

UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS



TESIS DOCTORAL

Reduction by stages in field theory

Reducción por etapas en teoría de campos

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Miguel Ángel Berbel López

DIRECTOR

Marco Castrillón López

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This thesis is dedicated to my beloved parents, Miguel and Estilita,
thank you for your endless love, support, and encouragement.

Agradecimientos

En primer lugar me gustaría agradecer a mi familia por su apoyo vital, que ha sido esencial para poder llevar a término esta tesis.

- A mis padres, Miguel y Estilita, por inculcarme el hábito de trabajo y el valor de la educación, por las facilidades que me han dado durante toda mi vida académica, pero sobre todo por su cariño y apoyo a diario.
- A mis tíos, Leoncio y Teodoro, por alentarme a perseverar en los momentos de desánimo. A mis tíos Ángel, Conchi y Ángel, por su continuo afecto.
- A mis abuelos Lázaro y Florentino. Aunque nos dejaron antes del inicio de esta tesis, me han acompañado en este camino, pues han sido una referencia para mí.

Estos años han sido intensos, tanto por la naturaleza del propio doctorado como por las situaciones rocambolescas que nos ha hecho vivir la emergencia sanitaria. Quiero dar las gracias a los amigos y compañeros que me han ayudado a desconectar en momentos de estrés, a los que han compartido conmigo la alegría de resolver un problema y a los que me han empujado en momentos de frustración. En particular:

- A Pedro, Jorge y Borja, por 16 años de aventuras en común. Por los buenos recuerdos en la calle Luisa Fernanda y los paseos aleatorios en Madrid.
- A José y Lucía, por compartir las *alegrías* y *miserias* del doctorado. Su positividad y cercanía siempre hicieron todo fácil. A Álvaro, por su amabilidad.
- A mis compañeros de doctorado de la Universidad Complutense de Madrid por los seminarios, cursos y *sobremesas de doctorado*. Sin pretender ser exhaustivo, a Mauro, Guille, el otro Guille, Edu, el otro Edu, Blanca, Ángel, Juan, David, Javi, Laura y Dani.
- A mis compañeros del doble grado, Dani, Alejandro y Mateo, por las enriquecedoras experiencias que nos unieron esos apasionantes años.

- A mis compañeros de clase de francés del CSIM, Esther, Jorge y Sofía, por sus edificantes discusiones. A Yannick, por ser un profesor inspirador.

Me gustaría agradecer a todas las personas de las que he aprendido en mi paso por la universidad. En particular:

- A mis profesores de los grados en Matemáticas y en Física por mostrarme la belleza de estas disciplinas, las diferencias filosóficas que las separan y los problemas que las unen. En especial, a Miguel Domínguez Vázquez, Elena Martín Peinador, Rosario Martínez Herrero y Jesús M. Ruiz Sancho, por impulsarme a iniciar una carrera investigadora.
- A los profesores del departamento de Álgebra, Geometría y Topología, por generar un ambiente de trabajo estimulante. A Ángela, por atendernos siempre cuidadosamente.
- A los organizadores y miembros de la Red Temática de Geometría, Mecánica y Control, por facilitar el intercambio de los avances científicos en esta área.

Je suis très reconnaissant envers François Gay-Balmaz pour son accueil et hospitalité pendant mon stage au Laboratoire de Météorologie Dynamique de l'École Normale Supérieure. Merci beaucoup pour choisir une thématique sur mesure. Ces trois mois-là ont été magnifiques.

Por supuesto, quiero agradecer a Marco todo su trabajo durante estos años y la confianza que ha depositado en mí. Es evidente que, sin su guía y su ayuda para profundizar en los problemas y las lecturas que tan atinadamente me ha recomendado, esta tesis no existiría. Sin embargo, me gustaría destacar que trabajar con Marco ha sido una experiencia enriquecedora en aspectos que trascienden las matemáticas. Además de haber aprendido de él el oficio de investigador en matemáticas, espero haberme quedado con una parte de su curiosidad por una gran diversidad de temas, su serenidad para afrontar los imprevistos y la cordialidad con la que trata a todos los que se acercan a su despacho.

Miguel Á. Berbel

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Resumen

La mecánica geométrica investiga cómo las técnicas globales y covariantes de la geometría diferencial pueden utilizarse para estudiar una amplia variedad de problemas de la mecánica clásica que, a primera vista, pueden parecer desconectados. En este paradigma podemos estudiar sólidos rígidos, movimientos planetarios, vehículos rodados, control de enjambres de cuadricópteros y robots esféricos entre muchos otros sistemas.

Una de las ideas fundamentales de esta área de investigación afirma que la simetría de los sistemas físicos da lugar con bastante frecuencia a cantidades conservadas y versiones reducidas, potencialmente más sencillas, del problema. Un número considerable de sistemas presentan simetrías con naturaleza notablemente diferenciada e impulsan la tarea de realizar por etapas su proceso de reducción. Primero, se obtiene un sistema reducido utilizando las simetrías de un tipo para, posteriormente, reducir el sistema resultante mediante las simetrías de otra naturaleza. Este proceso iterativo ya ha sido profusamente estudiado en mecánica clásica tanto Lagrangiana como Hamiltoniana.

Además, hay muchas otras situaciones, de índole puramente geométrica, que debido a su naturaleza variacional, encajan y participan perfectamente de estas ideas. Así, la formulación matemática tiene por tanto una vertiente puramente geométrica que es muy valiosa y que, enriquece y se enriquece de los modelos mecánicos mencionados.

El objetivo principal de esta tesis es proporcionar un esquema de reducción por etapas en el contexto de la teoría clásica de campos, una generalización de la mecánica clásica en la que los sistemas físicos evolucionan respecto a un espacio multidimensional concreto en vez de hacerlo respecto al tiempo (espacio unidimensional). Algunos ejemplos de teorías clásicas de campos incluyen la dinámica de fluidos, medios elásticos, relatividad general y cadenas moleculares. Nos centraremos en la formulación geométrica y lagrangiana de la teoría de campos. Sin embargo, el caso particular de la mecánica clásica y la reducción por etapas en teorías de campos hamiltonianas también serán explorados.

En el contexto de la mecánica lagrangiana, se caracterizan geométricamente los objetos de la categoría de fibrados de Lagrange–Poincaré empleados para la reducción por etapas. Se responde afirmativamente al problema abierto sobre si la reducción puede realizarse en toda la categoría de fibrados de Lagrange–Poincaré. Asimismo, estos resultados se conectan con

la reducción por etapas hamiltoniana. Se establece un teorema de Noether para sistemas mecánicos en fibrados de Lagrange–Poincaré. Finalmente, estos resultados se aplican a sólidos rígidos con rotores, un atractivo ejemplo cuya bibliografía era extensa pero incompleta.

En el contexto de la teoría de campos lagrangiana, se propone una nueva categoría de fibrados para realizar la reducción por etapas. En esta nueva categoría quedan incluidos como casos particulares de objetos los fibrados jet de primer orden, los espacios covariantes de configuración reducidos y los fibrados de Lagrange–Poincaré que aparecen en mecánica. En primer lugar, se introduce la dinámica para estos sistemas, posteriormente se realiza la reducción de Lagrange–Poincaré para los sistemas simétricos y finalmente se explica el proceso de iteración. Se obtiene una condición de reconstrucción para cada etapa de reducción y se formula un teorema de Noether. Para ilustrar estos resultados se consideran cadenas de espinos y cadenas de moléculas con rotores.

Respecto a la teoría de campos hamiltoniana, se obtiene un procedimiento de reducción de la formulación mediante corchetes de la teoría de campos polisimpléctica y multisimpléctica. Esto generaliza la reducción de Poisson–Poincaré desde la mecánica a la teoría de campos. Dada la categoría de fibrados adecuada, este procedimiento podría volverse iterativo y proporcionar los primeros resultados hacia una teoría de reducción por etapas en teoría de campos hamiltoniana.

En conclusión, la reducción por etapas puede utilizarse para examinar diversos sistemas tanto en mecánica como en teoría de campos. La reducción de Lagrange–Poincaré y los fibrados de Lagrange–Poincaré son una herramienta espléndida para este propósito desde el punto de vista lagrangiano. La reducción de Poisson–Poincaré parece ser igualmente importante desde el punto de vista hamiltoniano.

Abstract

Geometric mechanics delves on how the global and coordinate-independent techniques of differential geometry can be used to study a wide variety of problems in classical mechanics which, at first sight, seem to be unrelated. Rigid bodies, planetary motion, rolling vehicles, control of swarms of quadrotors, sphere robots, among many other systems can be studied within this paradigm.

One of the cores ideas of this area of research is that symmetries of the physical system do quite often produce conserved quantities and lead to a reduced (and potentially simpler) version of the problem. A fair number of mechanical systems present symmetries with a remarkably distinct nature and prompt the task to carry their reduction procedure by stages. First, a reduced system is obtained using the symmetries of one kind and, afterwards, the resulting system is reduced using the symmetries of the other nature. This iterative process has already been studied with profusion in both, Lagrangian and Hamiltonian classical mechanics.

Furthermore, there are many situations of purely geometric character, that, due to their variational nature, fit perfectly and play a fundamental role within these ideas. Thus, this mathematical formulation has a very valuable purely geometric perspective that has an enriching interplay with the aforementioned mechanical models.

The main objective of this thesis is to provide a scheme of reduction by stages in the context of classical field theory, a generalization of classical mechanics, in which physical systems evolve along a specific (multi-dimensional) space rather than with (one-dimensional) time. Particularly interesting instances of classical field theories include fluid dynamics, elastic media, general relativity, and molecular strands. We focus on the Lagrangian geometric formulation of field theory. Nonetheless, the particular case of classical mechanics and reduction by stages in Hamiltonian field theory are also explored.

In the context of Lagrangian mechanics, the objects of the category of Lagrange–Poincaré bundles for reduction by stages are geometrically characterized. We answer affirmatively to the open question of whether reduction can be done in the whole category of Lagrange–Poincaré bundles. In addition, these results are linked with Hamiltonian reduction by stages. A Noether theorem is established for mechanical systems in Lagrange–Poincaré bundles.

Finally, these results are applied to rigid bodies with rotors, a compelling example whose bibliography was disperse and incomplete.

In the context of Lagrangian field theory, a novel category of bundles is proposed to perform reduction by stages. This new category includes as particular cases of objects, first order jet bundles, reduced covariant configuration spaces, and the Lagrange–Poincaré bundles appearing in mechanics. First, dynamics are introduced in this setting, then Lagrange–Poincaré reduction is carried for symmetric systems, and, finally, the iteration of this process is explained. A reconstruction condition for each stage of reduction is obtained and a Noether theorem is formulated. To illustrate these results, spin chains and molecular strands with rotors are considered.

Concerning Hamiltonian field theory, we obtain a reduction procedure for the bracket formulation of polysymplectic and multisymplectic field theory. This is a generalization of Poisson–Poincaré reduction from mechanics to field theory. Provided the adequate category of bundles, this procedure could be iterative and provide first results toward Hamiltonian reduction by stages in field theory.

In conclusion, reduction by stages can be used to examine diverse in both, mechanical systems, and field theories. Lagrange–Poincaré reduction and Lagrange–Poincaré categories of bundles are a magnificent tool for this purpose in the Lagrangian picture and Poisson–Poincaré reduction seems to be equally important in the Hamiltonian analogue.

Introduction

It is undeniable that contemporary mathematics and classical mechanics have mutually influenced one another. This is especially true in the case of differential geometry. For instance; pseudo-Riemannian geometry is intimately related to gravity and general relativity, calculus of variations is rooted in the Lagrangian formulation of mechanics, symplectic geometry and Poisson brackets are motivated by Hamiltonian mechanics, and fiber bundles provide a magnificent setting to explore field theories.

The so-called research area of *geometric mechanics* studies geometric techniques and their application to mechanical problems [1, 2, 77]. Some renowned examples of mechanical systems are rigid bodies (possibly with internal rotors), Euler's top, the Toda lattice [12, 14], Newton's cradle, skates, rattle-backs [11] and underwater vehicles [58], among many others. This appealing interplay between mechanics and differential geometry is still much alive. Particular problems appearing nowadays in engineering and robotics, such as describing the movement of a quadrotor drone, controlling swarms of them [26, 41] or controlling spherical robots [36], reflect how new problems in mechanics arise in our current time, and they confront us with new challenges.

But why does differential geometry merges so nicely with mechanics? Very often, these new challenges appeal to the global and coordinate-independent properties of the involved mechanical systems. As such they require the use of geometric methods. By making use of differential geometry and global analysis, one may unveil the underlying structures of the involved systems and provide a unifying framework for many seemingly disparate physical systems.

One of the most useful core ideas of geometric mechanics is the notion that the symmetries of the system quite often provide conservation laws and leads to a "reduced" (and potentially simpler) version of the problem [67]. Symmetries are mathematically modeled by Lie group actions on the space of configurations of the problem mapping a solution of the system onto a new solution. For example, in the Lagrangian formulation of mechanics, we consider a Lie group G of symmetries acting (freely and properly) on the configuration manifold Q . As shown in [23], this action is lifted to the tangent bundle TQ to define a re-

duced Lagrangian and variational principle in the quotient $(TQ)/G$. This procedure is called Lagrange–Poincaré reduction. Similarly, on the Hamiltonian framework [66], the Poisson structure and the Hamiltonian on T^*Q are projected to $(T^*Q)/G$. This approach contributes to understand the origin of the different equations of motion of each system (See [38] for a fascinating example) and to find adapted numerical methods of simulation [10, 51, 57, 78, 33].

The ideas present in geometric mechanics can be broaden, up to different extents, to *field theory*, a generalization of classical mechanics for systems where the configuration evolves not with (one-dimensional) time but along a certain (multi-dimensional) space. Well-known examples of classical field theories are gauge theories (in particular, electromagnetic fields), fluid dynamics [3, 40], general relativity [37], and molecular strands [31]. In general, the mathematical description of a field theory is the study of sections of a particular bundle. In fact, systems described by sections of bundles with no-prescribed time evolution, such as covariant fields on space-times or geometric theories as harmonic maps, are also included in this versatile panorama.

Outset

This thesis focuses on reduction by stages, a procedure that facilitates studying problems in classical mechanics when the Lie group G of symmetries of the system contains transformations of a remarkably distinct nature. Indeed, there are many occasions when there exists a normal subgroup of symmetries N encoding one type of information which for mathematical or physical reasons is different to the remaining symmetries in G/N . In this case, the action of the full-symmetry group G is split and the reduction process can be organized as a concatenation of simpler steps: first reduction by N and then by the quotient G/N . Prototypical examples include underwater vehicles (see [58]) and rigid bodies with rotors (see [77]). By way of illustration, in the last instance, the symmetry group contains information of the internal rotors as well as the global symmetry of the body. These two symmetries present (physically and geometrically) substantially different traits.

The problem was first tackled in the realm of Lagrangian mechanics in [23]. As the quotient of a tangent bundle by the action of a Lie group is not in general another tangent bundle, the iteration of Lagrange–Poincaré reduction implies reducing variational principles and Lagrangians defined in a different kind of phase spaces. Thus, a new category of vector bundles, the so-called Lagrange–Poincaré bundles, which is closed under reduction, is introduced in [23] to iterate the reduction process. The results of [23] were highly influential and later extended to include non-holonomic systems, Lie algebroids and discretization. Some of these references are [34, 35, 52, 70].

Reduction by stages has also received important attention in Hamiltonian mechanics, where the two main procedures of reduction show a surprisingly different behavior with respect to iteration. On the one hand, as seen in [66], the quotient of a Poisson manifold by a free and proper Lie group canonical action is another Poisson manifold. Therefore, this process, called Poisson–Poincaré reduction, is iterative and reduction by stages is virtually straightforward. On the other hand, Marsden–Weinstein reduction for symplectic manifolds depends on the momentum map [62, 69] and it is not iterative: the quotient of a symplectic manifold is in general a Poisson manifold. As such, reduction by stages with this procedure is much more involved as shown in [63] and the references therein.

The main aim of this thesis is to explore how reduction by stages can be generalized to the Lagrangian geometric formulation of field theory. Yet, this thesis also explores the particular case of classical mechanics and works towards reduction by stages in Hamiltonian field theory.

Outline

Chapter 1 includes a quick overview of some results of fiber bundles which are relevant in this thesis. Please note that the exposition is not intended as a first approximation to the topic. Afterwards, the geometric formulation of Lagrangian mechanics and Lagrange–Poincaré reduction are explained in Section 1.2, which includes some of the results shown in [23, 67]. The geometric formulation of Lagrangian field theory is described in Section 1.3. The results in [32] about Lagrange–Poincaré reduction for Lagrangian field theories are summarized. The later results were historically preceded by Euler–Lagrange reduction in [17, 18] and Lagrange–Poincaré reduction in principal bundles shown in [20]. These are recovered as particular cases of the theory in Section 1.3.

Chapter 2 revolves around the category $\mathfrak{LP}$ of Lagrange–Poincaré bundles introduced in [23]. The reduction process explained in [23] is restricted to the subcategory $\mathfrak{RS}$ of Lagrange–Poincaré bundles obtained by reduction of tangent bundles. Whether or not this reduction procedure could be done in the whole category was left as an open problem. Chapter 2 answers positively to this question and completes the reduction scheme in the whole $\mathfrak{LP}$ category. In addition, Chapter 2 provides a geometric characterization of the Lie bracket on the sections of an $\mathfrak{LP}$ -bundle and a Noether theorem on $\mathfrak{LP}$ -bundles in which the usual notion of the conservation law is now transformed into a drift equation. The relationship between reduction in the Lagrange–Poincaré category and Poisson reduction is also studied.

Chapter 3 develops two applications of the Lagrange–Poincaré reduction in Chapter 2. Section 3.1 presents an alternative way, with respect to the classical approaches in [21, 48], of studying systems depending on parameters. Section 3.2 explores the possibilities of re-

duction by stages in a rigid body with rotors. This system has symmetries with two different sources, respectively related to the rigid body and the rotors, which can be considered by stages, in any order, or altogether.

Chapter 4 constructs the Lagrange–Poincaré category $\mathbf{FT}\mathfrak{L}\mathfrak{P}$ for field theories, defines its dynamics and a Lagrange–Poincaré reduction. The bundles of this category include as particular cases 1-jet bundles and the reduced bundles in [32]. The Lagrange–Poincaré bundles of the category $\mathfrak{L}\mathfrak{P}$ used in mechanics are also a particular case of this bundles. Funnily enough, some results on $\mathfrak{L}\mathfrak{P}$ bundles shown in Chapter 2 are used in the construction of $\mathbf{FT}\mathfrak{L}\mathfrak{P}$. As in the case of $\mathfrak{L}\mathfrak{P}$, the main advantage of this category is that it is closed under reduction, thus, it allows to perform reduction by stages in Lagrangian field theory. Special attention is devoted to the reconstruction problem. The possibility to construct a solution of the original problem from a solution of the reduced problem is characterized by the flatness of a connection.

Chapter 5 exemplifies Lagrange–Poincaré reduction in field theory. Section 5.1 is devoted to the $SO(3)$ -strand with broken symmetry that models a charged molecular or spin strand subjected to a uniform electric field. Section 5.2 analyzes a molecular strand composed by a chain of rigid bodies (as it is done in [31]) such that each body has one or more rotors, as in Section 3.2. Thus, this example involves reduction by stages. Simple proteins of amino acids could fit in this model of linked molecules with a rotating side chain.

Chapter 6 overviews multisymplectic and polysymplectic field theory as well as the covariant bracket formalism used in [19, 64]. Observe that we present multisymplectic and polysymplectic bundles appearing in field theory rather than general polysymplectic and multisymplectic manifolds. Afterwards, Poisson–Poincaré reduction in field theory is developed. This procedure is linked with the Lagrange–Poincaré reduction in [32] and the example in Section 5.1 is revisited to compare both procedures. At first sight it may seem that this chapter is uncorrelated to the rest of the thesis. However, the objective of establishing Poisson–Poincaré reduction in field theory is geared towards reduction by stages in Hamiltonian field theory.

Outcome

The results of this thesis are available in the following manuscripts:

- [6] Berbel, M. A. and Castrillón López, M. (2022b). Reduction (by stages) in the whole Lagrange–Poincaré category. *Adv. Theor. Math. Phys.*, 26(8)
- [7] Berbel, M. A. and Castrillón López, M. (2023). Rigid body with rotors and reduction by stages. *Reviews in Mathematical Physics*, 35(1)

- [4] Berbel, M. A. and Castrillón López, M. (2020). Lagrangian reduction by stages in field theory. arXiv:2007.14854
- [5] Berbel, M. A. and Castrillón López, M. (2022a). Poisson–Poincaré reduction for field theories. arXiv:2210.14732
- [8] Berbel, M. A. and Gay-Balmaz, F. (In preparation). Geometry of general relativistic Lagrangian continuum theories: Non abelian fluids

Paper [6] compiles results in Chapter 2 and Section 3.1 on Lagrange–Poincaré bundles in mechanics. Paper [7] is devoted to rigid bodies with rotors and covers the contents of Section 3.2. Paper [4] is developed in Chapter 4 and Section 5.2. The results of Chapter 6 on Poisson–Poincaré reduction for field theory are available in [5]. Finally, the preprint [8] in preparation was initiated during my research stay at the École Normale Supérieure de Paris under the supervision of François Gay-Balmaz and will expand the results on the dynamics of fluids with non-abelian charges in connection with Chapter 4.

Outlook

There are diverse future lines of research related with the results of this thesis:

- **Hamiltonian Reduction by Stages in Field Theory** So far, the results on reduction by stages in the Hamiltonian picture are exclusive to mechanics. A myriad of Hamiltonian formulations of field theory have been proposed together with reduction procedures with the intent to extend Marsden–Weinstein reduction to classical field theory (for instance, [15, 30, 43, 60]). There is a consensus that much work is still needed to obtain satisfactory results in this context, let alone, in reduction by stages. Some recent works concerning multisymplectic and polysymplectic reduction are [9, 16, 27, 61]. The Poisson–Poincaré reduction proposed in Chapter 6 does not need the selection of a specific momentum map and could be iterated if the adequate category of fiber bundles is introduced in the style of [23] and [6]. This shall provide results on reduction by stages of Hamiltonian field theories. The covariant Poisson bracket introduced in [39] could also be relevant to this goal.

Another promising line of research related to multisymplectic spaces is the reinterpretation of the correspondence principle via a Poisson bracket structure on the space of solutions of the equations of motion of first order Hamiltonian field theories as studied in [24, 25, 79].

- **Lagrangian Continuum Theories** Reduction of variational principles is used in [40, 38] to study non-relativistic and relativistic Lagrangian continuum theories. This geometric formalism explains the origin of the equations of motion present in fluid dynamics and elasticity. There is a great resemblance between the reduction of these variational principles and the techniques in [32], thus it seems that techniques of reduction by stages in Chapter 4 can be applied to study the dynamics of fluids with charged particles.
- **Nonlinear Splittings and Lagrange–Poincaré Reduction** Nonlinear Splittings are a generalization of Ehresmann connections recently introduced in [45] which has proven to be interesting in Finsler Geometry [44] and provides a reduction procedure of Lagrangian mechanical systems via the construction of a so-called subduced Lagrangian. As seen in [45], a nonlinear splitting can be associated to any fiber-regular Lagrangian. This can have major implications in the Lagrange–Poincaré reduction used throughout this thesis, as it requires selecting an arbitrary connection to obtain a splitting of the quotient of a tangent bundle. In addition, the relation of Lagrange–Poincaré reduction with the subduced Lagrangian and Routhian reduction can be studied in both, classical mechanics and field theory.
- **Numerical Methods** Lagrangian reduction by stages in mechanics has already been discretized to provide numerical methods [34, 35]. The same is true for multisymplectic field theory [81]. Numerical integrators could also be constructed for reduction by stages in field theory and the bracket formulation used in Chapter 6.
- **Hamel Equations** The local versions of the equations obtained in the reduction procedures used in this thesis consider that the local coordinates in the jet bundle $J^1 E$ are obtained from adapted coordinates in the fiber bundle $\pi_{Q,E} : E \rightarrow Q$. It is appealing to study these equations without this requirement. To do so, it is particularly relevant the Hamel’s formalism developed for field theory in [75, 76].

Chapter 1

Preliminaries: Variational Calculus

1.1 Fiber Bundles

The aim of this section is to introduce some notations and elemental notions on smooth fiber bundles that will be used throughout this thesis. Most of the results in this section can be found in more detail in classical references like [50, 55, 71].

Definition 1.1.1 A **fiber bundle** is a 4-tuple (E, Q, π, F) , where E, Q, F are smooth manifolds and $\pi : E \rightarrow Q$, is a locally trivial smooth map. That is, for any $q \in Q$ there is an open neighborhood U of q and a diffeomorphism φ_U such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times F \\ \downarrow \pi & \swarrow \pi_1 & \\ U & & \end{array}$$

where $\pi_1 : U \times F \rightarrow U$ is the projection to the first factor. E is called **total space**, Q the **base space**, and $E_q = \pi^{-1}(q) \cong F$, the **fiber** over q . The map φ_U is called **trivialization**.

Definition 1.1.2 A **section** is a map $s : Q \rightarrow E$ such that $\pi \circ s = \text{id}_Q$. The set of sections of (E, Q, π, F) is denoted $\Gamma(E \rightarrow Q)$ or simply $\Gamma(E)$.

Given two trivializations $\varphi_{U_\alpha}, \varphi_{U_\beta}$ such that $U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \emptyset$, a *change of trivializations* is a map

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi_{U_\beta} \circ \varphi_{U_\alpha}^{-1} : U_{\alpha\beta} \times F \rightarrow U_{\alpha\beta} \times F \\ (q, f) &\mapsto (q, \tilde{\varphi}_{\alpha\beta}(q, f)) = (q, g_{\alpha\beta}(q)(f)), \end{aligned}$$

where $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow \text{Diff}(F)$ is called *transition map*. If F has any additional structure, the image of these transition maps may be required to lie within a distinguished subgroup of $\text{Diff}(F)$ called *structure group*.

1.1.1 Vector Bundles

Definition 1.1.3 A **vector bundle** of rank r is a fiber bundle $(V, Q, \pi, \bar{V})$, where $\bar{V}$ is a real vector space of dimension r , whose structure group is $Gl(\bar{V}) \simeq Gl(r, \mathbb{R}) < \text{Diff}(\bar{V})$.

The fibers of a vector bundle are endowed with a structure of vector space from the structure on $\bar{V}$.

Definition 1.1.4 A **vector bundle morphism** from vector bundle $V_1 \rightarrow Q_1$ to vector bundle $V_2 \rightarrow Q_2$ is a pair of smooth maps $(\bar{f}, f)$ such that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\bar{f}} & V_2 \\ \downarrow \pi_{Q_1, V_1} & & \downarrow \pi_{Q_2, V_2} \\ Q_1 & \xrightarrow{f} & Q_2 \end{array}$$

commutes and the restriction of $\bar{f}$ to each fiber is a linear map. It is said that $\bar{f}$ covers f .

Example 1.1.5 The **tangent bundle** $TQ = \sqcup_{q \in Q} T_q Q$ of a manifold Q is a vector bundle of rank $n = \dim(Q)$ with projection $\pi_{Q, TQ} : TQ \rightarrow Q$ such that $\pi_{Q, TQ}(T_q Q) = q$.

Example 1.1.6 Given a fiber bundle $(E, Q, \pi_{Q, E}, F)$, the **vertical subbundle** VE is a vector bundle over E with fibers

$$V_e E = \ker(T_e \pi_{Q, E}) = \{v \in T_e E \mid T_e \pi_{Q, E}(v) = 0\}, \quad e \in E.$$

The vertical bundle $VE \subset TE$ is a distribution on E that provides a notion of verticality. Given $q = \pi_{E, Q}(e)$, $V_e E = T_e(E_q)$. In the particular case that $E \rightarrow Q$ is a vector bundle, $VE \rightarrow Q$ is a vector bundle and there is a canonical isomorphism,

$$\begin{aligned} \text{vl}_E : 2E \equiv E \times_Q E &\rightarrow VE \\ (v_q, w_q) &\mapsto \text{vl}_E(v_q, w_q) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (v_q + \varepsilon w_q), \end{aligned}$$

called **vertical lift**, which is fiber linear over Q .

Definition 1.1.7 An *affine connection* on a vector bundle $\pi : V \rightarrow Q$ is a map

$$\begin{aligned}\nabla : \mathfrak{X}(Q) \times \Gamma(V) &\rightarrow \Gamma(V) \\ (X, s) &\mapsto \nabla_X s\end{aligned}$$

such that

(i) for any $s \in \Gamma(E)$, $X_1, X_2 \in \mathfrak{X}(G)$ and $f_1, f_2 \in \mathcal{C}^\infty(Q)$,

$$\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s;$$

(i) for any $s_1, s_2 \in \Gamma(V)$, $X \in \mathfrak{X}(Q)$ and $f_1, f_2 \in \mathcal{C}^\infty(Q)$,

$$\nabla_X (f_1 s_1 + f_2 s_2) = X(f_1) s_1 + f_1 \nabla_X s_1 + X(f_2) s_2 + f_2 \nabla_X s_2,$$

where $X(f)$ denotes the derivative of f in the direction of the field X .

An affine connection ∇ defines a *horizontal subbundle* HV with $H_v V = ds(T_q Q)$, where $s : Q \rightarrow V$ is a (local) section such that $(\nabla_X s)_q = 0$ for all $X \in T_q Q$. This bundle splits $T_v V = H_v V \oplus V_v V$, for any $v \in V$. Furthermore, an affine connection defines a covariant derivative for curves $v(t) \in V$ as

$$\frac{Dv(t)}{Dt} = \nabla_{\dot{q}(t)} s,$$

where $q(t) = (\pi_{Q,V} \circ v)(t)$ and s is a local section on a neighborhood of $q(t)$ such that $v(t) = s(q(t))$. In fact, there is a bijection between affine connections and covariant derivatives on V .

We denote by $\Omega^p(Q, V) = \Gamma(\bigwedge^p T^* Q \otimes V)$ the space of V -valued p -differential forms on Q . Unlike real-valued differential forms on Q , the space

$$\Omega(Q, V) = \bigoplus_p \Omega^p(Q, V),$$

of V -valued differential forms on Q is not an algebra with respect to the wedge product. Yet, it is a $\Omega(Q)$ -module with the following wedge product

$$\omega \wedge \eta(X_1, \dots, X_{p+q}) = \sum_{\sigma} \text{sign}(\sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}),$$

where $\omega \in \Omega^p(Q)$, $\eta \in \Omega^q(Q, V)$ and the sum is over all permutations σ of $p + q$ elements.

Given an affine connection ∇ , there is a unique operator,

$$d_{\nabla} : \Omega^p(Q, V) \rightarrow \Omega^{p+1}(Q, V),$$

called *exterior covariant derivative*, such that

- (i) for any $s \in \Omega^0(Q, V) = \Gamma(V)$, $d_{\nabla}s = \nabla s$;
- (ii) for any $\omega \in \Omega^{\bullet}(Q)$, $\eta \in \Omega^q(Q, V)$ satisfies the Leibniz identity,

$$d_{\nabla}(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge (d_{\nabla}\eta).$$

The intrinsic formula of this operator is

$$\begin{aligned} d_{\nabla}(\eta)(X_1, \dots, X_{q+1}) &= \sum_{i < j} (-1)^{(i+j)} \eta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}) \\ &\quad + \sum_{i=1}^{q+1} (-1)^{(i+1)} \nabla_{X_i} (\eta(X_1, \dots, \hat{X}_i, \dots, X_{q+1})), \end{aligned}$$

where $\hat{X}_i$ means that X_i is omitted. In general, $d_{\nabla}^2 = d_{\nabla} \circ d_{\nabla} \neq 0$.

Definition 1.1.8 The *curvature* of an affine connection $k_{\nabla} \in \Omega^2(Q, \text{End}(V))$ is

$$k_{\nabla}(X_1, X_2)s = \nabla_{X_1} \nabla_{X_2} s - \nabla_{X_2} \nabla_{X_1} s - \nabla_{[X_1, X_2]} s,$$

where $X_1, X_2 \in \mathfrak{X}(Q)$ and $s \in \Gamma(V)$.

The curvature is interpreted as the failure to get $d_{\nabla}^2 = 0$ since $d_{\nabla}^2(s)(X_1, X_2) = k_{\nabla}(X_1, X_2)(s)$.

1.1.2 Principal Bundles

Definition 1.1.9 A *G-principal bundle* is a fiber bundle obtained from a free and proper left action $\Phi : G \times Q \rightarrow Q$ of a Lie group G on a manifold Q . In that instance, the quotient Q/G is also a manifold and we define $\pi_{Q/G, Q} : Q \rightarrow Q/G$.

Definition 1.1.10 A *principal connection* $\mathcal{A}$ on $Q \rightarrow Q/G$ is a 1-form on Q taking values on $\mathfrak{g}$, the Lie algebra of G , such that:

- (i) $\mathcal{A}(\xi_q^Q) = \xi$, for any $\xi \in \mathfrak{g}$, $q \in Q$,
- (ii) $\Phi_g^* \mathcal{A} = \text{Ad}_g \circ \mathcal{A}$,

where Ad denotes the adjoint action of G on $\mathfrak{g}$, and

$$\xi_q^Q = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot q \in T_q Q,$$

is the infinitesimal generator of $\xi \in \mathfrak{g}$ at $q \in Q$.

A principal connection splits the tangent space as $T_q Q = H_q Q \oplus V_q Q$, for all $q \in Q$, where

$$H_q Q = \ker \mathcal{A}_q = \{v \in T_q Q \mid \mathcal{A}_q(v) = 0\}, \quad q \in Q,$$

and $HQ = \sqcup_{q \in Q} H_q Q$ the *horizontal subbundle*. Let $x = \pi_{Q/G, Q}(q) \in Q/G$, $H_q Q$ is isomorphic to $T_x(Q/G)$, through $T_q \pi_{Q/G, Q}$. The inverse is called *horizontal lift* through $\mathcal{A}$ at q and is denoted by $H_{\mathcal{A}}^q$. In addition, given $v_x \in T_x(Q/G)$, we denote $(v_x)_q^h = H_{\mathcal{A}}^q(v_x)$.

Definition 1.1.11 The *curvature* of a principal connection $\mathcal{A}$ is the $\mathfrak{g}$ -valued 2-form

$$\mathcal{B}(v, w) = d\mathcal{A}(\text{Hor}(v), \text{Hor}(w)),$$

where $v, w \in T_q Q$ and $\text{Hor}(v)$ is the projection of $v \in T_q Q$ to $H_q Q$.

Definition 1.1.12 Let $\pi_{Q/G, Q} : Q \rightarrow Q/G$ be a G -principal bundle and let F be a manifold on which G acts on the left. Consider left action of G on $Q \times F$ such that for all $g \in G$ and any $(q, f) \in Q \times F$, $g \cdot (q, f) = (g \cdot q, g \cdot f)$. The **associated bundle** to $Q \rightarrow Q/G$ by the action of G on F has total space $E = (Q \times F)/G$, base space Q/G , fibers diffeomorphic to F and projection

$$\begin{aligned} \pi_{Q/G, E} : (Q \times F)/G &\rightarrow Q/G \\ [q, f]_G &\mapsto \pi_{Q/G, Q}(q). \end{aligned}$$

A principal connection $\mathcal{A}$ on $\pi : Q \rightarrow Q/G$ defines a horizontal distribution $HE \subset TE$ complementary to the vertical bundle

$$T_{[q, f]_G} E = V_{[q, f]_G} E \oplus H_{[q, f]_G} E, \quad [q, f]_G \in E,$$

as

$$H_{[q, f]_G} E = d_q \psi_f(H_q Q),$$

where $\psi_f : q \in Q \mapsto [q, f]_G \in E$. In case E is a vector bundle, the distribution hereby obtained gives rise to an affine connection on E , and hence, to a covariant derivative.

Example 1.1.13 The *adjoint bundle* to $Q \rightarrow Q/G$ is the associated bundle $(Q \times \mathfrak{g})/G$ by the adjoint action of G on $\mathfrak{g}$. We shall denote it by $\text{Ad}Q$, or $\tilde{\mathfrak{g}}$ provided that the principal bundle is implicit on the discussion. The elements are denoted by $[q, \xi]_G$, where $q \in Q$, $\xi \in \mathfrak{g}$. Remarkably, $\text{Ad}Q \rightarrow Q/G$ is a Lie algebra bundle with a fiberwise Lie bracket given by

$$[[q, \xi_1]_G, [q, \xi_2]_G] = [q, [\xi_1, \xi_2]]_G, \quad [q, \xi_1]_G, [q, \xi_2]_G \in \text{Ad}Q_x, \quad x = \pi_{Q/G, Q}(q).$$

The principal connection $\mathcal{A}$ on $Q \rightarrow Q/G$ defines an affine connection on the vector bundle $\text{Ad}Q \rightarrow Q/G$, denoted $\nabla^{\mathcal{A}}$ whose covariant derivative along curves

$$\frac{D[q(t), \xi(t)]_G}{Dt} = [q(t), \dot{\xi}(t) - [\mathcal{A}(\dot{q}(t)), \xi(t)]]_G.$$

In addition, the curvature of $\mathcal{A}$ can be seen as a 2-form on Q/G with values in the adjoint bundle as for any $X_1, X_2 \in T_x(Q/G)$

$$\tilde{\mathcal{B}}(X_1, X_2) = [q, \mathcal{B}((X_1)_q^h, (X_2)_q^h)]_G.$$

1.1.3 Quotient of Vector Bundles.

Given a vector bundle $V \rightarrow Q$, we say that an action Φ of a Lie group G on V is a vector bundle action if for all $g \in G$, $\Phi_g : V \rightarrow V$ are vector bundle isomorphisms. Then, if the action induced on Q is free and proper, there is a vector bundle structure on $V/G \rightarrow Q/G$ with operations

$$[v_q]_G + [w_q]_G = [v_q + w_q]_G \text{ and } \lambda[v_q]_G = [\lambda v_q]_G,$$

where $[v_q]_G, [w_q]_G \in V/G$ stand for the equivalence classes of $v_q, w_q \in V_q$ and $\lambda \in \mathbb{R}$. In the diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi_{V/G, V}} & V/G \\ \downarrow \pi_{Q, V} & & \downarrow \pi_{Q/G, V/G} \\ Q & \xrightarrow{\pi_{Q/G, Q}} & Q/G \end{array}$$

$\pi_{Q, V}$ and $\pi_{Q/G, V/G}$ are projections of vector bundles, $\pi_{Q/G, Q}$ is the projection of a G -principal bundle and $\pi_{V/G, V}$ is a surjective vector bundle homomorphism.

Suppose that $V \rightarrow Q$ has an affine connection ∇ or, equivalently, a covariant derivative $Dv(t)/Dt$ of curves $v(t)$ in V . Let $q(t)$ be a curve in Q and denote $q_0 = q(t_0)$ a fixed value of

the curve, the *horizontal lift* of $q(t)$ through $v \in V_{q_0}$ at t_0 is an horizontal curve in V (that is, its covariant derivative vanishes) denoted $q_v^h(t)$ such that $\pi_{Q,V} \circ q_v^h = q$ and $q_v^h(t_0) = v$.

Let $v(t)$ be a curve in V , $q(t) = \pi_{Q,V}(v(t))$, $q_0 = q(t_0)$ and $x(t) = \pi_{Q/G,Q}(q(t))$. If $q_h(t)$ is the horizontal lift through $q_0 \in Q$ of $x(t)$ with respect to $\mathcal{A}$, we define $g_{q_0}(t)$ in G such that

$$q(t) = g_{q_0}(t)q_h(t).$$

Consider the curve $v_h(t) = g_{q_0}^{-1}(t)v(t)$ in V , then

$$\begin{aligned} \frac{D}{Dt} \Big|_{t=t_0} v(t) &= \frac{D}{Dt} \Big|_{t=t_0} g_{q_0}(t)v_h(t) = \frac{D}{Dt} \Big|_{t=t_0} g_{q_0}(t)v_h(t_0) + \frac{D}{Dt} \Big|_{t=t_0} g_{q_0}(t_0)v_h(t) \\ &= \frac{D}{Dt} \Big|_{t=t_0} g_{q_0}(t)v(t_0) + \frac{D}{Dt} \Big|_{t=t_0} v_h(t) = \left(\dot{g}_{q_0}(t_0) \right)_{v(t_0)}^V + \frac{D}{Dt} \Big|_{t=t_0} v_h(t) \end{aligned}$$

Thus, the covariant derivative $Dv(t)/Dt$ at t_0 can be decomposed in horizontal and vertical components

$$\frac{D^{(\mathcal{A},H)}}{Dt} \Big|_{t=t_0} v(t) = \frac{D}{Dt} \Big|_{t=t_0} v_h(t), \quad \frac{D^{(\mathcal{A},V)}}{Dt} \Big|_{t=t_0} v(t) = \left(\dot{g}_{q_0}(t_0) \right)_{v(t_0)}^V.$$

Consequently, given $X \in \mathfrak{X}(Q)$ and $v \in \Gamma(Q, V)$ we can define

$$\nabla_X^{(\mathcal{A},H)} v(q_0) = \frac{D^{(\mathcal{A},H)}}{Dt} \Big|_{t=t_0} v(t), \quad \nabla_X^{(\mathcal{A},V)} v(q_0) = \frac{D^{(\mathcal{A},V)}}{Dt} \Big|_{t=t_0} v(t), \quad (1.1)$$

where $v(t) = v(q(t))$ and $q(t)$ is an integral curve of X in Q such that $q_0 = q(t_0)$.

1.1.4 Jet Bundles

Definition 1.1.14 Let $\pi_{Q,E} : E \rightarrow Q$ be a fiber bundle. Two local sections $s : U \rightarrow E$, $s' : U' \rightarrow E$ **represent the same jet** $j_q^1 s$ at $q \in U \cap U' \subset Q$ if and only if they agree to first order at $q \in Q$, that is, $s(q) = s'(q)$ and $T_q s = T_q s'$.

Agreement to first order is a relation of equivalence. We denote $J_e^1 E$ the space of classes of equivalence of local sections s such that $s(q) = e \in E$.

Definition 1.1.15 The **first jet bundle** $J^1 E$ of $E \rightarrow Q$ is a fiber bundle over E with total space

$$J^1 E = \bigsqcup_{e \in E} J_e^1 E.$$

An alternative description of the fibers of $J^1 E$ as

$$J_e E = \{\gamma : T_q Q \rightarrow T_e E \mid \gamma \text{ is linear, } T_e \pi_{Q,E} \circ \gamma = \text{id}_{T_q Q}\},$$

shows that $J^1 E$ is an affine bundle modeled over the vector bundle $\pi_{Q,E}^* TQ \otimes VE$. Given fiber coordinates (x^i, y^a) on E , (x^i, y^a, y_i^a) are fiber coordinates on $J^1 E$.

Definition 1.1.16 Given $s \in \Gamma(E \rightarrow Q)$, its **first jet prolongation**, $j^1 s$, is a section of $J^1 E \rightarrow Q$ such that $j^1 s(q) = T_q s \in J_{s(q)}^1 E$. A section of the bundle $J^1 E \rightarrow Q$ which is the first jet prolongation of a section of $E \rightarrow Q$ is said to be **holonomic**.

Definition 1.1.17 An **Ehresmann connection** on $E \rightarrow Q$ is a distribution $\Lambda \subset TE$ such that for every $e \in E$, $T_e E = V_e E \oplus \Lambda_e$.

The distribution Λ determines *horizontal vectors* that complement the vertical subbundle VE . An Ehresmann connection on E can be defined via a section $\gamma(e)$ on $J^1 E \rightarrow E$ setting $\Lambda_e = \text{Im}(\gamma(e))$. In analogy to principal connections, Λ_e is isomorphic to $T_q Q$, through $T_e \pi_{Q,E}$ and the inverse defines a *horizontal lift*, hor_Λ^q . In fact, a principal connection on $Q \rightarrow Q/G$ is a G -invariant Ehresmann connection, in the sense that the distribution is invariant under $T\Phi_g$ for any $g \in G$. Thus, a principal connection is a section of $(J^1 Q)/G \rightarrow Q$.

1.2 Lagrangian Formalism in Mechanics

1.2.1 Lagrangian Mechanics

The Lagrangian formulation of mechanics provides a method to study physical systems using a variational principle. The dynamics of the system are determined via a *Lagrangian function* $L : TQ \rightarrow \mathbb{R}$, defined on the tangent bundle to a manifold Q called the *configuration space*.

Given $I = [t_0, t_1] \subset \mathbb{R}$, the space of differentiable curves from I to Q is denoted as $\Omega(Q)$. Given $q_i \in Q$ with $i = 0, 1$; the space of curves $q(t)$ on Q such that $q(t_0) = q_0$ is denoted $\Omega(Q; q_0)$ and the space of curves such that $q(t_i) = q_i$ for both $i = 0, 1$ is $\Omega(Q; q_0, q_1)$.

Definition 1.2.1 A **deformation** of a curve $q(t)$ on Q is a differentiable function $\tilde{q} : I \times (-\epsilon, \epsilon) \rightarrow Q$ such that $\tilde{q}(t, \epsilon = 0) = q(t)$ for every $t \in I$.

We shall write $q(t, \epsilon)$ instead of $\tilde{q}(t, \epsilon)$ to simplify the notation. Any deformation defines a *variation*,

$$\delta q(t) = \left. \frac{\partial q(t, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0}.$$

The variation of a curve $q(t)$ can be equivalently defined as a section of TQ whose domain is the image of the curve. Variations obtained from deformations such that for any $\varepsilon_* \in (-\varepsilon, \varepsilon)$, $t \rightarrow q(t, \varepsilon_*)$ is a curve in $\Omega(Q; q_0, q_1)$, satisfy $\delta q(t_i) = 0$ for $i = 0, 1$. The set of these fixed-endpoints variations of $q(t)$ will be denoted as Δ_q .

Definition 1.2.2 The *action* defined by a Lagrangian function $L : TQ \rightarrow \mathbb{R}$ on the set $\Omega(Q; q_0, q_1)$ is the map $\mathcal{S}(L) : \Omega(Q; q_0, q_1) \rightarrow \mathbb{R}$ such that

$$\mathcal{S}(L)(q) = \int_{t_0}^{t_1} L(\dot{q}(t)) dt,$$

where $\dot{q}(t)$ denotes the natural lift of $q(t)$ to TQ .

Definition 1.2.3 A curve $q(t) \in \Omega(Q; q_0, q_1)$ is said to be *critical* with respect to the action if for any smooth deformation $\{q_\varepsilon(t)\}_{\varepsilon \in \mathbb{R}}$ such that for all ε , $q_\varepsilon(t) \in \Omega(Q; q_0, q_1)$; we have that

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{S}(q_\varepsilon(t)) = 0.$$

Principle 1.2.4 (Hamilton) A *solution* of the Lagrangian system (TQ, L) defined on the configuration bundle Q by $L : TQ \rightarrow \mathbb{R}$ is a curve $q(t) \in \Omega(Q; q_0, q_1)$ critical with respect to $\mathcal{S}(L)$.

Critical curves satisfy the renowned *Euler–Lagrange equations*, which in standard coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$ on TQ can be written as

$$\frac{\partial L}{\partial q^i}(q, \dot{q}) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) \right) = 0, \quad \forall i = 1, \dots, n. \quad (1.2)$$

Observe that $L(q, \dot{q})$ refers to the evaluation of a Lagrangian at point in TQ with coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$ introduced by a local chart on Q . These coordinates are often grouped as $(q, \dot{q})$. In contrast, $L(\dot{q}(t))$ denotes the composition with a curve $\dot{q}(t)$ on TQ lifted from a curve $q(t)$ on Q . Indeed, $\mathbb{R} \times TQ \rightarrow \mathbb{R}$ can be regarded as $J^1(\mathbb{R}, Q)$, the *jet bundle* of classes of curves from $\mathbb{R}$ to Q . The curve $\dot{q}(t)$ on TQ is just the curve $[q]^{(1)}(t)$, the class of $q(t)$ in the jet bundle. We shall denote $\ell\Omega(Q; q_0, q_1)$ the set of curves on TQ obtained by lifting curves in $\Omega(Q; q_0, q_1)$. Similarly, for $k \geq 2$, the k -th order tangent bundle $T^{(k)}Q$ is defined, $\mathbb{R} \times T^{(k)}Q$ is identified with the k -order jet bundle $J^k(\mathbb{R}, Q)$ and the lift of a curve $q(t)$ to $T^{(k)}Q$ is denoted by $[q]^{(k)}(t)$.

The local expression in (1.2) depends heavily on a chosen set of coordinates. However, Euler-Lagrange equations are intrinsic. To prove this, we shall apply the chain rule to the

composition

$$\begin{aligned}\mathbb{R} &\rightarrow TQ \rightarrow \mathbb{R} \\ \varepsilon &\mapsto \dot{q}(t, \varepsilon) \mapsto L(\dot{q}(t, \varepsilon))\end{aligned}$$

while having a notion of verticality and horizontality on $T(TQ)$. Provided an arbitrary affine connection ∇ on TQ , there is a decomposition $T_{\dot{q}(t, \varepsilon)}(TQ) = H_{\dot{q}(t, \varepsilon)}(TQ) \oplus V_{\dot{q}(t, \varepsilon)}(TQ)$ and

$$\frac{\partial \dot{q}}{\partial \varepsilon} = \text{Hor} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right) \oplus \text{Ver} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right).$$

Similarly, the dual connection on T^*Q provides the decomposition

$$\frac{\partial L}{\partial \dot{q}} = \text{Hor} \left(\frac{\partial L}{\partial \dot{q}} \right) \oplus \text{Ver} \left(\frac{\partial L}{\partial \dot{q}} \right).$$

In short, the introduction of an arbitrary connection ∇ on TQ allows to write the chain rule expression,

$$\frac{d}{d\varepsilon} L(\dot{q}(t, \varepsilon)) = \left\langle \text{Hor} \left(\frac{\partial L}{\partial \dot{q}} \right), \text{Hor} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right) \right\rangle + \left\langle \text{Ver} \left(\frac{\partial L}{\partial \dot{q}} \right), \text{Ver} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right) \right\rangle, \quad (1.3)$$

where every term is an intrinsic geometric object. Indeed, from the definition of a covariant derivative with respect to an affine connection,

$$\text{Ver} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right) = \frac{D\dot{q}}{D\varepsilon}.$$

In addition,

$$\text{Ver} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\delta L}{\delta \dot{q}},$$

the fiber derivative of $L : TQ \rightarrow \mathbb{R}$. Furthermore, the horizontal lift introduced by ∇ identifies $H_{\dot{q}(t, \varepsilon)}(TQ)$ with $T_{q(t, \varepsilon)}Q$. Then,

$$\text{Hor} \left(\frac{\partial \dot{q}}{\partial \varepsilon} \right) = \frac{\partial q}{\partial \varepsilon}, \quad \text{Hor} \left(\frac{\partial L}{\partial \dot{q}} \right) = d^{h, \nabla} L;$$

where $d^{h, \nabla} L$ is the *horizontal differential* of L , that is, the differential of L restricted to the horizontal vectors determined by ∇ . To match widespread notation in the bibliography (see

Definition 3.3.3 in [23] and [6, Section 3]) we shall write

$$\text{Hor} \left(\frac{\partial L}{\partial \dot{q}} \right) \cong \frac{\partial L}{\partial q} : H_{\dot{q}(t,\varepsilon)}(TQ) \cong T_{q(t,\varepsilon)}Q \rightarrow \mathbb{R}$$

Yet, it should be noted that $\partial L/\partial q$ is not a partial derivative but rather *the covariant derivative* of L at $(q_0, \dot{q}_0) \in TQ$ in the direction of $v \in T_{q_0}Q$ defined via the affine connection ∇ as

$$\frac{\partial L}{\partial q}(q_0, \dot{q}_0)(v) = \left. \frac{d}{ds} \right|_{s=0} L(u(s)),$$

where $u(s)$ is the horizontal lift through $(q(0), \dot{q}(0))$ of a curve $q(s)$ such that $q(0) = q_0$ and $\dot{q}(0) = v$. Given the above identifications, equation (1.3) is

$$\frac{d}{d\varepsilon} L(\dot{q}(t, \varepsilon)) = \left\langle \frac{\partial L}{\partial q}, \frac{\partial q}{\partial \varepsilon} \right\rangle + \left\langle \frac{\delta L}{\delta \dot{q}}, \frac{D\dot{q}}{D\varepsilon} \right\rangle.$$

Proposition 1.2.5 (Euler-Lagrange) *Given a Lagrangian system (TQ, L) , and ∇ an affine connection on $TQ \rightarrow Q$, the **Euler–Lagrange operator**,*

$$\begin{aligned} \mathcal{EL}(L) : T^{(2)}Q &\rightarrow T^*Q \\ [q]^{(2)} &\mapsto \frac{\partial L}{\partial q}(\dot{q}(t)) - \frac{D}{Dt} \left(\frac{\delta L}{\delta \dot{q}}(\dot{q}(t)) \right), \end{aligned}$$

is the unique bundle map such that for any fixed-endpoints deformation $q(t, \varepsilon)$ of a curve $q(t)$ in $\Omega(Q; q_0, q_1)$, we have

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{S}(q_\varepsilon(t)) = \int_{t_0}^{t_1} \mathcal{EL}(L)([q]^{(2)}) \cdot \delta q.$$

Remark 1.2.6 *Proposition 1.2.5 provides an intrinsic description of the Euler–Lagrange equations in (1.2). A curve $q(t) \in \Omega(Q; q_0, q_1)$ is critical if and only if $\mathcal{EL}(L)([q]^{(2)}) = 0$. From the local expression (1.2) the Euler–Lagrange operator is independent of the chosen connection.*

1.2.2 Lagrange–Poincaré Reduction

We now consider a connected Lie group G acting freely and properly on the configuration manifold Q from the left. Then, the quotient space Q/G is a manifold and $\pi : Q \rightarrow Q/G$ is a G -principal bundle. The natural induced action on TQ is a vector bundle action. If we

assume that the Lagrangian L is G -invariant with respect to this action, the quotient map,

$$\begin{aligned} l : (TQ)/G &\rightarrow \mathbb{R} \\ [v_q]_G &\mapsto l([v_q]_G) = L(v_q). \end{aligned}$$

is called the *reduced Lagrangian*. The unreduced variational problem stated in Principle 1.2.4 can also be dropped to the quotient bundle $(TQ)/G \rightarrow Q/G$. On one hand, the set of *admissible curves* in $(TQ)/G$ is obtained projecting curves in $\ell\Omega(Q; q_0, q_1)$. In this way, admissible curves are $[q]_G$, for $q \in \Omega(Q; q_0, q_1)$, or using jet notation, $[q^{(1)}]_G$. On the other hand, the set of *admissible variations* is comprised of projections $[\delta q]_G = [\delta q^{(1)}]_G$ of variations $\delta q \in \Delta_q$. This process is known as the *Lagrange–Poincaré reduction* and chapter 2 will explore how to iterate this process.

Remark 1.2.7 *Throughout this thesis, we shall always assume that the Lie group of symmetries is connected. This assumption is quite usual in reduction theory as discrete groups are more involved and entail a completely different approach. Generalization of the new results of this thesis to discrete groups of symmetries is an interesting topic for future work.*

Lemma 1.2.8 *Given $\mathcal{A}$, a principal connection on $Q \rightarrow Q/G$, the map*

$$\begin{aligned} \alpha_{\mathcal{A}}^{TQ} : TQ/G &\rightarrow T(Q/G) \oplus \tilde{\mathfrak{g}} \\ [v_q]_G &\mapsto T\pi(v_q) \oplus [q, \mathcal{A}(v_q)]_G, \end{aligned}$$

where $v_q \in T_q Q$, is a well defined vector bundle isomorphism whose inverse is

$$\begin{aligned} (\alpha_{\mathcal{A}}^{TQ})^{-1} : T(Q/G) \oplus \tilde{\mathfrak{g}} &\rightarrow TQ/G \\ u_x \oplus [q, \xi]_G &\mapsto [(u_x)_q^h + \xi_q^Q]_G, \end{aligned}$$

where $u_x \in T_x(Q/G)$ and $q \in \pi^{-1}(x)$.

Proof. To test if $\alpha_{\mathcal{A}}^{TQ}$ is well defined it is necessary to check, $\alpha_{\mathcal{A}}^{TQ}([gv_q]_G) = \alpha_{\mathcal{A}}^{TQ}([v_q]_G)$ for any $g \in G$. Clearly, $T\pi(gv_q) = T\pi(v_q)$ and

$$[gq, \mathcal{A}(gv_q)]_G = [gq, \text{Ad}_g(\mathcal{A}(v_q))]_G = [q, \mathcal{A}(v_q)]_G.$$

Similarly it must be checked that $(\alpha_{\mathcal{A}}^{TQ})^{-1}$ is well defined, that is, for any $g \in G$,

$$(\alpha_{\mathcal{A}}^{TQ})^{-1}(u_x \oplus [gq, \text{Ad}_g(\xi)]_G) = (\alpha_{\mathcal{A}}^{TQ})^{-1}(u_x \oplus [q, \xi]_G).$$

Indeed, as the horizontal distribution of a G -principal bundle is G -invariant, $(u_x)^h_{gq} = g(u_x)^h_q$. Since $(\text{Ad}_g \xi)^Q_{gq} = g\xi^Q_q$,

$$\begin{aligned} (\alpha_{\mathcal{A}}^{TQ})^{-1}(u_x \oplus [gq, \text{Ad}_g(\xi)]_G) &= [(u_x)^h_{gq} + (\text{Ad}_g \xi)^Q_{gq}]_G = [g(u_x)^h_q + g\xi^Q_q]_G \\ &= (\alpha_{\mathcal{A}}^{TQ})^{-1}(\dot{x} \oplus [q, \xi]_G). \end{aligned}$$

Finally, from

$$\begin{aligned} (\alpha_{\mathcal{A}}^{TQ})^{-1} \circ \alpha_A([v_q]_G) &= (\alpha_{\mathcal{A}}^{TQ})^{-1}(T\pi(v_q) \oplus [q, \mathcal{A}(\dot{q})]_G) = [T\pi(v_q)^h_q + \mathcal{A}(\dot{q})q]_G \\ &= [\text{Hor}(v_q) + \text{Ver}(v_q)]_G = [v_q]_G. \end{aligned}$$

we conclude that $(\alpha_{\mathcal{A}}^{TQ})^{-1} \circ \alpha_{\mathcal{A}}^{TQ} = \text{id}_{(TQ)/G}$. $\square$

Given a curve $q \in \Omega(Q; q_0, q_1)$, we define the curves

$$\dot{x}(t) = T\pi(\dot{q}(t)), \quad \bar{\xi}(t) = [q(t), \mathcal{A}(\dot{q}(t))]_G,$$

so that the admissible curves $[\dot{q}]_G(t) = [\dot{q}(t)]_G$ of the reduced variational problem on TQ/G are identified with $\dot{x}(t) \oplus \bar{\xi}(t)$ via $\alpha_{\mathcal{A}}$. As the notation suggests, $\dot{x}(t)$ coincides with the tangent lift of $x(t) = \pi(q(t))$. In other words, the set of curves for the variational problem of the reduced Lagrangian l lies in

$$\ell\Omega(Q/G; x_0, x_1) \oplus_{\Omega(Q/G; x_0, x_1)} \Omega(\tilde{\mathfrak{g}}; x_0, x_1),$$

where $x_0 = \pi(q_0)$, $x_1 = \pi(q_1)$ and $\Omega(\tilde{\mathfrak{g}}; x_0, x_1)$ is the space of curves in $\tilde{\mathfrak{g}}$ with endpoints whose projections are x_0, x_1 .

