \pm\infty$ the null surfaces inflate. This is in a way a runaway configuration, since $\hat{\omega}$ ends up growing indefinitely. Similar considerations apply to case 1 for $u \to 0, \infty$. The main difference for $\hat{F}_{\alpha\beta}$ with the solutions previously studied is that now there is not a linear term in $\bar{F}_{\alpha\beta}$. Note, as a particular case, that if $a_1 = 1$ and $a_2 = -\frac{1}{2}$, $\hat{F}_{\alpha\beta}$ in (5.6) vanishes, so that the gravitational field does not react back on the EM background.

VI. CONSTITUTIVE RELATIONS AND COMPARISON WITH NONCOMMUTATIVE ELECTROMAGNETIC PLANE WAVES IN FLAT SPACETIME

The complete EM field, which we denote by $F_{\alpha\beta}$, is the sum of the order-zero $\bar{F}_{\alpha\beta}$ and order-one $\hat{F}_{\alpha\beta}$ contributions. For the sake of concreteness, we will consider the case discussed in Sec. VA and furthermore we will take (4.2) and (4.4) as background. Similar considerations apply to other cases. Putting together the background EM contributions and the noncommutative corrections in Eq. (5.3), we obtain

$$D = F_{ux} = \bar{E} + \frac{\hat{\omega}}{2}\bar{E} - \frac{a_1}{4}(\theta \cdot \bar{F})\bar{E} - (1 - a_1)\frac{b^2 \sigma^2}{\kappa^2} e_x \tag{6.1}$$

$$H = F_{uy} = \bar{B} + \frac{\hat{\omega}}{2}\bar{B} - \frac{a_1}{4}(\theta \cdot \bar{F})\bar{B} - (1 - a_1)\frac{b^2 \sigma^2}{\kappa^2} e_y \tag{6.2}$$

as the only nonvanishing components of the complete EM field. These define the displacement **D** and induction **H** vectors in the same way that the components of $\bar{F}_{\alpha\beta}$ define the electric **E** and magnetic **B** background fields. Indeed, in a coordinate system (t, x, y, z), **E** and **B** have Cartesian components $E_i = \bar{F}_{0i}$ and $B_i = -\bar{F}_{jk}$ with ijk a cyclic permutation of 123. Similarly, for the components of **D** and **H** we have $D_i = F_{0i}$ and $H_i = -F_{jk}$. Equations (6.1) and (6.2) are thus constitutive relations and can be easily inverted. Let us analyze the terms occurring in them.

The first term in Eqs. (6.1) and (6.2) is the EM background $\bar{F}_{\alpha\beta}$ contribution. The second one arises from $\hat{\omega}\bar{F}_{\alpha\beta}$ in Eq. (5.3) and is a susceptibility contribution due to gravity, with susceptibility coefficient $\hat{\omega}$. This contribution is inhomogeneous ($\hat{\omega}$ is a function of u) and depends on the properties of the gravitational field ($\hat{\omega}$ is proportional to $\theta \cdot \bar{f}$, with $\bar{f}_{\alpha\beta}$ determined by the pp-wave background geometry). The third term in (6.1) and (6.2) comes from $(\theta \cdot \bar{F})\bar{F}_{\alpha\beta}$ in (5.3), is quadratic in the background EM field, and has its origin in the Seiberg-Witten map, since this was used to construct the classical action and hence enters the Einstein and field equations. Even more, the order-one contribution to the field strength provided by the Seiberg-Witten map

$$\theta^{\mu\nu}[\bar{F}_{\mu\alpha}\bar{F}_{\nu\beta} - (\partial_\mu \bar{F}_{\alpha\beta})\bar{A}_\nu]$$

can be recast for our pp-wave background geometry as

$$(\theta \cdot \bar{F})\bar{F}_{\alpha\beta} - \frac{1}{\sqrt{-g}}\bar{\nabla}_\mu(\theta^{\mu\nu}\bar{A}_\nu \bar{F}_{\alpha\beta}),$$

which, modulo the total derivative in the second term, is the nonlinear contribution that we are discussing. Finally, the last contribution in (6.1) and (6.2) can be regarded as a constant polarization/induction that can be tuned at will by adjusting a_1. This contribution is dynamically generated and arises because, as already mentioned in Sec. V B, $2\hat{K}_{\alpha\beta}$ and $\hat{H}_{\alpha\beta}$ cannot be energetically distinguished from each other and both satisfy the order-one field equation (3.10). Even if one takes $\phi = 0\,(\frac{\pi}{2})$, so that the magnetic (electric) component $\bar{B}\,(\bar{E})$ of the EM background is zero, the complete EM field acquires through this term an induction (displacement) component proportional to $e_y\,(e_x)$.

