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On the number of customers served in the $M/G/1$ retrial queue: first moments and maximum entropy approach

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Abstract

In this paper we present general results on the number of customers, I, served during the busy period in an $M/G/1$ retrial system. Its analysis in terms of Laplace transforms has been previously discussed in the literature. However, this solution presents important limitations in practice; in particular, the moments of I cannot be obtained by direct differentiation. We propose a direct method of computation for the second moment of I and also for the probability of $k, k \leqslant 4$, customers being served in a busy period. Then, the maximum entropy principle approach is used to estimate the true distribution of I according to the available information.

Scope and purpose

We consider an $M/G/1$ queue with retrials. Retrial queueing systems are characterized by the fact that, an arriving customer who finds the server busy is obliged to leave the service area and return later to repeat his request after some random time. We deal with I, the number of customers served during the busy period of a retrial queue, and obtain closed expressions for its main characteristics, which will be employed in order to estimate the true distribution of this random variable.

Keywords: Retrial queue; Busy period; Laplace transform; Maximum entropy principle

1. Introduction

Many queueing situations such as computer and communication systems have the feature that customers who find all servers busy upon arrival are obliged to leave the service area and to

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come back to the system after a random amount of time. Such customers are not allowed to queue so, repeated attempts for service from the pool of unsatisfied customers (called "orbit" or "retrial group") are superimposed on the normal stream of arrivals of first attempts. The most classical application of a retrial queue arises from telephone traffic theory where a subscriber whose call receive a busy signal is not allowed to await the termination of the busy condition. In this context each blocked call generates a source of repeated requests for service independently of the rest of calls in orbit. A review of the main results, literature, and discussion of practical situations where retrial queues arise can be found in the book by Falin and Templeton [1]; in addition, accessible bibliography can be found in [2] and [3].

In this paper, we consider the $M/G/1$ queue with exponential retrial times and analyze I, the number of customers served during the busy period L; it should be noted that the evolution of a retrial queue is described in terms of an alternating sequence of periods in which the server is free or busy. So, the busy period is defined as the time between a primary arrival finding the server idle with no customers in orbit, and a departure in which the system returns first to this state. In teletraffic applications of retrial queues only the service zone is observable but, this is not the situation in computational applications where repeated attempts are made (in a collective sense) by the service facility. However, the difficulty in observing the retrial group does not alter the fact that its analysis can be done from a theoretical point of view (see Artalejo and Falin [4]). As occurs in classical queues, the study of the main system characteristics (limit distribution, waiting time, busy period, number of customers served) is the key for understanding the whole system performance. In particular, the limit distribution and the waiting time enlighten about the physical behaviour of the system from a general perspective or from the customer's point of view, respectively. Busy period is particularly important related to regenerative processes. For our specific interest we would like to point out that, the random variable I can be seen as a measure of information about the level of congestion of the system, as the limiting probabilities do.

The analysis of I and L, has been the subject matter of previous articles by Choo and Conolly [5], Falin [6], Artalejo and Falin [4], and Artalejo and Lopez-Herrero [7]. The joint Laplace transform of the busy period and the number of customers served in this period was obtained in [6] by using an approach based on the method of collective marks. Artalejo and Falin [4] were deep into the structure of the busy period by investigating two characteristics of the orbit, namely, the orbit busy period and the orbit idle period. Artalejo and Lopez-Herrero [7] gave a closed expression for $E[L^2]$ by analyzing a set of transient taboo probabilities. When exponential service times are considered Choo and Conolly [5] developed a method for the numerical solution for the moments of L.

It should be noted that the existing literature on the number of customers served during the busy period for an $M/G/1$ retrial queue provides theoretical solutions in terms of its Laplace transforms, even when we consider the classical model $M/G/1$. Since these results involve complex integral expressions, the Laplace transform of the service time distribution and the Laplace transform of I in the classical $M/G/1$ system without repeated attempts, they are not convenient for practical applications. However, the expected number of customers served in a busy period, $E[I]$, follows easily from the theory of regenerative processes. Thus, our main objective in this paper is to analyze the random variable I, obtaining explicit expressions for its main characteristics, which will be useful from an applied perspective. In particular, our main goals are as follows:

(a) We obtain explicit formulae for the probabilities of 1, 2, 3 and 4 customers being served in a busy period, and also an explicit expression for the second order moment $E[I^2]$ of the $M/G/1$ retrial queue.
(b) We employ the principle of maximum entropy to estimate the distribution of I; i.e., $P\{I=k\}, k \geqslant 1$.

It should be pointed out that knowledge of the first two moments is essential to provide measures of the variability of I such as the variance $Var[I](=E[I^2]-E^2[I])$ and the coefficient of variation $C_I(=\sqrt{Var[I]}/E[I])$, and also to study inference problems. To know the values of $E[I]$ and $E[I^2]$ it is essential for developing accurate approximations based on the maximum entropy formalism.

The organization of the paper is as follows. In Section 2 we describe the mathematical model. In Section 3 we propose a method of obtaining the probabilities $p_k=P\{I=k\}$, when $k=1,2,3,4$, also we employ a "catastrophe process" to derive a direct method of calculation for $E[I]$ and $E[I^2]$. Numerical results for different service times distributions and sets of parameters are shown. In Section 4 we summarize the methodology of the maximum entropy principle. Approximated probability distributions, using one or both moments, are presented for different service times distributions and various ranges of parameter values.

In the future we would like to extend our study to more complex queueing models: retrial models with batch arrivals and/or server vacations structures (cf. [8] and [9]).

2. The mathematical model

We consider a single server queueing system to which primary customers arrive according to a Poisson stream of rate λ. Any customer who finds the server busy upon arrival must leave the service area in order to seek for service again at subsequent epochs, until he finds the server free. Between trials the customer is said to be in orbit. Delays between retrials of each customer in orbit are independent and identically exponentially distributed with rate $\mu > 0$. Service times are independent and identically distributed random variables with probability distribution function $B(x)$ (with $B(0)=0$), density function $b(x)$, Laplace–Stieltjes transform $\beta(\theta)$, and β_1 and β_2 are the first and second order moments of service time. We assume that interarrival times, service times, and retrial times are mutually independent random variables.

