A Crossing–Symmetric Phase
for $AdS_5 \times S^5$ Strings

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Abstract

We propose an all-order perturbative expression for the dressing phase of the $AdS_5 \times S^5$ string S-matrix at strong coupling. Moreover, we are able to sum up large parts of this expression. This allows us to start the investigation of the analytic structure of the phase at finite coupling revealing a few surprising features. The phase obeys all known constraints including the crossing relation and it matches with the known physical data at strong coupling. In particular, we recover the bound states of giant magnons recently found by Hofman and Maldacena as poles of the scattering matrix. At weak coupling our proposal seems to differ with gauge theory. A possible solution to this disagreement is the inclusion of additional pieces in the phase not contributing to crossing, which we also study.

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1   Introduction

The quantum description of type IIB string theory on $AdS_5 \times S^5$ remains a challenge because quantisation of the Metsaev-Tseytlin action [1] in the conformal gauge faces a number of intricate problems. Insight into a way to overcome this obstacle has arisen along the last years from the observation that the classical sigma model for the string on $AdS_5 \times S^5$ is integrable [2]. Integrability of the string implies that it admits a Lax connection, and allows a resolution of the spectrum of classical strings in terms of spectral curves [3]. The integral equations satisfied by the spectral density for the monodromy of the Lax connection remind of a thermodynamic limit of some Bethe equations, and a set of discrete Bethe equations for the quantum string sigma model were in fact suggested in [4, 5]. Integrable structures also arise on the gauge theory side of the AdS/CFT correspondence because the leading planar dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang-Mills has been identified with the Hamiltonian of an integrable spin chain [6]. Moreover, integrability has also been shown to hold at higher loops in some restricted sectors [7–10]. Assuming that integrability holds, Bethe equations have been proposed as an efficient means to describe the spectrum of $\mathcal{N} = 4$ Yang-Mills operators. For further details and references we would like to refer the reader to the reviews [11, 12].

The asymptotic Bethe ansätze for gauge and string theory are very similar, and the asymptotic S-matrices on each side of the correspondence differ simply by a scalar factor [10,5]. Actually, this is as much as they could possibly differ: The two-excitation S-matrix of an infinite spin chain system with the $\mathfrak{psu}(2, 2|4)$ symmetry that characterises both string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills can be fixed up to a scalar factor [13]

$$\mathcal{S}_{12} = S_{12}^0 \, \mathcal{S}_{12}^{\mathfrak{su}(2|2)} \, \mathcal{S}_{12}^{\mathfrak{su}(2|2)\prime} \; . \tag{1.1}$$

The spin chain vacuum breaks the $\mathfrak{psu}(2, 2|4)$ symmetry algebra down to $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$, where $\mathbb{R}$ represents a shared central charge [11]. In order to describe elementary excitations of the chain, it is necessary to extend the unbroken symmetry algebra with two central charges. The symbol $\mathcal{S}_{12}^{\mathfrak{su}(2|2)}$ denotes the uniquely fixed flavour structure of the S-matrix for each centrally extended $\mathfrak{su}(2|2)$ sector.

The determination of the scattering matrix by the symmetries up to a scalar factor is not unique to the AdS/CFT chain, but a generic fact in integrable systems [14]. In order to determine the dressing factor, additional dynamical information, such as crossing symmetry for relativistic systems, is required. The status of crossing symmetry in the AdS/CFT context is not a priori clear since the dispersion relation of elementary excitations does not have precise relativistic invariance. However it has been argued that crossing symmetry should still hold for strings on $AdS_5 \times S^5$ [15]. A strong argument in favour is that a purely algebraic implementation of crossing symmetry based on the underlying Hopf algebra structure of integrable systems, known to work in well studied examples, leads to a non-trivially consistent picture [15, 16]. Furthermore, it was shown in [13] that the constraint from crossing symmetry is equivalent to a certain bootstrap condition implying that a particle-hole pair should scatter trivially. Moreover, the classical string phase factor [4] plus its one-loop string sigma model quantum correction [17] have been shown to satisfy crossing to the appropriate order [18]. Recently, a function

satisfying crossing has been presented in [19]. The aim of this work is a proposal for a general dressing factor which obeys the crossing relation found in [15] and agrees with the available perturbative data from string theory.

The plan of the paper is as follows. In section 2 we will review the description of elementary excitations developed in [13] and the formulation of crossing symmetry [15] in order to make our presentation self-contained. We will end this section with a discussion of the separate kinematical regimes that characterise the strong and weak coupling limits. In section 3 we will describe the dressing phase of the scattering matrix in terms of a perturbative series at strong coupling and propose a concrete expression for the coefficients that govern the series. These coefficients are a natural extension of those determining the classical dressing phase [4] and its one-loop correction [17]. We will provide evidence that the proposed series satisfies crossing symmetry. An analytic expression is presented in section 4 and argued to represent the resummed series. Using this result we are able to identify the bound states of giant magnons recently found in [20] as poles of the scattering matrix. Although our dressing phase was constructed to satisfy the main physical requirements on the string side, such as crossing symmetry, we should stress that it does not correctly connect with gauge theory in the weak coupling regime. A possible way out is the addition of a homogeneous solution of the crossing equation to the dressing phase. The study of homogeneous solutions is addressed in section 5. In section 6 we discuss our results and comment on the many open issues. The paper concludes with two appendices that collect useful formulae for the weak and strong coupling expansions.

2 Particle Model

We will start by reviewing the model of physical excitations above a half-BPS vacuum state. This section describes the setup as well as the basic definitions and conventions to be used in later sections of this paper.

2.1 Setup

States in type IIB string theory are naturally described by a set of 8 bosonic and 8 fermionic excitations propagating on a circle. For $\mathcal{N} = 4$ gauge theory the setup is similar except that the circle is replaced by a periodic spin chain [21]. The vacuum state is a protected half-BPS state in both cases and each particle has an associated momentum p_k. Due to the compactness of the circle or the spin chain, the spectrum of states is discrete. Discreteness is achieved by imposing quantisation conditions on the particle momenta.

However, it is more convenient to replace the circle with an infinite line. This relaxes the quantisation condition of particle momenta and makes the spectrum continuous. To recover the circle we need to impose periodicity conditions on the multi-particle wave function, the so-called Bethe equations. The Bethe equations rely on the scattering matrix $\mathcal{S}$ of particles on the infinite line [10].

The symmetry of the full model is $\mathfrak{psu}(2, 2|4)$ and a subalgebra $\mathbb{R} \ltimes \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$

preserves the particle numbers [11]. The external automorphism is the $\mathfrak{so}(6)$ charge J and the central charge C measures the energy of a state. On the infinite line, the residual algebra enlarges by two central charges $\mathbb{R} \ltimes \mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ [13]. These central charges describe the momentum of a particle.

2.2 Particles

A particle is described by its momentum p, energy C (alias the $\mathfrak{su}(2|2)$ central charge) and its flavour. There are sixteen particle flavours which form a multiplet of the residual symmetry. The momentum and energy are related by the dispersion relation [22]

$$4C^2 - 16g^2 \sin^2(\tfrac{1}{2}p) = 1 \ , \tag{2.1}$$

where g is proportional to the square root of the 't Hooft coupling constant,

$$g = \frac{\sqrt{\lambda}}{4\pi} \ . \tag{2.2}$$

This dispersion relation is in fact an atypicality condition for a short multiplet of the residual algebra [13] and thus appears to be protected from quantum corrections.

This equation is neither a standard lattice nor a standard relativistic dispersion relation, but it shares features of both: It is periodic in the momentum p, i.e. it has the Brillouin zones of a discrete system. It is also relativistic if we consider $\sin(\tfrac{1}{2}p)$ (rather than p) to be the relevant relativistic momentum. These two properties square nicely with the observation that the kinematic space of the elementary excitations defines a complex torus [15]. The torus has two non-trivial cycles, let us call them "real" and "imaginary". The real cycle corresponds to periodicity of the momentum p for a lattice model. The imaginary cycle corresponds to periodicity of the mass shell condition (2.1) for imaginary relativistic momentum, i.e. $(2C)^2 + (4ig \sin(\tfrac{1}{2}p))^2 = 1$ defines a unit circle.

We will use complex variables $x^\pm$ to codify the momentum p and energy C of physical excitations via

$$e^{ip} = \frac{x^+}{x^-} \ , \qquad C = \frac{1}{2} + \frac{ig}{x^+} - \frac{ig}{x^-} \ . \tag{2.3}$$

These two variables are subject to the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \ , \tag{2.4}$$

which is equivalent to the dispersion relation (2.1). Furthermore we would like to introduce the auxiliary variable u as

$$u = x^+ + \frac{1}{x^+} - \frac{i}{2g} = x^- + \frac{1}{x^-} + \frac{i}{2g} \ . \tag{2.5}$$

Finally, we shall present two relevant discrete symmetries. One of them is parity which maps the particle variables as follows

$$x^\pm \mapsto -x^\mp, \qquad p \mapsto -p, \qquad C \mapsto C, \qquad u \mapsto -u \ . \tag{2.6}$$

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For the definition of crossing symmetry we will furthermore need the antipode map between particles and particle-holes given by

$$x^\pm \mapsto \frac{1}{x^\pm}, \qquad p \mapsto -p, \qquad C \mapsto -C, \qquad u \mapsto u. \qquad (2.7)$$

2.3   Scattering and Crossing

The pairwise scattering matrix $\mathcal{S}_{12}$ for the above particles was derived in [13]. It is fully constrained by symmetry up to a scalar prefactor S_{12}^0. We will not need its full form here, but only consider the scalar factor. It will be convenient for us to use the definition in terms of the dressing factor $\sigma_{12} = \sigma(x_1^\pm, x_2^\pm)$ and dressing phase θ_{12} introduced in [5]

$$S_{12}^0 = (\sigma_{12})^2 \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-}, \qquad \sigma_{12} = \exp(i\theta_{12}) . \qquad (2.8)$$

The aim of the present paper is to derive an expression for the dressing phase θ_{12} consistent with string theory. Note that in our conventions the dressing phase appears with a factor of $+2i$ in the exponent of the $\mathfrak{psu}(2,2|4)$ scattering matrix

$$\mathcal{S}_{12} \sim \exp(2i\theta_{12}) . \qquad (2.9)$$

A constraint on the form of σ_{12} is gained by imposing crossing symmetry [15]. Crossing symmetry relates the two-particle S-matrix with the S-matrix for a particle and a particle-hole. Assuming crossing symmetry holds, the following constraint on the dressing factor $\sigma_{12} = \sigma(x_1^\pm, x_2^\pm)$ was derived in [15],

$$\sigma_{12} \, \sigma_{\bar{1}2} = h_{12} , \qquad (2.10)$$

where the bar stands for the replacement of a particle by a particle-hole, cf. (2.7). The function h is given by [18]

$$h_{12} = \frac{x_2^-}{x_2^+} \frac{x_1^- - x_2^+}{x_1^+ - x_2^+} \frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-} . \qquad (2.11)$$

The crossing relation (2.10) looks superficially puzzling because the l.h.s. is naively symmetric under the particle-hole interchange, while the r.h.s. is not. It was shown in [15] that the operation $x^\pm \mapsto \bar{x}^\pm = 1/x^\pm$ corresponds to a displacement on the imaginary cycle of the complex torus by half a period. Therefore applying twice this operation we move once around a non-trivial cycle of the torus, which can result in a non-trivial monodromy, i.e. $x^\pm \mapsto \bar{x}^\pm \mapsto \bar{\bar{x}}^\pm$. This is indeed the case since (2.11) implies

$$\sigma_{\bar{\bar{1}}2} = \frac{h_{\bar{\bar{1}}2}}{h_{12}} \sigma_{12} \neq \sigma_{12} . \qquad (2.12)$$

The operation $x^\pm \mapsto \bar{\bar{x}}^\pm$, which is superficially the identity map, has the interpretation of a change in Riemann sheet for the function $\sigma_{12} = \sigma(x_1^\pm, x_2^\pm)$. It is therefore instructive to split the dressing factor into an "odd", an "even" and a "homogeneous" part, $\sigma =$

