Gauge reduction in covariant field theory

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Abstract

In this work we develop a Lagrangian reduction theory for covariant field theories with local symmetries and, more specifically, with gauge symmetries. We model these symmetries by using a Lie group fiber bundle acting fiberwisely on the corresponding configuration bundle. In order to reduce the variational principle, we utilize generalized principal connections, a type of Ehresmann connections that are equivariant by the fiberwise action. After obtaining the reduced equations, we give the reconstruction condition and we relate the vertical reduced equation with the Noether theorem. Lastly, we illustrate the theory by applying it to several examples, including the classical case (Lagrange-Poincaré reduction) and electromagnetism.

1 Introduction

Reduction by symmetries has played a major role in geometric mechanics since it was first introduced with its modern approach [1, 23, 24, 28]. The key idea is to use the symmetry group of the system to obtain a reduced set of equations on a space of lower dimension. In the Lagrangian setting, this reduced space is the quotient of the configuration manifold by the symmetry group, and the reduced equations come from a reduced variational principle.

The reduction results in Geometric Mechanics [9, 10, 22] have been extended successfully to the realm of Classical Field Theory [6, 15] and, in particular, to covariant field theories [5, 7, 8, 11], where the configuration manifold is substituted by a fiber bundle whose base space models the space-time. In previous works, reduction by global symmetries (also known

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in the Physics literature as global gauge symmetries) has been investigated. In this case a Lie group G acts upon the configuration bundle Y so that the configuration bundle of the reduced problem will be Y/G. In this situation, every element $g \in G$ induces a diffeomorphism on the whole space $\Phi_g : Y \to Y$. However, there is a wide variety of physical systems with *local symmetries*, i.e., when the symmetry group depends on the point of the space-time (also known in the Physics literature as local gauge symmetries). This is modelled by a Lie group fiber bundle acting fiberwisely on the configuration bundle. The main instance of this situation are gauge theories [12, 19]. In such theories, Utiyama's theorem [14, 29], as well as its generalization to interaction of gauge fields with matter [2], ensure that the Lagrangian density only depends on the curvature (force), instead of depending on the principal connection itself (potential). The standard Field Theory reduction scheme (Lagrange-Poincaré reduction) does not apply and a new framework is needed for these essential theories.

This paper addresses the construction of a general Lagrangian reduction procedure for (covariant) field theories with local symmetries, following the idea introduced in [26] for a very particular case. The reduction of the variational principle relies in the use of the generalized principal connections investigated in [18], that is, Ehresmann connections that are equivariant by the fiberwise action. Our theory of local symmetries extends the classical case investigated in [11] for global symmetries, as we will see below.

Reduction theory has also been analyzed in the discrete setting for both mechanics [3,4,20, 21] and field theory [30], yielding geometric integrators for the reduced theories. Nevertheless, we do not address this matter in this work.

We organise the paper as follows. In Section 2, we recall the geometric tools needed to develop our theory. Namely, fibered actions of Lie group fiber bundles and the natural connections arising on the corresponding quotient bundles, generalized principal connections. Section 3 is devoted to study the geometry of the quotient of the first jet of the configuration bundle by the fibered action, and to build an affine connection on it. Next, in Section 4 we compute the reduced variations and we apply it to the reduced Lagrangian, thus obtaining the main result of this work: the reduced field equations. After that, we investigate the reconstruction condition in Section 5, and we relate the vertical reduced equation with Noether theorem in Section 6. At last, in Section 7 we demonstrate our theory by applying it to several examples. In particular, we study the classical case (global symmetries), the full jet symmetry, electromagnetism in vacuum –as well as p-form electromagnetism–, and symmetry breaking of gauge theories with product groups.

In the following, every manifold or map is assumed to be smooth, meaning C^∞, unless otherwise stated. In addition, every fiber bundle $\pi_{Y,X}\colon Y \to X$ is assumed to be locally trivial and is denoted by $\pi_{Y,X}$. Given $x \in X$, $Y_x = \pi_{Y,X}^{-1}(\{x\})$ denotes the fiber over x. The space of (smooth) global sections of $\pi_{Y,X}$ is denoted by $\Gamma(\pi_{Y,X})$. In particular, vector fields on a manifold X are denoted by $\mathfrak{X}(X) = \Gamma(\pi_{TX,X})$, where TX is the tangent bundle of X. Likewise, the space of local sections on an open set $\mathcal{U} \subset X$ is denoted by $\Gamma(\mathcal{U}, \pi_{Y,X})$. The derivative, or tangent map, of a map $f \in C^\infty(X, X')$ between the manifolds X and X' is denoted by $(df)_x\colon T_xX \to T_{f(x)}X'$ for each $x \in X$. When working in local coordinates, we assume the Einstein summation convention for repeated indices. A compact interval will be denoted by $I = [a, b]$.

2 Preliminaries

This section is devoted to introduce the main geometric tools used in the forthcoming development. Namely, we recall the theory of generalized principal bundles and connections that was introduced in [18]. Here we present the results without proofs, since they appear in the aforementioned paper.

2.1 Actions of Lie group bundles

A *Lie group fiber bundle* with typical fiber a Lie group G is a fiber bundle $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ such that for any point $x \in X$ the fiber $\mathcal{G}_x$ is equipped with a Lie group structure and there is a neighborhood $\mathcal{U} \subset X$ and a diffeomorphism $\mathcal{U} \times G \to \pi_{\mathcal{G},X}^{-1}(\mathcal{U})$ preserving the Lie group structure fiberwisely.

Note that the map $1\colon X \to \mathcal{G}$ that assigns the identity element $1_x \in \mathcal{G}_x$ to each $x \in X$ is a global section (called the *unit section*) of $\pi_{\mathcal{G},X}$. Any Lie group bundle defines a Lie algebra bundle $\pi_{\mathfrak{g},X}\colon \mathfrak{g} \to X$ as the vector bundle whose fiber $\mathfrak{g}_x$ at each $x \in X$ is the Lie algebra of $\mathcal{G}_x$. That is, $\mathfrak{g} = 1^*(V\mathcal{G})$, where $V\mathcal{G} \subset T\mathcal{G}$ is the vertical bundle of $\pi_{\mathcal{G},X}$, i.e. the kernel of $(\pi_{\mathcal{G},X})_*$. We consider subgroups of Lie group bundles in the following sense.

Definition 2.1. *A* Lie group subbundle *of a Lie group bundle $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ is a Lie group bundle $\pi_{\mathcal{H},X}\colon \mathcal{H} \to X$ such that $\mathcal{H}$ is a submanifold of $\mathcal{G}$ and $\mathcal{H}_x$ is a Lie subgroup of $\mathcal{G}_x$ for each $x \in X$. It is said to be* closed *if $\mathcal{H}_x$ is a closed Lie subgroup of $\mathcal{G}_x$ for every $x \in X$.*

Let $\pi_{Y,X}$ be a fiber bundle and $\pi_{\mathcal{G},X}$ be a Lie group fiber bundle. We denote by $Y \times_X \mathcal{G}$ the corresponding fibered product, which is also a fiber bundle over X.

Definition 2.2. *A* (right) fibered action *of $\pi_{\mathcal{G},X}$ on $\pi_{Y,X}$ is a bundle morphism $\Phi\colon Y \times_X \mathcal{G} \to Y$ covering the identity id_X such that $\Phi(y, hg) = \Phi(\Phi(y, h), g)$ and $\Phi(y, 1_x) = y$, for all $(y, g), (y, h) \in Y \times_X \mathcal{G}$, $\pi_{\mathcal{G},X}(y) = x$.*

For the sake of simplicity, we will denote $\Phi(y, g) = y \cdot g$ and we will say that $\pi_{\mathcal{G},X}$ acts fiberwisely on the right on $\pi_{Y,X}$. Note that it induces a right action on each fiber, $\Phi_x = \Phi|_{Y_x \times \mathcal{G}_x}\colon Y_x \times \mathcal{G}_x \to Y_x$. The fibered action is said to be *free* if $y \cdot g = y$ for some $(y, g) \in Y \times_X \mathcal{G}$ implies that $g = 1_x$, $x = \pi_{Y,X}(y)$. In the same way, it is said to be *proper* if the bundle morphism $Y \times_X \mathcal{G} \ni (y, g) \mapsto (y, y \cdot g) \in Y \times_X Y$ is proper. If Φ is free and proper, so is each action Φ_x, since the fibers of a bundle are closed.

As the fibered action is vertical (i.e. it covers the identity id_X), we may regard the quotient space $Y/\mathcal{G}$ as the disjoint union of the quotients of the fibers by the induced actions, that is,

$$Y/\mathcal{G} = \bigsqcup_{x \in X} Y_x/\mathcal{G}_x = \left\{[y]_{\mathcal{G}} = (x, [y]_{\mathcal{G}_x})\colon x \in X, y \in Y_x\right\},$$

The following diagram is commutative:

$$\begin{array}{ccccccc} Y & \xrightarrow{\pi_{Y,X}} & X & \qquad & y & \longmapsto & x \\ & {\scriptstyle \pi_{Y,Y/\mathcal{G}}} \searrow \quad \nearrow {\scriptstyle \pi_{Y/\mathcal{G},X}} & & & & \searrow \quad \nearrow & \\ & Y/\mathcal{G} & & & & [y]_{\mathcal{G}} & \end{array} \tag{1}$$

Proposition 2.1. *If $\pi_{\mathcal{G},X}$ acts on $\pi_{Y,X}$ freely and properly, then $Y/\mathcal{G}$ admits a unique smooth structure such that*

(i) $\pi_{Y,Y/\mathcal{G}}$ is a fiber bundle with typical fiber G.

(ii) $\pi_{Y/\mathcal{G},X}$ is a fibered manifold, i.e. a surjective submersion.

If we fix $x \in X$, $y_0 \in Y_x$ and $g_0 \in \mathcal{G}_x$, we can consider the maps

$$\begin{array}{rccc} \Phi_{y_0}\colon & \mathcal{G}_x & \longrightarrow & Y_x, \\ & g & \longmapsto & y_0 \cdot g \end{array} \qquad \begin{array}{rccc} \Phi_{g_0}\colon & Y_x & \longrightarrow & Y_x \\ & y & \longmapsto & y \cdot g_0 \end{array}$$

In the same way, denote by $L_{g_0}\colon \mathcal{G}_x \to \mathcal{G}_x$ and $R_{g_0}\colon \mathcal{G}_x \to \mathcal{G}_x$ the left and right multiplication by $g_0 \in \mathcal{G}_x$, respectively. *Infinitesimal generators* (or *fundamental fields*) are defined in the same fashion as in classical actions of Lie groups. Namely, for each ξ belonging to the Lie algebra $\mathfrak{g}_x$ of $\mathcal{G}_x$, then $\xi^* \in \mathfrak{X}(Y_x)$ is defined as

$$\xi^*_y = \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\xi) = (d\Phi_y)_{1_x}(\xi), \qquad y \in Y_x \tag{2}$$

Fundamental vector fields are $\pi_{Y,Y/\mathcal{G}}$-vertical, i.e. $\xi^*_y \in V_yY$ for each $y \in Y_x$, where $VY = \ker(\pi_{Y,Y/\mathcal{G}})_*$ is the vertical bundle of $\pi_{Y,Y/\mathcal{G}}$. Of course, they are also $\pi_{Y,X}$-vertical.

Lemma 2.1. *Let $\pi_{\mathfrak{g},X}$ be the Lie algebra bundle of $\pi_{\mathcal{G},X}$. The following map is a vertical isomorphism of vector bundles over Y:*

$$\begin{array}{ccc} Y \times_X \mathfrak{g} & \longrightarrow & VY \\ (y,\xi) & \longmapsto & \xi^*_y \end{array} \tag{3}$$

In addition, for any $(g,\xi) \in \mathcal{G} \times_X \mathfrak{g}$, we have:

$$(\Phi_g)_*(\xi^*) = Ad_{g^{-1}}(\xi)^*. \tag{4}$$

2.2 Lie group bundle connections

Recall that an Ehresmann connection (see for example [17]) on a fiber bundle $\pi_{Z,X}\colon Z \to X$ is a fiber map $TZ \to VZ = \ker(\pi_{Z,X})_*$ such that its restriction to VZ is the identity. Similarly, we can regard an Ehresmann connection as a distribution $HZ \subset TZ$ complementary to VZ. Finally, an Ehresmann connection is also a section of the jet bundle $\pi_{J^1Z,Z}\colon J^1Z \to Z$.

If $\pi_{\mathcal{G},X}$ is a Lie group bundle, an Ehresmann connection $\nu\colon T\mathcal{G} \longrightarrow V\mathcal{G}$ can be also regarded as a bundle map (denoted by the same letter for the sake of simplicity)

$$\nu\colon T\mathcal{G} \longrightarrow \mathfrak{g}, \qquad U_g \longmapsto \left(dR_{g^{-1}}\right)_g (\nu(U_g))$$

Definition 2.3. *A* Lie group bundle connection *on* $\pi_{\mathcal{G},X}$ *is an Ehresmann connection* $\nu\colon T\mathcal{G} \to \mathfrak{g}$ *satisfying*

(i) $\ker \nu_{1_x} = (d1)_x(T_xX)$ *for each* $x \in X$*.*

(ii) For every $(g,h) \in \mathcal{G} \times_X \mathcal{G}$ *and* $(U_g, U_h) \in T_g\mathcal{G} \times_{T_xX} T_h\mathcal{G}$*,* $x = \pi_{\mathcal{G},X}(g)$*, then:*

$$\nu\left((dM)_{(g,h)}(U_g,U_h)\right) = \nu(U_g) + Ad_g\left(\nu(U_h)\right),$$

where $M\colon \mathcal{G}\times_X \mathcal{G} \to \mathcal{G}$ *is the fiber multiplication map.*

The geometric interpretation of Lie group bundle connections is provided by the following results. We denote by ${}^{\nu}||$ the parallel transport of ν and by $Hor^{\nu}_g\colon T_xX \to T_g\mathcal{G}$ its horizontal lift at any $g \in \mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$.

Proposition 2.2. *Let* ν *be an Ehresmann connection on* $\pi_{\mathcal{G},X}$ *such that* $\ker \nu_{1_x} = (d1)_x(T_xX)$ *for each* $x \in X$*. Then* ν *is a Lie group connection if and only if for any curve* $x\colon I \to X$ *we have*

$${}^{\nu}||_{x(a)}^{x(b)}(gh) = \left({}^{\nu}||_{x(a)}^{x(b)} g\right)\left({}^{\nu}||_{x(a)}^{x(b)} h\right), \qquad g,h \in \mathcal{G}_{x(a)}. \tag{5}$$

Consequently,

$${}^{\nu}||_{x(a)}^{x(b)}\, g^{-1} = \left({}^{\nu}||_{x(a)}^{x(b)}\, g\right)^{-1}, \qquad {}^{\nu}||_{x(a)}^{x(b)}\, 1_{x(a)} = 1_{x(b)}. \tag{6}$$

Proposition 2.3. *Let* ν *be an Ehresmann connection on* $\pi_{\mathcal{G},X}$ *and consider the corresponding jet section* $\hat{\nu} \in \Gamma(\pi_{J^1\mathcal{G},\mathcal{G}})$*. Then* ν *is a Lie group bundle connection if and only if*

(i) $\hat{\nu} \circ 1 = j^1 1 = d1$

(ii) $\hat{\nu}(gh) = \hat{\nu}(g)\,\hat{\nu}(h)$ *for each* $(g,h) \in \mathcal{G}\times_X \mathcal{G}$*.*

Lie group connections induce linear connections on the corresponding Lie algebra bundle.