At any arbitrary time t, the system state can be described by means of a bidimensional process $Y(t)=(C(t),N(t))$, where $C(t)$ is 0 or 1 according to whether the server is free or busy, and $N(t)$ denotes the number of customers in orbit at time t. It is well known that the stability condition $\rho=\lambda\beta_1 < 1$ guarantees that the limiting distribution of the process $Y(t)=(C(t),N(t))$ exists and is positive. The transitions among states in the process $Y(t)$ are described in Fig. 1.

3. Analysis of I and computation of $E[I^2]$

Let us assume that at time $t=0$ the system is empty, i.e., $C(0)=N(0)=0$, and one primary customer arrives just at time $t=0$. Our objective is to analyze the distribution of I, the number of customers served during the interval of time starting at this epoch and ending at the next

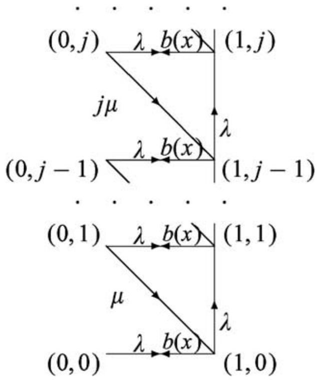

Fig. 1. State space and transitions.

departure epoch at which the process $Y(t)$ returns first to the state $(0,0)$, i.e., customers served during the busy period, L.

Next, we determine the exact values of the mass probability function $p_k = P\{I = k\}$, when $k \in \{1,2,3,4\}$, by direct probability statements. To obtain closed expressions for higher states is really cumbersome as we will see.

When $k \leqslant 4$, the probabilities p_k admit the following expressions given in terms of the service times Laplace transform, $\beta(s)$.

$$p_1 = \beta(\lambda), \tag{3.1}$$

$$p_2 = -\beta'(\lambda)\beta(\lambda)\frac{\lambda\mu}{\lambda+\mu}, \tag{3.2}$$

$$p_3 = -\beta'(\lambda)\beta^2(\lambda)\frac{\lambda^2\mu}{(\lambda+\mu)^2} + (\beta'(\lambda))^2\beta(\lambda)\frac{\lambda^2\mu^2}{(\lambda+\mu)^2} + \beta''(\lambda)\beta^2(\lambda)\frac{\lambda^2\mu^2}{(\lambda+\mu)(\lambda+2\mu)} \tag{3.3}$$

$$\begin{aligned} p_4 = &-\beta'''(\lambda)\beta^3(\lambda)\frac{\lambda^3\mu^3}{(\lambda+3\mu)(\lambda+2\mu)(\lambda+\mu)} + \beta''(\lambda)\beta^3(\lambda)\frac{\lambda^3\mu^2}{(\lambda+2\mu)^2(\lambda+\mu)} \\ &+\beta''(\lambda)\beta^3(\lambda)\frac{\lambda^3\mu^2}{(\lambda+2\mu)(\lambda+\mu)^2} - \beta''(\lambda)\beta'(\lambda)\beta^2(\lambda)\frac{2\lambda^3\mu^3}{(\lambda+2\mu)^2(\lambda+\mu)} \\ &+(\beta'(\lambda))^2\beta^2(\lambda)\frac{2\lambda^3\mu^2}{(\lambda+2\mu)(\lambda+\mu)^2} - \beta''(\lambda)\beta'(\lambda)\beta^2(\lambda)\frac{2\lambda^3\mu^3}{(\lambda+2\mu)(\lambda+\mu)^2} \\ &+(\beta'(\lambda))^2\beta^2(\lambda)\frac{2\lambda^3\mu^2}{(\lambda+\mu)^3} - \beta'(\lambda)\beta^3(\lambda)\frac{\lambda^3\mu}{(\lambda+\mu)^3} - (\beta'(\lambda))^3\beta(\lambda)\frac{\lambda^3\mu^3}{(\lambda+\mu)^3}. \end{aligned} \tag{3.4}$$

In order to prove (3.1)–(3.4) we use the probabilities $k_n, n \geqslant 0$, associated with the event "n primary customers arrive during a service time". Note that

$$k_n = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \, dB(t) = (-1)^n \frac{\lambda^n}{n!} \beta^{(n)}(\lambda), \quad n \geqslant 0. \tag{3.5}$$

Now, observe that $I = 1$ if and only if there are not primary arrivals during the first service time so, (3.1) follows easily. The event associated with p_2 can occur if and only if one primary customer arrives during the first service time, joins the orbit, from there goes into the server and no other one primary customer arrives to the service facility at any time during this period; i.e.,

$$p_2 = k_1 \frac{\mu}{\lambda + \mu} k_0,$$

which agrees with (3.2) by using (3.5).

To prove (3.3) and (3.4) we reason in a similar way, taking into account that there are different possibilities for serving three or four customers during the busy period. Namely, three and eleven, different possibilities respectively, depending on the epochs of the primary arrivals. Note that as the number of customers served increases its achievement looks more cumbrous due to the number of possibilities leading up to that event.

In order to determine the second moment, $E[I^2]$, we introduce an additional Poisson process with rate s, for some fixed $s > 0$. And we assume that this process is independent of our queueing system performance. In what follows we refer to this process as "catastrophe process". Usually, this methods of collective marks allows us to get formulas for Laplace–Stieltjes transforms directly. In fact, we will arrive to an ordinary differential equation which will be the key for determining the expression for the joint transform of the busy period L and the number of customers served I, and moreover for $E[I]$ and $E[I^2]$.

Suppose that at the epoch of the ith service completion there are $n \geqslant 1$ customers in orbit, no catastrophe has occurred till this moment, and the busy period has not come to an end. Let $P_n^{(i)}(s)$ denote the probability of that event. And define $\Pi_i(s)$ as the probability that during the busy period no catastrophe occurs and i customers are served.