$\sigma^{\mathrm{odd}}\sigma^{\mathrm{even}}\sigma^{\mathrm{hom}}$, where the odd part is responsible for generating the monodromy in the double crossing relation (2.12), while the even part is a homogeneous solution of double crossing. These factors individually obey the crossing relations

$$\sigma_{12}^{\mathrm{odd}}\,\sigma_{\bar{1}2}^{\mathrm{odd}} = h_{12}^{\mathrm{odd}}, \qquad \sigma_{12}^{\mathrm{even}}\,\sigma_{\bar{1}2}^{\mathrm{even}} = h_{12}^{\mathrm{even}}, \qquad \sigma_{12}^{\mathrm{hom}}\,\sigma_{\bar{1}2}^{\mathrm{hom}} = 1 \; , \qquad (2.13)$$

with the odd and even parts of the crossing function

$$h_{12}^{\mathrm{odd}} = \sqrt{\frac{h_{12}}{h_{\bar{1}2}}} = \sqrt{\frac{x_1^- - x_2^+}{x_1^+ - x_2^+}\,\frac{x_1^+ - x_2^-}{x_1^- - x_2^-}\,\frac{1 - 1/x_1^+ x_2^+}{1 - 1/x_1^- x_2^+}\,\frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-}} \; ,$$

$$h_{12}^{\mathrm{even}} = \sqrt{h_{12}\,h_{\bar{1}2}} = \frac{x_2^-}{x_2^+}\sqrt{\frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-}} = \frac{x_2^-}{x_2^+}\sqrt{\frac{u_1 - u_2 - i/g}{u_1 - u_2 + i/g}} \; , \quad (2.14)$$

where u is the crossing-invariant variable (2.5). Clearly, the relations (2.13,2.14) are equivalent to the full crossing relation (2.10). The general solution of the crossing relation is not unique and consequently includes a homogeneous part. In the absence of further physical constraints this homogeneous piece can be chosen arbitrarily.

2.4   Limits

Before we consider solutions to these equations, we shall investigate the strong-coupling and weak-coupling regimes. In these limits, the kinematic space of particles splits up into disconnected regions. These regions give rise to different kinds of particles with different properties. They will play an important role in perturbative representations of the phase. Here we will only present a list of such regimes. Explicit formulae can be found in App. A.

At strong coupling, the kinematic space of particles splits up into four interesting regions. For later convenience we will denote these four regimes by MT (Metsaev-Tseytlin plane-wave excitations [1]), HM (Hofman-Maldacena regime [20]) and GKPr, GKPl (Gubser-Klebanov-Polyakov flat space limit [23] with distinct right and left-movers):

$$g \to \infty \quad \Longrightarrow \quad \text{particle} \in \begin{cases} \text{MT} & \text{if } p = \mathcal{O}(1/g^1) \; , \\ \text{GKPr} & \text{if } p = \mathcal{O}(1/g^{1/2}) \text{ and } p > 0 \; , \\ \text{GKPl} & \text{if } p = \mathcal{O}(1/g^{1/2}) \text{ and } p < 0 \; , \\ \text{HM} & \text{if } p \in (0, 2\pi) = \mathcal{O}(1/g^0) \; . \end{cases} \qquad (2.15)$$

Particles within different regimes can, in principle, scatter with themselves or with other types of particles, but it is expected that their scattering phase is suppressed by powers of the coupling constant. The MT elementary excitations and HM giant magnons also serve as constituents for Frolov-Tseytlin spinning strings [24] and Gubser-Klebanov-Polyakov spinning strings [23], respectively.

At weak coupling we find two regions for particles with real momenta. These correspond to magnons and magnon-holes:

$$g \to 0 \quad \Longrightarrow \quad \text{particle} \in \begin{cases} \text{magnon} & \text{if } C > 0 \; , \\ \text{hole} & \text{if } C < 0 \; . \end{cases} \qquad (2.16)$$

An additional complication is that both at strong and at weak coupling some branch points of the phase, for example the ones that will be discussed in section 3.4, move outside the kinematical regime and thus become inaccessible. The associated monodromies will then lead to additional integer labels for the particles in a perturbative treatment. For instance, at strong coupling the real period grows infinitely large with respect to the imaginary one. In this way periodicity along the real axes is lost and the momentum p is confined to a specific region. It will turn out that a shift by 2π can change the dressing factor. Therefore we have to specify for all particles what multiple of 2π we are considering in order to pin down the phase. Similarly, the two types of particles at weak coupling are related by the antipode map. The double antipode map is non-trivial and the dressing phase does change under it. Therefore we have to distinguish between magnons and their images under the double antipode map. As we shall see, both at strong and weak coupling, there will be more discrete choices to be made which lead to additional labels for the distinct regimes of particles. For the sake of clarity we shall not write out these labels explicitly.

3   Crossing-Symmetric Series

In this section we will search for a general solution to the crossing equation of the form

$$\theta = \sum_{n=0}^{\infty} \theta^{(n)} + \theta^{\text{hom}} \,, \tag{3.1}$$

where the summands $\theta^{(n)}$ represent a n-loop contribution in the perturbative string world sheet theory at strong coupling. We will postpone the discussion of the homogeneous piece to a later section and focus in what follows on one particular solution of the crossing relation given by the $\theta^{(n)}$.

3.1   Series Representation

A reasonably general form of the dressing factor σ is [4, 25]

$$\sigma_{12} = \exp i\theta_{12}, \qquad \theta_{12} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} \big(q_r(x_1^\pm)\, q_s(x_2^\pm) - q_s(x_1^\pm)\, q_r(x_2^\pm) \big) \,, \tag{3.2}$$

where the magnon charges are defined as

$$q_r(x^\pm) = \frac{i}{r-1} \left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right) \,, \tag{3.3}$$

and $c_{r,s}$ are some real coefficients depending on the 't Hooft coupling constant.

Before we proceed, let us motivate why the above form of the phase is useful: First of all, the phase is defined purely in terms of magnon charges which form a natural basis of conserved quantities. Secondly, zero-momentum particles representing symmetry generators have a trivial dressing factor. In addition, the first derivative of the phase

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around zero momentum vanishes. These two properties are required for the correct realisation of $\mathfrak{psu}(2,2|4)$ symmetry, see e.g. [5]. Thirdly, the phase is naturally doubly periodic on the complex torus. The form (3.2) thus represents a basis of periodic two-parameter functions with a couple of desired additional symmetry properties. It can also be viewed as a mode decomposition for functions on a torus; it is somewhat similar to a Fourier decomposition but with two periods.[1] Fourth, an analysis of perturbative integrable spin chains gives (3.2) as the most general expression [25]. Although this is not directly applicable to string theory models, we consider it a valid indication. Finally, this form collaborates nicely with the scattering of bound states in the bootstrap approach [26].

The form of the above phase (3.2,3.3) suggests to write it as a symmetrisation of a function $\chi(x_1, x_2)$ [18]

$$
\begin{aligned}
\theta_{12} = & +\chi(x_1^+, x_2^+) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+) + \chi(x_1^-, x_2^-) \\
& - \chi(x_2^+, x_1^+) + \chi(x_2^-, x_1^+) + \chi(x_2^+, x_1^-) - \chi(x_2^-, x_1^-) \ .
\end{aligned} \tag{3.4}
$$

We will generally use this definition of θ in terms of χ. In order to match with (3.2,3.3) we have to set

$$
\chi(x_1, x_2) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \frac{-c_{r,s}}{(r-1)(s-1)} \frac{1}{x_1^{r-1} x_2^{s-1}} \ . \tag{3.5}
$$

3.2   Strong-Coupling Expansion

The coefficients $c_{r,s}$ in (3.2) are known to first order at strong coupling [4, 27, 17, 28]. At leading order they are given by [4]

$$
c_{r,s}^{(0)} = g \, \delta_{r+1,s} \ . \tag{3.6}
$$

The first quantum correction turns out to be [17]

$$
c_{r,s}^{(1)} = \frac{(-1)^{r+s} - 1}{\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} \ , \tag{3.7}
$$

which follows from a one-loop comparison with spinning string energies [27, 17, 28].

It was shown in [18] that the first two contributions can be summed up to analytic expressions. The leading order contribution is given by

$$
\chi^{(0)}(x_1, x_2) = -\frac{g}{x_2} + g \left(-x_1 + \frac{1}{x_2} \right) \log \left(1 - \frac{1}{x_1 x_2} \right) \ , \tag{3.8}
$$

and the first order reads

$$
\begin{aligned}
\chi^{(1)}(x_1, x_2) = & -\frac{1}{2\pi} \operatorname{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi} \operatorname{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \\
& + \frac{1}{2\pi} \operatorname{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi} \operatorname{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \ ,
\end{aligned} \tag{3.9}
$$

---

[1]Let us note that, although the function is periodic by definition, infinite sums may lead to branch cuts which may render the analytic continuation of the function aperiodic. We shall be interested in this type of analytic continuation.

where $\text{Li}_2(z)$ is the dilogarithm function.[2]

These contributions have recently been shown to satisfy the crossing relation up to order $\mathcal{O}(1/g^3)$ for MT excitations [18]. Furthermore in [19] it was argued that the $n = 1$ contribution to the phase is actually sufficient to satisfy exactly the odd crossing relation (2.13,2.14) for finite values of the coupling. We shall provide a full proof of that statement in section 3.4. It was also demonstrated that in order to satisfy the original crossing relation (2.10) further even-n contributions are needed. We may therefore identify

$$\theta^{\text{odd}} = \theta^{(1)} \quad \text{and} \quad \theta^{\text{even}} = \sum_{n=0}^{\infty} \theta^{(2n)}. \tag{3.10}$$

3.3 Proposal

The central result of this paper is a proposal for the coefficients $c_{r,s}^{(n)}$ in (3.2) with even $n \geq 2$ such that the crossing relation (2.10) is satisfied. These take the following form

$$c_{r,s}^{(n)} = \frac{1}{g^{n-1}} \frac{\left((-1)^{r+s} - 1\right) \text{B}_n}{4 \cos(\frac{1}{2}\pi n) \, \Gamma[n+1] \, \Gamma[n-1]}$$

$$\times (r-1)(s-1) \frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]} \frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]} , \tag{3.11}$$

where B_n denotes the n-th Bernoulli number. Note that $c_{r,s}^{(n)} = 0$ if $r + s$ is even or if $n \geq s - r + 3$. The factor $(r-1)(s-1)$ cancels the denominators in q_r and q_s, and the sum over r and s can be performed easily. For every $\theta^{(n)}$ we find some rational function in $x_{1,2}^{\pm}$. For the first two terms in the expansion of χ we obtain the following rational functions

$$\chi^{(2)}(x_1, x_2) = -\frac{x_2}{24g(x_1 x_2 - 1)(x_2^2 - 1)},$$

$$\chi^{(4)}(x_1, x_2) = -\frac{x_2^3 + 4x_2^5 - 9x_1 x_2^6 + x_2^7 + 3x_1^2 x_2^7 - 3x_1 x_2^8 + 3x_1^2 x_2^9}{720g^3(x_1 x_2 - 1)^3(x_2^2 - 1)^5} . \tag{3.12}$$

We have obtained similar expressions up to $\chi^{(12)}$, but they are too bulky to be presented here.

A few features of the coefficients are worth mentioning: The fact that the odd Bernoulli numbers are zero relates nicely to the fact that odd-n contributions are not required for a solution of crossing.[3] The notable exception is $n = 1$ with $\text{B}_1 = -\frac{1}{2}$. Remarkably, the properly regularised expression for (3.11) with $n = 1$ yields *precisely* (3.7)! Also the leading order coefficients (3.6) are contained in (3.11) as the regularised contribution at $n = 0$.[4] Therefore (3.11) can be considered a natural extension of (3.6,3.7) to higher orders. Finally we mention that each coefficient $c_{r,s}(g)$ has a finite expansion in $1/g$ with the last contribution at $n = s - r + 1$.