Remark 1.2.9 *Since the projection of $\bar{\xi}(t)$ to Q/G is $x(t)$, the curve $\dot{x}(t) \oplus \bar{\xi}(t)$ can be recovered from $\bar{\xi}(t)$. Then, the set of admissible reduced curves is exactly $\Omega(\tilde{\mathfrak{g}}; x_0, x_1)$. However, to keep track of the curve on the reduced configuration manifold Q/G we shall adhere to the notation,*

$$\ell\Omega(Q/G; x_0, x_1) \oplus \Omega(\tilde{\mathfrak{g}}; x_0, x_1),$$

for the set of admissible curves and, $\dot{x}(t) \oplus \bar{\xi}(t)$, for admissible curves.

With respect to the set of admissible variations, we have that the first factor of the projection of $\delta[q]^{(1)}$ is simply $\delta[x]^{(1)}$, the lift of the variation of $x(t)$. However, the induced variation of $\bar{\xi}(t)$ is more involved. Since the affine connection $\nabla^{\mathcal{A}}$ on the vector bundle $\tilde{\mathfrak{g}} \rightarrow Q/G$ provides the splitting $T\tilde{\mathfrak{g}} = V\tilde{\mathfrak{g}} \oplus H\tilde{\mathfrak{g}} \cong 2\tilde{\mathfrak{g}} \oplus T(Q/G)$, given a deformation $\bar{\xi}(t, \varepsilon)$

of $\bar{\xi}(t)$ we have that,

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) &= \text{Ver} \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) \right) + \text{Hor} \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) \right) \\ &= \frac{D}{D\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) + \text{hor}_{\nabla}^{\xi(t)} \left(T\pi \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) \right) \right) \\ &= \frac{D}{D\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) + \text{hor}_{\nabla}^{\xi(t)} \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \pi(\bar{\xi}(t, \varepsilon)) \right) \\ &= \frac{D}{D\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) + \text{hor}_{\nabla}^{\xi(t)}(\delta x), \end{aligned}$$

where δx is the variation of $x(t) = \pi(\bar{\xi}(t))$ defined by $x(t, \varepsilon) = \pi(\bar{\xi}(t, \varepsilon))$. To deduce the induced variation of $\bar{\xi}(t)$, one studies the cases where δq is vertical or horizontal with respect to the projection $\pi : Q \rightarrow Q/G$ and the connection $\mathcal{A}$.

- In the case that δq is vertical, that is, $\delta q(t) = \eta(t) \frac{Q}{q(t)}$ with $\eta : I \rightarrow \mathfrak{g}$ and $\eta(t_0) = \eta(t_1) = 0$, we have that $\delta x = 0$. Then, as proved in [23],

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) = \frac{D}{D\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) = \frac{D}{Dt} \bar{\eta}(t) + [\bar{\xi}(t), \bar{\eta}(t)],$$

where $\bar{\eta}(t) = [q(t), \eta(t)]_G$.

- In the case that δq is horizontal, that is, $\mathcal{A}(\delta q) = 0$. Then,

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \bar{\xi}(t, \varepsilon) = \tilde{\mathcal{B}}_x(\delta x(t), \dot{x}(t)) + \text{hor}_{\nabla}^{\xi(t)}(\delta x)$$

and it is worth noting that, even if δq is horizontal, $d\bar{\xi}(t, \varepsilon)/d\varepsilon|_{\varepsilon=0}$ is not horizontal due to a vertical term coming from the curvature of the connection.

As any variation δq can be decomposed as the sum of its vertical and horizontal components, we conclude that the set of admissible variations of the reduced variational problem for a curve $\dot{x}(t) \oplus \bar{\xi}(t) \in \ell\Omega(Q/G; x_0, x_1) \oplus \Omega(\tilde{\mathfrak{g}}; x_0, x_1)$ is comprised of

$$\delta \dot{x} \oplus \delta \bar{\xi} = \delta \dot{x} \oplus \left(\frac{D}{Dt} \bar{\eta} + [\bar{\xi}, \bar{\eta}] + \tilde{\mathcal{B}}_x(\delta x, \dot{x}) \right)$$

where $\delta \dot{x}$ is the lift of a free variation $\delta x \in \Delta_x$ and $\bar{\eta}(t)$ is a curve in $\tilde{\mathfrak{g}}$ such that $\pi_{Q/G, \tilde{\mathfrak{g}}}(\bar{\eta}(t)) = x(t)$ and $\bar{\eta}(t_0) = \bar{\eta}(t_1) = 0$. The set of variations of $\dot{x}$ obtained from the lift of a variation $\delta x \in \Delta_x$ is denoted Δ_x^ℓ and $\Delta_{\dot{x} \oplus \bar{\xi}}^\ell \subset \Delta_{\dot{x} \oplus \bar{\xi}}$ is the subset of allowed variations of $\dot{x} \oplus \bar{\xi}$.

Remark 1.2.10 In order to showcase the curve $x(t)$ in the reduced configuration space Q/G and its variation as in Remark 1.2.9, the notation $\delta \bar{\xi}$ is reserved for $D\bar{\xi}(t, \varepsilon)/D\varepsilon|_{\varepsilon=0}$, the

vertical part of the variation of $\bar{\xi}(t)$. Similarly, in the following, given a vector bundle $V \rightarrow Q$ with an affine connection ∇ and deformation $v(t, \varepsilon)$ of a curve $v(t)$ on V we denote

$$\delta v = \left. \frac{D}{D\varepsilon} \right|_{\varepsilon=0} v(t, \varepsilon) \in 2V.$$

Once we understand both the set of admissible curves and variations, there is a reduced variational principle equivalent to the Hamilton Principle 1.2.4.

Principle 1.2.11 (reduced Hamilton) Let $L : TQ \rightarrow \mathbb{R}$ be a G -invariant Lagrangian on the principal bundle $Q \rightarrow Q/G$; $\mathcal{A}$ be a principal connection and $l : T(Q/G) \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$ be the reduced Lagrangian obtained after the identification of $(TQ)/G$ with $T(Q/G) \oplus \tilde{\mathfrak{g}}$ via $\alpha_{\mathcal{A}}$ as in Lemma 1.2.8.

A curve $u(t) = \dot{x}(t) \oplus \bar{\xi}(t) \in \ell\Omega(Q/G; x_0, x_1) \oplus \Omega(\tilde{\mathfrak{g}}, x_0, x_1)$ is said to be critical for the reduced Hamilton principle if it is a critical value of the action,

$$\mathcal{S}(l)(u) = \int_{t_0}^{t_1} l(u) dt,$$

defined by l with respect to variations $\delta u \in \Delta_{\dot{x} \oplus \bar{\xi}}^{\ell}$, that is,

$$\delta u = \delta \dot{x} \oplus \delta \bar{\xi} = \delta \dot{x} \oplus \left(\frac{D\bar{\eta}}{Dt} + [\bar{\xi}, \bar{\eta}] + \tilde{\mathcal{B}}_x(\delta x, \dot{x}) \right),$$

for any variation $\delta x \in \Delta_x$ and any curve $\bar{\eta}$ on $\mathfrak{g}$ such that $\bar{\eta}(t_0) = \bar{\eta}(t_1) = 0$.

Proposition 1.2.12 (Lagrange-Poincaré) Let $L : TQ \rightarrow \mathbb{R}$ be a G -invariant Lagrangian on the principal bundle $Q \rightarrow Q/G$; $\mathcal{A}$ be a principal connection and $l : T(Q/G) \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$ be the reduced Lagrangian. Given an affine connection ∇ on Q/G , the **Lagrange-Poincaré operator**,

$$\mathcal{LP}(l) : T^{(2)}Q/G \simeq T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}} \rightarrow T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

which can be split in two components as

$$\mathcal{LP}(l) = \text{Hor}(\mathcal{LP})(l) \oplus \text{Ver}(\mathcal{LP})(l) : T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}} \rightarrow T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

where

$$\text{Hor}(\mathcal{LP})(l) : T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}} \rightarrow T^*(Q/G)$$

$$[x]^{(2)} \oplus [\bar{\xi}]^{(1)} \mapsto \frac{\partial l}{\partial x}(\dot{x}, \bar{\xi}) - \frac{D}{Dt} \frac{\partial l}{\partial \dot{x}}(\dot{x}, \bar{\xi}) - \left\langle \frac{\partial l}{\partial \bar{\xi}}(\dot{x}, \bar{\xi}), \tilde{\mathcal{B}}_q(\dot{x}, \cdot) \right\rangle,$$

$$\begin{aligned} \text{Ver}(\mathcal{L}\mathcal{P})(l) : T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}} &\rightarrow \tilde{\mathfrak{g}}^* \\ [x]^{(2)} \oplus [\bar{\xi}]^{(1)} &\mapsto \text{ad}_{\bar{\xi}}^* \frac{\partial l}{\partial \bar{\xi}}(\dot{x}, \bar{\xi}) - \frac{D}{Dt} \frac{\partial l}{\partial \bar{\xi}}(\dot{x}, \bar{\xi}); \end{aligned}$$

is the unique bundle map such that for any deformation $(\dot{x} \oplus \bar{\xi})(t, \varepsilon)$ inducing an admissible variation $\delta \dot{x} \oplus \delta \bar{\xi} \in \Delta_{\dot{x} \oplus \bar{\xi}}^\ell$, we have

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{S}((\dot{x} \oplus \bar{\xi})(t, \varepsilon)) = \int_{t_0}^{t_1} \mathcal{L}\mathcal{P}(l)([x]^{(2)} \oplus [\bar{\xi}]^{(1)}) \cdot (\delta x \oplus \delta \bar{\xi}) dt.$$

Proposition 1.2.12 provides intrinsic equations for curves $(\dot{x} \oplus \bar{\xi})(t)$ that are solutions of the reduced Lagrangian system $(T(Q/G) \oplus \tilde{\mathfrak{g}}, l)$. These so called *Lagrange–Poincaré equation* are decomposed in horizontal and vertical components with respect to $\mathcal{A}$. The horizontal Lagrange–Poincaré equations are similar to the Euler–Lagrange equations except for an additional curvature term. The usual physical interpretation of this term is as a force term, sometimes called magnetic term. The vertical Lagrange–Poincaré equations are related with the conservation of quantities in the presence of symmetries.

1.2.3 Noether Current and Vertical Equations

Definition 1.2.13 Given a G -invariant Lagrangian $L : TQ \rightarrow \mathbb{R}$, the **Noether current** is the map

$$\begin{aligned} J : TQ &\rightarrow \mathfrak{g}^* \\ \dot{q} &\mapsto J(\dot{q})(\eta) = \left\langle \frac{\partial L}{\partial \dot{q}}(\dot{q}), \eta_q^Q \right\rangle. \end{aligned}$$

Theorem 1.2.14 (Noether) Let $q(t)$ be a curve on Q solution of the Euler–Lagrange equations of L , then for any $\eta \in \mathfrak{g}$

$$\left\langle \frac{d}{dt} J(\dot{q}(t)), \eta \right\rangle = \frac{d}{dt} \langle J(\dot{q}(t)), \eta \rangle = 0. \quad (1.4)$$

In other words, the Noether current is preserved along solutions of the Lagrangian system.

Proof. Since L is G -invariant, choosing $\exp(s\eta) \in G$, we have $L(\dot{q}) = L(\exp(s\eta)\dot{q})$ for all $s \in \mathbb{R}$. Differentiation with respect to s provides

$$0 = \left\langle \frac{\partial L}{\partial \dot{q}}, \text{Hor}(\eta_q^{TQ}) \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \text{Ver}(\eta_q^{TQ}) \right\rangle = \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_q^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_q^{TQ} \right\rangle.$$

Then, the evolution of the Noether current along $\dot{q}(t)$ is

$$\begin{aligned} \frac{d}{dt} \langle J(\dot{q}(t)), \eta \rangle &= \frac{d}{dt} \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_{q(t)}^Q \right\rangle = \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{D}{Dt} \eta_{q(t)}^Q \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_{\dot{q}(t)}^{TQ} \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle - \left\langle \frac{\partial L}{\partial q}, \eta_{q(t)}^Q \right\rangle = \mathcal{E} \mathcal{L}(L)([q]^{(2)}) \cdot (\eta_{q(t)}^Q), \end{aligned}$$

which is zero if $q(t)$ solution of the Euler–Lagrange equations of L . $\square$

The Noether current J is G -equivariant with respect to the lifted action on TQ and the co-adjoint action on $\mathfrak{g}^*$. Indeed,

$$\begin{aligned} J(g\dot{q})(\text{Ad}_g \eta) &= \left\langle \frac{\partial L}{\partial \dot{q}}(g\dot{q}(t)), (\text{Ad}_g \eta)_{gq}^Q \right\rangle \\ &= \frac{d}{ds} \Big|_{s=0} L(g\dot{q} + s \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} g \exp(\varepsilon \eta) g^{-1} \cdot g\dot{q}) \\ &= \frac{d}{ds} \Big|_{s=0} L(\dot{q} + g^{-1}s \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} g \exp(\varepsilon \eta) g^{-1} \cdot g\dot{q}) \\ &= \frac{d}{ds} \Big|_{s=0} L(\dot{q} + s\eta_q^Q) = \left\langle \frac{\partial L}{\partial \dot{q}}(\dot{q}(t)), \eta_q^Q \right\rangle = J(\dot{q})(\eta). \end{aligned}$$

The Noether current can be considered as a bundle map $J : TQ \rightarrow Q \times \mathfrak{g}$ covering id_Q . As $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$, the G -equivariance of the Noether current provides a reduced object:

Definition 1.2.15 The *reduced Noether current* is the bundle map

$$\begin{aligned} j : T(Q/G) \oplus \tilde{\mathfrak{g}} &\rightarrow \tilde{\mathfrak{g}} \\ (\dot{x}, \bar{\xi}) &\mapsto j(\dot{x}, \bar{\xi})(\bar{\eta}) = J(\dot{x}_q^h + \xi_q^Q)(\eta), \end{aligned}$$

where $\xi, \eta \in \mathfrak{g}$ such that $\bar{\xi} = [q, \xi]$ and $\bar{\eta} = [q, \eta]$.

Remark 1.2.16 The reduced Noether current coincides with the vertical derivative of the reduced Lagrangian

$$\begin{aligned} j(\dot{x}, \bar{\xi})(\bar{\eta}) &= J(\dot{x}_q^h + \xi_q^Q)(\eta) = \frac{d}{ds} \Big|_{s=0} L(\dot{x}_q^h + \xi_q^Q + s\eta_q^Q) \\ &= \frac{d}{ds} \Big|_{s=0} l(\dot{x} \oplus \bar{\xi} + s(0 \oplus \bar{\eta})) = \left\langle \frac{\partial l}{\partial \bar{\xi}}, \bar{\eta} \right\rangle. \end{aligned}$$

Proposition 1.2.17 *A curve $q(t)$ in Q preserves the Noether current of a G -invariant Lagrangian $L : TQ \rightarrow \mathbb{R}$ if and only if the curve $[\dot{q}]_G$ in $TQ/G \cong T(Q/G) \oplus \mathfrak{g}$ satisfies the vertical Lagrange–Poincaré equations.*

Proof. The previous remark links the derivative of the Noether current with the vertical derivative of the reduced Lagrangian:

$$\left\langle \frac{d}{dt} J(\dot{q}(t)), \eta \right\rangle = \frac{d}{dt} \langle J(\dot{q}(t)), \eta \rangle = \frac{d}{dt} \langle j(\dot{x}(t), \bar{\xi}(t)), \bar{\eta}(t) \rangle = \frac{d}{dt} \left\langle \frac{\partial l}{\partial \bar{\xi}}(\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle.$$

From the definition of the covariant derivative on a dual bundle,

$$\begin{aligned} & \frac{d}{dt} \left\langle \frac{\partial l}{\partial \bar{\xi}}(\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial l}{\partial \bar{\xi}} \right) (\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\xi}}(\dot{x}(t) \oplus \bar{\xi}(t)), \frac{D\bar{\eta}(t)}{Dt} \right\rangle, \end{aligned}$$

and from the definition of the associated affine connection on the adjoint bundle,

$$\frac{D\bar{\eta}(t)}{Dt} = [q(t), -[A(\dot{q}), \eta(t)] + \dot{\eta}]_G = [q(t), -[\xi(t), \eta(t)]]_G = -[\bar{\xi}(t), \bar{\eta}(t)].$$

Finally, from the last three equations, we get

$$\begin{aligned} \left\langle \frac{d}{dt} J(\dot{q}(t)), \eta \right\rangle &= \left\langle \frac{D}{Dt} \left(\frac{\partial l}{\partial \bar{\xi}} \right) (\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle + \left\langle \frac{\partial l}{\partial \bar{\xi}}(\dot{x}(t) \oplus \bar{\xi}(t)), \frac{D\bar{\eta}(t)}{Dt} \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial l}{\partial \bar{\xi}} \right) (\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle - \left\langle \text{ad}_{\bar{\xi}(t)}^* \frac{\partial l}{\partial \bar{\xi}}(\dot{x}(t) \oplus \bar{\xi}(t)), \bar{\eta}(t) \right\rangle \\ &= \text{Ver}(\mathcal{L}\mathcal{P})(l)\bar{\eta}(t). \end{aligned}$$

Thus, $q(t)$ preserves the Noether current if and only if $\bar{\eta}(t)$ is a solution of the vertical Lagrange–Poincaré equations. $\square$

1.3 Lagrangian Formalism in Field Theory

1.3.1 Lagrangian Field Theory

Field theory is a generalization of mechanics in the sense that the objects of study are no longer curves $q(t)$ on a manifold Q , that is, sections of the trivial bundle $\mathbb{R} \times Q \rightarrow Q$. Instead, we shall study sections of a general fiber bundle $P \rightarrow M$.

Definition 1.3.1 A first order **Lagrangian density** is a smooth bundle map over M ;

$$\mathcal{L} : J^1 P \rightarrow \bigwedge^n T^* M,$$

where $n = \dim(M)$.

Suppose that M is oriented and $v \in \Gamma(\bigwedge^n T^* M)$ is a volume form, then the function $L : J^1 P \rightarrow \mathbb{R}$ such that $\mathcal{L} = Lv$ is called a *Lagrangian*.

Definition 1.3.2 Let $U \subset M$ be an open subset whose closure $\bar{U}$ is compact and $\rho : \bar{U} \rightarrow P$ be a smooth local section. A **compactly supported vertical variation** of ρ is a local section $\delta\rho : \bar{U} \rightarrow VP$ such that

$$\delta\rho(x) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \rho_\varepsilon(x),$$

where $\rho_\varepsilon : \bar{U} \rightarrow P$ is a smooth deformation such that $\rho_0 = \rho$ and $\rho_\varepsilon|_{\partial U} = \rho|_{\partial U}$.

Remark 1.3.3 Recall that a function $f : C \subset M \rightarrow N$ from an arbitrary subset C of a manifold M , to a manifold N is said to be smooth if for any $x \in C$ there is an open neighborhood U_x of x and a smooth extension $f_x : U_x \rightarrow N$ of f .

Since it has been imposed on Definition 1.3.2 that $\delta\rho : \bar{U} \rightarrow VP$, for a fixed value of ε , $\rho_\varepsilon : \bar{U} \rightarrow P$ is a local section. Observe that for any compactly supported vertical variation $\delta\rho(x) \in V_{\rho(x)}P$ and $\delta\rho|_{\partial U} = 0$. In fact, there is a vertical vector field $V \in \Gamma(VP \rightarrow P)$ such that $V(\rho(x)) = 0$ for all $x \in \partial U$ and $\rho_\varepsilon = \phi_\varepsilon^V \circ \rho$, where ϕ_ε^V is the flow of V . Let $j^1 V$ be the 1-jet lift of vector field V as seen in [42, 74],

$$\delta j^1 \rho(x) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} j^1 \rho_\varepsilon(x) = j^1 V(j^1 \rho(x)), \quad (1.5)$$

and $\delta j^1 \rho$ is a smooth vertical deformation of the jet prolongation $j^1 \rho : \bar{U} \rightarrow J^1 P$.

Definition 1.3.4 A local section $\rho : \bar{U} \subset M \rightarrow P$ is a **critical section** of the variational problem defined by $\mathcal{L} = Lv$ if

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_U \mathcal{L}(j^1 \rho_\varepsilon) = 0$$

for all compactly supported vertical variations, ρ_ε , of ρ .

Principle 1.3.5 (Hamilton, Field Theory) A **solution** of the Lagrangian system $(J^1 P, \mathcal{L})$ is a critical section $\rho : M \rightarrow P$.

(Local) critical sections satisfy the local *Euler–Lagrange equations*, which in coordinates (x^i, y^A, y_i^A) on $J^1 P$, can be written as

$$\frac{\partial L}{\partial y^A}(j^1 \rho) - \frac{\partial}{\partial x^i} \left(\frac{\partial L}{\partial y_i^A}(j^1 \rho) \right) = 0, \quad \forall A = 1, \dots, \dim(P) - n, \quad (1.6)$$

provided that $d^n x = dx^1 \wedge \dots \wedge dx^n$ is a volume form on $\bar{U}$. As in Section 1.2, the Euler–Lagrange equations can be written in an implicit way via an auxiliary affine connection ∇^P on $TP \rightarrow P$. Then, Variational Principle 1.3.5 is equivalent to equation:

$$\frac{\delta L}{\delta \rho} - \operatorname{div}^P \frac{\delta L}{\delta j^1 \rho} = 0, \quad (1.7)$$

where $\delta L / \delta j^1 \rho \in TM \otimes_P VP^*$ is the fiber derivative in $J^1 P$, div^P is defined for VP^* -valued fields using the restriction of connection ∇^P to a connection on $VP \rightarrow P$ and $\delta L / \delta \rho$ is the horizontal differential with respect to an affine connection $\nabla^{J^1 P}$ on $J^1 P$ induced by ∇^P . Observe that $TM \otimes_P VP^*$ is a vector bundle over P and the factor TM is an abuse of notation to denote the pullback bundle $\pi_{M,P}^* TM$ over P . In successive chapters, $TM \otimes VP^*$ shall be written whenever the base space of a product bundle is clear from the context.

We shall now focus on the construction of $\nabla^{J^1 P}$. Given ∇^P an affine connection on $TP \rightarrow P$ and ∇^M an affine connection on $TM \rightarrow M$, there is a affine connection on $T^*M \otimes_P TP$, the vector bundle that models $J^1 P$. Indeed, just consider the dual connection of ∇^M and the product connection with ∇^P . A general, non necessarily affine, connection $\nabla^{J^1 P}$ on $J^1 P$ is obtained from the connection on $T^*M \otimes_P TP$ after adequate composition with the natural contact mappings on $J^1 P$. From [53, Lemma 3.1], the local expression of this connection is

$$\begin{aligned} \bar{\Gamma}_{Bi}^A &= \Gamma_{BC}^A y_i^C + \Gamma_{Bi}^A - y_k^A (\Gamma_{BC}^k y_i^C + \Gamma_{Bi}^k), \\ \bar{\Gamma}_{ji}^A &= \Gamma_{jC}^A y_i^C + \Gamma_{ji}^A - y_k^A (\Gamma_{jC}^k y_i^C + \Gamma_{ji}^k). \end{aligned}$$

where $\bar{\Gamma}$ and Γ denote the coefficients of $\nabla^{J^1 P}$ and ∇^P respectively. If ∇^P is projectable to ∇^M , the connection on $\nabla^{J^1 P}$ is affine and it is determined by ∇^P alone. Further discussion on how $\nabla^{J^1 P}$ can be obtained is found [53].

1.3.2 Lagrange–Poincaré Reduction in Field Theory

Reduction of a system with symmetry in the context of Lagrangian field theory has already been studied. Reduction of principal bundles, called Euler–Poincaré reduction, is described in [18] and was later expanded in [17] to Lagrange–Poincaré reduction of principal bundles by a subgroup of the structure group. An analogue to Lagrange–Poincaré reduction as introduced in Section 1.2.2 for mechanical systems was introduced in [20] in the case of principal bundles and later in [32] in the case of a general fiber bundle. We shall review some results introduced in [32].

Let $\Phi : G \times P \rightarrow P$ be a free and proper left action on P such that

$$\pi_{M,P} \circ \Phi_g = \pi_{M,P} \quad \forall g \in G$$

Then $P \rightarrow \Sigma = P/G$ is a G -principal bundle, the induced action of G on M is trivial and $\Sigma \rightarrow M$ is a fiber bundle with projection $\pi_{M,\Sigma}$ defined as $\pi_{M,\Sigma}(y) = s$ for all $s \in \Sigma$ and $y \in P$ such that $\pi_{M,\Sigma}(s) = \pi_{M,P}(y)$.

There is a lifted free and proper left action of G on $J^1 P$. As a result, $J^1 P \rightarrow (J^1 P)/G$ is a principal bundle. A principal connection $\mathcal{A}$ on $P \rightarrow \Sigma$ introduces a splitting of $(J^1 P)/G$ similar fashion to the splitting of $(TP)/G$ introduced in Proposition 1.2.8.

Proposition 1.3.6 [32, Section 2.2] *Given $\mathcal{A}$, a principal connection on $P \rightarrow P/G$, the map*

$$\begin{aligned} \alpha_{\mathcal{A}} : (J^1 P)/G &\rightarrow J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P) \\ [\gamma_y]_G &\mapsto T\pi_{\Sigma,P} \circ \gamma_y \oplus [\rho, \mathcal{A} \circ \gamma_y]_G \end{aligned} \quad (1.8)$$

is a well defined bundle isomorphism whose inverse is

$$\begin{aligned} \alpha_{\mathcal{A}}^{-1} : J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P) &\rightarrow (J^1 P)/G \\ \beta_s \oplus l_s &\mapsto [\text{hor}_{\mathcal{A}}^y \circ \beta_s + \zeta_y \circ l_s]_G, \end{aligned}$$

where $\text{hor}_{\mathcal{A}}^y$ is the horizontal lift to $y \in P$ through $\mathcal{A}$ and ζ_y is the map

$$\begin{aligned} \zeta_y : (\text{Ad} P)_s &\rightarrow V_y P, \\ [y', \eta]_G &\mapsto \xi_y^P, \end{aligned}$$

provided that $[y', \eta]_G = [y, \xi]_G$.

Let ρ be section of $P \rightarrow M$ and $\sigma = \pi_{\Sigma, P} \circ \rho = [\rho]_G$ a section of $\Sigma \rightarrow M$,

$$\alpha_{\mathcal{A}}([j^1 \rho]_G) = j^1 \sigma \oplus [\rho, \rho^* \mathcal{A}]_G.$$

We shall denote $\bar{\rho} = [\rho, \rho^* \mathcal{A}]_G$.

Given a G -invariant Lagrangian $L : J^1 P \rightarrow \mathbb{R}$ and its reduced Lagrangian $l : J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P) \rightarrow \mathbb{R}$, the main result in [32] provides a reduced variational principle equivalent to the variational Principle 1.3.5 defined by L . This procedure is called Lagrange–Poincaré reduction for field theoretical covariant Lagrangians.

Theorem 1.3.7 [32, Theorem 3.5] *The following are equivalent:*

- (i) *The section ρ is critical for the variational problem defined by $\mathcal{L} = Lv$.*
- (ii) *The section ρ is a solution of the Euler-Lagrange equations (1.2.5) for Lv .*
- (iii) *The reduced section $j^1 \sigma \oplus \bar{\rho}$ is critical for the reduced variational principle defined by $l : J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P) \rightarrow \mathbb{R}$, that is,*

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_M l(j^1 \sigma_{\varepsilon} \oplus \bar{\rho}_{\varepsilon}) v = 0$$

for variations $j^1 \sigma_{\varepsilon} \oplus \bar{\rho}_{\varepsilon}$ such that $\delta \bar{\rho} = \nabla^{\mathcal{A}} \bar{\eta} - [\bar{\eta}, \bar{\rho}] - \tilde{\mathcal{B}}(\delta \sigma, T \sigma)$, where $\delta \sigma$ is an arbitrary compactly supported vertical variation of σ , $\bar{\eta}$ is an arbitrary section of $\text{Ad} P \rightarrow M$ such that $\pi_{\Sigma, \text{Ad} P} \bar{\eta} = \sigma$. Recall that $\nabla^{\mathcal{A}}$ is the connection on $\text{Ad} P$ induced by $\mathcal{A}$ as defined in §1.1.

- (iv) *The reduced section $j^1 \sigma \oplus \bar{\rho}$ satisfies the Lagrange–Poincaré equations;*

$$\text{ad}^* \frac{\delta l}{\delta \bar{\rho}} - \text{div}^{\nabla} \frac{\delta l}{\delta \bar{\rho}} = 0, \quad (1.9)$$

$$\frac{\delta l}{\delta \sigma} - \text{div}^{\Sigma} \frac{\delta l}{\delta j^1 \sigma} = \langle \frac{\delta l}{\delta \bar{\rho}}, i_{T_{\sigma}} \tilde{\mathcal{B}} \rangle, \quad (1.10)$$

where $\delta l / \delta j^1 \sigma \in T M \otimes_{\Sigma} V \Sigma^*$ and $\delta l / \delta \rho \in T M \otimes_{\Sigma} \text{Ad} P^*$ are fiber derivatives in $J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P)$, div^{Σ} is defined for $V \Sigma^*$ -valued fields using the restriction of connection ∇^{Σ} to a torsion-free connection on $V \Sigma \rightarrow \Sigma$. Finally $\delta l / \delta \sigma$ is the horizontal differential with respect to the affine connection on $J^1 \Sigma \oplus (T^* M \otimes_{\Sigma} \text{Ad} P)$ obtained as the sum of $\nabla^{J^1 \Sigma}$ and ∇^L a connection on $T^* M \otimes_{\Sigma} \text{Ad} P$ from $\nabla^{\mathcal{A}}$ and a connection ∇^M on $T M \rightarrow M$.

A generalization of this result that allows the iteration of the Lagrange–Poincaré reduction process is the main objective of chapter 4

The attentive reader may have noticed that $\nabla^{\mathcal{A}}$ is a connection on $\text{Ad}P \rightarrow \Sigma$ and, consequently, acts on sections of $\text{Ad}P \rightarrow \Sigma$ while $\bar{\eta}$ is a section of $\text{Ad}P \rightarrow M$. We shall now explain how to extend a connection to derive this kind of sections.

Let $V \rightarrow P$ be a vector bundle with connection ∇ and $P \rightarrow M$ a fiber bundle, given $f : M \rightarrow V$ a section of $V \rightarrow M$, define the ∇ -derivative of f with respect to $u_x \in T_x M$ as

$$\tilde{\nabla}_{u_x} f = \left. \frac{D^\nabla}{Dt} \right|_{t=0} f(c(t)) \in V_{\pi_{P,V} f(x)},$$

where $c(t)$ is a curve in M such that $\dot{c}(0) = u_x$, D^∇/Dt is the usual covariant derivative associated to ∇ and $t \mapsto f(c(t))$ is a curve in V . Since f is a section on $V \rightarrow M$,

$$\text{id}_M = \pi_{M,V} \circ f = \pi_{M,P} \circ \pi_{P,V} \circ f$$

and $\rho = \pi_{P,V} \circ f$ is a section of $P \rightarrow M$. Then, $f(c(t)) = f \circ \pi_{M,P} \circ \rho(c(t))$ projects to curve $\rho(c(t))$ in P with $D\rho(c(t))/dt|_{t=0} = T_x \rho(u_x)$ and

$$\tilde{\nabla}_{u_x} f = \left. \frac{D^\nabla}{Dt} \right|_{t=0} f(c(t)) = \nabla_{T_x \rho(u_x)} \bar{f},$$

where $\bar{f}$ is a local section of $V \rightarrow P$ around $\rho(x)$ such that $\bar{f}|_{\text{im}(\rho(M))} = f \circ \pi_{M,P}$. Sometimes we will use the abuse of notation $f = \bar{f}$.

Chapter 2

The Lagrange-Poincaré Category in Mechanics

2.1 The Lagrange-Poincaré Category, $\mathfrak{LP}$

2.1.1 The Lagrange-Poincaré Category

Iteration of the Lagrange-Poincaré reduction process introduced in subsection 1.2.2 has an immediate difficulty: the original Lagrangian L is defined on TQ , the tangent bundle of the configuration space Q , while the reduced Lagrangian l is defined on $TQ/G \cong T(Q/G) \oplus \mathfrak{g}$ which need not be a tangent bundle itself. Hence, reduction of l involves the formulation of Lagrangian mechanics in a wider category of bundles stable by the action of groups and consistent with the Lagrange-Poincaré reduction presented in subsection 1.2.2. To that end, the category $\mathfrak{LP}$ was introduced in [23].

Definition 2.1.1 *The category of **Lagrange-Poincaré bundles** is denoted by $\mathfrak{LP}$ and is defined as follows:*

1. The **objects** of $\mathfrak{LP}$ are vector bundles $\pi_{Q,TQ} \oplus \pi_{Q,V} : TQ \oplus V \rightarrow Q$ where $\pi_{Q,TQ} : TQ \rightarrow Q$ is the tangent bundle of a manifold Q , and $\pi_{Q,V} : V \rightarrow Q$ is a vector bundle with the following additional structure:
 - (a) a Lie algebra on each fiber of V , denoted by $[\cdot, \cdot]$, such that V is a Lie algebra bundle;
 - (b) a V -valued 2-form ω on Q ;
 - (c) a covariant derivative D/Dt for curves in V (equivalently a connection ∇ on V);

(d) the bilinear operator defined by

$$[X_1 \oplus w_1, X_2 \oplus w_2] = [X_1, X_2] \oplus (\nabla_{X_1} w_2 - \nabla_{X_2} w_1 - \omega(X_1, X_2) + [w_1, w_2]),$$

is a Lie bracket on sections $X \oplus w \in \Gamma(TQ \oplus V)$. Note that $[X_1, X_2]$ denotes the Lie bracket of vector fields while $[w_1, w_2]$ denotes the Lie bracket in the fibers of V .

2. The **morphisms** between two Lagrange-Poincaré bundles $TQ_i \oplus V_i$, $i = 1, 2$ with structures $[\cdot, \cdot]_i$, ω_i and D_i/Dt are vector bundle morphisms $f : TQ_1 \oplus V_1 \rightarrow TQ_2 \oplus V_2$ such that:

- (a) $f(TQ_1) \subset TQ_2$ and $f|_{TQ_1} = Tf_0$, where $f_0 : Q_1 \rightarrow Q_2$ is the function induced by f in the base spaces;
- (b) $f(V_1) \subset V_2$ and $f|_{V_1}$ commutes with the additional structure. That is, given $v, v' \in (\pi_{Q_1, V_1})^{-1}(q)$, $X, X' \in (\pi_{Q_1, TQ_1})^{-1}(q)$ and a curve $v(t)$ in V_1 :

$$f([v, v']_1) = [f(v), f(v')]_2, \quad (2.1)$$

$$f(\omega_1(X, X')) = \omega_2(f(X), f(X')), \quad (2.2)$$

$$f\left(\frac{D_1 v(t)}{Dt}\right) = \frac{D_2 f(v(t))}{Dt}. \quad (2.3)$$

Example 2.1.2 Tangent bundles are a special case of Lagrange-Poincaré bundles for which $V = 0$ is the trivial vector bundle. The Lie bracket on $\Gamma(TQ)$ is simply the Jacobi-Lie bracket for vector fields, and the morphisms between them are the tangent lift of functions.

Example 2.1.3 Lie Algebras are a special case of Lagrange-Poincaré bundles for which the base space is a point.

Example 2.1.4 Given a principal connection $\mathcal{A}$ on $Q \rightarrow Q/G$, the Atiyah algebroid TQ/G is identified with the $\mathfrak{g}$ -bundle $T(Q/G) \oplus \mathfrak{g}$ where the Lie bracket on $\mathfrak{g}$ is defined as in section 1.1, the $\mathfrak{g}$ -valued 2-form on Q/G is the curvature $\tilde{\mathcal{B}}$ of $\mathcal{A}$, and the covariant derivative on $\mathfrak{g}$ induced by $\nabla^{\mathcal{A}}$, the associated affine connection on $\mathfrak{g}$ provided by $\mathcal{A}$. Furthermore, given two sections $X_i \oplus \bar{\xi}_i \in \Gamma(T(Q/G) \oplus \mathfrak{g}) \equiv \mathfrak{X}^\infty(Q/G) \oplus \Gamma(\mathfrak{g})$ the expression

$$[X_1 \oplus \bar{\xi}_1, X_2 \oplus \bar{\xi}_2] = [X_1, X_2] \oplus (\nabla_{X_1}^{\mathcal{A}} \bar{\xi}_2 - \nabla_{X_2}^{\mathcal{A}} \bar{\xi}_1 - \tilde{\mathcal{B}}(X_1, X_2) + [\bar{\xi}_1, \bar{\xi}_2]),$$

where the bracket $[X_1, X_2]$ is the Jacobi-Lie bracket for tangent fields on Q/G , coincides with the quotient Lie bracket on $\Gamma((TQ)/G)$.

Condition 1.(d) is not deduced from the previous three conditions 1.(a), 1.(b), 1.(c). In fact, condition 1.(d) can be rewritten in a clearer way imposing geometrical relations between $[\cdot, \cdot]$, ω , and ∇ as follows.

Proposition 2.1.5 *Let $\pi_{Q,TQ} \oplus \pi_{Q,V} : TQ \oplus V \rightarrow Q$, where $\pi_{Q,TQ} : TQ \rightarrow Q$ is the tangent bundle of a manifold Q and $\pi_{Q,V} : V \rightarrow Q$ is a vector bundle, satisfying properties 1.(a), 1.(b), and 1.(c). Then, the expression*

$$[X_1 \oplus w_1, X_2 \oplus w_2] = [X_1, X_2] \oplus (\nabla_{X_1} w_2 - \nabla_{X_2} w_1 - \omega(X_1, X_2) + [w_1, w_2]) \quad (2.4)$$

defines a Lie bracket on sections $X \oplus w \in \Gamma(TQ \oplus V)$ if and only if

$$(d') \quad d_{\nabla} \omega = 0;$$

$$(e') \quad \nabla_X [w_1, w_2] = [\nabla_X w_1, w_2] + [w_1, \nabla_X w_2] \text{ for all } X \in \mathfrak{X}(Q) \text{ and all } w_1, w_2 \in \Gamma(V);$$

$$(f') \quad k_{\nabla}(X, Y)w = -[\omega(X, Y), w] \text{ for all } X, Y \in \mathfrak{X}(Q) \text{ and all } w \in \Gamma(V).$$

Proof. Expression (2.4) is clearly $\mathbb{R}$ -bilinear on $\Gamma(TQ \oplus V)$. It is skew symmetric since

$$[X \oplus w, X \oplus w] = [X, X] \oplus (\nabla_X w - \nabla_X w - \omega(X, X) + [w, w]) = 0.$$

Thus, (2.4) defines a Lie Bracket if and only if it satisfies the Jacobi identity, that is, if the expression:

$$\begin{aligned} & [X_1 \oplus w_1, [X_2 \oplus w_2, X_3 \oplus w_3]] = \\ &= \left[X_1 \oplus w_1, [X_2, X_3] \oplus \left(\nabla_{X_2} w_3 - \nabla_{X_3} w_2 - \omega(X_2, X_3) + [w_2, w_3] \right) \right] \\ &= [X_1, [X_2, X_3]] \oplus \left(\nabla_{X_1} \nabla_{X_2} w_3 - \nabla_{X_1} \nabla_{X_3} w_2 - \nabla_{X_1} \omega(X_2, X_3) + \nabla_{X_1} [w_2, w_3] \right. \\ &\quad \left. - \nabla_{[X_2, X_3]} w_1 - \omega(X_1, [X_2, X_3]) + [w_1, \nabla_{X_2} w_3 - \nabla_{X_3} w_2 - \omega(X_2, X_3) + [w_2, w_3]] \right) \end{aligned}$$

is cyclic. This is equivalent to ask if

$$\begin{aligned} C(X_1, w_1, X_2, w_2, X_3, w_3) &= \nabla_{X_2} \nabla_{X_3} w_1 - \nabla_{X_3} \nabla_{X_2} w_1 - \nabla_{X_1} \omega(X_2, X_3) + \nabla_{X_1} [w_2, w_3] \\ &\quad - \nabla_{[X_2, X_3]} w_1 - \omega(X_1, [X_2, X_3]) + [w_3, \nabla_{X_1} w_2] - [w_2, \nabla_{X_1} w_3] - [w_1, \omega(X_2, X_3)] \end{aligned}$$

is cyclic. Assume that $C(X_1, w_1, X_2, w_2, X_3, w_3)$ is cyclic. For $w_1 = w_2 = w_3 = 0$ and arbitrary $X_1, X_2, X_3 \in \mathfrak{X}(Q)$, we have

$$C(X_i, 0, X_j, 0, X_k, 0) = -\nabla_{X_i} \omega(X_j, X_k) - \omega(X_i, [X_j, X_k]),$$

for all $i, j, k \in \{1, 2, 3\}$. Hence,

$$\begin{aligned} 0 &= C(X_1, 0, X_2, 0, X_3, 0) + C(X_2, 0, X_3, 0, X_1, 0) + C(X_3, 0, X_1, 0, X_2, 0) \\ &= -\nabla_{X_1}\omega(X_2, X_3) - \omega(X_1, [X_2, X_3]) - \nabla_{X_2}\omega(X_3, X_1) - \omega(X_2, [X_3, X_1]) \\ &\quad - \nabla_{X_3}\omega(X_1, X_2) - \omega(X_3, [X_1, X_2]) = -d_{\nabla}\omega(X_1, X_2, X_3), \end{aligned}$$

and condition 1.(d') holds. For $w_1 = 0, X_2 = X_3 = 0$, we have

$$C(X_1, 0, 0, w_2, 0, w_3) = [w_3, \nabla_{X_1}w_2] - [w_2, \nabla_{X_1}w_3] + \nabla_{X_1}[w_2, w_3],$$

as well as $C(0, w_3, X_1, 0, 0, w_2) = C(0, w_2, 0, w_3, X_1, 0) = 0$. Hence,

$$\begin{aligned} 0 &= C(X_1, 0, 0, w_2, 0, w_3) + C(0, w_3, X_1, 0, 0, w_2) + C(0, w_2, 0, w_3, X_1, 0) \\ &= \nabla_{X_1}[w_2, w_3] - [\nabla_{X_1}w_2, w_3] - [w_2, \nabla_{X_1}w_3], \end{aligned}$$

and condition 1.(e') holds. For $X_1 = 0, w_2 = w_3 = 0$, we have

$$C(0, w_1, X_2, 0, X_3, 0) = \nabla_{X_2}\nabla_{X_3}w_1 - \nabla_{X_3}\nabla_{X_2}w_1 - \nabla_{[X_2, X_3]}w_1 - [w_1, \omega(X_2, X_3)],$$

as well as $C(X_3, 0, 0, w_1, X_2, 0) = C(X_2, 0, X_3, 0, 0, w_1) = 0$. Hence,

$$\begin{aligned} 0 &= C(0, w_1, X_2, 0, X_3, 0) + C(X_3, 0, 0, w_1, X_2, 0) + C(X_2, 0, X_3, 0, 0, w_1) \\ &= \nabla_{X_2}\nabla_{X_3}w_1 - \nabla_{X_3}\nabla_{X_2}w_1 - \nabla_{[X_2, X_3]}w_1 - [w_1, \omega(X_2, X_3)] \\ &= k_{\nabla}(X_2, X_3)w_1 + [\omega(X_2, X_3), w_1], \end{aligned}$$

and condition 1.(f') holds.

Conversely, assume that conditions 1.(d'), 1.(e') and 1.(f') hold true. These imply that $C(X_1, 0, X_2, 0, X_3, 0)$, $C(X_1, 0, 0, w_2, 0, w_3)$, and $C(0, w_1, X_2, 0, X_3, 0)$ are cyclic. Then,

$$\begin{aligned} C(X_1, w_1, X_2, w_2, X_3, w_3) &= C(X_1, 0, X_2, 0, X_3, 0) + C(X_1, 0, 0, w_2, 0, w_3) \\ &\quad + C(0, w_1, X_2, 0, X_3, 0). \end{aligned}$$

is cyclic. □

2.1.2 Reduction of Lagrange–Poincaré Bundles

We shall state the adequate notion of symmetry and reduction in the category $\mathfrak{LP}$.

Definition 2.1.6 An *action in the category* $\mathfrak{LP}$ of a Lie group G on an object $TQ \oplus V$ of $\mathfrak{LP}$ is a vector bundle action $\Phi : G \times (TQ \oplus V) \rightarrow TQ \oplus V$ such that for all $g \in G$, $\Phi_g : TQ \oplus V \rightarrow TQ \oplus V$ is a $\mathfrak{LP}$ isomorphism.

Definition 2.1.6 implies that the additional structures $[\cdot, \cdot]$, ω , ∇ are invariant by Φ_g and quotient structures can be defined in the sense specified below.

A Lie bracket $[\cdot, \cdot]$ on V is said to be invariant if $g[v_1, v_2] = [gv_1, gv_2]$, for each $g \in G$ and each $v_1, v_2 \in V$ such that $\pi_{Q,V}(v_1) = \pi_{Q,V}(v_2)$. Then, the expression

$$[[v_1]_G, [v_2]_G]_G = [[v_1, v_2]]_G$$

defines a *quotient Lie bracket* on V/G .

A V -valued 2-form ω on Q is said to be invariant if $g\omega(X_1, X_2) = \omega(gX_1, gX_2)$ for each $g \in G$ and each $X_1, X_2 \in TQ$ such that $\pi_{Q,TQ}(X_1) = \pi_{Q,TQ}(X_2)$. An invariant form defines a *quotient generalized form* as

$$[\omega]_G([X_1]_G, [X_2]_G) = [\omega(X_1, X_2)]_G$$

for all $X_1, X_2 \in TQ$ such that $\pi_{Q,TQ}(X_1) = \pi_{Q,TQ}(X_2)$. Observe that $[\omega]_G$ is not a form on Q/G since it is a skew-symmetric bilinear form on the fibers of TQ/G rather than on the fibers of $T(Q/G)$. In fact, identifications $[X_1]_G \equiv T\pi(X_1) \oplus \bar{\xi}$, $[X_2]_G \equiv T\pi(X_2) \oplus \bar{\eta}$ via the isomorphism $\alpha_{\mathcal{A}}^{TQ}$ provide

$$\begin{aligned} [\omega]_G([X_1]_G, [X_2]_G) &= [\omega]_G(T\pi(X_1), T\pi(X_2)) \\ &\quad + [\omega]_G(T\pi(X_1), \bar{\eta}) + [\omega]_G(\bar{\xi}, T\pi(X_2)) + [\omega]_G(\bar{\xi}, \bar{\eta}), \end{aligned}$$

where only the term $[\omega]_G(T\pi(X_1), T\pi(X_2))$ is a 2-form on Q/G .

Let $X = Y \oplus \bar{\xi} \in \mathfrak{X}(Q/G) \oplus \Gamma(\bar{\mathfrak{g}}) \simeq \Gamma(TQ/G)$, using the identification (1.2.8). There is a unique G -invariant vector field $\tilde{X} \in \Gamma^G(TQ)$ on Q projecting to X . Furthermore, $\tilde{X} = Y^h \oplus W$ with $Y^h \in \mathfrak{X}(TQ)$ the horizontal lift of Y and W the unique vertical G -invariant vector field such that for all $x \in Q/G$, $\bar{\xi}(x) = [q, \mathcal{A}(W(q))]_G$ with $q \in \pi_{Q/G,Q}^{-1}(x)$. Then, for $[v]_G \in \Gamma(Q/G, V/G)$ with $v \in \Gamma^G(Q, V)$ a G -invariant section, we define the *quotient connection* by

$$[\nabla^{(\mathcal{A})}]_{G,Y \oplus \bar{\xi}}[v]_G = [\nabla_{\tilde{X}} v]_G,$$

the *horizontal quotient connection* is defined by

$$[\nabla^{(\mathcal{A},H)}]_{G,Y \oplus \bar{\xi}}[v]_G = [\nabla_{Y^h} v]_G = [\nabla_{\tilde{X}}^{(\mathcal{A},H)} v]_G \quad (2.5)$$

and the *vertical quotient connection* is defined by

$$[\nabla^{(\mathcal{A},V)}]_{G,Y\oplus\bar{\xi}}[v]_G = [\nabla_W v]_G = [\xi_v^V]_G, \quad (2.6)$$

where $\bar{\xi}$ satisfies $\bar{\xi} = [\pi_{Q,V}(v), \xi]_G$ and $\nabla^{(\mathcal{A},H)}$, $\nabla^{(\mathcal{A},V)}$ are the derivatives introduced in 1.1.3. Note that these are not connections in the usual sense as derivation is performed with respect to sections of TQ/G instead of sections of $T(Q/G)$. In particular, this is called a $(TQ)/G$ -connection in the context of Lie algebroids [59]. Yet, the horizontal quotient connection can be thought as a usual connection since it only depends on $T(Q/G) \subset TQ/G$.

Using these quotient structures, it was proved that the category $\mathfrak{LP}$ is stable by reduction:

Theorem 2.1.7 [23, Theorem 6.2.10] *Let $\pi_{Q,TQ} \oplus \pi_{Q,V} : TQ \oplus V \rightarrow Q$ be an object of $\mathfrak{LP}$ with additional structures $[\cdot, \cdot]$, ω , ∇ . Let $\Phi : G \times (TQ \oplus V) \rightarrow TQ \oplus V$ be an action in the category $\mathfrak{LP}$ and $\mathcal{A}$ a principal connection on $Q \rightarrow Q/G$. Then, the vector bundle*

$$T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$$

with additional structures $[\cdot, \cdot]^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ and $\nabla^{\tilde{\mathfrak{g}}}$ in $\tilde{\mathfrak{g}} \oplus (V/G)$ given by

$$\begin{aligned} \nabla_X^{\tilde{\mathfrak{g}}}(\bar{\xi} \oplus [v]_G) &= \nabla_X^{\mathcal{A}} \bar{\xi} \oplus ([\nabla^{(\mathcal{A},H)}]_{G,X}[v]_G - [\omega]_G(X, \bar{\xi})), \\ \omega^{\tilde{\mathfrak{g}}}(X_1, X_2) &= \tilde{\mathcal{B}}^{\mathcal{A}}(X_1, X_2) \oplus [\omega]_G(X_1, X_2), \\ [\bar{\xi}_1 \oplus [v_1]_G, \bar{\xi}_2 \oplus [v_2]_G]^{\tilde{\mathfrak{g}}} &= [\bar{\xi}_1, \bar{\xi}_2] \oplus ([\nabla^{(\mathcal{A},V)}]_{G,\bar{\xi}_1}[v_2]_G - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\xi}_2}[v_1]_G \\ &\quad - [\omega]_G(\bar{\xi}_1, \bar{\xi}_2) + [v_1]_G, [v_2]_G)_G \end{aligned}$$

*is an object of the $\mathfrak{LP}$ category, the **reduced bundle** with respect to the group G and the connection $\mathcal{A}$.*

2.1.3 The subcategory $\mathfrak{RT}$

A relevant subcategory of $\mathfrak{LP}$ is the category $\mathfrak{RT}$ of reduced tangent bundles. This is the smallest subcategory of $\mathfrak{LP}$ that contains tangent bundles and is closed under the quotient operation in 2.1.7. The following example showcases that $\mathfrak{RT}$ is not the whole category $\mathfrak{LP}$.

Let Z be an abelian Lie group, $\mathfrak{z}$ be its abelian Lie algebra, and $P \rightarrow M$ be a Z -principal bundle. Since the adjoint action is trivial, the adjoint bundle $\tilde{\mathfrak{z}} \rightarrow M$ is a trivial Lie algebra vector bundle. Let $V \rightarrow M$ be a non-trivial vector bundle equipped with a trivial fiberwise Lie bracket. Then the bundle $TM \oplus V$ cannot belong to the subcategory $\mathfrak{RT}$ of reduced

tangent bundles. However, it can be seen as an object in $\mathfrak{LP}$, considering the adequate triple $[\cdot, \cdot], \omega$ and ∇ . Firstly, we choose a flat connection ∇ on V . There are instances of non-trivial vector bundles with flat connections, all of them over non-simply connected manifolds. As ∇ is flat and the Lie bracket on the fibers is trivial, conditions 1.(e') and 1.(f') are clearly satisfied. With respect to 1.(d'), we take any closed 2-form ω with respect to the covariant derivative defined by ∇ (that is, a representative of the cohomology with values in V defined by the covariant differential). In particular, we can even choose $\omega = 0$. In short, the problems posed in $TM \oplus V$ with trivial $[\cdot, \cdot]$, flat connection ∇ and a form ω chosen as above are set in $\mathfrak{LP}$ and not in the subcategory $\mathfrak{RT}$.

A similar construction can be given in a wider class of fiber bundles. Let G be any non-abelian Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{z}$ be the center of $\mathfrak{g}$. If $P \rightarrow M$ is a G -principal bundle, then the adjoint bundle $\tilde{\mathfrak{g}} \rightarrow M$ has a trivial subbundle with fiber dimension equal to the dimension of the center. Indeed, the subbundle

$$\tilde{\mathfrak{z}} = \{[p, B]_G : p \in P, B \in \mathfrak{z}\}$$

can be identified with $M \times \mathfrak{z}$ as the adjoint action in the center is trivial. In addition, at each point, the subbundle is the center of the Lie algebra on the fibers of $\tilde{\mathfrak{g}}$. Furthermore, suppose that $\mathfrak{g}$ is a reductive Lie algebra, that is, $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{s}$ is semisimple. Accordingly, the adjoint bundle can be decomposed as $\tilde{\mathfrak{g}} = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{s}}$. If we replace $\tilde{\mathfrak{z}}$ by a non-trivial vector bundle $V \rightarrow M$ equipped with a trivial bracket, the bundle $TQ \oplus V \oplus \tilde{\mathfrak{s}} \rightarrow M$ does not belong to the subcategory $\mathfrak{RT}$ even though it is an element of $\mathfrak{LP}$ with a convenient choice of connection ∇ and 2-form ω .

2.2 The Lagrangian Formalism in $\mathfrak{LP}$

2.2.1 Lagrangian mechanics in the category $\mathfrak{LP}$

Lagrangian systems can be defined on Lagrange–Poincaré bundles as described below.

Definition 2.2.1 A *Lagrangian* on $TQ \oplus V \in \mathfrak{LP}$ is a function

$$L : TQ \oplus V \rightarrow \mathbb{R}.$$

Denote by $\ell\Omega(Q) \oplus \Omega(V)$ the space of curves in $TQ \oplus V$ of the form $\dot{q}(t) \oplus v(t)$ where $v(t)$ is a curve in V , $q(t) = \pi_{Q,V}(v(t))$ is a curve in Q and $\dot{q}(t)$ the tangent lift of $q(t)$ to TQ . In addition, $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$ denotes the curves within $\ell\Omega(Q) \oplus \Omega(V)$ such that $q(t)$ has endpoints q_0, q_1 .

Definition 2.2.2 The *action* of $L : TQ \oplus V \rightarrow \mathbb{R}$ is defined on $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$ as

$$\mathcal{S}(L)(\dot{q} \oplus v) = \int_{t_0}^{t_1} L(\dot{q}(t), v(t)) dt.$$

Let $\Delta_{\dot{q} \oplus v}^\ell \subset \Delta_{\dot{q} \oplus v}$ be the set of variations of the form $\delta\dot{q} \oplus \delta v$ with $\delta\dot{q} \in \Delta_q^\ell$ and

$$\delta v = \frac{Dw}{Dt} + [v, w] + \omega_q(\delta q, \dot{q}),$$

where $w(t)$ is a curve in V with zero endpoints and $\pi_{Q,V}(w(t)) = q(t)$.

Principle 2.2.3 A curve $\dot{q}(t) \oplus v(t) \in \ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$ is a **solution** of the Lagrangian system $(TQ \oplus V, L)$ defined by $L : TQ \oplus V \rightarrow \mathbb{R}$ if for any $\delta(\dot{q} \oplus v) \in \Delta_{\dot{q} \oplus v}^\ell$ we have that

$$0 = d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{S}(L)((\dot{q} \oplus v)(t, \varepsilon)),$$

where $(\dot{q} \oplus v)(t, \varepsilon)$ is a deformation of $\dot{q} \oplus v$ inducing $\delta\dot{q} \oplus \delta v$.

Variational principle 2.2.3 for Lagrangians defined on $\mathfrak{LP}$ bundles can be translated into equivalent equations.