Proposition 2.4. *Let* $x\colon I \to X$ *be a smooth curve. Then the map* ${}^{\mathfrak{g}}||_{x(a)}^{x(b)}\colon \mathfrak{g}_{x(a)} \to \mathfrak{g}_{x(b)}$ *defined as*

$${}^{\mathfrak{g}}||_{x(a)}^{x(b)}\xi = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} {}^{\nu}||_{x(a)}^{x(b)} \exp(\epsilon\,\xi), \quad \xi \in \mathfrak{g}_{x(a)}$$

is a linear parallel transport on $\pi_{\mathfrak{g},X}$*.*

Denoting by $\nabla^{\mathfrak{g}}$ the linear connection corresponding to this parallel transport ${}^{\mathfrak{g}}||$, it can be checked that

$$\nabla^{\mathfrak{g}}\xi = \left.\frac{d}{dt}\right|_{t=0} \nu \circ d\exp(t\xi), \qquad \xi \in \Gamma(\pi_{\mathfrak{g},X}).$$

2.3 Generalized principal connections

Let $\pi_{Y,X}\colon Y \to X$ be a fiber bundle on which a Lie group bundle $\pi_{\mathcal{G},X}\colon \mathcal{G} \to X$ acts freely and properly on the right. We denote by $\Phi\colon Y \times_X \mathcal{G} \to Y$ the fibered action.

Definition 2.4. *Let ν be an Ehresmann connection on the Lie group bundle $\pi_{\mathcal{G},X}$. A* generalized principal connection *on $\pi_{Y,Y/\mathcal{G}}$ associated to ν is a form*[1] $\omega \in \Omega^1(Y,\mathfrak{g})$ *satisfying:*

(i) (Complementarity) $\omega_y(\xi_y^*) = \xi$ for every $(y,\xi) \in Y \times_X \mathfrak{g}$.

(ii) (Ad-equivariance) For each $(y,g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $x = \pi_{Y,X}(x)$, then:

$$\omega_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y,U_g)\right) = Ad_{g^{-1}}\left(\omega_y(U_y) + \nu(U_g)\right).$$

We denote by $Hor_y^\omega\colon T_{[y]_\mathcal{G}}(Y/\mathcal{G}) \to T_yY$ the *horizontal lifting* given by ω at $y \in Y$. The next result gives a geometric interpretation of the above definition in terms of the parallel transports ${}^\nu||$ and ${}^\omega||$, in the same vein as Proposition 2.2.

Proposition 2.5. *Let ν be an Ehresmann connection on $\pi_{\mathcal{G},X}$ and $\omega \in \Omega^1(Y,\mathfrak{g})$ be an Ehresmann connection on $\pi_{Y,Y/\mathcal{G}}$. Then ω is a generalized principal connection associated to ν if and only if for any curve $\gamma\colon I \to Y/\mathcal{G}$, the corresponding parallel transports satisfy*

$${}^\omega||_{\gamma(a)}^{\gamma(t)}(y\cdot g) = \left({}^\omega||_{\gamma(a)}^{\gamma(t)} y\right)\cdot\left({}^\nu||_{x(a)}^{x(t)} g\right), \qquad g \in \mathcal{G}_{x(a)}, \quad y \in Y_{\gamma(a)}, \quad t \in I, \tag{7}$$

where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$.

The *cuvature* of ω (see, for example [17, §9.4]) is the 2-form $\Omega \in \Omega^2(Y,\mathfrak{g})$ defined as:

$$\Omega(U_1,U_2) = -\omega\left([U_1 - \omega(U_1)^*, U_2 - \omega(U_2)^*]\right), \qquad U_1,U_2 \in \mathfrak{X}(Y).$$

The linear connection $\nabla^\mathfrak{g}$ on $\pi_{\mathfrak{g},X}$ enables us to express the curvature as follows.

Proposition 2.6. *Let $d^\mathfrak{g}$ be the exterior covariant derivative*[2] *associated to $\nabla^\mathfrak{g}$. Then*

$$\Omega\left(U_1,U_2\right) = d^\mathfrak{g}\omega\left(U_1^h,U_2^h\right), \qquad U_1,U_2 \in \mathfrak{X}(Y)$$

The *adjoint bundle* of the action of $\pi_{Y,X}$ is defined to be the quotient $\tilde{\mathfrak{g}} = (Y \times_X \mathfrak{g})/\mathcal{G}$ by the (right) fibered action

$$\begin{array}{ccc} (Y \times_X \mathfrak{g}) \times_X \mathcal{G} & \longrightarrow & Y \times_X \mathfrak{g} \\ ((y,\xi),g) & \longmapsto & (y\cdot g, Ad_{g^{-1}}(\xi)). \end{array} \tag{8}$$

It is a vector bundle over $Y/\mathcal{G}$ equipped with a Lie algebra bundle structure. As in the case of (standard) principal connections, it is possible to regard the curvature as a 2-form on the base space $Y/\mathcal{G}$ with values in $\tilde{\mathfrak{g}}$.

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[1]In fact, ω takes values on the vector bundle $\pi_{Y\times_X\mathfrak{g},Y}$, which is the pull-back of $\pi_{\mathfrak{g},X}$ by $\pi_{Y,X}$. Abusing the notation, we denote the pull-back bundle by the same symbol.

[2]The *exterior covariant derivative* of a linear connection ∇^E on a vector bundle $\pi_{E,X}$ is an operator in the family of E-valued forms on X, $d^E\colon \Omega^\bullet(X,E) \longrightarrow \Omega^{\bullet+1}(X,E)$. For a 1-form $\alpha \in \Omega^1(X,E)$ it is given by

$$d^E\alpha(U_1,U_2) = \nabla_{U_1}^E(\alpha(U_2)) - \nabla_{U_2}^E(\alpha(U_1)) - \alpha([U_1,U_2]), \qquad U_1,U_2 \in \mathfrak{X}(X).$$

Definition 2.5. *The* reduced curvature *of ω is the 2-form $\tilde{\Omega} \in \Omega^2(Y/\mathcal{G}, \tilde{\mathfrak{g}})$ given by*

$$\tilde{\Omega}_{[y]_\mathcal{G}}(U_1, U_2) = \left[y, \Omega_y\left(Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right)\right]_\mathcal{G}$$

for each $[y]_\mathcal{G} \in Y/\mathcal{G}$ and $U_1, U_2 \in T_{[y]_\mathcal{G}}(Y/\mathcal{G})$, where $y \in Y$ is such that $\pi_{Y,Y/\mathcal{G}}(y) = [y]_\mathcal{G}$.

The reduced curvature is well-defined, i.e. it does not depend on the choice of $y \in Y$. Indeed, let $g \in \mathcal{G}_x$, where $x = \pi_{Y,X}(y)$, $u_i = (d\pi_{Y/\mathcal{G},X})_{[y]_\mathcal{G}}(U_i) \in T_xX$ for $i = 1, 2$ and $\gamma \in \Gamma(\pi_{\mathcal{G},X})$ be such that $\gamma(x) = g$ and $\nu_{\gamma(x)} \circ (d\gamma)_x = 0$. Using [18, Proposition 3.10] we obtain

$$Hor_{y\cdot g}^\omega(U_i) = (d\Phi)_{(y,g)}\left(Hor_y^\omega(U_i), (d\gamma)_x(u_i)\right), \qquad i = 1, 2.$$

Hence, we have

$$\begin{aligned}
\left[y \cdot g, \Omega_{y\cdot g}\left(Hor_{y\cdot g}^\omega(U_1), Hor_{y\cdot g}^\omega(U_2)\right)\right]_\mathcal{G} &= \left[y \cdot g, -\omega_{y\cdot g}\left(\left[Hor_{y\cdot g}^\omega(U_1), Hor_{y\cdot g}^\omega(U_2)\right]\right)\right]_\mathcal{G} \\
&= \left[y \cdot g, -\omega_{y\cdot g}\left(\left[(d\Phi)_{(y,g)}\left(Hor_y^\omega(U_1), (d\gamma)_x(u_1)\right), (d\Phi)_{(y,g)}\left(Hor_y^\omega(U_2), (d\gamma)_x(u_2)\right)\right]\right)\right]_\mathcal{G} \\
&\overset{(\star)}{=} \left[y \cdot g, -\omega_{y\cdot g}\left((d\Phi)_{(y,g)}\left(\left[Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right], \left[(d\gamma)_x(u_1), (d\gamma)_x(u_2)\right]\right)\right)\right]_\mathcal{G} \\
&= \left[y \cdot g, -\omega_{y\cdot g}\left((d\Phi)_{(y,g)}\left(\left[Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right], (d\gamma)_x([u_1, u_2])\right)\right)\right]_\mathcal{G} \\
&= \left[y \cdot g, -Ad_{g^{-1}}\left(\omega_y\left(\left[Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right]\right)\right)\right]_\mathcal{G} \\
&= \left[y, -\omega_y\left(\left[Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right]\right)\right]_\mathcal{G} \\
&= \left[y, \Omega_y\left(Hor_y^\omega(U_1), Hor_y^\omega(U_2)\right)\right]_\mathcal{G},
\end{aligned} \tag{9}$$

where we have used that $[(d\gamma)_x(u_1), (d\gamma)_x(u_2)] = (d\gamma)_x([u_1, u_2])$.

3 Geometry of the reduced configuration space

Let $\pi_{\mathcal{G},X}$ be a Lie group bundle endowed with a Lie group bundle connection ν. Then it defines an identification of jet bundle $J^1\mathcal{G} \to X$ with the vector bundle on which it is modelled, that is,

$$\begin{aligned}
\Theta_\nu : \quad J^1\mathcal{G} &\longrightarrow \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}) \\
j_x^1\eta &\longmapsto (\eta(x), \nu \circ (d\eta)_x).
\end{aligned} \tag{10}$$

Note that $J^1\mathcal{G} \to X$ is also a Lie group bundle (for example, cf. [12, §3, Theorem 1]). This structure may be transferred to $\mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$ via the above identification, yielding the following fiber multiplication

$$(g, \xi_x)(h, \eta_x) = (gh, \xi_x + Ad_g \circ \eta_x), \qquad (g, \xi_x), (h, \eta_x) \in \mathcal{G}_x \times (T_x^*X \otimes \mathfrak{g}_x), \quad x \in X.$$

Henceforth, let $\pi_{Y,X}$ be a fiber bundle on which $\pi_{\mathcal{G},X}$ acts fiberwisely, freely and properly, and $\pi_{H,X}$ be a closed Lie group subbundle $H \subset J^1\mathcal{G}$ of $\pi_{J^1\mathcal{G},X}$ such that $\pi_{J^1\mathcal{G},\mathcal{G}}(H) = \mathcal{G}$. Besides, suppose that $\pi_{H,\mathcal{G}}$ is an affine subbundle of $\pi_{J^1\mathcal{G},\mathcal{G}}$. Identification (10) enables us to regard

$$\Theta_\nu(H) \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}),$$

We denote $\Theta_\nu(H)$ by the same symbol H, since the context will clarify the distinction between the two objects. If ν satisfies $\hat{\nu}(\mathcal{G}) \subset H$, where $\hat{\nu} \in \Gamma(\pi_{J^1\mathcal{G},\mathcal{G}})$ is the corresponding jet field, then the condition for $\pi_{H,\mathcal{G}}$ to be an affine subbundle of $J^1\mathcal{G} \to \mathcal{G}$ is equivalent to be a vector subbundle of $\mathcal{G} \times_X (T^*X \otimes \mathfrak{g}) \to \mathcal{G}$.

Proposition 3.1. *Let $H \subset J^1\mathcal{G}$ be an affine Lie group subbundle and ν be a Lie group bundle connection on $\pi_{\mathcal{G},X}$ such that $\hat{\nu}(\mathcal{G}) \subset H$. Taking into account the identification* (10), *there exists a vector subbundle $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$ such that*

$$H = \mathcal{G} \times_X \mathfrak{H}$$

Furthermore, $\mathfrak{H}$ is Ad-invariant, i.e. $Ad_g(\mathfrak{H}_x) \subset \mathfrak{H}_x$ for each $x \in X$ and $g \in \mathcal{G}_x$,

Proof. Given the subgroup $H \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$, we consider the bundle $\mathfrak{H} \to X$, by $\mathfrak{H}_x = H_{1_x}$, $x \in X$. Given any element $(g, \eta) \in H_g$, $\pi_{\mathcal{G},X}(g) = x$, if we consider the group operation by $(g^{-1}, 0) \in H$, we have

$$(g, \eta)(g^{-1}, 0) = (1_x, \eta) \in H,$$

so that $\eta \in \mathfrak{H}_x$.

For the second part, observe that for each $\eta \in \mathfrak{H}_x$ and $g \in \mathcal{G}_x$ we have

$$(g, 0)(1_x, \eta) = (g, Ad_g(\eta)) \in \mathcal{G}_x \times \mathfrak{H}_x.$$

□

Analogous to the adjoint bundle defined in (8), we can consider the tensor product $T^*X \otimes \tilde{\mathfrak{g}}$ as the quotient

$$T^*X \otimes \tilde{\mathfrak{g}} = \frac{Y \times_X (T^*X \otimes \mathfrak{g})}{\mathcal{G}} \longrightarrow Y/\mathcal{G}$$

by letting $\mathcal{G}$ act trivially on T^*X. As the vector subbundle $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$ obtained in Proposition 3.1 is Ad-equivariant, the action of $\mathcal{G}$ on $Y \times_X (T^*X \otimes \mathfrak{g})$ restricts to $Y \times_X \mathfrak{H}$ and we may consider the corresponding quotient

$$\tilde{\mathfrak{H}} = \frac{Y \times_X \mathfrak{H}}{\mathcal{G}} \subset T^*X \otimes \tilde{\mathfrak{g}} \longrightarrow Y/\mathcal{G} \tag{11}$$

In addition, $\pi_{\tilde{\mathfrak{H}},Y/\mathcal{G}}$ is a vector subbundle of $\pi_{T^*X\otimes\tilde{\mathfrak{g}},Y/\mathcal{G}}$, so it makes sense to consider the corresponding quotient vector bundle

$$\frac{T^*X \otimes \tilde{\mathfrak{g}}}{\tilde{\mathfrak{H}}} \longrightarrow Y/\mathcal{G} \tag{12}$$

We denote its elements by $[\![y, \xi_x]\!]_{\tilde{\mathfrak{H}}}$, where $y \in Y_x$, $\xi_x \in (T^*X \otimes \mathfrak{g})_x$ and $x \in X$.

On the other hand, the first jet extension of the fibered action Φ turns out to be a right fibered action of $\pi_{J^1\mathcal{G},X}$ on $\pi_{J^1Y,X}$

$$\begin{array}{rccc} \Phi^{(1)}: & J^1Y \times_X J^1\mathcal{G} & \longrightarrow & J^1Y \\ & (j^1_x s, j^1_x \gamma) & \longmapsto & j^1_x(\Phi(s, \gamma)) \end{array} \tag{13}$$

where, for (local) sections $s \in \Gamma(\pi_{Y,X})$ and $\gamma \in \Gamma(\pi_{\mathcal{G},X})$, we have the (local) section $\Phi(s,\gamma)(x) = \Phi(s(x),\gamma(x))$, $x \in X$. If we regard 1-jets as differentials of sections at a point, that is, $j^1_x s \equiv (ds)_x$ and $j^1_x\gamma \equiv (d\gamma)_x$, we can see Φ^1 as

$$\Phi^{(1)}(j^1_x s, j^1_x\gamma) = (d\Phi)_{(s(x),\gamma(x))} \circ (ds_x, d\gamma_x) : T_xX \to T_{s(x)\cdot\gamma(x)}Y$$

The following result studies the geometry of the quotient of $\pi_{J^1Y,X}$ by the fibered action (13) when restricted to the Lie group subbundle $H \subset J^1\mathcal{G}$.

Theorem 3.1. *Let $\omega \in \Omega^1(Y,\mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to the Lie group connection ν on $\pi_{\mathcal{G},X}$ and consider a Lie group subbundle $H = \mathcal{G} \times_X \mathfrak{H}$ as in Proposition 3.1. Then the following map is a bundle isomorphism over $Y/\mathcal{G}$:*

$$\begin{array}{ccc} J^1Y/H & \longrightarrow & J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \\ [j^1_x s]_H & \longmapsto & \left(j^1_x\sigma_s, \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x\right]\!\right]_{\tilde{\mathfrak{H}}}\right), \end{array} \tag{14}$$

where $\sigma_s = [s]_{\mathcal{G}} = \pi_{Y,Y/\mathcal{G}} \circ s \in \Gamma(\pi_{Y/\mathcal{G},X})$.