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[2]We have made use of the identity $\text{Li}_2(z) + \text{Li}_2(1 - z) = \frac{1}{6}\pi^2 - \log(z)\log(1 - z)$ to absorb all the terms bilinear in logarithms that appear in [18].

[3]In fact this is not straightforward because (3.11) contains $\cos(\frac{1}{2}\pi n)$ in the denominator, which is zero for odd n. Thus the coefficients are ambiguous for odd n, and we may only define them to be zero. We will return to this issue in section 5.

[4]The contribution with $r = 1$, $s = 2$ is zero for all n with the exception of $n = 0$ where it is defined

3.4 Proof of Odd Crossing

We now turn towards confirming the crossing relation for our proposed series. In this section we will prove that the odd crossing relation (2.13,2.14) is satisfied by $\theta^{(1)}$ alone. We will address the proof of this statement in two steps. First we will show that $\theta^{(1)}$ satisfies double crossing (2.12), and afterwards we will turn to the odd crossing relation.

Double Crossing. Under double crossing the $x^{\pm}$ variables are mapped to themselves: $x^{\pm} \mapsto 1/x^{\pm} \mapsto x^{\pm}$. We therefore have to investigate the monodromies of the phase. We will discard shifts by multiples of 2π because they will drop out after exponentiating the phase.

The phase $\theta^{(1)}$ is composed from dilog functions, therefore let us review its monodromies first. It has the following two: When z is taken once around $z = 1$ (counter-clockwise) the analytic continuation of $\text{Li}_2(z)$ shifts by

$$\oint_{z=1} d\,\text{Li}_2(z) = -2\pi i \log(z) . \tag{3.13}$$

Likewise for circles around $z = \infty$ it shifts by the same amount in the opposite direction,

$$\oint_{z=\infty} d\,\text{Li}_2(z) = +2\pi i \log(z) . \tag{3.14}$$

Altogether the sum of shifts for all points cancels as it should.

Equipped with these formulae we can now consider the monodromies of $\chi^{(1)}$. The relevant points are those where the argument of the dilog is 1 or ∞. This happens at $x_1 = \infty$, $x_2 = \pm 1, 0$ or $x_1 = x_2$. The monodromies at $x_2 = \pm 1$ are

$$\oint_{x_2=\pm 1} d\chi^{(1)}(x_1, x_2) = \pm i \log \frac{x_1 - 1/x_2}{x_1 - x_2} . \tag{3.15}$$

More explicitly this means that the monodromies at $\sqrt{x_2} = +1$ and $\sqrt{x_2} = -1$ both take the above value with the $+$ sign. Similarly for $\sqrt{x_2} = \pm i$ and the $-$ sign. The monodromy for $x_1 = x_2$ however needs to be split into the two cases $\sqrt{x_2} = +\sqrt{x_1}$ and $\sqrt{x_2} = -\sqrt{x_1}$ for which we get opposite monodromies

$$\oint_{\sqrt{x_2}=\pm\sqrt{x_1}} d\chi^{(1)}(x_1, x_2) = \pm i \log \frac{1 + 1/\sqrt{x_1}\sqrt{x_2}}{1 - 1/\sqrt{x_1}\sqrt{x_2}} . \tag{3.16}$$

The potential monodromies at $x_1 = \infty$ and $x_2 = 0$ both cancel out.

We are finally in the position to consider double crossing of $\theta_{12}^{(1)}$. The monodromy (3.16) cannot contribute here because it is symmetric under the interchange of x_1 and x_2 whereas $\theta_{12}^{(1)}$ is anti-symmetric. In other words, the monodromies for $\sqrt{x_1}$ circling around $\pm\sqrt{x_2}$ cancel out between each term $\chi^{(1)}(x_1, x_2)$ and the corresponding term

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ambiguously. Adding this contribution with $c_{1,2}^{(0)} = g$ to the sum (3.5) solves h^{even} without the only term which makes direct reference to $x^{\pm}$, cf. x_2^-/x_2^+.

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$-\chi^{(1)}(x_2, x_1)$ in (3.4). We are thus left with (3.15). The monodromies of θ_{12} for $x_1^+ = \pm 1$ and $x_1^- = \pm 1$ are

$$\oint_{x_1^+ = \pm 1} d\theta_{12}^{(1)} = \mp i \log \frac{x_2^- - x_1^+}{x_2^+ - x_1^+} \frac{x_2^+ - 1/x_1^+}{x_2^- - 1/x_1^+} \ ,$$

$$\oint_{x_1^- = \pm 1} d\theta_{12}^{(1)} = \mp i \log \frac{x_2^+ - x_1^-}{x_2^- - x_1^-} \frac{x_2^- - 1/x_1^-}{x_2^+ - 1/x_1^-} \ . \tag{3.17}$$

Now we need to investigate what path $x_1^\pm$ takes under the double crossing map. Let us parametrise the momentum p using Jacobi's amplitude function am with elliptic modulus k and rapidity variable z,

$$p = 2\,\mathrm{am}(z, k), \qquad k = 4ig \ . \tag{3.18}$$

Then the half-periods of the torus are given by

$$\omega_1 = 2\,\mathrm{K}(k), \qquad \omega_2 = 2i\,\mathrm{K}(\sqrt{1 - k^2}) - 2\,\mathrm{K}(k) \ . \tag{3.19}$$

We note that p increases by 4π under a shift of the full real period $2\omega_1$. Therefore this parametrisation of the torus is a double covering. A single covering can be achieved as well, but at the cost of expressions which are substantially longer than e.g. (3.18).

Double crossing takes the rapidity z once around the imaginary period of the torus. The direction is not immediately obvious, it could be either of the two maps $z \mapsto z \pm 2\omega_2$. Let us for convenience assume the positive sign. To define a path between the two points we shall furthermore assume that the real part of z remains constant. Then it turns out that for $\mathrm{Re}\, z \in (-\frac{1}{4}\omega_1, +\frac{1}{4}\omega_1) + \omega_1\mathbb{Z}$ the variables $x^\pm$ circle clockwise around $+1$. Conversely, for $\mathrm{Re}\, z \in (\frac{1}{4}\omega_1, \frac{3}{4}\omega_1) + \omega_1\mathbb{Z}$ they circle counterclockwise around $+1$. Here we take -1 as the reference point for defining the outside of the path.

The overall monodromy is therefore

$$\theta_{\bar{1}2}^{(1)} - \theta_{12}^{(1)} = \pm i \log(h_{12}^{\mathrm{odd}})^2. \tag{3.20}$$

The plus sign is valid for $\mathrm{Re}\, z \in (-\frac{1}{4}\omega_1, +\frac{1}{4}\omega_1) + \omega_1\mathbb{Z}$ (roughly speaking for $p \approx 2\pi\mathbb{Z}$) and agrees literally with (2.12). The minus sign holds for $\mathrm{Re}\, z \in (\frac{1}{4}\omega_1, \frac{3}{4}\omega_1) + \omega_1\mathbb{Z}$ (roughly speaking for $p \approx \pi + 2\pi\mathbb{Z}$) and we should therefore reverse the crossing path for these momenta in order to achieve agreement with (2.12). This is not a problem as the definition of the crossing relation in fact allows both signs in $z \mapsto z \pm \omega_2$. Nevertheless, it would be illuminating to find out why we have to choose different orientations depending on the particle momentum. Alternatively, we could specify a path such that $x^\pm$ always circle clockwise around $+1$. This completes the proof that $\theta^{(1)}$ satisfies the double crossing relation (2.12).

Odd Crossing. Above we have investigated the structure of monodromies of the function $\chi^{(1)}(x_1, x_2)$. This was sufficient for the proof of double crossing because the double antipode map takes x to itself. The regular crossing relation on the other hand maps x

non-trivially and therefore monodromies are not sufficient for proving the full relation. Nevertheless they are essential for our understanding:

We have seen for instance that there are monodromies at $x_k = \pm 1$ and that the monodromies at $\sqrt{x_1} = \pm\sqrt{x_2}$ cancel in the combination $\chi^{(1)}(x_1, x_2) - \chi^{(1)}(x_2, x_1)$. Thus, the symmetrised combination has a simpler analytic structure and one should be able to simplify the expression (3.9). We obtain the following form

$$\chi(x_1, x_2) - \chi(x_2, x_1) = \psi(q_1 - q_2) + \frac{\mathrm{Li}_2(x_2) - \mathrm{Li}_2(-x_2) - \mathrm{Li}_2(x_1) + \mathrm{Li}_2(-x_1)}{2\pi} , \quad (3.21)$$

with the auxiliary function $\psi(q)$

$$\psi(q) = \frac{1}{2\pi} \mathrm{Li}_2\big(1 - e^{iq}\big) - \frac{1}{2\pi} \mathrm{Li}_2\big(1 - e^{iq+i\pi}\big) - \frac{i}{2} \log\big(1 - e^{iq+i\pi}\big) + \frac{\pi}{8} , \quad (3.22)$$

and where q_k is related to x_k as follows

$$e^{iq} = \frac{x+1}{x-1} . \quad (3.23)$$

Note that all the terms besides $\psi(q_1 - q_2)$ in (3.21) depend on either x_1 or x_2 only and thus they cancel out in the dressing phase (3.4)

$$\theta_{12}^{(1)} = \psi(q_1^+ - q_2^+) - \psi(q_1^+ - q_2^-) - \psi(q_1^- - q_2^+) + \psi(q_1^- - q_2^-) . \quad (3.24)$$

Here $q^\pm$ are related to $x^\pm$ as in (3.23).

A very tedious method to compare (3.21) to (3.9) is to apply dilogarithm identities such as the Abel identity.[5] However, it is much easier to confirm that (3.21) has the right monodromies. Here the variable q comes into play. It shifts by $+2\pi$ for a full clockwise rotation around $x = +1$ w.r.t. $x = -1$. This matches nicely with the monodromies of the dilogs in $\psi(q)$, cf. (3.13), namely

$$\mathrm{Li}_2\big(1 - e^{iq+2\pi in}\big) = \mathrm{Li}_2\big(1 - e^{iq}\big) - 2\pi in \log\big(1 - e^{iq}\big) . \quad (3.25)$$

In this new notation, the double crossing map reads

$$\psi(q + 2\pi) = \psi(q) - i \log \frac{1 - e^{iq}}{1 + e^{iq}} = \psi(q) - i \log\big(-i \tan(\tfrac{1}{2}q)\big) , \quad (3.26)$$

where

$$i \tan(\tfrac{1}{2}q_1 - \tfrac{1}{2}q_2) = \frac{x_1 - x_2}{1 - x_1 x_2} . \quad (3.27)$$

After multiplying the various terms for $x_{1,2}^\pm$ in (3.4) this function becomes

$$\frac{x_1^+ - x_2^+}{1 - x_1^+ x_2^+} \frac{1 - x_1^+ x_2^-}{x_1^+ - x_2^-} \frac{1 - x_1^- x_2^+}{x_1^- - x_2^+} \frac{x_1^- - x_2^-}{1 - x_1^- x_2^-} = \frac{1}{(h_{12}^{\mathrm{odd}})^2} , \quad (3.28)$$

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[5]We have used 16 Abel identities to show the equivalence, but for a better choice of identities there may be a shorter path.

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in agreement with the double crossing relation (2.13,2.14).