The nonlinear contribution in Eqs. (6.1) and (6.2) is responsible for harmonic generation similar to that in nonlinear optics. To understand this, we consider this contribution together with the last one in (6.1) and (6.2), and take $\phi(\sigma u) = \sigma u$ in the background EM field. Doing so, we obtain

$$\text{3rd} + \text{4th contributions} = \left(\frac{a_1}{2} - 1\right)\frac{b^2\sigma^2}{\kappa^2}\begin{bmatrix} e_x \\ e_y \end{bmatrix} + \frac{a_1}{2}\frac{b^2\sigma^2}{\kappa^2}\begin{bmatrix} e_y\sin(2\sigma u) - e_x\cos(2\sigma u) \\ e_x\sin(2\sigma u) + e_y\cos(2\sigma u) \end{bmatrix}.$$

The first term on the right-hand side is again a constant polarization/induction contribution, whereas the second one describes an EM plane wave propagating in the same direction as the background wave but with twice its frequency. Hence, harmonic generation and constant polarization/induction cannot be distinguished energetically from each other and one can "move" between them by tuning a_1.

It is important to remark that our solutions for the complete EM field, though similar, are of a different type to those discussed in the literature for flat spacetime [9,15]. The difference goes beyond the explicit form of the solutions and affects the nature of the EM waves. The susceptibility contribution due to gravity that we have found is absent in the flat spacetime solutions. The nonlinear contribution, having its origin in the Seiberg-Witten map, is present in both cases. Finally, whereas in our solution there is a dynamically generated constant polarization/induction at order one in $\theta^{\alpha\beta}$, in the flat spacetime solutions presented in [9,15] there is not such a contribution. One must bear in mind, though, that the argument above for harmonic generation shows that the nonlinear contribution can be split into a constant piece and a higher harmonic term. The difference is that whereas in our case the coefficient of the constant contribution is $(a_1 - 2)/2$, with a_1 arbitrary, in flat spacetime it takes the value $-1/2$.

By contrast, the flat spacetime solution for the EM field has an order-zero constant contribution. This generates through the nonlinear term a longitudinal component for the electric and magnetic fields which propagates at different velocity than the transverse components [9,16]. None of this happens in our case, since such an order-zero contribution cannot occur in our background solution. The reason for this is that if a constant contribution is added to a given background solution $\bar{F}_{\alpha\beta}$, the Einstein equation does not hold and such a background is not acceptable.

VII. CONCLUSION AND OUTLOOK

In this paper we have considered the noncommutative coupling of gravity to an EM field as described by Seiberg-Witten–type formal series in the noncommutativity parameters $\theta^{\alpha\beta}$. We have constructed a very large class of null pp-waves that solve the corresponding field equations up to order one in $\theta^{\alpha\beta}$. These solutions can thus be regarded as the first-order noncommutative contributions to full noncommutative Einstein-Maxwell pp-waves. To date, there are several extensions of noncommutative gravity [3,4,6,8], all of them sharing the property that corrections to the Einstein-Hilbert action start at order two in $\theta^{\alpha\beta}$. Barring thus the eventuality that some specific symmetries might get lost through the coupling to a SW gauge field, our results do not depend on the particular model chosen for noncommutative gravity.

In our solutions, the noncommutativity parameters $\theta^{\alpha\beta}$ enter the pp-wave metric through a conformal factor $\hat{\omega}$ that depends on the null coordinate and which is obtained by solving a linear second order differential equation. As a result, the distance between points in the same null surface is modified and grows indefinitely for asymptotic values of the null coordinate. Concerning the EM field, it receives types of noncommutative corrections: a susceptibility contribution caused by gravity, a constant polarization/induction contribution that can be tuned at will, and a nonlinear contribution similar to those in nonlinear optics.