Proof. For $j^1_x s \in J^1Y$ and $j^1_x\eta \in H$, let s and η be (local) sections defining those jet elements. On one hand, it is clear that $\sigma_{\Phi(s,\eta)} = \sigma_s$. On the other,

$$\begin{aligned} \left[\!\left[\Phi(s,\eta)(x), \omega_{\Phi(s,\eta)(x)} \circ (d\hat{\Phi}(s,\eta))_x\right]\!\right]_{\tilde{\mathfrak{H}}} &= \left[\!\left[s(x)\cdot\eta(x), \omega_{s(x)\cdot\eta(x)} \circ (d\Phi)_{(s(x),\eta(x))} \circ ((ds)_x, (d\eta)_x)\right]\!\right]_{\tilde{\mathfrak{H}}} \\ &= \left[\!\left[s(x)\cdot\eta(x), Ad_{\eta(x)^{-1}}\left(\omega_{s(x)} \circ (ds)_x + \nu \circ (d\eta)_x\right)\right]\!\right]_{\tilde{\mathfrak{H}}} \\ &= \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x + \nu \circ (d\eta)_x\right]\!\right]_{\tilde{\mathfrak{H}}} \\ &= \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x\right]\!\right]_{\tilde{\mathfrak{H}}}. \end{aligned}$$

Subsequently, the map is well defined. A straightforward computation shows that the inverse map is

$$\begin{array}{ccc} J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} & \longrightarrow & J^1Y/H \\ \left(j^1_x\sigma, [\![y,\xi_x]\!]_{\tilde{\mathfrak{H}}}\right) & \longmapsto & \left[Hor^\omega_y \circ (d\sigma)_x + (\xi_x)^*_y\right]_H \end{array}$$

where $Hor^\omega_y \colon T_{[y]_{\mathcal{G}}}(Y/\mathcal{G}) \to T_yY$ is the horizontal lifting given by ω at $y \in Y$. It is well defined thanks to [18, Lemma 3.1, Proposition 3.10].

□

3.1 Connections on the reduced spaces

Let ω be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$ and ∇^X be a linear connection on $\pi_{TX,X}$. These connections induce linear connections on the reduced bundles $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$ and $\pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},Y/\mathcal{G}}$ as follows. Consider the linear connection $\nabla^{\mathfrak{g}}$ on $\pi_{\mathfrak{g},X}$ induced by ν (cf. Proposition 2.4). As above, denote by ${}^\nu||$, ${}^{\mathfrak{g}}||$ and ${}^\omega||$ the corresponding parallel transports. It is easy to check from the definition that the connections ν and $\nabla^{\mathfrak{g}}$ satisfy the following compatibility relation:

$${}^{\mathfrak{g}}||_{x(a)}^{x(b)} Ad_g(\xi) = Ad_{{}^\nu||_{x(a)}^{x(b)} g}\left({}^{\mathfrak{g}}||_{x(a)}^{x(b)}\xi\right), \qquad g \in \mathcal{G}_{x(a)}, \quad \xi \in \mathfrak{g}_{x(a)} \tag{15}$$

for every curve $x\colon I \to X$. This and Proposition 2.5 ensure that the following parallel transport is well defined.

Proposition 3.2. *The assignment that each curve $\gamma\colon I \to Y/\mathcal{G}$ corresponds to the map*

$$\begin{array}{rccc} {}^{\tilde{\mathfrak{g}}}\big|\big|_{\gamma(a)}^{\gamma(b)}\colon & \tilde{\mathfrak{g}}_{\gamma(a)} & \longrightarrow & \tilde{\mathfrak{g}}_{\gamma(b)} \\ & [y,\xi]_{\mathcal{G}} & \longmapsto & \left[{}^{\omega}\big|\big|_{\gamma(a)}^{\gamma(b)} y, {}^{\mathfrak{g}}\big|\big|_{\alpha(a)}^{\alpha(b)}\xi\right]_{\mathcal{G}} \end{array}$$

where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$ is a linear parallel transport on $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$.

We denote by $\nabla^{\tilde{\mathfrak{g}}}$ the corresponding linear connection on $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$ and by $\nabla^{\tilde{\mathfrak{g}}}/dt$ the corresponding covariant derivative.

Lemma 3.1. *Let $\nabla^{\mathfrak{H}}$ be a linear connection on $\pi_{\mathfrak{H},X}$. The assignment that each curve $x\colon I \to X$ corresponds to the map*

$$ {}^{H}\big|\big|_{x(a)}^{x(b)}\colon H_{x(a)} \to H_{x(b)}, \qquad {}^{H}\big|\big|_{x(a)}^{x(b)}(g,\eta_x) = \left({}^{\nu}\big|\big|_{x(a)}^{x(b)} g, {}^{\mathfrak{H}}\big|\big|_{x(a)}^{x(b)}\eta_x\right), \qquad (g,\eta_x) \in H_{x(a)}$$

is a linear parallel transport on $\pi_{H,X}$.

Now we extend $\nabla^{\mathfrak{H}}$ to a linear connection $\nabla^{\otimes}$ on $T^*X \otimes \mathfrak{g}$ and we suppose that it is *compatible* with ν, that is, that the relation (15) holds for ${}^{\otimes}\big|\big|$ instead of ${}^{\mathfrak{g}}\big|\big|$. We are ready to define a linear connection on the reduced bundle.

Proposition 3.3. *The assignment that each curve $\gamma\colon I \to Y/\mathcal{G}$ corresponds to the map*

$$\begin{array}{rccc} \big|\big|_{\gamma(a)}^{\gamma(b)}\colon & \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)_{\gamma(a)} & \longrightarrow & \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)_{\gamma(b)} \\ & [\![y,\xi_x]\!]_{\tilde{\mathfrak{H}}} & \longmapsto & \left[\!\!\left[{}^{\omega}\big|\big|_{\gamma(a)}^{\gamma(b)} y, {}^{\otimes}\big|\big|_{x(a)}^{x(b)} \xi_x\right]\!\!\right]_{\tilde{\mathfrak{H}}}, \end{array}$$

where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$, is a linear parallel transport on $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$

Proposition 2.5 and Equation (15), together with the Ad-invariance of $\mathfrak{H}$, guarantee that it is well-defined. Besides, linearity comes from the linearity of $\nabla^{\otimes}$. We denote by ∇ the linear connection on $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$ associated to $\big|\big|$ and by ∇/dt the corresponding covariant derivative.

4 Reduction of the variational principle

4.1 Calculus of variations and reduced Lagrangian

Let $\pi_{Y,X}\colon Y \to X$ be a fiber bundle. A *(first order) Lagrangian density* on $\pi_{Y,X}$ is a bundle morphism

$$\mathcal{L}\colon J^1Y \longrightarrow \textstyle\bigwedge^n T^*X$$

covering the identity on X, where $n = \dim X$. Assuming that X is orientable and $v \in \Omega^n(X)$ is a volume form, we can write $\mathcal{L} = Lv$ for certain $L\colon J^1Y \to \mathbb{R}$ called *Lagrangian.*

Henceforth, we suppose that X is compact. The *action functional* defined by $\mathcal{L}$ is

$$\mathcal{S}(s) = \int_X \mathcal{L}\left(j^1 s\right), \qquad s \in \Gamma\left(\pi_{Y,X}\right)$$

A *variation* of a section $s \in \Gamma\left(\pi_{Y,X}\right)$ is a 1-parameter family

$$\{s_t\} = \{s_t \in \Gamma\left(\pi_{Y,X}\right) : t \in (-\epsilon, \epsilon)\}$$

such that $s_0 = s$, where $\delta s = ds_t/dt|_{t=0} \in \Gamma\left(\pi_{s^*TY,X}\right)$ is the *infinitesimal variation.* In what follows, we consider only $\pi_{Y,X}$-vertical variations, that is, $d\pi_{Y,X} \circ \delta s = 0$.

Definition 4.1. *A section $s \in \Gamma\left(\pi_{Y,X}\right)$ is* critical for the variational problem defined by $\mathcal{L}$ *if the variation of the corresponding action functional*[3] *vanishes for every vertical variation of s, that is,*

$$\delta\mathcal{S}(s) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{S}(s_t) = \left.\frac{d}{dt}\right|_{t=0} \int_X \mathcal{L}\left(j^1 s_t\right) = 0, \qquad \delta s \in \Gamma\left(\pi_{s^*TY,X}\right)$$

A well-known fact is that $s \in \Gamma\left(\pi_{Y,X}\right)$ is critical for $\mathcal{S}$ if and only if it satisfies the Euler-Lagrange equations for the Lagrangian L, i.e. $\mathcal{EL}(L)\left(j^2 s\right) = 0$, where $\mathcal{EL}(L)\colon J^2Y \to V^*Y = (\ker \pi_{Y,X})^*$ is the *Euler-Lagrange operator* (see [8, §2.4]).

We now assume that L is invariant with respect to an affine Lie subgroup bundle H of $J^1\mathcal{G} \to X$ such that $\pi_{J^1\mathcal{G},\mathcal{G}}(H) = \mathcal{G}$, by a fiberwise action $\Phi : Y \times \mathcal{G} \to Y$ of a Lie group bundle $\pi_{\mathcal{G},X}$ on $\pi_{Y,X}$. That is, we have

$$L\left(\Phi^{(1)}\left(j_x^1 s, j_x^1 \eta\right)\right) = L\left(j_x^1 s\right), \qquad \forall \left(j_x^1 s, j_x^1 \eta\right) \in J^1Y \times_X H.$$

This enables us to define the *dropped* or *reduced Lagrangian,*

$$l\colon J^1Y/H \longrightarrow \mathbb{R},$$

as $l\left([j_x^1 s]_H\right) = L\left(j_x^1 s\right)$.

Let $\omega \in \Omega^1(Y, \mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$ and suppose that the corresponding section $\hat{\nu} \in \Gamma(\pi_{J^1\mathcal{G},G})$ satisfies $\hat{\nu}(\mathcal{G}) \subset H$. Thanks to Theorem 3.1, we may regard the reduced Lagrangian as defined on $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}$. Given $s \in \Gamma\left(\pi_{Y,X}\right)$, the corresponding *reduced section* is

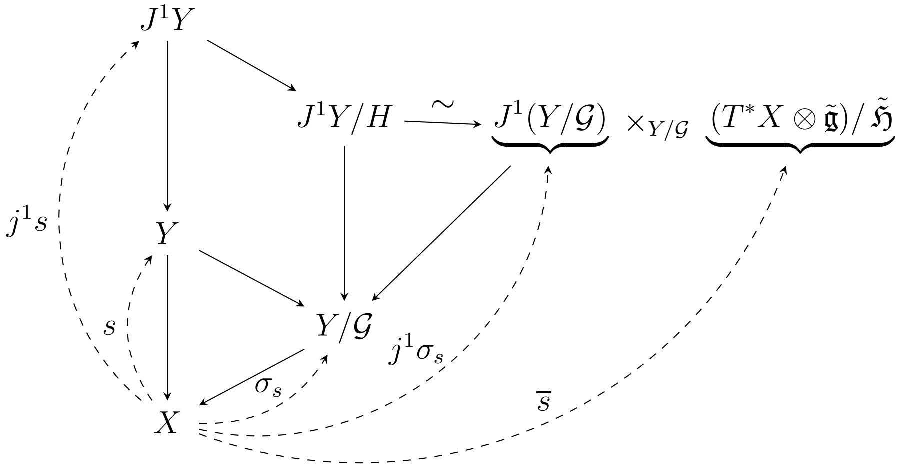
$$\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}} \in \Gamma\left(\pi_{\left(T^*X \otimes \tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},X}\right)$$

Observe that the projection $\pi_{\left(T^*X \otimes \tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s} \in \Gamma\left(\pi_{Y/\mathcal{G},X}\right)$ is nothing but the quotient section $\sigma_s = [s]_{\mathcal{G}} = \pi_{Y,Y/\mathcal{G}} \circ s$.

[3]The variation of $\mathcal{S}$ only depends on the infinitesimal variation. This means that if $\{s_t\}$ and $\{s'_t\}$ are two variations of s such that $\delta s = \delta s'$, then $d\mathcal{S}(s_t)/dt|_{t=0} = d\mathcal{S}(s'_t)/dt|_{t=0}$

A variation $\{s_t\}$ of s induces a variation $\{\overline{s}_t = [\![s_t, s_t^*\omega]\!]_{\tilde{\mathfrak{H}}}\}$ of the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}}$. By construction, $L(j^1 s_t) = l(j^1(\sigma_s)_t, \overline{s}_t)$ for every $t \in (-\epsilon, \epsilon)$. Therefore:

$$\left.\frac{d}{dt}\right|_{t=0} \int_X L\left(j^1 s_t\right) v = \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v \tag{16}$$



Remark 4.1. *The calculus of variations described above is straightforwardly extended to a non-compact base manifold X by considering compactly supported variations. In other words, given a section $s \in \Gamma(\pi_{Y,X})$, the only variations δs of s allowed are those satisfying*

$$\{x \in X : \delta s(x) \neq 0\} \subset \mathcal{U}$$

for some open subset $\mathcal{U} \subset X$ with compact closure $\overline{\mathcal{U}}$. Observe that, in particular $\delta s = 0$ on the boundary $\partial\mathcal{U}$.

4.2 Reduced variations on $(T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}$

In this section we compute the variation $\delta\bar{s}$ of of the reduced section $\bar{s} \in \Gamma(\pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},X})$ induced by a variation δs of an unreduced section $s \in \Gamma(\pi_{Y,X})$. More particularly, we are interested in the vertical part

$$\delta^\nabla \overline{s}(x) = \delta\overline{s}(x)^v = \left.\frac{\nabla \overline{s}_t(x)}{dt}\right|_{t=0}, \qquad x \in X,$$

of that reduced variation with respect to the connection ∇ on $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$ built in Proposition 3.3. For that we analyze below the corresponding expression of $\delta^\nabla\overline{s}$ when the variation $\delta s = ds_t/dt|_{t=0}$ is vertical or horizontal with respect to ω. The expression of $\delta^\nabla\overline{s}$ will be the combination of both terms.