Before we turn towards the odd crossing relation, let us investigate unitarity and parity invariance of the above expression (3.21). Both translate to $\psi(q)$ being an odd function in q which is not manifest. To prove it, we have to use the dilog identity

$$\text{Li}_2(1 - e^{iq}) + \text{Li}_2(1 - e^{-iq}) = \tfrac{1}{2}q^2 \tag{3.29}$$

to flip the sign of the exponents in the dilogs. Before we can do this in the second term in (3.22) we have to shift via (3.25). The remainder of the proof reads as follows

$$\begin{aligned}
\psi(q) + \psi(-q) &= +\frac{q^2}{4\pi} - \frac{(q + \pi)^2}{4\pi} - \frac{i}{2}\log(1 + e^{iq}) + \frac{i}{2}\log(1 + e^{-iq}) + \frac{\pi}{4} \\
&= +\frac{q^2}{4\pi} - \frac{(q + \pi)^2}{4\pi} + \frac{q}{2} + \frac{\pi}{4} = 0 \ .
\end{aligned} \tag{3.30}$$

Finally, we can attack the odd crossing relation. The discussion at the end of the double crossing proof leads to the conclusion that the variable q as defined in (3.23) shifts by $+2\pi$ under double crossing. The sign for shifts of q has to be positive in all cases. This is in contradistinction to the shift in the above rapidity variable z for which the path on the universal cover of the torus has to be chosen carefully. Thus q appears to be a more fundamental quantity than z. Under regular crossing e^{iq} maps to $-e^{iq}$ and thus q has to shift by $+\pi$ to match with the above. Let us see how $\psi(q)$ behaves under such a shift. After cancelling an intermediate term and shifting according to (3.25) we find

$$\begin{aligned}
\psi(q) + \psi(q + \pi) &= \frac{i}{2}\log(1 - e^{iq}) - \frac{i}{2}\log(1 + e^{iq}) + \frac{\pi}{4} \\
&= -\frac{i}{2}\log(i\cot(\tfrac{1}{2}q)) + \frac{\pi}{4} \ .
\end{aligned} \tag{3.31}$$

In analogy to (3.26) this proves the odd crossing relation.

As the odd part of the crossing relation is solved, we can focus on the even part for the remainder of this paper.

3.5 Confirmation of Even Crossing

We will now provide evidence that our proposed coefficients (3.11) satisfy also the even crossing symmetry. For the n-loop expressions, n even, we find the following contributions to the crossing relation

$$\begin{aligned}
\theta_{12}^{(0)} + \theta_{\bar{1}2}^{(0)} &= i\log\frac{x_2^+}{x_2^-} + g\Delta\log\frac{\Delta^2 + 1/g^2}{\Delta^2} + i\log\frac{\Delta + i/g}{\Delta - i/g} \ , \\
\theta_{12}^{(n)} + \theta_{\bar{1}2}^{(n)} &= -\frac{i^n \mathrm{B}_n}{n(n-1)\,g^{n-1}}\left(\frac{2}{\Delta^{n-1}} - \frac{1}{(\Delta + i/g)^{n-1}} - \frac{1}{(\Delta - i/g)^{n-1}}\right) \ , \quad (3.32)
\end{aligned}$$

with $\Delta = u_1 - u_2$. These expressions are exact, therefore they apply in any of the strong-coupling regimes, but we have made use of the defining identity of $x^\pm$ (2.4). Although

we do not have a general proof of the last expression in (3.32), we have confirmed it up to $n = 12$. The even crossing phase to compare to reads

$$- i \log h_{12}^{\text{even}} = i \log \frac{x_2^+}{x_2^-} + \frac{i}{2} \log \frac{\Delta + i/g}{\Delta - i/g} \, . \tag{3.33}$$

Our claim is that

$$- i \log h^{\text{even}} = \sum_{n=0}^{N} \left(\theta_{12}^{(2n)} + \theta_{\bar{1}2}^{(2n)} \right) + \mathcal{O}(1/g^{2N+3}) \, , \tag{3.34}$$

for any upper limit of the sum N. Assuming the validity of (3.32) it is easy to verify (3.34) to very large values of N. However, the sum in (3.34) is in fact problematic if we set $N = \infty$ to confirm the exact crossing relation. In that case we cannot, a priori, interchange the expansion in $1/g$ and the summation. This is related to the fact that (3.32,3.34) is not a pure power series. In fact, it is straightforward to show that the sum does not converge for arbitrarily large values of g. The reason is that the Bernoulli numbers B_n grow like $n!$ and thus faster than a^n for any number a. An alternative path for testing crossing symmetry is to Borel sum the above series. Remarkably, this leads to precise agreement with (3.33). We believe that this is convincing evidence for the validity of our proposed solution.

It is conceivable that the lack of convergence of (3.32) also implies that the original series defining the phase (3.4,3.5,3.11) is problematic. In order to investigate the phase, in particular its analytic structure, we would therefore benefit very much from a more explicit representation of the sum. This will be the topic of the next section. In App. B we will present another representation of the series which is most useful for weak coupling.

4 Crossing-Symmetric Function

In this section we will present an analytic expression for the resummed series which allows us to study the structure of singularities in the phase. In particular, we find exact expressions for the bound state poles of giant magnons recently derived in [20].

4.1 Claim

Our claim is that the proper analytic expression corresponding to the above series is

$$\chi^{\text{even}}(x_1, x_2) = \lim_{N \to \infty} \left[\frac{g}{2x_1} \log \frac{g x_2}{N} - \frac{i}{4} \sum_{n=1}^{N} \log \frac{1 - 1/x_1 x^{(+2n)}(x_2)}{1 - 1/x_1 x^{(-2n)}(x_2)} \right] \tag{4.1}$$
$$+ \frac{g}{2} \left(-\frac{1}{x_1} - \frac{1}{x_2} \right) + \frac{g}{2} \left(-x_1 - x_2 + \frac{1}{x_1} + \frac{1}{x_2} \right) \log \left(1 - \frac{1}{x_1 x_2} \right) \, .$$

Here N is a cut-off parameter which should be taken to infinity. In that limit, the first term correctly regularises the logarithmically divergent sum. The terms on the second line are such that they give no contribution to the physical phase θ^{even} via (3.4), but

they are necessary for reproducing correctly the expected behaviour of the function χ in the series representation given in the previous section. Note also that the explicit appearance of the cut-off N in the first term is of this sort. The divergence of the sum would thus cancel out the full phase θ^{even} even without introducing a regularisation. The analytic function χ^{even} is expressed in terms of the new quantities $x^{(n)}$, which are related to x as

$$x^{(n)} + \frac{1}{x^{(n)}} - x - \frac{1}{x} = \frac{in}{2g} . \tag{4.2}$$

4.2 Gluing Two Infinite Genus Surfaces

Before we compare the function χ^{even} to the series expression of the previous section, we would like to investigate its analytic structure. First, consider the map $\mathbb{C} \to \mathbb{C}^\infty$

$$x \mapsto \left(\ldots, x^{(-2)}, x^{(-1)}, x^{(0)}, x^{(+1)}, x^{(+2)}, \ldots \right) . \tag{4.3}$$

This map has branch points where $x + 1/x + in/2g = \pm 2$ for any integer n. When x moves once around one of these branch points, the component $x^{(n)}$ is mapped to its inverse, which is the other solution of (4.2). All the other components will remain unchanged. The only exception is the map $x^{(0)}(x)$ which we shall define as the identity map

$$x^{(0)}(x) := x . \tag{4.4}$$

This is possible because the branch points degenerate for $n = 0$. The analytic completion of the map thus has infinitely many branch points with distinct monodromies and is defined on a Riemann surface of infinite genus. The function χ^{even} is essentially defined with x_2 on this Riemann surface because for every even n we have made a choice between $x^{(n)}$ and its inverse. The fact that we only require even n does not alter the picture qualitatively, therefore let us stick to the more general surface.

Next we need to consider $\theta_{12}^{\text{even}}$ which consists of terms of the sort $f(x^+) - f(x^-)$, cf. (3.4). The problem is now that these functions are defined on two different Riemann surfaces. In general, the combined function would be defined on the product of two infinite-genus surfaces. Nevertheless, the two variables x^+ and x^- are related in a special way such that all the branch points of $f(x^+)$ and $f(x^-)$ coincide. Therefore the monodromies of the two functions are not unrelated. Namely, if we move x^+ once around a branch point that inverts $x^{(n-1)}(x^+)$ then x^- has to move around the corresponding branch point that inverts $x^{(n+1)}(x^-)$. Consequently, it is consistent to make a specific choice

$$x^{(n-1)}(x^+) = \left(x^{(n+1)}(x^-) \right)^{s^{(n)}} , \tag{4.5}$$

where $s^{(n)} = \pm 1$ determines whether the $x^{(n)}$'s are inversely related or not.[6] Moreover we are forced to make this choice in order to define the function $f(x^+) - f(x^-)$ properly because variations of $x^\pm$ cannot flip the signs $s^{(n)}$. In conclusion, the combination $f(x^+) - f(x^-)$ requires us to fix the signs $s^{(n)}$ and is then defined on an infinite-genus surface.

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[6]Notice that (4.5) for $n = 1$ is not in conflict with (4.4). Since x^+ and x^- are related by (2.4), moving x^- around the branch cut of $x^{(2)}(x^-)$ will also invert x^+. This does not need to affect $x^{(n)}(x^+)$ for $n \neq 0$, because they depend of the combination $x + 1/x$.

In contrast, the function $f(x) - f(y)$ with uncorrelated x, y is uniquely defined on the product of two infinite-genus surfaces without sign ambiguities.

It is important to stress that we should consider not only the $x^{\pm}$'s but also the signs $s^{(n)}$, at least for n odd, as kinematic parameters of the particle. Particles with different signs are not equivalent to each other, which manifests in a different scattering behaviour. For simplicity of notation, using (4.5), we can systematically write $x^{(n-1)}(x_k^+)$ in terms of $x^{(n+1)}(x_k^-)$. Consequently we shall define a single set of kinematic parameters $x_k^{(n)}$ which are related to $x^{(n)}(x_k^{\pm})$ by

$$x^{(n)}(x_k^-) = x_k^{(n-1)}, \qquad x^{(n)}(x_k^+) = (x_k^{(n+1)})^{s_k^{(n+1)}}. \tag{4.6}$$

with $x_k^{(-1)} = x_k^-$, but not necessarily $x_k^{(+1)} = x_k^+$.

We should consider the transformation of the new parameters under the discrete symmetries of the system, in particular parity and the antipode map, cf. (2.6,2.7). The antipode simply maps the $x_k^{(n)}$ to their inverse

$$x_k^{(n)} \mapsto 1/x_k^{(n)}. \tag{4.7}$$

Parity maps $x^{\pm} \mapsto -x^{\mp}$ and it is consistent to define $x^{(n)}(-x) = -x^{(-n)}(x)$. Hence parity will act on the $x_k^{(n)}$ and on the signs $s^{(n)}$ as

$$x_k^{(n)} \mapsto -(x_k^{(-n)})^{s_k^{(n)}}, \qquad s^{(n)} \mapsto s^{(-n)}, \tag{4.8}$$

implying that particles with $s^{(+n)} \neq s^{(-n)}$ do not map to themselves under parity.

After making the choice of signs $s_{1,2}^{(n)}$ for each particle we can compute the phase $\theta_{12}^{\text{even}}$

$$\begin{aligned}
\theta_{12}^{\text{even}} = &-\frac{i}{2} \log \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-} + \frac{g}{2}\left(\frac{1}{x_1^+} - \frac{1}{x_1^-}\right) \log \frac{x_2^+}{x_2^-} - \frac{g}{2}\left(\frac{1}{x_2^+} - \frac{1}{x_2^-}\right) \log \frac{x_1^+}{x_1^-} \\
&- \frac{i}{4} \sum_{n=-\infty}^{\infty} \text{sign}(2n-1) \frac{1 - s_1^{(2n-1)}}{2} \log \frac{x_2^- - x_1^{(2n-1)}}{x_2^+ - x_1^{(2n-1)}} \frac{x_2^+ - 1/x_1^{(2n-1)}}{x_2^- - 1/x_1^{(2n-1)}} \\
&- \frac{i}{4} \sum_{n=-\infty}^{\infty} \text{sign}(2n-1) \frac{1 - s_2^{(2n-1)}}{2} \log \frac{x_1^+ - x_2^{(2n-1)}}{x_1^- - x_2^{(2n-1)}} \frac{x_1^- - 1/x_2^{(2n-1)}}{x_1^+ - 1/x_2^{(2n-1)}},
\end{aligned} \tag{4.9}$$

This phase is consistent with unitarity if we notice that when exchanging the particles we should interchange the momenta as well as the signs. It has also the right behaviour under parity. Parity invariance implies that the overall scattering phase should change sign under (4.8). The phase (4.9) indeed fulfils this property. We can now consider the crossing relation. It is straightforward to verify the even crossing relation (2.13,2.14). In fact all the terms on the second and third line in (4.9) represent homogeneous solutions of crossing.