It would also be interesting to investigate the relation of the model and the solutions presented here with the exact Seiberg-Witten maps studied in Ref. [17] and their Dirac-Born-Infeld low energy effective actions. Another question lying ahead is the generalization to higher dimensions, especially in relation to compactified extra dimensions, for these may lower the energy scale at which gravity, and hopefully the noncommutativity scale, is felt [18].

We note that, so far, noncommutative corrections to conventional pp-waves at order one in $\theta^{\alpha\beta}$ are not known for pure gravity. This is in part due to the fact that noncommutative corrections in the available noncommutative generalizations of general relativity start at order two [4,8]. Technically, the EM sector that we have introduced provides an order-one source for gravity, which in turn reacts back modifying the order-one Seiberg-Witten noncommutative correction to the EM field. The intractability of the order-two corrections to the Einstein-Hilbert action makes one look for other ways to approach noncommutativity corrections to general relativity solutions. The result obtained here, namely, that noncommutativity goes into the metric through a conformal factor, suggests approaching noncommutative pp-waves in pure gravity by considering the Seiberg-Witten limit in string scenarios, with the dilaton accounting for noncommutativity.

ACKNOWLEDGMENTS

Partial support by the BMFB, Germany, under Project No. 05HT4PSA/6 and from CICyT and UCM-CAM, Spain, through Grants No. FIS2005-02309 and No. 910770 is acknowledged.

[1] S. Doplicher, K. Fredenhagen, and J.E. Roberts, Commun. Math. Phys. **172**, 187 (1995).

[2] R.J. Szabo, hep-th/0606233.

[3] J.W. Moffat, Phys. Lett. B **491**, 345 (2000); A.H. Chamseddine, Commun. Math. Phys. **218**, 283 (2001); H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, Phys. Rev. D **68**, 044015 (2003); H. Nishino and S. Rajpoot, Phys. Lett. B **532**, 334 (2002); D.V. Vassilevich, Nucl. Phys. **B715**, 695 (2005); A. Kobakhidze, hep-th/0603132; C. Deliduman, hep-th/0607096; E. Harikumar and V.O. Rivelles, hep-th/0607115; A.H. Chamseddine, Int. J. Geom. Methods Mod. Phys. **3**, 149 (2006); Phys. Lett. B **504**, 33 (2001).

[4] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, and J. Wess, Classical Quantum Gravity **22**, 3511 (2005); P. Aschieri, M. Dimitrijević, F. Meyer, and J. Wess, Classical Quantum Gravity **23**, 1883 (2006).

[5] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.

[6] L. Alvarez-Gaumé, F. Meyer, and M.A. Vázquez-Mozo, Nucl. Phys. **B753**, 92 (2006).

[7] P. Nicolini, A. Smailagic, and E. Spalucci, Phys. Lett. B **632**, 547 (2006); T.G. Rizzo, J. High Energy Phys. 09 (2006) 021.

[8] P. Mukherjee and A. Saha, Phys. Rev. D **74**, 027702 (2006); X. Calmet and A. Kobakhidze, Phys. Rev. D **74**, 047702 (2006).

[9] Z. Guralnik, R. Jackiw, S.Y. Pi, and A.P. Polychronakos, Phys. Lett. B **517**, 450 (2001).

[10] D. Colladay and V.A. Kostelecký, Phys. Rev. D **58**, 116002 (1998); J.M. Gracia-Bondía, F. Lizzi, F. Ruiz Ruiz, and P. Vitale, Phys. Rev. D **74**, 025014 (2006); **74**, 029901(E) (2006

[11] V. Gayral, J.M. Gracia-Bondía, and F. Ruiz Ruiz, Nucl. Phys. **B727**, 513 (2005).

[12] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).

[13] J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 49.

[14] R. Debever and M. Cahen, C.R. Acad. Sci. (Paris) **251**, 1160 (1960). See also Ref. [12], p. 555.

[15] Y. Abe, R. Banerjee, and I. Tsutsui, Phys. Lett. B **573**, 248 (2003); N. Chatillon and A. Pinzul, hep-ph/0607243.

[16] R.-G. Cai, Phys. Lett. B **517**, 457 (2001).

[17] R. Banerjee and H.S. Yang, Nucl. Phys. **B708**, 434 (2005).

[18] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **436**, 257 (1998); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).