Proposition 2.2.4 Let $L : TQ \oplus V \rightarrow \mathbb{R}$ a Lagrangian and

$$\mathcal{S}(L)(\dot{q} \oplus v) = \int_{t_0}^{t_1} L(\dot{q}(t), v(t)) dt$$

its action on curves $\dot{q}(t) \oplus v(t)$ within $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$. There exists a unique bundle map

$$\mathcal{LP}(L) : T^{(2)}Q \oplus 2V \rightarrow T^*Q \oplus V^*$$

such that, for each variation $\delta\dot{q} \oplus \delta v$ in $\Delta_{\dot{q} \oplus v}^\ell$,

$$d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) = \int_{t_0}^{t_1} \mathcal{LP}(L)([q]^{(2)} \oplus [v]^{(1)}) \cdot (\delta q \oplus \delta v).$$

The bundle map $\mathcal{LP}(L)$ is called **Lagrange–Poincaré operator** and since it takes values in a direct sum, it can be decomposed into two terms: the **horizontal** Lagrange–Poincaré operator $\text{Hor}(\mathcal{LP})(L)$ and the **vertical** Lagrange–Poincaré operator $\text{Ver}(\mathcal{LP})(L)$. Their

expressions are

$$\begin{aligned}\text{Ver}\mathcal{L}\mathcal{P}(L) : T^{(2)}Q \oplus 2V &\rightarrow T^*Q \\ [q]^{(2)} \oplus [v]^{(1)} &\mapsto \frac{\partial L}{\partial q} - \frac{D}{Dt} \frac{\partial L}{\partial \dot{q}} - \left\langle \frac{\partial L}{\partial v}, \omega_q(\dot{q}, \cdot) \right\rangle, \\ \text{Hor}\mathcal{L}\mathcal{P}(L) : T^{(2)}Q \oplus 2V &\rightarrow V^* \\ [q]^{(2)} \oplus [v]^{(1)} &\mapsto \text{ad}_v^* \frac{\partial L}{\partial v} - \frac{D}{Dt} \frac{\partial L}{\partial v}.\end{aligned}$$

Proof. In analogy to the explicit deduction of the Euler–Lagrange equations presented in Section 1.2, the chain rule is applied to the factorization,

$$\begin{aligned}\mathbb{R} &\rightarrow TQ \oplus V \rightarrow \mathbb{R} \\ \varepsilon &\mapsto (\dot{q} \oplus v)(t, \varepsilon) \mapsto L((\dot{q} \oplus v)(t, \varepsilon)),\end{aligned}$$

to obtain

$$\frac{dL(\dot{q} \oplus v)(t, \varepsilon)}{d\varepsilon} = \left\langle \frac{\partial L}{\partial(\dot{q} \oplus v)}, \frac{\partial(\dot{q} \oplus v)(t, \varepsilon)}{\partial\varepsilon} \right\rangle,$$

where $\partial\dot{q} \oplus v(t, \varepsilon)/\partial\varepsilon \in T_{\dot{q} \oplus v(t, \varepsilon)}(TQ \oplus V)$, and $\partial L/\partial(\dot{q} \oplus v) \in T_{\dot{q} \oplus v(t, \varepsilon)}^*(TQ \oplus V)$. Notation $\langle \cdot, \cdot \rangle$ stands for their natural pairing but it shall be omitted wherever a compact notation is needed. An arbitrary connection on $TQ \rightarrow Q$ and the connection ∇ on V define a connection on $TQ \oplus V$. The later connection provides a split of $T_{\dot{q} \oplus v(t, \varepsilon)}(TQ \oplus V)$ into

$$\begin{aligned}\text{Hor}_{\dot{q} \oplus v(t, \varepsilon)}(TQ \oplus V) \oplus \text{Ver}_{\dot{q} \oplus v(t, \varepsilon)}^1(TQ \oplus V) \oplus \text{Ver}_{\dot{q} \oplus v(t, \varepsilon)}^2(TQ \oplus V) \\ \cong T_{q(t, \varepsilon)}Q \oplus T_{q(t, \varepsilon)}Q \oplus V_{q(t, \varepsilon)}.\end{aligned}$$

Then,

$$\begin{aligned}d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{S}(L)((\dot{q} \oplus v)(t, \varepsilon)) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \left(\int_{t_0}^{t_1} L((\dot{q} \oplus v)(t, \varepsilon)) dt \right) \\ &= \int_{t_0}^{t_1} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L((\dot{q} \oplus v)(t, \varepsilon)) dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} \frac{\partial q}{\partial \varepsilon} \Big|_{\varepsilon=0} + \frac{\partial L}{\partial \dot{q}} \frac{D\dot{q}}{d\varepsilon} \Big|_{\varepsilon=0} + \frac{\partial L}{\partial v} \frac{Dv}{D\varepsilon} \Big|_{\varepsilon=0} \right) dt.\end{aligned}$$

Performing integration by parts gives,

$$d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) = \int_{t_0}^{t_1} \left(\left(\frac{\partial L}{\partial q} - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \frac{\partial q}{\partial \varepsilon} \Big|_{\varepsilon=0} + \frac{\partial L}{\partial v} \frac{Dv}{D\varepsilon} \Big|_{\varepsilon=0} \right) dt.$$

As $\delta\dot{q} \oplus \delta v \in \Delta_{\dot{q} \oplus v}^\ell$, we have that

$$\frac{Dv}{D\varepsilon} = \frac{Dw}{Dt} + [v, w] + \omega_q(\delta q, \dot{q}),$$

where $w(t)$ is a curve in V with null endpoints and $\pi_{Q,V}(w(t)) = q(t)$. Hence;

$$d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) = \int_{t_0}^{t_1} \left(\left(\frac{\partial L}{\partial q} - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q + \frac{\partial L}{\partial v} \left(\frac{Dw}{Dt} + [v, w] + \omega_q(\delta q, \dot{q}) \right) \right) dt$$

Integration by parts is performed again and provides

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial v} \frac{Dw}{Dt} dt = \int_{t_0}^{t_1} \left(\frac{d}{dt} \left(\frac{\partial L}{\partial v} w \right) - \frac{D}{Dt} \left(\frac{\partial L}{\partial v} \right) w \right) dt = \left[\frac{\partial L}{\partial v} w \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{D}{Dt} \left(\frac{\partial L}{\partial v} \right) w dt$$

As $w(t)$ has zero endpoints, the surface term above is zero and

$$\begin{aligned} d\mathcal{S}(L) \cdot (\delta\dot{q} \oplus \delta v) &= \\ &= \int_{t_0}^{t_1} \left(\left(\frac{\partial L}{\partial q} - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q - \frac{D}{Dt} \left(\frac{\partial L}{\partial v} \right) w + \text{ad}_v^* \frac{\partial L}{\partial v} w - \frac{\partial L}{\partial v} \omega_q(\dot{q}, \delta q) \right) dt \\ &= \int_{t_0}^{t_1} \left(\left(\frac{\partial L}{\partial q} - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q - \frac{\partial L}{\partial v} \omega_q(\dot{q}, \delta q) + \left(\text{ad}_v^* \frac{\partial L}{\partial v} - \frac{D}{Dt} \left(\frac{\partial L}{\partial v} \right) \right) w \right) dt \\ &= \int_{t_0}^{t_1} \mathcal{L}\mathcal{P}(L)([q]^{(2)} \oplus [v]^{(1)}) \cdot (\delta q \oplus \delta v), \end{aligned}$$

where the definition of the Lagrange-Poincaré operator, $\mathcal{L}\mathcal{P}(L)$, has been used. $\square$

Remark 2.2.5 Example 2.1.4 states that $T(Q/G) \oplus \tilde{\mathfrak{g}}$ is a Lagrange-Poincaré bundle. Thus, Proposition 2.2.4 generalizes Proposition 1.2.12.

Proposition 2.2.4 implies that a curve $\dot{q}(t) \oplus v(t)$ of $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V, q_0, q_1)$ satisfies the variational Principle 2.2.3 if and only if it satisfies the Lagrange-Poincaré equations

$$\frac{\partial L}{\partial q} - \frac{D}{Dt} \frac{\partial L}{\partial \dot{q}} - \left\langle \frac{\partial L}{\partial v}, \omega_q(\dot{q}, \cdot) \right\rangle = 0, \quad (2.7)$$

$$\text{ad}_v^* \frac{\partial L}{\partial v} - \frac{D}{Dt} \left(\frac{\partial L}{\partial v} \right) = 0. \quad (2.8)$$

Example 2.2.6 We specify an example of Lagrangian system in the Lagrange-Poincaré bundle introduced in Subsection 2.1.3. Let $L : TM \oplus V \rightarrow \mathbb{R}$ be the Lagrangian given by

$$L(q, \dot{q}, v) = g(\dot{q}, \dot{q}) + h(v, v),$$

where g is a Riemannian metric on M and h is a vector bundle metric on V . The Lagrange–Poincaré equations are

$$\frac{D\dot{q}}{Dt} = h(v, \omega(\dot{q}, \cdot)), \quad (2.9)$$

$$\frac{D^\nabla v}{Dt} = 0. \quad (2.10)$$

Equation (2.9) provides the Newtonian dynamics of a particle on M under a force defined by v and ω . This equation reduces to the simple geodesic equation when $\omega = 0$. Equation (2.10) is just the parallel transport of v along $q(t)$.

2.2.2 Reduction of Variations

Given an action of a Lie group G on an element $TQ \oplus V$ in the category $\mathfrak{LP}$ and a G -invariant Lagrangian on $TQ \oplus V$, there is a reduction process analogous to the one presented in subsection 1.2.2 for Lagrangian systems on tangent bundles. This process was established for Lagrangian systems on objects of the subcategory $\mathfrak{RT}$ in [23], where it was left as an open question if it could be extended to the whole category $\mathfrak{LP}$. In the following, we shall answer affirmatively to this question.

Given a connection $\mathcal{A}$ on the principal bundle $Q \rightarrow Q/G$, there exists an identification

$$\begin{aligned} \alpha_{\mathcal{A}}^{TQ \oplus V} : (TQ \oplus V)/G &\rightarrow T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G) \\ [\dot{q} \oplus v]_G &= \alpha_{\mathcal{A}}^{TQ}(\dot{q}) \oplus [v]_G. \end{aligned}$$

Furthermore, $\alpha_{\mathcal{A}}^{TQ \oplus V}$ applied to sections is a Lie algebra homomorphism between $\Gamma((TQ \oplus V)/G)$ equipped with the quotient Lie bracket of $\Gamma(TQ \oplus V)$ and $\Gamma(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G))$ with the Lie bracket induced by the additional structures of Theorem 2.1.7.

Definition 2.2.7 Given a G -invariant Lagrangian $L : TQ \oplus V \rightarrow \mathbb{R}$, the **reduced Lagrangian**,

$$L^{(G)} : T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G) \rightarrow \mathbb{R},$$

is the quotient map of L after identification via $\alpha_{\mathcal{A}}^{TQ \oplus V}$.

The aforementioned reduction process in the category $\mathfrak{LP}$ consists on solving the variational problem posed by the Lagrangian system $(L^{(G)}, T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G))$ instead of the one posed by the Lagrangian system $(L, TQ \oplus V)$. To establish the equivalence of these variational principles, a lemma of reconstruction of curves is needed.

Lemma 2.2.8 *Let π_G be the projection of $TQ \oplus V$ to $(TQ \oplus V)/G$, the map*

$$\begin{aligned} \Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G) : \Omega(TQ \oplus V) &\rightarrow \Omega(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G) \\ \gamma(t) &\mapsto (\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)(\gamma(t)) \end{aligned}$$

restricted to $\ell\Omega(Q; q_0) \oplus \Omega(V; q_0)$ is injective and the image of its restriction is

$$\ell\Omega(Q/G; x_0) \oplus \Omega(\tilde{\mathfrak{g}}; x_0) \oplus \Omega(V/G; x_0).$$

Proof. For any curve $\dot{x}(t) \oplus \tilde{\xi}(t) \oplus [v]_G(t)$ such that $x(t_0) = x_0$, from [23, Lemma 6.4.2] there is a unique curve $q \in \Omega(Q; q_0)$ such that $\Omega(\alpha_{\mathcal{A}}^{TQ} \circ \pi_G)(q) = \dot{x} \oplus \tilde{\xi}$. As fibers of V are isomorphic to the fibers of V/G , there is a unique curve $v \in \Omega(V)$ such that $\pi_{Q,V} v(t) = q(t)$ and $\pi_{V/G,V}(v(t)) = [v]_G(t)$ for any t . Hence, $q \oplus v$ is the seeked curve. $\square$

From Lemma 2.2.8, $\Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)$ is a bijection between $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$, set of curves considered in $(L, TQ \oplus V)$, and $\ell\Omega(Q/G; x_0, x_1) \oplus \Omega(\tilde{\mathfrak{g}}; x_0, x_1) \oplus \Omega(V/G; x_0, x_1)$, set of curves considered in $(L^{(G)}, T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G))$. A similar result for allowed variations requires a careful study of the geometry of the reduced variations in $\tilde{\mathfrak{g}} \oplus V/G$.

Theorem 2.2.9 *Let $\dot{q} \oplus v$ be a curve in $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$ and $\dot{x} \oplus \tilde{\xi} \oplus [v]_G = \alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G(\dot{q} \oplus v)$. Then, the map*

$$\begin{aligned} T_{\dot{q} \oplus v} \Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G) : \Delta_{\dot{q} \oplus v}^{\ell} &\rightarrow \Delta_{\dot{x} \oplus \tilde{\xi} \oplus [v]_G}^{\ell} \\ \delta \dot{q} \oplus \delta v &\mapsto \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)(\dot{q}(t, \varepsilon) \oplus v(t, \varepsilon)), \end{aligned}$$

where $\dot{q}(t, \varepsilon) \oplus v(t, \varepsilon)$ is any deformation of $\dot{q} \oplus v$ inducing the variation $\delta \dot{q} \oplus \delta v$, is a linear isomorphism from $\Delta_{\dot{q} \oplus v}^{\ell}$ onto $\Delta_{\dot{x} \oplus \tilde{\xi} \oplus [v]_G}^{\ell}$.

Proof. The argument is separated in different steps.

(1). We write explicitly the variations $\Delta_{\dot{x} \oplus \tilde{\xi} \oplus [v]_G}^{\ell}$, that is, the allowed variations of a curve $\dot{x} \oplus \tilde{\xi} \oplus [v]_G$ in $T(Q/G) \oplus (\tilde{\mathfrak{g}} \oplus V/G)$ with additional structures $[\cdot]_{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ and $\nabla^{\tilde{\mathfrak{g}}}$. Whereas $\delta \dot{x}$ is the lift of a free variation δx , the variation $\delta \tilde{\xi} \oplus \delta [v]_G$ is

$$\delta \tilde{\xi} \oplus \delta [v]_G = \frac{D^{\tilde{\mathfrak{g}}}}{Dt}(\tilde{\eta} \oplus [w]_G) + [\tilde{\xi} \oplus [v]_G, \tilde{\eta} \oplus [w]_G]^{\tilde{\mathfrak{g}}} - \omega^{\tilde{\mathfrak{g}}}(\dot{x}, \delta x),$$

where $\bar{\eta} \oplus [w]_G$ is a curve in $\tilde{\mathfrak{g}} \oplus V/G$ with null endpoints such that $\pi_{Q/G, \tilde{\mathfrak{g}} \oplus V/G}(\bar{\eta}(t) \oplus [w]_G(t)) = x(t)$. From the explicit expression of the additional structures in Theorem 2.1.7,

$$\begin{aligned} \frac{D^{\tilde{\mathfrak{g}}}}{Dt}(\bar{\eta} \oplus [w]_G) &= \nabla_{\dot{x}}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [w]_G) = \nabla_{\dot{x}} \bar{\eta} \oplus ([\nabla^{(\mathcal{A}, H)}]_{G, \dot{x}}[w]_G - [\omega]_G(\dot{x}, \bar{\eta})); \\ \omega^{\tilde{\mathfrak{g}}}(\dot{x}, \delta x) &= \tilde{\mathcal{B}}^{\mathcal{A}}(\dot{x}, \delta x) \oplus [\omega]_G(\dot{x}, \delta x); \\ [\bar{\xi} \oplus [v]_G, \bar{\eta} \oplus [w]_G]^{\tilde{\mathfrak{g}}} &= [\bar{\xi}, \bar{\eta}] \oplus ([\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}[w]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}}[v]_G \\ &\quad - [\omega]_G(\bar{\xi}, \bar{\eta}) + [v]_G, [w]_G)_G. \end{aligned}$$

Hence, the variation of $\bar{\xi}$ is

$$\delta \bar{\xi} = \frac{D\bar{\eta}}{Dt} + [\bar{\xi}, \bar{\eta}] - \tilde{\mathcal{B}}^{\mathcal{A}}(\dot{x}, \delta x),$$

and the variation of $[v]_G$ is

$$\begin{aligned} \delta[v]_G &= [\nabla^{(\mathcal{A}, H)}]_{G, \dot{x}}[w]_G - [\omega]_G(\dot{x}, \bar{\eta}) + [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}[w]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}}[v]_G \\ &\quad - [\omega]_G(\bar{\xi}, \bar{\eta}) + [v]_G, [w]_G)_G - [\omega]_G(\dot{x}, \delta x). \end{aligned}$$

(2). We now prove that

$$T_{\dot{q} \oplus v} \Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)(\Delta_{\dot{q} \oplus v}^{\ell}) \subset \Delta_{x \oplus \bar{\xi} \oplus [v]_G}^{\ell}.$$

A variation in $\Delta_{\dot{q} \oplus v}^{\ell}$ is obtained as the derivative at $\varepsilon = 0$ of a deformation $\dot{q}(t, \varepsilon) \oplus v(t, \varepsilon)$ of $\dot{q}(t) \oplus v(t)$ such that for each ε , $\dot{q}_{\varepsilon}(t) \oplus v_{\varepsilon}(t)$ is a curve belonging to $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$. We study the variation in $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G$ obtained from the deformation

$$\dot{x}(t, \varepsilon) \oplus \bar{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G = \alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G(\dot{q}(t, \varepsilon) \oplus v(t, \varepsilon)).$$

As seen in Subsection 1.2.2, the variation δx is free and $\delta \dot{x} \in \Delta_x^{\ell}$. Furthermore, observe that the horizontal part of

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\bar{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G)$$

coincides with the horizontal lift of δx . Consequently, we calculate directly the covariant derivative with respect to the connection $\nabla^{\tilde{\mathfrak{g}}}$ of $\tilde{\mathfrak{g}} \oplus V/G$. In turn, this derivative can be

expressed in terms of the connection as

$$\left. \frac{D\tilde{g}}{D\varepsilon} \right|_{\varepsilon=0} (\tilde{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G) = \nabla_{\delta x}^{\tilde{g}}(\tilde{\xi} \oplus [v]_G).$$

Since

$$\begin{aligned} \nabla_{\delta x}^{\tilde{g}}(\tilde{\xi} \oplus [v]_G) &= \nabla_{\delta x}^{\mathcal{A}} \tilde{\xi} \oplus ([\nabla^{(\mathcal{A}, H)}]_{G, \delta x} [v]_G - [\omega]_G(\delta x, \tilde{\xi})) \\ &= \nabla_{\delta x}^{\mathcal{A}} \tilde{\xi} \oplus ([\nabla^{(\mathcal{A})}]_{G, \delta x \oplus \tilde{\eta}} [v]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\eta}} [v]_G - [\omega]_G(\delta x, \tilde{\xi})) \\ &= \nabla_{\delta x}^{\mathcal{A}} \tilde{\xi} \oplus ([\nabla^{(\mathcal{A})}]_{G, \delta q} [v]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\eta}} [v]_G - [\omega]_G(\delta x, \tilde{\xi})), \end{aligned}$$

where $\tilde{\eta}$ is a curve in $\tilde{g}$ such that $\delta q = \delta x \oplus \tilde{\eta}$. Then, for $\varepsilon = 0$

$$\left. \frac{D\tilde{g}}{D\varepsilon} \right|_{\varepsilon=0} (\tilde{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G) = \frac{D\tilde{\xi}}{D\varepsilon} \oplus \left(\left[\frac{D}{D\varepsilon} v(t, \varepsilon) \right]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\eta}} [v]_G - [\omega]_G(\delta x, \tilde{\xi}) \right).$$

As seen in Section 1.2.2,

$$\left. \frac{D}{D\varepsilon} \right|_{\varepsilon=0} \tilde{\xi} = \frac{D\tilde{\eta}}{Dt} + [\tilde{\xi}, \tilde{\eta}] - \mathcal{B}^{\mathcal{A}}(\dot{x}, \delta x).$$

Since $\delta \dot{q} \oplus \delta v$ is an allowed variation of $TQ \oplus V$,

$$\delta v = \left. \frac{D}{D\varepsilon} \right|_{\varepsilon=0} v = \frac{Dw}{Dt} + [v, w] + \omega_q(\delta q, \dot{q}),$$

where $w(t)$ is a curve in V such that $w(t_0) = w(t_1) = 0$ and $\pi_{Q, V} \circ w = q$. We shall study now the projection to V/G of each of the terms of δv :

$$\begin{aligned} \left[\frac{Dw}{Dt} \right]_G &= [\nabla_{\dot{q}}^{\mathcal{A}} w]_G = [\nabla^{(\mathcal{A})}]_{G, \dot{x} \oplus \tilde{\xi}} [w]_G = [\nabla^{(\mathcal{A}, H)}]_{G, \dot{x}} [w]_G + [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\xi}} [w]_G; \\ [[v, w]]_G &= [v]_G, [w]_G; \\ [\omega(\dot{q}, \delta q)]_G &= [\omega]_G([\dot{q}]_G, [\delta q]_G) = [\omega]_G(\dot{x} \oplus \tilde{\xi}, \delta x \oplus \tilde{\eta}) \\ &= [\omega]_G(\dot{x}, \delta x) + [\omega]_G(\tilde{\xi}, \delta x) + [\omega]_G(\dot{x}, \tilde{\eta}) + [\omega]_G(\tilde{\xi}, \tilde{\eta}). \end{aligned}$$

Altogether,

$$\begin{aligned} \left[\frac{Dv}{D\varepsilon} \right]_{\varepsilon=0} \Big|_G &= [\nabla^{(\mathcal{A}, H)}]_{G, \dot{x}} [w]_G + [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\xi}} [w]_G + [v]_G, [w]_G \\ &\quad - [\omega]_G(\dot{x}, \delta x) + [\omega]_G(\delta x, \tilde{\xi}) - [\omega]_G(\dot{x}, \tilde{\eta}) - [\omega]_G(\tilde{\xi}, \tilde{\eta}). \end{aligned} \quad (2.11)$$

Finally, equation (2.11) is substituted into the covariant derivative of $\tilde{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G$ to obtain the variation $\delta\tilde{\xi} \oplus \delta[v]_G$ through $\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G$:

$$\begin{aligned} \delta\tilde{\xi} \oplus \delta[v]_G &= \frac{D\tilde{\mathfrak{g}}}{D\varepsilon} (\tilde{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G) \\ &= \left(\frac{D\tilde{\eta}}{Dt} + [\tilde{\xi}, \tilde{\eta}] - \tilde{\mathcal{B}}^{\mathcal{A}}(\dot{x}, \delta x) \right) \oplus ([\nabla^{(\mathcal{A}, H)}]_{G, \dot{x}}[w]_G + [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\xi}}[w]_G \\ &\quad - [\nabla^{(\mathcal{A}, V)}]_{G, \tilde{\eta}}[v]_G + [v]_G, [w]_G]_G - [\omega]_G(\dot{x}, \delta x) - [\omega]_G(\dot{x}, \tilde{\eta}) - [\omega]_G(\tilde{\xi}, \tilde{\eta})) \end{aligned}$$

and it coincides with the variation of $\tilde{\xi}(t, \varepsilon) \oplus [v(t, \varepsilon)]_G$ obtained in the previous step from $\tilde{\eta} \oplus [w]_G$ in $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G$.

(3). The map $T_{\dot{q} \oplus v} \Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)$ does not depend on the chosen deformation and is clearly linear. Then, it only remains to prove its bijectivity. Let $\dot{x}(t, \varepsilon) \oplus \tilde{\xi}(t, \varepsilon) \oplus [v]_G(t, \varepsilon)$ be an allowed deformation of $\dot{x}(t) \oplus \tilde{\xi}(t) \oplus [v]_G(t)$. From Lemma 2.2.8, for each ε there is a unique curve $\dot{q}_\varepsilon(t) \oplus v_\varepsilon(t)$ in $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V, q_0, q_1)$ such that its image by $\Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)$ is $\dot{x}(t, \varepsilon) \oplus \tilde{\xi}(t, \varepsilon) \oplus [v]_G(t, \varepsilon)$. Since $\dot{x}(t, 0) \oplus \tilde{\xi}(t, 0) \oplus [v]_G(t, 0) = \dot{x}(t) \oplus \tilde{\xi}(t) \oplus [v]_G(t)$, uniqueness implies that $\dot{q}_0(t) \oplus v_0(t) = \dot{q}(t) \oplus v(t)$ and $q(t, \varepsilon) = q_\varepsilon(t)$ is a deformation of $\dot{q}(t) \oplus v(t)$. This allows to define an inverse function of $T_{\dot{q}} \Omega(\alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G)$ deriving with respect to ε at $\varepsilon = 0$. $\square$

Reduction of Lagrangian systems on objects of the category $\mathfrak{LP}$ can be now stated as follows.

Theorem 2.2.10 *Given G -invariant Lagrangian $L : TQ \oplus V \rightarrow \mathbb{R}$ and a curve $\dot{q}(t) \oplus v(t)$ in $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$, the following are equivalent*

(i) *The curve $\dot{q}(t) \oplus v(t)$ is a critical point of the action $\int_{t_0}^{t_1} L(\dot{q}(t), v(t))dt$ with allowed variations $\Delta_{\dot{q} \oplus v}^\ell$.*

(ii) *The curve $\dot{q}(t) \oplus v(t)$ satisfies the Lagrange–Poincaré equations*

$$\mathcal{LP}(L)(\dot{q} \oplus v) = 0.$$

(iii) *The curve*

$$\dot{x}(t) \oplus \tilde{\xi}(t) \oplus [v]_G(t) = \alpha_{\mathcal{A}}^{TQ \oplus V} \circ \pi_G(\dot{q}(t) \oplus v(t))$$

in $\ell\Omega(Q/G; x_0, x_1) \oplus \Omega(\tilde{\mathfrak{g}}; x_0, x_1) \oplus \Omega(V/G; x_0, x_1)$ is a critical point of the action $\int_{t_0}^{t_1} L^{(G)}(\dot{x}(t), \tilde{\xi}(t), [v]_G(t))dt$ with allowed variations $\Delta_{\dot{x} \oplus \tilde{\xi} \oplus [v]_G}^\ell$.

(iv) The curve $\dot{x}(t) \oplus \bar{\xi} \oplus [v]_G(t)$ satisfies the Lagrange-Poincaré equations

$$\mathcal{L}\mathcal{P}(L^{(G)})(\dot{x} \oplus \bar{\xi} \oplus [v]_G) = 0.$$

2.2.3 Reduction by Stages in the category $\mathfrak{LP}$

Next, the equivalence of variational principles induced by an isomorphism in the category $\mathfrak{LP}$ is explained. Afterwards, we shall see that this implies that reduction by stages is equivalent to direct reduction in the whole category $\mathfrak{LP}$.

Proposition 2.2.11 *Let $TQ_1 \oplus V_1$ and $TQ_2 \oplus V_2$ be Lagrange-Poincaré bundles and $f : TQ_1 \oplus V_1 \rightarrow TQ_2 \oplus V_2$ an isomorphism in the category $\mathfrak{LP}$. Let L_1 and L_2 be Lagrangians defined respectively in these bundles such that $L_1 = L_2 \circ f$. Then a curve $\dot{q}_1 \oplus v_1$ satisfies the variational principle for L_1 if and only if $f(\dot{q}_1 \oplus v_1)$ satisfies the variational principle for L_2 .*

Proof. It is enough to exhibit that f induces a bijection between the sets of allowed curves $\ell\Omega(Q_1; q_0, q_1) \oplus \Omega(V_1; q_0, q_1)$, and $\ell\Omega(Q_2; f_0(q_0), f_0(q_1)) \oplus \Omega(V_2; f_0(q_0), f_0(q_1))$. As well as a bijection between the sets of variations $\Delta_{\dot{q} \oplus v}^\ell$ and $\Delta_{f(\dot{q}) \oplus f(v)}^\ell$.

On one hand, observe that

$$f(\dot{q} \oplus v) = f(\dot{q}) \oplus f(v) = T f_0(\dot{q}) \oplus f(v).$$

Since both, $T f_0(\dot{q})$ and $f(v)$, project to $f_0(q)$, we conclude that $f(\dot{q} \oplus v)$ is an allowed curve in $TQ_2 \oplus V_2$ and

$$\ell\Omega(Q_1; q_0, q_1) \oplus \Omega(V_1; q_0, q_1) \hookrightarrow \ell\Omega(Q_2; f_0(q_0), f_0(q_1)) \oplus \Omega(V_2; f_0(q_0), f_0(q_1)).$$

On the other hand, let $\delta\dot{q} \oplus \delta v \in \Delta_{\dot{q} \oplus v}^\ell$, that is,

$$\delta v = \frac{D_1 w}{Dt} + [v, w]_1 - \omega_{1,q}(\dot{q}, \delta q),$$

and let $q(t, \varepsilon) \oplus v(t, \varepsilon)$ be a deformation producing this variation. Then,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(\dot{q}(t, \varepsilon)) = T f_0 \left(\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \dot{q}(t, \varepsilon) \right) = T f_0(\delta q) = f(\delta q),$$

and

$$\begin{aligned} \left. \frac{D_2}{D\varepsilon} \right|_{\varepsilon=0} f(v(t, \varepsilon)) &= f \left(\left. \frac{D_1}{D\varepsilon} \right|_{\varepsilon=0} v(t, \varepsilon) \right) = f \left(\frac{D_1 w}{Dt} + [v, w]_1 - \omega_{1,q}(\dot{q}, \delta q) \right) \\ &= \frac{D_2(f(w))}{Dt} + [f(v), f(w)]_2 - \omega_{2,f(q)}(f(\dot{q}), f(\delta q)), \end{aligned}$$

where we have used that f commutes with the additional structures of $TQ_1 \oplus V_1$ and $TQ_2 \oplus V_2$. The variation found lies in $\Delta_{f(\dot{q}) \oplus f(v)}^\ell$ and, consequently, $\Delta_{\dot{q} \oplus v}^\ell \hookrightarrow \Delta_{f(\dot{q}) \oplus f(v)}^\ell$.

Finally, as f is an isomorphism, the opposite injections are obtained analogously from f^{-1} . $\square$

We now suppose that $L : TQ \oplus V$ is a Lagrangian invariant by the action of G in $TQ \oplus V$, N is a normal subgroup of G , and $K = G/N$ is the quotient group. Since the category $\mathfrak{LP}$ is closed under reduction, it is possible to reduce L by the group N and afterwards reduce by the group K . The natural question is whether this is equivalent to directly reduce by G or not.

Definition 2.2.12 Let $\mathcal{A}_N$ be a principal connection on $Q \rightarrow Q/N$, $\mathcal{A}_{G/N}$ be a principal connection on $Q/N \rightarrow (Q/N)/(G/N)$, and $\mathcal{A}_G$ be a principal connection on $Q \rightarrow Q/G$. These connections are said to be **compatible** if for all $v_q \in T_q Q$ and all $q \in Q$

$$\mathcal{A}_G(v_q) = 0 \Leftrightarrow \mathcal{A}_N(v_q) = 0 \text{ and } \mathcal{A}_{G/N}(T\pi_N(v_q)) = 0.$$

Proposition 2.2.13 [23] (Section 6.3) If the connections $\mathcal{A}_G$, $\mathcal{A}_N$ and $\mathcal{A}_{G/N}$ are compatible, the map

$$\beta_{(\mathcal{A}_N, \mathcal{A}_{G/N}, \mathcal{A}_G)}^{TQ \oplus V} = \alpha_{\mathcal{A}_{G/N}}^{T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N)} \circ [\alpha_{\mathcal{A}_N}^{TQ \oplus V}]_{G/N} \circ i_{(G/N)}^{TQ \oplus V} \circ (\alpha_{\mathcal{A}_G}^{TQ \oplus V})^{-1},$$

where $i_{(G/N)}^{TQ \oplus V} : (TQ \oplus V)/G \rightarrow ((TQ \oplus V)/N)/(G/N)$ denotes the natural identification, is a $\mathfrak{LP}$ isomorphism from $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$ onto

$$T((Q/N)/(G/N)) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}} \oplus (V/N))/(G/N),$$

where $\mathfrak{n}$ is the Lie algebra of N and $\mathfrak{k}$ is the Lie algebra of K .

Clearly,

$$L^{(G)} = (L^{(N)})^{(K)} \circ \beta_{(\mathcal{A}_N, \mathcal{A}_{G/N}, \mathcal{A}_G)}^{TQ \oplus V}$$

and Proposition 2.2.11 concludes that $L^{(G)}$ and $(L^{(N)})^{(K)}$ pose equivalent problems. More accurately,

Theorem 2.2.14 *Let $N < G$ be a normal subgroup and $K = G/H$. Given G -invariant Lagrangian $L : TQ \oplus V \rightarrow \mathbb{R}$ and a curve $\dot{q}(t) \oplus v(t)$ in $\ell\Omega(Q; q_0, q_1) \oplus \Omega(V; q_0, q_1)$, the following are equivalent:*

(i) *The curve*

$$\dot{y}(t) \oplus \bar{\eta}(t) \oplus [v]_N(t) = \alpha_{\mathcal{A}_N}^{TQ \oplus V} \circ \pi_N(\dot{q}(t) \oplus v(t))$$

in $\ell\Omega(Q/N; y_0, y_1) \oplus \Omega(\tilde{\mathfrak{n}}; y_0, y_1) \oplus \Omega(V/N; y_0, y_1)$ is a critical point of the action $\int_{t_0}^{t_1} L^{(N)}(\dot{y}(t), \bar{\eta}(t), [v]_N(t)) dt$ with allowed variations $\Delta_{y \oplus \bar{\eta} \oplus [v]_N}^\ell$.

(ii) *The curve $\dot{y}(t) \oplus \bar{\eta}(t) \oplus [v]_N(t)$ satisfies the Lagrange–Poincaré equations $\mathcal{L}\mathcal{P}(L^{(N)})(\dot{y} \oplus \bar{\eta} \oplus [v]_N) = 0$.*

(iii) *The curve*

$$\dot{z}(t) \oplus \bar{\kappa}(t) \oplus [\bar{\eta}]_K(t) \oplus [[v]_N]_K(t) = \alpha_{\mathcal{A}_{G/N}}^{T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus V/N} \circ \pi_{G/N}(\dot{y}(t) \oplus \bar{\eta}(t) \oplus [v]_N(t))$$

in $\ell\Omega((Q/N)/K; z_0, z_1) \oplus \Omega(\tilde{\mathfrak{k}}; z_0, z_1) \oplus \Omega(\tilde{\mathfrak{n}}/K; z_0, z_1) \oplus \Omega((V/N)/K; z_0, z_1)$ is a critical point of the action

$$\int_{t_0}^{t_1} (L^{(N)})^{(K)}(\dot{z}(t), \bar{\kappa}(t), [\bar{\eta}]_K(t), [[v]_N]_K(t)) dt$$

with allowed variations $\Delta_{z \oplus \kappa \oplus [\bar{\eta}]_K, [[v]_N]_K}^\ell$.

(iv) *The curve $\dot{z}(t) \oplus \bar{\kappa}(t) \oplus [\bar{\eta}]_K(t) \oplus [[v]_N]_K(t)$ satisfies the Lagrange–Poincaré equations $\mathcal{L}\mathcal{P}((L^{(N)})^{(K)})(\dot{z} \oplus \bar{\kappa} \oplus [\bar{\eta}]_K \oplus [[v]_N]_K) = 0$.*

In turn, each of these statements are equivalent to each of the statements listed in Theorem 2.2.10.

2.2.4 Noether Current and Vertical Equations

This Subsection is devoted to show that the standard Noether current is not a constant of motion for Lagrangians defined on $\mathfrak{LP}$ -bundles. Yet, the drift of this current reduces to the new vertical equation appearing in each step of the reduction.

Definition 2.2.15 *Let G be a Lie group acting by $\mathfrak{LP}$ isomorphisms on the Lagrange–Poincaré bundle $TQ \oplus V$, and let $L : TQ \oplus V \rightarrow \mathbb{R}$ be a G -invariant Lagrangian. The **Noether current** is a function $J : TQ \oplus V \rightarrow \mathfrak{g}^*$*

$$J(\dot{q} \oplus v)(\eta) = \left\langle \frac{\partial L}{\partial \dot{q}}(\dot{q} \oplus v), \eta_q^Q \right\rangle, \quad (2.12)$$

for any $\dot{q} \oplus v \in TQ \oplus V$ and any $\eta \in \mathfrak{g}$.

Proposition 2.2.16 *Let $L : TQ \oplus V \rightarrow \mathbb{R}$ be a Lagrangian invariant under the action in the Lagrange–Poincaré category $\mathfrak{LP}$ of a Lie group G , and $\dot{q}(t) \oplus v(t)$ be a curve in $TQ \oplus V$ satisfying the Lagrange–Poincaré equations. Then the derivative of the Noether current along the critical curve satisfies*

$$\frac{d}{dt}J(\dot{q}(t) \oplus v(t))(\eta) = - \left\langle \frac{\partial L}{\partial v}(\dot{q}(t) \oplus v(t)), \omega(\dot{q}(t), \eta_{q(t)}^Q) + \eta_{v(t)}^V \right\rangle \quad (2.13)$$

for all $\eta \in \mathfrak{g}$.

Proof. As L is G -invariant, choosing $\exp(s\eta) \in G$, we have $L(\dot{q} \oplus v) = L(\exp(s\eta)\dot{q} \oplus \exp(s\eta)v)$ for all $s \in \mathbb{R}$. Differentiation at $s = 0$ provides:

$$\begin{aligned} 0 &= \left\langle \frac{\partial L}{\partial q}, \text{Hor} \left(\eta_{\dot{q} \oplus v}^{TQ \oplus V} \right) \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \text{Ver}^1(\eta_{\dot{q} \oplus v}^{TQ \oplus V}) \right\rangle + \left\langle \frac{\partial L}{\partial v}, \text{Ver}^2(\eta_{\dot{q} \oplus v}^{TQ \oplus V}) \right\rangle \\ &= \left\langle \frac{\partial L}{\partial q}, \eta_q^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_{\dot{q}}^{TQ} \right\rangle + \left\langle \frac{\partial L}{\partial v}, \eta_v^V \right\rangle. \end{aligned}$$

Then, the evolution of the Noether current along $\dot{q}(t) \oplus v(t)$ is

$$\begin{aligned} \frac{d}{dt}J(\dot{q}(t) \oplus v(t))(\eta) &= \frac{d}{dt} \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_{q(t)}^Q \right\rangle = \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{D\eta_{q(t)}^Q}{Dt} \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_{\dot{q}(t)}^{TQ} \right\rangle \\ &= \left\langle \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right), \eta_{q(t)}^Q \right\rangle - \left\langle \frac{\partial L}{\partial q}, \eta_{q(t)}^Q \right\rangle - \left\langle \frac{\partial L}{\partial v}, \eta_{v(t)}^V \right\rangle \\ &= - \left\langle \frac{\partial L}{\partial v}(\dot{q}(t) \oplus v(t)), \omega(\dot{q}(t), \eta_{q(t)}^Q) + \eta_{v(t)}^V \right\rangle, \end{aligned}$$

where it has been used that $\text{Hor}(\mathcal{LP})(L)(\dot{q}(t) \oplus v(t))(\eta_{q(t)}^Q) = 0$. □

Remark 2.2.17 *The proof of Proposition 2.2.16 does not make use of the vertical equation $\text{Ver}(\mathcal{LP})(L)(\dot{q}(t) \oplus v(t)) = 0$. Thus, the result is valid for curves $\dot{q}(t) \oplus v(t)$ that just solve the horizontal equation, $\text{Hor}(\mathcal{LP})(L)(\dot{q}(t) \oplus v(t)) = 0$.*

The Noether current defined by an G -invariant Lagrangian is G -equivariant. Hence, it induces a quotient function.

Definition 2.2.18 *The reduced Noether current is a function*

$$j : T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G \rightarrow \tilde{\mathfrak{g}}^*$$

$$(\dot{x}, \bar{\xi}, [v]) \mapsto j(\dot{x}, \bar{\xi}, [v]) = [q, J(\dot{q}, v)]_G,$$

where $\pi_{TQ \oplus V, T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G}(\dot{q}, v) = (\dot{x}, \bar{\xi}, [v])$.

The drift (2.13) of the Noether current J in $TQ \oplus V$ projects to $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$. The drift of the reduced Noether curvature is

$$\frac{d}{dt} j(\dot{x}, \bar{\xi}, [v]) \bar{\eta} = - \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\omega]_G(\dot{x} \oplus \bar{\xi}, \bar{\eta}) + [\eta_{v(t)}^V]_G \right\rangle, \quad (2.14)$$

where $\bar{\eta} = [q(t), \eta]_G$, $\eta \in \mathfrak{g}$, along the projection of critical curves of the Lagrangian system $(L, TQ \oplus V)$, that is, along critical curves $(\dot{x}, \bar{\xi}, [v])$ of the reduced Lagrangian system $(L^{(G)}, T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G)$. We shall relate this reduced drift to the vertical Lagrange–Poincaré equation of the reduced Lagrangian system.

Remark 2.2.19 *The vertical equation of $(L^{(G)}, T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G)$ is*

$$\left\langle \frac{D\tilde{\mathfrak{g}}}{Dt} \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \oplus \frac{\partial L^{(G)}}{\partial [v]} \right), \bar{\eta} \oplus [u] \right\rangle = \left\langle \text{ad}_{\bar{\xi} \oplus [v]}^* \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \oplus \frac{\partial L^{(G)}}{\partial [v]} \right), \bar{\eta} \oplus [u] \right\rangle,$$

where $\bar{\eta} \oplus [u] \in \tilde{\mathfrak{g}} \oplus V/G$. This can be rewritten using the explicit expressions of Theorem 2.1.7 as

$$\begin{aligned} & \left\langle \frac{D}{Dt} \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \right), \bar{\eta} \right\rangle + \left\langle \left[\frac{D^{(\mathcal{A}, H)}}{Dt} \right] \left(\frac{\partial L^{(G)}}{\partial [v]} \right), [u] \right\rangle + \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\omega]_G(\dot{x}, \bar{\eta}) \right\rangle \\ &= \left\langle \frac{\partial L^{(G)}}{\partial \bar{\xi}}, [\bar{\xi}, \bar{\eta}] \right\rangle + \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}[u] - [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}}[v] - [\omega]_G(\bar{\xi}, \bar{\eta}) + [[v], [u]] \right\rangle. \end{aligned}$$

Taking alternatively, $[u] = 0$ and $\bar{\eta} = 0$, the vertical Lagrange–Poincaré equation splits into two: A **new vertical equation** coming from the reduction step

$$\left\langle \frac{D}{Dt} \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \right), \bar{\eta} \right\rangle = \left\langle \text{ad}_{\bar{\xi}}^* \frac{\partial L^{(G)}}{\partial \bar{\xi}}, \bar{\eta} \right\rangle - \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}}[v] + [\omega]_G(\dot{x} \oplus \bar{\xi}, \bar{\eta}) \right\rangle \quad (2.15)$$

and a **reduced vertical equation** coming from the (unreduced) vertical equations in $TQ \oplus V$,

$$\left\langle \left[\frac{D^{(\mathcal{A}, H)}}{Dt} \right] \left(\frac{\partial L^{(G)}}{\partial [v]} \right), [u] \right\rangle = \left\langle \text{ad}_{[v]}^* \frac{\partial L^{(G)}}{\partial [v]}, [u] \right\rangle + \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}[u] \right\rangle. \quad (2.16)$$

Proposition 2.2.20 *The evolution of the reduced current given in (2.14) is equivalent to the group (2.15) of the vertical Lagrange–Poincaré equation in $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G$.*

Proof. Given a curve $\dot{q}(t) \oplus v(t)$ in $TQ \oplus V$ and its reduced curve $\dot{x}(t) \oplus \bar{\xi}(t) \oplus [v](t)$ in $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus V/G$, it can be checked that,

$$\left\langle \frac{d}{dt} J(\dot{q}(t)), \eta \right\rangle = \frac{d}{dt} \left\langle j(\dot{x}(t), \bar{\xi}(t)), \bar{\eta}(t) \right\rangle = \left\langle \frac{D}{Dt} \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \right), \bar{\eta}(t) \right\rangle - \left\langle \text{ad}_{\bar{\xi}(t)}^* \frac{\partial L^{(G)}}{\partial \bar{\xi}}, \bar{\eta}(t) \right\rangle,$$

in an analogous way as proven in Proposition 1.2.17 for Lagrangian systems on tangent bundles. Thus, the drift equation (2.14) is rewritten as:

$$\left\langle \frac{D}{Dt} \left(\frac{\partial L^{(G)}}{\partial \bar{\xi}} \right), \bar{\eta}(t) \right\rangle - \left\langle \text{ad}_{\bar{\xi}(t)}^* \frac{\partial L^{(G)}}{\partial \bar{\xi}}, \bar{\eta}(t) \right\rangle = - \left\langle \frac{\partial L^{(G)}}{\partial [v]}, [\omega]_G(\dot{x} \oplus \bar{\xi}, \bar{\eta}) + [\eta_{v(t)}^V] \right\rangle.$$

Since $[\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}}[v] = [\eta_{v(t)}^V]$, the above expression is the new vertical equation (2.15) obtained in the reduction process. $\square$

2.3 The Poisson category $\mathfrak{LP}^*$

We shall now study the dual category of $\mathfrak{LP}$.

Definition 2.3.1 *The objects of the category $\mathfrak{LP}^*$ are bundles $\pi_{Q, T^*Q} \oplus \pi_{Q, V^*} : T^*Q \oplus V^* \rightarrow Q$, such that, the dual vector bundle $TQ \oplus V \rightarrow Q \in \mathfrak{LP}$.*

The vector bundle $V^* \rightarrow Q$ is the dual of a Lie algebra vector bundle $V \rightarrow Q$, equipped with an affine connection ∇ . Then, there is an affine connection on $V^* \rightarrow Q$ provided by the dual connection of ∇ . This connection will also be denoted by ∇ . From condition 1.(d) in definition 2.1.1, the Lie algebra on the fibers of V , connection ∇ , and the V -valued 2-form on Q , ω , define a Lie bracket on sections of $TQ \oplus V$. The dual bundles in $\mathfrak{LP}^*$ are Poisson manifolds.

Proposition 2.3.2 [23, Theorem 8.1.3] *If $\pi_{Q, T^*Q} \oplus \pi_{Q, V^*} : T^*Q \oplus V^* \rightarrow Q$ is the dual bundle of $TQ \oplus V \rightarrow Q \in \mathfrak{LP}$, then there is a unique Poisson bracket*

$$\{, \} : C^\infty(T^*Q \oplus V^*) \times C^\infty(T^*Q \oplus V^*) \rightarrow C^\infty(T^*Q \oplus V^*)$$

characterized by its restriction to affine functions as

$$(i) \quad \{\bar{f}, \bar{g}\} = 0 \text{ for all } f, g \in C^\infty(Q);$$

$$(ii) \quad \{\bar{f}, \theta(X \oplus w)\} = \overline{X(f)};$$

$$(iii) \quad \{\theta(X_1 \oplus w_1), \theta(X_2 \oplus w_2)\} = -\theta([X_1 \oplus w_1, X_2 \oplus w_2]);$$

where

- for any f in $C^\infty(Q)$, $\bar{f} = f \circ (\pi_{Q, T^*Q} \oplus \pi_{Q, V^*})$ in $C^\infty(T^*Q \oplus V^*)$;
- for $X \oplus w \in \Gamma(TQ \oplus V)$, we define $\theta(X \oplus w)$ in $C^\infty(T^*Q \oplus V^*)$ as

$$\theta(X \oplus w)(p \oplus v) = \langle p, X \rangle + \langle v, w \rangle,$$

$$\text{at } p \oplus v \in T^*Q \oplus V^*.$$

This Poisson bracket behaves well under reduction by stages:

Proposition 2.3.3 [23, Theorem 8.2.1] *A $\mathfrak{LP}$ action of a Lie group on an element $TQ \oplus V \in \mathfrak{LP}$ naturally induces an action on $T^*Q \oplus V^* \in \mathfrak{LP}^*$ such that the projection $T^*Q \oplus V^* \rightarrow T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \oplus V^*$ is a Poisson map with respect to the Poisson brackets defined in Proposition 2.3.2.*

The explicit expression of the Poisson bracket of Proposition 2.3.2 was obtained in [22] for Lagrange-Poincaré bundles of the type $T(Q/G) \oplus \tilde{\mathfrak{g}}$. The generalization to the whole $\mathfrak{LP}$ category is given hereunder:

Proposition 2.3.4 *If $\pi_{Q, T^*Q} \oplus \pi_{Q, V^*} : T^*Q \oplus V^* \rightarrow Q$ is an element of the category $\mathfrak{LP}^*$ dual to $TQ \oplus V \rightarrow Q \in \mathfrak{LP}$, then the Poisson bracket characterized in Proposition 2.3.2 is*

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} + \left\langle v, \omega \left(\frac{\partial f}{\partial p}, \frac{\partial g}{\partial p} \right) \right\rangle + \left\langle v, \left[\frac{\partial g}{\partial v}, \frac{\partial f}{\partial v} \right] \right\rangle. \quad (2.17)$$

Proof. It is necessary to prove that this expression defines a Poisson bracket on $T^*Q \oplus V^*$ and that it has the required properties for affine functions. Skew-symmetry and alternativity of $\{, \}$ are a consequence of the skew-symmetry and alternativity of ω and the Lie bracket on V . Since $\{, \}$ does only depend on the differential of f, g it suffices to prove the Jacobi identity for affine functions instead of doing so for general functions $f, g \in C^\infty(T^*Q \oplus V^*)$.

We first prove that $\{, \}$ has the required properties for affine functions in order to coincide with the bracket in Proposition 2.3.2. For any $\bar{f}$, with $f \in C^\infty(Q)$, $\partial \bar{f} / \partial p = \partial \bar{f} / \partial v = 0$, then $\{\bar{f}, \bar{g}\} = 0$ for all $f, g \in C^\infty(Q)$. For any $\theta(X \oplus w)$, $\partial \theta(X \oplus w) / \partial p = X$ and $\partial \theta(X \oplus w) / \partial v = w$. Thus,

$$\{\bar{f}, \theta(X \oplus w)\} = \frac{\partial f}{\partial q} \frac{\partial \theta(X \oplus w)}{\partial p} = X(\bar{f}) = \overline{X(f)}.$$

Finally, to obtain $\{\theta(X_1 \oplus w_1), \theta(X_2 \oplus w_2)\}$ is necessary to find $\partial\theta(X \oplus w)/\partial q$. This can be done using local coordinates in $T^*Q \oplus V^*$,

$$\theta(X \oplus w) = X^i(q^1 \dots q^n)p_i + w^\alpha(q^1 \dots q^n)v_\alpha,$$

where $(q^i, p_i), i = 1 \dots n = \dim Q$ are coordinates in T^*Q and $v_\alpha, \alpha = 1 \dots m$ are independent local sections on V . Then,

$$\frac{\partial\theta(X \oplus w)}{\partial q^j} = \frac{\partial X^i}{\partial q^j}p_i + \left(\frac{\partial w^\alpha}{\partial q^j} + \Gamma_{\beta j}^\alpha w^\beta \right) v_\alpha,$$

and it follows that

$$\left\langle \frac{\partial\theta(X_1 \oplus w_1)}{\partial q}, X_2 \right\rangle = \theta(X_1 \circ X_2) + \theta(\nabla_{X_2} w_1).$$

Hence,

$$\begin{aligned} & \{\theta(X_1 \oplus w_1), \theta(X_2 \oplus w_2)\} \\ &= \left\langle \frac{\partial\theta(X_1 \oplus w_1)}{\partial q}, X_2 \right\rangle - \left\langle \frac{\partial\theta(X_2 \oplus w_2)}{\partial q}, X_1 \right\rangle + \langle \nu, \omega(X_1, X_2) - [w_1, w_2] \rangle \\ &= -\theta([X_1, X_2])(p, \nu) + \theta(\nabla_{X_2} w_1 - \nabla_{X_1} w_2)(p, \nu) + \theta(\omega(X_1, X_2) - [w_1, w_2])(p, \nu) \\ &= -\theta([X_1 \oplus w_1, X_2 \oplus w_2]) \end{aligned}$$

These properties imply the Jacobi identity for affine functions, and hence, for all. Indeed;

$$\{\bar{f}, \{\bar{g}, \bar{h}\}\} + \{\bar{g}, \{\bar{h}, \bar{f}\}\} + \{\bar{h}, \{\bar{f}, \bar{g}\}\} = 0 + 0 + 0 = 0;$$

$$\begin{aligned} & \{\bar{f}, \{\bar{g}, \theta(X \oplus w)\}\} + \{\bar{g}, \{\theta(X \oplus w), \bar{f}\}\} + \{\theta(X \oplus w), \{\bar{f}, \bar{g}\}\} \\ &= \{\bar{f}, \overline{X(g)}\} + \{\bar{g}, -\overline{X(f)}\} + 0 = 0. \end{aligned}$$

Furthermore, given $\bar{f}$ and two functions $\theta(X_i \oplus w_i)$,

$$\begin{aligned} & \{\bar{f}, \{\theta(X_1 \oplus w_1), \theta(X_2 \oplus w_2)\}\} \\ &+ \{\theta(X_1 \oplus w_1), \{\theta(X_2 \oplus w_2), \bar{f}\}\} + \{\theta(X_2 \oplus w_2), \{\bar{f}, \theta(X_1 \oplus w_1)\}\} \\ &= \{\bar{f}, -\theta([X_1 \oplus w_1, X_2 \oplus w_2])\} + \{\theta(X_1 \oplus w_1), -\overline{X_2(f)}\} + \{\theta(X_2 \oplus w_2), \overline{X_1(f)}\} \\ &= -[X_1, X_2](f) + \overline{X_1(X_2(f))} - \overline{X_2(X_1(f))} = 0. \end{aligned}$$

For three functions $\theta(X_i \oplus w_i)$, the Jacobi identity is based on the Jacobi identity for the Lie bracket on $\Gamma(TQ \oplus V)$

$$\begin{aligned} & \{ \theta(X_1 \oplus w_1), \{ \theta(X_2 \oplus w_2), \theta(X_3 \oplus w_3) \} \} \\ &= - \{ \theta(X_1 \oplus w_1), \theta([X_2 \oplus w_2, X_3 \oplus w_3]) \} = \theta([X_1 \oplus w_1, [X_2 \oplus w_2, X_3 \oplus w_3]]) \end{aligned}$$

At last, the Leibniz identity is easily obtained from (2.17). $\square$

Proposition 2.3.4 provides an intrinsic definition of the $\mathfrak{LP}^*$ without an explicit notion of duality with respect to the Lagrange–Poincaré category.:

Definition 2.3.5 *The objects of $\mathfrak{LP}^*$ are bundles $T^*Q \oplus V^* \rightarrow Q$, such that $V^* \rightarrow Q$ is the dual of a Lie algebra vector bundle $V \rightarrow Q$, equipped with an affine connection ∇ , Q has a 2-form ω taking values in V , and the expression (2.17) is a Poisson bracket.*

We finally study the dynamical equations defined by the Poisson bracket in the dual Lagrange–Poincaré category. Let $H : T^*Q \oplus V^* \rightarrow \mathbb{R}$ be a Hamiltonian on an element of $\mathfrak{LP}^*$. A simple computation shows that the Hamiltonian field defined by the bracket (2.17) in Proposition 2.3.4 is

$$X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} + \left\langle v, \omega \left(\cdot, \frac{\partial H}{\partial p} \right) \right\rangle, \text{ad}^*_{\frac{\partial H}{\partial v}} v \right).$$

A curve $(p(t), v(t))$ in $T^*Q \oplus V^*$, projecting to a curve $q(t)$, is an integral of X_H if and only if

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \frac{Dp}{Dt} &= -\frac{\partial H}{\partial q} + \left\langle v, \omega \left(\cdot, \frac{\partial H}{\partial p} \right) \right\rangle, \\ \frac{D^\nabla v}{Dt} &= \text{ad}^*_{\frac{\partial H}{\partial v}} v. \end{aligned}$$

These equations are called the *Hamilton-Poincaré equations* for H .

Given a Lagrangian $L : TQ \oplus V \rightarrow \mathbb{R}$ in the Lagrange–Poincaré category, we define the Legendre map $\mathbb{F}L : TQ \oplus V \rightarrow T^*Q \oplus V^*$ in the usual way as the fiber derivative of L . We write this as

$$(q, \dot{q}, v) \mapsto (q, p = \partial L / \partial \dot{q}, v = \partial L / \partial v).$$

If $\mathbb{F}L$ is a diffeomorphism, we define the Hamiltonian $H : T^*Q \oplus V^* \rightarrow \mathbb{R}$ as

$$\begin{aligned} H : T^*Q \oplus V^* &\rightarrow \mathbb{R} \\ (p, \nu) &\mapsto \langle p, \dot{q} \rangle + \langle \nu, v \rangle - L(\dot{q} \oplus v), \end{aligned}$$

where $(\dot{q}, v) = \mathbb{F}L^{-1}(p, \nu)$. Then, Lagrange–Poincaré equations and Hamilton–Poincaré equations are equivalent. Reduction using momentum techniques as in [63], that is, tracking the symplectic leaves structure, should be a subject of future work. Specially, under the light of results in Section 2.2.4.

Chapter 3

Applications of Lagrange–Poincaré Reduction in Mechanics

3.1 Lagrangian Depending on a Parameter

3.1.1 Reduction of a Variational Problem

We present, as an example, Lagrangian systems that depend on several parameters. These systems are not defined on tangent bundles, yet, they can be solved via ad hoc reduction of an appropriate variational principle, by means of Lagrange multipliers as in [21] or, in special cases, using semi-direct groups as in [48]. This subsection contextualizes the example and the next one shall show how to treat these Lagrangian systems within the framework of Lagrange-Poincaré bundles.

Let $L : T(G \times Q) \times V^* \rightarrow \mathbb{R}$ be a Lagrangian function where G is a Lie group, Q is a manifold, and V^* is the dual of a vector space V . The variable in V^* is understood as a fixed parameter. More precisely, for each $a_0 \in V^*$, we are looking for curves $(g(t), q(t), a_0)$ in $G \times Q \times V^*$ which are critical points of the action

$$\int_{t_0}^{t_1} L([g]^{(1)}, [q]^{(1)}, a_0) dt$$

with restrictions on variations given by $\delta g(t_i) = \delta q(t_i) = 0$ for $i = 0, 1$ and a_0 fixed ($\delta a_0 = 0$). For the sake of simplicity, hereafter $[g]^{(1)}$ and $[q]^{(1)}$ will be respectively denoted $(g, \dot{g})$ and $(q, \dot{q})$. Consider a representation of G on V , as well as the induced dual representation on V^* . Then, G acts on $T(G \times Q) \times V^*$ as follows

$$k(g, q, \dot{g}, \dot{q}, a_0) = (kg, q, k\dot{g}, \dot{q}, ka_0)$$

for all $k \in G$ and $(g, q, \dot{g}, \dot{q}, a_0) \in T(G \times Q) \times V^*$. Assume that L is G -invariant and a reduced Lagrangian can be defined as

$$\begin{aligned} l : \mathfrak{g} \times TQ \times V^* &\rightarrow \mathbb{R} \\ (\xi, q, \dot{q}, a) &\mapsto l(\xi, q, \dot{q}, a) = L(e, q, \xi, \dot{q}, a) \end{aligned}$$

Therefore, for all $g \in G$, all $q \in Q$ and all $a_0 \in V^*$,

$$L(g, q, \dot{g}, \dot{q}, a_0) = l(\xi, q, \dot{q}, a),$$

where $\xi = g^{-1}\dot{g}$ and $a = g^{-1}a_0$. The variational problem posed by a G -invariant Lagrangian, L , is reduced as follows.

Principle 3.1.1 *Solutions of the variational problem posed by l are curves $(\xi(t), q(t), a(t))$ in $\mathfrak{g} \times TQ \times V^*$, where $\xi(t) = g^{-1}(t)\dot{g}(t)$ and $a(t) = g^{-1}(t)a_0$, which are critical elements for the action defined by l with restricted variations*

$$\delta\xi = \dot{\eta} + [\xi, \eta],$$

where η is a curve in $\mathfrak{g}$ such that $\delta\eta(t_i) = 0$ for $i = 0, 1$; $\delta q(t_i) = 0$, for $i = 0, 1$; and

$$\delta a = -\eta_a^{V^*},$$

satisfying the additional condition

$$\dot{a} + \eta_a^{V^*} = 0. \quad (3.1)$$

Condition (3.1) is required to impose $\dot{a}_0 = 0$. Principle 3.1.1 leads to equations

$$-\frac{D}{Dt} \left(\frac{\partial l}{\partial \xi} \right) + \text{ad}_\xi^* \frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial a} \diamond a = 0, \quad (3.2)$$

$$\frac{\partial l}{\partial q} - \frac{D}{Dt} \left(\frac{\partial l}{\partial \dot{q}} \right) = 0, \quad (3.3)$$

where we denote $(b \diamond a)(\eta) = -\langle \eta_a^{V^*}, b \rangle$, for all $\eta \in \mathfrak{g}$, $a \in V^*$, and $b \in V$. Thus, these Lagrange–Poincaré equations together with condition, $\dot{a} + \eta_a^{V^*} = 0$ solve, directly from the reduction of a variational principle, the problem of an invariant Lagrangian depending on a parameter.

