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Causality of a wave equation and invariance of its hyperbolicity conditions

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A characterization of the noncausal behavior of a covariant wave equation is given in terms of the invariance of the hyperbolicity conditions. Some examples in which the Dirac field is not causal are studied.

I. INTRODUCTION

Velo and Zwanziger showed in 1969 that a Lorentz-covariant theory does not automatically satisfy the theory of special relativity.[1] They found that the Rarita-Schwinger field propagates faster than light in an external electromagnetic field, which implies a violation of causality in the Einstein sense. Later on the same phenomenon was found in other cases.[2] In spite of this problem they were able to present a quantum interpretation of the Rarita-Schwinger equation for weak enough electromagnetic fields.

In this paper we show that there is a close relation between causality and invariance of the hyperbolicity conditions on the external fields. This allows an easy characterization of the equations which can exhibit superluminality. These results are contained in Sec. II.

In Sec. III we present a generalization of the concept of the hyperbolicity equation which is useful for time-reversal-invariant equations. When the Rarita-Schwinger equation ceases to be hyperbolic in the usual since (for strong enough fields), it is still hyperbolic in the new sense, which is naturally invariant. This allows one to understand more deeply the Velo-Zwanziger phenomenon.

In Sec. IV we give some examples which show that the Dirac equation can present noncausal behavior with derivative couplings to external fields. Finally in Sec. V we state the conclusions.

II. CAUSALITY AND THE INVARIANCE OF THE HYPERBOLICITY CONDITIONS

In the case of the Rarita-Schwinger equation in an external electromagnetic field, Velo and Zwanziger showed[1] that the equation which determines the normals to the characteristic surfaces is

$$D(n)=(n^2)^4\left[n^2+\left(\frac{2e}{3m^2}\right)^2(\tilde{F}\cdot n)^2\right]^4=0\,, \qquad (1)$$

where $\tilde{F}$ is the dual of the electromagnetic tensor. The equation will be hyperbolic[3] if for any unit space vector $\vec{n}$ all the values of n_0 solutions of (1) are real, and the theory will be causal if all satisfy $|n_0|\leq 1$. The reason is that this guarantess that in any system the causes will precede the effects. Velo and Zwanziger considered the "weak-field case" in which

$$\left(\frac{2e}{3m^2}\vec{B}\right)^2<1\,, \qquad (2)$$

where $\vec{B}$ is the magnetic field. Curiously enough, this turns out to be one of the hyperbolicity conditions, as is proved in Sec. IV, and it is not Lorentz invariant. This is really surprising since there always exist inertial frames in which (2) does not hold. This implies that the equation is hyperbolic in some systems but not in others. As the causal character of a theory is closely related to the hyperbolicity of its equation and cannot depend on the selected inertial frame, it seems that the Rarita-Schwinger equation in an external field cannot be causal. In fact, Velo and Zwanziger found that this is the case because some of the characteristic velocities are greater than the velocity of light. We will show in the following that this situation is completely general. Any field equation whose hyperbolicity condition is not invariant exhibits superluminal behavior and is not causal. In other words, to construct a relativistic theory covariant equations are not enough. It is necessary that the hyperbolicity condition be invariant. The reason is, perhaps, that the formulation of the wave propagation can break the invariance of the theory by choosing a special direction whose coordinate is taken as the evolution parameter.

Let us consider the general-relativistic form-invariant equation for arbitrary spin

$$[\Gamma^\mu(u)\partial_\mu+B(u)]\psi(x)=0\,, \qquad (3)$$

where u is an external field, Γ^μ and B are $m\times m$ matrices, and ψ is a field with m components, which transforms as follows, under a transformation (a,Λ) of the Poincaré group $\mathcal{P}=T_4\dot{\times}\mathcal{L}$:

$$\psi'(x)=S(\Lambda)\psi(\Lambda^{-1}(x-a))\,,$$

$$S(\Lambda)^{-1}\Gamma^\mu(u)S(\Lambda)=\Lambda^\mu{}_\nu\Gamma^\nu(u')\,,$$

$$S(\Lambda)^{-1}B(u)S(\Lambda)=B(u'),$$

where $S(\Lambda)$ is a representation of the Lorentz group $\mathfrak{L}$.