Lemma 4.1. *If $\{s_t\}$ is a $\pi_{Y,Y/\mathcal{G}}$-vertical variation, that is, $\delta s = \xi_s^*$ for some $\xi \in \Gamma(\pi_{\tilde{\mathfrak{g}},X})$, then we can suppose that it is of the form $s_t = s \cdot \exp(t\,\xi)$. In that case, $d/dt|_{t=0}\, Ad_{\exp(t\,\xi)} \circ s_t^*\omega = \nabla^{\tilde{\mathfrak{g}}}\xi$.*

Proof. The first statement is because the variation of the functional only depends on the infinitesimal variation. The second part is a computation:

$$\begin{aligned}
\tfrac{d}{dt}\big|_{t=0} Ad_{\exp(t\,\xi)} \circ s_t^*\omega &= \tfrac{d}{dt}\big|_{t=0} s_t^*\omega + \tfrac{d}{dt}\big|_{t=0} Ad_{\exp(t\,\xi)} \circ s^*\omega \\
&= \tfrac{d}{dt}\big|_{t=0} \omega_{\Phi(s,\exp(t\,\xi))} \big((d\Phi)_{(s,\exp(t\,\xi))} (ds, d\exp(t\,\xi))\big) + ad(\xi)(s^*\omega) \\
&= \tfrac{d}{dt}\big|_{t=0} Ad_{\exp(-t\,\xi)} (s^*\omega + \nu \circ d\exp(t\,\xi)) + [\xi, s^*\omega] \\
&= \tfrac{d}{dt}\big|_{t=0} \nu \circ d\exp(t\,\xi) + \tfrac{d}{dt}\big|_{t=0} Ad_{\exp(-t\,\xi)} \circ \nu \circ d1 \\
&\overset{(\star)}{=} \tfrac{d}{dt}\big|_{t=0} \nu \circ d\exp(t\,\xi) \\
&= \nabla^{\mathfrak{g}}\xi
\end{aligned}$$

Equality $(\star)$ comes from property (i) in the definition of Lie group connection. □

Proposition 4.1 ($\delta^\nabla \bar{s}$ for vertical variations)**.** *If $\{s_t\}$ is $\pi_{Y,Y/\mathcal{G}}$-vertical, that is, $\delta s = \xi_s^*$ for some $\xi \in \Gamma(\pi_{\mathfrak{g},X})$, then:*

$$\delta^\nabla \overline{s}(x) = \left[\!\left[s(x), \left(\nabla^{\mathfrak{g}}\xi\right)(x) \right]\!\right]_{\tilde{\mathfrak{H}}}, \qquad x \in X$$

Proof. The reduced variations are

$$\overline{s}_t(x) = [\![s(x) \cdot \exp(t\,\xi(x)), (s_t^*\omega)_x]\!]_{\tilde{\mathfrak{H}}} = \left[\!\left[s(x), Ad_{\exp(t\,\xi(x))} \circ (s_t^*\omega)_x \right]\!\right]_{\tilde{\mathfrak{H}}}.$$

Since $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}_t = \sigma_s$ for all $t \in (-\epsilon, \epsilon)$, we have that $\{\overline{s}_t\}$ is a $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$-vertical variation and $\nabla\overline{s}_t(x)/dt|_{t=0} = d\overline{s}_t(x)/dt|_{t=0}$. Subsequently, the previous Lemma results in

$$\delta^\nabla \overline{s}(x) = \left[\!\left[s(x), \tfrac{d}{dt}\big|_{t=0} Ad_{\exp(t\,\xi(x))} \circ (s_t^*\omega)_x \right]\!\right]_{\tilde{\mathfrak{H}}} = [\![s(x), (\nabla^{\mathfrak{g}}\xi)(x)]\!]_{\tilde{\mathfrak{H}}}.$$

□

Lemma 4.2. *If $\{s_t\}$ is ω-horizontal, i.e. $\omega(\delta s) = 0$, then the horizontal component of $\delta\bar{s}$ with respect to ∇ is*

$$\delta\overline{s}(x)^h = \frac{d}{dt}\bigg|_{t=0} [\![s_t(x), (s^*\omega)_x]\!]_{\tilde{\mathfrak{H}}}$$

for each $x \in X$.

Proof. Let $Hor^\nabla_{\overline{s}(x)} \colon T_{\sigma_s(x)}(Y/\mathcal{G}) \to T_{\overline{s}(x)}\big((T^*X \otimes \tilde{\mathfrak{g}})/\,\tilde{\mathfrak{H}}\big)$ be the horizontal lift given by ∇ at $\overline{s}(x)$. Then $\delta\overline{s}(x)^h = Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x))$, since $(d\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}})_{\overline{s}(x)} \circ \delta\overline{s}(x) = \delta\sigma_s(x)$. By construction of ∇, we have[4]:

$$Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x)) = (dq)_{(s(x),(s^*\omega)_x)} \left(Hor^\omega_{s(x)}(\delta\sigma_s(x)), Hor^{\nabla^\otimes}_{(s^*\omega)_x}(0_x) \right)$$

where $q \colon Y \times_X (T^*X \otimes \mathfrak{g}) \longrightarrow (T^*X \otimes \tilde{\mathfrak{g}})/\,\tilde{\mathfrak{H}}$ is the quotient projection. The horizontality of $\delta s(x)$ yields $Hor^\omega_{s(x)}(\delta\sigma_s(x)) = \delta s(x)$, and it is clear that $Hor^{\nabla^\otimes}_{(s^*\omega)_x}(0_x) = 0_{(s^*\omega)_x}$. Hence,

$$\delta\overline{s}(x)^h = Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x)) = \frac{d}{dt}\bigg|_{t=0} (q \circ \gamma)(t) = \frac{d}{dt}\bigg|_{t=0} [\![s_t(x), (s^*\omega)_x]\!]_{\tilde{\mathfrak{H}}}$$

where for the computation of the differentail dq, we consider the curve $\gamma \colon (-\epsilon, \epsilon) \to Y \times_X (T^*X \otimes \mathfrak{g})$, $\gamma(t) = (s_t(x), (s^*\omega)_x)$, since $\gamma(0) = (s(x), (s^*\omega)_x)$ and $\gamma'(0) = (\delta s(x), 0_x)$. □

[4] Recall that $(d\pi_{Y/\mathcal{G},X})_{\sigma_s(x)}(\delta\sigma_s(x)) = 0$, since the variation $\{s_t\}$ is $\pi_{Y,X}$-vertical.

Proposition 4.2 ($\delta^\nabla \bar{s}$ for horizontal variations)**.** *If* $\{s_t\}$ *is* ω*-horizontal, i.e.* $\omega(\delta s) = 0$, *then*

$$\delta^\nabla \overline{s}(x) = \left[\!\left[s(x), \Omega_{s(x)}\left(\delta s(x), (ds)_x\right)\right]\!\right]_{\tilde{\mathfrak{H}}}, \qquad x \in X.$$

Proof. Thanks to the previous Lemma:

$$\begin{aligned}
\delta^\nabla \overline{s}(x) &= \delta\overline{s}(x) - \delta\overline{s}(x)^h \\
&= \left.\frac{d}{dt}\right|_{t=0} [\![s_t(x), (s_t^*\omega)_x]\!]_{\tilde{\mathfrak{H}}} - \left.\frac{d}{dt}\right|_{t=0} [\![s_t(x), (s^*\omega)_x]\!]_{\tilde{\mathfrak{H}}} \\
&= \left[\!\left[s(x), \left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x \right]\!\right]_{\tilde{\mathfrak{H}}}.
\end{aligned}$$

Since the formula that we are proving is local, we can suppose that our bundles are trivial, that is, $\mathcal{G} = X \times G$, $\mathfrak{g} = X \times \mathfrak{g}$ and $Y = Y/\mathcal{G} \times G$, where $\mathfrak{g}$ is the Lie algebra of G. We can thus regard ω as a 1-form on Y with values in $\mathfrak{g}$. Then

$$\left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x = s^*\pounds_{\delta s}\omega = s^*(i_{\delta s}d\omega + d(i_{\delta s}\omega)) = d\omega_{s(x)}(\delta s, ds)$$

where $\pounds$ is the Lie derivative and $\omega(\delta s) = 0$. Then

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x &= d\omega_{s(x)}(\delta s, (ds)^h) + d\omega_{s(x)}(\delta s, (ds)^v) = d\omega_{s(x)}(\delta s, (ds)^h) = d^{\mathfrak{g}}\omega(\delta s, ds) \\
&= \Omega_{s(x)}(\delta s, ds),
\end{aligned}$$

since $d\omega_{s(x)}(\delta s, (ds)^v) = \delta s(\omega((ds)^v)) - \omega_{s(x)}([\delta s, (ds)^v]) = 0$. Indeed, we are working on a trivialization, so we can write $(ds)^v = \hat{\xi}^*$ for some $\hat{\xi} \in \mathfrak{g}$. Thus, $\omega((ds)^v) = \omega\left(\hat{\xi}^*\right) = \hat{\xi}$ and $\delta s(\hat{\xi}) = 0$. In addition, we have

$$\begin{aligned}
[\delta s, (ds)^v] &= \left(\pounds_{(ds)^v}\delta s\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} (d\Phi)_{(s,g(t))}\left(\delta s, 0_{g(t)}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} U_t \\
&= U
\end{aligned}$$

where $g\colon (-\epsilon, \epsilon) \to G$ is defined as $g(t) = \exp\left(t\hat{\xi}\right)$. Note that $U \in T_sY$ is $\pi_{Y,Y/\mathcal{G}}$-horizontal since so is each $Z_t \in T_{s\cdot g(t)}Y$:

$$\omega_{s\cdot g(t)}\left((d\Phi)_{s,g(t)}\left(\delta s, 0_{g(t)}\right)\right) = Ad_{g(t)^{-1}}\left(\omega_s(\delta s(x)) + \nu(0_{g(t)})\right) = 0$$

□

Since every arbitrary variation δs can be split into its $\pi_{Y,Y/\mathcal{G}}$-vertical and horizontal parts, we obtain the following result.

Corollary 4.1. *Let δs be a variation of a section $s \in \Gamma(\pi_{Y,X})$ and consider the induced variation of the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}} \in \Gamma(\pi_{\left(T^*X \otimes \tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},X})$. Then:*

$$\delta^{\nabla}\overline{s} = \left[\!\!\left[s, \nabla^{\mathfrak{g}}\xi + \Omega\left(\delta s, ds\right)\right]\!\!\right]_{\tilde{\mathfrak{H}}}, \qquad \xi = \omega(\delta s)$$

At last, we express the reduced variation in terms of the reduced section. To do so, we define the operator $\overline{\nabla}^{\tilde{\mathfrak{g}}} \colon \Gamma(\pi_{\tilde{\mathfrak{g}},X}) \to \Gamma(\pi_{T^*X \otimes \tilde{\mathfrak{g}},X})$ as

$$\left(\overline{\nabla}^{\tilde{\mathfrak{g}}}_U \tilde{\xi}\right)(x) = \left.\frac{\nabla^{\tilde{\mathfrak{g}}}\left(\tilde{\xi} \circ \gamma\right)(t)}{dt}\right|_{t=0}, \qquad \tilde{\xi} \in \Gamma(\pi_{\tilde{\mathfrak{g}},X}), \quad U \in \mathfrak{X}(X), \quad x \in X, \tag{17}$$

where $\gamma \colon (-\epsilon, \epsilon) \to X$ is a curve such that $\gamma'(0) = U(x)$. By definition, it is clear that

$$\overline{\nabla}^{\tilde{\mathfrak{g}}}[y, \xi]_{\mathcal{G}} = [y, \nabla^{\mathfrak{g}}\xi]_{\mathcal{G}}, \qquad [y, \xi]_{\mathcal{G}} \in \Gamma(\pi_{\tilde{\mathfrak{g}},X})$$

Corollary 4.2. *Let δs be a variation of a section $s \in \Gamma(\pi_{Y,X})$ and consider the induced variation of the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}}$. Then*

$$\delta^{\nabla}\overline{s}(x) = \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{H}}}$$

where $\tilde{\xi}(x) = [s(x), \omega_{s(x)}(\delta s(x))]_{\mathcal{G}}$.

4.3 Variations on $J^1(Y/\mathcal{G})$

The induced variations $\delta\sigma_s$ of $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$ are just the projection of the variations of s, that is

$$\delta\sigma_s = d\pi_{Y,Y/\mathcal{G}}(\delta s).$$

In particular, these reduced variations are free, with no particular constraints (in contrast with the constraints for $\delta\bar{s}$ analyzed in the previous section). For later convenience, we now analyze the vertical part of the 1-jet lift of $\delta\sigma_s$ to $J^1(Y/\mathcal{G})$ with respect to a suitable connection.

Let $\nabla^{Y/\mathcal{G}}$ be a linear connection on the tangent bundle $T(Y/\mathcal{G}) \to Y/\mathcal{G}$ and consider the operator $\overline{\nabla}^{Y/\mathcal{G}} \colon \Gamma(\pi_{T(Y/\mathcal{G}),X}) \to \Gamma(\pi_{T^*X \otimes T(Y/\mathcal{G}),X})$ defined as

$$\left(\overline{\nabla}^{Y/\mathcal{G}}_U \alpha\right)(x) = \left.\frac{\nabla^{Y/\mathcal{G}}(\alpha \circ \gamma)(t)}{dt}\right|_{t=0}, \qquad \alpha \in \Gamma(\pi_{T(Y/\mathcal{G}),X}), \quad U \in \mathfrak{X}(X), \quad x \in X, \tag{18}$$

where $\gamma \colon (-\epsilon, \epsilon) \to X$ is such that $\gamma'(0) = U(x)$. The following Lemma is an adaptation of [11, Corollary 3.4].

Lemma 4.3. *Let $\{s_t\}$ be a variation of a section $s \in \Gamma(\pi_{Y,X})$ and consider the induced variation of $\sigma_s \in \Gamma(\pi_{Y/\mathcal{G},X})$. Then:*

$$\left.\frac{\nabla^{Y/\mathcal{G}} d(\sigma_s)_t(U_x)}{dt}\right|_{t=0} = \left(\overline{\nabla}^{Y/\mathcal{G}}_U \delta\sigma_s\right)(x) + T^{Y/\mathcal{G}}\left(\delta\sigma_s(x), j^1_x\sigma_s(U_x)\right), \qquad x \in X, \quad U_x \in T_xX,$$

where $U \in \mathfrak{X}(X)$ is such that $U(x) = U_x$, and $T^{Y/\mathcal{G}} \in \mathcal{T}^1_2(Y/\mathcal{G})$ is the torsion tensor *of $\nabla^{Y/\mathcal{G}}$.*

The connection $\nabla^{Y/\mathcal{G}}$ is said to be *projectable* on a linear connection ∇^X on $\pi_{TX,X}$ if the following diagram is commutative:

$$\begin{array}{ccc} T\left(T(Y/\mathcal{G})\right) & \xrightarrow{\nu^{Y/\mathcal{G}}} & T(Y/\mathcal{G}) \\ {\scriptstyle d(d\pi_{Y/\mathcal{G},X})}\downarrow & & \downarrow{\scriptstyle d\pi_{Y/\mathcal{G},X}} \\ T(TX) & \xrightarrow{\nu^X} & TX \end{array}$$

where $\nu^{Y/\mathcal{G}}$ and ν^X are the vertical projections of the connections $\nabla^{Y/\mathcal{G}}$ and ∇^X, respectively. Consider the vector bundle[5]

$$\pi_{V,Y/\mathcal{G}}\colon V = T^*X \otimes T(Y/\mathcal{G}) \to Y/\mathcal{G}.$$

A section $\rho \in \Gamma(\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}})$ can be regarded as a section $\rho \in \Gamma(\pi_{V,Y/\mathcal{G}})$ since $J^1(Y/\mathcal{G}) \subset V$. Likewise, ρ can be regarded as a connection on $\pi_{Y/\mathcal{G},X}$. The next proposition is an adaptation of [16, Theorem 3.1, Lemma 3.1] to our case. See also [11, Equations (3.13), (3.14)].

Proposition 4.3. *If $\nabla^{Y/\mathcal{G}}$ is projectable onto ∇^X, then it induces an affine connection $\nabla^{J^1(Y/\mathcal{G})}$ in $\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}$ given by*

$$\nabla_Z^{J^1(Y/\mathcal{G})}\rho = (id \otimes \nu^\rho) \circ \nabla_Z^V \rho, \qquad Z \in \mathfrak{X}(Y/\mathcal{G}), \quad \rho \in \Gamma(\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}),$$

where $id\colon T^*X \to T^*X$ is the identity map, ν^ρ is the vertical projection associated to ρ, and ∇^V is the linear connection induced on $V = T^*X \otimes T(Y/\mathcal{G}) \to Y/\mathcal{G}$ by the tensor product of the connections $\nabla^{Y/\mathcal{G}}$ and ∇^X.