It is interesting to obtain these equations from different perspectives. For example, in [48], Euler–Poincaré reduction is performed in the particular case that Q is a point and, accordingly, $L : TG \times V^* \rightarrow \mathbb{R}$. In general, the equations can be also obtained combining Lagrange–Poincaré reduction and Lagrange multipliers (see [21] and [23]) for Lagrangians

$$\begin{aligned} L^V : T(G \times Q \times V^* \times V) &\rightarrow \mathbb{R} \\ (g, q, a, b, \dot{g}, \dot{q}, \dot{a}, \dot{b}) &\mapsto L(g, q, \dot{g}, \dot{q}, a) + \langle \dot{a} + g^{-1}\dot{g}a, b \rangle. \end{aligned}$$

3.1.2 Reduction of a Lagrange–Poincaré Bundle

To provide a different approach within the Lagrange–Poincaré category $\mathfrak{LP}$, we define the Lagrangian

$$\begin{aligned} \bar{L} : T(G \times Q) \oplus \tilde{V}^* \oplus \tilde{V} &\rightarrow \mathbb{R} \\ (g, q, \dot{g}, \dot{q}, a_0, b_0) &\mapsto L(g, q, \dot{g}, \dot{q}, a_0) + \langle a_0, b_0 \rangle, \end{aligned}$$

where $\tilde{V} = (G \times Q) \times V$ is a trivial vector bundle endowed with the correspondingly trivial connection, ∇ ; and a trivial Lie bracket, $[\cdot, \cdot] = 0$, in the fibers. The dual bundle $\tilde{V}^*$ is similarly equipped with a trivial Lie bracket. These structures, together with the constantly zero $\tilde{V}^* \oplus \tilde{V}$ -valued 2-form on $G \times Q$ makes $T(G \times Q) \oplus \tilde{V}^* \oplus \tilde{V}$ a Lagrange–Poincaré bundle and, hence, there is a notion of Lagrange–Poincaré equations for $\bar{L}$.

As the chosen 2-form is zero, the horizontal Lagrange–Poincaré equations (2.8) of $\bar{L}$ depends only on horizontal derivatives of $\bar{L}$. These coincide with the horizontal derivatives of L , and consequently, the horizontal Lagrange–Poincaré equation of $\bar{L}$ coincide with the Euler–Lagrange equations for the original Lagrangian L . Furthermore, as $\tilde{V}^* \oplus \tilde{V}$ has two factors, there are two sets of vertical Lagrange–Poincaré equations (2.7). The set of vertical equations relative to $\tilde{V}$ imposes that a_0 is a fixed parameter

$$0 = \frac{D}{Dt} \left(\frac{\partial \bar{L}}{\partial b_0} \right) = \dot{a}_0,$$

while the set of vertical equations relative to $\tilde{V}^*$ gives the evolution of the auxiliary variable b_0 :

$$0 = \frac{D}{Dt} \left(\frac{\partial \bar{L}}{\partial a_0} \right) = \frac{D}{Dt} \left(\frac{\partial L}{\partial a_0} \right) + \dot{b}_0.$$

The Lagrange–Poincaré equations of $\bar{L}$ reduce to equations (3.1), (3.2), and (3.3) via the reduction process explained in Section 2.2. To this end, we discuss the action of G on $T(G \times Q) \oplus \tilde{V}^* \oplus \tilde{V}$ and the resulting quotient bundle in $\mathfrak{LP}$. Since $T(G \times Q) \oplus \tilde{V}^* \oplus \tilde{V} \cong$

$T(G \times Q) \oplus (G \times Q \times V^*) \oplus (G \times Q \times V)$ the action of $k \in G$ can be explicated as

$$k \cdot ((g, q, \dot{g}, \dot{q}) \oplus (g, q, b_0) \oplus (g, q, a_0)) = (kg, q, k\dot{g}, \dot{q}) \oplus (kg, q, hb_0) \oplus (kg, q, ka_0).$$

Furthermore, the following isomorphism describes the quotient bundle

$$\begin{aligned} TQ \oplus \tilde{\mathfrak{g}} \oplus \tilde{V}^*/G \oplus \tilde{V}/G &\cong TQ \oplus (Q \times \mathfrak{g}) \oplus (Q \times V^*) \oplus (Q \times V) \\ (q, \dot{q}) \oplus [(e, q), \xi]_G \oplus [g, q, b_0]_G \oplus [g, q, a_0]_G &\longleftrightarrow (q, \dot{q}) \oplus (q, \xi) \oplus (q, b) \oplus (q, a), \end{aligned}$$

where $b = g^{-1}b_0$, and $a = g^{-1}a_0$.

The bundle $G \times Q \rightarrow Q$ is equipped with the trivial connection that induces in the adjoint bundle $\tilde{\mathfrak{g}} \cong Q \times \mathfrak{g}$ the trivial covariant derivative

$$\frac{D}{Dt}(q(t), \xi(t)) = (q(t), \dot{\xi}(t)).$$

This connection is flat, that is, $\tilde{\mathcal{R}} = 0$, and $Q \times \mathfrak{g}$ has a fiber-wise Lie bracket given by $[(q, \xi_1), (q, \xi_2)] = (q, [\xi_1, \xi_2])$. The factor $(Q \times V^*) \oplus (Q \times V) \cong Q \times V \times V^*$ has a trivial Lie bracket and a constantly zero $(Q \times V \times V^*)$ -valued 2 form on Q coming from the respective structures in $\tilde{V}^* \oplus \tilde{V}$. However, reduction of the trivial connection on $\tilde{V}^* \oplus \tilde{V}$,

$$\frac{D}{Dt}(g(t), q(t), a_0(t), b_0(t)) = (g(t), q(t), \dot{a}_0(t), \dot{b}_0(t)).$$

to a connection on $\tilde{V}^*/G \oplus \tilde{V}/G \cong (Q \times V^*) \oplus (Q \times V)$ is somewhat trickier: It requires to separate the horizontal and vertical component as explained in Subsection 1.1.3. As seen in Subsection 2.1.2, this splitting results in the horizontal and vertical quotient connections defined by (2.5) and (2.6). For any section $[v]_G(q) = (q, a(q), b(q))$ of $Q \times V \times V^* \rightarrow Q$, the horizontal quotient connection is

$$[\nabla^{(H)}]_{G, (q, \dot{q}, \xi)} [v]_G = (q, \dot{q}(a), \dot{q}(b)),$$

where $\dot{q}(a)$ is the partial derivative of $a : Q \rightarrow V$ in the direction $\dot{q}$, and the vertical quotient connection is

$$[\nabla^{(V)}]_{G, (q, \dot{q}, \xi)} [v]_G = (q, \xi_{a(q)}^V, \xi_{b(q)}^V).$$

Direct application of Theorem 2.1.7 endows the quotient bundle,

$$TQ \oplus (Q \times \mathfrak{g}) \oplus (Q \times V^*) \oplus (Q \times V) \cong TQ \oplus (Q \times \mathfrak{g} \times V^* \times V);$$

with the $\mathfrak{LP}$ -bundle structure determined by

$$\nabla_{(q,\dot{q})}^{\tilde{\mathfrak{g}}}(q, \xi, a, b) = \nabla_{(q,\dot{q})}^{\mathcal{A}}(q, \xi) \oplus [\nabla^{(H)}]_{G,(q,\dot{q})}(q, a, b) = (q, \dot{q}(\xi), \dot{q}(a), \dot{q}(b));$$

$$\omega^{\tilde{\mathfrak{g}}} = \tilde{\mathcal{B}} \oplus [\omega]_G = 0;$$

$$[(q, \xi_1, a_1, b_1), (q, \xi_2, a_2, b_2)]^{\tilde{\mathfrak{g}}} = (q, [\xi_1, \xi_2], (\xi_1)_{a_2}^{V^*} - (\xi_2)_{a_1}^{V^*}, (\xi_1)_{b_2}^V - (\xi_2)_{b_1}^V).$$

Finally, since L is G -invariant and $\langle ga_0, gb_0 \rangle = \langle a_0, b_0 \rangle$, $\bar{L}$ is G -invariant and the reduced Lagrangian is

$$\begin{aligned} \bar{l} : TQ \oplus (Q \times \mathfrak{g} \times V^* \times V) &\rightarrow \mathbb{R} \\ (q, \dot{q}, \xi, a, b) &\mapsto l(q, \dot{q}, \xi, a) + \langle a, b \rangle. \end{aligned}$$

Its horizontal Lagrange–Poincaré equation is

$$0 = \frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial \dot{q}} \right) - \frac{\partial \bar{l}}{\partial q} = \frac{D}{Dt} \left(\frac{\partial l}{\partial \dot{q}} \right) - \frac{\partial l}{\partial q},$$

which coincides with equation (3.3), while its vertical Lagrange–Poincaré equation is

$$0 = -\frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial (\xi, a, b)} \right) + \text{ad}_{(\xi, a, b)}^* \left(\frac{\partial \bar{l}}{\partial (\xi, a, b)} \right)$$

Applying this expression to any variation $(\delta \xi, \delta a, \delta b) \in \mathfrak{g} \times V^* \times V$ we have

$$\begin{aligned} &\left\langle -\frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial (\xi, a, b)} \right) + \text{ad}_{(\xi, a, b)}^* \left(\frac{\partial \bar{l}}{\partial (\xi, a, b)} \right); \delta \xi, \delta a, \delta b \right\rangle = \\ &= -\left\langle \frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial \xi} \right), \delta \xi \right\rangle - \left\langle \frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial a} \right), \delta a \right\rangle - \left\langle \frac{D}{Dt} \left(\frac{\partial \bar{l}}{\partial b} \right), \delta b \right\rangle \\ &\quad + \left\langle \frac{\partial \bar{l}}{\partial (\xi, a, b)}; [\xi, \delta \xi], (\xi)_{\delta a}^{V^*} - (\delta \xi)_a^{V^*}, (\xi)_{\delta b}^V - (\delta \xi)_b^V \right\rangle \\ &= -\left\langle \frac{D}{Dt} \left(\frac{\partial l}{\partial \xi} \right), \delta \xi \right\rangle - \left\langle \frac{D}{Dt} \left(\frac{\partial l}{\partial a} \right) + \dot{b}, \delta a \right\rangle - \langle \dot{a}, \delta b \rangle + \left\langle \frac{\partial l}{\partial \xi}, [\xi, \delta \xi] \right\rangle \\ &\quad + \left\langle \frac{\partial l}{\partial a}, (\xi)_{\delta a}^{V^*} - (\delta \xi)_a^{V^*} \right\rangle + \left\langle b, (\xi)_{\delta a}^{V^*} - (\delta \xi)_a^{V^*} \right\rangle + \left\langle a, (\xi)_{\delta b}^V - (\delta \xi)_b^V \right\rangle \\ &= \left\langle -\frac{D}{Dt} \left(\frac{\partial l}{\partial \xi} \right) + \text{ad}_{\xi}^* \left(\frac{\partial l}{\partial \xi} \right) - \frac{\partial l}{\partial a} \diamond a, \delta \xi \right\rangle \\ &\quad + \left\langle -\frac{D}{Dt} \left(\frac{\partial l}{\partial a} \right) - \dot{b} - \xi_{\frac{\partial l}{\partial a}}^V - \xi_b^V, \delta a \right\rangle - \left\langle \dot{a} + \xi_a^{V^*}, \delta b \right\rangle, \end{aligned}$$

where it has been repeatedly used that $\langle a, \xi_b^V \rangle + \langle \xi_a^{V*}, b \rangle = 0$ for all $\xi \in \mathfrak{g}$ and the abuse of notation $\langle a, b \rangle = \langle b, a \rangle$. As the variations $\delta\xi, \delta a, \delta b$ are free, we recover (3.1), (3.2) as well as the evolution of the auxiliary variable b .

3.2 Rigid Body with Rotors

3.2.1 Euler–Poincaré Reduction

A *rigid body with rotors* is a mechanical system consisting of a rigid body that has moving pieces which are rotating around different axes. See Figure 3.1. The literature about these systems is ample. A purely theoretical approach is found in [68], yet a plethora of references focus in applications (for instance [13], [28], [56], [73] as well as the references therein). Indeed, the applications of the system are very valuable and can be found in many different situations, with particular emphasis in controllability of robots [72]. However this profuse bibliography is scattered, incomplete and somewhat confusing. In the following, we extensively describe the rigid body with rotors in an ordered manner implementing the theoretical models developed in Chapter 2.

We shall restrict ourselves to the case in which there are three rotors whose axes are the principal axes of inertia, although the example could be easily generalised to any number of rotors at different positions. The configuration bundle of such system is $Q = SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, encoding the spatial rotation of both the rigid body and the three rotors respectively. We denote an arbitrary point in Q either $(R, \theta_1, \theta_2, \theta_3)$ or simply (R, θ) .

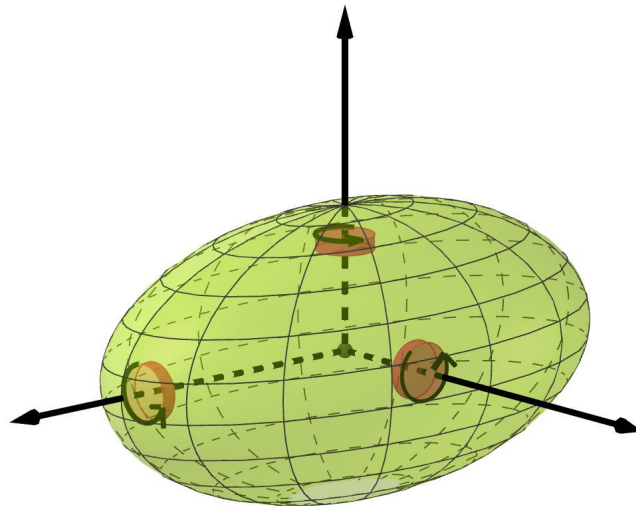


Figure 3.1 Rigid body with rotors in the principal axes of inertia

The Lagrangian, L , for a free rigid body with rotors is a real function defined on TQ as

$$L = \frac{1}{2} \langle R^{-1} \dot{R}, I R^{-1} \dot{R} \rangle + \frac{1}{2} \langle R^{-1} \dot{R} + \dot{\theta}, K(R^{-1} \dot{R} + \dot{\theta}) \rangle \quad (3.4)$$

where $(R, \theta, \dot{R}, \dot{\theta}) \in TQ$, I represents the inertia tensor of the rigid solid, K is the inertia tensor of the rotors, and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^3$. Lie algebras $(\mathfrak{so}(3), [\cdot, \cdot])$ and $(\mathbb{R}^3, \times)$ are identified via the hat map.

Since the configuration bundle $Q = SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is a Lie group, there is an smooth action of Q on itself by left translations which is free and proper. According to the theory of Euler-Poincaré reduction in classical references [1, 2, 67], the action lifts to TQ and the lifted action is free and proper. In addition, the quotient of TQ by this action is a smooth manifold identified with the Lie algebra via

$$\begin{aligned} T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) / (SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) &\rightarrow \mathfrak{so}(3) \oplus \mathbb{R}^3 \\ [R, \theta, \dot{R}, \dot{\theta}] &\mapsto (\Omega, \Omega_r) \end{aligned}$$

where $\Omega = R^{-1} \dot{R}$ is the angular velocity of the rigid body and $\Omega_r = \dot{\theta}$ is the rotor angular velocity. As the free Lagrangian is $(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)$ -invariant, there is a reduced Lagrangian, $\ell : \mathfrak{so}(3) \oplus \mathbb{R}^3 \rightarrow \mathbb{R}$, defined as

$$\begin{aligned} \ell(\Omega, \Omega_r) &= \frac{1}{2} \langle \Omega, I \Omega \rangle + \frac{1}{2} \langle \Omega + \Omega_r, K(\Omega + \Omega_r) \rangle \\ &= \frac{1}{2} \langle \Omega, (I + K) \Omega \rangle + \frac{1}{2} \langle \Omega_r, K \Omega_r \rangle + \langle \Omega, K \Omega_r \rangle. \end{aligned}$$

The evolution of the system is described by the reduced Euler-Poincaré equations

$$\frac{D}{Dt} \left(\frac{\partial \ell}{\partial \Omega} \oplus \frac{\partial \ell}{\partial \Omega_r} \right) = \text{ad}_{(\Omega, \Omega_r)}^* \left(\frac{\partial \ell}{\partial \Omega} \oplus \frac{\partial \ell}{\partial \Omega_r} \right), \quad (3.5)$$

where the coadjoint action refers to the Lie bracket on $\mathfrak{so}(3) \oplus \mathbb{R}^3$ such that

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = (\alpha_1 \times \alpha_2, 0),$$

for any two elements $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ in $\mathfrak{so}(3) \oplus \mathbb{R}^3$. Note that (3.5) is an equation in $(\mathfrak{so}(3) \oplus \mathbb{R}^3)^*$ as $\partial \ell / \partial \Omega = \langle (I + K) \Omega, \cdot \rangle + \langle K \Omega_r, \cdot \rangle$ and $\partial \ell / \partial \Omega_r = \langle K \Omega_r, \cdot \rangle + \langle K \Omega, \cdot \rangle$ both belong to that dual vector space. Then, equation (3.5) applied to an element $(\alpha, \beta) \in \mathfrak{so}(3) \oplus \mathbb{R}^3$ can be written as

$$\langle (I + K) \dot{\Omega}, \alpha \rangle + \langle K \dot{\Omega}_r, \alpha \rangle + \langle K \dot{\Omega}_r, \beta \rangle + \langle K \dot{\Omega}, \beta \rangle = \langle (I + K) \Omega + K \Omega_r, \Omega \times \alpha \rangle$$

As a consequence, the reduced equations of motion are

$$(I + K)\dot{\Omega} + K\dot{\Omega}_r = ((I + K)\Omega + K\Omega_r) \times \Omega \quad (3.6)$$

$$K\dot{\Omega}_r + K\dot{\Omega} = 0 \quad (3.7)$$

where (3.7) can be understood as the conservation of the rotors' momentum.

Euler–Poincaré reduction has been performed above for a Lagrangian system describing a free rigid body. The symmetry of the whole group $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is used in this process of reduction. However, it is interesting to reduce by the subgroup of symmetry of the rigid body, $SO(3)$, and reduce afterwards by the rest of group of symmetry, $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. This splitting is consistent with the different nature of the symmetries, one dealing with the main body and the other with the rotors. This is a paradigmatic situation for which the theory of reduction by stages initiated in [23] and expanded in Chapter 2.

As seen in Section 2.2, reduction by stages can be performed in different ways, either by choosing a different subgroup to begin the process or by choosing different affine connections along the way. In the following, different processes of reduction by stages are analyzed and compared.

3.2.2 Reduction by Stages Initiated by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$

We shall reduce by stages the free rigid body with rotors, first reducing by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, and later on by $SO(3)$. In order to do the first step of reduction we choose a principal connection $\mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta}) = R^{-1}\dot{R} + \dot{\theta}$ on the principal bundle $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow SO(3)$ and we define the total angular velocity $\eta = \Omega + \Omega_r = \mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta})$. This choice of connection coincides with the mechanical connection induced by the Riemannian metric on $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$\langle\langle (\dot{R}, \dot{\theta}), (\dot{Q}, \dot{\phi}) \rangle\rangle_{(R,\theta)} = \langle R^{-1}\dot{Q}, I R^{-1}\dot{R} \rangle + \langle R^{-1}\dot{Q} + \dot{\phi}, K(R^{-1}\dot{R} + \dot{\theta}) \rangle$$

and it is driven by the resulting simplification of the reduced Lagrangian. We apply this connection to the identification

$$\begin{aligned} \frac{T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)}{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} &\rightarrow T(SO(3)) \oplus \tilde{\mathbb{R}}^3 \rightarrow T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3) \\ [R, \theta, \dot{R}, \dot{\theta}] &\mapsto (R, \dot{R}, [R, \theta, \eta]) \mapsto (R, \dot{R}, \eta) \end{aligned}$$

where the right side is obtained by trivializing $\tilde{\mathbb{R}}^3$ as $SO(3) \times \mathbb{R}^3$. The bundle $T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3)$ is an object in the category $\mathfrak{L}\mathfrak{P}$ with trivial Lie bracket in the fibers of $SO(3) \times$

$\mathbb{R}^3 \rightarrow SO(3)$, trivial covariant derivative and a $\mathbb{R}^3$ -valued 2-form on $SO(3)$, $\tilde{\mathcal{B}}_R(\dot{R}, \dot{Q}) = (R^{-1}\dot{Q}) \times (R^{-1}\dot{R})$ induced by

$$\begin{aligned}\mathcal{B}_{R,\theta}((\dot{R}, \dot{\theta}), (\dot{Q}, \dot{\phi})) &= -\mathcal{A}_{R,\theta}([\text{Hor}(\dot{R}, \dot{\theta}), \text{Hor}(\dot{Q}, \dot{\phi})]) \\ &= -\mathcal{A}_{R,\theta}([\dot{R}, -R^{-1}\dot{R}], [\dot{Q}, -R^{-1}\dot{Q}]) \\ &= -\mathcal{A}_{R,\theta}(\dot{R}R^{-1}\dot{Q} - \dot{Q}R^{-1}\dot{R}, 0) = (R^{-1}\dot{Q}) \times (R^{-1}\dot{R}).\end{aligned}$$

The system obtained after the first step of reduction is defined in this Lagrange–Poincaré bundle and is determined by the reduced Lagrangian;

$$l_{1,S}(R, \dot{R}, \eta) = \frac{1}{2}\langle R^{-1}\dot{R}, IR^{-1}\dot{R} \rangle + \frac{1}{2}\langle \eta, K\eta \rangle. \quad (3.8)$$

We now study the equations of motion of the reduced system. On one hand, since the group $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is abelian and $\frac{\partial l_{1,S}}{\partial \eta} = \langle K\eta, \cdot \rangle$, the vertical Lagrange–Poincaré equation

$$\frac{D}{Dt} \left(\frac{\partial l_{1,S}}{\partial \eta} \right) - \text{ad}_{\eta}^* \left(\frac{\partial l_{1,S}}{\partial \eta} \right) = 0$$

can be written as

$$K\dot{\eta} = 0, \quad (3.9)$$

which, as mentioned in Subsection 3.2.1, can be interpreted as the conservation of the momentum of the rotors. On the other hand, since

$$\left\langle \frac{\partial l_{1,S}}{\partial \eta}, \tilde{\mathcal{B}}_R(\dot{R}, \dot{Q}) \right\rangle = \langle K\eta, (R^{-1}\dot{Q}) \times (R^{-1}\dot{R}) \rangle = \langle R^{-1}\dot{Q}, (R^{-1}\dot{R}) \times K\eta \rangle,$$

the horizontal Lagrange–Poincaré equation,

$$\frac{D}{Dt} \left(\frac{\partial l_{1,S}}{\partial \dot{R}} \right) - \frac{\partial l_{1,S}}{\partial R} + \left\langle \frac{\partial l_{1,S}}{\partial \eta}, \tilde{\mathcal{B}}_R(\dot{R}, \cdot) \right\rangle = 0,$$

is written as

$$I\dot{\Omega} - I\Omega \times \Omega - (K\eta) \times \Omega = 0 \quad (3.10)$$

which is the equation of a rigid body with inertia tensor I with an additional force term $(K\eta) \times \Omega$ coming from the curvature of the connection used to reduce.

The second step of reduction uses the action of $SO(3)$ in $T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3)$ given by $\Gamma \cdot (R, \dot{R}, \eta) = (\Gamma R, \Gamma \dot{R}, \eta)$ together with the connection $\mathcal{A}_R^{SO(3)}(\dot{R}) = R^{-1}\dot{R} = \Omega$. This

gives the isomorphism

$$T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3)/SO(3) \rightarrow \mathfrak{so}(3) \times \mathbb{R}^3$$

$$[R, \dot{R}, \eta] \mapsto (\Omega, \eta).$$

As the base space resulting from this step of reduction is a single point, the induced connection $\nabla^{\tilde{\mathfrak{so}}(3)}$ is trivial and the 2-form $\omega^{\tilde{\mathfrak{so}}(3)} = 0$. It can be seen as well that $[\nabla^{(\mathcal{A}, H)}]_{SO(3)}$ is trivial and consequently the action of $SO(3)$ in $T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3)$ is horizontal in the sense that $[\nabla^{(\mathcal{A}, V)}]_{SO(3), \Gamma}[\eta] = [\Gamma_\eta^V] = 0$ for all $\Gamma \in SO(3)$. Therefore, given $\Omega_1, \Omega_2 \in \mathfrak{so}(3)$ and $\eta_1, \eta_2 \in \mathbb{R}^3$ the induced bracket is

$$\begin{aligned} [(\Omega_1, \eta_1), (\Omega_2, \eta_2)]^{\tilde{\mathfrak{so}}(3)} &= [\Omega_1, \Omega_2] \oplus \left([\nabla^{(\mathcal{A}^{SO(3)}, V)}]_{SO(3), \Omega_1}[\eta_2] \right. \\ &\quad \left. - [\nabla^{(\mathcal{A}^{SO(3)}, V)}]_{SO(3), \Omega_2}[\eta_1] - [\tilde{\mathcal{B}}]_R(\Omega_1, \Omega_2) + [\eta_1, \eta_2] \right) \\ &= (\Omega_1 \times \Omega_2) \oplus (\Omega_1 \times \Omega_2) \end{aligned}$$

In other words, the reduced configuration space obtained from the second step of reduction is the Lie algebra $\mathfrak{so}(3) \times \mathbb{R}^3$ with bracket $[\cdot, \cdot]^{\tilde{\mathfrak{so}}(3)}$. Recall that Lie algebras are a special case of bundles in $\mathfrak{LP}$ as seen in Example 2.1.3. In addition, the reduced Lagrangian is

$$l_{2,S}(\Omega, \eta) = \frac{1}{2} \langle \Omega, I\Omega \rangle + \frac{1}{2} \langle \eta, K\eta \rangle. \quad (3.11)$$

As the reduced space is a Lie algebra, there are no horizontal Lagrange–Poincaré equations. In contrast, the vertical equations are

$$\frac{D}{Dt} \left(\frac{\partial l_{2,S}}{\partial \Omega} \oplus \frac{\partial l_{2,S}}{\partial \eta} \right) = \text{ad}_{\Omega, \eta}^* \left(\frac{\partial l_{2,S}}{\partial \Omega} \oplus \frac{\partial l_{2,S}}{\partial \eta} \right).$$

Applying this equation to an arbitrary element $(\alpha, \beta) \in \mathfrak{so}(3) \times \mathbb{R}^3$ and using that $\partial l_{2,S}/\partial \Omega = \langle I\Omega, \cdot \rangle$, $\partial l_{2,S}/\partial \eta = \langle K\eta, \cdot \rangle$, the vertical equation can be written as

$$\langle I\dot{\Omega}, \alpha \rangle + \langle K\dot{\eta}, \beta \rangle = \langle I\Omega, \Omega \times \alpha \rangle + \langle K\eta, \Omega \times \alpha \rangle$$

Consequently, the following equations of motion are obtained:

$$I\dot{\Omega} = I\Omega \times \Omega + (K\eta) \times \Omega \quad (3.12)$$

and

$$K\dot{\eta} = 0. \quad (3.13)$$

Vertical equations from the second step of reduction (equations (3.12) and (3.13)) coincide with the horizontal and vertical ones of the first step of reduction (equations (3.9) and (3.10)) showing that the systems obtained in both steps of reduction are equivalent. This illustrates Theorem 2.2.14. Notice that equation (3.10) is an horizontal equation in the first step of reduction whereas in the second step it is obtained as the vertical equation (3.12). These descriptions are in turn equivalent to the set of equations obtained by Euler-Poincaré reduction in Subsection 3.2.1. Indeed, equation (3.7) coincides with equation (3.13) and equation (3.6) is the sum of (3.12) and (3.13).

The $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ -invariance of the Lagrangian in (3.4) is used in the first step of reduction and, as stated in Subsection 1.2.3, the associated Noether current is:

$$J_{1,S} : T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \rightarrow (\mathbb{R}^3)^*$$

$$(R, \theta, \dot{R}, \dot{\theta}) \mapsto \left(a \mapsto \left\langle \frac{\partial L}{\partial \dot{R}} \oplus \frac{\partial L}{\partial \dot{\theta}}, a_{(R,\theta)}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)} \right\rangle \right)$$

Since $\frac{\partial L}{\partial \dot{\theta}} = K(R^{-1} \dot{R} + \dot{\theta})$ and $a_{(R,\theta)}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)} = (R, \theta, 0, a)$, identification of $\mathbb{R}^3$ with its dual provides

$$J_{1,S} : T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \mathbb{R}^3$$

$$(R, \theta, \dot{R}, \dot{\theta}) \mapsto K(R^{-1} \dot{R} + \dot{\theta}). \quad (3.14)$$

This Noether current can be interpreted as the momentum of the rotors, it is conserved and its conservation is equivalent to equation (3.9). In turn, the $SO(3)$ -invariance of the reduced Lagrangian $l_{1,S}$ in (3.8) is used in the second step of reduction. As studied in Subsection 2.2.4, this symmetry has an associated Noether current

$$J_{2,S} : T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3) \rightarrow \mathfrak{so}(3)^*$$

$$(R, \dot{R}, \eta) \mapsto \left(b \mapsto \left\langle \frac{\partial l_{1,S}}{\partial \dot{R}}, b_R^{SO(3)} \right\rangle \right).$$

Since $b_R^{SO(3)} = (R, bR)$, the identification of $\mathfrak{so}(3)$ with its dual gives

$$J_{2,S} : T(SO(3)) \oplus (SO(3) \times \mathbb{R}^3) \rightarrow \mathfrak{so}(3)$$

$$(R, \dot{R}, \eta) \mapsto R I R^{-1} \dot{R} R^{-1}.$$

This Noether current is interpreted as the angular momentum of a rigid body with inertia tensor I . Due to the presence of the rotors this is not the total angular momentum of the

system and it is not preserved. Particularization of the drift of the Noether current in equation (2.13) shows that for every $b \in \mathfrak{so}(3)$

$$\frac{d}{dt} \langle J(R, \dot{R}, \eta), b \rangle = - \langle K\eta, (R^{-1}bR) \times R^{-1}R \rangle \quad (3.15)$$

which can be seen to be equivalent to equation (3.12). This is the vertical equation of the second step of reduction that was obtained as horizontal equation (3.10) in the first reduction, in accordance to Proposition 2.2.20.

3.2.3 Reduction by Stages Initiated by $SO(3)$

We shall now reduce by stages starting by the reduction with respect to $SO(3)$ and later by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. This will be done in two different ways. In the first method we shall use in each step the Maurer-Cartan connection as in [72], while in the second method a mechanical connection like the one presented in [68] will be used.

Reduction with Maurer-Cartan connection

Consider the natural lift to $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3))$ of the action of $SO(3)$ on itself on the second factor of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)$. The Maurer-Cartan form defines a principal connection $\mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta}) = \text{Ad}_R(R^{-1}\dot{R}) = \dot{R}R^{-1}$ on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3) \rightarrow \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Thus, there is a splitting

$$\begin{aligned} T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)/SO(3) &\rightarrow T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3) \\ [R, \theta, \dot{R}, \dot{\theta}] &\mapsto (\theta, \dot{\theta}, [R, \theta, \dot{R}R^{-1}]) \end{aligned}$$

which in turn can be trivialized to

$$\begin{aligned} T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3) &\rightarrow T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)) \\ (\theta, \dot{\theta}, [R, \theta, \dot{R}R^{-1}]) &\mapsto (\theta, \dot{\theta}, \Omega = R^{-1}\dot{R}), \end{aligned}$$

where $\Omega = R^{-1}\dot{R}$ as in Subsection 3.2.1. The reduced space $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3)$ is a bundle in $\mathfrak{RP}$ with covariant derivative in $\tilde{\mathfrak{so}}(3)$ given by

$$\frac{D}{Dt} [(R(t), \theta(t)), \alpha(t)] = [(R(t), \theta(t)), \dot{\alpha}(t) - \Omega \times \alpha(t)], \quad (3.16)$$

a Lie bracket in $\tilde{\mathfrak{so}}(3)$ induced by the usual cross product, and, since the Maurer-Cartan connection is integrable, the $\tilde{\mathfrak{so}}(3)$ -valued 2-form on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is zero. The trivialization

$T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ is also a bundle in $\mathfrak{RP}$ with constantly zero $(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ -valued 2-form, Lie bracket in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)$ induced by the usual cross product and trivial covariant derivative.

The reduced Lagrangian is

$$l_{1,O}(\theta, \dot{\theta}, \Omega) = \frac{1}{2} \langle \Omega, (I + K)\Omega \rangle + \frac{1}{2} \langle \dot{\theta}, K\dot{\theta} \rangle + \frac{1}{2} \langle \Omega, K\dot{\theta} \rangle, \quad (3.17)$$

which has a non-decoupled term depending on both, the rotor and the rigid body. This does not alter the fact that Lagrange-Poincaré equations can still be obtained. The vertical equation

$$\frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \Omega} \right) - \text{ad}_\Omega^* \left(\frac{\partial l_{1,O}}{\partial \Omega} \right) = 0$$

can be rewritten as $\langle (I + K)\dot{\Omega} + K\ddot{\theta}, \alpha \rangle = \langle (I + K)\Omega + K\dot{\theta}, \Omega \times \alpha \rangle$, where α is an arbitrary element of $\mathfrak{so}(3)$. That is, we have

$$(I + K)\dot{\Omega} + K\ddot{\theta} = ((I + K)\Omega + K\dot{\theta}) \times \Omega. \quad (3.18)$$

With respect to the horizontal equation,

$$\frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \dot{\theta}} \right) - \frac{\partial l_{1,O}}{\partial \theta} + \left\langle \frac{\partial l_{1,O}}{\partial \Omega}, \tilde{\mathcal{B}}_\theta(\dot{\theta}, \cdot) \right\rangle = 0,$$

since $l_{1,O}$ does not depend on θ and $\frac{\partial l_{1,O}}{\partial \dot{\theta}} = \langle K\dot{\theta} + K\Omega, \cdot \rangle$, it reads

$$K\ddot{\theta} + K\dot{\Omega} = 0. \quad (3.19)$$

The equations of motion (3.18) and (3.19) obtained in the first step of reduction are, respectively, equivalent to equations (3.6) and (3.7), obtained in the Euler-Poincaré reduction.

We now perform the second step of reduction. The group $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts by isomorphisms in $\mathfrak{RP}$ on $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ and for any $\varphi \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\varphi \cdot (\theta, \dot{\theta}, \Omega) = (\theta + \varphi, \dot{\theta}, \Omega)$. The connection used to reduce is $\mathcal{A}_\theta(\dot{\theta}) = \dot{\theta} = \Omega_r$, which induces the isomorphism

$$\begin{aligned} (T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))) / (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) &\rightarrow \mathbb{R}^3 \oplus \mathfrak{so}(3) \\ [\theta, \dot{\theta}, \Omega] &\mapsto (\Omega_r = \dot{\theta}, \Omega). \end{aligned}$$

The second reduced space is the Lagrange-Poincaré bundle $\mathbb{R}^3 \times \mathfrak{so}(3)$ over a point with trivial $\nabla^{\mathbb{R}^3}$ and $\omega^{\mathbb{R}^3} = 0$. However, in contrast to the second step of reduction in Subsection

3.2.2, the Lie Algebra on $\mathbb{R}^3 \times \mathfrak{so}(3)$ is $[(\Omega_{r,1}, \Omega_1), (\Omega_{r,2}, \Omega_2)]^{\mathbb{R}^3} = 0 \oplus (\Omega_1 \times \Omega_2)$ and the reduced Lagrangian is

$$l_{2,O}(\Omega_r, \Omega) = \frac{1}{2} \langle \Omega, (I + K)\Omega \rangle + \frac{1}{2} \langle \Omega_r, K\Omega_r \rangle + \frac{1}{2} \langle \Omega, K\Omega_r \rangle. \quad (3.20)$$

There are no horizontal equation, and the vertical equations

$$\frac{D}{Dt} \left(\frac{\partial l_{2,O}}{\partial \Omega_r} \oplus \frac{\partial l_{2,O}}{\partial \Omega} \right) = \text{ad}_{\Omega_r, \Omega}^* \left(\frac{\partial l_{2,O}}{\partial \Omega_r} \oplus \frac{\partial l_{2,O}}{\partial \Omega} \right) = 0$$

are

$$(I + K)\dot{\Omega} + K\dot{\Omega}_r = ((I + K)\Omega + K\Omega_r) \times \Omega. \quad (3.21)$$

$$K\dot{\Omega}_r + K\dot{\Omega} = 0. \quad (3.22)$$

These are the same as the Lagrange–Poincaré equations of the first step of reduction, labeled as (3.18) and (3.19), once the change of variable $\dot{\theta} = \Omega_r$ is done. As in Subsection 3.2.2, the horizontal equation of the first step of reduction is obtained as a vertical equation in the second step of reduction.

Reduction with the mechanical connection

We continue working with the same action of $SO(3)$ on $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3))$ as in the previous case. However, we shall use the connection

$$\mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta}) = \text{Ad}_R(R^{-1}\dot{R} + (I + K)^{-1}K\dot{\theta})$$

which is the mechanical connection induced on the principal bundle $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3) \rightarrow \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ by the metric on $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)$ that makes Lagrangian L in (3.4) a kinetic energy term. The isomorphism induced by this connection is

$$\begin{aligned} T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)/SO(3) &\rightarrow T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3) \\ [R, \theta, \dot{R}, \dot{\theta}] &\mapsto (\theta, \dot{\theta}, [R, \theta, \dot{R}R^{-1} + R(I + K)^{-1}K\dot{\theta}R^{-1}]) \end{aligned}$$

which can subsequently be trivialized to

$$\begin{aligned} T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3) &\rightarrow T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)) \\ (\theta, \dot{\theta}, [R, \theta, \mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta})]) &\mapsto (\theta, \dot{\theta}, \xi = \text{Ad}_{R^{-1}}(\mathcal{A}_{R,\theta}(\dot{R}, \dot{\theta}))), \end{aligned}$$

where a new variable $\xi = R^{-1}\dot{R} + (I + K)^{-1}K\dot{\theta} \in \mathfrak{so}(3)$ has been introduced.

The reduced space $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus \tilde{\mathfrak{so}}(3)$ is a Lagrange–Poincaré bundle with a Lie bracket in $\tilde{\mathfrak{so}}(3)$ induced by the cross product, and a covariant derivative given by

$$\frac{D}{Dt}[(R(t), \theta(t)), \alpha(t)] = [(R(t), \theta(t)), \dot{\alpha}(t) - (\dot{R}R^{-1} + R(I + K)^{-1}K\dot{\theta}R^{-1}) \times \alpha(t)].$$

The trivialization $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ is also a bundle in $\mathfrak{LP}$ with the same Lie bracket and the covariant derivative in $\tilde{\mathfrak{so}}(3)$ given by

$$\frac{D}{Dt}[(R(t), \theta(t)), \alpha(t)] = [(R(t), \theta(t)), \dot{\alpha}(t) - ((I + K)^{-1}K\dot{\theta}) \times \alpha(t)].$$

The $(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ -valued 2-form associated to the trivialization comes from the curvature of the mechanical connection:

$$\tilde{\mathcal{B}}_\theta(\dot{\theta}, \dot{\varphi}) = ((I + K)K\dot{\varphi}) \times ((I + K)K\dot{\theta}).$$

In addition, the reduced Lagrangian on $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ is

$$l_{1,O}(\theta, \dot{\theta}, \xi) = \frac{1}{2}\langle \xi, (I + K)\xi \rangle + \frac{1}{2}\langle K\dot{\theta}, (I + K)^{-1}I\dot{\theta} \rangle. \quad (3.23)$$

Observe that, in contrast to the approach with the Maurer–Cartan connection, this Lagrangian is decoupled in the sense that there is no term depending on both, the rigid body and the rotors. This is the main reason why the mechanical connection is used in [68], yet, the previous example shows that equations of motion can be written even for a coupled Lagrangian.

We now write explicitly the vertical Lagrange–Poincaré equations

$$\frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \xi} \right) - \text{ad}_\xi^* \left(\frac{\partial l_{1,O}}{\partial \xi} \right) = 0.$$

Let $\alpha \in \mathfrak{so}(3)$, then

$$\begin{aligned} \left\langle \frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \xi} \right), \alpha \right\rangle &= \frac{d}{dt} \left\langle \left(\frac{\partial l_{1,O}}{\partial \xi} \right), \alpha \right\rangle - \left\langle \frac{\partial l_{1,O}}{\partial \xi}, \frac{D\alpha}{Dt} \right\rangle = \\ &= \langle (I + K)\dot{\xi}, \alpha \rangle + \langle (I + K)\xi, \dot{\alpha} \rangle - \langle (I + K)\xi, \dot{\alpha} \rangle + \langle (I + K)\xi, ((I + K)^{-1}K\dot{\theta}) \times \alpha \rangle \\ &= \langle (I + K)\dot{\xi}, \alpha \rangle + \langle (I + K)\xi, ((I + K)^{-1}K\dot{\theta}) \times \alpha \rangle, \end{aligned}$$

and

$$\left\langle \text{ad}_\xi^* \left(\frac{\partial l_{1,O}}{\partial \xi} \right), \alpha \right\rangle = \langle (I + K)\xi, \xi \times \alpha \rangle = \langle ((I + K)\xi) \times \xi, \alpha \rangle.$$

Hence, the vertical Lagrange-Poincaré equation is

$$(I + K)\dot{\xi} = ((I + K)\xi) \times (\xi - (I + K)^{-1}K\dot{\theta}) = ((I + K)\xi) \times \Omega. \quad (3.24)$$

The explicit expression of the horizontal Lagrange-Poincaré equations,

$$\frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \dot{\theta}} \right) - \frac{\partial l_{1,O}}{\partial \theta} + \left\langle \frac{\partial l_{1,O}}{\partial \xi}, \tilde{\mathcal{B}}_{\theta}(\dot{\theta}, \cdot) \right\rangle = 0, \quad (3.25)$$

is more involved since the term $\partial l_{1,O}/\partial \theta$ does not denote a partial derivative but a covariant derivative of a function as introduced in Subsection 1.2.1. Thus, in spite that the Lagrangian does not depend on θ , $\partial l_{1,O}/\partial \theta$ is distinct from zero. We shall temporarily denote this covariant derivative as $\delta l_{1,O}/\delta \theta$ to distinguish it from the partial derivative. The covariant derivative of $l_{1,O}$ at point $(\theta, \dot{\theta}, \xi)$ applied to $(\varphi, \dot{\varphi})$ is

$$\frac{\delta l_{1,O}}{\delta \theta}(\theta, \dot{\theta}, \xi)(\varphi, \dot{\varphi}) = \left. \frac{d}{ds} \right|_{s=0} l_{1,O}(\varphi(s), u(s), v(s)),$$

where $\varphi(s)$ is a curve in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ such that $\varphi(0) = \varphi$, $\dot{\varphi}(0) = \dot{\varphi}$; $u(s)$ is the horizontal lift of $\varphi(s)$ to $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)$ through $(\theta, \dot{\theta})$ and $v(s)$ is the horizontal lift of $\varphi(s)$ to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)$ through (θ, ξ) . Then,

$$\begin{aligned} \frac{\delta l_{1,O}}{\delta \theta}(\theta, \dot{\theta}, \xi)(\varphi, \dot{\varphi}) &= \left. \frac{d}{ds} \right|_{s=0} l_{1,O}(\varphi(s), u(s), v(s)) \\ &= \left\langle \frac{\partial l_{1,O}}{\partial \theta}, \dot{\varphi}(s) \right\rangle + \left\langle \frac{\partial l_{1,O}}{\partial \dot{\theta}}, \dot{u}(s) \right\rangle + \left\langle \frac{\partial l_{1,O}}{\partial \xi}, \dot{v}(s) \right\rangle \\ &= \left\langle \frac{\partial l_{1,O}}{\partial \xi}, ((I + K)^{-1}K\dot{\varphi}(0)) \times v(0) \right\rangle \\ &= \langle (I + k)\xi, ((I + K)^{-1}K\dot{\varphi}) \times \xi \rangle, \end{aligned}$$

where we have used that $\partial l_{1,O}/\partial \theta = 0$, $u(s) = \dot{\theta}$ and $\dot{v}(s) = ((I + K)^{-1}K\dot{\varphi}(s)) \times v(s)$. Denoting again the covariant derivative of $l_{1,O}$ as $\partial l_{1,O}/\partial \theta$, we conclude that in (3.25);

$$\frac{\partial l_{1,O}}{\partial \theta}(\dot{\varphi}) = \langle \xi \times ((I + k)\xi), (I + K)^{-1}K\dot{\varphi} \rangle.$$

Since

$$\frac{D}{Dt} \left(\frac{\partial l_{1,O}}{\partial \dot{\theta}} \right) = \langle K(I + K)^{-1}I\ddot{\theta}, \cdot \rangle,$$

and

$$\left\langle \frac{\partial l_{1,0}}{\partial \xi}, \tilde{\mathcal{B}}_\theta(\dot{\theta}, \cdot) \right\rangle = \langle (I + K)\xi, ((I + K)^{-1}K\dot{\theta}) \times ((I + K)^{-1}K\dot{\theta}) \rangle,$$

the horizontal Lagrange-Poincaré equations applied to $\dot{\varphi} \in T_\theta (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)$ is written as

$$\begin{aligned} 0 &= \langle K(I + K)^{-1}I\ddot{\theta}, \dot{\varphi} \rangle - \langle \xi \times ((I + K)\xi), (I + K)^{-1}K\dot{\varphi} \rangle \\ &\quad + \langle (I + K)\xi, ((I + K)^{-1}K\dot{\varphi}) \times ((I + K)^{-1}K\dot{\theta}) \rangle \\ &= \langle I\ddot{\theta}, (I + K)^{-1}K\dot{\varphi} \rangle - \langle \xi \times ((I + K)\xi), (I + K)^{-1}K\dot{\varphi} \rangle \\ &\quad + \langle ((I + K)^{-1}K\dot{\varphi}), ((I + K)^{-1}K\dot{\theta}) \times ((I + K)\xi) \rangle \\ &= \langle I\ddot{\theta} - \xi \times ((I + K)\xi) + ((I + K)^{-1}K\dot{\theta}) \times ((I + K)\xi), (I + K)^{-1}K\dot{\varphi} \rangle \end{aligned}$$

Consequently,

$$I\ddot{\theta} = -((I + K)\xi) \times \Omega. \quad (3.26)$$

The equations obtained in this first step of reduction with the mechanical connection are equivalent to the ones obtained with Euler-Poincaré reduction. In fact, equation (3.6) is the same as equation (3.24) written in terms of the new variable ξ and equation (3.26) is the difference between equations (3.6) and (3.7).

We finally proceed to the second step of reduction. The group $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts on $T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3))$ by isomorphisms in $\mathfrak{L}\mathfrak{P}$. For any $\varphi \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\varphi \cdot (\theta, \dot{\theta}, \Omega) = (\theta + \varphi, \dot{\theta}, \Omega)$. In addition, the connection used to reduce is $\mathcal{A}_\theta(\dot{\theta}) = \dot{\theta} = \Omega_r$ and induces the isomorphism

$$\begin{aligned} T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)) / (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) &\rightarrow \mathbb{R}^3 \oplus \mathfrak{so}(3) \\ [\theta, \dot{\theta}, \xi] &\mapsto (\Omega_r = \dot{\theta}, \xi). \end{aligned}$$

The second reduced space is the Lagrange-Poincaré bundle $\mathbb{R}^3 \times \mathfrak{so}(3)$ over a point with trivial $\nabla^{\mathbb{R}^3}$ and $\omega^{\mathbb{R}^3} = 0$. The covariant derivative in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)$ reduces to a trivial horizontal quotient connection, $[\nabla^{(\mathcal{A}, H)}]_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} \xi = \dot{\xi}$, and a vertical quotient connection, $[\nabla^{(\mathcal{A}, V)}]_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, \Omega_r} \xi = -((I + K)^{-1}K\Omega_r) \times \xi$. Hence, the second reduced space is the Lie algebra $\mathbb{R}^3 \times \mathfrak{so}(3)$ with the Lie bracket $[(\Omega_{r,1}, \xi_1), (\Omega_{r,2}, \xi_2)]^{\mathbb{R}^3}$ whose $\mathbb{R}^3$ component is zero and whose $\mathfrak{so}(3)$ component is

$$\begin{aligned} &[\nabla^{(\mathcal{A}, V)}]_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, \Omega_{r,1}} \xi_2 - [\nabla^{(\mathcal{A}, V)}]_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, \Omega_{r,2}} \xi_1 - [\tilde{\mathcal{B}}]_\theta(\Omega_{r,1}, \Omega_{r,2}) + [\xi_1, \xi_2] = \\ &\xi_2 \times ((I + K)^{-1}K\Omega_{r,1}) - \xi_1 \times ((I + K)^{-1}K\Omega_{r,2}) + ((I + K)^{-1}K\Omega_{r,2}) \times ((I + K)^{-1}K\Omega_{r,1}) + \xi_1 \times \xi_2. \end{aligned}$$

The reduced Lagrangian in $\mathbb{R}^3 \times \mathfrak{so}(3)$ is

$$l_{2,O}(\Omega_r, \xi) = \frac{1}{2} \langle \xi, (I + K)\xi \rangle + \frac{1}{2} \langle K\Omega_r, (I + K)^{-1} I\Omega_r \rangle \quad (3.27)$$

There are clearly no horizontal Lagrange–Poincaré equations, and the vertical equations

$$\frac{D}{Dt} \left(\frac{\partial l_{2,O}}{\partial \Omega_r} \oplus \frac{\partial l_{2,O}}{\partial \xi} \right) = \text{ad}_{\Omega_r, \xi}^* \left(\frac{\partial l_{2,O}}{\partial \Omega_r} \oplus \frac{\partial l_{2,O}}{\partial \Omega} \right) = 0$$

can be written more explicitly as

$$I\dot{\Omega}_r = -((I + K)\xi) \times \Omega, \quad (3.28)$$

$$(I + K)\dot{\xi} = ((I + K)\xi) \times \Omega, \quad (3.29)$$

which are the same as the Lagrange–Poincaré equations obtained in the first step of reduction after the change of variable $\dot{\theta} = \Omega_r$.

3.2.4 Comparison

We have studied different reduction procedures on the free rigid body with rotors. In Subsection 3.2.1, Euler–Poincaré reduction is performed using the whole symmetry group $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. In Subsection 3.2.2 equivalent equations of motion are obtained reducing first by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ and then by $SO(3)$. This shows that the free rigid body with rotors can be interpreted as a free rigid body with a force coming from the curvature term. In Subsection 3.2.3, reduction by stages is carried out in the reverse order using either the Maurer–Cartan connection or the mechanical connection of the system. In the first case the Lagrangian (3.20) is more complicated but the structure of the $\mathfrak{LP}$ -bundles involved is simple, while the mechanical connection simplifies the Lagrangian (3.27) at the cost of complicating the $\mathfrak{LP}$ -bundles involved. A diagram with the different reduction procedures performed above on the rigid body, as well as their equations, is shown in Figure 3.2. Certainly, the equations of motion obtained in each method are equivalent in virtue of Theorem 2.2.14.

However, the Noether currents obtained in each step depends on the method of reduction. When reduction by stages of the Lagrangian L in (3.4) begins with $SO(3)$ -invariance as in Subsection 3.2.3, the associated Noether current is

$$J_{1,O} : T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \mathfrak{so}(3)^* \\ (R, \theta, \dot{R}, \dot{\theta}) \mapsto \left(b \mapsto \left\langle \frac{\partial L}{\partial \dot{R}} \oplus \frac{\partial L}{\partial \dot{\theta}}, b_{(R, \theta)}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)} \right\rangle \right).$$

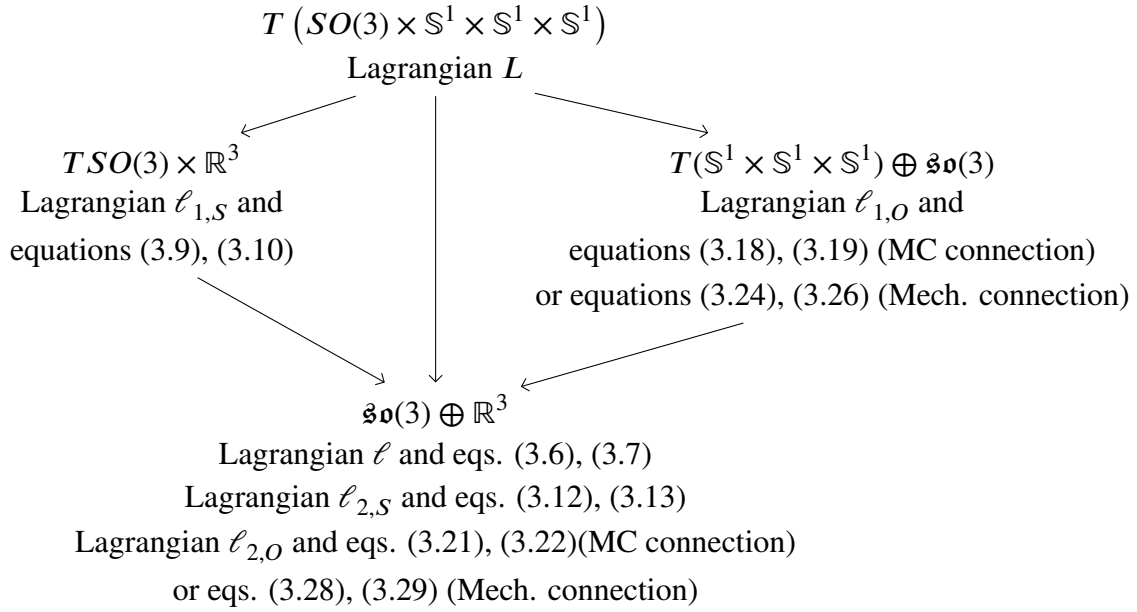


Figure 3.2 Diagram of different reduction procedures on the rigid body with rotors

Since $\partial L / \partial R = R(I + K)R^{-1} \dot{R} + RK\dot{\theta}$ and $b_{(R,\theta)}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times SO(3)} = (R, \theta, bR, 0)$, identification of $\mathfrak{so}(3)$ with its dual gives

$$J_{1,O} : T(T(SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1)) \rightarrow \mathfrak{so}(3) \quad (3.30)$$

$$(R, \theta, \dot{R}, \dot{\theta}) \mapsto R(I + K)R^{-1} \dot{R}R^{-1} + RK\dot{\theta}R^{-1} = R\xi R^{-1}. \quad (3.31)$$

This Noether current can be interpreted as the total momentum of the rigid body with rotors and its conservation is equivalent to vertical equations (3.18) and (3.24). Observe that this current is different $J_{1,S}$ in equation (3.14) obtained in Subsection 3.2.2. In fact, both currents may be obtained in Euler–Poincaré reduction.

The Noether current $J_{1,O}$ does not depend on the connection used in the reduction process. In contrast, the Noether current found in second step of reduction does. In this example, if the Maurer–Cartan connection is used in the first step, the Noether current obtained from the Lagrangian (3.17) is

$$J_{2,MC} : T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)) \rightarrow \mathbb{R}^3$$

$$(\theta, \dot{\theta}, \Omega) \mapsto K\dot{\theta} + K\Omega,$$

after identification of $\mathbb{R}^3$ with its dual. This Noether current is the momentum of the rotors. As the Maurer–Cartan connection is flat and the action of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is horizontal on

$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3) \rightarrow \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, the second term in equation (2.13) is zero and the Noether current is preserved as the system evolves. As determined in Proposition 2.2.20, the conservation of this current is equivalent to the new vertical equation of the second step of reduction, (3.19). On the other hand, if the first reduction is carried using the mechanical connection, the Noether current obtained by the invariance of the Lagrangian (3.23) is

$$J_{2,\text{mech}} : T(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \oplus (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)) \rightarrow \mathbb{R}^3$$

$$(\theta, \dot{\theta}, \xi) \mapsto K(I + K)^{-1} I \dot{\theta}.$$

In this case, neither the connection is flat nor the action of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is horizontal. Hence, the drift of the Noether current is

$$\begin{aligned} \frac{d}{dt} \langle J_{2,\text{mech}}, (\theta, \dot{\theta}, \xi), a \rangle &= - \left\langle \frac{\partial l_{1,O}}{\partial \xi}, \tilde{\mathcal{B}}_{\theta}(\dot{\theta}, a_{\theta}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1}) + a_{\xi}^{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathfrak{so}(3)} \right\rangle \\ &= - \langle (I + K)\xi, ((I + K)^{-1} K a) \times ((I + K)^{-1} K \dot{\theta} - \xi) \rangle \\ &= \langle (I + K)^{-1} K a, \Omega \times (I + K)\xi \rangle. \end{aligned}$$

for every $a \in \mathbb{R}^3$. This drift is equivalent to equation (3.24) which again is the new vertical equation of the second step of reduction.

Further research related to the content of Chapters 2 and 3 may involve the study of techniques of reduction by stages for nonholonomic systems and their possible impact on the control of such systems. It seems particularly interesting to have a discussion in the fashion of this section for a *spherical robot*: a non-holonomic variant of this problem where the rigid body considered is an homogeneous sphere rolling along a plane without slipping and the symmetry group of the system is the product $SO(2) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The control of that system was already tackled in [36, 80].

Chapter 4

The Lagrange–Poincaré Category in Field Theory

4.1 Lagrangian Field Theory in $\text{FT}\mathfrak{LP}$

4.1.1 The $\text{FT}\mathfrak{LP}$ category

We now define the category of bundles such that reduction by stages can be performed in covariant Lagrangian field theory.