For generic values of the signs, the phase (4.9) gives rise to square root singularities in the scattering matrix. We cannot offer an explanation for this puzzling behaviour, however, there are two points to be remarked: On the one hand one might consider giving up manifest parity invariance and use the expression (B.11) for χ^{even} instead of

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(4.1) in which case there would be no fractional singularities. On the other hand, one could adjust the signs $s^{(n)}$ such that the square root singularities go away. This happens for scattering of parity self-conjugate particles with equal sign assignations $s_1^{(n)} = s_2^{(n)}$, as well as for particles with all signs equal, $s_1^{(n)} = s_1$, $s_2^{(n)} = s_2$. In these cases several terms appearing in (4.9) can be seen to cancel among themselves using

$$\left(x_k^\pm - x_j^{(n)}\right)\left(1 - 1/x_k^\pm x_j^{(n)}\right) = u_k - u_j - \frac{i(n \mp 1)}{2g}, \qquad (4.10)$$

and we are left with a reduced expression whose singularities lead to poles and zeros only. In particular a model whose particles have $s_k^{(n)} = s_k = \pm 1$ would be free from fractional singularities, which also leads to simple expressions for the dressing phase. We will argue in the next section that the two choices $s_k = \pm 1$ are related to MT excitations and HM giant magnons respectively.

When all $s^{(n)} = +1$ the total phase depends on $x_1^\pm$ and $x_2^\pm$ only

$$\theta_{12}^{\text{elem}} = -\frac{i}{2} \log \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-} + \frac{g}{2}\left(\frac{1}{x_1^+} - \frac{1}{x_1^-}\right) \log \frac{x_2^+}{x_2^-} - \frac{g}{2}\left(\frac{1}{x_2^+} - \frac{1}{x_2^-}\right) \log \frac{x_1^+}{x_1^-}, \quad (4.11)$$

and is thus naturally defined on a torus. This choice produces a minimal set of poles and zeros in the scattering matrix. We will therefore refer to the phase above as "elementary". It differs from the solution of the crossing relation proposed in [19] by a term $\delta\theta_{12} = \frac{1}{2}(C_1 p_2 - C_2 p_1)$. This piece $\delta\theta_{12}$ repairs some of the manifestly unphysical behaviour of the phase proposed in [19]. Since $(C, p) \mapsto -(C, p)$ under the antipode map, it is clear that $\delta\theta_{12}$ is a homogeneous solution of the crossing relation. Remarkably, the phase (4.11) has appeared previously in the context of light-cone gauge quantisation at leading order in [29].[7] As emphasised in this article, the rightmost term in (4.11) combines nicely with the phase contribution $-p_1 L$ from the Bethe equations. Together, the terms multiplying p_1 form the light-cone momentum $p_{+,2}$. Although the term manifestly removes periodicity of the phase by shifts of momenta by 2π, its appearance is actually useful because the length L is not a physical quantity, whereas the light-cone momentum is a charge under one of the Cartan generators of $\mathfrak{psu}(2, 2|4)$.

The choice $s^{(n)} = -1$ leads to the following result

$$\theta_{12}^{\text{giant}} = \theta_{12}^{\text{elem}} - \frac{i}{2} \sum_{n=-\infty}^{\infty} \text{sign}(2n-1) \log \frac{x_1^+ - x_2^{(2n-1)}}{x_1^- - x_2^{(2n-1)}} \frac{x_2^- - x_1^{(2n-1)}}{x_2^+ - x_1^{(2n-1)}}. \qquad (4.12)$$

It gives rise to an infinite array of additional poles and zeros in the scattering matrix. The proper definition of (4.12) needs the full-fledged, infinite-genus surface introduced above. We will connect it to the HM giant magnons and thus denote it by "giant".

Before we make this more explicit by considering the strong coupling limit of the phase, we would like to make a final comment regarding the choice of signs $s^{(n)}$ for the Bethe equations: Although all assignments of $s^{(n)}$ may be meaningful to distinguish different kinds of particles propagating on the infinite line, there should be one specific

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[7]We thank M. Staudacher as well as S. Frolov, M. Zamaklar for pointing this out to us.

choice to be used for the Bethe equations in [5]. This is analogous to the situation for bound states of magnons which can exist as elementary objects on the infinite line, but not on the circle. We consider it very likely that one of the above two choices (4.11,4.12) is the correct one. On the one hand, the "elementary" choice has minimal genus and would therefore lead to a relatively simple analytic structure of the Bethe equations. On the other hand, the "giant" choice naturally incorporates the giant magnon solitons and might be favourable from a physics standpoint. However, as already contemplated in [20], the giant magnon may turn out to be a composite object; below we will find some further indications strengthening this point of view. In that case, the elementary choice $s^{(n)} = +1$ would most likely be the correct one for the Bethe equations.

4.3   Strong Coupling

In this section we will compare the analytic expression (4.1) for χ^{even} to the perturbation series (3.5). The latter is defined at strong coupling where the definition of the former simplifies drastically.

At strong coupling the above infinite-genus surface degenerates into many disjoint regions. This is related to the fact that either $x^{(n)} = x + \mathcal{O}(1/g)$ or $x^{(n)} = 1/x + \mathcal{O}(1/g)$. Thus the function $\chi^{\text{even}}(x_1, x_2)$ does not have infinitely many branch cuts anymore. In other words, the branch points of the maps $x \mapsto x^{(n)}(x)$ have all moved to $1 + \mathcal{O}(1/\sqrt{g})$ where they cannot be used to change sheets individually.[8] Therefore for every $n \neq 0$ we can definitely choose between $x^{(n)} \approx x$ or $x^{(n)} \approx 1/x$.

Although the analytic function (4.1) intrinsically leads to the previous infinite set of strong coupling choices, only one of them directly connects with the perturbative series (3.5,3.11) for χ^{even}. This is the simplest case $x^{(n)} \approx x$. A restriction like this might seem unnatural, however recall that the perturbative sum does not converge literally. The interpretation we are advocating for in this section is that once we associate an analytic expression to the problematic sum, we unavoidably get an enlarged phase space.

The agreement between the case $x^{(n)} \approx x$ and the series (3.5,3.11) can be checked using the Euler-MacLaurin summation formula

$$\sum_{n=1}^{\infty} f(n/g) = g \int_0^{\infty} dz\, f(z) - \sum_{k=1}^{\infty} \frac{\text{B}_{2k}}{(2k)!\, g^{2k-1}}\, f^{(2k-1)}(0)\,, \tag{4.13}$$

where $f^{(k)}$ denotes the k-th derivative of $f(z)$ and we have assumed that the function vanishes at zero and infinity and in addition all its derivatives vanish at infinity. In our case $f(z)$ is defined via[9]

$$f(n/g) = -\frac{i}{4} \log \frac{1 - 1/x_1 x^{(+2n)}(x_2)}{1 - 1/x_1 x^{(-2n)}(x_2)}\,. \tag{4.14}$$

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[8]In fact there is a tail of branch points for $n \sim g$ which extends throughout the complex plane. This allows to invert contributions on a global scale for very large n, but not for finite n.

[9]Note that g does not appear explicitly in the function, cf. (4.2).

The integral by itself is divergent. The divergence is however cured by the counterterm in (4.1) and we should compute the following finite combination

$$\lim_{N \to \infty} \left[\frac{g}{2x_1} \log \frac{x_2}{N/g} + g \int_0^{N/g} dz \, f(z) \right]$$
$$= \frac{g}{2} \left(\frac{1}{x_1} - \frac{1}{x_2} \right) + \frac{g}{2} \left(-x_1 + x_2 - \frac{1}{x_1} + \frac{1}{x_2} \right) \log \left(1 - \frac{1}{x_1 x_2} \right) . \qquad (4.15)$$

Together with the terms on the second line in (4.1) we reproduce exactly $\chi^{(0)}$ of the AFS phase in (3.8). Finally each term in the sum over derivatives produces precisely $\chi^{(2k)}$, cf. (3.12), which we have confirmed up to $\chi^{(12)}$. Note that the presence of the Bernoulli numbers in the Euler-MacLaurin formula nicely fits their appearance in our proposed coefficients.

MT Regime. As we have chosen $x^{(n)} \approx x$, which applies to $x = x^+$ as well as $x = x^-$, all the $s^{(n)}$ in (4.5) will coincide being equal to either $+1$ or -1. The MT excitations are characterised by $x^+ \approx x^-$, see (A.1), and therefore we require all $s^{(n)} = +1$. The scattering phase then reduces to the simple analytic expression (4.11). This phase agrees with all the available data for spinning strings and near plane wave states, i.e. so far only the leading order in θ^{elem} [30, 29, 31] together with the odd contribution $\theta^{(1)}$ [17, 28].

HM Giant Magnons. The giant magnons have $x^+ \approx 1/x^-$, see (A.3), implying that this case should be represented by $s^{(n)} = -1$. The cancellations that took place for MT excitations do not occur now and we are left with an infinite array of poles and zeros in the scattering matrix, as can be seen in (4.12).

Based on the connection to the sine-Gordon model [32] it was shown in [20] that the semiclassical behaviour of the giant magnons was described by the AFS dressing phase [4]. The latter reads in this limit[10]

$$\theta_{12}^{(0)} = 2g \left(\cos(\tfrac{1}{2}p_1) - \cos(\tfrac{1}{2}p_2) \right) \log \frac{\sin^2\left(\frac{1}{4}(p_1 - p_2) \right)}{\sin^2\left(\frac{1}{4}(p_1 + p_2) \right)} + \mathcal{O}(1/g^0) . \qquad (4.16)$$

The leading phase (4.16) has branch cuts starting from $p_1 = \pm p_2$. It is natural to interpret these branch cuts as condensates of poles and zeros.[11] Notice that the square in the argument of the log implies that it has two branch cuts originating from $p_1 = p_2$, and correspondingly from $p_1 = -p_2$. One of them is associated with poles and the other with zeros depending on the sign of the prefactor of the log.

The poles of the scattering matrix have the interpretation of bound states. The bound states corresponding to the phase (4.16) were derived in [20]. Recalling that $u = 2\cos(\tfrac{1}{2}p)$, they appear at

$$u_1 - u_2 = \frac{in}{2g} + \mathcal{O}(1/g^2) . \qquad (4.17)$$

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[10]The sign conventions for the phase in [20] seem to be reversed from ours.
[11]We thank N. Dorey and J. Maldacena for discussions and explanations.

Indeed, the semiclassical counting of these states agrees with the discontinuity of the cut in (4.16). Namely, the prefactor of the log increases by $i/2$ between the positions of two adjacent bound states. We have shown above that the leading piece of χ^{even}, the integral term in the Euler-MacLaurin formula, leads to the AFS phase. Therefore the array of poles and zeros in (4.12) precisely reconstructs the branch cuts in (4.16): The exact scattering phase leads to double poles at

$$u_1 - u_2 = \frac{in}{g} \, . \tag{4.18}$$

This is not in contradiction with the results of [20]. Relation (4.17) was derived using semiclassical quantisation, and can only be trusted for the overall counting of states, but not for their precise positions. The result (4.17) should be understood as an average density of approximately $2g$ poles per imaginary unit of u. We find double poles with a density of exactly g which means that (4.17) and (4.18) are fully compatible.

In [20] it was raised the question about the fate of the bound states as the coupling decreases. From (4.9) we observe that there is nothing that prevents them from being present all the way to small coupling. Moreover, as the signs $s^{(n)}$ are stable, they must appear in the scattering of particles at weak coupling. This does not necessarily represent a problem for a smooth interpolation to gauge theory though: Firstly, the infinite genus of the function leads to various inequivalent weak-coupling limits (this problem is discussed in the next paragraph, although for strong coupling instead of weak coupling). Most of the poles could be invisible, their influence being however still present in the perturbative series. Secondly, it is not clear which configuration of signs $s^{(k)}$ connects to gauge theory magnons in the first place: the "elementary" choice, the "giant" choice or an altogether different choice. Finally, we do not understand the analytic structure of homogeneous solution, see section 5. Including the right homogeneous solution could, in principle, cure the disagreement between the asymptotic phases for both models.

