ARITHMETIC MOTIVIC POINCARÉ SERIES OF TORIC VARIETIES

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Abstract. The *arithmetic motivic Poincaré series* of a variety V defined over a field of characteristic zero, is an invariant of singularities which was introduced by Denef and Loeser by analogy with the Serre-Oesterlé series in arithmetic geometry. They proved that this motivic series has a rational form which specializes to the Serre-Oesterlé series when V is defined over the integers. This invariant, which is known explicitly for a few classes of singularities, remains quite mysterious. In this paper we study this motivic series when V is an affine toric variety. We obtain a formula for the rational form of this series in terms of the Newton polyhedra of the *ideals of sums of combinations* associated to the minimal system of generators of the semigroup of the toric variety. In particular, we deduce explicitly a finite set of candidate poles for this invariant.

Introduction

Let S denote an irreducible and reduced algebraic variety defined over a field k of characteristic zero. The set $H(S)$ of formal arcs of the form Spec $k[[t]] \to S$ can be given the structure of scheme over k (not necessarily of finite type). If $0 \in S$ we denote by $H(S)_0$ the subscheme of the arc space consisting on arcs in $H(S)$ with origin at 0. The set $H_m(S)$ of m-jets of S, of the form Spec $k[t]/(t^{m+1}) \to S$, has the structure of algebraic variety over k. By a theorem of Greenberg, the image of the space of arcs $H(S)$ by the natural morphism of schemes $j_m : H(S) \to H_m(S)$ which maps any arc to its m-jet, is a constructible subset of $H_m(S)$.

It follows from this that $j_m(H(S))$ defines a class $[j_m(H(S))]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ and also a class $\chi_c([H_m(S)]) \in K_0(\mathrm{CHMot}_k)$ in the Grothendieck ring of Chow motives, where $\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$ is the unique ring homomorphism, which maps the class of a smooth projective variety to its Chow motive (see [12, 14]). We denote by $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ the image of $K_0(\mathrm{Var}_k)$ by the homomorphism χ_c. We use the same symbol $\mathbf{L}$ to denote the class $[\mathbf{A}^1_k] \in K_0(\mathrm{Var}_k)$ and the class $\chi_c([\mathbf{A}^1_k]) \in K_0^{\mathrm{mot}}(\mathrm{Var}_k)$.

Denef and Loeser have defined various notions of motivic Poincaré series, motivated by some generating series in arithmetic geometry (see [8]). Assume for simplicity that the variety S is defined over the integers. We denote by p a prime number and by $\mathbf{Z}_p$ the p-adic integers. For every positive integer m, the symbol $N_{p,m}(S)$ denotes the number of rational points of S over $\mathbf{Z}/p^{m+1}\mathbf{Z}$ which can be lifted to rational points of S over $\mathbf{Z}_p$ by the projection induced by the natural map $\mathbf{Z}_p \to \mathbf{Z}/p^{m+1}\mathbf{Z}$. The *Serre-Oesterlé* series of S at the prime p is

$$P_p^S(T) = \sum_{m\geq 0} N_{p,m}(S) T^m \in \mathbf{Z}[[T]]$$

The definition of the *geometric motivic Poincaré series*,

$$P_{\mathrm{geom}}^S(T) = \sum_{m\geq 0} \chi_c([j_m(H)]) T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]]$$

is inspired by that of the Serre-Oesterlé series. However, there is no specialization of the series $P_{\mathrm{geom}}^S(T)$ into $P_p^S(T)$ in general (see [8]).