Definition 4.1.1 *Given M a manifold called **base space**, the category $\text{FT}\mathfrak{LP}(M)$ of **field theoretical Lagrange–Poincaré bundles** over M is defined as follows:*

1. The **objects** of $\text{FT}\mathfrak{LP}(M)$ are bundles of the form $J^1 P \oplus (T^* M \otimes_P V) \rightarrow P$, where $\pi_{M,P} : P \rightarrow M$ is a fiber bundle (not necessarily principal), $T^* M \rightarrow P$ is an abuse of notation to denote the pullback $\pi_{M,P}^* T^* M \rightarrow P$, and $V \rightarrow P$ is a vector bundle which is the vectorial part of an $\mathfrak{LP}$ -bundle. In other words, $TP \oplus V \rightarrow P$ is an $\mathfrak{LP}$ -bundle, which in turn is equivalent to the existence of

- (a) a Lie bracket, $[\cdot, \cdot]$, in the fibers of V ;
- (b) a V -valued 2-form ω on P ;
- (c) a linear connection, ∇ , on $V \rightarrow P$;
- (d) a Lie bracket operation on the sections $X \oplus w \in \Gamma(TP \oplus V)$ defined by

$$[X_1 \oplus w_1, X_2 \oplus w_2] = [X_1, X_2] \oplus \nabla_{X_1} w_2 - \nabla_{X_2} w_1 - \omega(X_1, X_2) + [w_1, w_2].$$

2. Let $J^1 P_1 \oplus (T^* M \otimes_{P_1} V_1) \rightarrow P_1$ and $J^1 P_2 \oplus (T^* M \otimes_{P_2} V_2) \rightarrow P_2$ be two field theoretical Lagrange–Poincaré bundles over M with structures $[\cdot, \cdot]_i, \nabla_i$ and ω_i on $V_i \rightarrow$

P_i , $i = 1, 2$. A **morphism**, $f : J^1 P_1 \oplus (T^* M \otimes_{P_1} V_1) \rightarrow J^1 P_2 \oplus (T^* M \otimes_{P_2} V_2)$ is a bundle map covering $f_0 : P_1 \rightarrow P_2$ that satisfies

- (a) $f_0 : P_1 \rightarrow P_2$ is a bundle map between $P_1 \rightarrow M$ and $P_2 \rightarrow M$ covering the identity on M ,
- (b) $f(J^1 P_1) \subset J^1 P_2$ and $f|_{J^1 P_1} = j^1 f_0$, the 1-jet extension of f_0 ,
- (c) $f(T^* M \otimes_{P_1} V_1) \subset T^* M \otimes_{P_2} V_2$ and $f|_{T^* M \otimes_{P_1} V_1} = \text{id}_{f_0} \otimes \bar{f}$, where

$$\begin{aligned} \text{id}_{f_0} \otimes \bar{f} : T^* M \otimes_{P_1} V_1 &\rightarrow T^* M \otimes_{P_2} V_2 \\ (y_1, \alpha) \otimes v &\mapsto (f_0(y_1), \alpha) \otimes \bar{f}(v) \end{aligned}$$

$(y_1, \alpha) \in (\pi_{MP}^* T^* M)_{y_1}$ and $\bar{f} : V_1 \rightarrow V_2$ is a vector bundle morphism covering f_0 and commuting with the structures on V_i given by $[\cdot, \cdot]_i, \nabla_i$ and ω_i on $V_i \rightarrow P_i$, $i = 1, 2$. More explicitly, given $w, w' \in (V_1)_{y_1}$, $X, X' \in (TP_1)_{y_1}$ and a curve $v(t)$ in V_1 ;

$$\bar{f}([w, w']_1) = [\bar{f}(w), \bar{f}(w')]_2, \quad (4.1)$$

$$\bar{f}(\omega_1(X, X')) = \omega_2(Tf_0(X), Tf_0(X')), \quad (4.2)$$

$$\bar{f}\left(\frac{D_1 v(t)}{Dt}\right) = \frac{D_2 \bar{f}(v(t))}{Dt}. \quad (4.3)$$

Remark 4.1.2 There are several special cases of objects in $\text{FT}\mathfrak{LP}(M)$ appearing in the present bibliography. For $V = 0$, we obtain 1-jet bundles used in covariant Lagrangian field theory (for example, see [42]). Another instance of object in $\text{FT}\mathfrak{LP}(M)$ is the quotient of a 1-jet bundle $J^1 P$ by a proper and free lifted action of a Lie group G on P , found to be isomorphic to $J^1(P/G) \oplus (T^* M \otimes_{(P/G)} \text{Ad} P)$ in [32]. In the case that $P \rightarrow M$ is G -principal bundle, the quotient $J^1 P/G$ is a $\text{FT}\mathfrak{LP}(M)$ bundle of the form $T^* M \otimes \text{Ad} P$, which is the vector bundle underlying the **affine bundle of connections** used in [18] to perform Euler–Poincaré reduction.

Finally, the particular case when $M = \mathbb{R}$ and $P = \mathbb{R} \times Q$, where Q is a manifold, gives the $\mathfrak{LP}$ bundle

$$J^1(\mathbb{R} \times Q) \oplus (T^* \mathbb{R} \otimes \text{Ad}(\mathbb{R} \times Q)) = T^* \mathbb{R} \otimes (TQ \oplus \text{Ad}(\mathbb{R} \times Q)) \simeq \mathbb{R} \times (TQ \oplus \text{Ad} P)$$

appearing when reducing time-dependent Lagrangians in classical mechanics.

Remark 4.1.3 Field theoretical Lagrange–Poincaré bundles in $\text{FT}\mathfrak{LP}(M)$ can be defined without direct reference to the category $\mathfrak{LP}$ studied on Chapter 2. Let $V \rightarrow P$ be a vector

bundle, provided the objects in 1.(a), 1.(b) and 1.(c), we impose conditions 1.(d'), 1.(e') and 1.(f') in Proposition 2.1.5 instead of condition 1.(d).

Bundles in $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ can be thought as $\mathfrak{L}\mathfrak{P}$ bundles. To that end, we shall define relevant subcategories of $\mathfrak{L}\mathfrak{P}$.

Definition 4.1.4 Given M , a manifold called base space, we define the **subcategory** $\mathfrak{L}\mathfrak{P}(M)$ of $\mathfrak{L}\mathfrak{P}$ whose objects are $\mathfrak{L}\mathfrak{P}$ bundles, $TP \oplus V$, such that P is the total space of a fiber bundle $P \rightarrow M$, and whose morphisms are $\mathfrak{L}\mathfrak{P}$ morphisms, $f = Tf_0 \oplus \bar{f} : TP_1 \oplus V_1 \rightarrow TP_2 \oplus V_2$, such that $f_0 : P_1 \rightarrow P_2$ is a bundle map over M covering id_M .

Proposition 4.1.5 The applications

$$\begin{aligned} \mathcal{F} : \mathfrak{L}\mathfrak{P}(M) &\rightarrow \text{FT}\mathfrak{L}\mathfrak{P}(M) \\ TP \oplus V &\mapsto J^1P \oplus (T^*M \otimes V), \end{aligned}$$

$$\begin{aligned} \mathcal{F} : \text{Hom}(\mathfrak{L}\mathfrak{P}(M)) &\rightarrow \text{Hom}(\text{FT}\mathfrak{L}\mathfrak{P}(M)) \\ Tf_0 \oplus \bar{f} &\mapsto j^1f_0 \oplus (\text{id}_{f_0} \otimes \bar{f}), \end{aligned}$$

where $\text{Hom}(\mathfrak{L}\mathfrak{P}(M))$ denotes the class of morphisms of $\mathfrak{L}\mathfrak{P}(M)$, define a covariant functor with inverse. Thus, the categories $\mathfrak{L}\mathfrak{P}(M)$ and $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ are isomorphic.

Proof. Let $TP \oplus V \in \mathfrak{L}\mathfrak{P}(M)$,

$$\mathcal{F}(\text{id}_{TP \oplus V}) = \mathcal{F}(T\text{id}_P \oplus \text{id}_V) = j^1(\text{id}_P) \oplus (\text{id}_{\text{id}_P} \otimes \text{id}_V) = \text{id}_{\mathcal{F}(TP \oplus V)}.$$

Furthermore, given $f, g \in \text{Hom}(\mathfrak{L}\mathfrak{P}(M))$ such that $g \circ f$ is well defined, it is easy to prove that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ as $T(g \circ f) = Tg \circ Tf$ and $j^1(g \circ f) = j^1g \circ j^1f$. These two properties show that $\mathcal{F}$ is a functor. There is an inverse functor since

$$\begin{aligned} \mathcal{G} : \text{FT}\mathfrak{L}\mathfrak{P}(M) &\rightarrow \mathfrak{L}\mathfrak{P}(M) \\ J^1P \oplus (T^*M \otimes V) &\mapsto TP \oplus V, \end{aligned}$$

$$\begin{aligned} \mathcal{G} : \text{Hom}(\text{FT}\mathfrak{L}\mathfrak{P}(M)) &\rightarrow \text{Hom}(\mathfrak{L}\mathfrak{P}(M)) \\ j^1f_0 \oplus (\text{id}_{f_0} \otimes \bar{f}) &\mapsto Tf_0 \oplus \bar{f}, \end{aligned}$$

is well defined, $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathfrak{L}\mathfrak{P}(M)}$, and $\mathcal{F} \circ \mathcal{G} = \text{id}_{\text{FT}\mathfrak{L}\mathfrak{P}(M)}$. □

Corollary 4.1.6 *The following three statements are equivalent: $j^1 f_0 \oplus (\text{id}_{f_0} \otimes \bar{f})$ is an isomorphism; $T f_0 \oplus \bar{f}$ is an isomorphism; and $\bar{f}$ is an isomorphism.*

Proof. The first two statements are equivalent since $\mathcal{F}$ is an isomorphism of categories. The third statement is equivalent to the others. Indeed, as $\bar{f}$ is a vector bundle morphism covering f_0 , $\bar{f}$ fully determines both, $j^1 f_0 \oplus (\text{id}_{f_0} \otimes \bar{f})$ and $T f_0 \oplus \bar{f}$. $\square$

We shall define the notion of an action of a group G on an object of $\text{FT}\mathfrak{L}\mathfrak{P}(M)$.

Definition 4.1.7 *An action in the category $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ of a group G on an object $J^1 P \oplus (T^* M \otimes V)$ of $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ is a differentiable action*

$$\Phi : G \times (J^1 P \oplus (T^* M \otimes V)) \rightarrow J^1 P \oplus (T^* M \otimes V)$$

such that for each $g \in G$, $\Phi_g : J^1 P \oplus (T^* M \otimes V) \rightarrow J^1 P \oplus (T^* M \otimes V)$ is an isomorphism in $\text{FT}\mathfrak{L}\mathfrak{P}(M)$. We will say that this action is free and proper if the induced action on P by the functions $(\Phi_g)_0$ is free and proper.

Proposition 4.1.8 *Let $J^1 P \oplus (T^* M \otimes_P V)$ be an object of $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ and let $[\cdot, \cdot]$, ∇ and ω be the structure in V . Let G be a Lie group acting freely and properly on $J^1 P \oplus (T^* M \otimes_P V)$ and $\mathcal{A}$ a connection on the principal bundle $P \rightarrow \Sigma = P/G$. The bundle*

$$J^1 \Sigma \oplus (T^* M \otimes_\Sigma (\text{Ad} P \oplus (V/G)))$$

with the structure $[\cdot, \cdot]^{\tilde{\mathfrak{g}}}$, $\nabla^{\tilde{\mathfrak{g}}}$ and $\omega^{\tilde{\mathfrak{g}}}$ on $\text{Ad} P \oplus V$ as in Proposition 2.1.7 is an object in $\text{FT}\mathfrak{L}\mathfrak{P}(M)$ diffeomorphic to $(J^1 P \oplus (T^* M \otimes_P V))/G$ via the bundle diffeomorphism

$$\begin{aligned} \alpha_{\mathcal{A}} : (J^1 P \oplus (T^* M \otimes V)) / G &\rightarrow J^1 \Sigma \oplus (T^* M \otimes (\text{Ad} P \oplus (V/G))) \\ [j_x^1 \rho \oplus ((y, \alpha), v)]_G &\mapsto \left(T_{\pi_{\Sigma, P}} \circ j_x^1 \rho, [y, \rho^* \mathcal{A}]_G \oplus ((s, \alpha), [v]_G) \right), \end{aligned} \quad (4.4)$$

where $y \in P$, $s = \pi_{\Sigma, P}(y) \in \Sigma$, $x = \pi_{M, P}(y) \in M$ and $j_x^1 \rho \in J_y^1 P$.

Proof. As G acts on $J^1 P \oplus (T^* M \otimes V)$, for each $g \in G$ there exists an isomorphism

$$\Phi_g = j^1 \left((\Phi_g)_0 \right) \oplus \left(\text{id}_{(\Phi_g)_0} \otimes \bar{\Phi}_g \right) : J^1 P \oplus (T^* M \otimes V) \rightarrow J^1 P \oplus (T^* M \otimes V)$$

of the category $\text{FT}\mathfrak{L}\mathfrak{P}(M)$. Therefore, by Corollary 4.1.6, G acts on $TP \oplus V = \mathcal{F}^{-1}(J^1 P \oplus (T^* M \otimes V))$ with isomorphisms

$$\mathcal{F}^{-1}(\Phi_g) = T(\Phi_g)_0 \oplus \bar{\Phi}_g : TP \oplus V \rightarrow TP \oplus V.$$

From Theorem 2.1.7, the quotient of $TP \oplus V$ by this action using connection $\mathcal{A}$ is isomorphic to another $\mathfrak{L}\mathfrak{P}$ bundle, $T\Sigma \oplus \text{Ad}P \oplus (V/G)$ with structures $[\cdot]^\mathfrak{A}$, $\nabla^\mathfrak{A}$ and $\omega^\mathfrak{A}$. This bundle happens to be in $\mathfrak{L}\mathfrak{P}(M)$ since $\Sigma \rightarrow M$, then, from Proposition 4.1.5

$$\mathcal{F}(T\Sigma \oplus \text{Ad}P \oplus (V/G)) = J^1\Sigma \oplus (T^*M \otimes (\text{Ad}P \oplus (V/G)))$$

is isomorphic to $(J^1P \oplus (T^*M \otimes_P V))/G$. Finally, it can be checked that $\alpha_{\mathcal{A}}$ is well defined and that

$$\begin{aligned} \alpha_{\mathcal{A}}^{-1} : J^1\Sigma \oplus (T^*M \otimes (\text{Ad}P \oplus (V/G))) &\rightarrow (J^1P \oplus (T^*M \otimes V)) / G \\ \beta_s \oplus \ell_s \oplus ((s, \alpha), [v]_G) &\mapsto [\text{hor}_{\mathcal{A}}^V \circ \beta_s + \zeta_y \circ \ell_s] \oplus ((y, \alpha), v_y)]_G, \end{aligned}$$

where $v_y \in \pi_{P,V}^{-1}(y) = V_y$ such that $[v_y]_G = [v]_G$ and $\zeta_y : (\text{Ad}P)_s \rightarrow V_y P$ defined by $\zeta_y([y, \xi]_G) = \xi_y^P$. $\square$

4.1.2 Variational problems in FT $\mathfrak{L}\mathfrak{P}$ bundles

Lagrangian field theory can be defined on FT $\mathfrak{L}\mathfrak{P}$ bundle as described below.

Definition 4.1.9 A *Lagrangian density* on $J^1P \oplus (T^*M \otimes_P V) \in \text{FT}\mathfrak{L}\mathfrak{P}$ is a smooth fiber map

$$\mathcal{L} : J^1P \oplus (T^*M \otimes_P V) \rightarrow \bigwedge^n TM,$$

where n the dimension of M . We will assume that M is orientable and we choose a volume form $v \in \bigwedge^n TM$, then, the Lagrangian density can be expressed as $\mathcal{L} = Lv$, where $L : J^1P \oplus (T^*M \otimes_P V) \rightarrow \mathbb{R}$ is a *Lagrangian function*.

Let $U \subset M$ be an open subset whose closure $\bar{U}$ is compact. We will only consider smooth sections $j^1\rho \oplus v : \bar{U} \rightarrow J^1P \oplus (T^*M \otimes_P V)$ such that $v \in \Gamma(\bar{U}, T^*M \otimes_P V)$ projects to a section $\rho = \pi_{P, T^*M \otimes_P V} \circ v \in \Gamma(\bar{U}, P)$, the 1-jet extension of which is $j^1\rho$. These sections are called *allowed sections*. We say that $j^1\rho \oplus v \in \Gamma(\bar{U}, J^1P \oplus (T^*M \otimes_P V))$ is a *critical section* for the variational problem defined by $\mathcal{L}$ if

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_U \mathcal{L}(j^1\rho_\varepsilon \oplus v_\varepsilon) = 0 \quad (4.5)$$

for all smooth *allowed variations* $j^1\rho_\varepsilon \oplus v_\varepsilon$. However, to define the set of allowed variations we shall first introduce a connection on $T^*M \otimes_P V$ from a connection ∇^M on $TM \rightarrow M$ and a connection ∇ on $V \rightarrow P$. A similar process was described in Subsection 1.3.2: given

$v \in \Gamma(P, T^*M \otimes_P V)$, $X \in \mathfrak{X}(P)$ and $u \in \mathfrak{X}(U)$, the connection ∇^L on $T^*M \otimes_P V \rightarrow P$ is given by

$$(\nabla_X^L v)(u) = \nabla_X(v(u)) - v\left(\nabla_{T\pi_{M,P}(X)}^M u\right). \quad (4.6)$$

Definition 4.1.10 An *allowed variation* of an allowed section, $j^1\rho \oplus v : \bar{U} \rightarrow J^1P \oplus (T^*M \otimes_P V)$, is a smooth map $j^1\rho(x, \varepsilon) \oplus v(x, \varepsilon) : \bar{U} \times I \rightarrow J^1P \oplus (T^*M \otimes_P V)$, where I is an open interval with $0 \in I$, such that:

1. for all $\varepsilon \in I$,

$$j^1\rho_\varepsilon(x) \oplus v_\varepsilon(x) : \bar{U} \rightarrow J^1P \oplus (T^*M \otimes_P V)$$

is an allowed section and $j^1\rho_\varepsilon \oplus v_\varepsilon|_{\partial U} = j^1\rho \oplus v|_{\partial U}$;

2. for $\varepsilon = 0$, $j^1\rho_\varepsilon(x) \oplus v_\varepsilon(x) = j^1\rho(x) \oplus v(x)$;

3. the variation of v is of the form

$$\delta v \equiv \left. \frac{D^{\nabla^L} v_\varepsilon}{D\varepsilon} \right|_{\varepsilon=0} = \tilde{\nabla}\mu - [\mu, v] + \rho^*(i_{\delta\rho}\omega), \quad (4.7)$$

where $\tilde{\nabla}$ is the ∇ -derivative on $(V, \nabla) \rightarrow P \rightarrow M$, ω is the V -valued 2-form on P of condition 1.(b), and $\mu \in \Gamma(\bar{U}, V)$ is an arbitrary section such that $\pi_{P,V} \circ \mu = \rho$ and $\mu|_{\partial U} = 0$.

Remark 4.1.11 Let $j^1\rho_\varepsilon \oplus v_\varepsilon$ be an allowed variation of $j^1\rho \oplus v$, the variation of $\rho_\varepsilon = \pi_{P, T^*M \otimes_P V} \circ v_\varepsilon$ is vertical in the sense that

$$\delta\rho(x) \equiv \left. \frac{d\rho_\varepsilon(x)}{d\varepsilon} \right|_{\varepsilon=0} \in V_{\rho(x)}P = \ker(T_{\rho(x)}\pi_{M,P}).$$

Consequently, $\delta j^1\rho(x) \in V_{\rho(x)}J^1P = \ker(T_{j^1\rho(x)}\pi_{M, J^1P})$.

Remark 4.1.12 Since $\rho = \pi_{P, T^*M \otimes_P V} \circ v$, an allowed section $(j^1\rho, v)$ is completely defined by $v \in \Gamma(\bar{U}, T^*M \otimes_P V)$. An analogous result was found in Subsection 2.2.1 concerning the category $\mathfrak{L}\mathfrak{P}$, where a curve $v(t)$ in V fully determines an allowed curve $\dot{q}(t) \oplus v(t)$ in $TQ \oplus V$.

Remark 4.1.13 Given $u_x \in T_x M$,

$$\delta v(u_x) \equiv \left. \frac{D^{\nabla^L} v_\varepsilon}{D\varepsilon} \right|_{\varepsilon=0}(u_x) = \left. \frac{D^{\nabla} v_\varepsilon(u_x)}{D\varepsilon} \right|_{\varepsilon=0} - v\left(\nabla_{T\pi_{M,P}(\delta\rho)}^M u\right) = \left. \frac{D^{\nabla} v_\varepsilon(u_x)}{D\varepsilon} \right|_{\varepsilon=0},$$

where u is a local section of $TM \rightarrow M$ around x such that $u(x) = u_x$ and the last equivalence is a consequence of $\delta\rho$ being vertical. Thus, the variation δv does not depend on the connection ∇^M on $TM \rightarrow M$.

We now find the variational equations defined by these sets of allowed sections and variations. For any smooth function $L : J^1P \oplus (T^*M \otimes_P V) \rightarrow \mathbb{R}$, the fiber derivatives are defined as

$$\left\langle \frac{\delta L}{\delta j^1\rho}(j^1\rho \oplus v), \alpha \right\rangle = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L((j^1\rho + \varepsilon\alpha) \oplus v); \quad (4.8)$$

$$\left\langle \frac{\delta L}{\delta v}(j^1\rho \oplus v), \beta \right\rangle = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L(j^1\rho \oplus (v + \varepsilon\beta)), \quad (4.9)$$

where $\alpha \in T^*M \otimes VP$ and $\beta \in T^*M \otimes V$. Therefore,

$$\frac{\delta L}{\delta j^1\rho}(j^1\rho \oplus v) \in TM \otimes VP^*, \quad \frac{\delta L}{\delta v}(j^1\rho \oplus v) \in TM \otimes V^*,$$

and if we compose them with a section $j^1\rho(x) \oplus v(x) \in \Gamma(\bar{U}, J^1P \oplus (T^*M \otimes V))$ we obtain $\delta L/\delta j^1\rho(x)$ in $\Gamma(\bar{U}, TM \otimes VP^*)$ and $\delta L/\delta v(x)$ in $\Gamma(\bar{U}, TM \otimes V^*)$.

As seen in the last paragraph of Subsection 1.3.1, an affine connection ∇^{J^1P} on $J^1P \rightarrow P$ is obtained from a projectable linear connection ∇^P on $TP \rightarrow P$. In addition, we define an affine connection, $\nabla^{J^1P} \oplus \nabla^L$, on $J^1P \oplus (T^*M \otimes_P V)$. Therefore, we define an horizontal derivative,

$$\left\langle \frac{\delta L}{\delta \rho}(j^1\rho \oplus v), X_y \right\rangle = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L(\zeta_{j^1\rho \oplus v}^h(\varepsilon)); \quad (4.10)$$

where $X_y \in T_yP$ and $\zeta(\varepsilon)$ is a curve in P such that $\dot{\zeta}(0) = X_y$ and $\zeta_{j^1\rho \oplus v}^h(\varepsilon)$ is the horizontal lift of $\zeta(\varepsilon)$ to $(J^1P \oplus (T^*M \otimes V))$ at $j^1\rho \oplus v$ using the connection $\nabla^{J^1P} \oplus \nabla^L$. Observe that,

$$\frac{\delta L}{\delta \rho}(j^1\rho \oplus v) \in T^*P.$$

We will also need a general notion of divergence of fields with values in a vector bundle.

Definition 4.1.14 Let $E \rightarrow P$ be a vector bundle with affine connection ∇ and let $P \rightarrow M$ be a fiber bundle, the **divergence** of $\chi \in \Gamma(M, TM \otimes E^*)$ is $\text{div}^\nabla \chi \in \Gamma(M, E^*)$ such that, for any $\eta \in \Gamma(M, E)$,

$$\text{div} \langle \xi, \eta \rangle = \langle \text{div}^\nabla \chi, \eta \rangle + \langle \chi, \tilde{\nabla} \eta \rangle,$$

where div is the divergence of a real vector field in M with respect to a volume form v .

In this Chapter, we will use the operators

$$\text{div}^\nabla : \Gamma(\bar{U}, TM \otimes V^*) \rightarrow \Gamma(\bar{U}, V^*);$$

$$\operatorname{div}^P : \Gamma(\bar{U}, TM \otimes VP^*) \rightarrow \Gamma(\bar{U}, VP^*);$$

respectively induced by the connection ∇ on $V \rightarrow P$, and the restriction of the connection ∇^P to $VP \subset TP$.

Finally, we define the *coadjoint operator* in this context as

$$\begin{aligned} \operatorname{ad}^* : \Gamma(\bar{U}, T^*M \otimes V) \times \Gamma(\bar{U}, TM \otimes V^*) &\rightarrow \Gamma(\bar{U}, V^*) \\ (v_1, v_2) &\mapsto \left(\mu \mapsto \operatorname{ad}_{v_1} v_2(\mu) = \langle v_2, [v_1, \mu] \rangle \right) \end{aligned}$$

for all $\mu \in \Gamma(\bar{U}, V)$.

Proposition 4.1.15 *Let $J^1P \oplus (T^*M \otimes V)$ be a $\text{FT}\mathfrak{L}\mathfrak{P}$ bundle with a Lagrangian density $\mathcal{L} : J^1P \oplus (T^*M \otimes V) \rightarrow \bigwedge^n TM$ and a volume form $v \in \bigwedge^n TM$, such that $\mathcal{L} = Lv$. Let ∇^P be a linear connection on $TP \rightarrow P$. Then, an allowed section $j^1\rho \oplus v \in \Gamma(\bar{U}, J^1P \oplus (T^*M \otimes V))$ is critical for the variational problem defined by $\mathcal{L}$ if and only if it satisfies the Lagrange–Poincaré equations:*

$$\operatorname{ad}^* \frac{\delta L}{\delta v} - \operatorname{div}^\nabla \frac{\delta L}{\delta v} = 0, \quad (4.11)$$

$$\frac{\delta L}{\delta \rho} - \operatorname{div}^P \frac{\delta L}{\delta j^1\rho} - \left\langle \frac{\delta L}{\delta j^1\rho}, i_{T_\rho} T^P \right\rangle = \left\langle \frac{\delta L}{\delta v}, i_{T_\rho} \omega \right\rangle, \quad (4.12)$$

where T^P is the torsion tensor of connection ∇^P .

Remark 4.1.16 *Since connection ∇^P is arbitrary, it is always to choose a connection without torsion and remove this term from equation (4.12).*

Proof. Using the derivatives defined by equations (4.8), (4.9) and (4.10), we rewrite the derivative of the action as;

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_U \mathcal{L}(j^1\rho_\varepsilon \oplus v_\varepsilon) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_U L(j^1\rho_\varepsilon \oplus v_\varepsilon)v = \\ &= \int_U \left\langle \frac{\delta L}{\delta \rho}(x), \left. \frac{d\rho_\varepsilon(x)}{d\varepsilon} \right|_{\varepsilon=0} \right\rangle v + \int_U \left\langle \frac{\delta L}{\delta j^1\rho}(x), \left. \frac{D^{\nabla^{J^1P}} j^1\rho_\varepsilon(x)}{D\varepsilon} \right|_{\varepsilon=0} \right\rangle v \\ &\quad + \int_U \left\langle \frac{\delta L}{\delta v}(x), \left. \frac{D^L v_\varepsilon(x)}{D\varepsilon} \right|_{\varepsilon=0} \right\rangle v. \end{aligned}$$

Since the variation of ρ is vertical and ∇^P is projectable,

$$\left. \frac{D^{\nabla^{J^1 P}} j^1 \rho_\varepsilon(x)}{D\varepsilon} \right|_{\varepsilon=0} (u_x) = \left. \frac{D^{\nabla^P} j^1 \rho_\varepsilon(x)(u_x)}{D\varepsilon} \right|_{\varepsilon=0},$$

for all $u_x \in T_x M$. Consider $\rho_\varepsilon(\gamma(t))$, where $\gamma(t)$ is a curve such that $\dot{\gamma}(0) = u_x$, then

$$\frac{D^{\nabla^P}}{D\varepsilon} \frac{d}{dt} \rho_\varepsilon(\gamma(t)) - \frac{D^{\nabla^P}}{Dt} \frac{d}{d\varepsilon} \rho_\varepsilon(\gamma(t)) = T^P \left(\frac{d}{d\varepsilon} \rho_\varepsilon(\gamma(t)), \frac{d}{dt} \rho_\varepsilon(\gamma(t)) \right),$$

and we conclude that

$$\left. \frac{D^{\nabla^{J^1 P}} j^1 \rho_\varepsilon(x)}{D\varepsilon} \right|_{\varepsilon=0} (u_x) = \tilde{\nabla}_{u_x}^P \delta \rho(x) + T^P(\delta \rho(x), T_x \rho(u_x)). \quad (4.13)$$

After substitution of equations (4.7) and (4.13), the derivative of the action is

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_U \mathcal{L}(j^1 \rho_\varepsilon \oplus v_\varepsilon) &= \int_U \left\langle \frac{\delta L}{\delta \rho}(x), \delta \rho(x) \right\rangle v + \int_U \left\langle \frac{\delta L}{\delta j^1 \rho}(x), \tilde{\nabla}^P \delta \rho(x) + T^P(\delta \rho(x), T \rho) \right\rangle v \\ &\quad + \int_U \left\langle \frac{\delta L}{\delta v}(x), \tilde{\nabla} \mu - [\mu, v] + \omega(\delta \rho(x), T \rho) \right\rangle v \\ &= \int_U \left\langle \frac{\delta L}{\delta \rho}(x) - \operatorname{div}^P \frac{\delta L}{\delta j^1 \rho}(x) - \left\langle \frac{\delta L}{\delta j^1 \rho}, i_{T \rho} T^P \right\rangle - \left\langle \frac{\delta L}{\delta \mu}, i_{T \rho} \omega \right\rangle, \delta \rho(x) \right\rangle v \\ &\quad + \int_U \left\langle -\operatorname{div}^\nabla \frac{\delta L}{\delta v}(x) + \operatorname{ad}_v, \mu \right\rangle v, \end{aligned}$$

where $\delta \rho|_{\partial U} = 0$ and $\mu|_{\partial U} = 0$ have been used to prove the second identity. Thus, $j^1 \rho(x) \oplus v(x)$ is critical if and only if the Lagrange–Poincaré equations are satisfied. $\square$

4.2 Lagrangian Reduction in FTQPB

4.2.1 Reduction by Stages in FTQPB

We shall see that a symmetric Lagrangian field theory defined on a bundle in FTQPB can be reduced. In fact, reduction by stages can be performed.

Let $J^1 P \oplus (T^* M \otimes V)$ be an object in FTQPB, G a Lie group acting freely and properly on $J^1 P \oplus (T^* M \otimes V)$ and $\mathcal{A}$ a connection on $P \rightarrow \Sigma = P/G$. We recall from Subsections

1.1.3 and 2.1.2 that the connection ∇ on $V \rightarrow P$ induces an affine connection $[\nabla^{(\mathcal{A}, H)}]_{G, Y}$ on $V/G \rightarrow \Sigma$, for $Y \in \mathfrak{X}(\Sigma)$. Hence, there is a $[\nabla^{(\mathcal{A}, H)}]_G$ -derivative on $V/G \rightarrow \Sigma \rightarrow M$ denoted as $[\tilde{\nabla}^{(\mathcal{A}, H)}]_G$. On the contrary, the vertical component of the reduced connection, $[\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}$, is not a connection since we derive with respect to $\bar{\xi} \in \Gamma(\text{Ad}P)$ instead of vector fields on Σ . Yet, given a section $[w]$ of $V/G \rightarrow M$, we define for all $x \in M$

$$[\nabla^{(\mathcal{A}, V)}]_{G, \bar{\xi}}[w](x) = [\xi_v^V]_G(x),$$

where $v \in V$ such that $\pi_{V/G, V}(v) = [w](x)$, and $\bar{\xi} = [y, \xi]_G$ with $y = \pi_{V, P}(v)$. This is well defined since $\bar{\xi} = [gy, \text{Ad}_g \xi]_G$ and $g\xi_v^V = (\text{Ad}_g \xi)_{gv}^V$.

Proposition 4.2.1 *Let $J^1 P \oplus (T^* M \otimes V)$ be an object in $\text{FT}\mathfrak{Q}\mathfrak{P}$, G a Lie group acting freely and properly on $J^1 P \oplus (T^* M \otimes V)$, ν a volume form on M and $L : J^1 P \oplus (T^* M \otimes V) \rightarrow \mathbb{R}$ a G -invariant Lagrangian. Consider a connection $\mathcal{A}$ on $P \rightarrow \Sigma = P/G$ and*

$$l : J^1 \Sigma \oplus (T^* M \otimes (\text{Ad}P \oplus (V/G))) \rightarrow \mathbb{R}$$

*the **reduced Lagrangian** induced in the quotient via the identification (4.4). Given a smooth local section $j^1 \rho \oplus \nu \in \Gamma(\bar{U}, J^1 P \oplus (T^* M \otimes V))$, we define the **reduced local section***

$$j^1 \varsigma \oplus \bar{\rho} \oplus [\nu]_G = \alpha_{\mathcal{A}} \circ \pi_G(j^1 \rho \oplus \nu),$$

where $\varsigma = \pi_{\Sigma, P} \circ \rho \in \Gamma(\bar{U}, \Sigma)$ and $\bar{\rho} = [\rho, \rho^* \mathcal{A}]_G$. Then, the following are equivalent:

- (i) *Section $j^1 \rho \oplus \nu \in \Gamma(\bar{U}, J^1 P \oplus (T^* M \otimes V))$ is critical for the variational problem defined by L in $J^1 P \oplus (T^* M \otimes V)$.*
- (ii) *Section $j^1 \rho \oplus \nu \in \Gamma(\bar{U}, J^1 P \oplus (T^* M \otimes V))$ satisfies the Lagrange–Poincaré equations given by L in $J^1 P \oplus (T^* M \otimes V)$.*
- (iii) *Section $j^1 \varsigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1 \Sigma \oplus (T^* M \otimes (\text{Ad}P \oplus (V/G))))$ is critical for the variational problem defined by l in $J^1 \Sigma \oplus (T^* M \otimes (\text{Ad}P \oplus (V/G)))$.*
- (iv) *Section $j^1 \varsigma \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1 \Sigma \oplus (T^* M \otimes (\text{Ad}P \oplus (V/G))))$ satisfies the Lagrange–Poincaré equations given by l in $J^1 \Sigma \oplus (T^* M \otimes (\text{Ad}P \oplus (V/G)))$.*

Proof. From Proposition 4.1.15, statements (i) and (ii) are equivalent. For the same reason statements (iii) and (iv) are equivalent. We will prove the result by checking that statements (i) and (iii) are equivalent.

Let $j^1\rho_\varepsilon \oplus v_\varepsilon$ be an allowed variation of $j^1\rho \oplus v$ and

$$j^1\zeta_\varepsilon \oplus \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon} = \alpha_{\mathcal{A}} \circ \pi_G(j^1\rho_\varepsilon \oplus v_\varepsilon),$$

the projection of the allowed variation in $J^1P \oplus (T^*M \otimes V)$. Since

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_U L(j^1\rho_\varepsilon \oplus v_\varepsilon) v &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_U l(j^1\zeta_\varepsilon \oplus \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}) v = \\ &= \int_U \left\langle \frac{\delta l}{\delta \zeta}(x), \frac{d\zeta_\varepsilon(x)}{d\varepsilon} \Big|_{\varepsilon=0} \right\rangle v + \int_U \left\langle \frac{\delta l}{\delta j^1\zeta}(x), \frac{D^{\nabla^{j^1\zeta}} j^1\zeta_\varepsilon(x)}{D\varepsilon} \Big|_{\varepsilon=0} \right\rangle v \\ &\quad + \int_U \left\langle \frac{\delta l}{\delta(\bar{\rho} \oplus [v]_G)}(x), \frac{D^L \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}(x)}{D\varepsilon} \Big|_{\varepsilon=0} \right\rangle v, \end{aligned}$$

the variational problem defined by L on $J^1P \oplus (T^*M \otimes V)$ is equivalent to consider variations of $j^1\zeta \oplus \bar{\rho} \oplus [v]_G$ obtained by projecting allowed variations of $j^1\rho \oplus v$. Then, for all $u_x \in T_x M$,

$$\begin{aligned} \frac{D^L \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}}{D\varepsilon} \Big|_{\varepsilon=0} (u_x) &= \frac{D^{\tilde{\mathfrak{g}}} \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}(u_x)}{D\varepsilon} \Big|_{\varepsilon=0} \\ &= \nabla_{\delta\zeta}^{\tilde{\mathfrak{g}}} \bar{\rho}_\varepsilon(u_x) \oplus ([\nabla^{(\mathcal{A},H)}]_{G,\delta\zeta}[v]_G(u_x) - [\omega]_G(\delta\zeta, \bar{\rho}_\varepsilon(u_x))) \\ &= \nabla_{\delta\zeta}^{\tilde{\mathfrak{g}}} \bar{\rho}_\varepsilon(u_x) \oplus ([\nabla^{(\mathcal{A})}]_{G,\delta\rho}[v]_G(u_x) - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}}[v]_G(u_x) - [\omega]_G(\delta\zeta, \bar{\rho}_\varepsilon(u_x))), \end{aligned}$$

where $\bar{\eta}$ is such that $\delta\rho = \delta\zeta \oplus \bar{\eta}$. From Lagrange–Poincaré reduction in jet bundles (see [32, Corollary 3.2]), we know that

$$\nabla_{\delta\zeta}^{\tilde{\mathfrak{g}}} \bar{\rho}_\varepsilon(u_x) = \tilde{\nabla}_{u_x} \bar{\eta} - [\bar{\eta}, \bar{\rho}(u_x)] + \tilde{\mathcal{B}}(\delta\zeta, T\zeta(u_x)).$$

On the other hand, as δv is an allowed variation

$$[\nabla^{(\mathcal{A})}]_{G,\delta\rho}[v]_G(u_x) = [\delta v(u_x)]_G = \left[\tilde{\nabla}_{u_x} \mu - [\mu, v(u_x)] + \omega(\delta\rho, T\rho(u_x)) \right]_G.$$

We rewrite this class in an alternative way. From the vertical component of the reduced connection, we define $[\tilde{\nabla}^{(\mathcal{A},V)}]_{G,u_x}[\mu]_G = [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}(u_x)}[\mu]_G$, in analogy to ∇ -derivatives. Then,

$$\begin{aligned} [\tilde{\nabla}_{u_x} \mu]_G &= [\nabla_{T\rho(u_x)} \mu]_G = [\nabla_{T\zeta(u_x) \oplus \bar{\rho}(u_x)} \mu]_G = [\nabla^{(\mathcal{A},H)}]_{G,T\zeta(u_x)}[\mu]_G + [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}(u_x)}[\mu]_G \\ &= [\tilde{\nabla}^{(\mathcal{A},H)}]_{G,u_x}[\mu]_G + [\tilde{\nabla}^{(\mathcal{A},V)}]_{G,u_x}[\mu]_G. \end{aligned}$$

We also have that $[\mu, v(u_x)]_G = [\mu]_G, [v(u_x)]_G$, and

$$\begin{aligned} [\omega(\delta\rho, T\rho(u_x))]_G &= [\omega(\delta\zeta \oplus \bar{\eta}, T\zeta(u_x) \oplus \bar{\rho}(u_x))]_G \\ &= [\omega]_G(\delta\zeta, T\zeta(u_x)) + [\omega]_G(\bar{\eta}, T\zeta(u_x)) + [\omega]_G(\delta\zeta, \bar{\rho}(u_x)) + [\omega]_G(\bar{\eta}, \bar{\rho}(u_x)). \end{aligned}$$

In conclusion, for projected allowed variations

$$\begin{aligned} \left. \frac{D^L \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}}{D\varepsilon} \right|_{\varepsilon=0} (u_x) &= \left(\tilde{\nabla}_{u_x} \bar{\eta} - [\bar{\eta}, \bar{\rho}(u_x)] + \tilde{\mathcal{B}}(\delta\zeta, T\zeta(u_x)) \right) \oplus \left([\tilde{\nabla}^{(\mathcal{A},H)}]_{G,u_x} [\mu]_G \right. \\ &\quad + [\tilde{\nabla}^{(\mathcal{A},V)}]_{G,u_x} [\mu]_G - [\mu]_G, [v(u_x)]_G - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}} [v]_G(u_x) \\ &\quad \left. + [\omega]_G(\delta\zeta, T\zeta(u_x)) + [\omega]_G(\bar{\eta}, T\zeta(u_x)) + [\omega]_G(\bar{\eta}, \bar{\rho}(u_x)) \right), \end{aligned}$$

where $\bar{\eta} \in \Gamma(\bar{U}, \text{Ad}P)$ such that $\pi_{\Sigma, \text{Ad}P}(\bar{\eta}) = \zeta$, $\bar{\eta}|_{\partial U} = 0$ and $[\mu]_G \in \Gamma(\bar{U}, V/G)$ such that $\pi_{\Sigma, V/G}([\mu]_G) = \zeta$, $[\mu]_G|_{\partial U} = 0$. To conclude, we shall see that these variations coincide with the allowed variations in $J^1\Sigma \oplus (T^*M \otimes (\text{Ad}P \oplus (V/G)))$. Any allowed variation of section $j^1\zeta \oplus \bar{\rho} \oplus [v]_G$ is of the kind

$$\left. \frac{D^L \bar{\rho}_\varepsilon \oplus [v]_{G,\varepsilon}}{D\varepsilon} \right|_{\varepsilon=0} (u_x) = \tilde{\nabla}_{u_x}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [\mu]_G) - [\bar{\eta} \oplus [\mu]_G, \bar{\rho} \oplus [v]_G(u_x)]_G^{\tilde{\mathfrak{g}}} + \omega^{\tilde{\mathfrak{g}}}(\delta\zeta, T\zeta u_x)$$

where $\bar{\eta} \oplus [\mu]_G \in \Gamma(\bar{U}, \text{Ad}P \oplus (V/G))$ such that $\pi_{\Sigma, \text{Ad}P \oplus (V/G)}(\bar{\eta} \oplus [\mu]_G) = \zeta$, and $\bar{\eta} \oplus [\mu]_G|_{\partial U} = 0$. The additional structure in $\text{Ad}P \oplus (V/G)$ is the one detailed in Theorem 2.1.7. Hence,

$$\begin{aligned} \tilde{\nabla}_{u_x}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [\mu]_G) &= \nabla_{T\zeta(u_x)}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [\mu]_G) \\ &= \nabla_{T\zeta(u_x)} \bar{\eta} \oplus \left([\nabla^{(\mathcal{A},H)}]_{G, T\zeta(u_x)} [\mu]_G - [\omega]_G(T\zeta(u_x), \bar{\eta}) \right) \\ &= \tilde{\nabla}_{u_x} \bar{\eta} \oplus \left([\tilde{\nabla}^{(\mathcal{A},H)}]_{G, u_x} [\mu]_G + [\omega]_G(\bar{\eta}, T\zeta(u_x)) \right). \end{aligned}$$

In addition,

$$\begin{aligned} [\bar{\eta} \oplus [\mu]_G, \bar{\rho} \oplus [v]_G(u_x)]_G^{\tilde{\mathfrak{g}}} &= [\bar{\eta}, \bar{\rho}(u_x)] \oplus \left([\nabla^{(\mathcal{A},V)}]_{G,\bar{\eta}} [v]_G(u_x) \right. \\ &\quad \left. - [\nabla^{(\mathcal{A},V)}]_{G,\bar{\rho}(u_x)} [\mu]_G - [\omega]_G(\bar{\eta}, \bar{\rho}(u_x)) + [\mu]_G, [v(u_x)]_G \right), \end{aligned}$$

and

$$\omega^{\tilde{\mathfrak{g}}}(\delta\zeta, T\zeta(u_x)) = \tilde{\mathcal{B}}(\delta\zeta, T\zeta(u_x)) \oplus \omega(\delta\zeta, T\zeta(u_x)).$$

These last three expressions prove that the allowed variations in $J^1\Sigma \oplus (T^*M \otimes (\text{Ad}P \oplus (V/G)))$ are the same as the projection of the allowed variations of the original space and, consequently, (i) and (iii) are equivalent statements. $\square$

Remark 4.2.2 *Let N be a normal subgroup of G , and $K = G/N$ the quotient group. We can reduce L by N and afterwards by K . Let $\mathcal{A}_N$ be a principal connection on $P \rightarrow P/N$ and $\mathcal{A}_{G/N}$ a principal connection on $P/N \rightarrow (P/N)/K$. From Definition 2.2.12, these connections are said to be compatible with respect to $\mathcal{A}$ if for all $u \in TP$,*

$$\mathcal{A}(u) = 0 \Leftrightarrow \mathcal{A}_N(u) = 0 \text{ and } \mathcal{A}_{G/N}(T\pi_{P/N,P}(u)) = 0.$$

In this case, Proposition 2.2.13 states that there exists a $\mathfrak{LP}$ -isomorphism from $T(P/G) \oplus \text{Ad}P \oplus (V/G)$ to

$$T((P/N)/K) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}} \oplus (V/N))/(G/N),$$

where $\mathfrak{n}$ is the Lie algebra of N , $\mathfrak{k}$ is the Lie algebra of K and $\tilde{\mathfrak{n}}, \tilde{\mathfrak{k}}$ their respective adjoint bundles. Propositions 4.2.1 and 4.1.15 imply that this process of reduction by stages can be performed in $\text{FT}\mathfrak{LP}$ and its equivalence to direct reduction by the whole group of symmetry. This will be exemplified in Section 5.2. For more details on reduction by stages, see Subsection 2.2.3, [23, Section 6.3] or [6, Section 3.4].

4.2.2 Reconstruction in $\text{FT}\mathfrak{LP}$

Proposition 4.2.1 states that the projection of a critical section for a G -invariant Lagrangian L is a critical section for the reduced lagrangian l . We shall focus on the reconstruction problem: Given a critical section $j^1\zeta \oplus \bar{\rho} \oplus [\nu]_G \in \Gamma(\bar{U}, J^1\Sigma \oplus (T^*M \otimes (\text{Ad}P \oplus (V/G))))$ for l , is there a critical section $j^1\rho \oplus \nu \in \Gamma(\bar{U}, J^1P \oplus (T^*M \otimes V))$ for L whose projection is $j^1\zeta \oplus \bar{\rho} \oplus [\nu]_G$?

Let $\varsigma(x) = \pi_{\Sigma, V/G}([\nu]_G(x)) = \pi_{\Sigma, \text{Ad}P}(\bar{\rho}(x))$. This section in $\Gamma(\bar{U}, \Sigma)$ defines the G -principal pull-back bundle $\varsigma^*P \rightarrow \bar{U} \subset M$

$$\varsigma^*P = \{(x, y) \in \bar{U} \times P \mid \pi_{\Sigma, P}(y) = \varsigma(x)\}$$

of the G -principal bundle $P \rightarrow \Sigma$. In addition, ς^*P can be identified with $P^\varsigma = \{y \in P \mid \pi_{\Sigma, P}(y) \in \varsigma(\bar{U})\}$ via $y \in P^\varsigma \mapsto (\pi_{M, P}(y), y) \in \varsigma^*P$. As $\varsigma(x) = \pi_{\Sigma, \text{Ad}P}(\bar{\rho}(x))$, the section $\bar{\rho}(x)$ can be interpreted as a section in $\Gamma(\bar{U}, \text{Ad}P^\varsigma)$ and there is an equivariant horizontal 1-form $\omega^{\bar{\rho}} \in \Omega^1(P^\varsigma, \mathfrak{g})$ such that for all $x \in \bar{U} \subset M$ and $u_x \in T_x M$,

$$\bar{\rho}(u_x) = [y, \omega^{\bar{\rho}}(u_y)]_G,$$

where $y \in P^\zeta$ and $u_y \in T_y P$ such that $T\pi_{M,P}(u_y) = u_x$. The connection $\mathcal{A}$ on $P \rightarrow \Sigma$ induces a connection $\mathcal{A}^\zeta$ on $P^\zeta \rightarrow \bar{U}$. Since the space of connections of a principal bundle is an affine space modeled over the space of equivariant 1-forms taking values in the adjoint bundle, we define on $P^\zeta \rightarrow \bar{U}$ the principal connection

$$\mathcal{A}^{\bar{\rho}} = \mathcal{A}^\zeta - \omega^{\bar{\rho}}.$$

Theorem 4.2.3 *Let $\mathcal{A}$ be a principal connection on $P \rightarrow \Sigma$, let $L : J^1 P \oplus (T^* M \otimes V) \rightarrow \mathbb{R}$ be a G -invariant Lagrangian defined in a FT $\mathfrak{L}\mathfrak{P}$ bundle and let $l : J^1 \Sigma \oplus (T^* M \otimes (\text{Ad} P \oplus (V/G))) \rightarrow \mathbb{R}$ be the reduced Lagrangian.*

If $j^1 \rho \oplus v \in \Gamma(\bar{U}, J^1 P \oplus (T^* M \otimes V))$ satisfies the Lagrange–Poincaré equations given by L , then the reduced section $j^1 \zeta \oplus \bar{\rho} \oplus [v]_G \in \Gamma(\bar{U}, J^1 \Sigma \oplus (T^* M \otimes (\text{Ad} P \oplus (V/G))))$ satisfies the Lagrange–Poincaré equations given by l and connection $\mathcal{A}^{\bar{\rho}}$ on $P^\zeta \rightarrow \Sigma$ is flat.

Conversely, given a solution $j^1 \zeta \oplus \bar{\rho} \oplus [v]_G \in \Gamma(\bar{U}, J^1 \Sigma \oplus (T^* M \otimes (\text{Ad} P \oplus (V/G))))$ of the Lagrange–Poincaré equations given by l such that $\mathcal{A}^{\bar{\rho}}$ is flat and has trivial holonomy over an open set containing $\bar{U}$, there is a family $\Phi_g(j^1 \rho \oplus v)$, $g \in G$, of solutions of the Lagrange–Poincaré equations given by L projecting to $j^1 \zeta \oplus \bar{\rho} \oplus [v]_G$. If the connection $\mathcal{A}^{\bar{\rho}}$ is flat one can always restrict it to an open simply connected set contained in U so that its holonomy on U is automatically zero.

Proof. Suppose that $j^1 \zeta \oplus \bar{\rho} \oplus [v]_G$ is the projection of $j^1 \rho \oplus v$, in particular, $\bar{\rho} = [\rho, \rho^* \mathcal{A}]_G$, where ρ is a section of $P \rightarrow M$. Observe that for all $y = \rho(x) \in P^\zeta$,

$$T_{\rho(x)} P^\zeta = T_x \rho(T_x M) \oplus \ker(T_{\rho(x)} \pi_{\Sigma, P}),$$

and any $v_y \in T_y P^\zeta$ can be written as $v_y = T_x \rho(v_x) + \xi_y^P$, where $v_x \in T_x M$ and $\xi \in \mathfrak{g}$. Then,

$$\begin{aligned} \mathcal{A}^{\bar{\rho}}(v_y) &= \mathcal{A}^\zeta(v_y) - \omega^{\bar{\rho}}(v_y) = \mathcal{A}(v_y) - \mathcal{A}(T_x \rho(T_y \pi_{M,P}(v_y))) \\ &= \mathcal{A}(T_x \rho(v_x)) + \mathcal{A}(\xi_y^P) - \mathcal{A}(T_x \rho(v_x)) = \xi. \end{aligned}$$

Consequently, the horizontal subbundle defined by $\mathcal{A}^{\bar{\rho}}$ is given by

$$H_{\rho(x)}^{\mathcal{A}^{\bar{\rho}}} P^\zeta = T_x \rho(T_x M),$$

the horizontal distribution is integrable, the integral leaves are

$$\{\Phi_g(\rho(x)) \mid x \in M\} = \Phi_g(\text{Im} \rho), \quad g \in G,$$

and $\mathcal{A}^{\bar{\rho}}$ is a flat connection on $P^\zeta \rightarrow M$.

Conversely, given $j^1\zeta \oplus \bar{\rho} \oplus [\nu]_G$ and $\zeta(x) = \pi_{\Sigma, V/G}([\nu]_G(x)) = \pi_{\Sigma, \text{Ad}P}(\bar{\rho}(x))$. Suppose that $\mathcal{A}^{\bar{\rho}}$ is flat and has trivial holonomy over an open set containing $\bar{U}$. The horizontal distribution of $\mathcal{A}^{\bar{\rho}}$ is integrable and the leaves cover the base. Since the holonomy is trivial each fiber intersects the leaf exactly once, that is, they are sections of $P^\zeta \rightarrow M$. Therefore, there is a family $\Phi_g(\rho(x))$ of sections of $P \rightarrow M$ that projects to ζ via $\pi_{M, \Sigma}$ and such that

$$[\rho, \rho^* \mathcal{A}]_G = [\rho, \rho^* \mathcal{A}^\rho + \omega^{\bar{\rho}}]_G = [\rho, \omega^{\bar{\rho}}]_G = \bar{\rho}.$$

Furthermore, there is a unique section $\nu(x)$ of $T^*M \oplus V \rightarrow M$ such that

$$\pi_{T^*M \oplus (V/G), T^*M \oplus V}(\nu(x)) = [\nu](x), \quad \pi_{P, T^*M \oplus V}(\nu(x)) = \rho(x).$$

In addition, $\Phi_g(\nu(x))$ is the unique section of $T^*M \oplus V \rightarrow M$ such that

$$\pi_{T^*M \oplus (V/G), T^*M \oplus V} \Phi_g(\nu(x)) = [\nu](x), \quad \pi_{P, T^*M \oplus V} \Phi_g(\nu(x)) = \Phi_g(\rho(x)).$$

Thus, the family of sections $\Phi_g(j^1\rho \oplus \nu) = j^1\Phi_g(\rho) \oplus \Phi_g(\nu)$, $g \in G$, projects to $j^1\zeta \oplus \bar{\rho} \oplus [\nu]_G$ and, by equivalence (iv) $\Rightarrow$ (ii) in Proposition 4.2.1, they are solutions of the Lagrange–Poincaré equations given by L . $\square$

The curvature of connection $\mathcal{A}^{\bar{\rho}}$ can be rewritten in terms of the curvature $\mathcal{B}$ of $\mathcal{A}$. Then, the flatness of $\mathcal{A}^{\bar{\rho}}$ gives the following *reconstruction condition*

$$\mathcal{B} - d^{\nabla^{\mathcal{A}}} \omega^{\bar{\rho}} - \omega^{\bar{\rho}} \wedge \omega^{\bar{\rho}} = 0, \quad (4.14)$$

where $d^{\nabla^{\mathcal{A}}}$ is the exterior derivative of $\mathfrak{g}$ -valued forms induced by the covariant derivative on $\text{Ad}P$, $\nabla^{\mathcal{A}}$, and the Cartan formula. This showcases that in Lagrangian field theory it is not always possible to construct a solution of the original system from a solution of the reduced Lagrangian. However, reconstruction is always possible in Lagrangian mechanics as indicated in Lemma 2.2.8. Indeed, if $M = \mathbb{R}$, every connection is flat as the base is one dimensional and the holonomy is trivial since $\mathbb{R}$ is simply connected.

4.2.3 The Noether Drift Law and Vertical Equations

In this section, we define a Noether current for symmetries in $\text{FT}\mathfrak{L}\mathfrak{P}$ bundles and prove that is not a constant of motion. Instead there is a drift of this current that reduces to the new vertical equation appearing in each step of the reduction. This phenomenon was already established in Subsection 2.2.4 for the case of mechanics.

Definition 4.2.4 Let $L : J^1 P \oplus (T^* M \otimes V) \rightarrow \mathbb{R}$ be a Lagrangian and G a Lie group acting freely and properly on $J^1 P \oplus (T^* M \otimes V)$ by isomorphisms in the category $\mathbf{FT\mathfrak{A}}$. The **Noether current** is the application

$$J : J^1 P \oplus (T^* M \otimes_P V) \rightarrow TM \otimes_P \mathfrak{g}^*$$

such that for all $j^1 \rho \oplus v \in J^1 P \oplus (T^* M \otimes_P V)$,

$$J(j^1 \rho \oplus v)((\rho, \alpha) \otimes (\rho, \eta)) = \frac{\delta L}{\delta j^1 \rho}(j^1 \rho \oplus v)((\rho, \alpha) \otimes \eta_\rho^P),$$

where $(\rho, \alpha) \otimes (\rho, \eta) \in T^* M \otimes_P \mathfrak{g}$.

Proposition 4.2.5 Suppose that L is G -invariant and let $j^1 \rho \oplus v(x)$ be a section satisfying the Lagrange–Poincaré equations. Then, the Noether current satisfies

$$\operatorname{div}((j^1 \rho \oplus v)^* J(\rho(x), \eta)) = - \left\langle \frac{\delta L}{\delta v}, \omega(T_x \rho(\bullet), \eta_y^P) + \eta_{v(\bullet)}^V \right\rangle, \quad (4.15)$$

where $\omega(T_x \rho(\bullet), \eta_y^P) + \eta_{v(\bullet)}^V \in T^* M \otimes V$ acts by replacing the slot $\bullet$ by an element in $T_x M$. Equation (4.15) is called the **Noether drift law**.

Proof. As the Lagrangian is G -invariant, for all $\eta \in \mathfrak{g}$,

$$\begin{aligned} 0 &= dL \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \exp(\varepsilon \eta)(j^1 \rho \oplus v) \right) = \\ &\left\langle \frac{\delta L}{\delta \rho}, \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \exp(\varepsilon \eta) \rho \right\rangle + \left\langle \frac{\delta L}{\delta j^1 \rho}, \frac{D^{\nabla^{j^1 P}}}{D\varepsilon} \Big|_{\varepsilon=0} \exp(\varepsilon \eta) j^1 \rho \right\rangle + \left\langle \frac{\delta L}{\delta v}, \frac{D^L}{D\varepsilon} \Big|_{\varepsilon=0} \exp(\varepsilon \eta) v \right\rangle \end{aligned}$$

Since $\exp(\varepsilon \eta) j^1 \rho = j^1(\exp(\varepsilon \eta) \rho)$, it is the lifted variation of the vertical variation $\exp(\varepsilon \eta) \rho$ of ρ . Then, for all $u_x \in TM$;

$$\frac{D^{\nabla^{j^1 P}} \exp(\varepsilon \eta) j^1 \rho}{D\varepsilon} \Big|_{\varepsilon=0} (u_x) = \frac{D^{\nabla^P} \exp(\varepsilon \eta) j^1 \rho(u_x)}{D\varepsilon} \Big|_{\varepsilon=0} = \tilde{\nabla}_{u_x}^P \eta_\rho^P + T^P(\eta_\rho^P, T_x \rho(u_x)).$$

Furthermore, $\exp(\varepsilon \eta) v$ is a vertical variation and

$$\frac{D^L \exp(\varepsilon \eta) v}{D\varepsilon} \Big|_{\varepsilon=0} (u_x) = \frac{D^\nabla \exp(\varepsilon \eta) v(u_x)}{D\varepsilon} \Big|_{\varepsilon=0} = \eta_{v(u_x)}^V.$$

In short, the G -invariance of L can be written as,

$$0 = \left\langle \frac{\delta L}{\delta \rho}, \eta_\rho^P \right\rangle + \left\langle \frac{\delta L}{\delta j^1 \rho}, \tilde{\nabla}_{u_x}^P \eta_\rho^P + T^P (\eta_\rho^P, T_x \rho(u_x)) \right\rangle + \left\langle \frac{\delta L}{\delta v}, \eta_{v(u_x)}^V \right\rangle. \quad (4.16)$$

Finally,

$$\begin{aligned} \operatorname{div}((j^1 \rho \oplus v)^* J(\rho(x), \eta)) &= \\ &= \operatorname{div} \left\langle \frac{\delta L}{\delta j^1 \rho} (j^1 \rho \oplus v(x)), \eta_{\rho(x)}^P \right\rangle = \left\langle \operatorname{div}^P \frac{\delta L}{\delta j^1 \rho}, \eta_\rho^P \right\rangle + \left\langle \frac{\delta L}{\delta j^1 \rho}, \tilde{\nabla}^P \eta_\rho^P \right\rangle \\ &= \left\langle \operatorname{div}^P \frac{\delta L}{\delta j^1 \rho}, \eta_\rho^P \right\rangle - \left\langle \frac{\delta L}{\delta \rho}, \eta_\rho^P \right\rangle - \left\langle \frac{\delta L}{\delta j^1 \rho}, T^P (\eta_\rho^P, T_x \rho(\bullet)) \right\rangle - \left\langle \frac{\delta L}{\delta v}, \eta_{v(\bullet)}^V \right\rangle \\ &= - \left\langle \frac{\delta L}{\delta v}, \omega (T \rho(\bullet), \eta_y^P) + \eta_{v(\bullet)}^V \right\rangle, \end{aligned}$$

where we have used relation (4.16) and Lagrange–Poincaré equations. $\square$

The Noether current is G -equivariant. Hence, it defines a quotient bundle map

$$j : J^1 \Sigma \oplus (T^* M \otimes_\Sigma (\operatorname{Ad} P \oplus (V/G))) \rightarrow T M \otimes_P \operatorname{Ad} P^*$$

such that for all $j^1 \varsigma \oplus \bar{\rho} \oplus [v]_G \in J^1 \Sigma \oplus (T^* M \otimes_\Sigma (\operatorname{Ad} P \oplus (V/G)))$, and all $(\varsigma, \alpha) \otimes \bar{\eta} \in T^* M \otimes_\Sigma \operatorname{Ad} P$,

$$\begin{aligned} j(j^1 \varsigma \oplus \bar{\rho} \oplus [v]_G)((\varsigma, \alpha) \otimes \bar{\eta}) &= \\ &= J(j^1 \rho \oplus v)((\rho, \alpha) \otimes (\rho, \eta)) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} L(j^1 \rho + \varepsilon((\rho, \alpha) \otimes \eta_\rho^P) \oplus v) \\ &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} l(j^1 \varsigma \oplus (\bar{\rho} + \varepsilon((\varsigma, \alpha) \otimes \bar{\eta}) \oplus [v]_G) = \frac{\delta l}{\delta \bar{\rho}}(j^1 \varsigma \oplus \bar{\rho} \oplus [v]_G)((\varsigma, \alpha) \otimes \bar{\eta}). \end{aligned}$$

Consequently, the *reduced Noether current* j coincides with $\frac{\delta l}{\delta \bar{\rho}}$.