Note that the phase $\theta_{12}^{\text{giant}}$ in (4.12) is defined on an infinite-genus surface. This has the interesting implication that there exist infinitely many inequivalent strong-coupling limits for it, although only the one considered above is directly associated with the series (3.5,3.11). These limits can be deformed into each other by going to finite coupling, changing the sheet, and going back to infinite coupling. We could for instance take the HM limit $x^+ \approx 1/x^-$ while making sure that $x^{(4n+1)} \approx 1/x^{(4n+3)} \approx x^{(-4n-1)} \approx 1/x^{(-4n-3)} \approx x^-$ for $n \geq 0$. By this staggering, adjacent poles and zeros will cancel each other in the strong coupling limit and only a few singularities near $n = 0$ will remain. The latter will however not contribute at order $\mathcal{O}(g)$ and the strong coupling limit becomes simply

$$\theta_{12}^{\text{giant}} = g\big(p_2 \sin(\tfrac{1}{2}p_1) - p_1 \sin(\tfrac{1}{2}p_2)\big) + \mathcal{O}(1/g^0) = \theta_{12}^{\text{elem}} + \mathcal{O}(1/g^0) \, . \tag{4.19}$$

This particular phase actually has the same leading $\mathcal{O}(g)$ contribution as the (unique) HM limit of the elementary phase in (4.11) with all $s^{(n)} = +1$. Notice that, correspondingly, the HM limit for a particle with $s^{(n)} = +1$ is not compatible with the choice $x^{(n)} \approx x^{(n+1)}$. Hence the elementary phase is not either directly representable by the series (3.5,3.11) in this kinematical regime. Although in the HM regime θ^{elem} and the

limit of θ^{giant} just considered coincide at leading order, the agreement between the two expressions will clearly break down at higher orders in $1/g$ and lead to the much richer structure of θ^{giant}.

4.4 Bound State Scattering

To understand better the additional terms in the general dressing phase (4.9) it will be instructive to consider scattering of bound states [33]. This will reveal that particles with $s^{(n)} \neq +1$ for any n may potentially correspond to some non-minimal bound states.

A bound state can be thought of as composed from m elementary particles. The particles are parametrised by $x_{1(k)}^{\pm}$, $k = 1, \ldots m$, with [33, 26]

$$x_{1(k)}^{+} = x_1^{(-m+2k)}, \qquad x_{1(k)}^{-} = x_1^{(-m+2k-2)}, \tag{4.20}$$

where the $x_1^{(k)}$ are parameters obeying (4.2). In particular, we shall denote the extremal parameters which define the multiplet under the residual symmetry algebra by

$$x_1^{-m} = x_{1(1)}^{-}, \qquad x_1^{+m} = x_{1(m)}^{+} . \tag{4.21}$$

The total dressing phase for the scattering of two such bound states is given by

$$\theta_{12} = \sum_{k=1}^{m} \sum_{l=1}^{n} \theta_{1(k)2(l)} . \tag{4.22}$$

When expressing the phase in terms of χ using (3.4) one finds that both sums telescope to [26]

$$\theta_{12} = +\chi(x_1^{+m}, x_2^{+n}) - \chi(x_1^{+m}, x_2^{-n}) - \chi(x_1^{-m}, x_2^{+n}) + \chi(x_1^{-m}, x_2^{-n})$$
$$- \chi(x_2^{+n}, x_1^{+m}) + \chi(x_2^{-n}, x_1^{+m}) + \chi(x_2^{+n}, x_1^{-m}) - \chi(x_2^{-n}, x_1^{-m}) . \tag{4.23}$$

When we now substitute the explicit expression (4.1) leading to the (even part of the) phase, we find in analogy to (4.11)

$$\theta_{12}^{\text{even}} = -\frac{i}{2} \log \frac{1 - 1/x_1^{-m} x_2^{+n}}{1 - 1/x_1^{+m} x_2^{-n}}$$
$$- \frac{i}{2} \sum_{k=1}^{m-1} \log \frac{1 - 1/x_1^{(+m-2k)} x_2^{+n}}{1 - 1/x_1^{(+m-2k)} x_2^{-n}} - \frac{i}{2} \sum_{l=1}^{n-1} \log \frac{1 - 1/x_1^{-m} x_2^{(+n-2l)}}{1 - 1/x_1^{+m} x_2^{(+n-2l)}}$$
$$+ \frac{g}{2} \left(\frac{1}{x_1^{+m}} - \frac{1}{x_1^{-m}} \right) \log \frac{x_2^{+n}}{x_2^{-n}} - \frac{g}{2} \left(\frac{1}{x_2^{+n}} - \frac{1}{x_2^{-n}} \right) \log \frac{x_1^{+m}}{x_1^{-m}} . \tag{4.24}$$

Here we have assumed $x^{(k-m)}(x_1^{+m}) = x^{(k+m)}(x_1^{-m}) = x_1^{(k)}$ and likewise for x_2. This is equivalent to setting all signs defined by the analogous to (4.5) in this more general case to $+1$. This answer is indeed consistent with substituting θ^{elem} from (4.11) in (4.22).

Superficially (4.23) suggests that all the intermediate $x_{1,2}^{(k)}$ do not matter for scattering of bound states. This is however not true as can be seen from (4.24). In particular, the

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expression does make a distinction between $x_{1,2}^{(k)}$ and $1/x_{1,2}^{(k)}$. The infinite genus of the function χ^{even} allows for the $x_{1,2}^{(k)}$ to be reintroduced; this feature has some interesting consequences:

Let us for instance briefly compare to the results in [26] for the total scattering factor for bound states. When we supplement the above dressing phase $\theta_{12}^{\text{even}}$ with the expression in [26] based on the BDS phase [22], we find

$$S_{12}^{\text{BDS+even}} = \frac{x_1^{+m} - x_2^{-n}}{x_1^{-m} - x_2^{+n}} \prod_{k=1}^{m-1} \frac{x_1^{(+m-2k)} - x_2^{-n}}{x_1^{(+m-2k)} - x_2^{+n}} \prod_{l=1}^{n-1} \frac{x_1^{+m} - x_2^{(+n-2l)}}{x_1^{-m} - x_2^{(+n-2l)}} \tag{4.25}$$

$$\times \exp\left[ig\left(\frac{1}{x_1^{+m}} - \frac{1}{x_1^{-m}}\right) \log \frac{x_2^{+n}}{x_2^{(-n)}} - ig\left(\frac{1}{x_2^{+n}} - \frac{1}{x_2^{-n}}\right) \log \frac{x_1^{+m}}{x_1^{-m}} \right].$$

One can observe that the double poles found in [26] may now split up into two separate poles depending on the intermediate $x_{1,2}^{(k)}$. For example, the equality of $x_1^{-m} = x_2^{(+n-2l)}$ does not necessarily imply $x_1^{(-m+2l)} = x_2^{+n}$ and the corresponding poles in the third and second terms might not overlap.

It is also curious to see that the terms involving intermediate parameters of one bound state, e.g. $x_1^{(k)}$, depend on the other bound state only via the extremal parameters, e.g. $x_2^{\pm n}$. This is in fact the same pattern as for the additional terms in (4.9), so let us write the analogous terms for this case explicitly

$$\delta\theta_{12}^{\text{even}} = -\frac{i}{4} \sum_{k=-\infty}^{\infty} \text{sign}(2k-1) \frac{1 - s_1^{(+m-2k)}}{2} \log \frac{x_1^{(+m-2k)} - x_2^{+n}}{x_1^{(+m-2k)} - x_2^{-n}} \frac{1/x_1^{(+m-2k)} - x_2^{-n}}{1/x_1^{(+m-2k)} - x_2^{+n}}$$

$$-\frac{i}{4} \sum_{l=-\infty}^{\infty} \text{sign}(2l-1) \frac{1 - s_2^{(+n-2l)}}{2} \log \frac{x_1^{-m} - x_2^{(+n-2l)}}{x_1^{+m} - x_2^{(+n-2l)}} \frac{x_1^{+m} - 1/x_2^{(+n-2l)}}{x_1^{-m} - 1/x_2^{(+n-2l)}}, \tag{4.26}$$

Interestingly, we can remove the explicit reference to all the intermediate $x_1^{(+m-2k)}$ by setting $s_1^{(+m-2k)} = -1$ for all $0 < k < m$. In that case, the terms on the first line in (4.25) combine with the ones in (4.26) to give

$$S_{12}^{\text{BDS+even}} = \frac{x_1^{+m} - x_2^{-n}}{x_1^{-m} - x_2^{+n}} \prod_{k=1}^{m-1} \sqrt{\frac{x_2^{-n}}{x_2^{+n}} \frac{u_1 - u_2 + i(m - 2k + n)/2g}{u_1 - u_2 + i(m - 2k - n)/2g}}$$

$$\times \prod_{l=1}^{n-1} \sqrt{\frac{x_1^{+m}}{x_1^{-m}} \frac{u_1 - u_2 + i(-n + 2l + m)/2g}{u_1 - u_2 + i(-n + 2l - m)/2g}} \times \ldots. \tag{4.27}$$

Here we have expressed all the intermediate parameters $x_1^{(k)}$ through u_1 and thus through the extremal parameters $x_1^{\pm m}$ via $u_1 = x_1^{-m} + 1/x_1^{-m} + im/2g$ only. The choice of intermediate signs $s_1^{(+m-2k)} = -1$ for $0 < k < m$ in bound states therefore lowers the genus of the scattering phase. If we also set $s_1^{(+m-2k)} = +1$ otherwise, we obtain a scattering phase with minimal genus one, i.e. it is defined on a complex torus just as the elementary phase. Furthermore, most of the square roots in the above expression

appear twice, such that there will almost be no fractional poles. Therefore this appears to be a natural choice for particles transforming in bigger representations of the residual symmetry such as the bound states.

What if we set the exterior sign $s_1^{(+m-2k)}$ for $k < 0$ or $k > m$ to -1? It activates explicit dependence of the phase on $x_1^{(+m-2k)}$ suggesting that the bound state becomes non-minimal. This view is reinforced by the fact that there are additional singularities at $x_1^{(+m-2k)}$ in the scattering phase which indicate the presence of some new substructure in the bound state. However, the new parameter lies outside the range of the constituent parameters in (4.20) so it is not expected to appear in the minimal case. A possible conclusion would be that the scattering phase belongs to a novel kind of extended bound state.

In any case, it seems that non-trivial signs $s^{(k)} = -1$ play a role especially for bound states. Along these lines, one might consider the choice $s^{(k)} = -1$ for all k to correspond to some bound state. As the giant magnons phase requires precisely this choice, one may draw the conclusion that the giant magnons represent some extended type of bound state. This agrees with the fact that its scattering phase has many more singularities than one might expect for fundamental particles. We however feel that more rigorous investigations are required to probe the nature of the signs, giant magnons and the proposed extended bound states.

We would also like to mention that the terms in (4.26) resemble the monodromies of the one-loop phase $\theta^{(1)}$ in (3.17) which may appear as ambiguities in the definition of phase for bound states. For the choice $x_1^\pm \to x_1^{(+m-2k)}$ and $x_2^\pm \to x_2^{\pm n}$ we find the same contributions but with a prefactor which is four times as large. Clearly, more work is needed to fully understand the analytic structure of the dressing phase, especially in the case of bound states.

5 Homogeneous Solutions

In the previous sections we have discussed a particular solution to the crossing relation (2.13). However, the weak-coupling limit of this solution apparently does not agree with planar $\mathcal{N} = 4$ Yang-Mills. Indeed, the series (3.5,3.11) can be continued to weak coupling without encountering negative powers of g. Each $\chi^{(n)}$, with $n > 1$, leads to a weak coupling contribution of order g^2 (see App. B), while the gauge theory phase vanishes at least at order g^4 [22]. In addition, $\chi^{(1)}$ (3.9) gives rise to a non-analytical weak-coupling contribution at order g^3. The fact that the phase we have proposed does not connect with gauge theory strongly points towards the need for additional pieces with a different weak-coupling behaviour. If crossing symmetry holds, they must correspond to homogeneous solutions (2.13). The study of these solutions will be the subject of this section.