Denef and Loeser studied the "motivic nature" of the series $P_p^S(T)$, passing through the Grothendieck ring $K_0(\mathrm{Field}_k)$ of ring formulas over k. First, by Greenberg's theorem for every

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m there exists a formula ψ_m over k such that, for any field extension $k \subset K$, the m-jets over k which can be lifted to arcs defined over K correspond to the tuples satisfying ψ_m in K. It follows that ψ_m defines an element $[\psi_m] \in K_0(\mathrm{Field}_k)$. Then, Denef and Loeser defined a ring homomorphism $\chi_f : K_0(\mathrm{Field}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$. This homomorphism can be seen as a generalization of χ_c, since the image by χ_f of the class of the ring formula defining a variety V coincides with the class $\chi_c([V])$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$. The *arithmetic motivic Poincaré series* of S is defined as

$$P^S_{\mathrm{ar}}(T) = \sum_{m \geq 0} \chi_f([\psi_m])T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]].$$

Denef proved the rationality of the series $P^S_p(T)$ using quantifier elimination results (see [4]). Denef and Loeser proved the rationality of the series $P^S_{\mathrm{geom}}(T)$ and $P^S_{\mathrm{ar}}(T)$ by using quantifier eliminations theorems, various forms of motivic integration and the existence of resolution of singularities (see [6, 7]).

If V is a variety defined over the integers and p is a prime number, the symbol $N_p(V)$ denotes the number of rational points of V over the field of p elements. Denef and Loeser proved that the result of applying the operator N_p to the motivic arithmetic series $P^S_{\mathrm{ar}}(T)$ provides the Serre-Oesterlé series $P^S_p(T)$ for almost all primes p.

If we fix the origin of the arcs in a fixed point $0 \in S$ we obtain local versions of these series $P^{(S,0)}_{\mathrm{ar}}(T)$ and $P^{(S,0)}_{\mathrm{geom}}(T)$, which are also rational (see [6, 7]). The rationality proofs in [6, 7] are qualitative in nature, in particular there is no conjecture on the significance of the terms appearing in the denominator of the rational form of the series $P^{(S,0)}_{\mathrm{ar}}(T)$ or in $P^{(S,0)}_{\mathrm{geom}}(T)$.

The rational form of the series $P^{(S,0)}_{\mathrm{ar}}(T)$ is known explicitly for a few classes of singularities. If $(S, 0)$ is an analytically irreducible germ of plane curve, the information provided by the series $P^{(S,0)}_{\mathrm{ar}}(T)$ is equivalent to the data of the Puiseux pairs (see [7]). In [19] Nicaise proved the equality of the geometric and arithmetic motivic Poincaré series in the case of varieties which admits a very special resolution of singularities, in particular for normal toric surfaces (see also [18, 17]). He gave a criterion for the equality $P^{(S,0)}_{\mathrm{ar}}(T) = P^{(S,0)}_{\mathrm{geom}}(T)$ for various classes of singularities and also an example of normal toric threefold $(S_0, 0)$ such that the series $P^{(S_0,0)}_{\mathrm{ar}}(T)$ and $P^{(S_0,0)}_{\mathrm{geom}}(T)$ are different. Some features of the motivic arithmetic series are studied for quasi-ordinary singularities in [21].

In this paper we describe the arithmetic motivic Poincaré series of an affine toric variety $Z^\Lambda = \mathrm{Spec}k[\Lambda]$, in terms of the semigroup Λ. We assume that Λ is a semigroup of finite type of a rank d lattice M (lattice of characters), which generates M as a group, and such that the cone $\mathbf{R}_{\geq 0}\Lambda$ contains no lines. In this situation there is a unique minimal system of generators $e_1, \dots, e_n$ of the semigroup Λ. The monomial ideal $(X^{e_i})_{i=1,\dots,n} \subset k[\Lambda]$ is maximal and defines the distinguished point $0 \in Z^\Lambda$. In this paper we consider other monomial ideals as the *logarithmic jacobian ideals* $\mathcal{J}_l$, generated by monomials of the form X^u for u in the set

$$\{e_{i_1} + \cdots + e_{i_l} \mid e_{i_1} \wedge \ldots \wedge e_{i_l} \neq 0\}$$

for $l = 1, \dots, d$ (see [2]), and the *ideals of sums of combinations* $\mathcal{C}_j$, defined by monomials X^w with w in the set

$$\left\{ e_{i_1} + \cdots + e_{i_j} \mid \{i_1, \dots, i_j\} \in \binom{\{1, \dots, n\}}{j} \right\},$$

where $\binom{\{1,\dots,n\}}{j}$ denotes the set of combinations of j elements of $\{1, \dots, n\}$, for $j = 1, \dots, n$.