Remark 4.2.6 . *The vertical Lagrange–Poincaré equations,*

$$\operatorname{div}^{\tilde{\mathfrak{g}}} \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [v]_G} \right) - \operatorname{ad}_{\bar{\rho} \oplus [v]_G}^* \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [v]_G} \right) = 0,$$

associated to the reduced bundle $J^1 \Sigma \oplus (T^* M \oplus (\operatorname{Ad} P \oplus (V/G)))$ is an equation in $\operatorname{Ad} P^* \oplus (V/G)^*$ acting on vectors $\bar{\eta} \oplus [u]_G \in \operatorname{Ad} P \oplus (V/G)$. We can decompose these equations

restricting to each of the factors on $\text{Ad}P \oplus (V/G)$ as in Remark 2.2.19. On one hand, from

$$\begin{aligned} \left\langle \text{div}^{\tilde{\mathfrak{g}}} \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \bar{\eta} \oplus [u]_G \right\rangle &= \\ &= \text{div} \left\langle \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \bar{\eta} \oplus [u]_G \right\rangle - \left\langle \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \tilde{\nabla}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [u]_G) \right\rangle \end{aligned}$$

and

$$\tilde{\nabla}^{\tilde{\mathfrak{g}}}(\bar{\eta} \oplus [u]_G) = \tilde{\nabla}^{\mathcal{A}} \bar{\eta} \oplus ([\tilde{\nabla}^{(\mathcal{A}, H)}]_G [u]_G - [\omega]_G (T\zeta, \bar{\eta})),$$

where $\text{div}^{\mathcal{A}}$ is the divergence of $\text{Ad}P^*$ -valued vector fields induced by connection $\nabla^{\mathcal{A}}$ on $\text{Ad}P$, we obtain that

$$\begin{aligned} \left\langle \text{div}^{\tilde{\mathfrak{g}}} \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \bar{\eta} \oplus [u]_G \right\rangle &= \\ &= \text{div} \left\langle \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \bar{\eta} \oplus [u]_G \right\rangle - \text{div} \left\langle \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle + \left\langle \text{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle \\ &\quad - \text{div} \left\langle \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle + \left\langle \text{div}^{(\mathcal{A}, H)} \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle + \left\langle \frac{\delta l}{\delta [\nu]_G}, [\omega]_G (T\zeta, \bar{\eta}) \right\rangle \\ &= \left\langle \text{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle + \left\langle \text{div}^{(\mathcal{A}, H)} \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle + \left\langle \frac{\delta l}{\delta [\nu]_G}, [\omega]_G (T\zeta, \bar{\eta}) \right\rangle. \end{aligned}$$

On the other hand;

$$\begin{aligned} \left\langle \text{ad}_{\bar{\rho} \oplus [\nu]_G}^* \left(\frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G} \right), \bar{\eta} \oplus [u]_G \right\rangle &= \left\langle \frac{\delta l}{\delta \bar{\rho}} \oplus \frac{\delta l}{\delta [\nu]_G}, [\bar{\rho} \oplus [\nu]_G, \bar{\eta} \oplus [u]_G] \right\rangle \\ &= \left\langle \frac{\delta l}{\delta \bar{\rho}}, [\bar{\rho}, \bar{\eta}] \right\rangle + \left\langle \frac{\delta l}{\delta [\nu]_G}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\rho}} [u]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}} [\nu]_G - [\omega]_G (\bar{\rho}, \bar{\eta}) + [[\nu]_G, [u]_G] \right\rangle \\ &= \left\langle \text{ad}_{\bar{\rho}}^* \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle + \left\langle \text{ad}_{[\nu]_G}^* \frac{\delta l}{\delta [\nu]_G}, [u]_G \right\rangle \\ &\quad + \left\langle \frac{\delta l}{\delta [\nu]_G}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\rho}} [u]_G - [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}} [\nu]_G - [\omega]_G (\bar{\rho}, \bar{\eta}) \right\rangle. \end{aligned}$$

Thus, restriction of the vertical Lagrange–Poincaré equations to $\text{Ad}P$ provides **new vertical equations** coming from the reduction step

$$\left\langle \text{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle = \left\langle \text{ad}_{\bar{\rho}}^* \frac{\delta l}{\delta \bar{\rho}}, \bar{\eta} \right\rangle - \left\langle \frac{\delta l}{\delta [\nu]_G}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}} [\nu]_G + [\omega]_G (T\zeta \oplus \bar{\rho}, \bar{\eta}) \right\rangle, \quad (4.17)$$

while restriction of the vertical Lagrange–Poincaré equations to V/G provides **reduced vertical equations**,

$$\left\langle \operatorname{div}^{(\mathcal{A}, H)} \frac{\delta l}{\delta [v]_G}, [u]_G \right\rangle = \left\langle \operatorname{ad}_{[v]_G}^* \frac{\delta l}{\delta [v]_G}, [u]_G \right\rangle + \left\langle \frac{\delta l}{\delta [v]_G}, [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\rho}} [u]_G \right\rangle, \quad (4.18)$$

obtained from the projection of the vertical Lagrange–Poincaré equations induced by L .

Proposition 4.2.7 *The drift of the Noether current along critical sections $j^1 \zeta \oplus \bar{\rho} \oplus [v]_G$ in equation (4.15) projects to the new vertical equations (4.17), where $\bar{\eta}(x) = [\rho(x), \eta]_G$.*

Proof. We rewrite the left hand side of equation (4.15):

$$\begin{aligned} \operatorname{div} \langle J(j^1 \rho \oplus v), (\rho, \eta) \rangle &= \operatorname{div} \langle j(j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), \bar{\eta} \rangle \\ &= \langle \operatorname{div}^{\mathcal{A}} j(j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), \bar{\eta} \rangle + \langle j(j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), \tilde{\nabla}^{\mathcal{A}} \bar{\eta} \rangle \\ &= \left\langle \operatorname{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}} (j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), \bar{\eta} \right\rangle - \langle j(j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), [\bar{\rho}, \bar{\eta}] \rangle \\ &= \left\langle \operatorname{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}} (j^1 \zeta \oplus \bar{\rho} \oplus [v]_G) - \operatorname{ad}_{\bar{\rho}}^* \frac{\delta l}{\delta \bar{\rho}} (j^1 \zeta \oplus \bar{\rho} \oplus [v]_G), \bar{\eta} \right\rangle. \end{aligned}$$

On the other hand, the right hand side of equation (4.15) projects to

$$- \left\langle \frac{\delta l}{\delta [v]_G}, [\omega]_G (T\zeta(\bullet) \oplus \bar{\rho}(\bullet), \bar{\eta}) + [\eta_{v(\bullet)}^V]_G \right\rangle,$$

and we conclude by observing that $[\eta_{v(\bullet)}^V]_G = [\nabla^{(\mathcal{A}, V)}]_{G, \bar{\eta}} [v]_G(\bullet)$. $\square$

Remark 4.2.8 *In the special case $V = 0$, the drift law becomes a conservation law expressed as a vanishing of a divergence. Indeed, we recover the Noether theorem for covariant invariant Lagrangians from equation (4.15). In addition, the conservation of this current is equivalent to the vertical Lagrange–Poincaré equation of the reduced Lagrangian:*

$$\operatorname{div}^{\mathcal{A}} \frac{\delta l}{\delta \bar{\rho}} - \operatorname{ad}_{\bar{\rho}}^* \frac{\delta l}{\delta \bar{\rho}} = 0.$$

Chapter 5

Applications of Lagrange–Poincaré Reduction in Field Theory

5.1 $SO(3)$ -Strand with Broken Symmetry

A G -strand is a map into a Lie group G , $g(s, t) : \mathbb{R}^2 \rightarrow G$, whose dynamics in $(s, t) \in \mathbb{R}^2$ may be obtained from Hamilton's principle for a G -invariant Lagrangian. These systems and their integrability were studied in [46, 47]. The equations of motion of these G -strands can be interpreted as the dynamics of a continuous spin chain. In the particular case $G = SO(3)$, the described system is a continuous chain of rigid bodies. In [31], the case $G = SE(3)$ is considered to allow this chain to be flexible and move in a tridimensional space. This system models charged *molecular strands*, a molecule that can be described by the rotation of repeating rigid charge conformations along a flexible filament. As the symmetry group of a system on a G -strand is defined to be G , Euler–Poincaré reduction can always be performed on G -strands. However, a plethora of examples arise for which the symmetry group is a subgroup of G . This phenomenon is called symmetry break. In this section, an $SO(3)$ -strand with broken symmetry is presented to illustrate Lagrange–Poincaré reduction introduced in Section 1.3 and extended in Chapter 4.

5.1.1 $SO(3)$ -Strand with $SO(2)$ Symmetry

Consider the trivial bundle $P = \mathbb{R}^2 \times SO(3)$ over $M = \mathbb{R}^2$ with projection $(s, t, R) \mapsto x = (s, t)$. This notation in $\mathbb{R}^2$ is chosen to suggest that one coordinate is spatial and the other is time. A Lagrangian system is defined on $J^1(\mathbb{R}^2 \times SO(3))$ by the volume form $v = ds \wedge dt$

and the Lagrangian function,

$$L(R, R_s, R_t) = -\frac{1}{2}\langle R^{-1}R_s, JR^{-1}R_s \rangle + \frac{1}{2}\langle R^{-1}R_t, IR^{-1}R_t \rangle - ee_3 \cdot R\chi, \quad (5.1)$$

where sub-indexes s and t denote partial derivatives, I is the inertia tensor of the rigid body, J is a tensor describing the opposition to the rotation of consecutive rigid bodies of the strand, $\chi \in \mathbb{R}^3$ is the magnetic dipole of the rigid body in the configuration reference, and $\mathbf{E} = ee_3$ is a uniform electric field in the direction e_3 of the spatial frame of reference. This system models the evolution of a molecular strand subjected to a uniform electric field. The strand is described by the rotation of rigid charge conformations with magnetic dipole along a rigid filament. Since

$$\frac{\partial L}{\partial R_s} = -RJ R^{-1} R_s, \quad (5.2)$$

$$\frac{\partial L}{\partial R_t} = RI R^{-1} R_t, \quad (5.3)$$

and $\partial L / \partial R = -ee_3 \times R\chi$, the Euler–Lagrange equations (1.7) for this system are,

$$-(R^{-1}R_s) \times J(R^{-1}R_s) - J(R^{-1}R_s)_s + (R^{-1}R_t) \times I(R^{-1}R_t) + I(R^{-1}R_t)_t + e(R^{-1}e_3) \times \chi = 0 \quad (5.4)$$

There is a left $SO(3)$ action on $P = \mathbb{R}^2 \times SO(3)$ by left translations on the second factor which is lifted to J^1P . Given $Q \in SO(3)$,

$$\begin{aligned} L(Q \cdot (R, R_s, R_t)) &= L(QR, QR_s, QR_t) = \\ &= -\frac{1}{2}\langle R^{-1}Q^{-1}QR_s, JR^{-1}Q^{-1}QR_s \rangle + \frac{1}{2}\langle R^{-1}Q^{-1}QR_t, IR^{-1}Q^{-1}QR_t \rangle - ee_3 \cdot QR\chi \\ &= -\frac{1}{2}\langle R^{-1}R_s, JR^{-1}R_s \rangle + \frac{1}{2}\langle R^{-1}R_t, IR^{-1}R_t \rangle - e(Q^{-1}e_3) \cdot R\chi, \end{aligned}$$

and the Lagrangian is not $SO(3)$ invariant. Yet, it is invariant by the action of rotations around the axis e_3 and the group of symmetry of the system is $SO(2) \simeq \mathbb{S}^1$.

The *heavy top* is a rigid body moving about a fixed point subject to a uniform gravitational field. This mechanical system has a configuration bundle $SO(3)$ and is described by a $SO(2)$ -invariant Lagrangian:

$$\tilde{L}(R, R_t) = \frac{1}{2}\langle R^{-1}R_t, IR^{-1}R_t \rangle - ee_3 \cdot R\chi, \quad (5.5)$$

where $e = -Mgl$, Mg is the magnitude of the gravitational force and χ is a unit vector along the segment of length l connecting the fixed point with the center of mass of the top. In the bibliography, this system is usually studied using an extended version of Euler–Poincaré

reduction for semidirect product $SE(3) = SO(3) \times \mathbb{R}^3$ rather than Lagrange–Poincaré reduction. See for instance [65, 48] and [49]. The resulting equations are:

$$I\omega_t + \omega \times I\omega - e\zeta \times \chi \quad (5.6)$$

$$\zeta_t = \zeta \times \omega, \quad (5.7)$$

where $\omega = R^{-1}R_t \in \mathfrak{so}(3)$ and $\zeta = R^{-1}e_3 \in \mathbb{R}^3$. The $SO(3)$ -strand with broken symmetry $SO(2)$ can alternatively be understood as a continuous chain of heavy tops in the same way as $SO(3)$ -strands can be understood as a chain of rigid bodies. Indeed, Euler–Lagrange equations (5.4) can be rewritten as

$$-\Omega \times J\Omega - J\Omega_s + \omega \times I\omega + I\omega_t + e\zeta \times \chi = 0 \quad (5.8)$$

where $\Omega = R^{-1}R_s$.

5.1.2 $SO(3)$ -Strand and Lagrange–Poincaré Reduction

We shall perform Lagrange–Poincaré reduction in the $SO(3)$ -strand with $SO(2) \simeq \mathbb{S}^1$ symmetry. Consider the principal bundle, $P \rightarrow P/\mathbb{S}^1$, with projection

$$P = \mathbb{R}^2 \times SO(3) \rightarrow \Sigma = P/\mathbb{S}^1 = \mathbb{R}^2 \times \mathbb{S}^2$$

$$(s, t, R) \mapsto (s, t, \zeta = R^{-1}e_3).$$

This bundle is essentially the Hopf fibration $\mathbb{S}^1 \hookrightarrow SO(3) \rightarrow SO(2)$. Let $v \in T_R SO(3)$ and identify $T_R SO(3)$ with $\mathbb{R}^3$ via the hat map as in Section 3.2. Then, v can be decomposed in parallel and perpendicular components with respect to e_3 as

$$v = \langle v, e_3 \rangle e_3 - e_3 \times (e_3 \times v). \quad (5.9)$$

A straightforward calculation provides $T\pi_{P,\Sigma}(v) = R^{-1}(e_3 \times v)$. Hence, the mechanical connection is $\mathcal{A}(v) = \langle v, e_3 \rangle$ and for any $v_\zeta \in T_\zeta \mathbb{S}^2$, the horizontal lift at R is $(v_\zeta)_R^h = -e_3 \times Rv_\zeta = R(-\zeta \times v_\zeta)$.

The bundle $J^1(\mathbb{R}^2 \times SO(3))/\mathbb{S}^1$ is identified with a $\text{FT}\mathfrak{LP}$ bundle via

$$\alpha_{\mathcal{A}} : J^1(\mathbb{R}^2 \times SO(3))/\mathbb{S}^1 \rightarrow J^1(\mathbb{R}^2 \times \mathbb{S}^2) \oplus (T^*\mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}))$$

$$j^1(x, R(x)) \mapsto j^1(x, \zeta(x)) \oplus (x, \zeta(x), \Xi(x)ds + \xi(x)dt),$$

where $\zeta = R^{-1}e_3$, $\Xi = \mathcal{A}(R_s) = \langle \zeta, R^{-1}R_s \rangle$ and $\xi = \mathcal{A}(R_t) = \langle \zeta, R^{-1}R_t \rangle$. As $\zeta \in \mathbb{S}^1$, we have that $\zeta_s \in T_\zeta \mathbb{S}^1$ and $\zeta_t \in T_\zeta \mathbb{S}^1$. Thus, ζ_s and ζ_t are both perpendicular to ζ as vectors in $\mathbb{R}^3$. In fact, when $\zeta(x)$ is the projection of a $(x, R(x)) \in \Gamma(\mathbb{R}^2 \times SO(3))$, partial derivatives of identity $e_3 = R\zeta$ provide

$$\zeta_s = -(R^{-1}R_s) \times \zeta, \quad \zeta_t = -(R^{-1}R_t) \times \zeta;$$

and decomposition of $R^{-1}R_s$ and $R^{-1}R_t$ with respect to ζ as in equation (5.9) reads as

$$R^{-1}R_s = \Xi\zeta + \zeta_s \times \zeta, \quad R^{-1}R_t = \xi\zeta + \zeta_t \times \zeta. \quad (5.10)$$

The reduced Lagrangian is

$$\begin{aligned} l(\zeta, \zeta_s, \zeta_t, \Xi, \xi) = & -\frac{1}{2}\Xi^2 \langle \zeta, J\zeta \rangle - \Xi \langle \zeta, J(\zeta_s \times \zeta) \rangle - \frac{1}{2} \langle \zeta_s \times \zeta, J(\zeta_s \times \zeta) \rangle \\ & + \frac{1}{2}\xi^2 \langle \zeta, I\zeta \rangle + \xi \langle \zeta, J(\zeta_t \times \zeta) \rangle + \frac{1}{2} \langle \zeta_t \times \zeta, I(\zeta_t \times \zeta) \rangle - e\zeta \cdot \chi, \end{aligned} \quad (5.11)$$

To obtain the equations of motion of the reduced system is necessary to calculate the fiber derivatives of l . On one hand, the fiber derivative (4.8) is

$$\frac{\delta l}{\delta(\zeta_s ds + \zeta_t dt)} = \frac{\delta l}{\partial \zeta_s} \frac{\partial}{\partial s} + \frac{\delta l}{\partial \zeta_t} \frac{\partial}{\partial t},$$

where

$$\frac{\delta l}{\partial \zeta_s} = -\zeta \times (\Xi J\zeta + J(\zeta_s \times \zeta)), \quad \frac{\delta l}{\partial \zeta_t} = \zeta \times (\xi I\zeta + I(\zeta_t \times \zeta)). \quad (5.12)$$

On the other hand, the fiber derivative in (4.9) is

$$\frac{\delta l}{\delta(\Xi ds + \xi dt)} = \frac{\delta l}{\partial \Xi} \frac{\partial}{\partial s} + \frac{\delta l}{\partial \xi} \frac{\partial}{\partial t},$$

where

$$\frac{\delta l}{\partial \Xi} = -\langle \zeta, \Xi J\zeta + J(\zeta_s \times \zeta) \rangle, \quad \frac{\delta l}{\partial \xi} = \langle \zeta, \xi I\zeta + I(\zeta_t \times \zeta) \rangle. \quad (5.13)$$

Since $\mathbb{S}^1$ is abelian and the connection considered on the adjoint bundle $\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ is trivial, the vertical Lagrange–Poincaré equations (4.11) particularize as

$$\frac{\partial}{\partial s} \left(\frac{\delta l}{\partial \Xi} \right) + \frac{\partial}{\partial t} \left(\frac{\delta l}{\partial \xi} \right) = 0.$$

From equations (5.13), the vertical Lagrange–Poincaré equations for an $SO(3)$ -strand with broken symmetry are

$$\begin{aligned} 0 = & -\Xi\zeta_s \times J\zeta - \zeta_s \times J(\zeta_s \times \zeta) - \Xi\zeta \times J\zeta_s - \zeta \times J(\zeta_{ss} \times \zeta) \\ & + \xi\zeta_t \times I\zeta + \zeta_t \times I(\zeta_t \times \zeta) + \xi\zeta \times I\zeta_t + \zeta \times I(\zeta_{tt} \times \zeta), \end{aligned} \quad (5.14)$$

which does coincide with the component of equation (5.8) in the direction of ζ after substitution of equations in (5.10). In addition, the horizontal Lagrange–Poincaré equations (4.12) for a Lagrangian on $J^1(\mathbb{R}^2 \times \mathbb{S}^2) \oplus (T^*\mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}))$ are

$$\frac{\delta l}{\delta \zeta} - \frac{D}{Ds} \frac{\delta l}{\partial \zeta_s} - \frac{D}{Dt} \frac{\delta l}{\partial \zeta_t} = \left\langle \frac{\delta l}{\delta(\Xi ds + \xi dt)}, i_{T_\zeta \tilde{\mathcal{B}}} \right\rangle, \quad (5.15)$$

whenever a torsion-free connection is chosen on $\mathbb{S}^2$. Recall that D/Ds and D/Dt are covariant derivatives on $\mathbb{S}^2$, and $\delta l/\delta \zeta$ is a horizontal derivative in the sense of (4.10) rather than a partial derivative. The horizontal derivative is taken with respect to a connection on $J^1(\mathbb{R}^2 \times \mathbb{S}^2) \oplus (T^*\mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}))$ built upon a linear connection on $\mathbb{S}^2$ and the trivial connection on $\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \mathbb{S}^2$. For instance, if the Levi–Civita connection on $\mathbb{S}^2$ is chosen,

$$\frac{\delta l}{\delta \zeta} = \frac{\partial l}{\partial \zeta} - \zeta \left\langle \zeta, \frac{\delta l}{\delta \zeta} \right\rangle. \quad (5.16)$$

From the Chern class of $P \rightarrow P/\mathbb{S}^1$, $\tilde{\mathcal{B}}(\zeta_s, \zeta \times \zeta_s) = -1$. Then, as $\{\zeta, \zeta_s, \zeta \times \zeta_s\}$ is an orthonormal basis, identification of $\mathbb{R}^3$ with $(\mathbb{R}^3)^*$ provides $\mathcal{B}(\zeta_s, \cdot) = -\zeta \times \zeta_s$. Altogether, after a straightforward but lengthy calculation, the horizontal Lagrange–Poincaré equations for an $SO(3)$ -strand with broken symmetry are

$$0 = \zeta \times ((p^s)_s + (\Xi\zeta + \zeta_s \times \zeta) \times p^s + (p^t)_t + (\xi\zeta + \zeta_t \times \zeta) \times p^t - e\zeta \times \chi), \quad (5.17)$$

where $p^s = -\Xi J\zeta - J(\zeta_s \times \zeta)$, and $p^t = \xi I\zeta + I(\zeta_t \times \zeta)$. Equation (5.17) does coincide with equation (5.8) in the orthogonal plane to ζ .

Let $j^1(x, \zeta(x)) \oplus (x, \zeta(x), \Xi(x)ds + \xi(x)dt) \in \Gamma(J^1(\mathbb{R}^2 \times \mathbb{S}^2) \oplus (T^*\mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R})))$. We shall now study if there exists $j^1(x, R(x)) \in \Gamma(J^1(\mathbb{R}^2 \times SO(3)))$ whose projection is the former. Since $\omega^{\tilde{R}} = \Xi ds + \xi dt$, we have

$$d^{\mathcal{A}} \omega^{\tilde{R}} = (\xi_s - \Xi_t) ds \wedge dt.$$

As the group of symmetry is abelian, $\omega^{\bar{R}} \wedge \omega^{\bar{R}} = 0$. Furthermore,

$$\tilde{\mathcal{B}}((0, \zeta_s), (0, \zeta_t)) = -\langle \zeta \times \zeta_s, \zeta_t \rangle ds \wedge dt = -\langle \zeta_s \times \zeta_t, \zeta \rangle ds \wedge dt = -\|\zeta_s \times \zeta_t\| ds \wedge dt. \quad (5.18)$$

Thus, the reconstruction equation (4.14) becomes

$$\xi_s - \Xi_t + \|\zeta_s \times \zeta_t\| = 0. \quad (5.19)$$

This condition can be easily interpreted. From $j^1(x, \zeta(x)) \oplus (x, \zeta(x), \Xi(x)ds + \xi(x)dt)$, we define a section of $T^*\mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathfrak{so}(3))$, $\Omega ds + \omega dt$ where,

$$\Omega = \Xi\zeta + \zeta_s \times \zeta, \quad (5.20)$$

$$\omega = \xi\zeta + \zeta_t \times \zeta, \quad (5.21)$$

Thus, $\zeta_s \times \zeta_t = (\zeta \times \Omega) \times (\zeta \times \omega) = \zeta \langle \zeta, \Omega \times \omega \rangle$ and

$$\begin{aligned} 0 &= \xi_s - \Xi_t + \|\zeta_s \times \zeta_t\| = \partial_s \langle \zeta, \omega \rangle - \partial_t \langle \zeta, \Omega \rangle + \langle \zeta, \Omega \times \omega \rangle \\ &= \langle \zeta_s, \omega \rangle + \langle \zeta, \omega_s \rangle - \langle \zeta_t, \Omega \rangle - \langle \zeta, \Omega_t \rangle + \langle \zeta, \Omega \times \omega \rangle \\ &= \langle \zeta \times \Omega, \omega \rangle + \langle \zeta, \omega_s \rangle - \langle \zeta \times \omega, \Omega \rangle - \langle \zeta, \Omega_t \rangle + \langle \zeta, \Omega \times \omega \rangle \\ &= \langle \zeta, \omega_s - \Omega_t + \Omega \times \omega \rangle. \end{aligned}$$

Consequently, if $j^1(x, \zeta(x)) \oplus (x, \zeta(x), \Xi(x)ds + \xi(x)dt)$ satisfies the reconstruction condition (5.19), $\Omega ds + \omega dt$ satisfies the ζ component of the reconstruction condition obtained in [46, Section 2.2] for the $SO(3)$ -strand without symmetry break.

5.2 Molecular Strand with Rotors

In this section we shall discuss a problem of strand dynamics as an example of the theory of reduction by stages in field theory (see Subsection 4.2.1). Our model is called the *molecular strand with rotors* and consists of a mobile base strand repeating the same configuration with different orientation. In turn, this configuration does have a moving piece that can rotate, called *rotor*. In this example we will suppose that each configuration has three rotors, each in the principal axes. However, the case with only one rotor or multiple rotors non necessarily along the principal axes follows directly from our description. The particular case of this model when there are no rotors has been the object of research, for example, in [31] following the same covariant Lagrangian approach.

This example is related to other systems already discussed in this thesis. The molecular strand with rotors can be thought as a continuum over a moving rod of rigid bodies with rotors. In this sense, it is a generalization to field theory of the system studied in Section 3.2. Furthermore, the molecular strand with rotors can be thought as a mobile version of the G -strands presented in Section 5.1, for the case $G = SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

5.2.1 First Stage of Reduction

The principal bundle that we shall use has base space $M = \mathbb{R}^2$ and total space $P = M \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. As in Section 5.1, the variables in $\mathbb{R}^2$ will be denoted by (s, t) . The Lie group $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts on P . The action of $(Q, \alpha) \in SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ on $(x, r, R, \theta) \in M \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, where $x = (s, t) \in \mathbb{R}^2$ and $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, is

$$(Q, \alpha) \cdot (x, r, R, \theta) = (x, Qr, QR, \theta + \alpha).$$

We shall first reduce by the normal subgroup $SO(3)$ and then by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. We denote $\Sigma = P/SO(3) = M \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, the first quotient space, and $(x, \rho, \theta) \in \Sigma$. The projection map is

$$\begin{aligned} \pi_{\Sigma, P} : M \times \mathbb{R}^3 \times SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 &\rightarrow M \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \\ (x, r, R, \theta) &\mapsto (x, \rho = R^{-1}r, \theta), \end{aligned}$$

whose derivative is

$$\begin{aligned} T\pi_{\Sigma, P} : TP &\rightarrow T\Sigma \\ (x, r, R, \theta, v_x, v_r, v_R, v_\theta) &\mapsto (x, \rho = R^{-1}r, \theta, v_x, v_\rho = R^{-1}v_r - R^{-1}v_R\rho, v_\theta). \end{aligned}$$

To identify $J^1 P/SO(3)$ as a FT $\mathfrak{g}\mathfrak{p}$ bundle we need a connection $\mathcal{A}$ on $P \rightarrow \Sigma$. In terms of the Maurer-Cartan connection, any connection $\mathcal{A}$ can be written as

$$\mathcal{A}(x, r, R, \theta, v_x, v_r, v_R, v_\theta) = v_R R^{-1} + \tilde{\mathcal{A}}(x, r, R, \theta, v_x, v_r, v_R, v_\theta),$$

where $\tilde{\mathcal{A}}$ is a $\mathfrak{so}(3)$ -valued 1-form such that

- (i) $\tilde{\mathcal{A}}(\eta_{x,r,R,\theta}^P) = 0$, for any $\eta \in \mathfrak{so}(3)$,
- (ii) $\tilde{\mathcal{A}}(Qv) = \text{Ad}_Q \tilde{\mathcal{A}}(v)$, for all $Q \in SO(3)$, $v \in TP$,

In addition, since $P \rightarrow M$ is trivial, $\text{Ad}P$ is trivialized as $\Sigma \times \mathfrak{so}(3)$ via the identification

$$[(x, r, R, \theta), \zeta]_{SO(3)} = [(x, R^{-1}r, e, \theta), \text{Ad}_{R^{-1}}\zeta]_{SO(3)} = (x, R\rho, \theta, \text{Ad}_{R^{-1}}\zeta).$$

Then, the covariant derivative on $\text{Ad}P$

$$\begin{aligned} \frac{D^{\mathcal{A}}}{D\tau}[x(\tau), r(\tau), R(\tau), \theta(\tau), \zeta(\tau)]_{SO(3)} &= \\ &= [x(\tau), r(\tau), R(\tau), \theta(\tau), \zeta_\tau - (R_\tau R^{-1} + \tilde{\mathcal{A}}(x, r, R, \theta, x_\tau, r_\tau, R_\tau, \theta_\tau)) \times \zeta_\tau]_{SO(3)}, \end{aligned}$$

where the sub-index τ denotes a derivative with respect to τ , induces on $\Sigma \times \mathfrak{so}(3)$ the covariant derivative

$$\frac{D^{\mathcal{A}}}{D\tau}(x(\tau), \rho(\tau), \theta(\tau), \zeta(\tau)) = (x(\tau), \rho(\tau), \theta(\tau), \zeta_\tau - \tilde{\mathcal{A}}(x, \rho, e, \theta, x_\tau, \rho_\tau, 0, \theta_\tau) \times \zeta_\tau). \quad (5.22)$$

We lift the section $(x, r(x), R(x), \theta(x))$ of $P \rightarrow M$ to a section of $J^1P \rightarrow M$,

$$(x, r(x), R(x), \theta(x), ds + dt, r_s ds + r_t dt, R_s ds + R_t dt, \theta_s ds + \theta_t dt),$$

where subindices on r , R and θ denote partial derivatives. To project this section to $J^1\Sigma \oplus (T^*M \otimes (\Sigma \times \mathfrak{so}(3)))$ we evaluate

$$\begin{aligned} \mathcal{A}(x, r, R, \theta, ds + dt, r_s ds + r_t dt, R_s ds + R_t dt, \theta_s ds + \theta_t dt) &= \\ &= R_s R^{-1} ds + R_t R^{-1} dt + \tilde{\mathcal{A}}(x, r, R, \theta, 1, r_s, R_s, \theta_s) ds + \tilde{\mathcal{A}}(x, r, R, \theta, 1, r_t, R_t, \theta_t) dt \end{aligned}$$

and

$$\begin{aligned} \text{Ad}_{R^{-1}} \mathcal{A}(x, r, R, \theta, ds + dt, r_s ds + r_t dt, R_s ds + R_t dt, \theta_s ds + \theta_t dt) &= \\ &= \Omega ds + \omega dt + \bar{\mathcal{A}}_s ds + \bar{\mathcal{A}}_t dt = \Xi ds + \xi dt, \end{aligned}$$

where $\Omega = R^{-1}R_s$, $\omega = R^{-1}R_t$,

$$\bar{\mathcal{A}}_s(x, \rho, \theta, 1, \rho_s, \theta_s) = \text{Ad}_{R^{-1}} \tilde{\mathcal{A}}(x, r, R, \theta, 1, r_s, R_s, \theta_s) = \tilde{\mathcal{A}}(x, \rho, e, \theta, 1, \rho_s, 0, \theta_s),$$

$\bar{\mathcal{A}}_t = \bar{\mathcal{A}}(x, \rho, \theta, 1, \rho_t, \theta_t)$, $\Xi = \Omega + \bar{\mathcal{A}}_s$ and $\xi = \omega + \bar{\mathcal{A}}_t$. Consequently, the bundle $J^1 P/SO(3)$ is identified with FT $\mathfrak{L}\mathfrak{P}$ bundle $J^1 \Sigma \oplus (T^* M \otimes (\Sigma \times \mathfrak{so}(3)))$ via

$$\begin{aligned} \pi_{\mathfrak{so}(3)} : J^1 P/G &\rightarrow J^1 \Sigma \oplus (T^* M \otimes (\Sigma \times \mathfrak{so}(3))) \\ j^1(x, r(x), R(x), \theta(x)) &\mapsto j^1(x, \rho(x), \theta(x)) \oplus (x, \rho(x), \theta(x), \Xi(x)ds + \xi(x)dt) \end{aligned}$$

and the reduced variables are $\rho, \rho_s, \rho_t, \theta, \theta_s, \theta_t, \Xi$ and ξ . From equation (4.11) in Proposition 4.1.15, the vertical Lagrange–Poincaré equations are:

$$\text{ad}^*_{\Xi ds + \xi dt} \frac{\delta l}{\delta(\Xi ds + \xi dt)} - \text{div}^\nabla \frac{\delta l}{\delta(\Xi ds + \xi dt)} = 0.$$

Hence, since

$$\text{ad}^*_{\Xi ds + \xi dt} \frac{\delta l}{\delta(\Xi ds + \xi dt)} = -\Xi \times \frac{\delta l}{\delta \Xi} - \xi \times \frac{\delta l}{\delta \xi},$$

and

$$\text{div}^\nabla \frac{\delta l}{\delta(\Xi ds + \xi dt)} = \partial_s \frac{\delta l}{\delta \Xi} - \bar{\mathcal{A}}_s \times \frac{\delta l}{\delta \Xi} + \partial_t \frac{\delta l}{\delta \xi} - \bar{\mathcal{A}}_t \times \frac{\delta l}{\delta \xi},$$

we conclude that the vertical Lagrange–Poincaré equations of a molecular strand with rotors are

$$0 = \partial_s \frac{\delta l}{\delta \Xi} + \partial_t \frac{\delta l}{\delta \xi} + \Omega \times \frac{\delta l}{\delta \Xi} + \omega \times \frac{\delta l}{\delta \xi}. \quad (5.23)$$

On the other hand, the horizontal Lagrange–Poincaré equations applied to variation $\delta \rho \oplus \delta \theta$ are

$$\begin{aligned} \left\langle \frac{\delta l}{\delta(\rho \oplus \theta)} - \partial_s \frac{\delta l}{\delta(\rho_s \oplus \theta_s)} - \partial_t \frac{\delta l}{\delta(\rho_t \oplus \theta_t)}, \delta \rho \oplus \delta \theta \right\rangle &= \\ &= \left\langle \frac{\delta l}{\delta(\Xi ds + \xi dt)}, \tilde{\mathcal{B}}((ds + dt, \rho_s ds + \rho_t dt, \theta_s ds + \theta_t dt), (0, \delta \rho, \delta \theta)) \right\rangle. \end{aligned}$$

Recall that, as seen in Subsection 1.2.1 and equation (4.10) in Subection 4.1.2, $\delta l / \delta(\rho \oplus \theta)$ is not a partial derivative, instead

$$\left\langle \frac{\delta l}{\delta(\rho \oplus \theta)}, \delta \rho \oplus \delta \theta \right\rangle = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} l \left(\gamma_{j^1 \rho \oplus j^1 \theta, \Xi ds + \xi dt}^h \right),$$

where $\gamma(\varepsilon)$ is a curve in $\Sigma = M \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with $\gamma'(0) = (0, \delta \rho, \delta \theta)$ and $\gamma_{j^1 \rho \oplus j^1 \theta, \Xi ds + \xi dt}^h(\varepsilon) = (\gamma(\varepsilon), j^1 \rho(0), j^1 \theta(0), v(\varepsilon))$ the horizontal lift to $J^1 \Sigma \oplus (T^* M \otimes (\Sigma \times \mathfrak{so}(3)))$ with $v(\varepsilon) = v_1(\varepsilon)ds + v_2(\varepsilon)dt$ such that $v(0) = \Xi ds + \xi dt$ and $v(\varepsilon)$ is horizontal.

From covariant derivative (5.22) on $\Sigma \times \mathfrak{so}(3)$, this means that

$$\dot{v}_1(\varepsilon)ds + \dot{v}_2(\varepsilon)dt = \mathcal{A}(0, \delta\rho, 0, \delta\theta) \times (v_1(\varepsilon)ds + v_2(\varepsilon)dt).$$

In particular, for $\varepsilon = 0$;

$$\dot{v}_1(0)ds + \dot{v}_2(0)dt = \mathcal{A}(0, \delta\rho, 0, \delta\theta) \times (\Xi ds + \xi dt),$$

and

$$\begin{aligned} \left\langle \frac{\delta l}{\delta(\rho \oplus \theta)}, \delta\rho \oplus \delta\theta \right\rangle &= \left\langle \frac{\partial l}{\partial \rho}, \delta\rho \right\rangle + \left\langle \frac{\partial l}{\partial \theta}, \delta\theta \right\rangle \\ &\quad + \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, \delta\theta) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, \delta\theta) \times \xi \right\rangle. \end{aligned}$$

It is possible to split the horizontal equations in Σ in two parts as in Remark 4.2.6. One related to $\mathbb{R}^3$, for which we make $\delta\theta = 0$;

$$\begin{aligned} \left\langle \frac{\partial l}{\partial \rho} - \partial_s \frac{\delta l}{\partial \rho_s} - \partial_t \frac{\delta l}{\partial \rho_t}, \delta\rho \right\rangle &+ \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, 0) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, \delta\rho, 0, 0) \times \xi \right\rangle = \\ &= \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{B}}((1, \rho_s, \theta_s), \delta\rho) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{B}}((1, \rho_t, \theta_t), \delta\rho) \right\rangle, \end{aligned} \quad (5.24)$$

and one related to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, for which $\delta\rho = 0$;

$$\begin{aligned} \left\langle \frac{\partial l}{\partial \theta} - \partial_s \frac{\delta l}{\partial \theta_s} - \partial_t \frac{\delta l}{\partial \theta_t}, \delta\theta \right\rangle &+ \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{A}}(0, 0, 0, \delta\theta) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{A}}(0, 0, 0, \delta\theta) \times \xi \right\rangle = \\ &= \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{B}}((1, \rho_s, \theta_s), \delta\theta) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{B}}((1, \rho_t, \theta_t), \delta\theta) \right\rangle. \end{aligned} \quad (5.25)$$

In conclusion, the equations of motion of the molecular strand with rotors after reduction by $SO(3)$ are equations (5.23), (5.24) and (5.25). Yet, they are only equivalent to the equations of motion of the original problem if considered together with the reconstruction equation studied in Subsection 4.2.2. Observe that in $\text{Ad}P$, $\omega^{\bar{p}} = R\Xi R^{-1}ds + R\xi R^{-1}dt$. Therefore,

$$d^{\mathcal{A}}\omega^{\bar{p}} = \text{Ad}_R(-\Xi_t + \tilde{\mathcal{A}}_t \times \Xi + \xi_s - \tilde{\mathcal{A}}_s \times \xi)ds \wedge dt,$$

and

$$\omega^{\bar{p}} \wedge \omega^{\bar{p}} = (R\Xi R^{-1}ds + R\xi R^{-1}dt) \wedge (R\Xi R^{-1}ds + R\xi R^{-1}dt) = \text{Ad}_R(\Xi \times \xi)ds \wedge dt.$$

The reconstruction equation (4.14) is then written as

$$\xi_s - \mathcal{A}_s \times \xi - \Xi_t + \mathcal{A}_t \times \Xi + \Xi \times \xi = \text{Ad}_R \tilde{\mathcal{B}} \left((0, \rho_s, \theta_s), (0, \rho_t, \theta_t) \right) = 0. \quad (5.26)$$

5.2.2 Second Stage of Reduction

The group $S = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ acts on $J^1 \Sigma \oplus (T^* M \otimes (\Sigma \times \mathfrak{so}(3)))$: for all $\alpha \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$,

$$\begin{aligned} \alpha \cdot (x, \rho, \theta, ds + dt, \rho_s ds + \rho_t dt, \theta_s ds + \theta_t dt) \oplus (x, \rho, \theta, \Xi ds + \xi dt) = \\ = (x, \rho, \theta + \alpha, ds + dt, \rho_s ds + \rho_t dt, \theta_s ds + \theta_t dt) \oplus (x, \rho, \theta + \alpha, \Xi ds + \xi dt). \end{aligned}$$

Let $K = \Sigma / (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) = M \times \mathbb{R}^3$. Since $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is abelian, the identification $\text{Ad} \Sigma = K \times \mathbb{R}^3$ is immediate and the principal connection on $\Sigma \rightarrow K$ is trivial. Consequently, we have the identification

$$\begin{aligned} \pi_S : J^1 \Sigma \oplus (T^* M \otimes (\Sigma \times \mathfrak{so}(3))) / S \rightarrow J^1 K \oplus (T^* M \otimes (K \times \mathbb{R}^3 \times \mathfrak{so}(3))) \\ [j^1(x, \rho(x), \theta(x)) \oplus (x, \rho, \theta, \Xi ds + \xi dt)]_S \mapsto j^1(x, \rho(x)) \oplus (x, \rho, \theta_s ds + \theta_t dt, \Xi ds + \xi dt), \end{aligned}$$

where $J^1 K \oplus (T^* M \otimes (K \times \mathbb{R}^3 \times \mathfrak{so}(3)))$ is a $\text{FT}\mathfrak{L}\mathfrak{P}$ bundle with extra structure given by;

$$\begin{aligned} \nabla_{v_x, v_\rho}^{\mathfrak{s}}(x, \rho, \beta(x, \rho), \zeta(x, \rho)) = \\ = \left(x, \rho, \beta_{v_x, v_\rho}, \zeta_{v_x, v_\rho} - \tilde{\mathcal{B}}((v_x, v_\rho, 0), (0, 0, \beta)) - \tilde{\mathcal{A}}(v_x, v_\rho, 0, 0) \times \zeta \right); \end{aligned}$$

the $\mathbb{R}^3 \times \mathfrak{so}(3)$ -valued 2-form $\omega^{\mathfrak{s}} = 0 \oplus [\tilde{\mathcal{B}}]_S$; and the Lie bracket

$$\begin{aligned} [(\beta_1, \zeta_1), (\beta_2, \zeta_2)]^{\mathfrak{s}} = \\ = 0 \oplus \left(-\mathcal{A}(0, 0, 0, \beta_1) \times \zeta_2 + \mathcal{A}(0, 0, 0, \beta_2) \times \zeta_1 - \tilde{\mathcal{B}}((0, 0, \beta_1), (0, 0, \beta_2)) + \zeta_1 \times \zeta_2 \right). \end{aligned}$$

From equation (4.11) in Proposition 4.1.15, the vertical Lagrange–Poincaré equations in the second stage of reduction are

$$\text{ad}_{(\theta_s, \Xi)ds + (\theta_t, \xi)dt}^* \frac{\delta l}{\delta(\theta_s ds + \theta_t dt, \Xi ds + \xi dt)} - \text{div}^{\mathfrak{s}} \frac{\delta l}{\delta(\theta_s ds + \theta_t dt, \Xi ds + \xi dt)} = 0. \quad (5.27)$$

Given $(\delta\theta, \delta\eta) \in \mathbb{R}^3 \times \mathfrak{so}(3)$, we obtain that, on one hand,

$$\begin{aligned} & \left\langle \text{ad}_{(\theta_s, \Xi)ds + (\theta_t, \xi)dt}^* \frac{\delta l}{\delta(\theta_s ds + \theta_t dt, \Xi ds + \xi dt)}, (\delta\theta, \delta\eta) \right\rangle \\ &= \left\langle \frac{\delta l}{\delta(\theta_s ds + \theta_t dt)} \oplus \frac{\delta l}{\delta(\Xi ds + \xi dt)}, [(\theta_s ds + \theta_t dt, \Xi ds + \xi dt), (\delta\theta, \delta\eta)]^{\mathfrak{g}} \right\rangle \\ &= \left\langle \frac{\delta l}{\delta \Xi}, -\mathcal{A}(0, 0, 0, \theta_s) \times \delta\eta + \mathcal{A}(0, 0, 0, \delta\theta) \times \Xi - \tilde{\mathcal{B}}((0, 0, \theta_s), (0, 0, a)) + \Xi \times \delta\eta \right\rangle \\ &\quad + \left\langle \frac{\delta l}{\delta \xi}, -\mathcal{A}(0, 0, 0, \theta_t) \times \delta\eta + \mathcal{A}(0, 0, 0, \delta\theta) \times \Xi - \tilde{\mathcal{B}}((0, 0, \theta_t), (0, 0, a)) + \xi \times \delta\eta \right\rangle \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left\langle \text{div}^{\mathfrak{g}} \frac{\delta l}{\delta(\theta_s ds + \theta_t dt, \Xi ds + \xi dt)}, (\delta\theta, \delta\eta) \right\rangle = \\ &= \left\langle \partial_s \frac{\delta l}{\delta \theta_s} + \partial_t \frac{\delta l}{\delta \theta_t}, \delta\theta \right\rangle + \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{B}}((0, \rho_s, 0), (0, 0, \delta\theta)) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{B}}((0, \rho_t, 0), (0, 0, \delta\theta)) \right\rangle \\ &\quad \oplus \left\langle \partial_s \frac{\delta l}{\delta \Xi} + \partial_t \frac{\delta l}{\delta \xi} + \mathcal{A}(0, \rho_s, 0, 0) \times \frac{\delta l}{\delta \Xi} + \mathcal{A}(0, \rho_t, 0, 0) \times \frac{\delta l}{\delta \xi}, \delta\eta \right\rangle. \end{aligned}$$

From these two expressions, it can be proven that the vertical Lagrange–Poincaré equations 5.27 can be split into two parts related to $\mathbb{R}^3$ and $\mathfrak{so}(3)$. These respectively coincide with equations (5.25) and (5.23), except for the term $\delta l / \delta \theta$ which is zero from the symmetry of the Lagrangian with respect to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The horizontal Lagrange–Poincaré equations are

$$\left\langle \frac{\delta l}{\delta \rho} - \partial_s \frac{\delta l}{\delta \rho_s} - \partial_t \frac{\delta l}{\delta \rho_t}, \delta \rho \right\rangle = \left\langle \frac{\delta l}{\delta \Xi}, \tilde{\mathcal{B}}((0, \rho_s, 0), \delta \rho) \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \tilde{\mathcal{B}}((0, \rho_t, 0), \delta \rho) \right\rangle. \quad (5.28)$$

As we did in in Subsection 5.2.1 in the first stage of reduction, we express the horizontal derivative in terms of partial and fiber derivatives as

$$\left\langle \frac{\delta l}{\delta \rho}, \delta \rho \right\rangle = \left\langle \frac{\partial l}{\partial \rho}, \delta \rho \right\rangle + \left\langle \frac{\delta l}{\delta \Xi}, \mathcal{A}(0, \delta \rho, 0, 0) \times \Xi \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \mathcal{A}(0, \delta \rho, 0, 0) \times \xi \right\rangle.$$

Therefore, the horizontal Lagrange–Poincaré equation (5.28) of the second stage coincides with equation (5.24). In conclusion, equations (5.23), (5.24) and (5.25) are also obtained as the equations of motion in the second stage of reduction. However, in the first stage equation (5.25) is an horizontal equation whereas in the second stage is a new vertical equation in the sense of Remark 4.2.6. Furthermore, the reconstruction equation from the second stage to

the first one is easily seen to be

$$\partial_t \theta_s = \partial_s \theta_t. \quad (5.29)$$

Remark 5.2.1 *The trivial Maurer-Cartan connection may not always be the more convenient for a particular Lagrangian. In mechanics, the study of a rigid body with rotors in [68] is an example where the appropriate connection adapted to the Lagrangian is the mechanical connection. This was the topic of discussion in Section 3.2.*

5.2.3 A Particular Lagrangian

We shall now study a particular Lagrangian in $J^1 P$;

$$\begin{aligned} L(r, r_s, r_t, R, R_s, R_t, \theta, \theta_s, \theta_t) = & \frac{1}{2} \langle r_t, r_t \rangle + \frac{1}{2} \langle R^{-1} R_t, I R^{-1} R_t \rangle \\ & + \frac{1}{2} \langle R^{-1} R_t + \theta_t, K(R^{-1} R_t + \theta_t) \rangle - U(R^{-1} R_s, \theta_s, \|r\|), \end{aligned}$$

where I is the inertia tensor of the configuration of the strand, K is the inertia tensor of the rotors and U is a function called potential energy. This Lagrangian combines terms appearing in the Lagrangian given for the rigid body with rotors in [68] and Subsection 3.2, as well as, the Lagrangian proposed in [31] for the molecular strand. It is invariant by $SO(3) \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ and the dependence of the potential energy on r must be radial for this to hold. If the first stage of reduction is performed using the subgroup $SO(3)$ and the Maurer-Cartan connection, that is $\tilde{\mathcal{A}} = 0$, then the reduced Lagrangian is:

$$\begin{aligned} l(\rho, \rho_s, \rho_t, \theta, \theta_s, \theta_t, \Omega, \omega) = & \frac{1}{2} \langle \rho_t + \omega \times \rho, \rho_t + \omega \times \rho \rangle + \frac{1}{2} \langle \omega, I \omega \rangle \\ & + \frac{1}{2} \langle \omega + \theta_t, K(\omega + \theta_t) \rangle - U(\Omega, \theta_s, \|\rho\|), \end{aligned}$$

For this Lagrangian,

$$\begin{aligned} \frac{\partial l}{\partial \rho} &= (\rho_t + \omega \times \rho) \times \omega - \frac{\partial U}{\partial \rho} \frac{\rho}{\|\rho\|}, & \frac{\delta l}{\partial \rho_s} &= 0, & \frac{\delta l}{\partial \rho_t} &= \rho_t + \omega \times \rho, \\ \frac{\delta l}{\partial \theta_s} &= -\frac{\partial U}{\partial \theta_s}, & \frac{\delta l}{\partial \theta_t} &= K(\omega + \theta_t), \\ \frac{\delta l}{\delta \Xi} &= \frac{\partial l}{\partial \Omega} = -\frac{\partial U}{\partial \Omega}, & \frac{\delta l}{\delta \xi} &= \frac{\partial l}{\partial \omega} = \rho \times (\rho_t + \omega \times \rho) + (I + K)\omega + K\theta_t, \end{aligned}$$

The vertical Lagrange–Poincaré equations (5.23) takes the form

$$0 = \rho \times (\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \omega \times (\omega \times \rho)) \\ + (I + K)\omega_t + K\theta_{tt} + \omega \times ((I + K)\omega + K\theta_t) - \partial_s \frac{\partial U}{\partial \Omega} + \Omega \times \frac{\partial U}{\partial \Omega}, \quad (5.30)$$

while the horizontal Lagrange–Poincaré equations (5.24) and (5.25) particularize as

$$\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \omega \times (\omega \times \rho) = -\frac{\partial U}{\partial \rho} \frac{\rho}{\|\rho\|}, \quad (5.31)$$

$$K\omega_t + K\theta_{tt} = \partial_s \frac{\partial U}{\partial \theta_s}. \quad (5.32)$$

The reconstruction equation (5.26) becomes

$$\omega_s - \Omega_t \Omega \times \omega = 0. \quad (5.33)$$

Observe that on the left hand side of equation (5.31), the acceleration R_{tt} in the inertial fixed frame is reinterpreted as the sum of the acceleration ρ_{tt} in a non-inertial frame and additional acceleration terms. These additional terms have been historically interpreted as fictitious forces: $2\omega \times \rho_t$ is the Coliolis force, $\omega_t \times \rho$ is the Euler force and $\omega \times (\omega \times \rho)$ is the centrifugal force. Thus, equation (5.31), states that the time evolution of ρ is the same as the evolution of particle in a potential U seen from a non-inertial frame with angular rotation $\omega(s, t)$. As the potential U is radial, equation (5.30) can be rewritten as

$$0 = (I + K)\omega_t + K\theta_{tt} + \omega \times ((I + K)\omega + K\theta_t) - \partial_s \frac{\partial U}{\partial \Omega} + \Omega \times \frac{\partial U}{\partial \Omega},$$

after substitution of equation (5.31). Equation (5.32) is related to the momentum of the rotors and will appear as a new vertical equation in the next stage of reduction. In fact, the second stage of reduction defines the reduced Lagrangian

$$l(\rho, \rho_s, \rho_t, a, b, \Omega, \omega) = \frac{1}{2} \langle \rho_t + \omega \times \rho, \rho_t + \omega \times \rho \rangle + \frac{1}{2} \langle \omega, I\omega \rangle \\ + \frac{1}{2} \langle \omega + b, K(\omega + b) \rangle - U(\Omega, a, \langle \rho, \rho \rangle),$$

whose vertical Lagrange–Poincaré equations are;

$$0 = \rho \times (\rho_{tt} + 2\omega \times \rho_t + \omega_t \times \rho + \omega \times (\omega \times \rho)) \\ + (I + K)\omega_t + Kb_t + \omega \times ((I + K)\omega + Kb) - \partial_s \frac{\partial U}{\partial \Omega} + \Omega \times \frac{\partial U}{\partial \Omega}, \quad (5.34)$$

$$K\omega_t + Kb_t = \partial_s \frac{\partial U}{\partial a}. \quad (5.35)$$

The horizontal Lagrange–Poincaré equations are equations (5.31) and the reconstruction equation is simply $a_t = b_s$. This reconstruction condition makes the set of equations of motion in the first stage of reduction; (5.30), (5.31), and (5.32); equivalent to the equations of motion of motion of the second stage: (5.34), (5.35) and (5.31).

Chapter 6

Poisson-Poincaré Reduction

6.1 Hamiltonian Field Theory

The geometric formulations of Hamiltonian classical field theory are diverse and a complete generalization of the well-known Marsden–Weinstein reduction theorem in [62] is yet to be accomplished. Let alone, a field theory analogue to the Hamiltonian reduction by stages gathered in [63]. As studied in Section 2.3, the bundles in $\mathfrak{L}\mathfrak{P}^*$ are Poisson manifolds. Thus, a natural first step towards Hamiltonian reduction by stages in field theory is to establish a Poisson–Poincaré reduction procedure in field theory. The later is the main objective of this chapter.

6.1.1 Multisymplectic formalism

The aim of this section is to overview the multisymplectic and polysymplectic formalism of Hamiltonian field theory as well as their bracket formulation.

Definition 6.1.1 *The dual jet bundle $J^1 P^*$ of a fiber bundle $\pi_{M,P} : P \rightarrow M$ is a vector bundle over P such that for any $y \in P_x, x \in M$, the fiber is*

$$J_y^1 P^* = \text{Aff} \left(J_y^1 P, \bigwedge^n T_x^* M \right),$$

the set of affine maps from the jet bundle to the bundle of n -forms on M , where $n = \dim M$.

Provided a fiber chart (x^i, y^a) on P , we define fiber coordinates $(x^i, y^a, p_a^i, \tilde{p})$ on $J^1 P^*$ so that its elements have the expression:

$$y_i^a \mapsto (\tilde{p} + p_a^i y_i^a) d^n x,$$

where $d^n x = dx^1 \wedge \cdots \wedge dx^n$. Observe that the use of affine morphisms implies that $\text{rank}(J^1 P^*) = \text{rank}(J^1 P) + 1$. An alternative description of $J^1 P^*$ is provided by the sub-bundle Z of $\bigwedge^n T^* P$ whose fibers are

$$Z = \left\{ z \in \bigwedge^n T^* P \mid i_v i_w z = 0 \text{ for all } v, w \in VP \right\} \quad (6.1)$$

Any n -form $z \in Z$ can locally be written as $z = \tilde{p} d^n x + p_a^i dy_i^a \wedge d^{n-1} x_i$, where $d^{n-1} x_i = i_{\partial/\partial x^i} d^n x$. The mapping $\Psi : Z \rightarrow J^1 P^*$ defined by

$$\langle \Psi(z), j_x^1 s \rangle = s^* z \in \bigwedge^n T_x^* M,$$

is a vector bundle isomorphism whose localization in the coordinate systems previously introduced in Z and $J^1 P^*$ is the identity. In conclusion, both spaces are canonically isomorphic as vector bundles over P .

Definition 6.1.2 The *canonical n -form* $\Theta_\wedge$ on $\bigwedge^n T^* P$ is

$$\Theta_\wedge(z)(u_1, \dots, u_n) = z \left(T\pi_{P, \bigwedge^n T_x^* P} u_1, \dots, T\pi_{P, \bigwedge^n T_x^* P} u_n \right), \quad z \in Z, \quad (6.2)$$

where $u_i \in T_z(\bigwedge^n T^* P)$ for $i = 1, \dots, n$. The *canonical $(n+1)$ -form* on $\bigwedge^n T^* P$ is

$$\Omega_\wedge = -d\Theta_\wedge. \quad (6.3)$$

Definition 6.1.3 The *canonical multisymplectic n -form* Θ on Z is the pullback

$$\Theta = i_{Z, \bigwedge^n T^* P}^* \Theta_\wedge, \quad (6.4)$$

where $i_{Z, \bigwedge^n T^* P} : Z \rightarrow \bigwedge^n T^* P$ denotes the inclusion. The *canonical multisymplectic $(n+1)$ -form*

$$\Omega = -d\Theta. \quad (6.5)$$

The canonical n -form $(\Psi^{-1})^* \Theta$ and $(n+1)$ -form $(\Psi^{-1})^* \Omega$ on $J^1 P^*$ will again be denoted Θ and Ω respectively. The identification of $J^1 P^*$ and Z via Ψ justifies this abuse of notation. These forms have the following local expressions:

$$\Theta = p_a^i dy_i^a \wedge d^{n-1} x_i + \tilde{p} d^n x, \quad (6.6)$$

$$\Omega = dy^a \wedge dp_a^i \wedge d^{n-1} x_i - d\tilde{p} \wedge d^n x. \quad (6.7)$$

Remark 6.1.4 The pair $(J^1 P^*, \Omega)$ is an example of multisymplectic manifold called *canonical n -plectic structure* on $J^1 P^*$. A more detailed discussion of the multisymplectic formalism can be found on [42] and some advances on reduction of general multisymplectic manifolds are discussed in [9, 27, 30].