Corollary 4.3. *For any variation $\delta\sigma_s = d/dt|_{t=0}(\sigma_s)_t$ of a reduced variation σ_s, the vertical part with respect to the connection $\nabla^{J^1(Y/\mathcal{G})}$ of its 1-jet lift is given by*

$$\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s = \left.\frac{\nabla^{J^1(Y/\mathcal{G})} j^1(\sigma_s)_t}{dt}\right|_{t=0} = \overline{\nabla}^{Y/\mathcal{G}}\delta\sigma_s + T^{Y/\mathcal{G}}\left(\delta\sigma_s, d\sigma_s\right).$$

Proof. Given $x \in X$ and $U_x \in T_xX$, the tensor product connection ∇^V satisfies

$$\frac{\nabla^V j_x^1(\sigma_s)_t}{dt}(U_x) = \frac{\nabla^{Y/\mathcal{G}} j_x^1(\sigma_s)_t(U_x)}{dt} - (j_x^1\sigma_s)\left(\frac{\nabla^X U_x}{dt}\right) = \frac{\nabla^{Y/\mathcal{G}} j_x^1(\sigma_s)_t(U_x)}{dt}.$$

Since the projection of $j_x^1(\sigma_s)_t(U)_x$ by $d\pi_{Y/\mathcal{G},X}$ is constantly U_x, we have that $\nabla^{Y/\mathcal{G}} j_x^1(\sigma_s)_t(U_x)/dt$ is $\pi_{Y/\mathcal{G},X}$ vertical so that

$$\frac{\nabla^{Y/\mathcal{G}} j_x^1(\sigma_s)_t}{dt} = (id \otimes \nu^{j^1\sigma_s}) \circ \frac{\nabla^V j_x^1(\sigma_s)_t}{dt} = \frac{\nabla^V j_x^1(\sigma_s)_t}{dt},$$

and the proof is complete by Lemma 4.3. □

If the connection ∇^X is torsionless (and that will be our choice from now on), the formula above simply reads

$$\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s = \overline{\nabla}^{Y/\mathcal{G}}\delta\sigma_s.$$

[5]For the sake of simplicty, we are omitting the pull-back notation of $\pi_{T^*X,X}$ by $\pi_{Y/\mathcal{G},X}$.

4.4 Reduced equations

Let $\mathfrak{H}^\circ \subset TX \otimes \mathfrak{g}^*$ be the annhilator of $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$. Then, $\tilde{\mathfrak{H}}^\circ = (Y \otimes \mathfrak{H}^\circ)/\mathcal{G} \subset TX \otimes \tilde{\mathfrak{g}}^*$ will be the annihilator of $\tilde{\mathfrak{H}} \subset T^*X \otimes \tilde{\mathfrak{g}}$. This space is canonically isomorphic to the dual vector bundle

$$\tilde{\mathfrak{H}}^\circ \simeq \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)^*,$$

and the dual pairing is $\left\langle [y,\zeta_x]_\mathcal{G}, [\![y,\xi_x]\!]_{\tilde{\mathfrak{H}}} \right\rangle = \langle \zeta_x, \xi_x \rangle$ for each $[y,\zeta_x]_\mathcal{G} \in \tilde{\mathfrak{H}}^\circ$ and $[\![y,\xi_x]\!]_{\tilde{\mathfrak{H}}} \in (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}$.

On the other hand, note that $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \to Y/\mathcal{G}$ is an affine bundle, since $\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}$ affine and $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$ vector bundle respectively. Let $\nabla^\times$ be the affine connection induced by the connections $\nabla^{J^1(Y/\mathcal{G})}$ and ∇ on the corresponding bundles.

Definition 4.2. *Let $s \in \Gamma(\pi_{Y,X})$ and consider the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}}$, as well as $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$. The* partial derivatives *of the reduced Lagrangian*

$$l \colon J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \longrightarrow \mathbb{R}$$

are the sections

$$\frac{\delta l}{\delta \sigma_s} \in \Gamma\left(\pi_{T^*(Y/\mathcal{G}),X}\right), \quad \frac{\delta l}{\delta j^1\sigma_s} \in \Gamma\left(\pi_{TX\otimes V^*(Y/\mathcal{G}),X}\right), \quad \frac{\delta l}{\delta \overline{s}} \in \Gamma\left(\pi_{\tilde{\mathfrak{H}}^\circ,X}\right)$$

defined as

$$\begin{aligned}
\left\langle \frac{\delta l}{\delta \sigma_s}(x), U_x \right\rangle &= \left.\frac{d}{dt}\right|_{t=0} l\left(\gamma_U(t)^h(t)\right), \quad \forall U_x \in T_{\sigma_s(x)}(Y/\mathcal{G}),\\
\left\langle \frac{\delta l}{\delta j^1\sigma_s}(x), V_x \right\rangle &= \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s + t\,V_x, \overline{s}(x)\right), \quad \forall V_x \in T^*_xX \otimes V_{\sigma_s(x)}(Y/\mathcal{G}),\\
\left\langle \frac{\delta l}{\delta \overline{s}}(x), W_x \right\rangle &= \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s, \overline{s}(x) + t\,W_x\right), \quad \forall W_x \in \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)_{\sigma_s(x)}.
\end{aligned}$$

for each $x \in X$, where $\gamma_U(t)^h$ is the horizontal lift with respect to the connection $\nabla^\times$ of a curve $\gamma_U \colon (-\epsilon,\epsilon) \to Y/\mathcal{G}$ such that $\gamma'(0) = U_x$. As usual, $\langle\cdot,\cdot\rangle$ denotes the corresponding dual pairings.

We remark that, whereas the two latter derivatives are (intrinsic) fiber derivatives, the partial derivative $\delta l/\delta\sigma_s$ depends on the choice of the connections. Anyway, all of them are sections projecting onto σ_s.

Since $\delta l/\delta\overline{s}$ lies in the annihilator of $\tilde{\mathfrak{H}}$, then for each $[\![s,\xi]\!]_{\tilde{\mathfrak{H}}} \in \Gamma\left(\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},X}\right)$ the dual pairing satisfies

$$\left\langle \frac{\delta l}{\delta \overline{s}}, [\![s,\xi]\!]_{\tilde{\mathfrak{H}}} \right\rangle = \left\langle \frac{\delta l}{\delta \overline{s}}, [s,\xi]_\mathcal{G} \right\rangle \tag{19}$$

Indeed, let $x \in X$, $g \in \mathcal{G}_x$ and $\eta_x \in \mathfrak{H}_x$ then $[s(x)\cdot g, Ad_{g^{-1}} \circ \xi_x + \eta_x]_\mathcal{G} = [s(x),\xi_x]_\mathcal{G} + [s(x),\eta_x]_\mathcal{G}$ and $\left\langle \delta l/\delta\overline{s}(x), [s(x),\eta_x]_\mathcal{G} \right\rangle = 0$, since $[s(x),\eta_x]_\mathcal{G} \in \tilde{\mathfrak{H}}$.

Definition 4.3. *The* divergence *of the operator* $\overline{\nabla}^{\tilde{\mathfrak{g}}}$ *defined in* (17) *is minus the adjoint of* $\overline{\nabla}^{\tilde{\mathfrak{g}}}$*, i.e. the operator* $\operatorname{div}^{\tilde{\mathfrak{g}}}\colon \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right) \to \Gamma\left(\pi_{\tilde{\mathfrak{g}}^*,X}\right)$ *given by:*

$$\int_X \left\langle \zeta, \overline{\nabla}^{\tilde{\mathfrak{g}}}\xi \right\rangle v = -\int_X \left\langle \operatorname{div}^{\tilde{\mathfrak{g}}}\zeta, \xi \right\rangle v$$

for every $\zeta \in \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right)$ *and* $\xi \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$.

Analogously, the divergence of $\overline{\nabla}^{Y/\mathcal{G}}$ *is minus the adjoint of* (18) *restricted to vertical sections (the restriction is allowed since the linear connection* $\nabla^{Y/\mathcal{G}}$ *is projectable), that is,* $\operatorname{div}^{Y/\mathcal{G}}\colon \Gamma\left(\pi_{TX\otimes V^*(Y/\mathcal{G}),X}\right) \to \Gamma\left(\pi_{V^*(Y/\mathcal{G}),X}\right)$.

Theorem 4.1 (Reduced field equations)**.** *Let* $\pi_{Y,X}$ *be a fiber bundle over a compact manifold* X*,* $\pi_{\mathcal{G},X}$ *be a Lie group bundle,* $\pi_{\mathfrak{g},X}$ *be its Lie algebra bundle and* $H \subset J^1\mathcal{G}$ *be a Lie group subbundle such that* $H \simeq \mathcal{G} \times_X \mathfrak{H}$ *via* (10) *for some* $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$*. Suppose that* $\pi_{\mathcal{G},X}$ *acts fiberwisely, freely and properly on the right on* $\pi_{Y,X}$*. Let* $\omega \in \Omega^1(Y,\mathfrak{g})$ *be a generalized principal connection on* $\pi_{Y,Y/\mathcal{G}}$ *associated to a Lie group connection* ν *on* $\pi_{\mathcal{G},X}$*,* $\nabla^{\mathfrak{H}}$ *be a linear connection on* $\pi_{\mathfrak{H},X}$ *that can be extended to a compatible linear connection* $\nabla^{\otimes}$ *on* $\pi_{T^*X\otimes\mathfrak{g},X}$*, and* $\nabla^{Y/\mathcal{G}}$ *be a torsion free linear connection on* $\pi_{T(Y/\mathcal{G}),Y/\mathcal{G}}$ *projectable on a linear connection* ∇^X *on* $\pi_{TB,X}$*. Consider the induced connections built Section 3.1, as well as the affine connection* $\nabla^{\times}$ *on* $\pi_{J^1(Y/\mathcal{G})\times_{Y/\mathcal{G}}\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}}$.

Let $\mathcal{L} = Lv\colon J^1Y \to \bigwedge^n T^*X$ *be an* H*-invariant Lagrangian density and consider the corresponding reduced Lagrangian* $l\colon J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \to \mathbb{R}$*. For any section* $s \in \Gamma(\pi_{Y,X})$*, we denote by* $\overline{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}}$ *its reduced section and by* $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$.

Then the following assertions are equivalent:

(i) The variational principle $\delta \int_X \mathcal{L}(j^1 s) = 0$ *holds for arbitrary variations of* s.

(ii) The section s *satisfies the Euler-Lagrange equations for* L*, i.e.* $\mathcal{EL}(L)\left(j^1 s\right) = 0$.

(iii) The variational principle $\delta \int_X l(j^1\sigma_s, \overline{s})v = 0$ *holds for variations of the form:*

$$\delta^{\nabla}\overline{s} = \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{H}}}$$

where $\tilde{\xi} \in \Gamma(\pi_{\tilde{\mathfrak{g}},X})$ *is an arbitrary section and* $\delta\sigma_s$ *is an arbitrary variation of* σ_s.

(iv) The reduced section $\overline{s}$ *satisfies the* reduced field equations*:*

$$\begin{cases} \dfrac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega} \right\rangle \\ \operatorname{div}^{\tilde{\mathfrak{g}}}\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0 \end{cases}$$

Proof. The equivalence of (i) and (ii) is a well-known fact as we stated before. The equivalence of (i) and (iii) is a straightforward consequence of Equation (16) and Corollary 4.2. To complete the proof, we show the equivalence of (iii) and (iv).

Let δs be a $\pi_{Y,X}$-vertical variation of s and consider induced variations of σ_s and $\overline{s}$. Then:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v &= \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)\right) v \\ &= \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)^h\right) v + \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)^v\right) v \\ &= \int_X \left\langle \frac{\delta l}{\delta \sigma_s}, \delta\sigma_s \right\rangle v + \int_X \left\langle \frac{\delta l}{\delta j^1\sigma_s}, \delta^{J^1(Y/\mathcal{G})} j^1\sigma_s \right\rangle v + \int_X \left\langle \frac{\delta l}{\delta \overline{s}}, \delta^{\nabla}\overline{s} \right\rangle v, \end{aligned}$$

where the vertical parts $\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s$ and $\delta^{\nabla}\overline{s}$ are defined with the connections $\nabla^{J^1(Y/\mathcal{G})}$ and ∇ respectively. Making use of Corollaries 4.2 and 4.3 (with vanishing torsion) we get

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v &= \int_X \left\langle \frac{\delta l}{\delta \sigma_s}, \delta\sigma_s \right\rangle v + \int_X \left\langle \frac{\delta l}{\delta j^1\sigma_s}, \overline{\nabla}^{Y/\mathcal{G}} \delta\sigma_s \right\rangle v \\ &+ \int_X \left\langle \frac{\delta l}{\delta \overline{s}}, \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{h}}} \right\rangle v \end{aligned}$$

Thanks to (19), we may write:

$$\left\langle \frac{\delta l}{\delta \overline{s}}, \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{h}}} \right\rangle = \left\langle \frac{\delta l}{\delta \overline{s}}, \overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s) \right\rangle$$

Using this and the divergence operators defined above we get:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v &= \int_X \left\langle \frac{\delta l}{\delta \sigma_s}, \delta\sigma_s \right\rangle v + \int_X \left\langle -\operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right), \delta\sigma_s \right\rangle v \\ &+ \int_X \left\langle -\operatorname{div}^{\tilde{\mathfrak{g}}}\left(\frac{\delta l}{\delta \overline{s}}\right), \tilde{\xi} \right\rangle v + \int_X \left\langle -\left\langle \frac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega} \right\rangle, \delta\sigma_s \right\rangle v \end{aligned}$$

As a result, the variational principe of *(iii)* reads:

$$\int_X \left\langle \frac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right) - \left\langle \frac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega} \right\rangle, \delta\sigma_s \right\rangle v + \int_X \left\langle -\operatorname{div}^{\tilde{\mathfrak{g}}}\left(\frac{\delta l}{\delta \overline{s}}\right), \tilde{\xi} \right\rangle v = 0$$

for every section $\tilde{\xi} \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$ and every variation $\delta\sigma_s$ of σ_s. □

Remark 4.2. *The first equation is on $\pi_{V^*(Y/\mathcal{G}),X}$, and the second one on $\pi_{\tilde{\mathfrak{g}}^*,X}$. On the other hand, if the connection $\nabla^{Y/\mathcal{G}}$ has non-vanishing torsion, then the reduced equations are:*

$$\begin{cases} \dfrac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) + \left\langle \dfrac{\delta l}{\delta j^1\sigma_s}, \iota_{d\sigma_s} T^{Y/\mathcal{G}} \right\rangle = \left\langle \dfrac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega} \right\rangle, \\ \operatorname{div}^{\tilde{\mathfrak{g}}}\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0. \end{cases}$$

5 Reconstruction

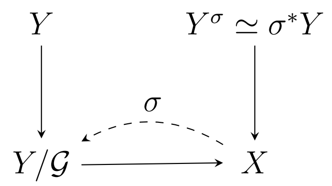
Let $\overline{s}$ be a section of the reduced bundle $(T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \to X$ and let

$$\sigma = \pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}$$

be the induced section of $Y/\mathcal{G} \to X$. We consider the subset

$$Y^\sigma = \pi^{-1}_{Y,Y/\mathcal{G}}(\sigma(X)) = \{y \in Y : \pi_{Y,Y/\mathcal{G}}(y) = \sigma(\pi_{Y,X}(y))\} \subset Y.$$

The action of $\mathcal{G}$ on Y restricts to Y^σ and $Y^\sigma/\mathcal{G} \simeq X$. In fact, we can regard Y^σ as a pull-back bundle $Y^\sigma \simeq \sigma^*Y \to X$ on which the Lie group bundle $\mathcal{G}$ acts transitively along the fibers:



In particular, the adjoint bundle of $Y^\sigma \to X$ is $\sigma^*\tilde{\mathfrak{g}} \to X$. From this point of view, the section $\overline{s}$ can be also considered as a section of the pull-backed bundle

$$\sigma^*((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}) = (T^*X \otimes \sigma^*\tilde{\mathfrak{g}})/\sigma^*\tilde{\mathfrak{H}} \to X.$$

The restriction $\sigma^*\omega$ of the connection form ω to Y^σ is a generalized principal connection on $Y^\sigma \to X$ associated to the Lie group connection ν. From [18, Proposition 3.11] we know that for any section $\tilde{\xi}$ of $T^*X \otimes \sigma^*\tilde{\mathfrak{g}} \to X$ the form $\sigma^*\omega - \tilde{\xi}$ is a generalized principal connection on $\sigma^*Y \to X$.

Proposition 5.1. *Let s be a section of $Y \to X$, which we regard s as a section of $\sigma^*Y \to X$, and $\sigma = \pi_{Y,Y/\mathcal{G}} \circ s$. Then the generalized principal connection defined as*

$$\omega^{\overline{s}} = \sigma^*\omega - \tilde{\xi}$$

is flat, where $\tilde{\xi} = [s, s^*\omega]_\mathcal{G} \in \Gamma(\pi_{T^*X\otimes\sigma^*\tilde{\mathfrak{g}},X})$.