The probabilities $P_n^{(i)}(s)$ satisfy the following recursion formulae:

$$P_n^{(1)}(s) = k_n(s), \quad n \geqslant 1, \tag{3.6}$$

$$P_n^{(i)}(s) = \sum_{m=1}^{n+1} P_m^{(i-1)}(s) \frac{m\mu}{s + \lambda + m\mu} k_{n-m+1}(s) + \sum_{m=1}^{n} P_m^{(i-1)}(s) \frac{\lambda}{s + \lambda + m\mu} k_{n-m}(s), \quad i \geqslant 2, \; n \geqslant 1, \tag{3.7}$$

where $k_n(s) = \int_0^\infty e^{-(s+\lambda)t} \frac{(\lambda t)^n}{n!} \, dB(t), \quad n \geqslant 0.$

Observe that probabilities $k_n(s)$, given in terms of the catastrophe process, are a generalization of the ones appearing above at the Eq. (3.5). In that sense, for each fixed $n \geqslant 0$, $k_n(s)$ is associated with the event "no catastrophe occurs and n primary customers arrive during the service time".

In order to prove (3.6) note that the event associated to $P_n^{(1)}(s)$ occurs if and only if the event related to $k_n(s)$ does. Eq. (3.7) describes the motion of the system between two successive completion epochs. Probabilities $m\mu/s+\lambda+m\mu$ and $\lambda/s+\lambda+m\mu$ indicate, respectively, whether the ith customer comes from the orbit or is a primary arrival.

The relation between $\Pi_i(s)$ and $P_1^{(i-1)}(s)$ is given by

$$\Pi_i(s) = \begin{cases} \beta(s+\lambda), & \text{if } i=1, \\ \dfrac{\mu}{s+\lambda+\mu} P_1^{(i-1)}(s)\beta(s+\lambda), & \text{if } i \geqslant 2. \end{cases} \tag{3.8}$$

Let us now introduce the generating functions

$$\varphi^{(i)}(s,z) = \sum_{n=1}^{\infty} z^n \frac{P_n^{(i)}(s)}{s+\lambda+n\mu}, \quad i \geqslant 1, \ z \in [0,1],$$

$$P^{(i)}(s,z) = \sum_{n=1}^{\infty} z^n P_n^{(i)}(s), \quad i \geqslant 1, \ z \in [0,1].$$

Then, we may write the following relation

$$P^{(i)}(s,z) = (s+\lambda)\varphi^{(i)}(s,z) + \mu z \frac{\partial}{\partial z}\varphi^{(i)}(s,z), \quad i \geqslant 1. \tag{3.9}$$

Now, working on Eqs. (3.6)–(3.8) it follows that the generating functions satisfy

$$P^{(1)}(s,z) = \beta(s+\lambda-\lambda z) - \beta(s+\lambda), \tag{3.10}$$

$$P^{(i)}(s,z) = \beta(s+\lambda-\lambda z)\left(\lambda\varphi^{(i-1)}(s,z) + \mu\frac{\partial}{\partial z}\varphi^{(i-1)}(s,z)\right) - \Pi_i(s), \quad i \geqslant 2. \tag{3.11}$$

Hence, from (3.9) and (3.10) we obtain

$$(s+\lambda)\varphi^{(1)}(s,z) + \mu z\frac{\partial}{\partial z}\varphi^{(1)}(s,z) = \beta(s+\lambda-\lambda z) - \beta(s+\lambda) \tag{3.12}$$

and from (3.9) and (3.11) we get

$$\begin{aligned}(s+\lambda)\varphi^{(i)}(s,z) &+ \mu z\frac{\partial}{\partial z}\varphi^{(i)}(s,z) \\ &= \beta(s+\lambda-\lambda z)\left(\lambda\varphi^{(i-1)}(s,z) + \mu\frac{\partial}{\partial z}\varphi^{(i-1)}(s,z)\right) - \Pi_i(s), \quad i \geqslant 2.\end{aligned} \tag{3.13}$$

Now we define the generating functions

$$\varphi(x,s,z)=\sum_{i=1}^{\infty}x^i\varphi^{(i)}(s,z),\quad x\in[0,1],$$

$$\Pi(x,s)=E[x^I\mathrm{e}^{-sL}]=\sum_{i=1}^{\infty}x^i\Pi_i(s),\quad x\in[0,1].$$

Taking into account Eqs. (3.12) and (3.13), we find that the generating functions $\varphi(x,s,z)$ and $\Pi(x,s)$ satisfy the following differential equation:

$$\mu(x\beta(s+\lambda-\lambda z)-z)\frac{\partial}{\partial z}\varphi(x,s,z)$$
$$=(s+\lambda-\lambda x\beta(s+\lambda-\lambda z))\varphi(x,s,z)-x\beta(s+\lambda-\lambda z)+\Pi(x,s), \tag{3.14}$$

which is an ordinary differential equation similar to (17) in [4] so, it is well known that the joint generating function of L and I is given by

$$\Pi(x,s)=\frac{\displaystyle\int_0^{\pi_\infty(x,s)}\frac{x\beta(s+\lambda-\lambda u)}{e(x,s,u)(x\beta(s+\lambda-\lambda u)-u)}\,\mathrm{d}u}{\displaystyle\int_0^{\pi_\infty(x,s)}\frac{\mathrm{d}u}{e(x,s,u)(x\beta(s+\lambda-\lambda u)-u)}},\quad s>0,\ x\in[0,1],$$

where $\pi_\infty(x,s)$ is the joint transform of the length of the busy period and the number of customers served in the classical $M/G/1$ queue, and

$$e(x,s,z)=\exp\left\{\frac{1}{\mu}\int_0^z\frac{s+\lambda-\lambda x\beta(s+\lambda-\lambda t)}{x\beta(s+\lambda-\lambda t)-t}\,\mathrm{d}t\right\},\quad z\in[0,\pi_\infty(x,s)].$$

At this point, we should mention the existence of numerical methods for inverting Laplace transforms, having numerous applications and widely used by many authors (see Abate et al. [10]). However, all these methods employ numerical evaluations for the Laplace transform over points in the complex plane and, it should be noted that the transform expression for $\Pi(x,s)$ has been obtained by assuming that its parameters take only real values (see Falin and Templeton [1], pp. 43–45). Consequently, the numerical inversion of $\Pi(x,s)$ needs first the analytical extension of this function to complex values. Instead of this approach we next consider an alternative one based on the generating function $\varphi(x,s,z)$ which will allow us to obtain a closed form expressions for $E[I]$ and $E[I^2]$ avoiding the derivatives on the transform, which always involve undetermined expressions difficult to solve by using L'Hôpital's rule.