This equation is hyperbolic in a frame K if all the roots $n_0^{(1)}, n_0^{(2)}, \ldots, n_0^{(m)}$ of the characteristic equation

$$Q(n,u)=|\Gamma^{\mu}(u)n_{\mu}|=0 \tag{4}$$

are real for any unit vector $\vec{n}$. This imposes some limitations on the field u, which we call hyperbolicity conditions.

We will prove the following theorem: The hyperbolicity conditions of (3) are invariant if and only if the characteristic velocities satisfy $n_0^2 \leq 1$ for any space direction, or in other words, if and only if the theory is causal.

Before proceeding to prove the theorem let us introduce some definitions. If s is a vector directed along the time axis of the frame K, $\Omega_0(s)$ is the set of all the external-field configurations such that the hyperbolicity conditions are satisfied.

N_u is the cone formed by the vectors n which are solutions of (4) for a given u. It is clear that $0 \in N_u$ and that if $n \in N_u$, then $\lambda n \in N_u$.

As $Q(n,u)$ is a polynomial of degree m in the vector n^{μ}, it can be written as

$$Q(n,u)=a(n_0-n_0^{(1)})\cdots(n_0-n_0^{(m)}) \tag{5}$$

and the set N_u is the union

$$N_u=\bigcup_1^m N_u^{(k)}, \tag{6}$$

where $N_u^{(k)}$ is the cone corresponding to the kth root.

If $u \in \Omega_0(s)$ any root $n_0^{(k)}$ of (4) defines a mapping of $s_\perp$ on $s_\parallel$ as follows:

$$s_\perp \to s_\parallel,$$

$$r \to \lambda(r)s,$$

where $r+\lambda(r)s \in N_u^{(k)}$. This mapping characterizes the hyperbolicity of the equation.

Given a cone C we define its hyperbolic interior inth C as the set of all the non-light vectors t such that to any $t' \in t_\perp$ we can associate $\lambda t \in t_\parallel$ such that $\lambda t + t' \in C$. It consists of the interior of C in the usual sense except for the lightlike vectors. It is clear that if $u \in \Omega_0(s)$, then

$$s \in \bigcap_1^m \mathrm{inth} N_u^{(k)} \equiv \mathrm{inth} N_u. \tag{7}$$

The inverse in also satisfied.

We proceed now to prove the theorem. Let us assume, first, that the hyperbolicity conditions are invariant. In that case, if $u \in \Omega_0(s)$ and Λ^{-1} is a Lorentz transformation then $\Lambda^{-1}u \in \Omega_0(s)$.

The equation

$$Q(r+\lambda(r)s, \Lambda^{-1}u)=0 \tag{8}$$

has a real solution $\lambda(r)$ for any $r \in s_\perp$. Its invariance implies

$$Q(r+\lambda(r)s, \Lambda^{-1}u)=Q(\Lambda(r+\lambda(r)s),u), \tag{9}$$

from which it follows that $u \in \Omega_0(\Lambda s)$ or, in other words, that Ω_0 does not depend on the elected frame. This has an important consequence: if $s \in \mathrm{inth}N_u$, then $\Lambda s \in \mathrm{inth}N_u$.

If N_0 is the light cone, $\mathrm{inth}N_0=\{\Lambda s; \Lambda \in \mathfrak{L}\}$ and this implies

$$\mathrm{inth}N_0 \subseteq \mathrm{inth}N_u \subseteq \mathrm{inth}N_u^{(k)}, \quad k=1,\ldots,m$$

which means that all the vectors in $N_u^{(k)}$ are spacelike or lightlike. The characteristic velocities satisfy then $n_0^{(k)2} \leq 1$ and the theory is causal.

To complete the proof of our theorem we will show that if the hyperbolicity condition is not invariant there exists superluminality, which is equivalent to proving that if the theory is causal the hyperbolicity condition must be invariant.