5.1 General Perturbative Solution

The form of the strong-coupling coefficients (3.11) suggest that the $c_{r,s}^{(n)}$ may be written as polynomials in r and s with the degree determined by the loop order n. It is not

difficult to find a general expression for $c_{r,s}^{\text{hom}}$ of this form

$$c_{r,s}^{\text{hom}} = \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} b_{m,n}\left(1-(-1)^{r+s}\right)\left((r-1)^{2n+1}(s-1)^{2m+1}-(r-1)^{2m+1}(s-1)^{2n+1}\right), \quad (5.1)$$

where $b_{m,n}$ are some arbitrary real coefficients which may depend on the coupling constant g. These contributions sum up to

$$\chi^{\text{hom}}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} b_{m,n}\left(\text{Li}_{-2m}(x_1)\,\text{Li}_{-2n}(x_2) - \text{Li}_{-2m}(-x_1)\,\text{Li}_{-2n}(-x_2)\right) + \dots \, ,$$

$$(5.2)$$

where the dots represent terms which are symmetric under the interchange of x_1 and x_2 and which consequently drop out in the full phase θ_{12}, cf. (3.4). Using the identity

$$\text{Li}_{-n}(1/x) + (-1)^n\,\text{Li}_{-n}(x) = -\delta_{n,0} \qquad \text{for } n \geq 0 \qquad (5.3)$$

it is straightforward to show that $\theta_{12}^{\text{hom}} + \theta_{\bar{1}2}^{\text{hom}} = 0$.

Let us now consider the analytic structure of θ^{hom}. Note that $\text{Li}_{-n}(x)$ is a polynomial in $1/(x-1)$ of degree n. Therefore the homogeneous solution corresponding to a single coefficient $b_{m,n}$ has multiple poles at $x_{1,2} = \pm 1$. Thus each non-zero coefficient $b_{m,n}$ gives rise to essential singularities in the phase factor σ_{12}^{hom}. The only way to get rid of them is by taking either no homogeneous terms, or infinitely many. Taking infinitely many terms may lift essential singularities, but usually at the price of branch cuts. However the appearance of branch cuts has the potential to destroy the homogeneous nature of the solutions. It is thus rather difficult to figure out suitable non-trivial solutions.

The most trivial solution, with $\theta^{\text{hom}} = 0$, contains a relatively small number of singularities, and could seem thus a reasonable solution within string theory. However, since it does not connect with gauge theory at weak coupling it needs to include additional pieces with a different weak-coupling behaviour if the AdS/CFT correspondence is correct. In the following section we will comment on a natural non-trivial homogeneous solution which could have a chance of being part of the correct physical answer.

5.2 Special Solution

A special perturbative solution of crossing is given by the odd-n contributions of (3.11) with $n \geq 3$

$$\theta^{\text{hom}} = \sum_{n=1}^{\infty} \theta^{(2n+1)}. \qquad (5.4)$$

Superficially it may seem that all these are zero due to the coefficient $\text{B}_n = 0$ for odd $n \geq 3$. However, the term $\cos(\frac{1}{2}\pi n)$ in the denominator also vanishes for odd $n \geq 3$ and therefore we need to regularise $c_{r,s}^{(n)}$. A natural extension of the Bernoulli numbers is to use the identity

$$\text{B}_n = -\frac{2\,\Gamma(n+1)\cos(\frac{1}{2}\pi n)}{(-2\pi)^n}\,\zeta(n) \qquad (5.5)$$

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to replace them by the Riemann zeta function. Then the coefficients $c_{r,s}^{(n)}$ can be seen to be anti-symmetric under the interchange of r and s. Furthermore, they are odd polynomials in $r-1$ as well as in $s-1$. Therefore the properties of $c_{r,s}^{(n)}$ agree with all the properties of the general homogeneous solution (5.1) and consequently the above θ^{hom} represents a special homogeneous solution.

As this homogeneous solution is a natural extension of the inhomogeneous one, it could point towards the correct physical answer. The first contribution of this type appears at three world sheet loops and reads

$$\chi^{(3)}(x_1, x_2) = -\frac{x_2 + x_1 x_2^2 + 6x_2^3 - 6x_1 x_2^4 + x_2^5 - 3x_1 x_2^6}{32g^2(x_1 x_2 - 1)^2(x_2^2 - 1)^3} \frac{\zeta(3)}{\pi^3}. \tag{5.6}$$

The higher-loop contributions have a similar form, however, we do not know the analytic structure of the sum χ^{hom}. Let us only mention that, in contrast to the even-n contributions, the expansion of $c_{r,s}^{\mathrm{hom}}(g)$ does not stop at g^{s-r} as in the case of $c_{r,s}^{\mathrm{even}}(g)$. Thus, adding this piece will substantially alter the weak-coupling behaviour.

5.3 Rational Solutions

In the preceding sections we have already seen a couple of explicit solutions to the homogeneous crossing equation. Let us collect and investigate them briefly here. None of these solutions will actually be of the perturbative form proposed in section 3.1.

One of the homogeneous solutions is proportional to

$$C_1 p_2 - C_2 p_1 = g\left(\frac{1}{x_1^+} - \frac{1}{x_1^-} - \frac{i}{2g}\right)\log\frac{x_2^+}{x_2^-} - g\left(\frac{1}{x_2^+} - \frac{1}{x_2^-} - \frac{i}{2g}\right)\log\frac{x_1^+}{x_1^-}. \tag{5.7}$$

In principle, this term could also be written using $x_{12}^{(\pm m)}$ instead of $x_{12}^\pm$. Nevertheless the function alters the strong-coupling limit substantially unless suppressed by sufficiently many powers of $1/g$ or if it appears in a suitable linear combination. On its own, we can exclude it.

It is also clear that the monodromies of the one-loop contribution (3.17) are homogeneous solutions as well. By themselves they violate unitarity,[12] but they can be symmetrised w.r.t. unitarity and one obtains the function h^{odd} governing the odd part of the crossing relation, cf. (2.14). A slight generalisation gives the following homogeneous solution

$$-\frac{i}{2}\log\frac{x_1^{(+m)} - x_2^{(+n)}}{x_1^{(+m)} - x_2^{(-n)}}\frac{x_1^{(-m)} - x_2^{(-n)}}{x_1^{(-m)} - x_2^{(+n)}}\frac{1 - 1/x_1^{(+m)}x_2^{(-n)}}{1 - 1/x_1^{(+m)}x_2^{(+n)}}\frac{1 - 1/x_1^{(-m)}x_2^{(+n)}}{1 - 1/x_1^{(-m)}x_2^{(-n)}}, \tag{5.8}$$

Note that this expression is somewhat reminiscent of the CDD poles [34] for ordinary S-matrices; here the position of the poles is determined by the two integer parameters m, n. The transformation of this solution under parity symmetry depends on how the

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[12]Note that the one-loop solution does not violate unitarity due to its branch cuts. More explicitly, the variables $q^\pm$, cf. (3.23), will be exchanged. The function $\psi(q)$ in (3.22) then produces extra instances of the monodromy terms and unitarity is recovered.

$x^{(k)}$ transform, i.e. to $-x^{(-k)}$ or to $-1/x^{(-k)}$. To preserve parity, one of the two $x^{(k)}$ needs to transform to $-x^{(-k)}$ and the other one to $-1/x^{(-k)}$. In the analytic expression for the even part of the phase (4.9) this was the case: The variable $x^{\pm}$ transforms without inverse while $x^{(2n-1)}$ transforms with inverse if $s^{(2n-1)} = -1$. If the sign was $s^{(2n-1)} = +1$ instead, the variable $x^{(2n-1)}$ would transform without inverse, but also the homogeneous term would be absent due to the prefactor. Similarly, the above term (5.8) should be activated only if it does not violate parity, i.e. if $s_1^{(\pm m)} = -s_2^{(\pm n)}$.

6   Conclusions and Outlook

In this paper we have constructed a dressing phase factor for the world sheet scattering matrix of type IIB string theory on $AdS_5 \times S^5$. The general expression that we propose solves the condition imposed by crossing symmetry on the dressing factor [15], and admits an expansion in the strong coupling regime. The main result of the paper is a proposal for the coefficients governing this series. The coefficients provide an explicit all-loop expansion of the dressing phase factor, and contain the leading order term [4] as well as the first quantum correction [27,17,28]. A direct two-loop test of this proposal would be highly desirable and it might even be feasible.

The structure of the perturbative series is not straight-forward because it does not converge properly. In order to support the proposed expansion, we have shown how the coefficients satisfy the different pieces in the crossing relation. In particular, in order to satisfy the odd piece of the crossing relation it suffices to consider the one-loop contribution to the phase. This was already suggested in [19] and the present work contains a proof of the statement. Moreover, we specify clearly how the antipode map must act in order to obey the correct crossing relation. The even piece of the crossing relation is satisfied when including the even terms in the loop expansion of the phase.

In addition, we have found an analytic expression for the resummed series. The strong-coupling limit of the perturbative series agrees with this analytic expression. Furthermore, the analytic form for the resummed series allows an analysis of the spectrum of bound states of giant magnons. Bound states arise as poles of the scattering matrix, and have been found for giant magnons in [20]. We have identified these bound states of giant magnons using the analytic form of the dressing phase. What complicates the discussion is that our analytic expression for the phase involves an arbitrary or even an infinite number of branch points. Specially the kinematical space for giant magnons becomes an infinite-genus surface. In contrast, there is also a minimal particle for which the phase merely requires a genus-one surface. In order to probe the structure of the dressing phase, we have furthermore considered scattering of bound states. This seems to point out toward the possibility that the giant magnon states of [20] are not elementary, but rather composites of some minimal particles. A better understanding of this issue as well as the analytic structure of the phase in general clearly deserves further study.

We have also presented an abridged study of homogeneous solutions to the crossing relation. We know only few physical constraints on the homogeneous piece of the crossing condition, and thus most of the homogeneous solutions can be introduced arbitrarily. A careful look at them is worthwhile because they might be at the root of a discrepancy

between our proposed phase and gauge theory: The weak coupling limit of the analytic phase disagrees with gauge theory, as opposed to the agreement in the strong-coupling regime with string theory. In particular, our perturbative phase includes homogeneous pieces which we were not able to sum up to an analytic expression. These homogeneous solutions most likely change the weak-coupling behaviour, and open the possibility for a cure of the disagreement in the gauge theory limit. Homogeneous solutions could also clarify the nature, or even the existence, of the fractional singularities in the scattering matrix for the general parity-invariant dressing phase that we have constructed. Further research on homogeneous solutions to the crossing condition could clarify the existence of a smooth interpolating function from the string to the gauge theory scattering matrices. A three-loop string theory calculation would verify or disprove the first homogeneous piece in our perturbative phase, but unfortunately this is most likely beyond the current computational abilities.

Finally let us note that in the present work we have not considered particles whose energy and momentum scale as $\lambda^{\pm 1/4}$ (GKP regime). An investigation of this kinematical regime at strong coupling may be particularly interesting because it contains some of the special points in the phase. Furthermore, it seems that the structure of the perturbation series should be changed which possibly enables different tests of our proposal. Another interesting class of states are the 'antiferromagnetic' states [35–37] whose study in the current framework might lead to further insight.

Acknowledgements

A Strong and Weak Coupling

In this appendix we collect useful expressions to parametrise the kinematics of particles in the strong and weak-coupling limits.

A.1 Strong Coupling

Let us start with strong coupling, $g \to \infty$. In this case there are four interesting and distinct regions for the kinematic space. This can most easily be seen by considering relation (2.4). For large values of g we can solve constraint (2.4) by setting either $x^+ \approx x^-$ or $x^+ \approx 1/x^-$ or $x^+ \approx x^- \approx \pm 1$. These regions remind curiously of the different types

of particles considered in [38].[13]

Note that we will use a relativistic rapidity variable ϑ (different in all four cases) to parametrise the momenta of particles. The energy and momentum must be periodic in shifts of ϑ by $2\pi i$ and real for real values of ϑ. Furthermore, increasing ϑ should increase the momentum.