6.1.2 Polysymplectic formalism

Definition 6.1.5 Given a fiber bundle $P \rightarrow M$, its **polysymplectic bundle** is the vector bundle over M

$$\Pi_P = \pi_{MP}^* TM \otimes_P V^* P \otimes_P \pi_{MP}^* \left(\bigwedge^n T^* M \right).$$

Unless the base space is not clear from the context, we shall simply write $\Pi_P = TM \otimes V^* P \otimes \bigwedge^n T^* M$ to avoid tedious notation. For every $p \in \Pi_P$, there are local coordinates (x^i, y^a, p_a^i) such that

$$p = p_a^i \frac{\partial}{\partial x^i} \otimes dy^a \otimes d^n x.$$

Observe that $\text{rank}(\Pi_P) = \text{rank}(J^1 P)$ as bundles over P . In addition, the fiber map

$$\begin{aligned} J^1 P^* &\rightarrow \Pi_P \\ \varphi &\mapsto \vec{\varphi}, \end{aligned}$$

where $\vec{\varphi} \in (\Pi_P)_y = (T_x^* M \otimes V_y P)^* \otimes \bigwedge^n T_x^* M$ is the linear map associated to the affine map $\varphi \in J_y^1 P^* = \text{Aff}(J_y^1 P, \bigwedge^n T_x^* M)$, is an affine bundle modeled on the rank-one vector bundle $\bigwedge^n T^* M \rightarrow M$.

Definition 6.1.6 A **Hamiltonian system** is a pair (Π_P, δ) where δ is a section of $J^1 P^* \rightarrow \Pi_P$.

The canonical forms on $J^1 P^*$ can be pulled back to Π_P to obtain $\Theta_\delta = \delta^* \Theta$ and $\Omega_\delta = -d\Theta_\delta$. In local coordinates, $\delta(x^i, y^a, p_a^i) = (x^i, y^a, p_a^i, H_\delta(x^i, y^a, p_a^i))$ and

$$\Theta_\delta = p_a^i dy^a \wedge d^{n-1} x_i + H_\delta(x^i, y^a, p_a^i) d^n x. \quad (6.8)$$

Definition 6.1.7 A section p of $\Pi_P \rightarrow M$ is a **solution** of the Hamiltonian system if for any vertical vector field X on Π_P ,

$$0 = p^* i_X \Omega_\delta = -p^* i_X d\Theta_\delta. \quad (6.9)$$

Given an Ehresmann connection Λ in $P \rightarrow M$, there is a natural section δ_Λ of $J^1 P^* \rightarrow \Pi_P$ such that for any $v_x \otimes \omega_y \otimes v \in T_x M \otimes V_y^* P \otimes \bigwedge^n T_x^* M$,

$$\delta_\Lambda(v_x \otimes \omega_y \otimes v) = \langle \omega_y, \text{Hor}(\cdot) \rangle \wedge i_{v_x} v \in Z_y \simeq (J^1 P^*)_y.$$

For any *Hamiltonian section* δ of $J^1 P^* \rightarrow \Pi_P$, there is a mapping $\mathcal{H} : \Pi_P \rightarrow \bigwedge^n T^* M$ called *Hamiltonian density* such that $\mathcal{H} = \delta - \delta_\Lambda$. Thus, a Hamiltonian system can equivalently be defined by a triple $(\Pi_P, \Lambda, \mathcal{H})$ where Λ is an Ehresmann connection and $\mathcal{H}$ a Hamiltonian density. The Hamilton equation (6.9) is locally equivalent to the Hamilton–Cartan equations

$$\frac{\partial H}{\partial p_a^i} = \frac{\partial y^a}{\partial x^i} - \Lambda_i^a, \quad -\frac{\partial H}{\partial y^a} = \frac{\partial p_a^i}{\partial x^i} + \frac{\partial \Lambda_i^b}{\partial y^a} p_b^i, \quad (6.10)$$

where Λ_i^a are the coefficients of the horizontal lift defined by Λ and $\mathcal{H} = H d^n x$.

Definition 6.1.8 An r -form F on $J^1 P^*$ is said to be **horizontal** if $i_u F = 0$ for any vertical tangent vector with respect to $J^1 P^* \rightarrow M$.

In local coordinates, $F = F_{i_1, \dots, i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$.

Definition 6.1.9 A horizontal r -form F is a **Poisson form** if there is a vertical $(n - r)$ -multivector field χ_F on $J^1 P^*$ such that

$$i_{\chi_F} \Omega = dF. \quad (6.11)$$

Given F a Poisson r -form and E a Poisson s -form their *Poisson bracket* is the $(r + s + 1 - n)$ -form on $J^1 P^*$,

$$\{F, E\} = (-1)^{r(s-1)} i_{\chi_E} i_{\chi_F} \Omega. \quad (6.12)$$

which, in turn, is a Poisson form [19]. To be exact, this operator is a generalized graded Poisson bracket with a modified Leibniz rule giving rise to a higher-order Gerstenhaber algebra. More details can be found in [54]. It has been proved in [19] that a Poisson form on $J^1 P^*$ projects to Π_P so that we can consider that Poisson forms are defined on Π_P . Poisson $(n - 1)$ -forms are key to express the Hamilton–Cartan equations (6.10) in terms of a bracket in the same way affine functions in a cotangent bundle play a fundamental role in Hamiltonian mechanics.

Theorem 6.1.10 [19, Proposition 5.2] A section p of $\Pi_P \rightarrow M$ is a solution of a given Hamiltonian system $(\Pi_P, \Lambda, \mathcal{H} = H v)$ if and only if for any horizontal Poisson $(n - 1)$ -form F the following equation holds true:

$$\{F, H\} v \circ p = d(p^* F) - (d^h F) \circ p, \quad (6.13)$$

where $d^h F$ is the horizontal differential of F with respect to the connection on Π_P .

The aforementioned connection on $\Pi_P \rightarrow M$ is obtained from the Ehresmann connection Λ on $P \rightarrow M$ and any Riemannian connection on M . The local expression of the horizontal lift of this connection is

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i} + \Lambda_i^a \frac{\partial}{\partial y^a} + \left(-\frac{\partial \Lambda_i^b}{\partial y^a} p_b^j + \Gamma_{ik}^j p_a^k - \Gamma_{ik}^k p_a^j \right) \frac{\partial}{\partial p_a^j}, \quad (6.14)$$

where Γ_{ij}^k are the Christoffel symbols of the Riemannian connection. As stated before, the main objective of this chapter is to present a reduction theory for this Poisson covariant formulation of the Hamilton equations.

6.2 The Reduced Polysymplectic Bundle

Given a fiber bundle $P \rightarrow M$ and a Lie group G acting freely, properly and vertically on P , we have $P \rightarrow P/G = \Sigma \rightarrow M$, where $P \rightarrow \Sigma$ is a principal bundle and $\Sigma \rightarrow M$ a fiber bundle. The action of G is canonically lifted to Π_P . We now study the reduced polysymplectic bundle Π_P/G and relate it with Π_Σ .

6.2.1 Splitting Induced by a Principal Bundle

The maps $\alpha_{\mathcal{A}}^{TP}$ and $(\alpha_{\mathcal{A}}^{TP})^{-1}$ introduced in Lemma 1.2.8 provide a splitting $(TP)/G \simeq T\Sigma \oplus \tilde{\mathfrak{g}}$. These maps are isomorphisms of vector bundles over Σ and, in turn, isomorphisms of bundles over M . The dual application of $(\alpha_{\mathcal{A}}^{TP})^{-1}$, $\tau_{\mathcal{A}}^{TP} = ((\alpha_{\mathcal{A}}^{TP})^{-1})^*$, similarly splits $(T^*P)/G$ in a vertical and horizontal part with respect to $\mathcal{A}$. Indeed,

$$\tau_{\mathcal{A}}^{TP} = [\text{hor}_{\mathcal{A}}^*]_G \oplus [\bar{J}]_G : (T^*P)/G \rightarrow T^*\Sigma \oplus \tilde{\mathfrak{g}}^* \quad (6.15)$$

where $\text{hor}_{\mathcal{A}}^* : T^*P \rightarrow \pi_{\Sigma, P}^* T^*\Sigma$ is the dual map of the horizontal lift $\text{hor}_{\mathcal{A}}$ and $[\bar{J}]_G : (T^*P)/G \rightarrow \tilde{\mathfrak{g}}^*$ is the function induced by the equivariant momentum map of the action on T^*P ,

$$\begin{aligned} \bar{J} : T^*P &\rightarrow \mathfrak{g}^* \\ \gamma_y &\mapsto \bar{J}(\gamma_y)(\eta) = \gamma_y(\eta_y^P). \end{aligned} \quad (6.16)$$

The map $\tau_{\mathcal{A}}^{TP}$ is an isomorphism of vector bundles over Σ whose inverse is

$$\begin{aligned} (\tau_{\mathcal{A}}^{TP})^{-1} : T^*\Sigma \oplus \tilde{\mathfrak{g}}^* &\rightarrow (T^*P)/G \\ \bar{\omega}_s \oplus \eta_s &\mapsto \left[\pi_{\Sigma,P}^* \bar{\omega}_s + \eta_s \circ [p, \mathcal{A}(\cdot)] \right]_G. \end{aligned}$$

Remark 6.2.1 *There are two different notions of vertical subspace of TP depending on which base space is considered. On one hand,*

$$V_{\pi_{M,P}}P = \{v \in TP, T\pi_{M,P}(v) = 0\} = \ker(T\pi_{M,P})$$

are the vertical vectors with respect to the fiber bundle $\pi_{M,P} : P \rightarrow M$. On the other hand, $V_{\pi_{\Sigma,P}}P = \ker(T\pi_{\Sigma,P})$, where vertical refers to the bundle $\pi_{\Sigma,P} : P \rightarrow \Sigma$. Hereafter, the notation VP will exclusively refer to $V_{\pi_{M,P}}P$.

Proposition 6.2.2 *The restricted map $\alpha_{\mathcal{A}}^{TP}|_{VP/G} : (VP)/G \rightarrow V\Sigma \oplus \tilde{\mathfrak{g}}$ is an isomorphism of vector bundles over Σ and a diffeomorphism of bundles over M , whose inverse is given by $(\alpha_{\mathcal{A}}^{TP})^{-1}|_{V\Sigma \oplus \tilde{\mathfrak{g}}}$.*

Proof. For every $v_y \in T_yP$,

$$T_y\pi_{M,P}(v_y) = T_s\pi_{M,\Sigma} \circ T_y\pi_{\Sigma,P}(v_y),$$

and $v_y \in V_yP$ if and only if $T\pi_{\Sigma,P}(v_y) \in V_s\Sigma$. Consequently, $\text{Im}(\alpha_{\mathcal{A}}^{TP}|_{VP/G}) \subseteq V\Sigma \oplus \tilde{\mathfrak{g}}$. For every $v_s \in T_s\Sigma$,

$$T_s\pi_{M,\Sigma}(v_s) = T_s\pi_{M,\Sigma} \circ T_y\pi_{\Sigma,P} \circ \text{hor}_{\mathcal{A}}^y(v_s) = T_y\pi_{M,P} \circ \text{hor}_{\mathcal{A}}^y(v_s),$$

and $v_s \in V_s\Sigma$ if and only if $\text{hor}_{\mathcal{A}}^y(v_s) \in V_yP$. Therefore, $\text{hor}_{\mathcal{A}}|_{V\Sigma} : V\Sigma \rightarrow VP$ and $\text{Im}((\alpha_{\mathcal{A}}^{TP})^{-1}|_{V\Sigma \oplus \tilde{\mathfrak{g}}}) \subseteq VP/G$. As $(\alpha_{\mathcal{A}}^{TP})^{-1} \circ \alpha_{\mathcal{A}}^{TP} = \text{Id}_{TP/G}$, we observe that

$$\text{Id}_{VP/G} = ((\alpha_{\mathcal{A}}^{TP})^{-1} \circ \alpha_{\mathcal{A}}^{TP})|_{VP/G} = (\alpha_{\mathcal{A}}^{TP})^{-1}|_{V\Sigma \oplus \tilde{\mathfrak{g}}} \circ \alpha_{\mathcal{A}}^{TP}|_{VP/G}$$

and conclude that both maps are in fact isomorphisms. $\square$

We shall denote $\alpha_{\mathcal{A}} : (VP)/G \rightarrow V\Sigma \oplus \tilde{\mathfrak{g}}$ and $\alpha_{\mathcal{A}}^{-1} : V\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow (VP)/G$ the restricted maps introduced in Proposition 6.2.2. Correspondingly, we shall denote their respective dual maps,

$$\tau_{\mathcal{A}}^{-1} : V^*\Sigma \oplus \tilde{\mathfrak{g}}^* \rightarrow (V^*P)/G, \quad \tau_{\mathcal{A}} : (V^*P)/G \rightarrow V^*\Sigma \oplus \tilde{\mathfrak{g}}^*.$$

These splittings induce a similar one in the polysymplectic bundle.

Proposition 6.2.3 *Given a principal connection $\mathcal{A}$ on the fiber bundle $P \rightarrow \Sigma$, the map*

$$\psi_{\mathcal{A}} = \text{id}_{TM} \otimes \tau_{\mathcal{A}} \otimes \text{id}_{\bigwedge^n T^*M} : \Pi_P/G \longrightarrow \Pi_{\Sigma} \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$$

is an isomorphism of vector bundles over Σ .

Proof. Since G acts vertically on $P \rightarrow M$, the action of G on both, TM and $\bigwedge^n T^*M$ is trivial. Thus, the quotient of the polysymplectic space $\Pi_P = TM \otimes_M V^*P \otimes_M \bigwedge^n T^*M$ is

$$\Pi_P/G = TM \otimes_{\Sigma} (V^*P)/G \otimes_{\Sigma} \bigwedge^n T^*M,$$

and $\tau_{\mathcal{A}}$ splits the second factor as desired. $\square$

We denote $\kappa = \psi_{\mathcal{A}} \circ \pi_{\Pi_P/G, \Pi_P}$, the projection of $\Pi_P \rightarrow \Pi_P/G$ using the identification $\psi_{\mathcal{A}}$. In the following, we obtain a local expression for κ and $T\kappa$ in an adapted coordinate system.

Suppose that the principal bundle $P \rightarrow \Sigma$ is trivializable and $P \simeq \Sigma \times G$. Let $\Sigma \rightarrow M$ be in turn trivializable and $\Sigma \simeq M \times F$, where F is diffeomorphic to the fibers. For any point $y \in P$, we will respectively denote its projection to Σ and M by $s = \pi_{\Sigma, P}(y)$ and $x = \pi_{M, P}(y)$. A trivialization such that $y = (s, e)$ can always be chosen. In coordinates we can express $s = (x^i, x^a)$ where $i = 1, \dots, n = \dim M$, $k = \dim G$, $m = \dim P > n + k$, and $a = 1, \dots, m - k - n$. Also, $x = \pi_{M, P}(y) = \pi_{M, \Sigma}(s)$ has coordinates $(x^1, \dots, x^i, \dots, x^n)$. Fixing a basis $\{B_1, \dots, B_k\}$ of $\mathfrak{g}$ we can choose a normal coordinate system in a neighborhood of (s, e) such that $v = dx^1 \wedge \dots \wedge dx^n$ is a volume form and the coordinates of any point (s, g) in such neighborhood are (x^i, x^a, y^α) where $y^\alpha(g)$, for $\alpha = 1, \dots, k$ are given by

$$g = \exp(y^\alpha(g)B_\alpha).$$

The induced coordinate system on Π_P has coordinates $(x^i, x^a, y^\alpha, p_\alpha^i, p_\alpha^a)$. Similarly, since the coordinates in Σ are given by (x^i, x^a) , the induced coordinate system on Π_{Σ} is $(x^i, x^a, \sigma_\alpha^i)$. The trivialization around $y = (s, e)$ defines local sections $\tilde{B}_\alpha$ in $\tilde{\mathfrak{g}} \rightarrow \Sigma$ given by $s \mapsto [(s, e), B_\alpha]_G$ as well as local sections $\tilde{B}^\alpha$ in $\tilde{\mathfrak{g}}^* \rightarrow \Sigma$ and $\tilde{B}_i^\alpha = \frac{\partial}{\partial x^i} \otimes \tilde{B}^\alpha \otimes v$ in $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$. The later set of local sections induces a local coordinate system (x^i, x^a, μ_α^i) on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$. The results of this chapter are not restricted to trivializable bundles: Since bundles are locally trivializable, we can define such sets of coordinates around any $y \in P$ and use them to prove local formulas.

In particular, the horizontal lift at y through $\mathcal{A}$, $\text{hor}_{\mathcal{A}}^y$, has a local expression given by

$$\begin{aligned} \text{hor}_{\mathcal{A}}^y : \pi_{\Sigma, P}^* T_s \Sigma &\rightarrow T_y P \\ \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial x^i} + A_i^\alpha(y) \frac{\partial}{\partial y^\alpha} \\ \frac{\partial}{\partial x^a} &\mapsto \frac{\partial}{\partial x^a} + A_a^\alpha(y) \frac{\partial}{\partial y^\alpha}, \end{aligned}$$

and we have the following result.

Proposition 6.2.4 *The local expression of the projection $\kappa : \Pi_P \rightarrow \Pi_\Sigma \oplus (TM \otimes \mathfrak{g}^* \otimes \bigwedge^n T^*M)$ is:*

$$\kappa(x^i, x^a, y^\alpha, p_a^i, p_\alpha^i) = (x^i, x^a, \sigma_a^i = p_a^i + p_\beta^i A_a^\beta(x^i, x^a, y^\alpha), \mu_\alpha^i = Z_\alpha^\beta(g^{-1}) p_\beta^i), \quad (6.17)$$

where $Z_\alpha^\beta(g)$ is such that

$$T_g L_{g^{-1}} \left(\frac{\partial}{\partial y^\alpha} \right)_g = Z_\alpha^\beta(g) \left(\frac{\partial}{\partial y^\beta} \right)_e.$$

In particular, for $y = (s, e)$; $\sigma_a^i = p_a^i + p_\alpha^i A_a^\alpha$ and $\mu_\alpha^i = p_\alpha^i$.

Proof. First we obtain the local expression of $\tau_{\mathcal{A}}^{-1} : T^* \Sigma \oplus \mathfrak{g}^* \rightarrow (T^*P)/G$. From the definition:

$$\tau_{\mathcal{A}}^{-1}(dx^a) = \pi_{\Sigma, P}^* dx^a = dx^a.$$

Let $y = (s, g)$ for every $v \in T_y P$,

$$\begin{aligned} \tau_{\mathcal{A}}^{-1}(\tilde{B}^\alpha)([v]_G) &= \langle \tilde{B}^\alpha, [(s, g), \mathcal{A}(s, g)(v)]_G \rangle = \langle \tilde{B}^\alpha, [(s, e), \text{Ad}_{g^{-1}} \circ \mathcal{A}(s, e)(v)]_G \rangle \\ &= \langle \tilde{B}^\alpha, [(s, e), \mathcal{A}(s, e) \circ T_g L_{g^{-1}}(v)]_G \rangle. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{A}(s, e) \circ T_g L_{g^{-1}} \left(\frac{\partial}{\partial x^a} \right)_g &= \mathcal{A}(s, e) \left(\frac{\partial}{\partial x^a} \right)_e = -A_a^\beta(s, e), \\ \mathcal{A}(s, e) \circ T_g L_{g^{-1}} \left(\frac{\partial}{\partial y^\beta} \right)_g &= \mathcal{A}(s, e) \left(Z_\beta^\alpha(g) \left(\frac{\partial}{\partial y^\alpha} \right)_e \right) = \delta_\beta^\gamma Z_\gamma^\alpha(g), \end{aligned}$$

we obtain that

$$\tau_{\mathcal{A}}^{-1}(\tilde{B}^\alpha) = -A_a^\alpha(s, e) dx^a + Z_\beta^\alpha(g) dy^\beta$$

and complete the local expression of $\tau_{\mathcal{A}}^{-1}$. Then, we obtain the local expression of the inverse map:

$$\begin{aligned}\tau_{\mathcal{A}} &: (T^*P)/G \rightarrow T^*\Sigma \oplus \tilde{\mathfrak{g}}^* \\ dx^a &\mapsto dx^a \\ dy^\beta &\mapsto Z_\alpha^\beta(g^{-1})\tilde{B}^\alpha + Z_\gamma^\beta(g^{-1})A_a^\gamma(s, e)dx^a = Z_\alpha^\beta(g^{-1})\tilde{B}^\alpha + A_a^\beta(s, g)dx^a\end{aligned}$$

Consequently,

$$\begin{aligned}\kappa\left(p_\beta^i \frac{\partial}{\partial x^i} \otimes dy^\alpha \otimes v\right) &= p_\beta^i A_a^\beta(s, g) \frac{\partial}{\partial x^i} \otimes dx^a \otimes v + p_\beta^i Z_\alpha^\beta(g^{-1}) \frac{\partial}{\partial x^i} \otimes \tilde{B}^\alpha \otimes v, \\ \kappa\left(p_a^i \frac{\partial}{\partial x^i} \otimes dx^a \otimes v\right) &= p_a^i \frac{\partial}{\partial x^i} \otimes dx^a \otimes v.\end{aligned}$$

and the local expression follows. $\square$

Lemma 6.2.5 *The transformation $Z_\alpha^\beta(g)$ can be expressed as*

$$Z_\alpha^\beta(g) = \sum_{n=1}^{\infty} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} c_n^\beta(-y^\gamma B_\gamma : y^\gamma B_\gamma + \varepsilon B_\alpha),$$

where c_n are the coefficients of the Baker-Campbell-Hausdorff formula

$$\exp X \exp Y = \exp \left(\sum_{n=1}^{\infty} c_n(X : Y) \right), \quad X, Y \in \mathfrak{g}.$$

Furthermore,

$$\frac{\partial Z_\alpha^\beta}{\partial y^\gamma} \Big|_{y=(s,e)} = \frac{1}{2} c_{\alpha\gamma}^\beta, \quad (6.18)$$

where $c_{\alpha\gamma}^\beta$ are the coefficients of the Lie bracket in $\mathfrak{g}$.

Proof. For the first part, observe that

$$\begin{aligned}T_g L_{g^{-1}} \left(\frac{\partial}{\partial y^\alpha} \right)_g &= T_g L_{g^{-1}} \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \exp(y^\gamma B_\gamma + \varepsilon B_\alpha) \right) \\ &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \exp(-y^\gamma B_\gamma) \exp(y^\gamma B_\gamma + \varepsilon B_\alpha) \\ &= \sum_{n=1}^{\infty} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} c_n^\beta(-y^\gamma B_\gamma : y^\gamma B_\gamma + \varepsilon B_\alpha) \left(\frac{\partial}{\partial y^\beta} \right)_e.\end{aligned}$$

Then,

$$\begin{aligned}
\left. \frac{\partial Z_\alpha^\beta}{\partial y^\gamma} \right|_{y=(s,e)} &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} Z_\alpha^\beta(\exp(\epsilon B_\gamma)) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \sum_{n=1}^{\infty} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} c_n^\beta(-\epsilon B_\gamma : \epsilon B_\gamma + \epsilon B_\alpha) \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} c_2^\beta(-\epsilon B_\gamma : \epsilon B_\gamma + \epsilon B_\alpha) \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \frac{1}{2} [-\epsilon B_\gamma, \epsilon B_\gamma + \epsilon B_\alpha]^\beta = -\frac{1}{2} c_{\gamma\alpha}^\beta = \frac{1}{2} c_{\alpha\gamma}^\beta,
\end{aligned}$$

where we have used that $c_n^\beta(-\epsilon B_\gamma : \epsilon B_\gamma + \epsilon B_\alpha)$ is an homogeneous polynomial of degree n with respect to ϵ and ϵ in order to obtain the second-to-last equivalence. $\square$

Lemma 6.2.6 *Let $E : \Pi_P \rightarrow \mathbb{R}$ be a G -invariant real function on Π_P , then*

$$0 = \frac{\partial E}{\partial y^\alpha} + \frac{1}{2} c_{\alpha\beta}^\gamma \frac{\partial E}{\partial p_\beta^i}. \quad (6.19)$$

Similarly, let $C : P \rightarrow \mathfrak{g}$ be a G -equivariant function on P , then

$$0 = \frac{\partial C^\gamma}{\partial y^\alpha} + \frac{1}{2} c_{\beta\alpha}^\gamma C^\beta. \quad (6.20)$$

Proof. Since E is invariant under the action of G on Π_P , for any $g \in G$,

$$E(x^i, x^a, 0, p_a^i, p_\beta^i) = E(g \cdot (x^i, x^a, 0, p_a^i, p_\beta^i)) = E(x^i, x^a, y^\lambda, p_a^i, Z_\beta^\gamma(g) p_\gamma^i).$$

In particular, for $g(\epsilon) = \exp(\epsilon B_\alpha)$, $y^\alpha = \epsilon$, $y^\beta = 0$ for $\beta \neq \alpha$ and

$$E(x^i, x^a, 0, p_a^i, p_\beta^i) = E(x^i, x^a, y^\lambda(\epsilon), p_a^i, Z_\beta^\gamma(g(\epsilon)) p_\gamma^i).$$

Derivation with respect to ϵ and equation (6.18) in Lemma 6.2.5 finish the first part of the lemma. In a similar way, as C is equivariant, for any $g \in G$,

$$C^\beta(x^i, x^a, y^\lambda) = Z_\gamma^\beta(g^{-1}) C^\gamma(x^i, x^a, 0).$$

In particular, it holds for $g(\epsilon) = \exp(\epsilon B_\alpha)$ and derivation with respect to ϵ provides (6.20). $\square$

Proposition 6.2.7 *The local expression of the differential of the projection $\kappa : \Pi_P \rightarrow \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ at point $y = (s, e)$ is*

$$\begin{aligned}
 T\kappa : T\Pi_P &\rightarrow T(\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)) \\
 \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial x^i} + \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^i} \frac{\partial}{\partial \sigma_b^j} \\
 \frac{\partial}{\partial x^a} &\mapsto \frac{\partial}{\partial x^a} + \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^a} \frac{\partial}{\partial \sigma_b^j} \\
 \frac{\partial}{\partial y^\alpha} &\mapsto -\frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma A_b^\beta \frac{\partial}{\partial \sigma_b^j} - \frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma \frac{\partial}{\partial \mu_\beta^j} \\
 \frac{\partial}{\partial p_a^i} &\mapsto \frac{\partial}{\partial \sigma_a^i} \\
 \frac{\partial}{\partial p_\alpha^i} &\mapsto A_b^\alpha \frac{\partial}{\partial \sigma_b^i} + \frac{\partial}{\partial \mu_\alpha^i}
 \end{aligned}$$

Proof. The derivative of the local expression in Proposition 6.2.4 can be easily obtained by applying Lemma 6.2.6 to the invariance of $\mathcal{A}$, the second part of Lemma 6.2.5 and evaluation at $y = (s, e)$. $\square$

6.2.2 Reduced Poisson $(n-1)$ -Forms

As stated in Theorem 6.1.10, horizontal Poisson $(n-1)$ -forms on Π_P can be used to describe the dynamics of a Hamiltonian system. Let V be any vector bundle over P or Σ , we introduce a linear operator between vector bundles

$$\begin{aligned}
 \langle \cdot, \cdot \rangle : (TM \otimes V^* \otimes \bigwedge^n T^*M) \otimes V &\rightarrow \bigwedge^{n-1} T^*M \\
 \left(\frac{\partial}{\partial x^i} \otimes w \otimes v, v \right) &\mapsto \langle w, v \rangle i_{\partial/\partial x^i} v,
 \end{aligned}$$

and for any $\gamma \in \Gamma(V)$, the $(n-1)$ -form θ_γ is defined on $q \in TM \otimes V^* \otimes \bigwedge^n T^*M$ by $(\theta_\gamma)_q = \langle q, \gamma \rangle$.

Proposition 6.2.8 [19, Proposition 4.3] *Any Poisson $(n-1)$ -form on Π_P can be written as*

$$F = \theta_X + \pi_{P, \Pi_P}^* \omega + \Upsilon,$$

where $X \in \mathfrak{X}^V(P) = \Gamma(VP)$ is a vertical vector field on P , ω is an horizontal $(n-1)$ -form on P and Υ is a closed horizontal $(n-1)$ -form on Π_P .

We shall see that these objects project to the reduced polysymplectic space.

Theorem 6.2.9 *The projection of a G -invariant horizontal Poisson $(n-1)$ -form on Π_P , $F = \theta_X + \pi_{P, \Pi_P}^* \omega + \Upsilon$ is an horizontal $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ of the kind*

$$f = \theta_Y + \theta_{\bar{\xi}} + \bar{\pi}^* \bar{\omega} + \bar{\Upsilon},$$

where $Y \oplus \bar{\xi} \in \Gamma(V\Sigma \oplus \tilde{\mathfrak{g}} \rightarrow M)$ such that $\alpha_{\mathcal{A}}([X]_G) = Y \oplus \bar{\xi}$, $\bar{\omega}$ is an horizontal $(n-1)$ -form on Σ , $\bar{\Upsilon}$ is a closed horizontal $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$, and $\bar{\pi} = \pi_{\Sigma, \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)}$.

Proof. Since F is invariant under the action of G on Π_P , X is invariant under the action of G on P , $\omega \in \Omega^{n-1}(P)$ is G -invariant and Υ is a G -invariant $(n-1)$ -form on Π_P . From the last statement, as Υ is horizontal there is a $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$, $\bar{\Upsilon}$ such that $\Upsilon = \kappa^* \bar{\Upsilon}$. Observe that $\bar{\Upsilon}$ is closed as needed. Furthermore, there is a $(n-1)$ -form on Σ , $\bar{\omega}$ such that $\omega = \pi_{\Sigma, P}^* \bar{\omega}$. Then,

$$\pi_{P, \Pi_P}^* \omega = \pi_{P, \Pi_P}^* \pi_{\Sigma, P}^* \bar{\omega} = \pi_{\Sigma, \Pi_P}^* \bar{\omega} = \kappa^* \bar{\pi}^* \bar{\omega}$$

and the second term of F projects to $\bar{\pi}^* \bar{\omega}$.

The connection $\mathcal{A}$ on $P \rightarrow \Sigma$ splits the vector field X in an horizontal and vertical part with respect to $P \rightarrow \Sigma$,

$$X = \text{Hor}(X) + \text{Ver}(X).$$

As X is invariant, there exists a vector field Y in Σ such that for any $y \in P$,

$$(\text{Hor}(X))_y = \text{hor}_{\mathcal{A}}^y(Y_s).$$

Moreover, from Proposition 6.2.2, $Y \in \mathfrak{X}(\Sigma)$ is vertical with respect to $\Sigma \rightarrow M$ as X is vertical with respect to $P \rightarrow M$. The function $\xi : P \rightarrow \mathfrak{g}$ defined by $\xi(y) = \mathcal{A}((\text{Ver}(Y))_y)$ is Ad -equivariant and induces a section $\bar{\xi}$ in $\tilde{\mathfrak{g}} \rightarrow \Sigma$. To conclude this proof we shall show that for any $y \in P$, $(\theta_X)_y = \kappa^*((\theta_Y)_\sigma + (\theta_{\bar{\xi}})_\mu)$. Indeed,

$$\begin{aligned} (\theta_X)_p &= \langle p, X \rangle = \langle p, \text{Hor}(X) \rangle + \langle p, \text{Ver}(X) \rangle = \langle p, \text{hor}_{\mathcal{A}}(Y) \rangle + \langle p, \text{Ver}(X) \rangle \\ &= \kappa^* \langle \sigma, Y \rangle + \kappa^* \langle \mu, [y, \mathcal{A}(\text{Ver}(X))]_G \rangle = \kappa^* (\theta_Y)_\sigma + \kappa^* (\theta_{\bar{\xi}})_\mu, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is used for the vector bundles VP , $V\Sigma$ and $\tilde{\mathfrak{g}}$. □

6.2.3 Connections

We now study how both, the Ehresmann connection Λ on $P \rightarrow M$ and the principal connection $\mathcal{A}$ on $P \rightarrow \Sigma$, are propagated to other bundles relevant to describe reduced Hamiltonian systems.

From [19, Proposition 5.1], stated at the end of Subsection 6.1.2, any Ehresmann connection Λ on $P \rightarrow M$ together with any Riemannian connection on M provide an Ehresmann connection Λ^{Π_P} on the bundle $\Pi_P \rightarrow M$. The horizontal lift in the local coordinates introduced in Subsection 6.2.1 is

$$\begin{aligned} \frac{\partial}{\partial x^i} \mapsto & \frac{\partial}{\partial x^i} + \Lambda_i^a \frac{\partial}{\partial x^a} + \Lambda_i^\alpha \frac{\partial}{\partial y^\alpha} + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} p_b^j - \frac{\partial \Lambda_i^\beta}{\partial x^a} p_\beta^j + \Gamma_{ik}^j p_a^k - \Gamma_{ik}^k p_a^j \right) \frac{\partial}{\partial p_a^j} \\ & + \left(-\frac{\partial \Lambda_i^b}{\partial y^\alpha} p_b^j - \frac{\partial \Lambda_i^\beta}{\partial y^\alpha} p_\beta^j + \Gamma_{ik}^j p_\alpha^k - \Gamma_{ik}^k p_\alpha^j \right) \frac{\partial}{\partial p_\alpha^j}, \end{aligned}$$

where $\Lambda_i^\alpha, \Lambda_i^a$ are the coefficients of Λ and Γ_{ij}^k are the Christoffel symbols of the Riemannian connection.

Assume that Λ is G -invariant in the sense that for every $v \in TM$

$$\text{hor}_\Lambda^{g \cdot y}(v) = T_y \Phi_g(\text{hor}_\Lambda^y(v)),$$

where $\text{hor}_\Lambda^y(v)$ denotes the horizontal lift of v at $y \in P$ through Λ . Then, there is an induced connection Λ' on $\Sigma \rightarrow M$ such that for every $v \in TM$,

$$\text{hor}_{\Lambda'}^s(v) = T_y \pi_{\Sigma, P}(\text{hor}_\Lambda^y(v))$$

provided that $y \in P_s$. The local expression of the horizontal lift of Λ' is $\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i} + \Lambda_i^a(x^j, x^b) \frac{\partial}{\partial x^a}$. We will abuse the notation and denote the induced connection Λ as well. Furthermore, particularization of equation (6.14) to the induced connection on Σ introduces a connection Λ^{Π_Σ} on $\Pi_\Sigma \rightarrow M$ whose horizontal lift is

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i} + \Lambda_i^a \frac{\partial}{\partial x^a} + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^j + \Gamma_{ik}^j \sigma_a^k - \Gamma_{ik}^k \sigma_a^j \right) \frac{\partial}{\partial \sigma_a^j}.$$

Definition 6.2.10 Let Λ be an Ehresmann connection on $P \rightarrow M$, $\tilde{\Lambda}$ an Ehresmann connection on $\Sigma \rightarrow M$ and $\mathcal{A}$ a principal connection on $P \rightarrow \Sigma$, they are **compatible** if for every $y \in P$, $\text{hor}_\Lambda^y = \text{hor}_\mathcal{A}^y \circ \text{hor}_{\tilde{\Lambda}}^s$.

Observe that if the connections are compatible, $\tilde{\Lambda}$ is the connection on $\Sigma \rightarrow M$ induced by Λ as

$$\text{hor}_{\Lambda'}^s = T_y \pi_{\Sigma, P} \circ \text{hor}_{\Lambda}^y = T_y \pi_{\Sigma, P} \circ \text{hor}_{\mathcal{A}}^y \circ \text{hor}_{\tilde{\Lambda}}^s = \text{hor}_{\tilde{\Lambda}}^s.$$

In local coordinates the compatibility condition is given by equations

$$\Lambda_i^\alpha = A_i^\alpha + A_b^\alpha \Lambda_i^b \quad (6.21)$$

$$\tilde{\Lambda}_i^b = \Lambda_i^b \quad (6.22)$$

We also provide a connection in the vector bundle $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow \Sigma$. On one hand, the principal connection $\mathcal{A}$ on $P \rightarrow \Sigma$ provides a linear connection on $\tilde{\mathfrak{g}} \rightarrow \Sigma$ and a dual connection on $\tilde{\mathfrak{g}}^* \rightarrow \Sigma$. On the other hand, the Riemannian connection Γ on M , thought as a linear connection on $TM \rightarrow M$, can be pulled back by $\pi_{M, \Sigma}$ to a linear connection on $\pi_{M, \Sigma}^* TM \rightarrow \Sigma$ whose horizontal lift is

$$\begin{aligned} \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial x^i} + \Gamma_{ij}^k(x^i) \dot{x}^j \frac{\partial}{\partial \dot{x}^k}, \\ \frac{\partial}{\partial x^a} &\mapsto \frac{\partial}{\partial x^a}. \end{aligned}$$

Furthermore, Γ provides a linear connection on $\bigwedge^n T^*M \rightarrow M$ which can be pulled back to another one in $\pi_{M, \Sigma}^* \bigwedge^n T^*M \rightarrow \Sigma$. Then, the connection $\nabla^{\mathcal{A}, \Gamma}$ on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow \Sigma$ obtained as the tensor product of the connections in each factor has the following horizontal lift:

$$\begin{aligned} \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial x^i} + c_{\beta\alpha}^\gamma A_i^\beta \mu_\gamma^j \frac{\partial}{\partial \mu_\alpha^j} + \left(\Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j \right) \frac{\partial}{\partial \mu_\alpha^j}, \\ \frac{\partial}{\partial x^a} &\mapsto \frac{\partial}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^j \frac{\partial}{\partial \mu_\alpha^j}. \end{aligned}$$

Remark 6.2.11 Consider the pullback bundle

$$\begin{array}{ccc} J^1 \Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) & \longrightarrow & TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \\ \downarrow \pi_1 & & \downarrow \\ J^1 \Sigma^* & \xrightarrow{\pi_{\Sigma, J^1 \Sigma^*}} & \Sigma \end{array}$$

The pullback connection $\pi_{\Sigma, J^1 \Sigma^*}^* \nabla^{\mathcal{A}, \Gamma}$ is defined on $\pi_1 : J^1 \Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow J^1 \Sigma^*$. We will denote this connection by $\nabla^{\mathcal{A}, \Gamma}$ as well. Similarly, the pullback connection $\pi_{\Sigma, \Pi_\Sigma}^* \nabla^{\mathcal{A}, \Gamma}$ is defined on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow \Pi_\Sigma$ and denoted $\nabla^{\mathcal{A}, \Gamma}$.

Observe that using Λ to lift a vector from M to Σ as well as the lift provided by $\nabla^{\mathcal{A},\Gamma}$ defines an Ehresmann connection $\Lambda^{\mathcal{A},\Gamma}$ on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow M$.

Proposition 6.2.12 *Let Λ , $\tilde{\Lambda}$ and $\mathcal{A}$ be compatible connections on $P \rightarrow M$, $\Sigma \rightarrow M$ and $P \rightarrow \Sigma$, the connection $\tilde{\Lambda}^{\mathcal{A},\Gamma}$ on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow M$ does not depend on $\mathcal{A}$.*

Proof. We will obtain the horizontal lift of a basic vector $\frac{\partial}{\partial x^i}$ to $\mu \in TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow M$ with coordinates (x^i, x^a, μ_α^j) . Firstly, $\frac{\partial}{\partial x^i}$ lifts to $\frac{\partial}{\partial x^i} + \tilde{\Lambda}_i^a \frac{\partial}{\partial x^a} \in T_s \Sigma$, where $s = (x^i, x^a)$. The later is then lifted via $\nabla^{\mathcal{A},\Gamma}$ to

$$\begin{aligned} \frac{\partial}{\partial x^i} + \tilde{\Lambda}_i^a \frac{\partial}{\partial x^a} + \left(\mu_\gamma^j c_{\beta\alpha}^\gamma (A_i^\beta + A_a^\beta \tilde{\Lambda}_i^a) + \Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j \right) \frac{\partial}{\partial \mu_\alpha^j} \\ = \frac{\partial}{\partial x^i} + \Lambda_i^a \frac{\partial}{\partial x^a} + \left(\mu_\gamma^j c_{\beta\alpha}^\gamma \Lambda_i^\beta + \Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j \right) \frac{\partial}{\partial \mu_\alpha^j}, \end{aligned}$$

where the compatibility equations (6.21) and (6.22) have been used. $\square$

Finally, an Ehresmann connection $\Lambda^{\Pi_\Sigma} \oplus \Lambda^{\mathcal{A},\Gamma}$ is defined on the reduced polysymplectic space $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow M$.

6.2.4 Poisson Forms

Our next objective is to establish a *Poisson bracket* on $\Pi_P/G \simeq \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ compatible with the Poisson structure on Π_P and the choice of connections in Subsection 6.2.3. This bracket consists of three components. The first component that we analyze is attached to the polysymplectic bundle Π_Σ . This bundle is equipped with a Poisson structure in the sense of [54]. However, this bracket will not appear as one of the ingredients when the reduction of the bracket of Π_P is performed. We require a structure that takes into account the fact that Π_Σ is twisted with the vector bundle $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$ and will need the connection $\nabla^{\mathcal{A},\Gamma}$ constructed in Subsection 6.2.3. We define this structure by introducing first the corresponding notion of Poisson forms through a series of preliminary definitions.

Definition 6.2.13 *An r -form f on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ is said to be $J^1\Sigma^*$ -horizontal if $i_u f = 0$ for any vertical tangent vector u with respect to $\pi_1 : J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow J^1\Sigma^*$*

In local coordinates these forms are a linear combination of basic r -forms on $J^1\Sigma^*$ with coefficients in $\mathcal{C}^\infty(J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M))$. Alternatively, a $J^1\Sigma^*$ -horizontal r -form on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ can be defined as a section in the pullback bundle

$$\begin{array}{ccc}
\pi_1^* (\bigwedge^r T^*(J^1\Sigma^*)) & \longrightarrow & \bigwedge^r T^*(J^1\Sigma^*) \\
\downarrow & & \downarrow \\
J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) & \xrightarrow{\pi_1} & J^1\Sigma^*
\end{array}$$

In general, the exterior derivative of a $J^1\Sigma^*$ -horizontal r -form is not a $J^1\Sigma^*$ -horizontal $(r+1)$ -form. Therefore, we shall use instead the horizontal differential $d^{h,J^1\Sigma^*}$ of the connection $\nabla^{\mathcal{A},\Gamma}$ in $\pi_1 : J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow J^1\Sigma^*$.

Proposition 6.2.14 *Let f be a $J^1\Sigma^*$ -horizontal r -form, then $d^{h,J^1\Sigma^*} f$ is a $J^1\Sigma^*$ -horizontal $(r+1)$ -form.*

Proof. Let f be a 0-form, that is, a differentiable function on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$. The horizontal differential of f is

$$\begin{aligned}
d^{h,J^1\Sigma^*} f = & \left(\frac{\partial f}{\partial x^i} + c_{\beta\alpha}^\gamma A_i^\beta \mu_\gamma^j \frac{\partial f}{\partial \mu_\alpha^j} + \left(\Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j \right) \frac{\partial f}{\partial \mu_\alpha^j} \right) dx^i \\
& + \left(\frac{\partial f}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^j \frac{\partial f}{\partial \mu_\alpha^j} \right) dx^a + \frac{\partial f}{\partial \sigma_a^i} d\sigma_a^i + \frac{\partial f}{\partial \tilde{\sigma}} d\tilde{\sigma}, \quad (6.23)
\end{aligned}$$

which is a $J^1\Sigma^*$ -horizontal 1-form, as it has no terms proportional to any $d\mu_\alpha^i$. Suppose now that f is a $J^1\Sigma^*$ -horizontal r -form, its horizontal differential is obtained by horizontal differentiation of its coefficients. Thus, there will be no terms in $d^{h,J^1\Sigma^*} f$ including the wedge with any $d\mu_\alpha^i$. $\square$

Definition 6.2.15 A $J^1\Sigma^*$ -horizontal s -multivector field on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ is a section of

$$\pi_1^* \left(\bigwedge^s T(J^1\Sigma^*) \right) \rightarrow J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M). \quad (6.24)$$

In local coordinates, these $J^1\Sigma^*$ -horizontal s -multivector fields are a linear combination of s -multivector fields on $J^1\Sigma^*$ with coefficients in $\mathcal{C}^\infty(J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M))$.

Definition 6.2.16 An r -form f on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ is said to be **horizontal** if $i_u f = 0$ for any vertical tangent vector u with respect to $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow M$

Observe that the definition of horizontal and $J^1\Sigma^*$ -horizontal forms only differs in the choice of bundle with respect to which u is a vertical tangent vector. In fact, any horizontal form is $J^1\Sigma^*$ -horizontal.

Definition 6.2.17 An s -multivector χ on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$ is said to be **vertical** if its contraction $i_\chi f$ with any horizontal s -form f on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M) \rightarrow M$ vanishes.

Definition 6.2.18 A horizontal r -form f on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$ is a **Poisson form** if there is a vertical $J^1\Sigma^*$ -horizontal $(n-r)$ -multivector field χ_f on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$ such that

$$i_{\chi_f} \Omega_\Sigma = d^{h, J^1\Sigma^*} f, \quad (6.25)$$

where Ω_Σ is the multisymplectic form on $J^1\Sigma^*$. Given a Poisson r -form f and a Poisson s -form h on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$, their **horizontal Poisson bracket** is

$$\{f, h\}_\Sigma = (-1)^{r(s-1)} i_{\chi_h} i_{\chi_f} \Omega_\Sigma \quad (6.26)$$

Remark 6.2.19 The multivector field χ_f associated to a Poisson form f on $J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$ may not be unique as χ_f is defined up to an element belonging to $\ker \Omega_\Sigma$. This lack of uniqueness differs from the symplectic setting in mechanics as the symplectic form is non-degenerate. The horizontal Poisson bracket in Definition 6.2.18 is unambiguous and does not depend on the choice of multivectors. Indeed, from equation (6.25) we observe that $\{f, h\}_\Sigma$ only depends on f through $d^{h, J^1\Sigma^*} f$. Since $i_{\chi_h} i_{\chi_f} \Omega_\Sigma = (-1)^{(n-r)(n-s)} i_{\chi_f} i_{\chi_h} \Omega_\Sigma$, an analogous argument shows that $\{f, h\}_\Sigma$ only depends on h through $d^{h, J^1\Sigma^*} h$.

In the following, some results are exhibited to justify that this is the first component of the searched Poisson bracket in $\Pi_\Sigma \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$.

Proposition 6.2.20 A function $f : J^1\Sigma^* \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M) \rightarrow \mathbb{R}$ is a Poisson 0-form if and only if f is projectable to $\Pi_\Sigma \oplus (TM \otimes \tilde{g}^* \otimes \bigwedge^n T^*M)$.

Proof. The local expression for $d^{h, J^1\Sigma^*} f$ is given in equation (6.23) and $\Omega_\Sigma = dx^a \wedge d\sigma_a^i \wedge v_i - d\tilde{\sigma} \wedge v$, where $v_i = i_{\partial/\partial x^i} v$. It can easily be checked that the n -multivector field

$$\begin{aligned} \chi = & - \left(\frac{\partial f}{\partial x^i} + c_{\beta\alpha}^\gamma A_i^\beta \mu_\gamma^j \frac{\partial f}{\partial \mu_\alpha^j} + \left(\Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j \right) \frac{\partial f}{\partial \mu_\alpha^j} \right) \frac{\partial}{\partial \tilde{\sigma}} \wedge v_i^* \\ & - \left(\frac{\partial f}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^j \frac{\partial f}{\partial \mu_\alpha^j} \right) \frac{\partial}{\partial \sigma_a^i} \wedge v_i^* + \frac{\partial f}{\partial \sigma_a^i} \frac{\partial}{\partial x^a} \wedge v_i^* + \frac{\partial f}{\partial \tilde{\sigma}} v^*, \end{aligned} \quad (6.27)$$

where $v^* = (\partial/\partial x^1) \wedge \cdots \wedge (\partial/\partial x^n)$, satisfies the condition (6.25). If f is projectable, $\partial f/\partial \tilde{\sigma} = 0$. Thus, χ is a vertical n -multivector and f is Poisson. Conversely, if f is Poisson with

vertical $J^1\Sigma^*$ -horizontal n -multivector χ_f satisfying the condition (6.25), χ_f does not have a term proportional to v^* . Then, $i_{\chi_f}\Omega_\Sigma$ does not have a term proportional to $d\tilde{\sigma}$. From condition (6.25), the same is true for $d^{h,J^1\Sigma^*}f$ and $\partial f/\partial\tilde{\sigma} = 0$. $\square$

Proposition 6.2.21 *If a horizontal r -form f , $r > 0$, on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ is Poisson, then it is projectable to $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$.*

Proof. In local coordinates, $f = f^{i_1 \dots i_s} v_{i_1 \dots i_s}$, where $s = n - r$ and $v_{i_1 \dots i_s} = i_{\partial/\partial x^{i_1}} \dots i_{\partial/\partial x^{i_s}} v$. Consequently,

$$\begin{aligned} d^{h,J^1\Sigma^*}f &= \left(\frac{\partial f^{i_2 \dots i_s j}}{\partial x^j} + \left(c_{\beta\alpha}^\gamma A_j^\beta \mu_\gamma^k + (\Gamma_{jl}^k \mu_\alpha^l - \Gamma_{jl}^l \mu_\alpha^k) \right) \frac{\partial f^{i_2 \dots i_s j}}{\partial \mu_\alpha^k} \right) v_{i_2 \dots i_s} \\ &\quad + \left(\frac{\partial f^{i_1 \dots i_s}}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^k \frac{\partial f^{i_1 \dots i_s}}{\partial \mu_\alpha^k} \right) dx^a \wedge v_{i_1 \dots i_s} \\ &\quad + \frac{\partial f^{i_1 \dots i_s}}{\partial \sigma_a^i} d\sigma_a^i \wedge v_{i_1 \dots i_s} + \frac{\partial f^{i_1 \dots i_s}}{\partial \tilde{\sigma}} d\tilde{\sigma} \wedge v_{i_1 \dots i_s} \end{aligned}$$

Let χ_f be the s -multivector field such that $i_{\chi_f}\Omega_\Sigma = d^{h,J^1\Sigma^*}f$. Then χ_f does not contain terms with component $\partial/\partial x^{i_1} \wedge \dots \wedge \partial/\partial x^{i_s}$ as that would imply that $i_{\chi_f}\Omega_\Sigma$ has terms with component $dx^a \wedge d\sigma_a^i \wedge v_{i,i_1 \dots i_s}$ not present in $d^{h,J^1\Sigma^*}f$. Consequently, $d^{h,J^1\Sigma^*}f$ does not have any term with component $d\tilde{\sigma} \wedge v_{i_1 \dots i_s}$, $\partial f^{i_1 \dots i_s}/\partial\tilde{\sigma} = 0$ and f is projectable. $\square$

From Proposition 6.2.21, we can consider that Poisson forms on $J^1\Sigma^* \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ are defined on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$. As the bracket of Poisson forms is also a Poisson form, the horizontal bracket $\{f, g\}_\Sigma$ will be hereafter considered a bracket of Poisson forms on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$. We should now focus on the bracket of $(n-1)$ -forms and functions, which is of special interest to study the dynamics of Hamiltonian systems.

Proposition 6.2.22 *Any Poisson $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ can be written as*

$$f = \theta_Y + \hat{\pi}_2^* \varpi + \tilde{Y},$$

where $Y : TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow V\Sigma$ is a bundle morphism covering Id_Σ , ϖ is an horizontal $(n-1)$ -form on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow M$, $\tilde{Y}$ is a closed horizontal $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$, and $\hat{\pi}_2 = \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$ is the projection to the second factor.

Proof. In local coordinates, $f = f^i v_i$ and

$$\begin{aligned} d^{h,J^1\Sigma^*} f &= \left(\frac{\partial f^i}{\partial x^i} + \left(c_{\beta\alpha}^\gamma A_i^\beta \mu_\gamma^j + (\Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j) \right) \frac{\partial f^i}{\partial \mu_\alpha^j} \right) v \\ &\quad + \left(\frac{\partial f^i}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^k \frac{\partial f^i}{\partial \mu_\alpha^k} \right) dx^a \wedge v_i + \frac{\partial f^i}{\partial \sigma_a^j} d\sigma_a^j \wedge v_i. \end{aligned}$$

As f is Poisson, there is a $J^1\Sigma^*$ -horizontal vector field

$$\chi_f = Y^a \frac{\partial}{\partial x^a} + Y_a^i \frac{\partial}{\partial \sigma_a^i} + W \frac{\partial}{\partial \tilde{\sigma}}$$

such that $i_{\chi_f} \Omega_\Sigma = d^{h,J^1\Sigma^*} f$. Since

$$i_{\chi_f} \Omega_\Sigma = Y^a d\sigma_a^i \wedge v_i - Y_a^i dx^a \wedge v_i - W v,$$

we obtain the following conditions for χ_f :

$$\begin{aligned} W &= - \left(\frac{\partial f^i}{\partial x^i} + \left(c_{\beta\alpha}^\gamma A_i^\beta \mu_\gamma^j + (\Gamma_{ik}^j \mu_\alpha^k - \Gamma_{ik}^k \mu_\alpha^j) \right) \frac{\partial f^i}{\partial \mu_\alpha^j} \right), \\ Y_a^i &= - \frac{\partial f^i}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^k \frac{\partial f^i}{\partial \mu_\alpha^k}, \quad \delta_j^i Y^a = \frac{\partial f^i}{\partial \sigma_a^j}. \end{aligned}$$

In conclusion, the general expression of a Poisson $(n-1)$ -form is $f = (\sigma_a^i Y^a + g^i) v_i$ for any functions Y^a, g^i on $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$. This allows to identify the first two terms in the proposition and the result follows since Poisson forms are defined up to a closed form. $\square$

From Theorem 6.2.9, the projection of a G -invariant Poisson $(n-1)$ -form on Π_P is a Poisson $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ of a particular kind where $Y : TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow V\Sigma$ projects to a vertical field $Y : \Sigma \rightarrow V\Sigma$ and $\varpi, (n-1)$ -form in $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M \rightarrow M$, can be written as:

$$\varpi = \theta_{\tilde{\xi}} + \pi_{\Sigma, TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M}^* \bar{\omega},$$

where $\tilde{\xi}$ is a section in $\tilde{\mathfrak{g}} \rightarrow \Sigma$ and $\bar{\omega}$ is an horizontal $(n-1)$ -form on Σ . We will say such Poisson $(n-1)$ -forms are *affine*.

The horizontal bracket of a Poisson $(n-1)$ -form f on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ and a Poisson function h can be easily expressed in local coordinates as

$$\{f, h\}_\Sigma = \left(\frac{\partial h}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^j \frac{\partial h}{\partial \mu_\alpha^j} \right) \frac{\partial f^i}{\partial \sigma_a^i} - \left(\frac{\partial f^i}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta \mu_\gamma^j \frac{\partial f^i}{\partial \mu_\alpha^j} \right) \frac{\partial h}{\partial \sigma_a^i} \quad (6.28)$$

from the local expressions in the proof of Propositions 6.2.20 and 6.2.22. This expression is similar to the usual Poisson bracket defined for forms on Π_Σ replacing partial derivatives $\partial/\partial x^a$ by their covariant ones through $\nabla^{\mathcal{A}, \Gamma}$ since the forms considered here are defined on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ and not just on Π_Σ .

6.2.5 Poisson Bracket

We now define a Poisson bracket on the reduced polysymplectic bundle $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ compatible with the bracket on Π_P . To study the two remaining components we introduce first some notation.

For any function $h \in \mathcal{C}^\infty(\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M))$ there are two fiber derivatives. On one hand

$$\frac{\delta h}{\delta \sigma} : \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow T^*M \otimes V\Sigma \otimes \bigwedge^n TM$$

defined by

$$\frac{\delta h}{\delta \sigma}(\sigma \oplus \mu)(\tau) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h((\sigma + \varepsilon\tau) \oplus \mu),$$

whose expression in local coordinates is

$$\frac{\partial h}{\partial \sigma_a^i} dx^i \otimes \frac{\partial}{\partial x^a} \otimes v. \quad (6.29)$$

On the other hand

$$\frac{\delta h}{\delta \mu} : \Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M) \rightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM$$

given by

$$\frac{\delta h}{\delta \mu}(\sigma \oplus \mu)(v) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h(\sigma \oplus (\mu + \varepsilon v)),$$

whose local expression is

$$\frac{\partial h}{\partial \mu_\alpha^i} dx^i \otimes \tilde{B}_\alpha \otimes v. \quad (6.30)$$

With this fiber derivative we define a second component of the bracket in Π_P/G attached to $TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M$. This is a Lie-Poisson bracket in field theories as the one in [19].

Definition 6.2.23 *Given a function $h \in \mathcal{C}^\infty(\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M))$ and a section $\bar{\xi}$ of $\tilde{\mathfrak{g}} \rightarrow \Sigma$ we define*

$$\{\bar{\xi}, h\}_{LP} = - \left\langle \mu, \left[\bar{\xi}, \frac{\delta h}{\delta \mu} \right] \right\rangle, \quad \mu \in TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M. \quad (6.31)$$

Finally, our bracket in $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ needs a deformation term defined by the curvature. More precisely,

$$\begin{aligned} \tilde{\mathcal{B}} : T\Sigma \otimes T^*M \otimes T\Sigma \otimes \bigwedge^n TM &\rightarrow T^*M \otimes \tilde{\mathfrak{g}} \otimes \bigwedge^n TM \\ \left(\frac{\partial}{\partial x^a}, dx^i \otimes \frac{\partial}{\partial x^b} \otimes v^* \right) &\mapsto dx^i \otimes \tilde{\mathcal{B}} \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right) \otimes v^*, \end{aligned}$$

where $\tilde{\mathcal{B}}$ is the 2-form defined on Σ by the curvature form $\mathcal{B}$ of the connection $\mathcal{A}$ in $P \rightarrow \Sigma$

Once we have the bracket steaming from Π_Σ , the Lie Poisson bracket and the deformation we can define the full bracket in the reduced space as follows.

Definition 6.2.24 *Let f be an affine Poisson $(n-1)$ -form $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$ and h a Poisson function on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^n T^*M)$, their Poisson bracket is*

$$\{f, h\} = \{f, h\}_\Sigma + \{\bar{\xi}, h\}_{LP} - \left\langle \mu, \tilde{\mathcal{B}} \left(Y, \frac{\delta h}{\delta \sigma} \right) \right\rangle, \quad (6.32)$$

where Y and $\bar{\xi}$ are respectively sections of $T\Sigma \rightarrow \Sigma$ and $\tilde{\mathfrak{g}} \rightarrow \Sigma$ that define f as in Theorem 6.2.9.

6.3 Poisson–Poincaré Reduction

6.3.1 Reduced Dynamics

In this Section Poisson-Poincaré reduction is presented in the multisymplectic and polysymplectic formalism. The reduced dynamics are described by the bracket constructed in Subsection 6.2.4.