If $u \in \Omega_0(s)$, there exists a Lorentz transformation Λ such that $\Lambda^{-1}u \notin \Omega_0(s)$ or equivalently, $u \notin \Omega_0(\Lambda s)$. This implies that

$$\Lambda s \notin \mathrm{inth}N_u$$

and

$$\mathrm{inth}N_0 \not\subseteq \mathrm{inth}N_u,$$

which means that some vector in N_u is timelike and the corresponding characteristic velocity is greater than one. In other words, we cannot have simultaneously causality and noninvariance of the hyperbolicity condition. This completes the proof.

III. CAUSALITY AND TIME-REVERSAL INVARIANCE

In physics the concept of hyperbolicity is defined with respect to a timelike coordinate which is taken as the evolution parameter. We will show now that, in some cases in which a field equation ceases to be hyperbolic in the usual sense (as in the Velo-Zwanziger case for large enough magnetic fields), it can still be considered hyperbolic in a more general sense, in which the evolution parameter can be a spacelike coordinate. This will be the case if the theory is invariant under time reversal.

Equation (3) is said to be invariant under the time-reversal transformation

$$t \to t'=-t, \quad \vec{x} \to \vec{x}'=\vec{x}, \quad u \to u' \tag{10}$$

if there exists a matrix T such that

$$\begin{aligned} T\Gamma_\mu^*(u)T^{-1} &= g_{\mu\mu}\Gamma_\mu(u'), \\ TB^*(u)T^{-1} &= -B(u'). \end{aligned} \tag{11}$$

The transformed field is $\psi'(x') = T\psi^*(x)$. If this is the case, the characteristic polynomial Q transforms as follows:

$$\begin{aligned} Q(n,u) &= |\Gamma^\mu(u)n_\mu| = |\Gamma^{\mu *}(u)n_\mu|^* \\ &= (-1)^m Q^*(n',u'). \end{aligned} \tag{12}$$

Q and Q^* can be written as

$$\begin{aligned} Q(n,u) &= a(u)\prod_1^m [n_0 - n_0^{(j)}(\vec{n},u)], \\ (-1)^m Q^*(n',u') &= a^*(u')\prod_1^m [n_0 + n_0^{(j)*}(\vec{n},u')]. \end{aligned} \tag{13}$$

If the equation is invariant under (10), $a(u) = a^*(u')$ and the sets $\{n_0^{(k)}(\vec{n},u)\}$ and $\{-n_0^{(k)*}(\vec{n},u')\}$ are the same one. This allows one to pair all the roots if m is even, while if m is odd there is at least one root such that $n_0^{(k)}(\vec{n},u) = -n_0^{(k)*}(\vec{n},u')$. We consider only the case of even m which is, in fact, the interesting one. In that case,

$$\begin{aligned} Q(n,u) = a(u)\prod_1^{m/2} &[n_0 - n_0^{(k)}(\vec{n},u)] \\ &\times [n_0 + n_0^{(k)*}(\vec{n},u')]. \end{aligned} \tag{14}$$

Since $n_0^{(k)}$ is homogeneous of first degree in $\vec{n}$ it is convenient to write

$$Q(n,u) = \prod_1^{m/2} \bar{g}^{(k)}(n,n), \tag{15}$$

where

$$\bar{g}^{(k)}(n,n) = a^{(k)}(u)[n_0 - n_0^{(k)}(\vec{n},u)][n_0 + n_0^{(k)}(\vec{n},u')]$$

and (16)

$$a(u) = a^{(1)}(u)\cdots a^{(m/2)}(u).$$

Let us assume that we can choose $a^{(k)}(u)$ in such a way that $\bar{g}^{(k)}(n,n)$ is invariant. This is the case in all the cases studied thus far.