First we define

$$a(z)=\varphi(1,0,z)=\sum_{i=1}^{\infty}\varphi^{(i)}(0,z).$$

Taking into account that the expected value $E[I]=\sum_{i=1}^{\infty} P\{I \geqslant i\}$ can be written in terms of the probabilities $P_n^{(i)}(0)$, we get the alternative expression

$$E[I]=1+\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} P_n^{(i)}(0)=1+\sum_{i=1}^{\infty} P^{(i)}(0,1).$$

And, using the relation between the generating functions $P^{(i)}(s,z)$ and $\varphi^{(i)}(s,z)$, shown at the Eq. (3.9), it follows

$$E[I]=1+\sum_{i=1}^{\infty}\left(\lambda\varphi^{(i)}(0,1)+\mu\frac{\partial}{\partial z}\varphi^{(i)}(0,1)\right),$$

that is,

$$E[I]=1+\lambda a(1)+\mu a'(1). \tag{3.15}$$

In order to determine the function $a(z)$ we put $s=0$ and $x=1$ on Eq. (3.14), and solving the corresponding differential equation we find that

$$a(z)=\frac{1}{\lambda}\left(\exp\left\{\frac{\lambda}{\mu}\int_0^z \frac{1-\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\,\mathrm{d}t\right\}-1\right),$$

$$a(1)=\frac{1}{\lambda}(\mathrm{e}^{\frac{\lambda}{\mu}J}-1),$$

$$a'(1)=\frac{\rho}{\mu(1-\rho)}\mathrm{e}^{\frac{\lambda}{\mu}J}, \tag{3.16}$$

where $J=\int_0^1 \frac{1-\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\,\mathrm{d}t$.

By substitution from (3.16) on (3.15) we obtain

$$E[I]=\frac{1}{1-\rho}\mathrm{e}^{\frac{\lambda}{\mu}J}, \tag{3.17}$$

which agrees with the result obtained by the theory of regenerative processes.

Next step is to define

$$\psi(x,s,z)=\frac{\partial}{\partial x}\varphi(x,s,z), \quad \text{and} \quad d(z)=\psi(1,0,z).$$

In order to get the second order moment note that $E[I^2]$ admits an expression given in terms of the generating functions $P^{(i)}(0,1)$ as follows:

$$E[I^2]=\sum_{i=1}^{\infty} i^2\,P(I=i)=\sum_{i=1}^{\infty}(2i-1)P(I\geqslant i)$$

$$=E[I]+2\sum_{i=1}^{\infty} iP^{(i)}(0,1).$$

Now, from Eq. (3.9) by using the generating function $\varphi(x,s,z)$ and its partial derivative $\psi(x,s,z)$ we get

$$E[I^2]=E[I]+2\left(\lambda\psi(1,0,1)+\mu\frac{\partial}{\partial z}\psi(1,0,1)\right),$$

or, equivalently

$$E[I^2]=E[I]+2(\lambda d(1)+\mu d'(1)). \tag{3.18}$$

Differentiating (3.14) with respect to x and setting $x=1$ and $s=0$, we obtain after some algebra

$$d'(z)=\frac{\lambda(1-\beta(\lambda-\lambda z))}{\mu(\beta(\lambda-\lambda z)-z)}d(z)+F(z), \tag{3.19}$$

where

$$F(z)=\frac{1}{\mu(\beta(\lambda-\lambda z)-z)}\left(\frac{e^{\frac{\lambda}{\mu}J}}{1-\rho}-\frac{(1-z)\beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)-z}\exp\left\{\frac{\lambda}{\mu}\int_0^z\frac{1-\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\,dt\right\}\right). \tag{3.20}$$

Since $d(0)=0$, we have

$$d(z)=\exp\left\{\frac{\lambda}{\mu}\int_0^z\frac{1-\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\,dt\right\}$$
$$\int_0^z\frac{1}{\mu(\beta(\lambda-\lambda t)-t)}\left(\frac{1}{1-\rho}\exp\left\{\frac{\lambda}{\mu}\int_t^1\frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}\,du\right\}-\frac{(1-t)\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\right)dt. \tag{3.21}$$

Finally, from the Eqs. (3.18)–(3.21) we get the explicit expression for $E[I^2]$,

$$E[I^2]=\frac{e^{\frac{\lambda}{\mu}J}}{1-\rho}\left\{\frac{1-\rho^2+2\frac{\lambda}{\mu}\rho+\lambda^2\beta_2}{(1-\rho)^2}+\int_0^1\frac{2\lambda}{\mu(\beta(\lambda-\lambda t)-t)}\right.$$
$$\left.\left(\frac{1}{1-\rho}\exp\left\{\frac{\lambda}{\mu}\int_t^1\frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}\,du\right\}-\frac{(1-t)\beta(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t}\right)dt\right\}. \tag{3.22}$$

Note that $E[I^2]$ tends to $(1-\rho^2+\lambda^2\beta_2)/(1-\rho)^3$, as $\mu\to\infty$, which agrees, as it could be expected, with the second order moment of the number of customers served during the busy period in the classic system $M/G/1$. numerical solution for the integrals appearing in (3.22) can be obtained by calling numerical recipes software (NRS) subroutine MIDPNT (See Press et al. [11]). In particular, the integrals computed in this paper have an accuracy level of $\varepsilon=10^{-6}$, i.e., we stop the computation when the difference between two successive steps is lesser than $\varepsilon=10^{-6}$.