Metsaev-Tseytlin Regime. The first class of solutions to (2.4) is

$$x^{\pm} = \coth(\tfrac{1}{2}\vartheta)\left(1 \pm \frac{i}{4g}\sinh\vartheta\right) + \mathcal{O}(1/g^2) , \tag{A.1}$$

The resulting momentum, energy and u-parameter are given by

$$p = \frac{\sinh\vartheta}{2g} + \mathcal{O}(1/g^2) , \quad C = \tfrac{1}{2}\cosh\vartheta + \mathcal{O}(1/g) , \quad u = 2\coth\vartheta + \mathcal{O}(1/g^2) . \tag{A.2}$$

Most importantly, the momentum $p = \mathcal{O}(1/g)$ is very small. This combination corresponds to particles which behave like elementary excitations [1] in the plane-wave limit, see [21]. They also serve as the quantum constituents for the Frolov-Tseytlin spinning string solutions [24], see [40].

Hofman-Maldacena Regime. The second solution is

$$x^{\pm} = -\tanh\vartheta \pm \frac{i}{\cosh\vartheta} + \mathcal{O}(1/g) = e^{\pm ip/2} + \mathcal{O}(1/g) \tag{A.3}$$

The resulting momentum, energy and u-parameter are given by

$$p = \pi + 2\arctan\sinh\vartheta + \mathcal{O}(1/g) , \quad C = \frac{2g}{\cosh\vartheta} + \mathcal{O}(1/g^0) , \quad u = -2\tanh\vartheta + \mathcal{O}(1/g) . \tag{A.4}$$

In this case the momentum is finite, $p = \mathcal{O}(1/g^0)$, and its range is given by $0 < p < 2\pi$. This is the limit investigated by Hofman and Maldacena [20] with "giant" magnons as excitations. It is also the region of the spinning string solutions first found in [41].

Gubser-Klebanov-Polyakov Regime. The last two solutions involve square roots of g,

$$x^{\pm} = s + \frac{se^{-s\vartheta} \pm ie^{s\vartheta}}{2\sqrt{g}} + \mathcal{O}(1/g) , \tag{A.5}$$

where $s = \pm 1$ distinguishes the two solutions. In this case we find

$$p = s\frac{e^{s\vartheta}}{\sqrt{g}} + \mathcal{O}(1/g^{3/2}) , \quad C = \sqrt{g}\,e^{s\vartheta} + \mathcal{O}(1/g^{1/2}) , \quad u = 2s - \frac{\sinh(2\vartheta)}{2g} + \mathcal{O}(1/g^2) . \tag{A.6}$$

Now the momentum is small, $p = \mathcal{O}(1/g^{1/2})$, but not as small as in the above case. This region comprises the states whose energy scales as $\sqrt{g} \sim \sqrt[4]{\lambda}$ studied by Gubser, Klebanov and Polyakov [23]. As found in [4], the particles split up in right movers with $s = +1$ and left movers with $s = -1$.

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[13] In general, it would be interesting to recover our proposal from a covariant framework as in [38,36,39] which has proved to work at least to the leading order.

A.2 Weak Coupling

For weak coupling $g \to 0$, we find two distinct regions where the particle momenta are real. We shall use the momentum p as the fundamental parameter.

Magnons. The standard magnons correspond to the solution

$$x^{\pm} = \frac{e^{\pm ip/2}}{2g \sin(\frac{1}{2}p)} + \mathcal{O}(g) \tag{A.7}$$

The resulting energy and u-parameter are given by

$$C = \tfrac{1}{2} + 4g^2 \sin^2(\tfrac{1}{2}p) + \mathcal{O}(g^4) , \quad u = \frac{\cot(\frac{1}{2}p)}{2g} + \mathcal{O}(g) . \tag{A.8}$$

Holes. The other relevant solution comprises magnon-holes with

$$x^{\pm} = -\frac{2g \sin(\frac{1}{2}p)}{e^{\mp ip/2}} + \mathcal{O}(g^3) \tag{A.9}$$

Their energy and u-parameter read

$$C = -\tfrac{1}{2} - 4g^2 \sin^2(\tfrac{1}{2}p) + \mathcal{O}(g^4) , \quad u = -\frac{\cot(\frac{1}{2}p)}{2g} + \mathcal{O}(g) . \tag{A.10}$$

B Weak-Coupling Expansion

At weak coupling and for standard magnons, the $x^{\pm}$ variables scale as $x^{\pm} \sim 1/g$. The combination $c_{r,s}^{(n)} q_r(x_1^{\pm}) q_s(x_2^{\pm})$ consequently scales as $g^{r+s-n-1}$. The lowest-order terms in χ^{even} therefore appear for n as large as possible, i.e. for $n = s - r + 1$. This means that for fixed r, s the lowest order is g^{2r-2} and globally it is $\mathcal{O}(g^2)$ with terms of $r = 2$ and arbitrary odd s contributing

$$\chi^{\mathrm{weak-LO}}(x_1, x_2) = -\sum_{s=3}^{\infty} \frac{c_{2,s}^{(s+1)}}{(s-1)x_1 x_2^{s-1}} = \sum_{n=2}^{\infty} \frac{i^n \mathrm{B}_n}{2ng^{n-1}x_1 x_2^n} . \tag{B.1}$$

This series is not well-defined due to the asymptotics of the Bernoulli numbers $\mathrm{B}_n \sim n!$. A standard procedure in field theory is to Borel sum the series. This can be done in the present case. Let us first define a function

$$H(x) = \log(igx) - \sum_{n=2}^{\infty} \frac{i^n \mathrm{B}_n}{n \, g^n x^n} \tag{B.2}$$

such that

$$\chi^{\mathrm{weak-LO}}(x_1, x_2) = \frac{g}{2x_1} \log(igx_2) - \frac{g}{2x_1} H(x_2) . \tag{B.3}$$

First of all we perform an inverse Laplace transformation assuming that igx has a positive real part. Then the sum can be performed due to improved convergence

$$H(x) = \log(igx) - \int_0^\infty dt \exp(-igxt) \sum_{n=2}^\infty \frac{B_n}{n!} t^{n-1}$$
$$= \log(igx) - \int_0^\infty dt \exp(-igxt) \left(\frac{1}{2}\coth(t/2) - \frac{1}{t}\right) . \tag{B.4}$$

Finally, we perform the Laplace transformation integral to recover an analytic function

$$H(x) = \Psi(igx) + \frac{1}{2igx} \tag{B.5}$$

involving the digamma function $\Psi(z) = \partial_z \log\Gamma(z)$. The expansion of this function for large and *positive igx* in fact agrees with the series (B.2). For large negative igx the function oscillates strongly which explains the divergence of the series.

We can now convert $\chi^{\text{weak}-\text{LO}}$ to the dressing phase using (3.4). It is curious to see that the digamma function appears in the combination $\Psi(igx^+) - \Psi(igx^-)$ and that at weak coupling $igx^- - igx^+ \approx 1$. Together, the two facts lead to a large cancellation between the digamma functions and one is left with the simple term i/gx^+. In any case, the resulting phase is non-zero at $\mathcal{O}(g^2)$. It is therefore clear that this result does not agree with planar gauge theory for which the phase is zero at least at $\mathcal{O}(g^4)$ [22, 8, 42].

Nevertheless, it may be important to find the higher-order corrections at weak coupling. It turns out that these can also be written in terms of the function $H(x)$ and its derivatives. We find that the correct expansion of χ^{even} is encoded in the function

$$T(x_1, x_2, t) = \sum_{r=2}^\infty \sum_{m=0}^\infty \sum_{k=0}^m \frac{g\,(-1)^{r-1}t^{r+2m-k-2}}{2(m-k)!\,k!\,(r+m-k-1)!\,x_1^{r-1}x_2^k} \tag{B.6}$$

which acts on $H(x_2)$ as follows

$$\chi^{\text{even}}(x_1, x_2) = \frac{g}{2x_1}\log(igx_2) + :T(x_1, x_2, \partial/\partial x_2): H(x_2) \tag{B.7}$$
$$+ \frac{g}{2}\left(-\frac{1}{x_1} - \frac{1}{x_2}\right) + \frac{g}{2}\left(-x_1 - x_2 + \frac{1}{x_1} + \frac{1}{x_2}\right)\log\left(1 - \frac{1}{x_1 x_2}\right) .$$

The bracket $: \ldots :$ implies normal ordering between the variable x_2 and its derivative operator $\partial/\partial x_2$. The equivalence between (B.7,B.6,B.5) and (3.5,3.11) can be checked straightforwardly using the identity

$$\sum_{k=0}^m \frac{(-1)^{k-1}(n+r+2m-k-3)!}{(m-k)!\,k!\,(r+m-k-1)!} = \frac{\Gamma(m+n-1)\,\Gamma(m+n+r-2)}{\Gamma(n-1)\,\Gamma(m+1)\,\Gamma(m+r)} \tag{B.8}$$

and substituting $m = (s-r+1-n)/2$ in the final expression.[14]

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[14]The terms with $n = 0$ are reproduced by the expression only up to terms which are symmetric under the interchange of x_1 and x_2. These are cancelled by the terms on the second line in (B.7).

Let us remark that the function T can be summed up using an integral of the Bessel function I_0

$$T(x_1, x_2, t) = -\frac{g}{2x_1} \exp\left(\frac{t}{x_2} - \frac{t}{x_1}\right) \int_0^1 dq \exp\left(\frac{t}{x_1} q\right) I_0\left(2t\sqrt{q}\right) . \tag{B.9}$$

The expansion of T for large x_1, x_2 and small t reads

$$T(x_1, x_2, t) = -\frac{g}{2x_1} + \frac{gt(x_2 - tx_1 x_2 - 2x_1)}{4x_1^2 x_2} + \ldots \tag{B.10}$$

in agreement with the series (B.6).

It is now not difficult to compare also the weak coupling expansion of the analytic expression (4.1) with the above procedure. We find perfect agreement with the expansion of the function

$$\chi^{\text{even}-\text{left}}(x_1, x_2) = \lim_{N \to \infty} \left[\frac{g}{2x_1} \log \frac{igx_2}{N} + \frac{i}{2} \sum_{n=1}^{N} \log\left(1 - \frac{1}{x_1 x_2^{(-2n)}}\right)\right] \tag{B.11}$$

$$+ \frac{g}{2}\left(-\frac{1}{x_1} - \frac{1}{x_2}\right) + \frac{g}{2}\left(\frac{i}{2g} - x_1 - x_2 + \frac{1}{x_1} + \frac{1}{x_2}\right) \log\left(1 - \frac{1}{x_1 x_2}\right) .$$

at the leading six orders. The expression (B.11) does not literally agree with (4.1), but only after symmetrising as follows

$$\chi^{\text{even}}(x_1, x_2) = \tfrac{1}{2}\chi^{\text{even}-\text{left}}(x_1, x_2) - \tfrac{1}{2}\chi^{\text{even}-\text{left}}(-x_1, -x_2) . \tag{B.12}$$

The reason for this additional step in comparing can be explained as follows: The exact expression has an essential singularity at $x_2 = \infty$ due to accumulation of singularities. Therefore the power series around $x_2 = \infty$ could possibly not converge. In performing the above Borel summation and Laplace transform we specified that the real part of igx_2 is positive. Effectively, this regularised the resummed expression such that the singularities approach $x_2 = \infty$ with negative igx_2. If we had chosen to use negative igx_2 in the Laplace transform, the resummed expression would have the singularities approaching $x_2 = \infty$ with positive igx_2. In other words the resummed expression is ambiguous and in (4.1) we chose to present the symmetrised expression which has manifest parity invariance. This matching provides further evidence for the agreement between the proposed coefficients (3.11) and the proposed analytic expression (4.1).

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