When there is no external field

$$\bar{g}_{\mu\nu}(0) = g_{\mu\nu}, \quad Q(n,0) = (n^2)^{m/2}.$$

This shows that $\bar{g}^{(k)}(u)$ plays the role of metric tensor associated with the kth mode of propagation. A particular example of $\bar{g}^{(k)}$ has been considered by Zwanziger[4] who uses the name "extraordinary metric tensor."

If $\bar{g}^{(k)}(n,n) = 0$ for $k = 1,\ldots,m/2$ implies that n_0 is real for any $\vec{n}$, the equation is hyperbolic. A very curious situation arises when this is not the case but all the tensors are of signature $(3-1)$ as $g_{\mu\nu}$. In that case it is clear that the $m/2$ modes evolve hyperbolically although the evolution parameter may be a spacelike coordinate and can be different for different modes. As we will see, this is what happens in the Velo-Zwanziger case. This is the reason for the following definition.

A field equation whose characteristic polynomial can be expressed as

$$Q(n,u) = \prod_1^{m/2} \bar{g}^{(k)}(n,n), \tag{17}$$

where $\bar{g}^{(k)}$ are symmetric tensors depending on the external fields u, is hyperbolic in a generalized sense if the tensors $\bar{g}^{(k)}$ are real and of signature $(3-1)$ and if the eigenvectors w_j of the problem

$$(\Gamma^\mu(u)n_\mu)w_j = 0, \quad j = 1,\ldots,m$$

are linearly independent.

In contrast to the usual definition, this one is clearly invariant. If an equation is hyperbolic in the sense of this definition in a field u, it will also be so in $u' = \Lambda u$ if $\Lambda \in \mathcal{L}$. This is a consequence of the invariance of the signature of the matrices. For this reason we could also call this property intrinsic hyperbolicity.

If the signature of $\bar{g}^{(k)}$ is $(+,-,-,-)$, $\forall\, k$, we define a propagation vector s of the kth mode by the property

$$\bar{g}^{(k)}(s,s) > 0.$$

If any timelike vector defines a propagation direction for all the modes, it is clear that if $\bar{g}^{(k)}(n,n) = 0$ for a value of k, then $n^2 \leq 0$ and the theory will be causal. If, on the other hand, there is a timelike direction s which does not define a propagation direction for the kth mode, there exists a timelike vector n such that $\bar{g}^{(k)}(n,n) = 0$ and the theory is not causal.

It follows that a theory will be causal if and only if

$$\bar{g}^{(l)}(u)(\Lambda s, \Lambda s) > 0, \quad \forall \Lambda \in \mathcal{L}, \quad l = 1,\ldots,m/2 \tag{18}$$

where s is a timelike vector or, equivalently,

$$\bar{g}^{(l)}(\Lambda u)(s,s) > 0, \quad \forall \Lambda \in \mathcal{L}, \quad l = 1,\ldots,m/2. \tag{19}$$

Choosing $s = (1,\vec{0})$ we obtain a very simple condition which guarantees the causality of the equation:

$$\bar{g}_{00}^{(l)}(\Lambda u) > 0, \quad \forall \Lambda \in \mathcal{L}, \quad l = 1,\ldots,m/2. \tag{20}$$

In the initial frame this condition is formulated as

$$\bar{g}_{00}^{(l)}(u) > 0 , \quad l = 1, \ldots, m/2 . \tag{21}$$

If this condition is Lorentz invariant, it implies (20) and the theory is causal. If this is not the case, there will be superluminality. In the case of signature $(-, +, +, +)$ we can make similar considerations with some obvious changes in the signs.

IV. SOME APPLICATIONS

We will now apply the preceding ideas to the Dirac equation in an external field showing some parallelism with the Rarita-Schwinger case.