Table 1
$E[I^2]$ versus λ and several hyperexponential service times

λ	$C_B = 1.25$	$C_B = 1.5$	$C_B = 1.75$
0.05	1.2019129	1.2039665	1.2063938
0.15	2.1406685	2.1673216	2.1989103
0.25	4.8616549	4.9663390	5.0927280
0.35	13.218555	13.467045	13.795339
0.45	42.078414	42.121634	42.510264
0.55	161.89714	156.42517	153.15780
0.65	824.38187	749.80652	697.75534
0.75	6748.9663	5535.8244	4732.6877
0.85	145733.40	98119.334	71710.198

Table 2
$E[I^2]$ versus λ and μ, hyperexponential service times with $(\beta_1, C_B) = (1, 1.25)$

μ	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
0.1	38.971470	24805.693	0.1820×10^{11}
0.25	8.7591958	403.79179	259282.89
0.5	4.8616549	80.277213	6748.9663
0.75	3.9560855	43.872232	1916.7752
1	3.5640991	31.857826	999.78017
5	2.7705186	13.902025	188.87035
10	2.6847866	12.446547	151.04730
50	2.6182176	11.379093	125.90938
100	2.6100209	11.251423	123.05021
250	2.6059329	11.188056	121.36308
∞	2.6018518	11.125	120.25

Following tables summarize numerical results on $E[I^2]$ for service times hyperexponentially and Erlang-k distributed. More precisely, in Table 1 we assume that the retrial rate is $\mu = 0.5$ and hyperexponential service times distribution, with $\beta_1 = 1$ and coefficient of variation $C_B = (\beta_2 - \beta_1^2)^{1/2}/\beta_1$ taking values 1.25, 1.5, and 1.75. The corresponding parameters of the density function $\sum_{i=1}^{2} p_i v_i \exp\{-v_i t\}$ given in terms of the fixed $\beta_1 = 1$ and C_B are

$$p_1 = \frac{1}{2}\left(1 + \sqrt{\frac{C_B^2 - 1}{C_B^2 + 1}}\right), \quad p_2 = 1 - p_1, \quad v_i = \frac{2p_i}{\beta_1}, \; i = 1, 2.$$

The arrival parameter λ is equal to ρ. So, Table 1 illustrates the effect on $E[I^2]$ by varying λ $(= \rho)$ for several levels of the coefficient of variation C_B. Numerical results show that $E[I^2]$ increases with increasing values of λ, and it is interesting to note that, as λ increases, $E[I^2]$ increases much faster in the smaller C_B.

In Table 2 we fix $\beta_1 = 1$, $C_B = 1.25$, and show the influence of the retrial rate μ when $\lambda = 0.25$, 0.5 and 0.75. From these results, we can observe that $E[I^2]$ increases with decreasing values of μ which agrees with the intuitive expectations. Moreover, numerical data when $\mu \to \infty$

Table 3
$E[I^2]$ versus λ and $k, \mu = 0.5$, Erlang service times with $\beta_1 = 1$

λ	$k = 2$	$k = 4$	$k = 6$
0.05	1.1987396	1.1979929	1.1977441
0.15	2.0997458	2.0901704	2.0869816
0.25	4.7079698	4.6735921	4.6622560
0.35	12.946556	12.910358	12.900598
0.45	43.206940	43.813209	44.049085
0.55	183.09583	192.37523	195.98221
0.65	1108.6021	1245.9711	1302.2596
0.75	12417.665	15895.435	17478.338
0.85	507347.71	862779.17	1063460.4

Table 4
$E[I^2]$ versus λ and μ, Erlang service times with $(\beta_1, k) = (1, 3)$

μ	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
0.1	41.952745	59811.591	0.2670×10^{13}
0.25	8.7308185	547.73744	1555064.5
0.5	4.6849918	87.481480	14554.246
0.75	3.7665444	43.705549	2931.4214
1	3.3728251	30.156599	1274.2822
5	2.5851293	11.333247	142.25116
10	2.5009640	9.9236158	104.59959
50	2.4357619	8.9065067	81.099298
100	2.4277430	8.7858848	78.513742
250	2.4229455	8.7141862	76.996995
∞	2.4197530	8.6666666	76.0

are consistent with the second order moment of the number of customers served during the busy period of the classic $M/G/1$.

In Tables 3 and 4, we consider service times distributed according to an Erlang-k random variable which give us coefficients of variation lesser than 1. In Table 3 we fix $\mu = 0.5$, $\beta_1 = 1$ and represent $E[I^2]$ for various choices of λ and k, it should be noted that the moment, as k gets large, increases faster as function of the arrival parameter λ.

In Table 4, for service times Erlang-3 with mean $\beta_1 = 1$, we show the effect of μ on $E[I^2]$ by keeping $\lambda = 0.25$, 0.5 and 0.75. As in the parallel analysis shown in Table 2, when the retrial rate μ increases, numerical data are consistent with the second moment of the number of customers served in a busy period of the classic $M/G/1$ queue. While when we consider decreasing values of μ the moment $E[I^2]$ increases.

4. Maximum entropy analysis

In the precedent section we get explicit expressions (Eqs. (3.17) and (3.22)) for $E[I]$ and $E[I^2]$ without using the mass distribution function of I. Even when the information about the distribution of the number of customers served is limited to knowledge of two moments we will develop accurate approximations based on the Principle of Maximum Entropy, PME. This principle provides a method of inference for estimating probability distributions, $\{p(S_n)\}$, given information expressed in terms of mean value constraints (see Kouvatsos [12] and the references therein) of the form

$$\sum_{S_n \in S} f_k(S_n) p(S_n) = F_k, \quad 0 \leqslant k \leqslant m < \infty, \tag{4.1}$$

where S is a discrete state space.

The PME states that, of all distributions satisfying the constraints supplied by the given information, the minimally prejudiced distribution $\{p(S_n)\}$ is the one that maximizes the Shannon's entropy functional

$$H(p) = -\sum_{S_n \in S} p(S_n) \ln(p(S_n)),$$

subject to the constraints given in (4.1).