Proof. To begin with, we have that $s(X) \subset Y^\sigma$ is an integral leaf of $\sigma^*\omega - \tilde{\xi}$. Indeed, it is easy to check that $(ds)_x(T_xX)$ is $\omega^{\overline{s}}$-horizontal for each $x \in X$. This means that $\mathrm{Curv}(\omega^{\overline{s}})_{s(x)} = 0$ for each $x \in X$, where $\mathrm{Curv}(\omega^{\overline{s}})$ is the curvature of $\omega^{\overline{s}}$.

An analogous computation to (9) yields

$$\left[s(x)\cdot g, \mathrm{Curv}(\omega^{\overline{s}})_{s(x)\cdot g}\right]_\mathcal{G} = \left[s(x), \mathrm{Curv}(\omega^{\overline{s}})_{s(x)}\right]_\mathcal{G} = 0, \qquad x \in X, \quad g \in \mathcal{G}_x,$$

whence $\mathrm{Curv}(\omega^{\overline{s}})_y = 0$ for every $y \in Y^\sigma$. □

Theorem 5.1 (Reconstruction)**.** *Let $\overline{s}$ be a section of $(T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \to X$ that is critical for the variational problem defined by l, and let $\sigma = \pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}$ be the induced section of $Y/\mathcal{G} \to X$. Let $\tilde{\xi}$ a section of $T^*X \otimes \sigma^*\tilde{\mathfrak{g}} \to X$ whose $\sigma^*\tilde{\mathfrak{H}}$-class is $\overline{s}$ and suppose that the connection*

$$\omega^{\overline{s}} = \sigma^*\omega - \tilde{\xi}$$

is flat and has trivial holonomy. Then the integral leaves of the $\omega^{\overline{s}}$ are critical sections of the variational problem defined by L. Furthermore, any critical section of L is obtained in this way.

Proof. The trivial holonomy of $\omega^{\overline{s}}$ implies that its integral leaves are sections of $\sigma^*Y \to X$ that project to the reduced section $\overline{s}$. According to Theorem 4.1, s is critital.

Conversely, if s is critical, by Proposition 5.1 it is an integral leaf of $\omega^{\overline{s}}$. This Proposition also ensures that $\omega^{\overline{s}}$ is flat. □

Corollary 5.1. *If X is simply connected, any connection has trivial holonomy and we have the following equivalence of equations*

$$\mathcal{EL}(L)(j^1s) = 0 \iff \begin{cases} \dfrac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega} \right\rangle \\ \operatorname{div}^{\tilde{\mathfrak{g}}}\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0 \\ \operatorname{Curv}(\omega^{\overline{s}}) = 0 \end{cases}$$

For non-simply connected manifolds, there are topological obstructions and examples of reduced sections that do not admit unreduced sections.

Remark 5.1. *Given a reduced section $\overline{s}$, there may be many sections $\tilde{\xi}$ such that $\overline{s} = \left[\tilde{\xi}\right]_{\sigma^*\tilde{\mathfrak{H}}}$ and $\operatorname{Curv}(\omega^{\overline{s}}) = 0$. Each choice will induce different solution s and the transitions between them are governed by symmetries, that is, sections of $\mathcal{G} \to X$ with 1-jets lying on $H \to X$.*

6 Noether's theorem

The well known Noether's theorem establishes that infinitesimal symmetries of the Lagrangian density yield preserved quantities for the dynamics of the system. The aim of this section is to show that the vertical part of the reduced equation is equivalent to the Noether's conservation law. As before, let $\mathcal{L} = Lv$ be an H-invariant Lagrangian density for some $H \simeq \mathcal{G} \times_X \mathfrak{H}$.

Definition 6.1. *An* infinitesimal symmetry *of $\mathcal{L}$ is a vector field $U \in \mathfrak{X}(J^1Y)$ such that $\pounds_U\mathcal{L} = 0$ or, equivalently, $\pounds_U\Theta_{\mathcal{L}} = 0$, where $\pounds$ denotes the Lie derivative and $\Theta_{\mathcal{L}} \in \Omega^n(J^1Y)$ is the Poincaré-Cartan form of $\mathcal{L}$.*

If $s \in \Gamma(\pi_{Y,X})$ is a critical section for $\mathcal{L}$ and $U \in \mathfrak{X}(J^1Y)$ is an infinitesimal symmetry of the Lagrangian density, the Noether's theorem gives (see, for example, [13]) the following conservation law

$$d\left(\left(j^1 s\right)^* \iota_U \Theta_{\mathcal{L}}\right) = 0. \tag{20}$$

In particular, if $\xi \in \Gamma(\pi_{\mathfrak{g},X})$ is such that its 1-jet extension falls in the Lie subalgebra

$$\mathfrak{g} \times_X \mathfrak{H} \subset \mathfrak{g} \times_X (T^*X \otimes \mathfrak{g}) = J^1\mathfrak{g}.$$

of $G \times \mathfrak{H} \simeq H \subset J^1\mathcal{G}$, thanks to the H-invariance, it is clear that $(\xi^*)^{(1)} \in \mathfrak{X}(J^1Y)$ is an infinitesimal symmetry of $\mathcal{L}$.

Theorem 6.1. *Let $L : J^1Y \to \mathbb{R}$ be a Lagrangian invariant under the action of a subgroup $H \subset J^1\mathcal{G}$ and let s be a section of $\pi_{Y,X}$. If $\bar{s} = [\![s, s^*\omega]\!]_{\tilde{\mathfrak{H}}}$ is the induced section of $\pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},X}$ and $l : J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \to \mathbb{R}$ is the reduced Lagrangian, then the Noether conservation law*

$$d((j^1s)^* i_{(\xi^*)^{(1)}} \Theta_{\mathcal{L}}) = 0,$$

for any section ξ of $\pi_{\mathfrak{g},X}$ such that $\xi^{(1)}$ falls in $\mathfrak{g} \times_X \mathfrak{H}$, if and only if the vertical reduced equation

$$\operatorname{div}^{\tilde{\mathfrak{g}}}\left(\frac{\delta l}{\delta \bar{s}}\right) = 0$$

is satisfied for $\bar{s}$.

Proof. The result being local, we work with coordinates. Some conditions on these coordinates will be imposed along the proof.

Let (x^μ, y^i, y^A) be bundle coordinates for $\pi_{Y,X}$ and consider the corresponding bundle coordinates $(x^\mu, y^i, y^A, v^i_\mu, v^A_\mu)$ for $\pi_{J^1Y,X}$. Suppose that they are chosen so that (x^μ, y^i) are bundle coordinates for $\pi_{Y/\mathcal{G},X}$ and, thus, (x^μ, y^i, v^i_μ) are bundle coordinates for $\pi_{J^1(Y/\mathcal{G}),X}$. Let $\{B_A \colon 1 \le A \le m\}$ be a basis of local sections of $\pi_{\mathfrak{g},X}$ and $\{B^A \colon 1 \le A \le m\}$ be its dual basis. For a fixed $x_0 = (x^\mu_0) \in X$, we suppose that, using these coordinates, we have

$$\mathfrak{H}_{x_0} = \operatorname{span}\{(dx^\mu)_{x_0} \otimes B_A(x_0) \colon 1 \le \mu \le r, 1 \le A \le s\} \subset T^*_{x_0}X \otimes \mathfrak{g}_{x_0} \tag{21}$$

for some r, $1 \le r \le n$, and s, $1 \le s \le m$. We start by finding the local expression of the infinitesimal symmetries of our Lagrangian density.

We also suppose that (y^A) are normal coordinates of G in a neighbourhood $\mathcal{U} \subset G$ of the identity element 1. This means that there exist $g^A_{BC} \in C^\infty(\mathcal{U} \times \mathcal{U})$, $1 \le A, B, C \le m$, such that

$$y^A(\hat{g}_1\hat{g}_2) = y^A(\hat{g}_1) + y^A(\hat{g}_2) + g^A_{BC}(\hat{g}_1, \hat{g}_2)\, y^B(\hat{g}_1)\, y^C(\hat{g}_2), \qquad 1 \le A \le m,$$

for each $\hat{g}_1, \hat{g}_2 \in \mathcal{U}$ such that $\hat{g}_1\hat{g}_2 \in \mathcal{U}$. Hence, the infinitesimal generators are given by

$$(B^*_B)_y = \left(\delta^A_B + g^A_{CB}(\hat{g}, 1) y^C(\hat{g})\right)(\partial_A)_y, \qquad y = (\sigma, \hat{g}) \in Y/\mathcal{G} \times \mathcal{U}, \quad 1 \le B \le m.$$

We denote by $\hat{g} \colon \mathbb{R}^m \to G$ the inverse of $(y^A) \colon G \to \mathbb{R}^m$ and define $f^C_{AB} \in C^\infty(\mathbb{R}^m)$, $1 \le A, B, C \le m$, as

$$f^C_{AB}(y^A) = g^C_{AB}(\hat{g}(y^A), 1)$$

Therefore, given $\xi = \xi^B B_B \in \Gamma(\pi_{\mathfrak{g},X})$, where $\xi^B \in C^\infty(X)$, $1 \leq B \leq m$, the previous expressions yield

$$\xi^*(x^\mu, y^i, y^A) = \xi^B(x^\mu) \left(\delta_B^A + f_{CB}^A(y^A) y^C\right) \partial_A.$$

From the formula of 1-jet lift of vector fields (for example, see, [27]), we get

$$(\xi^*)^{(1)} = \xi^B \left(\delta_B^A + f_{CB}^A y^C\right) \partial_A + \left(\partial_\mu \xi^B \left(\delta_B^A + f_{CB}^A y^C\right) + v_\mu^D \xi^B \left(\partial_D f_{CB}^A y^C + f_{DB}^A\right)\right) \partial_A^\mu,$$

for some functions $f_{AB}^C \in C^\infty(\mathbb{R}^m)$ and $\xi^A \in C^\infty(X)$, $1 \leq A, B, C \leq m$. In addition, since the jet extension of ξ is given by $j^1\xi \equiv d\xi = \partial_\mu \xi^A dx^\mu \otimes B_A \in \Gamma(\pi_{T^*X\otimes\mathfrak{g},X})$, the condition $\xi^{(1)} \in \Gamma(\pi_{\mathfrak{h},X})$ at x_0 reads

$$(\partial_\mu \xi^A)(x_0) = 0, \qquad r+1 \leq \mu \leq n, \quad s+1 \leq A \leq m.$$

Using the local expression of our generalized principal connection $\omega \in \Omega^1(Y, \mathfrak{g})$, there exist bundle coordinates $(x^\mu, y^i, v_\mu^i; w_\mu^A)$ for the reduced space, where the indices of w_μ^A go through $\mu \in \{r+1, \ldots, n\}$ and $A \in \{s+1, \ldots, m\}$, such that identification of Theorem 3.1 is given by

$$\begin{array}{ccc} J^1Y/H & \longrightarrow & J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} \\ \left[x^\mu, y^i, y^A, v_\mu^i, v_\mu^A\right]_H & \longmapsto & \left(x^\mu, y^i, v_\mu^i; w_\mu^A\right) \end{array}$$

and, in the fiber over x_0, they satisfy

$$w_\mu^A = \omega_\mu^A(x_0^\mu, y^i) + \omega_i^A(x_0^\mu, y^i) v_\mu^i + v_\mu^A, \qquad r+1 \leq \mu \leq n, \quad s+1 \leq A \leq m$$

for some functions $\omega_\mu^A, \omega_i^A \in C^\infty(Y/\mathcal{G})$, $r+1 \leq \mu \leq n$, $1 \leq i \leq k$, $s+1 \leq A \leq m$, being k the dimension of the fiber of $\pi_{Y/\mathcal{G},X}$.

Likewise, using the definition of reduced Lagrangian, i.e.

$$l(x^\mu, y^i, v_\mu^i; w_\mu^A) = L(x^\mu, y^i, y^A, v_\mu^i, v_\mu^A),$$

it is easy to find the local expression of the vertical equation. Then, local expression of the partial derivative of l is

$$\frac{\delta l}{\delta \overline{s}}(x^\mu) = \frac{\partial l}{\partial w_\mu^A} \left(x^\mu, s^i(x^\mu), \frac{\partial s^i(x^\mu)}{\partial x^\mu}; w_\mu^A(x^\mu)\right) \partial_\mu \otimes B^A,$$

and the, the vertical equation becomes

$$\operatorname{div}^{\mathfrak{g}} \left(\frac{\partial l}{\partial w_\mu^A} \partial_\mu \otimes B^A\right) = 0. \tag{22}$$

As $\operatorname{div}\langle\zeta, \xi\rangle = \left\langle \operatorname{div}^{\mathfrak{g}} \zeta, \xi \right\rangle + \langle \zeta, \nabla^{\mathfrak{g}} \xi \rangle$, for each $\zeta = \zeta_A^\mu \partial_\mu \otimes B^A \in \Gamma(\pi_{TX\otimes\mathfrak{g}^*,X})$ and $\xi = \xi^A B_A \in \Gamma(\pi_{\mathfrak{g},X})$. An easy computation using this expression shows that

$$\operatorname{div}^{\mathfrak{g}} \zeta = \left(\partial_\mu \zeta_A^\mu - \Gamma_{\mu,A}^B \zeta_B^\mu\right) B^A$$

for some functions $\Gamma_{\mu,A}^B \in C^\infty(X)$, $1 \leq \mu \leq n$, $1 \leq A, B \leq m$, such that

$$\nabla^{\mathfrak{g}} \xi = \left(\partial_\mu \xi^A + \Gamma_{\mu,B}^A \zeta^B\right) dx^\mu \otimes B_A, \qquad \xi = \xi^A B_A \in \Gamma(\pi_{\mathfrak{g},X})$$

We take bundle coordinates $\pi_{\mathfrak{g},X}$ so that $\nabla^{\mathfrak{g}}$ is flat at $x_0 = (x_0^\mu)$, i.e. $\Gamma^B_{\mu,A}(x_0^\mu) = 0$ for every $1 \leq \mu \leq n$ and $1 \leq A, B \leq m$. We thus get,

$$\partial_\mu \left(\frac{\partial l}{\partial w^A_\mu} \right) (x_0) B^A(x_0) = 0, \tag{23}$$

where we recall that the summation is for $s+1 \leq A \leq m$.

Using the expression for the coordinates w^A_μ at x_0 we have

- $$\frac{\partial L}{\partial v^A_\mu}(x_0) = \begin{cases} 0, & 1 \leq \mu \leq r, \quad 1 \leq A \leq s \\ \dfrac{\partial l}{\partial w^A_\mu}(x_0), & r+1 \leq \mu \leq n, \quad s+1 \leq A \leq m \end{cases}$$
- $$\frac{\partial L}{\partial v^i_\mu}(x_0) = \begin{cases} \dfrac{\partial l}{\partial v^i_\mu}(x_0), & 1 \leq \mu \leq r \\ \dfrac{\partial l}{\partial v^i_\mu}(x_0) + \displaystyle\sum_{A=s+1}^{m} \dfrac{\partial l}{\partial w^A_\mu}(x_0)\omega^A_i, & r+1 \leq \mu \leq n \end{cases}$$

In addition, if the volume form is given by $v = d^n x = dx^1 \wedge \ldots dx^n$, then the local expression of the Poincaré-Cartan form is

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^i_\mu} \left(dy^i - v^i_\mu dx^\mu \right) \wedge d^{n-1}x_\mu + \frac{\partial L}{\partial v^A_\mu} \left(dy^A - v^A_\mu dx^\mu \right) \wedge d^{n-1}x_\mu + L\, d^n x$$

where $d^{n-1}x_\mu = \iota_{\partial_\mu} d^n x$ so that

$$\begin{aligned} \iota_{(\xi^*)^{(1)}} \Theta_{\mathcal{L}} \left(x^\mu, y^i, y^A, v^i_\mu, v^A_\mu \right) &= \frac{\partial L}{\partial v^A_\mu} \left(x^\mu, y^i, y^A, v^i_\mu, v^A_\mu \right) \xi^B(x^\mu) \left(\delta^A_B + f^A_{CB}(y^A) y^C \right) d^{n-1}x_\mu \\ &= \frac{\partial l}{\partial w^A_\mu} \left(x^\mu, y^i, v^i_\mu; w^A_\mu \right) \xi^B(x^\mu) \left(\delta^A_B + f^A_{CB}(y^A) y^C \right) d^{n-1}x_\mu \end{aligned}$$

The last condition on the coordinates is the requirement that $s(x_0) = (\sigma_s(x_0), 1) \in Y/\mathcal{G} \times G$, i.e. we have $s^A(x_0^\mu) = 0$, $1 \leq A \leq m$. Therefore,

$$(j^1 s)^* \left(\iota_{(\xi^*)^{(1)}} \Theta_{\mathcal{L}} \right) (x_0^\mu) = \frac{\partial l}{\partial w^A_\mu} \left(x_0^\mu, s^i(x_0^\mu), \partial_\mu s^i(x_0^\mu); w^A_\mu(x_0^\mu) \right) \xi^A(x_0^\mu) d^{n-1}x_\mu$$

and the Noether conservation law is

$$d \left(\frac{\partial l}{\partial w^A_\mu} d^{n-1}x_\mu \right)_{x_0} = 0, \qquad s+1 \leq A \leq m. \tag{24}$$

This equation is the same as

$$\partial_\mu \left(\frac{\partial l}{\partial w^A_\mu} \right) (x_0) = 0, \qquad s+1 \leq A \leq m$$

that is equivalent to (23). □

7 Examples

In this final section we discuss several applications of the reduction theory developed. The first one consists of recovering the case of global symmetries treated in the literature. Then we describe the reduced equations when the system is invariant by the whole jet bundle, $J^1\mathcal{G}$. The third example is devoted to study electromagnetism in vacuum, as well as it extension to p-forms. Finally, we study the proccess of symmetry breaking by product groups in pure gauge theories.