The minimal electromagnetic coupling or the Pauli coupling does not affect the principal part of the Dirac equation. The theory is hyperbolic with characteristic velocities ± 1, as in the free case. This is in sharp contrast to the higher-spin case in which the existence of constraints may make the hyperbolicity noninvariant, as was shown by Velo and Zwanziger. Nevertheless, we consider a more general class of coupling to an external field which may lead to superluminality. The minimal coupling consists of the substitution

$$i\partial_\mu \to i\partial_\mu - eA_\mu .$$

Let us consider the general substitution

$$\gamma^\mu \to \Gamma^\mu = \gamma^\mu + \Delta^\mu , \tag{22}$$

where Δ^μ depend on the exterior fields and vanish in the free case. The more general expression for Δ^μ is

$$\Delta^\mu = A^\mu + B^\mu{}_\nu \gamma^\nu + C^\mu{}_{\nu\lambda}\sigma^{\nu\lambda} + D^\mu{}_\nu \gamma^5\gamma^\nu + E^\mu \gamma^5 , \tag{23}$$

where the fields $A, B, \ldots$ must have the appropriate tensorial character.

Let us consider some simple cases.

Case (a):

$$\Gamma^\mu = \gamma^\mu + \lambda F^\mu{}_\nu \gamma^\nu + \lambda' \tilde{F}^\mu{}_\nu \gamma^5 \gamma^\nu , \tag{24}$$

where λ, λ' are coupling constant, F is an antisymmetric tensor field, and $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. The characteristic polynomial is

$$Q(n, F) = [n^2 + (\lambda n F)^2 + (\lambda' n \tilde{F})^2]^2 - (\tfrac{1}{2}\lambda\lambda' n^2 F \cdot \tilde{F})^2 , \tag{25}$$

which can be written as

$$Q(n, F) = \bar{g}^{(1)}(n, n)\bar{g}^{(2)}(n, n) , \tag{26}$$

where

$$\bar{g}_{\mu\nu}^{(1,2)} = g_{\mu\nu} - \lambda^2 F_{\mu\lambda}F^\lambda{}_\nu - \lambda'^2 \tilde{F}_{\mu\lambda}\tilde{F}^\lambda{}_\nu \pm \tfrac{1}{2}\lambda\lambda' g_{\mu\nu} F^{\lambda\rho}\tilde{F}_{\lambda\rho} . \tag{27}$$

We require n_0 to be real for any $\vec{n}$, so that the characteristic equation is satisfied:

$$Q(n, F) = 0 . \tag{28}$$

It is equivalent to require that

$$\bar{g}^{(l)}(n, n) = 0 , \quad l = 1 \text{ or } 2 . \tag{29}$$

These are second-order algebraic equations for n_0, which have real solutions for any n in the following cases:

(1) If $\lambda^2 + \lambda'^2 = 0$, one coupling constant is real and the other is purely imaginary; one obtains

$$Q(n, F) = \{[1 + \lambda^2(\vec{B}^2 - \vec{E}^2)]^2 + 4\lambda^4(\vec{E} \cdot \vec{B})^2\} n^4 . \tag{30}$$

The equation is causal $n_0 = \pm 1$ for any $F_{\mu\nu}$ which does not make $Q(n, F)$ singular.

(2) If $\lambda\lambda'$ is real, we obtain hyperbolicity if the following is satisfied,

$$\bar{g}_{00}^{(l)} = 1 - \lambda^2\vec{E}^2 - \lambda'^2\vec{B}^2 \pm 2\lambda\lambda'\vec{E} \cdot \vec{B} \neq 0 , \quad l = 1, 2 \tag{31}$$

and if the symmetric matrix h defined as

$$h_{ij}^{(l)} = \bar{g}_{0i}^{(l)}\bar{g}_{0j}^{(l)} - \bar{g}_{00}^{(l)}\bar{g}_{ij}^{(l)} , \quad l = 1, 2 \tag{32}$$

is positive definite, which is equivalent to the following invariant conditions:

$$\begin{aligned} I_{1,2} = {} & [1 \pm 2\lambda\lambda'(\vec{E} \cdot \vec{B})^2][1 + (\lambda^2 - \lambda'^2)(\vec{B}^2 - \vec{E}^2)] \\ & - (\lambda^2 + \lambda'^2)^2(\vec{E} \cdot \vec{B})^2 \\ & + (\lambda\lambda')^2[4(\vec{E} \cdot \vec{B})^4 - (\vec{E}^2 - \vec{B}^2)^2] > 0 , \end{aligned} \tag{33}$$

$$J_{1,2} = 2 + (\lambda^2 - \lambda'^2)(\vec{B}^2 - \vec{E}^2) \pm 4\lambda\lambda'\vec{E} \cdot \vec{B} > 0 . \tag{34}$$

The theory will be causal if the condition (31) is invariant or, equivalently, if the equation

$$\bar{g}_{00}^{(l)} = 0 , \quad l = 1, 2 \tag{35}$$

expresses an invariant condition on the external fields.