Theorem 6.3.1 *The projection map*

$$\kappa : \Pi_P \longrightarrow \Pi_P/G \cong \Pi_\Sigma \oplus \left(TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M \right)$$

is a Poisson map in the sense that for any G -invariant Poisson $(n-1)$ -form F and any G -invariant Poisson function H on Π_P

$$\{F, H\} = \kappa^* \{f, h\}, \quad (6.33)$$

where the bracket on the left hand side is defined by (6.12), f is a Poisson $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$ such that $\kappa^* f = F$, h is a function such that $\kappa^* h = H$, and the bracket on the right hand side is defined by (6.32).

Proof. In the coordinate system introduced in Π_P , we have the local expression

$$\{F, H\} = \frac{\partial F^i}{\partial x^a} \frac{\partial H}{\partial p_a^i} - \frac{\partial F^i}{\partial p_a^i} \frac{\partial H}{\partial x^a} + \frac{\partial F^i}{\partial y^\alpha} \frac{\partial H}{\partial p_\alpha^i} - \frac{\partial F^i}{\partial p_\alpha^i} \frac{\partial H}{\partial y^\alpha}$$

As F and H are G -invariant, using the differential of projection κ obtained in Proposition 6.2.7 gives a function r on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$

$$\begin{aligned} r = & \frac{\partial f^i}{\partial x^a} \frac{\partial h}{\partial \sigma_a^i} + \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^a} \frac{\partial f^i}{\partial \sigma_b^j} \frac{\partial h}{\partial \sigma_a^i} - \frac{\partial f^i}{\partial \sigma_a^i} \frac{\partial h}{\partial x^a} - \frac{\partial f^i}{\partial \sigma_a^i} \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^a} \frac{\partial h}{\partial \sigma_b^j} \\ & + \left(-\frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma A_b^\beta \frac{\partial f^i}{\partial \sigma_b^j} - \frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma \frac{\partial f^i}{\partial \mu_\beta^j} \right) \left(\frac{\partial h}{\partial \mu_\alpha^i} + A_a^\alpha \frac{\partial h}{\partial \sigma_a^i} \right) \\ & - \left(\frac{\partial f^i}{\partial \mu_\alpha^i} + A_a^\alpha \frac{\partial f^i}{\partial \sigma_a^i} \right) \left(-\frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma A_b^\beta \frac{\partial h}{\partial \sigma_b^j} - \frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^j} \right), \end{aligned}$$

such that $\{F, H\} = \kappa^* r$. We shall see that $r = \{f, h\}$ as desired. Since f is the projection of F , from Theorem 6.2.9, f is an affine Poisson form and $\partial f^i / \partial \sigma_a^j = Y^a \delta_j^i$ and $\partial f^i / \partial \mu_\alpha^j = \xi^\alpha \delta_j^i$. Therefore,

$$\begin{aligned} r = & \frac{\partial f^i}{\partial x^a} \frac{\partial h}{\partial \sigma_a^i} - \frac{\partial f^i}{\partial \sigma_a^i} \frac{\partial h}{\partial x^a} + \mu_\gamma^j \left(\frac{\partial A_b^\gamma}{\partial x^a} - \frac{\partial A_a^\gamma}{\partial x^b} \right) \frac{\partial f^i}{\partial \sigma_b^j} \frac{\partial h}{\partial \sigma_a^i} + \mu_\gamma^j c_{\beta\alpha}^\gamma A_a^\beta A_b^\alpha \frac{\partial h}{\partial \sigma_a^i} \frac{\partial f^i}{\partial \sigma_b^j} \\ & + \mu_\gamma^j c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^j} \frac{\partial f^i}{\partial \mu_\alpha^i} + \mu_\gamma^j c_{\beta\alpha}^\gamma A_b^\alpha \frac{\partial h}{\partial \mu_\beta^j} \frac{\partial f^i}{\partial \sigma_b^j} + \mu_\gamma^j c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \sigma_a^i} \frac{\partial f^i}{\partial \mu_\alpha^j} \\ = & \left(\frac{\partial f^i}{\partial x^a} + \mu_\gamma^j c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial f^i}{\partial \mu_\alpha^j} \right) \frac{\partial h}{\partial \sigma_a^i} - \frac{\partial f^i}{\partial \sigma_a^i} \left(\frac{\partial h}{\partial x^a} + \mu_\gamma^j c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \mu_\alpha^j} \right) \\ & + \mu_\gamma^j \left(\frac{\partial A_b^\gamma}{\partial x^a} - \frac{\partial A_a^\gamma}{\partial x^b} + c_{\beta\alpha}^\gamma A_a^\beta A_b^\alpha \right) \frac{\partial f^i}{\partial \sigma_b^j} \frac{\partial h}{\partial \sigma_a^i} + \mu_\gamma^j c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^j} \frac{\partial f^i}{\partial \mu_\alpha^i}, \end{aligned}$$

where the three terms of $\{f, h\}$ can be identified. $\square$

Consider the following map related with the lift of the Ehresmann connection Λ on $\Sigma \rightarrow M$:

$$\begin{aligned}\bar{\Lambda} : \Sigma &\rightarrow T^*M \otimes T\Sigma \otimes \bigwedge^n TM \\ s &\mapsto dx^i \otimes \left(\delta_i^j \frac{\partial}{\partial x^j} + \Lambda_i^a(s) \frac{\partial}{\partial x^a} \right) \otimes v^*,\end{aligned}$$

Lemma 6.3.2 *Let F be a G -invariant Poisson $(n-1)$ -form on Π_P and f the affine Poisson $(n-1)$ -form on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$ such that $\kappa^* f = F$ as in Theorem 6.2.9, then*

$$d^h F = \kappa^* (d^h f - \langle \mu, \bar{\mathcal{B}}(Y, \bar{\Lambda}) \rangle v), \quad (6.34)$$

where $d^h F$ is the horizontal derivative with respect to the Ehresmann connection Λ^{Π_P} on $\Pi_P \rightarrow M$ and $d^h f$ is the horizontal derivative with respect to the Ehresmann connection $\Lambda^{\Pi_\Sigma} \oplus \Lambda^{\mathcal{A}, \Gamma}$.

Proof. The local expression of the horizontal derivative of an $(n-1)$ -form $F = F^i v_i$ is

$$\begin{aligned}d^h F &= \left(\frac{\partial F^i}{\partial x^i} + \Lambda_i^a \frac{\partial F^i}{\partial x^a} + \Lambda_i^\alpha \frac{\partial F^i}{\partial y^\alpha} + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} p_b^j - \frac{\partial \Lambda_i^\beta}{\partial x^a} p_\beta^j + \Gamma_{ik}^j p_a^k - \Gamma_{ik}^k p_a^j \right) \frac{\partial F^i}{\partial p_a^j} \right. \\ &\quad \left. + \left(-\frac{\partial \Lambda_i^b}{\partial y^\alpha} p_b^j - \frac{\partial \Lambda_i^\beta}{\partial y^\alpha} p_\beta^j + \Gamma_{ik}^j p_\alpha^k - \Gamma_{ik}^k p_\alpha^j \right) \frac{\partial F^i}{\partial p_\alpha^j} \right) v\end{aligned}$$

Since F is Poisson, F^i depends on p_a^j or p_α^j only when $i = j$. In addition, as Λ is G -invariant, $\partial \Lambda_i^a / \partial y^\alpha = 0$ for $i = 1, \dots, n$; $a = 1, \dots, m - k - n$; $\alpha = 1 \dots k$. Thus,

$$d^h F = \left(\frac{\partial F^i}{\partial x^i} + \Lambda_i^a \frac{\partial F^i}{\partial x^a} + \Lambda_i^\alpha \frac{\partial F^i}{\partial y^\alpha} + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} p_b^j - \frac{\partial \Lambda_i^\beta}{\partial x^a} p_\beta^j \right) \frac{\partial F^i}{\partial p_a^j} - \frac{\partial \Lambda_i^\beta}{\partial y^\alpha} p_\beta^j \frac{\partial F^i}{\partial p_\alpha^j} \right) v.$$

Similarly, as Λ is G -invariant, Lemma 6.2.6 can be applied to any $\partial \Lambda_i^\gamma / \partial y^\alpha$ for $i = 1, \dots, n$; $\alpha, \gamma = 1 \dots k$. Furthermore as F is G -invariant, using the derivative of κ in Proposition 6.2.7

we have that

$$\begin{aligned}
d^h F = \kappa^* & \left(\frac{\partial f^i}{\partial x^i} + \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^i} \frac{\partial f^i}{\partial \sigma_b^j} + \Lambda_i^a \frac{\partial f^i}{\partial x^a} + \mu_\gamma^j \frac{\partial A_b^\gamma}{\partial x^a} \Lambda_i^a \frac{\partial f^i}{\partial \sigma_b^j} \right. \\
& - \frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma A_b^\beta \Lambda_i^\alpha \frac{\partial f^i}{\partial \sigma_b^j} - \frac{1}{2} \mu_\gamma^j c_{\beta\alpha}^\gamma \Lambda_i^\alpha \frac{\partial f^i}{\partial \mu_\beta^j} \\
& \left. + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^j + \frac{\partial \Lambda_i^b}{\partial x^a} \mu_\gamma^j A_b^\gamma - \frac{\partial \Lambda_i^\beta}{\partial x^a} \mu_\beta^j \right) \frac{\partial f^i}{\partial \sigma_a^j} + \frac{1}{2} \mu_\beta^j c_{\gamma\alpha}^\beta \Lambda_i^\gamma \left(\frac{\partial f^i}{\partial \mu_\alpha^j} + A_b^\alpha \frac{\partial f^i}{\partial \sigma_b^j} \right) \right) v.
\end{aligned}$$

Regrouping for different partial derivatives of f^i ,

$$\begin{aligned}
d^h F = \kappa^* & \left(\frac{\partial f^i}{\partial x^i} + \Lambda_i^a \frac{\partial f^i}{\partial x^a} + \mu_\gamma^j c_{\beta\alpha}^\gamma \Lambda_i^\beta \frac{\partial f^i}{\partial \mu_\alpha^j} \right. \\
& \left. + \left(-\frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^j + \mu_\gamma^j \left(\frac{\partial A_a^\gamma}{\partial x^i} + \frac{\partial A_a^\gamma}{\partial x^b} \Lambda_i^b + \frac{\partial \Lambda_i^b}{\partial x^a} A_b^\gamma - \frac{\partial \Lambda_i^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \Lambda_i^\alpha \right) \right) \frac{\partial f^i}{\partial \sigma_a^j} \right) v \\
= \kappa^* & (d^h f) + \kappa^* \left(\mu_\gamma^i \left(\frac{\partial A_a^\gamma}{\partial x^i} + \frac{\partial A_a^\gamma}{\partial x^b} \Lambda_i^b + \frac{\partial \Lambda_i^b}{\partial x^a} A_b^\gamma - \frac{\partial \Lambda_i^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \Lambda_i^\alpha \right) Y^a \right) v
\end{aligned}$$

where the specific structure of Poisson $(n-1)$ -form f has been used. To finish the proof, it suffices to show that the last term of the later expression is a curvature term;

$$\begin{aligned}
& \frac{\partial A_a^\gamma}{\partial x^i} + \frac{\partial A_a^\gamma}{\partial x^b} \Lambda_i^b + \frac{\partial \Lambda_i^b}{\partial x^a} A_b^\gamma - \frac{\partial \Lambda_i^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \Lambda_i^\alpha \\
& = \frac{\partial A_a^\gamma}{\partial x^i} + \frac{\partial A_a^\gamma}{\partial x^b} \Lambda_i^b - \frac{\partial A_b^\gamma}{\partial x^a} \Lambda_i^b + \frac{\partial(\Lambda_i^b A_b^\gamma)}{\partial x^a} - \frac{\partial \Lambda_i^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \Lambda_i^\alpha \\
& = \frac{\partial A_a^\gamma}{\partial x^i} - \frac{\partial A_i^\gamma}{\partial x^a} + \Lambda_i^b \left(\frac{\partial A_a^\gamma}{\partial x^b} - \frac{\partial A_b^\gamma}{\partial x^a} \right) + \frac{\partial(A_i^\gamma + \Lambda_i^b A_b^\gamma - \Lambda_i^\gamma)}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta \Lambda_i^\alpha \\
& = \frac{\partial A_a^\gamma}{\partial x^i} - \frac{\partial A_i^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta A_i^\alpha + \Lambda_i^b \left(\frac{\partial A_a^\gamma}{\partial x^b} - \frac{\partial A_b^\gamma}{\partial x^a} - c_{\beta\alpha}^\gamma A_a^\beta A_b^\alpha \right) \\
& \quad + \frac{\partial(A_i^\gamma + \Lambda_i^b A_b^\gamma - \Lambda_i^\gamma)}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta (A_i^\alpha + A_b^\alpha \Lambda_i^b - \Lambda_i^\alpha).
\end{aligned}$$

From the compatibility equation (6.21), the last two terms vanish. Finally, the other two terms are a local expression of the curvature $\mathcal{B}$ applied to $\partial/\partial x^a$ and the lift of $\partial/\partial x^i$ from M to $s \in \Sigma$. $\square$

Theorem 6.3.3 *For any section p of $\Pi_P \rightarrow M$, let $\sigma \oplus \mu = \kappa \circ p$ be the reduced section of*

$$\Pi_P/G \cong \Pi_\Sigma \oplus \left(TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M \right) \rightarrow \Sigma.$$

Then, the following are equivalent:

(i) *for every Poisson $(n-1)$ -form F on Π_P , the following identity holds true:*

$$\{F, H\}_V = d(F \circ p) - d^h F \circ p. \quad (6.35)$$

(ii) *for every affine Poisson $(n-1)$ -form f on $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$:*

$$\{f, h\}_V = d(f \circ (\sigma \oplus \mu)) - d^h f \circ (\sigma \oplus \mu) + \langle \mu, \bar{\mathcal{B}}(Y, \bar{\Lambda}) \rangle_V, \quad (6.36)$$

where the bracket is defined by equation (6.32).

Proof. From Theorem 6.3.1, κ is Poisson and $\{F, H\} = \kappa^* \{f, h\}$. Furthermore, from Lemma 6.3.2, $d^h F = \kappa^* (d^h f - \langle \mu, \bar{\mathcal{B}}(Y, \bar{\Lambda}) \rangle)$. Since every affine Poisson $(n-1)$ -form f can be obtained by reduction of a Poisson $(n-1)$ -form F on Π_P , we only need to prove that $d(F \circ p) = d(f \circ (\sigma \oplus \mu))$. As F is horizontal, for any $v_1, \dots, v_{n-1}$ vectors in $T_x M$

$$\begin{aligned} (p^* F)_x(v_1, \dots, v_{n-1}) &= F_{p(x)}(T_x p(v_1), \dots, T_x p(v_{n-1})) = F_{p(x)}(v_1, \dots, v_{n-1}) \\ &= (F \circ p)_x(v_1, \dots, v_{n-1}). \end{aligned}$$

In an analogous way, $(\sigma \oplus \mu)^* f = f \circ (\sigma \oplus \mu)$. Consequently,

$$F \circ p = p^* F = p^* \kappa^* f = (\sigma \oplus \mu)^* f = f \circ (\sigma \oplus \mu),$$

coincide as $(n-1)$ -forms on M and $d(F \circ p) = d(f \circ (\sigma \oplus \mu))$. □

Recall the notion of divergence of fields with values in a vector bundles introduced in Definition 4.1.14. In particular, $\text{div}^{\mathcal{A}}$, the divergence for $\text{Ad} P^*$ -valued vector fields induced by connection $\nabla^{\mathcal{A}}$ on $\text{Ad} P$.

Theorem 6.3.4 *In the same setting as in Theorem 6.3.3, the following statement is equivalent to (i) and (ii):*

(iii) the section $\sigma \oplus \mu$ satisfies the equations:

$$\operatorname{div}^{\mathcal{A}} \mu = \operatorname{ad}_{\frac{\delta h}{\delta \mu}}^* \mu, \quad (6.37)$$

$$\{\theta_Y, h\}_{\Sigma^V} = d(\theta_Y \circ (\sigma \oplus \mu)) - d^h \theta_Y \circ (\sigma \oplus \mu) + \left\langle \mu, \bar{\mathcal{B}} \left(Y, \bar{\Lambda} + \frac{\delta h}{\delta \sigma} \right) \right\rangle_V, \quad (6.38)$$

for every θ_Y Poisson $(n-1)$ -form on $\Pi_{\Sigma} \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$ defined by $Y \in \Gamma(V\Sigma \rightarrow M)$ as in Theorem 6.2.9.

Proof. Let f be an affine Poisson $(n-1)$ -form described by $Y \oplus \bar{\xi}$ as in Theorem 6.2.9. In the case that $\bar{\xi} = 0$, we have that $f = \theta_Y$ and equation (6.36) in Theorem 6.3.3 takes the form

$$\{\theta_Y, h\}_{\Sigma^V} - \left\langle \mu, \bar{\mathcal{B}} \left(Y, \frac{\delta h}{\delta \sigma} \right) \right\rangle_V = d(\theta_Y \circ (\sigma \oplus \mu)) - d^h \theta_Y \circ (\sigma \oplus \mu) + \left\langle \mu, \bar{\mathcal{B}}(Y, \bar{\Lambda}) \right\rangle_V, \quad (6.39)$$

from which equation (6.38) is obtained. In the case $Y = 0$, $f = \xi^\alpha \mu_\alpha^i v_i$ and equation (6.36) has the local expression

$$\begin{aligned} & \left(\frac{\partial \xi^\alpha}{\partial x^a} \mu_\alpha^i + \mu_\gamma^i c_{\beta\alpha}^\gamma A_a^\beta \xi^\alpha \right) \frac{\partial h}{\partial \sigma_a^i} + \mu_\gamma^i c_{\beta\alpha}^\gamma \xi^\alpha \frac{\partial h}{\partial \mu_\beta^i} = \\ & = \frac{\partial \xi^\alpha}{\partial x^i} \mu_\alpha^i + \frac{\partial \xi^\alpha}{\partial x^a} \mu_\alpha^i \frac{\partial x^a}{\partial x^i} + \xi^\alpha \frac{\partial \mu_\alpha^i}{\partial x^i} - \left(\frac{\partial \xi^\alpha}{\partial x^i} \mu_\alpha^i + \frac{\partial \xi^\alpha}{\partial x^a} \mu_\alpha^i \Lambda_i^a + \xi^\alpha \mu_\gamma^i c_{\beta\alpha}^\gamma \Lambda_i^\beta \right). \end{aligned}$$

As $\bar{\xi} \in \Gamma(\operatorname{Ad}P)$ is arbitrary, the terms proportional to ξ^α provide

$$\mu_\gamma^i c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^i} + \mu_\gamma^i c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \sigma_a^i} = \frac{\partial \mu_\alpha^i}{\partial x^i} - \mu_\gamma^i c_{\beta\alpha}^\gamma \Lambda_i^\beta, \quad (6.40)$$

while the terms proportional to $\partial \xi^\alpha / \partial x^a$ provide

$$\frac{\partial h}{\partial \sigma_a^i} = \frac{\partial x^a}{\partial x^i} - \Lambda_i^a. \quad (6.41)$$

Substitution of equation (6.41) into equation (6.40) gives

$$\mu_\gamma^i c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^i} = \frac{\partial \mu_\alpha^i}{\partial x^i} - \mu_\gamma^i c_{\beta\alpha}^\gamma A_i^\beta - \mu_\gamma^i c_{\beta\alpha}^\gamma A_b^\beta \frac{\partial x^b}{\partial x^i},$$

which is the local expression of equation (6.37). To this point, we have proved that equation (6.36) in statement (ii) is equivalent to the system of equations defined by (6.37), (6.38) and

(6.41). To end the proof, we must show that (6.41) is a consequence of equation (6.38). Indeed, equation (6.38) is written in local coordinates from

$$\{\theta_Y, h\}_\Sigma = \frac{\partial Y^b}{\partial x^a} \sigma_b^i \frac{\partial h}{\partial \sigma_a^i} - \left(\frac{\partial h}{\partial x^a} + \mu_\gamma^i c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \mu_\alpha^i} \right) Y^a,$$

$$d(\theta_Y \circ (\sigma \oplus \mu)) = \sigma_a^i \frac{\partial Y^a}{\partial x^i} + \frac{\partial x^b}{\partial x^i} \frac{\partial Y^a}{\partial x^b} \sigma_a^i + Y^a \frac{\partial \sigma_a^i}{\partial x^i},$$

and

$$d^h \theta_Y \circ (\sigma \oplus \mu) = \sigma_a^i \frac{\partial Y^a}{\partial x^i} + \Lambda_i^b \sigma_a^i \frac{\partial Y^a}{\partial x^b} - \frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^i Y^a.$$

While the curvature term has the following local expression:

$$\bar{\mathcal{B}}(Y, \cdot) = -Y^a \left(\frac{\partial A_a^\gamma}{\partial x^i} - \frac{\partial A_i^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_i^\beta A_a^\alpha \right) dx^i - Y^a \left(\frac{\partial A_a^\gamma}{\partial x^b} - \frac{\partial A_b^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_b^\beta A_a^\alpha \right) dx^b.$$

As $Y \in \mathfrak{X}(\Sigma)$ is arbitrary, Y^a and $\partial Y^b / \partial x^a$ in the local expression of equation (6.38) are arbitrary. As before, equation (6.41) easily follows from the terms proportional to $\partial Y^b / \partial x^a$ while the terms proportional to Y^a provide:

$$\begin{aligned} - \left(\frac{\partial h}{\partial x^a} + \mu_\gamma^i c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \mu_\alpha^i} \right) &= \frac{\partial \sigma_a^i}{\partial x^i} + \frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^i \\ &- \mu_\gamma^i \left(\frac{\partial A_a^\gamma}{\partial x^i} - \frac{\partial A_i^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_i^\beta A_a^\alpha + \left(\frac{\partial h}{\partial \sigma_b^i} + \Lambda_i^b \right) \left(\frac{\partial A_a^\gamma}{\partial x^b} - \frac{\partial A_b^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_b^\beta A_a^\alpha \right) \right). \end{aligned}$$

□

Remark 6.3.5 When $P \rightarrow M$ is a principal bundle and the reduction is performed by its structure group G , we have $\Sigma = M$, and equation (6.38) is trivial, that is, the only reduced equation is (6.37). This is the case studied in [19] and (6.37) is the so-called Lie-Poisson equation.

With respect to equation (6.38), we could regard it as the bracket formulation of the Hamilton equations on the polysymplectic bundle of $\Sigma \rightarrow M$, now defined by a new bracket $\{, \}_\Sigma$ together with an additional curvature term. Both perturbations collect the fact that the Hamiltonian is actually defined on a product bundle $\Pi_\Sigma \oplus (TM \otimes \tilde{\mathfrak{g}}^* \otimes \bigwedge^{n-1} T^*M)$.

In short, Theorem 6.3.4 provides the bracket formulation of the vertical and horizontal equations of a reduced Hamiltonian system in analogy with the vertical and horizontal Lagrange–Poincaré equations obtained in Lagrangian reduction [32]. We explore this

in Subsection 6.3.2. Observe that this splitting is also found in the different instances of Lagrange–Poincaré reduction exhibited in previous chapters.

Remark 6.3.6 The following local expressions of the Poisson-Poincaré equations are gathered from the proof of Theorem 6.3.4,

$$\frac{\partial \mu_\alpha^i}{\partial x^i} - \mu_\gamma^i c_{\beta\alpha}^\gamma A_i^\beta - \mu_\gamma^i c_{\beta\alpha}^\gamma A_b^\beta \frac{\partial x^b}{\partial x^i} = \mu_\gamma^i c_{\beta\alpha}^\gamma \frac{\partial h}{\partial \mu_\beta^i}, \quad (6.42)$$

$$\frac{\partial h}{\partial \sigma_a^i} = \frac{\partial x^a}{\partial x^i} - \Lambda_i^a, \quad (6.43)$$

$$\begin{aligned} - \left(\frac{\partial h}{\partial x^a} + \mu_\gamma^i c_{\beta\alpha}^\gamma A_a^\beta \frac{\partial h}{\partial \mu_\beta^i} \right) &= \frac{\partial \sigma_a^i}{\partial x^i} + \frac{\partial \Lambda_i^b}{\partial x^a} \sigma_b^i \\ &- \mu_\gamma^i \left(\frac{\partial A_a^\gamma}{\partial x^i} - \frac{\partial A_i^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta A_i^\alpha + \frac{\partial x^b}{\partial x^i} \left(\frac{\partial A_a^\gamma}{\partial x^b} - \frac{\partial A_b^\gamma}{\partial x^a} + c_{\beta\alpha}^\gamma A_a^\beta A_b^\alpha \right) \right). \end{aligned} \quad (6.44)$$

6.3.2 Legendre Transformation

The covariant Lagrange–Poincaré reduction described in Section 1.3 is the Lagrangian picture of the Poisson–Poincaré reduction developed in this chapter.

Definition 6.3.7 Let $\mathcal{L} : J^1 P \rightarrow \bigwedge^n T^* M$ be a Lagrangian density and $L : J^1 P \rightarrow \mathbb{R}$ such that $\mathcal{L} = Lv$, the covariant **Legendre transformation**, $\mathbb{F}\mathcal{L} : J^1 P \rightarrow J^1 P^*$ is the first-order vertical Taylor approximation to $\mathcal{L}$,

$$\mathbb{F}\mathcal{L}(j^1 s)(j^1 s') = \mathcal{L}(j^1 s) + \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}(j^1 s + \varepsilon(j^1 s' - j^1 s)). \quad (6.45)$$

Since $\dim(J^1 P^*) = \dim(J^1 P) + 1$, the Legendre transformation is not a diffeomorphism.

Definition 6.3.8 A Lagrangian density $\mathcal{L} : J^1 P \rightarrow \bigwedge^n T^* M$ is **hyper-regular** if the linear Legendre transformation

$$\widehat{\mathbb{F}\mathcal{L}}(j^1 s)(\vartheta) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}(j^1 s + \varepsilon \vartheta), \quad (6.46)$$

where $\vartheta \in T^* M \otimes VP$, defines a global diffeomorphism between $J^1 P$ and Π_P .

A Lagrangian system defined by an hyper-regular Lagrangian $\mathcal{L}$ is equivalent to the Hamiltonian system $(\Pi_P, \mathbb{F}\mathcal{L} \circ \widehat{\mathbb{F}\mathcal{L}}^{-1})$. For instance, a unified Lagrangian-Hamiltonian formalism

for hyper-regular systems is introduced in [29]. Provided an Ehresmann connection Λ on $P \rightarrow M$, the system is described by the Hamiltonian

$$\mathcal{H}(p) = \left\langle p, \widehat{\mathbb{F}\mathcal{L}}^{-1}(p) - \Lambda \right\rangle - \mathcal{L} \circ \widehat{\mathbb{F}\mathcal{L}}^{-1}(p) \quad (6.47)$$

which in local coordinates can be written as $\mathcal{H} = H_V$, where $H = p_a^i(y_i^a - \Lambda_i^a) - L$. The map

$$\begin{aligned} \widehat{\mathbb{F}\mathcal{H}} : \Pi_P &\rightarrow T^*M \otimes VP \\ p &\mapsto \widehat{\mathbb{F}\mathcal{H}}(p)(\varrho) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{H}(p + \varepsilon \varrho v) \end{aligned}$$

is called *inverse Lagrange transformation* since

$$\begin{array}{ccc} J^1 P & \xrightarrow{\widehat{\mathbb{F}\mathcal{L}}} & \Pi_P \\ F_\Lambda \downarrow & \swarrow \widehat{\mathbb{F}\mathcal{H}} & \\ T^*M \otimes VP & & \end{array}$$

where F_Λ is the linearization of the affine bundle $J^1 P \rightarrow P$ taking Λ as the zero section.

We shall see that Poisson–Poincaré reduction provides a reduced Legendre transformation. Suppose that $\mathcal{L}$ is G -invariant. On one hand, the linear Legendre transformation $\widehat{\mathbb{F}\mathcal{L}} : J^1 P \rightarrow \Pi_P$ is G -equivariant: for every $g \in G$

$$\widehat{\mathbb{F}\mathcal{L}}(gj^1 s)(g\vartheta) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(gj^1 s + \varepsilon(g\vartheta)) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(j^1 s + \varepsilon\vartheta) = \widehat{\mathbb{F}\mathcal{L}}(j^1 s)(\vartheta).$$

Thus, there exists a reduced map $[\widehat{\mathbb{F}\mathcal{L}}]_G : J^1 P/G \rightarrow \Pi_P/G$. On the other hand, the reduced Lagrangian l defines the map

$$\begin{aligned} \widehat{\mathbb{F}l} : J^1 P/G &\rightarrow \Pi_P/G \\ [j^1 s] &\mapsto \widehat{\mathbb{F}l}([j^1 s])([\vartheta]) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} l([j^1 s] + \varepsilon[\vartheta]) v. \end{aligned}$$

Then, we have that $\widehat{\mathbb{F}l} = [\widehat{\mathbb{F}\mathcal{L}}]_G$ since

$$\begin{aligned} [\widehat{\mathbb{F}\mathcal{L}}]_G([j^1 s])([\vartheta]) &= \widehat{\mathbb{F}\mathcal{L}}(j^1 s)(\vartheta) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}(j^1 s + \varepsilon\vartheta) \\ &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} l([j^1 s + \varepsilon\vartheta]) v = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} l([j^1 s] + \varepsilon[\vartheta]) v = \widehat{\mathbb{F}l}([j^1 s])([\vartheta]). \end{aligned}$$

Furthermore, the Hamiltonian in equation (6.47) is G -invariant and there exists a reduced Hamiltonian $h : \Pi_P/G \rightarrow \bigwedge^n T^*M$. It is similarly proven that $\widehat{\mathbb{F}\mathcal{H}}$ is G -equivariant and $\widehat{\mathbb{F}h} = [\widehat{\mathbb{F}\mathcal{H}}]_G$.

Suppose that the Ehresmann connection Λ is G -invariant, then there is a G -invariant section of $J^1P \rightarrow P$ which reduces to a section $[\Lambda]$ of $J^1P/G \rightarrow \Sigma$. Then, the linearization map F_Λ is G -equivariant and $[F_\Lambda]_G = F_{[\Lambda]}$, the linearization of J^1P/G taking $[\Lambda]$ as the zero section. In addition,

$$\begin{aligned} h([p]) &= H(p) = \left\langle [p], [\widehat{\mathbb{F}\mathcal{L}}^{-1}(p) - \Lambda] \right\rangle v^* - l\left([\widehat{\mathbb{F}\mathcal{L}}^{-1}(p)]\right) \\ &= \left\langle [p], \widehat{\mathbb{F}l}^{-1}([p]) - [\Lambda] \right\rangle v^* - l(\widehat{\mathbb{F}l}^{-1}([p])) \end{aligned} \quad (6.48)$$

and h can be obtained from $\widehat{\mathbb{F}l}$ and $[\Lambda]$. In conclusion, the following diagram commutes

$$\begin{array}{ccc} J^1P/G & \xrightarrow{\widehat{\mathbb{F}l}} & \Pi_P/G \\ F_{[\Lambda]} \downarrow & \swarrow \widehat{\mathbb{F}h} & \\ T^*M \otimes (VP)/G & & \end{array}$$

The map $\widehat{\mathbb{F}l}$ is called *reduced Legendre transform* and $\widehat{\mathbb{F}h}$ is called *reduced reverse Legendre transform*.

Given a compatible principal connection $\mathcal{A}$ on $P \rightarrow \Sigma$ in the sense of Definition 6.2.10 and adapted local coordinates as in Section 6.2, $\mathbb{F}l$ has the following local expression

$$\mathbb{F}l(y_i^a, v_i^\alpha) = \left(\sigma_a^i = \frac{\partial l}{\partial y_i^a}(y_i^a, v_i^\alpha), \mu_\alpha^i = \frac{\partial l}{\partial v_i^\alpha}(y_i^a, v_i^\alpha) \right),$$

and the reduced Hamiltonian in equation (6.48) is

$$h(x^i, x^a, \sigma_a^i, \mu_\alpha^i) = -l(x^i, x^a, y_i^a, v_i^\alpha) + \sigma_a^i(y_i^a - \Lambda_i^a) + \mu_\alpha^i v_i^\alpha, \quad (6.49)$$

where $(y_i^a, v_i^\alpha) = \widehat{\mathbb{F}l}^{-1}(\sigma_a^i, \mu_\alpha^i)$. The Poisson–Poincaré equations (6.42), (6.43) and (6.44) for h in (6.49) provide the Lagrange–Poincaré equations for l . More precisely, since equations (6.43) for that hamiltonian imposes that the solution of the Lagrangian system must be an holonomic section, the resulting equations can be interpreted as a field theoretic counterpart of the implicit Lagrange–Poincaré equations provided in [84] in mechanics. Obtaining these implicit Lagrange–Poincaré equations by reduction of systems as those introduced in [83] may prove to be an interesting future work.

6.3.3 Example: $SO(3)$ -strand with Broken Symmetry

We now present a Hamiltonian system corresponding to an $SO(3)$ -strand in which the $SO(3)$ symmetry is broken in one specific direction e_3 . This system was already studied in Section 5.1 from the Lagrangian perspective.

The model is defined on the trivial bundle $P = \mathbb{R}^2 \times SO(3) \rightarrow \mathbb{R}^2$ with projection $(s, t, R) \mapsto x = (s, t)$ and the polysymplectic bundle

$$\Pi_P = T\mathbb{R}^2 \otimes V^*(\mathbb{R}^2 \times SO(3)) \otimes \bigwedge^2 T^*\mathbb{R}^2 = T\mathbb{R}^2 \otimes T^*SO(3) \otimes \bigwedge^2 T^*\mathbb{R}^2.$$

Any section of $\Pi_P \rightarrow \mathbb{R}^2$ can be written as

$$\frac{\partial}{\partial s} \otimes p^s(x) \otimes (dt \wedge ds) + \frac{\partial}{\partial t} \otimes p^t(x) \otimes (dt \wedge ds),$$

where $p^s, p^t : \mathbb{R}^2 \rightarrow T^*SO(3)$. The Hamiltonian system is defined on Π_P via the trivial Ehresmann connection $\Lambda : T\mathbb{R}^2 \rightarrow TP$ on the product bundle $P = \mathbb{R}^2 \times SO(3) \rightarrow \mathbb{R}^2$ and the Hamiltonian function;

$$H(R, p^s, p^t) = -\frac{1}{2} \langle RJ^{-1}R^{-1}p^s, p^s \rangle + \frac{1}{2} \langle RI^{-1}R^{-1}p^t, p^t \rangle + ee_3 \cdot R\chi, \quad (6.50)$$

where I is the inertia tensor of the rigid body describing the configuration charge, J is a tensor describing the opposition to the rotation of consecutive rigid bodies, $\chi \in \mathbb{R}^3$ is the electric dipole of the rigid body in the configuration reference, and $\mathbf{E} = ee_3$ is a uniform electric field in the direction e_3 of the spatial frame of reference. Since

$$\frac{\partial H}{\partial p^s} = -RJ^{-1}R^{-1}p^s, \quad (6.51)$$

$$\frac{\partial H}{\partial p^t} = RI^{-1}R^{-1}p^t, \quad (6.52)$$

and $\partial H / \partial R = ee_3 \times R\chi$. The local Hamilton–Cartan equations for this system are:

$$R_s = -RJ^{-1}R^{-1}p^s, \quad (6.53)$$

$$R_t = RI^{-1}R^{-1}p^t, \quad (6.54)$$

$$-\partial_s p^s - \partial_t p^t = ee_3 \times R\chi. \quad (6.55)$$

There is a left $SO(3)$ action on $P = \mathbb{R}^2 \times SO(3)$ by left translations on the second factor which is lifted to Π_P . Since for any $Q \in SO(3)$,

$$\begin{aligned} H(Q \cdot (R, p^s, p^t)) &= H(QR, Qp^s, Qp^t) = \\ &= -\frac{1}{2}\langle QRJ^{-1}R^{-1}Q^{-1}Qp^s, Qp^s \rangle + \frac{1}{2}\langle QRJ^{-1}R^{-1}Q^{-1}Qp^t, Qp^t \rangle + ee_3 \cdot QR\chi \\ &= -\frac{1}{2}\langle RJ^{-1}R^{-1}p^s, p^s \rangle + \frac{1}{2}\langle RJ^{-1}R^{-1}p^t, p^t \rangle + eQ^{-1}e_3 \cdot R\chi, \end{aligned}$$

the Hamiltonian system is not $SO(3)$ -invariant. Yet, it is invariant by the action of rotations around the axis e_3 . In other words, the Hamiltonian system is $\mathbb{S}^1$ -invariant. As shown in Section 5.1, $P = \mathbb{R}^2 \times SO(3) \rightarrow \Sigma = P/\mathbb{S}^1 = \mathbb{R}^2 \times \mathbb{S}^2$ is a principal bundle with connection $\mathcal{A}(v) = \langle v, e_3 \rangle$, where $v \in T_R SO(3) \simeq \mathbb{R}^3$. This connections allows to identify $\Pi_P/\mathbb{S}^1$ with $\Pi_\Sigma \oplus (T\mathbb{R}^2 \otimes \mathbb{R} \otimes \bigwedge^2 T^*\mathbb{R}^2) = \Pi_\Sigma \oplus T^*\mathbb{R}^2$ as in Proposition 6.2.3. We shall now obtain the reduced multimomenta with the projection κ defined in that Proposition. Let $\alpha \in T_R^* SO(3)$ and $v_\zeta \in T_\zeta \mathbb{S}^2$,

$$\langle \text{hor}_{\mathcal{A}}^*(\alpha), v_\zeta \rangle = \langle \alpha, R(-\zeta \times v_\zeta) \rangle = \langle v_\zeta, \zeta \times R^{-1}\alpha \rangle.$$

Thus, $\text{hor}_{\mathcal{A}}^*(\alpha) = \zeta \times R^{-1}\alpha$. Let $a \in \mathbb{R}$, an element in the Lie algebra of $\mathbb{S}^1$. Then,

$$\langle J(\alpha), a \rangle = \langle \alpha, a_R \rangle = a \langle \alpha, (e_3)_R \rangle = a \langle R^{-1}\alpha, (R^{-1}e_3)_e \rangle = a \langle R^{-1}\alpha, \zeta \rangle.$$

Thus, $J(\alpha) = \langle R^{-1}\alpha, \zeta \rangle$. In conclusion, the multimomenta p^s, p^t in Π_P reduce to

$$\sigma^s = \zeta \times R^{-1}p^s, \quad \sigma^t = \zeta \times R^{-1}p^t, \quad (6.56)$$

$$\mu^s = \langle \zeta, R^{-1}p^s \rangle, \quad \mu^t = \langle \zeta, R^{-1}p^t \rangle. \quad (6.57)$$

Observe that for $i = s, t$, $R^{-1}p^i$ can be decomposed in a parallel and perpendicular to ζ in terms of the reduced multimomenta σ^i, μ^i . Indeed,

$$R^{-1}p^i = \mu^i \zeta - \zeta \times \sigma^i,$$

and substitution into H provides the reduced Hamiltonian

$$\begin{aligned} h(\zeta, \sigma^s, \sigma^t, \mu^s, \mu^t) &= -\frac{1}{2}(\mu^s)^2 \langle \zeta, J^{-1}\zeta \rangle - \mu^s \langle \sigma^s, \zeta \times J^{-1}\zeta \rangle - \frac{1}{2} \langle \zeta \times \sigma^s, J^{-1}(\zeta \times \sigma^t) \rangle \\ &\quad + \frac{1}{2}(\mu^t)^2 \langle \zeta, I^{-1}\zeta \rangle + \mu^t \langle \sigma^t, \zeta \times I^{-1}\zeta \rangle + \frac{1}{2} \langle \zeta \times \sigma^t, I^{-1}(\zeta \times \sigma^s) \rangle + e\zeta \cdot \xi. \end{aligned} \quad (6.58)$$

The equations of motion are given by the local Poisson–Poincaré equations gathered in Remark 6.3.6;

$$\zeta_s = -\zeta \times J^{-1}(\mu^s \zeta - \zeta \times \sigma^s), \quad (6.59)$$

$$\zeta_t = \zeta \times I^{-1}(\mu^t \zeta - \zeta \times \sigma^t), \quad (6.60)$$

$$\partial_s \sigma^s + \partial_t \sigma^t = \zeta \times \sigma^s \langle \zeta, J^{-1}(\mu^s \zeta - \zeta \times \sigma^s) \rangle - \zeta \times \sigma^t \langle \zeta, I^{-1}(\mu^t \zeta - \zeta \times \sigma^t) \rangle - e \chi^\parallel, \quad (6.61)$$

$$\partial_s \mu^s + \partial_t \mu^t = 0, \quad (6.62)$$

where $\chi^\parallel = \chi - \zeta \langle \xi, \zeta \rangle$ is the ortogonal projection of χ to $T_\zeta \mathbb{S}^2$. Equations (6.59) and (6.60) are the particularization of equation (6.43), and equation (6.62) is the vertical equation obtained from (6.42). The horizontal equation (6.61) is obtained from (6.44) using that $\mathcal{B}(\zeta_t, \cdot) = -\zeta \times \zeta_t$. In the special case $I = J = \text{Id}_{\mathbb{R}^3}$ and $\chi = 0$, equations (6.60–6.62) are the decomposition in terms parallel and perpendicular to ζ of the equations of motion of the classical chiral model for the $SO(3)$ -strand and the equations for harmonic maps from $\mathbb{R}^2$ to $SO(3)$ studied in [82].

As seen in Subsection 6.3.2, the Hamiltonian in equation (6.50) can be obtained applying the Legendre transform $H = \langle p^s, R_s \rangle + \langle p^t, R_t \rangle - L$ to the hyperregular $\mathbb{S}^1$ -invariant Lagrangian $L : J^1(\mathbb{R}^2 \times SO(3)) \rightarrow \mathbb{R}$ in equation (5.1). In addition, the momenta p^s, p^t can be calculated from the vertical derivatives of the Lagrangian L ,

$$p^s = \frac{\partial L}{\partial \Xi} = -\Xi J \zeta - J(\zeta_s \times \zeta), \quad (6.63)$$

$$p^t = \frac{\partial L}{\partial \xi} = \xi I \zeta + I(\zeta_t \times \zeta). \quad (6.64)$$

These momenta were already used in Section 5.1 to conveniently express the horizontal Lagrange–Poincaré equation (5.17).

In turn, the reduced hamiltonian h can be deduced applying the reduced Legendre transform $h = \langle \sigma^s, \zeta_s \rangle + \langle \sigma^t, \zeta_t \rangle + \mu^s \Xi + \mu^t \xi - l$ to the reduced Lagrangian $l : J^1(\mathbb{R}^2 \times \mathbb{S}^2) \oplus (T^* \mathbb{R}^2 \otimes (\mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R})) \rightarrow \mathbb{R}$ in equation (5.11). The momenta $\sigma^s, \sigma^t, \mu^s, \mu^t$ can be obtained from vertical derivatives of l ,

$$\sigma^s = \frac{\partial l}{\partial \zeta_s} = -\zeta \times (\Xi J \zeta + J(\zeta_s \times \zeta)), \quad \sigma^t = \frac{\partial l}{\partial \zeta_t} = \zeta \times (\xi I \zeta + I(\zeta_t \times \zeta)), \quad (6.65)$$

$$\mu^s = \frac{\partial l}{\partial \Xi} = -\langle \zeta, \Xi J \zeta + J(\zeta_s \times \zeta) \rangle, \quad \mu^t = \frac{\partial l}{\partial \xi} = \langle \zeta, \xi I \zeta + I(\zeta_t \times \zeta) \rangle, \quad (6.66)$$

where $\Xi = \mathcal{A}(R_s)$ and $\xi = \mathcal{A}(R_t)$. These were already calculated in equations (5.12) and (5.13) to obtain the Lagrange–Poincaré equations.

Bibliography

- [1] Abraham, R. and Marsden, J. E. (1978). *Foundations of Mechanics*, volume 364 of *Advanced Book Program, Reading, Mass.* Benjamin/Cummings Publishing Co., Inc., 2 edition.
- [2] Arnold, V. (1989). *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2 edition.
- [3] Balachandran, A. P., Borchardt, S., and Stern, A. (1978). Lagrangian and hamiltonian descriptions of yang-mills particles. *Phys. Rev. D*, 17:3247–3256.
- [4] Berbel, M. A. and Castrillón López, M. (2020). Lagrangian reduction by stages in field theory. arXiv:2007.14854.
- [5] Berbel, M. A. and Castrillón López, M. (2022a). Poisson–Poincaré reduction for field theories. arXiv:2210.14732.
- [6] Berbel, M. A. and Castrillón López, M. (2022b). Reduction (by stages) in the whole Lagrange–Poincaré category. *Adv. Theor. Math. Phys.*, 26(8).
- [7] Berbel, M. A. and Castrillón López, M. (2023). Rigid body with rotors and reduction by stages. *Reviews in Mathematical Physics*, 35(1).
- [8] Berbel, M. A. and Gay-Balmaz, F. (In preparation). Geometry of general relativistic Lagrangian continuum theories: Non abelian fluids.
- [9] Blacker, C. (2021). Reduction of multisymplectic manifolds. *Lett. Math. Phys.*, 111(3):Paper No. 64, 30.
- [10] Bloch, A., Colombo, L., and Jiménez, F. (2019). The variational discretization of the constrained higher-order Lagrange–Poincaré equations. *Discrete Contin. Dyn. Syst.*, 39(1):309–344.
- [11] Bloch, A. M. (2015). *Nonholonomic mechanics and control*, volume 24 of *Interdisciplinary Applied Mathematics*. Springer, New York, second edition. With the collaboration of J. Bailieul, P. E. Crouch, J. E. Marsden and D. Zenkov, With scientific input from P. S. Krishnaprasad and R. M. Murray.
- [12] Bloch, A. M., Brockett, R. W., and Ratiu, T. S. (1990). A new formulation of the generalized Toda lattice equations and their fixed point analysis via the momentum map. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):477–485.

- [13] Bloch, A. M., Krishnaprasad, P. S., Marsden, J. E., and Ratiu, T. S. (1994). Dissipation induced instabilities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 11(1):37–90.
- [14] Bloch, A. M. and Rojo, A. G. (2010). Sorting: the Gauss thermostat, the Toda lattice and double bracket equations. In *Three decades of progress in control sciences*, pages 35–48. Springer, Berlin.
- [15] Bursztyn, H., Cabrera, A., and Iglesias, D. (2015). Multisymplectic geometry and Lie groupoids. In *Geometry, mechanics, and dynamics*, volume 73 of *Fields Inst. Commun.*, pages 57–73. Springer, New York.
- [16] Capriotti, S., Díaz, V. A., Andrés, E. G.-T., and Mestdag, T. (2022). Cotangent bundle reduction and routh reduction for polysymplectic manifolds.
- [17] Castrillón López, M., García, P. L., and Rodrigo, C. (2013). Euler-Poincaré reduction in principal bundles by a subgroup of the structure group. *J. Geom. Phys.*, 74:352–369.
- [18] Castrillón López, M., García Pérez, P. L., and Ratiu, T. S. (2001). Euler-Poincaré reduction on principal bundles. *Lett. Math. Phys.*, 58(2):167–180.
- [19] Castrillón López, M. and Marsden, J. E. (2003). Some remarks on Lagrangian and Poisson reduction for field theories. *J. Geom. Phys.*, 48(1):52–83.
- [20] Castrillón López, M. and Ratiu, T. S. (2003). Reduction in principal bundles: covariant Lagrange-Poincaré equations. *Comm. Math. Phys.*, 236(2):223–250.
- [21] Cendra, H. and Marsden, J. E. (1987). Lin constraints, Clebsch potentials and variational principles. *Phys. D*, 27(1-2):63–89.
- [22] Cendra, H., Marsden, J. E., Pekarsky, S., and Ratiu, T. S. (2003). Variational principles for Lie-Poisson and Hamilton-Poincaré equations. volume 3, pages 833–867.
- [23] Cendra, H., Marsden, J. E., and Ratiu, T. S. (2001). Lagrangian reduction by stages. *Mem. Amer. Math. Soc.*, 152(722):1–108.
- [24] Ciaglia, F. M., Di Cosmo, F., Ibort, A., Marmo, G., Schiavone, L., and Zampini, A. (2022a). The geometry of the solution space of first order hamiltonian field theories I: from particle dynamics to free electrodynamics. arXiv: 2208.14136.
- [25] Ciaglia, F. M., Di Cosmo, F., Ibort, A., Marmo, G., Schiavone, L., and Zampini, A. (2022b). The geometry of the solution space of first order hamiltonian field theories II: non-abelian gauge theories. arXiv: 2208.14155.
- [26] Colombo, L. J. and Dimarogonas, D. V. (2020). Motion feasibility conditions for multi-agent control systems on Lie groups. *IEEE Trans. Control Netw. Syst.*, 7(1):493–502.
- [27] de Lucas, J., Gràcia, X., Rivas, X., Román-Roy, N., and Vilariño, S. (2022). Reduction and reconstruction of multisymplectic Lie systems. *J. Phys. A*, 55(29):Paper No. 295204, 34.
- [28] Doroshin, A. V. (2014). Homoclinic solutions and motion chaotization in attitude dynamics of a multi-spin spacecraft. *Commun. Nonlinear Sci. Numer. Simul.*, 19(7):2528–2552.

- [29] Echeverría-Enríquez, A., López, C., Marín-Solano, J., Muñoz Lecanda, M. C., and Román-Roy, N. (2004). Lagrangian-Hamiltonian unified formalism for field theory. *J. Math. Phys.*, 45(1):360–380.
- [30] Echeverría-Enríquez, A., Muñoz Lecanda, M. C., and Román-Roy, N. (2018). Remarks on multisymplectic reduction. *Rep. Math. Phys.*, 81(3):415–424.
- [31] Ellis, D. C. P., Gay-Balmaz, F., Holm, D. D., Putkaradze, V., and Ratiu, T. S. (2010). Symmetry reduced dynamics of charged molecular strands. *Arch. Ration. Mech. Anal.*, 197(3):811–902.
- [32] Ellis, D. C. P., Gay-Balmaz, F., Holm, D. D., and Ratiu, T. S. (2011). Lagrange-Poincaré field equations. *J. Geom. Phys.*, 61(11):2120–2146.
- [33] Fernández, J., Juchani, M., and Zuccalli, M. (2022). Discrete connections on principal bundles: the discrete Atiyah sequence. *J. Geom. Phys.*, 172:Paper No. 104417, 27.
- [34] Fernández, J., Tori, C., and Zuccalli, M. (2016). Lagrangian reduction of discrete mechanical systems by stages. *J. Geom. Mech.*, 8(1):35–70.
- [35] Fernández, J., Tori, C., and Zuccalli, M. (2020). Lagrangian reduction of nonholonomic discrete mechanical systems by stages. *J. Geom. Mech.*, 12(4):607–639.
- [36] Gajbhiye, S. and Banavar, R. N. (2016). Geometric modeling and local controllability of a spherical mobile robot actuated by an internal pendulum. *Internat. J. Robust Nonlinear Control*, 26(11):2436–2454.
- [37] Gaset, J. and Román-Roy, N. (2018). Multisymplectic unified formalism for Einstein-Hilbert gravity. *J. Math. Phys.*, 59(3):032502, 39.
- [38] Gay-Balmaz, F. (2022). Geometry of general relativistic Lagrangian continuum theories. Part I: reduced variational principles and junction conditions for hydrodynamics and elasticity. arXiv:2202.06560.
- [39] Gay-Balmaz, F., Marrero, J. C., and Martínez, N. (2022). A new canonical affine bracket formulation of Hamiltonian classical field theories of first order. arXiv:2209.08736.
- [40] Gay-Balmaz, F., Marsden, J. E., and Ratiu, T. S. (2012). Reduced variational formulations in free boundary continuum mechanics. *J. Nonlinear Sci.*, 22(4):463–497.
- [41] Goodman, J. and Colombo, L. (2022). Geometric control of two quadrotors carrying a rigid rod with elastic cables. *J. Nonlinear Sci.*, 32(5):Paper No. 65, 31.
- [42] Gotay, M. J., Isenberg, J., Marsden, J. E., and Montgomery, R. (2004). Momentum maps and classical relativistic fields. part I: Covariant field theory. arXiv: 9801019.
- [43] Günther, C. (1987). The polysymplectic Hamiltonian formalism in field theory and calculus of variations. I. The local case. *J. Differential Geom.*, 25(1):23–53.
- [44] Hajdu, S. and Mestdag, T. (2021). Homogeneous nonlinear splittings and finsler submersions. arXiv:2111.04411.

- [45] Hajdú, S. and Mestdag, T. (2022). Nonlinear splittings on fibre bundles. *Anal. Math. Phys.*, 12(1):Paper No. 14, 28.
- [46] Holm, D. D. and Ivanov, R. I. (2014). Matrix G -strands. *Nonlinearity*, 27(6):1445–1469.
- [47] Holm, D. D., Ivanov, R. I., and Percival, J. R. (2012). G -strands. *J. Nonlinear Sci.*, 22(4):517–551.
- [48] Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). The Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.*, 137(1):1–81.
- [49] Holm, D. D., Schmah, T., and Stoica, C. (2009). *Geometric mechanics and symmetry*, volume 12 of *Oxford Texts in Applied and Engineering Mathematics*. Oxford University Press, Oxford.
- [50] Husemoller, D. (1994). *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition.
- [51] Iglesias, D., Marrero, J. C., de Diego, D. M., and Martínez, E. (2008). Discrete non-holonomic Lagrangian systems on Lie groupoids. *J. Nonlinear Sci.*, 18(3):221–276.
- [52] Iglesias, D., Marrero, J. C., Martín de Diego, D., Martínez, E., and Padrón, E. (2007). Reduction of symplectic Lie algebroids by a Lie subalgebroid and a symmetry Lie group. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 3:Paper 049, 28.
- [53] Janyška, J. and Modugno, M. (1996). Relations between linear connections on the tangent bundle and connections on the jet bundle of a fibred manifold. *Arch. Math. (Brno)*, 32(4):281–288.
- [54] Kanatchikov, I. V. (1998). Canonical structure of classical field theory in the polymomentum phase space. *Rep. Math. Phys.*, 41(1):49–90.
- [55] Kobayashi, S. and Nomizu, K. (1963). *Foundations of differential geometry. I*, volume 15 of *Intersci. Tracts Pure Appl. Math.* Interscience Publishers, New York, NY.
- [56] Krishnaprasad, P. S. and Berenstein, C. A. (1984). On the equilibria of rigid spacecraft with rotors. *Systems Control Lett.*, 4(3):157–163.
- [57] Leok, M. and Zhang, J. (2011). Discrete Hamiltonian variational integrators. *IMA J. Numer. Anal.*, 31(4):1497–1532.
- [58] Leonard, N. E. and Marsden, J. E. (1997). Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry. *Phys. D*, 105(1-3):130–162.
- [59] Li, S., Stern, A., and Tang, X. (2017). Lagrangian mechanics and reduction on fibered manifolds. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 13:Paper No. 019, 26.
- [60] Madsen, T. B. and Swann, A. (2013). Closed forms and multi-moment maps. *Geom. Dedicata*, 165:25–52.
- [61] Marrero, J. C., Román-Roy, N., Salgado, M., and Vilariño, S. (2015). Reduction of polysymplectic manifolds. *J. Phys. A*, 48(5):055206, 43.

- [62] Marsden, J. and Weinstein, A. (1974). Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130.
- [63] Marsden, J. E., Misiolek, G., Ortega, J.-P., Perlmutter, M., and Ratiu, T. S. (2007). *Hamiltonian Reduction by Stages*, volume 1913 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 1 edition.
- [64] Marsden, J. E., Montgomery, R., Morrison, P. J., and Thompson, W. B. (1986). Co-variant Poisson brackets for classical fields. *Ann. Physics*, 169(1):29–47.
- [65] Marsden, J. E., Rațiu, T. S., and Weinstein, A. (1984). Semidirect products and reduction in mechanics. *Trans. Amer. Math. Soc.*, 281(1):147–177.
- [66] Marsden, J. E. and Ratiu, T. (1986). Reduction of Poisson manifolds. *Lett. Math. Phys.*, 11(2):161–169.
- [67] Marsden, J. E. and Ratiu, T. S. (1999). *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, volume 17 of *Texts in Applied Mathematics*. Springer, New York, NY, 2 edition.
- [68] Marsden, J. E. and Scheurle, J. (1993). The reduced Euler-Lagrange equations. In *Dynamics and control of mechanical systems (Waterloo, ON, 1992)*, volume 1 of *Fields Inst. Commun.*, pages 139–164. Amer. Math. Soc., Providence, RI.
- [69] Marsden, J. E. and Weinstein, A. (2001). Comments on the history, theory, and applications of symplectic reduction. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 1–19. Birkhäuser, Basel.
- [70] Mestdag, T. (2005). Lagrangian reduction by stages for non-holonomic systems in a Lie algebroid framework. *J. Phys. A*, 38(47):10157–10179.
- [71] Michor, P. W. (2008). *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.
- [72] Nayak, A., Banavar, R. N., and Maithripala, D. H. S. (2018). Almost-global tracking for a rigid body with internal rotors. *Eur. J. Control*, 42:59–66.
- [73] Puta, M., Tudoran, R., and Tudoran, R. (2005). Poisson manifolds and Bermejo-Fairén construction of Casimirs. *Tensor (N.S.)*, 66(1):59–70.
- [74] Saunders, D. J. (1989). *The geometry of jet bundles*, volume 142 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.
- [75] Shi, D., Berchenko-Kogan, Y., Zenkov, D. V., and Bloch, A. M. (2017). Hamel’s formalism for infinite-dimensional mechanical systems. *J. Nonlinear Sci.*, 27(1):241–283.
- [76] Shi, D., Zenkov, D. V., and Bloch, A. M. (2020). Hamel’s formalism for classical field theories. *J. Nonlinear Sci.*, 30(4):1307–1353.
- [77] Siegel, C. L. and Moser, J. K. (1995). *Lectures on celestial mechanics*. Classics in Mathematics. Springer-Verlag, Berlin. Translated from the German by C. I. Kalme, Reprint of the 1971 translation.

- [78] Simoes, A. A., Marrero, J. C., and de Diego, D. M. (2022). Exact discrete Lagrangian mechanics for nonholonomic mechanics. *Numer. Math.*, 151(1):49–98.
- [79] Souriau, J. M. (1997). *Structure of dynamical systems*, volume 149 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA.
- [80] Svinin, M., Morinaga, A., and Yamamoto, M. (2013). On the dynamic model and motion planning for a spherical rolling robot actuated by orthogonal internal rotors. *Regul. Chaotic Dyn.*, 18(1-2):126–143.
- [81] Tran, B. and Leok, M. (2022). Multisymplectic Hamiltonian variational integrators. *Int. J. Comput. Math.*, 99(1):113–157.
- [82] Uhlenbeck, K. (1989). Harmonic maps into Lie groups: classical solutions of the chiral model. *J. Differential Geom.*, 30(1):1–50.
- [83] Vankerschaver, J., Yoshimura, H., and Leok, M. (2012). The Hamilton-Pontryagin principle and multi-Dirac structures for classical field theories. *J. Math. Phys.*, 53(7):072903, 25.
- [84] Yoshimura, H. and Marsden, J. E. (2009). Dirac cotangent bundle reduction. *J. Geom. Mech.*, 1(1):87–158.