The maximum entropy approximation, if existing, can be carried out by using Lagrange's method of undetermined multipliers and leads to the solution

$$\hat{p}(S_n) = (F_0)^{-1} \exp\left\{ -\beta_0 - \sum_{k=1}^{m} f_k(S_n)\beta_k \right\},$$

where $\{\beta_k\}$, $k = 1, 2, \ldots, m$, are the Lagrangian multipliers determined from the set of constraints, and β_0 is a Lagrangian multiplier determined by the normalization constraint; i.e.,

$$\exp\{\beta_0\} = \sum_{S_n \in S} \exp\left\{ -\sum_{k=1}^{m} f_k(S_n)\beta_k \right\}.$$

It can be seen that the Lagrangian multipliers $\{\beta_k\}$ satisfy the following relations:

$$-\frac{\partial \beta_0}{\partial \beta_k} = F_k, \quad 1 \leqslant k \leqslant m,$$

but, unfortunately it is not generally possible to get an explicit solution and it is necessary to employ numerical methods.

Shore [13] and El-Affendi and Kouvatsos [14] have obtained information theoretic inference approximations of the most usual queueing systems. Rego and Szpankowski [15] show an equivalence between using the entropy maximization method with a one moment constraint and assuming exponentiality. In the last decade, Arizono et al. [16], Kouvatsos and Tabet-Aouel [17] have extended the PME to more specific queues: $M/M/s$, $G/G/1$ with priority classes, etc. Also information theoretic approximations of the steady state distribution in an $M/G/1$ system were obtained in [18] for a retrial queue, and in [19] by assuming a finite population arriving at the service facility under the so-called quasi-random input.

Next, we determine the maximum entropy distribution of I by using its known moments and probabilities.

For each fixed $k \in \{0,1,2,3,4\}$ let us define

$$m_i^k = \sum_{j=k+1}^{\infty} j^i p_j,$$

where $i \in \{0,1,2\}$ and $p_j = P\{I = j\}$.

Note that, for any fixed $k \in \{0,1,2,3,4\}$, we can write

$$m_0^k = 1 - I_{\{k>0\}} \sum_{j=1}^{k} p_j, \tag{4.2}$$

$$m_1^k = E[I] - I_{\{k>0\}} \sum_{j=1}^{k} j p_j, \tag{4.3}$$

$$m_2^k = E[I^2] - I_{\{k>0\}} \sum_{j=1}^{k} j^2 p_j, \tag{4.4}$$

these will constitute the set of constraints appearing at (4.1) for our particular situation.

Hence, Shannon's entropy functional has the following form:

$$H(p) = -\sum_{j=1}^{\infty} p_j \ln(p_j)$$

and its maximization subject to (4.2) and one or both moments constraints, by using Lagrange's multipliers for each fixed $k \in \{0,1,2,3,4\}$, leads to a solution of the form

$$\hat{p}_j^{k,n} = \exp\left\{ -\beta_0^k - \sum_{i=1}^{n} j^i \beta_i^k \right\}, \quad j \geqslant k+1, \tag{4.5}$$

where $n \in \{1,2\}$.

Following section is devoted to the first order approximation, where it is possible to get explicit solutions for the Lagrangians β_0^k and β_1^k.

4.1. The first order approximation

Applying general theory, for any fixed $k \in \{0,1,2,3,4\}$ we get a solution of the form $\hat{p}_j = xy^j$, $j > k$. So, the probability distribution maximizing Shannon's entropy subject to the constraints (4.2) and (4.3) is given by

$$\hat{p}_j^{k,1} = \frac{(m_0^k)^2}{(m_1^k - k m_0^k)} \left(\frac{m_1^k - (k+1) m_0^k}{m_1^k - k m_0^k} \right)^{j-(k+1)}, \quad j \geqslant k+1. \tag{4.6}$$

Note that (4.6) gives a decreasing sequence of probabilities for the number of customers served during the busy period.

In that sense, Table 5 shows different one-moment approximations for the I probability distribution when service times are exponentially distributed with rate $\sigma = 1$, depending on the number of known probabilities used in the maximization of Shannon's entropy functional. Due

Table 5
First order approximation when $\lambda = 0.2$, and $\mu = 1.0$.

j	p_j	Simulation	$\hat{p}_j^{0,1}$	$\hat{p}_j^{1,1}$	$\hat{p}_j^{2,1}$	$\hat{p}_j^{3,1}$	$\hat{p}_j^{4,1}$
1	0.8333333	(0.83026, 0.84178)	0.7650823	0.8333333	0.8333333	0.8333333	0.8333333
2	0.0964506	(0.08903, 0.09811)	0.1797314	0.0904668	0.0964506	0.0964506	0.0964506
3	0.0367373	(0.03331, 0.03899)	0.0422221	0.0413614	0.0351204	0.0367373	0.0367373
4	0.0165012	(0.01568, 0.01983)	0.0099187	0.0189104	0.0175540	0.0159738	0.0165012
5	—	(0.00582, 0.00850)	0.0023301	0.0086458	0.0087739	0.0083522	0.0078564
6	—	(0.00347, 0.00624)	0.0005474	0.0039529	0.0043854	0.0043671	0.0042208
7	—	(0.00122, 0.00264)	0.0001286	0.0018072	0.0021919	0.0022834	0.0022676
8	—	(0.00033, 0.00128)	0.0000302	0.0008263	0.0010956	0.0011939	0.0012183
9	—	(0.00043, 0.00144)	0.0000071	0.0003778	0.0005476	0.0006243	0.0006545
10	—	(0.00000, 0.00043)	0.0000017	0.0001727	0.0002737	0.0003264	0.0003516
$E(I)$	1.307049	1.304654	1.307049	1.307049	1.307049	1.307049	1.307049
$E(I^2)$	2.504001	2.493240	2.109705	2.438402	2.482482	2.496033	2.500817
Shannon entropy		0.6551453	0.7125385	0.6622653	0.6613183	0.6611692	0.6611374

to the optimization constraints the expected number of customers served and the estimated probabilities $\hat{p}_j^{k,1}$, $j \leqslant k$, agree with their corresponding real values.

In order to establish comparisons among obtained results we draw together 95% confidence intervals for the probabilities obtained from simulation. The corresponding simulated moments and Shannon's entropy are determined from point estimates, p_j^S, defined as the relative frequency of the number of busy periods in which j customers are served during an overall quantity covering 20,000 busy periods.