From the study of $\bar{g}_{00}^{(l)}$ it is clear that the theory will be causal in the following cases:

(a) if $\lambda(\lambda')$ vanishes and $\lambda'(\lambda)$ is purely imaginary for any $F_{\mu\nu}$ satisfying (33) and (34), and

(b) if $\lambda\lambda' \neq 0$ and λ and λ' are purely imaginary for any $F_{\mu\nu}$ satisfying (33) and (34).

In all the other cases the theory is not causal.

If we make use of the definition of hyperbolicity in the general sense, it suffices to require $\bar{g}^{(l)}$ to have the negative determinant

$$\det \bar{g}^{(l)} = -I_l{}^2 < 0 , \quad l = 1, 2 \tag{36}$$

which is satisfied if

$$I_l \neq 0 , \quad l = 1, 2 . \tag{37}$$

The equation obtained by Velo and Zwanziger for the Rarita-Schwinger field coupled with an external electromagnetic potential exhibits noncausal behavior, in close correspondence to our tensor coupling of the Dirac equation with $\lambda = 0$ and $\lambda' = \frac{2}{3} e m^{-2}$.

The hyperbolicity conditions for the noncausal modes of propagation are

$$\bar{g}_{00}^{(1,2)} = 1 - \left(\frac{2e}{3m^2}\vec{B}\right)^2 > 0\,, \tag{38}$$

$$\begin{aligned} I &= I_{(1,2)} \\ &= 1 + \left(\frac{2e}{3m^2}\right)^2 (\vec{E}^2 - \vec{B}^2) - \left(\frac{2e}{3m^2}\right)^4 (\vec{E}\cdot\vec{B})^2 > 0\,. \end{aligned} \tag{39}$$

Note that in this case the condition (33) implies (34). The first condition is clearly noninvariant in close relation to the noncausality of the equation.

If we pass from a frame where (38) and (39) hold to another where (38) does not hold, then the new time axis ceases to be a direction of propagation. The new one lies in the plane formed by the time axis and the perpendicular to $\vec{E}$ and $\vec{B}$, as a long but straightforward calculation shows. In a given Lorentz frame we can choose the following direction of propagation:

$$v = \left(1, -\left(\frac{2e}{3m^2}\right)^2 \frac{\vec{E}\times\vec{B}}{1 + \tau + (2e/3m^2)^2 \vec{E}^2}\right),$$

where

$$\tau = -\left(\frac{2e}{3m^2}\right)^2 T^{00} + \left[\left(\frac{2e}{3m^2}\right)^4 (T^{00})^2 + I\right]^{1/2}$$

is the eigenvalue of v being considered as an eigenvector of the matrix $\bar{g}$. Note that v does not transform as a vector, and $T^{00} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$.

If $I < 0$ the equation is not hyperbolic in the time direction but it is so in a spacelike direction. Let us take as an example the case wherein $\vec{E}$ and $\vec{B}$ are parallel along the z axis. In this case $\bar{g}_{\mu\nu}$ is diagonal:

$$\begin{aligned} \bar{g}_{\mu\nu} = \Bigg(1 - \left(\frac{2e}{3m^2}\vec{B}\right)^2, -\left[1 + \left(\frac{2e}{3m^2}\vec{E}\right)^2\right], \\ -\left[1 + \left(\frac{2e}{3m^2}\vec{E}\right)^2\right], -1 + \left(\frac{2e}{3m^2}\vec{B}\right)^2\Bigg). \end{aligned} \tag{40}$$

In passing from the weak-field case $[(2e/3m^2)\vec{B}]^2 < 1$ to the strong-field case $[(2e/3m^2)\vec{B}]^2 > 1$, the coordinate of propagation passes from t to z. In the last case waves propagate along the space direction defined by the magnetic field.