We study the motivic arithmetic series $P^{(Z^\Lambda,0)}_{\mathrm{ar}}(T)$ by extending the approach we used in [2, 3] to describe the geometric motivic Poincaré series of toric and quasi-ordinary singularities.

By convenience we explain the methods and results first when the variety Z^Λ is normal. The set $j_m(H(Z^\Lambda)_0)$ of m-jets of arcs through $(Z^\Lambda, 0)$ is constructible; it is a finite disjoint union of locally closed subsets of the form $j_m(H^*_\nu)$ (see [2]). Here H^*_ν denotes the set of arcs through $(Z^\Lambda, 0)$ which have generic point in the torus and a given order $\nu \in M^*$. The set H^*_ν is an orbit of the natural action of the arc space of the torus on the arc space of the toric variety Z^Λ (see [15, 16]).

We describe the class, denoted by $\chi_f([j_m(H_\nu^*)]_f)$, of the formula defining the locally closed subset $j_m(H_\nu^*)$ in terms of the Newton polyhedra of the logarithmic jacobian ideals and the degree of certain Galois cover. This Galois cover reflects the relation between the initial coefficients of the arcs in H_ν^* and the initial coefficients of the m-jets in $j_m(H_\nu^*)$, see Section 5.

A key point in the description of the rational form of the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ is that using the ideals $\mathcal{C}_j$ we can refine a finite partition of the set of possible pairs $\{(\nu,m)\}$, which was defined in [2] to describe the sum of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$. If (ν,m) and (ν',m') belong to the same subset of this refinement then the degrees of the Galois covers associated to $j_m(H_\nu^*)$ and $j_{m'}(H_{\nu'}^*)$ coincide (see Sections 6 and 7). Using these partitions we decompose the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ as a sum of a finite number of contributions. The main result is a formula for the rational form of $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ (see Theorem 11.4 and Corollary 10.4). The proofs pass by the results on the generating function of the projection of the set of integral points in the interior of a rational polyhedral cone (see [2]). The denominator of $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ is a finite product of terms of the form $1-\mathbf{L}^aT^b$ with $a\geq 0$ and $b>0$, which are determined explicitly in terms of the ideals of sums of combinations $\mathcal{C}_j$. The integers a and b can be described in terms of the orders of vanishing of the ideals $\mathcal{C}_j$ and $\mathcal{J}_l$ at the codimension one orbits of various toric modifications given by the Newton polyhedra of the ideals $\mathcal{C}_j$ (see Remark 10.8). In the normal toric case we obtain a formula for $P_{\mathrm{ar}}^{Z^\Lambda}(T)$ in terms of arithmetic motivic series at the distinguished points of the orbits.

In the non-normal case, we obtain in a similar way a formula for the rational form of $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ and the factors of its denominator. The main difference is that we have to consider contributions of jets of arcs with generic point in the various orbits of Z^Γ. We deduce a formula for the difference $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)-P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ and we give a criterion for the equality of these two series which generalizes the one given by Nicaise in [18] (see Proposition 10.5 and Theorem 10.6).

The paper is organized as follows. In Sections 1 and 2 we introduce the Grothendieck rings, the arc and jet spaces and the motivic Poincaré series. The notations on toric varieties, their monomial ideals and their arcs are introduced in Sections 3 and 4. The computation of the class $\chi_f([j_m(H_\nu^*)]_f)$ is given in Section 5. Sections 6 and 7 deal with the partitions associated to sequences of monomial ideals. The main results are stated and proved in Sections 8, 9 and 10. In the case of normal toric varieties some features of the computation can be simplified (see Section 11). We discuss some examples in Section 12.