As could be expected, the performance of $\{\hat{p}_j^{k,1}\}$ gets better by employing the additional information given by the assumed probabilities, which can be observed point-wise with respect to simulation confidence intervals and also by comparison between the second moment exact value and the corresponding approximated ones.

4.2. The second order approximation

For the second order approximation it is not possible to obtain an explicit solution of Lagrange's parameters, however, a computational tractable approximation is presented.

One standard method for seeking the Lagrangian multiplier β_1^k and β_2^k appearing at (4.5) is to find a solution of the following system

$$f_i(\beta_1^k, \beta_2^k) = \sum_{j=k+1}^{N} \left(j^i - \frac{m_i^k}{m_0^k} \right) \exp\left\{ -\sum_{n=1}^{2} \beta_n^k \left(j^n - \frac{m_n^k}{m_0^k} \right) \right\} = 0, \quad i = 1, 2, \tag{4.7}$$

where N is the truncated orbit size.

Equations in (4.7) are implicit and non linear. To solve them we can use a generalization of the method proposed in [20], reducing this problem to find β_1^k and β_2^k minimizing the potential function

$$F(\beta_1^k,\beta_2^k)=\ln\sum_{j=k+1}^{N}\exp\left\{-\sum_{i=1}^{2}\beta_i^k\left(j^i-\frac{m_i^k}{m_0^k}\right)\right\}. \tag{4.8}$$

Another possibility in order to solve (4.7) consists in determining β_1^k and β_2^k minimizing the convex function

$$G(\beta_1^k,\beta_2^k)=\alpha f_1^2(\beta_1^k,\beta_2^k)+(1-\alpha)f_2^2(\beta_1^k,\beta_2^k), \tag{4.9}$$

for any fixed $\alpha\in(0,1)$.

This latter choice has a supplementary advantage; namely, the optimum pair $(\hat{\beta}_1^k,\hat{\beta}_2^k)$ satisfies $G(\hat{\beta}_1^k,\hat{\beta}_2^k)=0$.

In order to compute the minimum of (4.8) or (4.9) we have employed Nelder and Mead's method (see Bunday [21]). This method does not employ derivatives avoiding the algorithmic singularity problems of the Hessian of $F(\beta_1^k,\beta_2^k)$. A complete discussion of this technical problem can be found in [22].

In the rest of the section we present numerical examples to demonstrate the effectiveness of the PME method. The truncated orbit size, N, is chosen as $P\{I>N\}\leqslant 10^{-4}$, with the help of Tchebychev's inequality. We also draw together results obtained from the simulation of approximately 20,000 busy periods for each considered model. These results include moments and Shannon's entropy determined from point estimates, and also 95% confidence intervals of the probabilities obtained from simulation.

In Table 6 we show different performances of the second order approximation using 0, 1, 2, 3 and 4 probabilities from an $M/M/1$ retrial queue with service rate $\sigma=1$, arrival rate $\lambda=0.2$ $(\rho=0.2)$ and retrial rate $\mu=1$.

The additional information given by the used probabilities makes both reduce the entropy of the system and improve the PME estimates in the following sense: when the number of assumed probabilities gets large more point PME estimates, $\hat{p}_j^{k,2}$, are inside of the corresponding simulated 95% confidence intervals.

If we compare first and second order approximations, in Tables 5 and 6, we can observe that Shannon's entropy decreases when we introduce more constraints and use more probabilities; and this reduction becomes more or less significant according to the values of the used probabilities; i.e., the entropy reduction, when performances $\{\hat{p}_j^{k-1,i}\}$ and $\{\hat{p}_j^{k,i}\}$ are compared, is more significant for large values of p_k.

Results shown in Table 7 correspond to an $M/M/1$ retrial queue with service rate $\sigma=1$, arrival rate $\lambda=0.8$ $(\rho=0.8)$ and retrial rate $\mu=4$. The congestion level has considerably increased, with respect to the previous model, and consequently the expected values $E[I]$, $E[I^2]$ and its Shannon's entropy do it. Moreover, the direct probabilities, p_k, $k\leqslant 4$, represent around 76.8% of the whole distribution, which is also an important reduction in comparison with the precedent situation.

In Table 7, we present numerical results obtained from simulation, a first order approximation with no use of direct probabilities and the second order approximation using all direct probabilities. The rest of first and second order PME performances have an intermediate behaviour. It can be observed that the reduction in Shannon's entropy is bigger than the one observed in

Table 6
Second order approximation when $\lambda = 0.2$, and $\mu = 1.0$

j	p_j	*Simulation*	$\hat{p}_j^{0,2}$	$\hat{p}_j^{1,2}$	$\hat{p}_j^{2,2}$	$\hat{p}_j^{3,2}$	$\hat{p}_j^{4,2}$
1	0.8333333	(0.83026, 0.84178)	0.7665402	0.8333333	0.8333333	0.8333333	0.8333333
2	0.0964506	(0.08903, 0.09811)	0.1784312	0.0916106	0.0964506	0.0964506	0.0964506
3	0.0367373	(0.03331, 0.03899)	0.0419310	0.0407480	0.0356282	0.0367373	0.0367373
4	0.0165012	(0.01568, 0.01983)	0.0099478	0.0183893	0.0173268	0.0161022	0.0165012
5	—	(0.00582, 0.00850)	0.0023826	0.0084202	0.0085337	0.0083011	0.0079228
6	—	(0.00347, 0.00624)	0.0005761	0.0039117	0.0042565	0.0043058	0.0041966
7	—	(0.00122, 0.00264)	0.0001406	0.0018438	0.0021501	0.0022472	0.0022360
8	—	(0.00033, 0.00128)	0.0000347	0.0008818	0.0010999	0.0011801	0.0011984
9	—	(0.00043, 0.00144)	0.0000086	0.0004279	0.0005698	0.0006235	0.0006461
10	—	(0.00000, 0.00043)	0.0000022	0.0002106	0.0002990	0.0003315	0.0003503
$E(I)$	1.307049	1.304654	1.307049	1.307049	1.307049	1.307049	1.307049
$E(I^2)$	2.504001	2.493240	2.504001	2.504001	2.504001	2.504001	2.504001
Shannon entropy		0.6551453	0.7106791	0.6618424	0.6612086	0.6611484	0.6611300