This phenomenon is formally similar to the interchange of the time and the real coordinates when one crosses the Schwarzschild radius[5] in general relativity. It is interesting to remark that time-reversal invariance implies that λ' is real and λ is purely imaginary.

Case (*b*):

$$\Gamma^\mu = \gamma^\mu + \lambda A_\nu \sigma^{\mu\nu} + \lambda' A^\mu\,, \tag{41}$$

where λ and λ' are coupling constants and A_μ is a vector field.

The characteristic polynomial is

$$Q(n,A) = [(1 + \lambda^2 A^2) n^2 - (\lambda^2 + \lambda'^2)(n\cdot A)^2]^2 \tag{42}$$

which can be written as

$$Q(n,A) = (n\bar{g}n)^2\,, \tag{43}$$

where

$$\bar{g}_{\mu\nu} = (1 + \lambda^2 A^2) g_{\mu\nu} - (\lambda^2 + \lambda'^2) A_\mu A_\nu\,. \tag{44}$$

The hyperbolicity conditions are obtained in the same way as in the tensor coupling:

$$(1 + \lambda^2 A^2)\bar{g}_{00} > 0\,, \tag{45}$$

$$I = (1 + \lambda^2 A^2)(1 - \lambda'^2 A^2) > 0\,. \tag{46}$$

If $1 + \lambda^2 A^2 > 0$, the signature of the $\bar{g}$ tensor is $(+,-,-,-)$ and if $1 + \lambda^2 A^2 < 0$, the signature is $(-,+,+,+)$. This explains the appearance of the factor $1 + \lambda^2 A^2$ in (45). The change of the signature of the tensor $\bar{g}$, depending on the values of the external fields, did not appear in the tensor coupling unless $I = 0$. In this case the signature of $\bar{g}$ was $(0,-,-,0)$. From the study of (45) and (46) we obtain the same cases (1) and (2) as in the tensor coupling.

Case (*c*):

$$\Gamma^\mu = (1 + \lambda\phi)\gamma^\mu + \lambda'\phi'\gamma^5\gamma^\mu\,, \tag{47}$$

where ϕ is a scalar field, ϕ' is a pseudoscalar field, and λ, λ' are coupling constants.

The characteristic polynomial is

$$Q(n, \phi, \phi') = [(1 + \lambda\phi)^2 - (\lambda'\phi')^2]^2 n^4\,. \tag{48}$$

The hyperbolicity condition is

$$(1 + \lambda\phi)^2 - (\lambda'\phi')^2 \neq 0\,. \tag{49}$$

In this case the equation is causal and the characteristic velocities are $n_0 = \pm 1$. The external fields do not disturb the propagation character of the initial equation.

V. SUMMARY AND CONCLUSIONS

We summarize with the following three points:

(1) We have shown that the necessary and sufficient condition for a relativistic form-invariant equation to be causal is the Lorentz invariance of hyperbolicity conditions. The superluminality found by Velo and Zwanziger and other authors in the Rarita-Schwinger equation and other equations will appear whenever the hyperbolicity conditions are not invariant. In other words, to

obtain a relativistic theory we need two elements: form invariance of the field equations and invariance of the hyperbolicity conditions.

(2) In some cases it is useful to extend the definition of hyperbolicity in a way that is naturally invariant. It happens that some equations are not longer hyperbolic in the usual sense, when passing to a new Lorentz frame or when the exterior fields are strong enough. However, they are still hyperbolic in the generalized sense, the evolution parameter being a spacelike coordinate which could depend on the propagation mode. This is what happens in the Velo-Zwanziger case.

(3) The Dirac equation can exhibit superluminality with derivative couplings to external fields. Some examples were given.

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