7.1 Classical case

The reduction by global symmetries given in [11] can be recovered as a particular case of our theory. Namely, let $\pi_{Y,X}$ be a fiber bundle and G be a Lie group acting freely and properly on the right on Y:

$$\begin{array}{rrcl} R\colon & Y \times G & \longrightarrow & Y \\ & (y,\hat{g}) & \longmapsto & R(y,\hat{g}) = R_{\hat{g}}(y) = \phi_y(\hat{g}) = y \cdot \hat{g} \end{array}$$

Suppose that $\pi_{Y,X}(y \cdot \hat{g}) = \pi_{Y,X}(y)$ for every $y \in Y$ and $\hat{g} \in G$. This action may be regarded as a fibered one of the trivial Lie group bundle $\mathcal{G} = X \times G$ by setting $\Phi(y,g) = R_{\hat{g}}(y) = y \cdot \hat{g}$ for each $(y,g) \in Y \times_X \mathcal{G}$, with $g = (x,\hat{g})$. It is clear that the fibered quotient and the usual quotient agree, i.e. $Y/\mathcal{G} \simeq Y/G$.

Recall that $\pi_{Y,Y/G}$ is a principal bundle and the Lie algebra bundle of $\pi_{\mathcal{G},X}$ is $\mathfrak{g} = X \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. From [18, Proposition 4.1.] we know that a generalized principal connection $\omega \in \Omega^1(Y,\mathfrak{g})$ on $\pi_{Y/\mathcal{G}}$ associated to the trivial connection ν_0 on $\pi_{\mathcal{G},X}$ is simply a principal connection $\hat{\omega} \in \Omega^1(Y,\mathfrak{g})$. Thanks to the bijective correspondence between (local) sections of $\pi_{\mathcal{G},X}$ and (local) functions $X \to G$, the identification (10) for ν_0 may be written as

$$J^1\mathcal{G} \simeq G \times T^*X \otimes \mathfrak{g}, \qquad j^1_x\gamma \longmapsto \left(\hat{\gamma}(x), d(R_{\hat{\gamma}(x)^{-1}} \circ \hat{\gamma})_x\right),$$

where $\gamma = (\mathrm{id}_X, \hat{\gamma})$. Let H be the Lie group subbundle of $\pi_{J^1\mathcal{G},X}$ corresponding to locally constant functions. Then

$$H \simeq G \times T^*X \otimes \{0\} \simeq X \times G = \mathcal{G}$$

The first jet extension of the fibered action restricted to $\pi_{H,X}$ (recall Equation (13)) and the jet extension of the Lie group action (cf. [11, Equation 2.7]) yield the same quotient:

$$J^1Y/H \simeq J^1Y/G \tag{25}$$

Since $\mathfrak{H} = \{0\}$, the quotient (12) is isomorphic to

$$\begin{array}{rcc} (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} & \stackrel{\sim}{\longrightarrow} & T^*X \otimes \mathrm{ad}(Y) \\ \left[\!\left[y,(x,\hat{X}_x) \right]\!\right]_{\tilde{\mathfrak{H}}} & \longmapsto & \left[y,\hat{X}_x\right]_G \end{array} \tag{26}$$

where $\mathrm{ad}(Y) = (Y \times \mathfrak{g})/G$ is the adjoint bundle of $\pi_{Y,Y/G}$.

Theorem 7.1. *Let $\hat{\omega} \in \Omega^1(Y, \mathfrak{g})$ be a principal connection on $\pi_{Y,Y/G}$ and consider the corresponding generalized principal connection $\omega \in \Omega^1(Y, \mathfrak{g})$ on $\pi_{Y,Y/\mathcal{G}}$ associated to the canonical connection ν_0 on $\pi_{\mathcal{G},X}$. Then the fiber diffeomorphism given in Theorem 3.1 reduces to the identification given in [11, Equation (2.8)] under the previous identifications, i.e.*

$$\begin{array}{rcl} J^1Y/G & \longrightarrow & J^1(Y/G) \times_{Y/G} T^*X \otimes \mathrm{ad}(Y) \\ {[j^1_x s]_G} & \longmapsto & \left(j^1_x\sigma_s, \left[s(x), \hat{\omega}_{s(x)} \circ (ds)_x\right]_G\right) \end{array}$$

Let $\hat{\omega} \in \Omega^1(Y, \mathfrak{g})$ be a principal connection on $\pi_{Y,Y/G}$, $\nabla_0^{\mathfrak{g}}$ be the canonical connection on $\pi_{\mathfrak{g},X}$ and ∇^X be a linear connection on $\pi_{TX,X}$. They yield a linear connection ∇ on $\pi_{\left(T^*X\otimes\tilde{\mathfrak{g}}\right)/\tilde{\mathfrak{H}},Y/\mathcal{G}} \simeq \pi_{T^*X\otimes\mathrm{ad}(Y),Y/G}$, as we have seen in Proposition 3.3. Using bases of local sections of these vector bundles, it can be seen that ∇ agrees with the linear connection defined in [11, Equation (3.19)]. In addition, let $\nabla^{\hat{\omega}}$ be the linear connection on $\pi_{\mathrm{ad}(Y),Y/G}$ associated to $\hat{\omega}$ (cf. [11, Equation (2.6)]). It descends to an operator $\overline{\nabla}^{\hat{\omega}} : \Gamma(\pi_{\mathrm{ad}(Y),X}) \to \Gamma(\pi_{T^*X\otimes\mathrm{ad}(Y),X})$ as in [11, Equation (3.7)], whose *divergence* is denoted by:

$$\mathrm{div}^{\hat{\omega}} \colon \Gamma(\pi_{TX\otimes\mathrm{ad}(Y)^*,X}) \longrightarrow \Gamma(\pi_{\mathrm{ad}(Y)^*,X})$$

Lastly, let $\nabla^{Y/\mathcal{G}}$ be a torsion free linear connection on $\pi_{T(Y/\mathcal{G}),Y/\mathcal{G}}$ projectable onto ∇^X and consider the induced affine connection $\nabla^{J^1(Y/\mathcal{G})}$ on $\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}$.

Let $L\colon J^1Y \to \mathbb{R}$ be a G-invariant Lagrangian and $l\colon J^1(Y/G) \times_{Y/G} T^*X \otimes \mathrm{ad}(Y) \to \mathbb{R}$ be the reduced Lagrangian. As in the general theory, for the sake of simplicity we suppose that X is compact. Let $s \in \Gamma(\pi_{Y,X})$ and consider the reduced section $\overline{s} = [s, s^*\hat{\omega}]_G \in \Gamma\left(\pi_{T^*X\otimes\mathrm{ad}(Y),X}\right)$. Observe that $\mathfrak{H}^\perp = TX \otimes \mathfrak{g}^*$ and, hence, $Y \times_{\mathcal{G}} \mathfrak{H}^\perp \simeq TX \otimes \mathrm{ad}(Y)^*$. Given a reduced section $\overline{s} \in \Gamma(\pi_{T^*X\otimes\mathrm{ad}(Y),X})$, the coadjoint representation of $\mathfrak{g}^*$ induces a map:

$$ad^*_{\overline{s}} \colon \Gamma(\pi_{TX\otimes\mathrm{ad}(Y)^*,X}) \longrightarrow \Gamma(\pi_{\mathrm{ad}(Y)^*,X}),$$

which is well defined for every $\overline{\zeta} \in \Gamma(\pi_{TX\otimes\mathrm{ad}(Y)^*,X})$ such that:

$$\pi_{TX\otimes\mathrm{ad}(Y)^*,Y/G} \circ \overline{\zeta} = \pi_{T^*X\otimes\mathrm{ad}(Y),Y/G} \circ \overline{s}.$$

Theorem 7.2. *Let $s \in \Gamma(\pi_{Y,X})$ and $\overline{s} = [s, s^*\hat{\omega}]_G \in \Gamma\left(\pi_{T^*X\otimes\mathrm{ad}(Y),X}\right)$ be the reduced section. Then the reduced equations for $\overline{s}$ given in Theorem 4.1 are equivalent to the Lagrange-Poincaré equations given in [11, Theorem 3.5], i.e.*

$$\begin{cases} \dfrac{\delta l}{\delta\sigma_s} - \mathrm{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta\overline{s}}, \iota_{d\sigma_s}\hat{B} \right\rangle \\ \mathrm{div}^{\hat{\omega}}\left(\dfrac{\delta l}{\delta\overline{s}}\right) - \mathrm{ad}^*_{\overline{s}}\left(\dfrac{\delta l}{\delta\overline{s}}\right) = 0 \end{cases}$$

where $\hat{B} \in \Omega^2(Y/G, \mathrm{ad}(Y))$ is the reduced curvature associated to $\hat{\omega}$.

7.2 Full jet symmetry

Let $\pi_{\mathcal{G},X}$ be a Lie group bundle endowed with a Lie group connection ν, and suppose that it acts (on the right) on a fiber bundle $\pi_{Y,X}$ freely and properly. In this example we consider a Lagrangian $L\colon J^1Y \to \mathbb{R}$ which is invariant by the whole jet bundle, that is,

$$H = J^1\mathcal{G} \overset{\nu}{\simeq} \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}).$$

It is clear that in this case we have $(T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}} = X$. Consequently, the identification of Theorem 3.1 can be performed without fixing a generalized principal connection:

$$\begin{array}{ccc} J^1Y/J^1\mathcal{G} & \longrightarrow & J^1(Y/\mathcal{G}) \\ [j^1_x s]_{J^1\mathcal{G}} & \longmapsto & j^1_x \sigma_s \end{array} \tag{27}$$

In the same way, since the reduced section $\overline{s}$ vanishes, the second equation of *(iv)* in Theorem 4.1 does not appear. Therefore, the reduced equations are the usual *Euler-Lagrange equations* for the reduced Lagrangian Lagrangian $l\colon J^1(Y/\mathcal{G}) \to \mathbb{R}$:

$$\frac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1 \sigma_s}\right) = 0$$

Of course, to write them we have fixed a linear connection $\nabla^{Y/\mathcal{G}}$ on $\pi_{T(Y/\mathcal{G}),Y/\mathcal{G}}$ projectable onto a linear connection ∇^X on $\pi_{TX,X}$.

7.3 Electromagnetism in vacuum

To describe electromagnetism in vacuum as a geometric Yang-Mills theory, let (X, g) be a 4-dimensional, compact, oriented semi-Riemannian manifold with volume form $v_g \in \Omega^4(X)$. The configuration bundle of this theory is the bundle of connections of a fixed a principal $U(1)$-bundle $\pi_{P,X}$:

$$Y \equiv C(P) = J^1P \big/ U(1) \longrightarrow X$$

It is an affine bundle modelled on $\pi_{T^*X,X}$, since the adjoint bundle is trivial in this case: $ad(P) = (P \times \mathfrak{u}(1))/U(1) \simeq X \times \mathbb{R}$, where $\mathfrak{u}(1) \simeq \mathbb{R}$ is the Lie algebra of $U(1)$.

Definition 7.1. *The Lagrangian density for electromagnetism in vacuum is* $\mathcal{L} = L\, v_g$ *with:*

$$L\left(j^1 A\right) = g\left(F_A, F_A\right), \qquad A \in \Gamma(\pi_{C(P),X}),$$

where $F_A \in \Omega^2(X)$ *is the* reduced curvature *of the principal connection* A.

This Lagrangian density is invariant by gauge transformations of connections. In the same way that principal connections on $\pi_{P,X}$ are in a bijective correspondence with sections of $\pi_{C(P),X}$, gauge transformations on $\pi_{P,X}$ are in a bijective correspondence with sections of $\pi_{Ad(P),X}$, where $Ad(P) = (P \times U(1))/U(1) \simeq X \times U(1)$ is the associated bundle given by the action $(p, g) \cdot h = (p \cdot h, g)$ for each $p \in P$ and $g, h \in U(1)$.

In order to extend gauge transformations to the family of principal connections, we need to take first derivatives, i.e. we need to consider $J^1Ad(P) \simeq J^1(X, U(1))$. From the local expression of gauge transformations of connections, we obtain the following right fibered action: $A_x \cdot j^1_x g = A_x + (dg)_x\, g(x)^{-1}$, for each $(A_x, j^1_x g) \in C(P) \times_X J^1(X, U(1))$. Note that the value $g(x) \in U(1)$ does not play any role and $(dg)_x\, g(x)^{-1} \in T_x^*X \otimes \mathfrak{u}(1) \simeq T_x^*X$. Subsequently, this fibered action induces another one:

$$\begin{array}{rcl} C(P) \times_X T^*X & \longrightarrow & C(P) \\ (A_x, \alpha_x) & \longmapsto & A_x + \alpha_x \end{array} \tag{28}$$

It can be straightforwardly extended to the first jets, $J^1C(P) \times_X J^1(T^*X) \to J^1C(P)$. By virtue of this, the Lie group bundle of symmetries for electromagnetism is the cotangent bundle of X with the additive structure:

$$\mathcal{G} \equiv T^*X \longrightarrow X$$

Second derivatives of gauge transformations correspond to the Lie group subbundle $J^2(X, U(1)) \subset J^1\left(J^1(X, U(1))\right)$. One can readily obtain the corresponding subbundle of $J^1(T^*X)$:

$$H = \left\{ j_x^1\alpha \in J^1(T^*X) \colon (d\alpha)_x = 0 \right\} \subset J^1(T^*X)$$

Note that $C(P)/T^*X = X$. In order to study the quotient $J^1C(P)/H$ we need to consider generalized principal connections on $\pi_{C(P),X}$. The following lemma can be proved using local coordinates.