Table 7
Approximations when $\lambda = 0.8$, $\mu = 4.0$ and exponential service times

j	p_j	p_j^S	*Conf.Interv.*	$\hat{p}_j^{0,1}$	$\hat{p}_j^{4,2}$
1	0.5555555	0.5520329	(0.54590, 00.55816)	0.1449559	0.5555555
2	0.1143118	0.1166372	(0.11267, 0.12060)	0.1239437	0.1143118
3	0.0597646	0.0593596	(0.05643, 0.06228)	0.1059773	0.0597646
4	0.0384817	0.0378314	(0.03546, 0.04019)	0.0906153	0.0384817
5	—	0.0266745	(0.02467, 0.02867)	0.0774801	0.0147125
6	—	0.0211353	(0.01934, 0.02292)	0.0662488	0.0136805
7	—	0.0166175	(0.01527, 0.01821)	0.0566457	0.0127258
8	—	0.0133962	(0.01196, 0.01483)	0.0484345	0.0118420
9	—	0.0121390	(0.01077, 0.01350)	0.0414137	0.0110239
10	—	0.0103319	(0.00907, 0.01155)	0.0354105	0.0102661
$E(I)$	6.898648	7.010449		6.898648	6.898648
$E(I^2)$	424.2972	426.6816		88.28404	424.2972
Entropy	—	2.109846		2.855067	2.113762

Tables 5 and 6. On the other hand, over the second order approximation we can observe that the estimates $\hat{p}_j^{4,2}$, $j = 5, 6, 7, 8$, are not inside the confidence interval obtained from simulation results, however the accuracy of Shannon's entropy gives evidence of the goodness of the PME approach when it is considered as a whole. We would to point out here that these estimated can be improved if we relaxed the condition on the choice of the orbit truncated size, N; i.e., $P\{I > N\} \leqslant 10^{-2}$.

In Tables 8 and 9 we summarize approximation results for retrial models with service times hyperexponential and Erlang-k distributed. We keep the congestion level fixed at $\rho = 0.2$

Table 8
Approximations when $\lambda = 0.2$, $\mu = 1.0$ and Hyperexponential service times with $(\beta_1, C_B) = (1, 1.25)$

j	p_j	p_j^S	*Conf.Interv.*	$\hat{p}_j^{0,1}$	$\hat{p}_j^{4,2}$
1	0.8393283	0.8411948	(0.83585, 0.84653)	0.7659144	0.8393283
2	0.0911059	0.0889572	(0.08479, 0.09311)	0.1792895	0.0911059
3	0.0352928	0.0360771	(0.03334, 0.03881)	0.0419691	0.0352928
4	0.0162408	0.0164735	(0.01459, 0.01835)	0.0098243	0.0162408
5	—	0.0077425	(0.00644, 0.00904)	0.0022997	0.0080860
6	—	0.0048322	(0.00379, 0.00587)	0.0005383	0.0043883
7	—	0.0018670	(0.00121, 0.00252)	0.0001260	0.0024114
8	—	0.0014826	(0.00089, 0.00207)	0.0000295	0.0013417
9	—	0.0006589	(0.00025, 0.00106)	0.0000069	0.0007558
10	—	0.0002745	(0.00001, 0.00054)	0.0000016	0.0004312
$E(I)$	1.305629	1.299654		1.305629	1.305629
$E(I^2)$	2.550163	2.479490		2.103705	2.550163
Entropy	—	0.6437603		0.7104791	0.6507964

Table 9
Approximations when $\lambda = 0.2$, $\mu = 1.0$ and Erlang-k service times with $(\beta_1, k) = (1, 3)$

j	p_j	p_j^S	*Conf.Interv.*	$\hat{p}_j^{0,1}$	$\hat{p}_j^{4,2}$
1	0.8239746	0.8257800	(0.82014, 0.83141)	0.7638824	0.8239746
2	0.1060835	0.1061262	(0.10154, 0.11070)	0.1803661	0.1060835
3	0.0381591	0.0380892	(0.03522, 0.04095)	0.0425876	0.0381591
4	0.0164165	0.0158273	(0.01395, 0.01770)	0.0100557	0.0164165
5	—	0.0068891	(0.00563, 0.00814)	0.0023743	0.0076349
6	—	0.0033591	(0.00247, 0.00424)	0.0005606	0.0037936
7	—	0.0019927	(0.00130, 0.00268)	0.0001324	0.0019075
8	—	0.0008540	(0.00039, 0.00131)	0.0000312	0.0009705
9	—	0.0004555	(0.00011, 0.00080)	0.0000074	0.0004998
10	—	0.0002847	(0.00001, 0.00056)	0.0000017	0.0002604
$E(I)$	1.309102	1.302379		1.309102	1.309102
$E(I^2)$	2.449592	2.416818		2.118394	2.449592
Entropy	—	0.6670832		0.7155071	0.6755083

by choosing adequate parameters. More precisely, we assume that the arrival rate $\lambda = 0.2$, the mean of service times is $\beta_1 = 1$ and the retrial rate is $\mu = 1$ in both situations. We consider an hyperexponential service times distribution with coefficient of variation $C_B = 1.25$ in Table 8 and Erlang-3 in Table 9. For each model, we also present simulation results joined with a first order approximation without using additional information, and a second order approximation using p_k, $k \leqslant 4$. Again, the remainder performances of first and second order give intermediate results between the ones shown here.

We can observe that random variable I has a similar behaviour in both situations, and also when we compare results in case of exponential service times (Tables 5 and 6). Note that all these models have the same traffic intensity and, consequently, the level of congestion agrees. Moreover, we can see from Tables 1–4 that mean values present slight differences when we consider moderate levels of congestion (say $\rho < 0.5$) which gives similar mean value constraints for the maximum entropy formalism. This fact, joined with the independence on the maximum entropy formalism of service time distribution (once the mean value constraints are fixed), explains why there are resemblances among PME approximations appearing at Tables 5, 6, 8 and 9.

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