Lemma 7.1. *Let ν be a linear connection on $\pi_{T^*X,X}$, then it is Lie group connection. In addition, it induces an affine connection ω on $\pi_{C(P),X}$ which is a generalized principal connection associated to ν.*

Noting that $\mathfrak{g} = T^*X$, identification (10) reads $J^1(T^*X) \overset{\nu}{\simeq} T^*X \times_X (T^*X \otimes T^*X)$. If ν is such that $\hat{\nu}(T^*X) \subset H$ when regarded as a section $\hat{\nu} \in \Gamma\left(\pi_{J^1(T^*X),T^*X}\right)$, using local coordinates it can be seen that:

$$H \overset{\nu}{\simeq} T^*X \times_X \bigvee^2 T^*X \tag{29}$$

Proposition 7.1. *Let ω be the affine connection on $\pi_{C(P),X}$ induced by the linear connection ν on $\pi_{T^*X,X}$ and suppose that $\hat{\nu}(T^*X) \subset H$. Then the isomorphism of Theorem 3.1 is the identification of Utiyama's theorem (up to a minus sign), that is to say:*

$$\begin{array}{rcl} J^1C(P)/H & \longrightarrow & \bigwedge^2 T^*X \\ [j_x^1 A]_H & \longmapsto & \mathrm{Alt}\,(A^*\omega)_x = -(F_A)_x \end{array}$$

Proof. Observe that the fibered action (8) reads:

$$\begin{array}{rcl} (C(P) \times_X T^*X) \times_X T^*X & \longrightarrow & C(P) \times_X T^*X \\ ((A_x, \xi_x), \eta_x) & \longmapsto & (A_x + \eta_x, \xi_x) \end{array}$$

Subsequently, $T^*X \otimes \tilde{\mathfrak{g}} = T^*X \otimes T^*X$ and $\tilde{\mathfrak{H}} = \bigvee^2 T^*X$. Using this and recalling that $C(P)/T^*X \simeq X$, the isomorphism of Theorem 3.1 becomes:

$$J^1C(P)/H \simeq (T^*X \otimes T^*X) / \bigvee^2 T^*X \simeq \bigwedge^2 T^*X$$

The last isomorphism is the *skew-symmetrization*, i.e. $\mathrm{Alt}\colon T^*X \otimes T^*X \to \bigwedge^2 T^*X$.

To conclude, using local coordinates again it can be checked that $\mathrm{Alt}(A^*\omega) = -F_A$ for each $A \in \Gamma(\pi_{C(P),X})$. □

Therefore, for each $F_A \in \Gamma\left(\pi_{\bigwedge^2 T^*X,X}\right)$, the reduced Lagrangian is given by:

$$l(F_A) = g(F_A, F_A)$$

In this case, we only have one partial derivative[6]:

$$\frac{\delta l}{\delta F_A} = 2\,\iota_{F_A} g \in \Gamma\left(\pi_{\bigwedge^2 TX,X}\right)$$

Let $\nabla^*\colon \Gamma(\pi_{T^*X,X}) \to \Gamma(\pi_{T^*X\otimes T^*X,X})$ be the covariant derivative on $\pi_{T^*X,X}$ corresponding to the linear connection ν and $\operatorname{div}^*\colon \Gamma(\pi_{TX\otimes TX,X}) \to \Gamma(\pi_{TX,X})$ be its divergence. From Theorem 4.1 we know that if $A \in \Gamma(\pi_{C(P),X})$ is a solution of the Euler-Lagrange equations for L, then the reduced section $F_A \in \Gamma\left(\pi_{\bigwedge^2 T^*X,X}\right)$ satisfies the following reduced equation:

$$\operatorname{div}^*\left(\iota_{F_A} g\right) = 0 \tag{30}$$

Recall that the *Hodge star operator* $\star\colon \Omega^\bullet(X) \to \Omega^{4-\bullet}(X)$ is defined implicitly as:

$$\alpha \wedge \star\beta = g(\alpha, \beta)\, v_g, \qquad \alpha, \beta \in \Omega^\bullet(X),$$

and it satisfies $\star\star = (-1)^{p(4-p)}\epsilon(g)$ on $\Omega^p(X)$, being $\epsilon(g)$ the parity of the signature of g.

Theorem 7.3 (Maxwell equations)**.** *In the above conditions, the reduced equation* (30) *is equivalent to the* Maxwell equation in vacuum*, that is:*

$$d^\star F_A = 0,$$

where $d^\star = \star\, d\, \star\colon \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ *is the codifferential.*

The equivalence is proved in local charts, making use of the isomorphisms $\sharp\colon T^*X \to TX$ and $\flat\colon TX \to T^*X$ implicitly defined as

$$g\left(\alpha^\sharp, U\right) = g\left(\alpha, U_\flat\right) = \alpha(U), \qquad (\alpha, U) \in T^*X \times_X TX$$

Theorem 7.4 (Reconstruction)**.** *Let* $\mathcal{U} \subset X$ *be a simply connected coordinate domain for* X *which is a trivializing set for* $\pi_{C(P),X}$ *and let* $F \in \Gamma\left(\mathcal{U}, \pi_{\bigwedge^2 T^*X,X}\right)$ *be a solution of the Maxwell's equations in vacuum. Then there exists a solution* $A \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right)$ *of the Euler-Lagrange equations for* L *such that* $F = F_A$ *if and only if the following* compatibility condition *holds:*

$$dF = 0.$$

This is a consequence of the Bianchi identity for the curvature of a principal connection, the fact that closed forms and exact forms agree on a simply connected manifold, and the equivalence given in Theorem 4.1.

In short, we have the following local equivalence for sections $A \in \Gamma(\pi_{C(P),X})$:

$$\mathcal{EL}(L)\left(j^1 A\right) = 0 \qquad \Longleftrightarrow \qquad \begin{cases} d^\star F_A = 0 \\ dF_A = 0 \end{cases}$$

[6]We have that $\left(\bigwedge^2 T^*X\right)^* = \bigwedge^2 TX = \left(\bigvee^2 T^*X\right)^\perp = \left((T^*X \otimes T^*X)\big/\bigvee^2 T^*X\right)^*$

7.3.1 p-form electromagnetism

The construction above can be generalized to p-form electromagnetism (see for example [25]). Fixed $p \geq 0$, we consider $(p+1)$-forms on X that play the role of connections in classical electromagnetism. Therefore, the configuration bundle and the Lie group bundle of symmetries are both $Y = \mathcal{G} = \bigwedge^{p+1} T^*X$. The fibered action is given by the addition:

$$\begin{array}{ccc} \bigwedge^{p+1} T^*X \times_X \bigwedge^{p+1} T^*X & \longrightarrow & \bigwedge^{p+1} T^*X \\ (A_x, \alpha_x) & \longmapsto & A_x + \alpha_x \end{array}$$

A generalized principal connection $\omega \in \Omega^1\left(\bigwedge^{p+1} T^*X, \bigwedge^{p+1} T^*X\right)$ on $\pi_{\bigwedge^{p+1} T^*X,X}$ is just a linear connection on that vector bundle, and it is associated to itself. The corresponding isomorphism (10) is

$$\begin{array}{ccc} J^1\left(\bigwedge^{p+1} T^*X\right) & \longrightarrow & \bigwedge^{p+1} T^*X \oplus \left(T^*X \otimes \bigwedge^{p+1} T^*X\right) \\ j_x^1\alpha & \longmapsto & (\alpha(x), (\alpha^*\omega)_x) \end{array}$$

Analogous to classical electromagnetism, we pick the Lie group subbundle of closed forms and we restrict the previous isomorphism to it:

$$H = \left\{ j_x^1\alpha \in J^1\left(\bigwedge^{p+1} T^*X\right) : (d\alpha)_x = 0 \right\} \overset{\omega}{\simeq} \bigwedge^{p+1} T^*X \oplus \left(T^*X \vee \bigwedge^{p+1} T^*X\right)$$

Of course, $Y/\mathcal{G} = X$. Taking this into account and using local coordinates, it can be shown that the identification of Theorem 3.1 reads (compare to Proposition 7.1):

$$\begin{array}{ccc} J^1\left(\bigwedge^{p+1} T^*X\right)/\,H & \longrightarrow & \bigwedge^{p+2} T^*X \\ [j_x^1 A]_H & \longmapsto & \mathrm{Alt}(A^*\omega)_x = -(dA)_x \end{array}$$

The Yang-Mills Lagrangian $L\colon J^1\left(\bigwedge^{p+1} T^*X\right) \to \mathbb{R}$ is defined as:

$$L\left(j^1 A\right) = g(dA, dA), \qquad A \in \Gamma\left(\pi_{\bigwedge^{p+1} T^*X,X}\right)$$

It is H-invariant, so we may consider the reduced Lagrangian $l\colon \bigwedge^{p+2} T^*X \to \mathbb{R}$. Namely, it is given by:

$$l(C) = g(C, C), \qquad C \in \Gamma\left(\pi_{\bigwedge^{p+2} T^*X,X}\right)$$

Fixed $C \in \Gamma\left(\pi_{\bigwedge^{p+2} T^*X,X}\right)$, the partial derivative is $\delta l/\delta C = 2\,\iota_C g \in \Gamma\left(\pi_{\bigwedge^{p+2} TX,X}\right)$ and, hence, the reduced equation is:

$$\mathrm{div}^*\left(\iota_C g\right) = 0$$

In the previous expression, $\mathrm{div}^*\colon \Gamma\left(\pi_{TX\otimes\bigwedge^{p+1} TX}\right) \to \Gamma\left(\pi_{\bigwedge^{p+1} TX,X}\right)$ is the divergence of the linear connection ω. Similarly to classical electromagnetism, this reduced equation is equivalent to the Maxwell equations:

$$d^\star C = 0$$

7.4 Symmetry breaking by product groups

In this last example, we consider a gauge theory whose structure group is a direct product and we suppose that the gauge symmetry is broken to the subgroup given by one of the factors. More specifically, let $\pi_{P,X}$ be a principal G-bundle with

$$G = H \times U(1),$$

being H a semisimple Lie group. We consider the Lie subgroup $G_0 = \{1\} \times U(1) \simeq U(1)$. Since action of G on G_0 by conjugation is trivial, we have $Ad(P) = (P \times G)/G \simeq (P \times H)/G \times U(1)$. Hence, we may write:

$$J^1 Ad(P) \simeq J^1 \left(\frac{P \times H}{G} \right) \times_X J^1(X, U(1))$$

As a usual gauge theory, the configuration bundle is the bundle of connections of $\pi_{P,X}$, that is, $\pi_{C(P),X}$. However, we suppose that the symmetry is broken to G_0, i.e. we only consider gauge transformations coming from elements of this subgroup. In other words, we restrict the right fibered action $C(P) \times_X J^1 Ad(P) \to C(P)$ to the Lie group subbundle $J^1(X, U(1)) \simeq \{1\} \times_X J^1(X, U(1)) \subset J^1 Ad(P)$. A quick computation shows that it is given by:

$$\begin{array}{ccc} C(P) \times_X J^1(X, U(1)) & \longrightarrow & C(P) \\ (A_x, j_x^1 g) & \longmapsto & A_x + (dg)_x g(x)^{-1} \end{array}$$

In the same manner as in electromagnetism in vacuum, $\mathcal{G}_0 = T^*X$ may be taken as the Lie group bundle of symmetries:

$$\begin{array}{ccc} C(P) \times_X T^*X & \longrightarrow & C(P) \\ (A_x, \alpha_x) & \longmapsto & A_x + \alpha_x \end{array} \tag{31}$$

A trivialization of $\pi_{P,X}$ enables us to prove the following:

Lemma 7.2. *Let $P_0 = P/G_0$ (which is a principal H-bundle over X), $\pi_{C(P_0),X}$ be the corresponding bundle of connections and $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ be the Lie algebra of $G = H \times U(1)$. Then there exists a bundle isomorphism:*

$$\begin{array}{ccc} C(P)/T^*X & \longrightarrow & C(P_0) \\ [A_x]_{T^*X} & \longmapsto & (A_0)_x \end{array}$$

On the other hand, from the jet extension of the fibered action (31), we are only interested in elements coming from $J^2(X, U(1)) \subset J^1(J^1(X, U(1)))$, which correspond to the Lie group subbundle:

$$H = \left\{ j_x^1 \alpha \in J^1(T^*X) \colon (d\alpha)_x = 0 \right\} \subset J^1(T^*X)$$

Let ν be a linear connection on $\pi_{T^*X,X}$ such that $\hat{\nu}(T^*X) \subset H$ and note that the Lie algebra bundle of $\mathcal{G}_0 = T^*X$ is again $\mathfrak{g}_0 = T^*X$. Hence, $\tilde{\mathfrak{g}}_0 = (C(P) \times_X T^*X)/T^*X \simeq C(P_0) \times_X T^*X = \pi^*_{C(P_0),X}(T^*X)$. Equation (29) is also valid for this case, so we have $\mathfrak{H} = \bigvee^2 T^*X$. A slight modification of the proof of Proposition 7.1 gives the following result.

Proposition 7.2. *Let $\omega \in \Omega^1(C(P), T^*X)$ be a generalized principal connection on $\pi_{C(P),C(P_0)}$ associated to ν. Then the identification of Theorem 3.1 reads:*

$$\begin{array}{ccc} J^1C(P)/H & \longrightarrow & J^1C(P_0) \times_X C(P_0) \times_X \bigwedge^2 T^*X \\ [j^1_x A]_H & \longmapsto & (j^1_x A_0, A_0(x), -(F_A)_x) \end{array}$$

Thanks to this identification, for each section $A \in \Gamma(\pi_{C(P),X})$ we define the reduced section as

$$\overline{A} = (A_0, F_A) \in \Gamma\left(\pi_{C(P_0)\times_X \bigwedge^2 T^*X, X}\right)$$

Let $L\colon J^1C(P) \to \mathbb{R}$ be an H-invariant Lagrangian density and consider the reduced Lagrangian, $l\colon J^1C(P_0)\times_X C(P_0)\times_X \bigwedge^2 T^*X \to \mathbb{R}$. Let ∇^0 be a torsion free linear connection on $T(C(P_0))$ projectable onto a linear connection ∇^X on TX. We know that it induces an affine connection $\nabla^{(1)}$ on $\pi_{J^1C(P_0),C(P_0)}$. In addition, we assume that ν is the dual connection of ∇^X. These connections induce an affine connection on the reduced space, as described in Proposition 3.3. Let $\overline{A} = (A_0, F_A) \in \Gamma\left(\pi_{C(P_0)\times_X \bigwedge^2 T^*X, X}\right)$ be a section of the reduced space, then the partial derivatives of the reduced Lagrangian are[7]:

$$\frac{\delta l}{\delta A_0} \in \Gamma\left(\pi_{T^*(C(P_0)),X}\right), \quad \frac{\delta l}{\delta j^1 A_0} \in \Gamma\left(\pi_{TX\otimes V^*(C(P_0)),X}\right), \quad \frac{\delta l}{\delta F_A} \in \Gamma\left(\pi_{\bigwedge^2 TX, X}\right)$$

Let $\nabla^*\colon \Gamma(\pi_{T^*X,X}) \to \Gamma(\pi_{T^*X\otimes T^*X,X})$ be the covariant derivative associated to ν and denote by $\operatorname{div}^*\colon \Gamma(\pi_{TX\otimes TX,X}) \to \Gamma(\pi_{TX,X})$ its divergence. Likewise, let $\operatorname{div}^0\colon \Gamma(\pi_{TX\otimes V^*(C(P_0)),X}) \to \Gamma(\pi_{V^*(C(P_0)),X})$ be the divergence of the operator $\overline{\nabla}^0$ defined from ∇^0 in (18). The reduced equations are straightforwardly obtained from *(iv)* of Theorem 4.1 applied to this case.

Theorem 7.5. *In the above conditions, let $A \in \Gamma(\pi_{C(P),X})$ be a solution of the Euler-Lagrange equations for L. Then the reduced section $\overline{A} = (A_0, F_A) \in \Gamma(\pi_{C(P_0)\times_X \bigwedge^2 T^*X, X})$ satisfies the following reduced equations:*

$$\begin{cases} \dfrac{\delta l}{\delta A_0} - \operatorname{div}^0\left(\dfrac{\delta l}{\delta j^1 A_0}\right) = \left\langle \dfrac{\delta l}{\delta F_A}, \iota_{dA_0}\tilde{\Omega} \right\rangle \\ \operatorname{div}^*\left(\dfrac{\delta l}{\delta F_A}\right) = 0 \end{cases}$$

where $\tilde{\Omega} \in \Omega^2(C(P_0), T^*X)$ is the reduced curvature of ω.

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[7]Recall that $\left(\bigwedge^2 T^*X\right)^* = \bigwedge^2 TX = \left((T^*X \otimes T^*X)\big/ \bigvee^2 T^*X\right)^*$.

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