1. Grothendieck rings of varieties and of ring formulas

The *Grothendieck ring* $K_0(\mathrm{Var}_k)$ *of* k*-varieties* is the free abelian group of isomorphism classes $[X]$ of k-varieties X modulo the relations $[X]=[X']+[X\setminus X']$ if X' is closed in X, and where the product is defined by $[X][X']=[X\times X']$. We denote by $\mathbf{L}:=[\mathbf{A}_k^1]$ the class of the affine line. If C is a constructible subset of some variety X, i.e. a disjoint union of finitely many locally closed subvarieties A_i of X, then $[C]\in K_0(\mathrm{Var}_k)$ is well defined as $[C]:=\sum_i[A_i]$ independently of the representation. Bittner proved, using the weak factorization theorem, that the ring $K_0(\mathrm{Var}_k)$ is generated by classes of smooth projective k-varieties, modulo relations of the form $[W]-[E]=[X]-[Y]$, where $Y\subset X$ is a closed subvariety, and W is the blowing up of X along Y with exceptional divisor E (see [1]).

There exists a unique ring homomorphism:

$$\chi_c : K_0(\mathrm{Var}_k)\to K_0(\mathrm{CHMot}_k), \tag{1}$$

which maps the class of a smooth projective variety over k to its *Chow motive*, where $K_0(\mathrm{CHMot}_k)$ denotes the *Grothendieck ring of the category of Chow motives* over k (with coefficients in $\mathbf{Q}$). This result, which is due to Guillet and Soulé [12] and Guillén and Navarro Aznar [14], can be seen also in terms of Bittner's result. We refer to [12, 14, 1] for details and to [22] for an introduction to the notion of motives. We denote by $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ the image of $K_0(\mathrm{Var}_k)$ in $K_0(\mathrm{CHMot}_k)$ under this homomorphism. Notice that the image of $\mathbf{L}$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$, which we denote with the same symbol, is not a zero divisor in $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ since it is a unit in $K_0(\mathrm{CHMot}_k)$.

A *ring formula* ψ over k is a first order formula in the language of k-algebras and free variables $x_1, \dots, x_n$, that is, the formula ψ is built from boolean combinations ("and", "or", "not") of polynomial equations over k and existential and universal quantifiers. The *Grothendieck ring* $K_0(\mathrm{Field}_k)$ *of ring formulas over* k, is generated by symbols $[\psi]$, where ψ is a ring formula over k, subject to the relations $[\psi_1 \vee \psi_2] = [\psi_1] + [\psi_2] - [\psi_1 \wedge \psi_2]$ if ψ_1 and ψ_2 have the same free variables, and $[\psi_1] = [\psi_2]$ if there exists a ring formula Ψ over k such that, when interpreted in any field $K \supseteq k$ provides the graph of a bijection between the tuples of elements of K satisfying ψ_1 and those satisfying ψ_2. The ring multiplication is induced by the conjunction of formulas in disjoint sets of variables (see [7]). Denef and Loeser defined a ring homomorphism

$$\chi_f : K_0(\mathrm{Field}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}. \tag{2}$$

They proved that this homomorphism is characterized by two conditions. The first one is that for any ring formula ψ which is a conjunction of polynomial equations over k, the element $\chi_f([\psi])$ is equal to the class $\chi_c([V])$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ of the variety V defined by ψ. The second condition, which is more technical, expresses that certain relations should hold in terms of unramified Galois coverings over k. We refer to [7, 8] for the precise statement. In the simplest case it implies the following:

Example 1.1. (see [8] Example 6.4.3) If $n \geq 1$ is a fixed integer, k is a field containing all n-th roots of unity and ψ is the ring formula $\psi : (\exists y)(x = y^n \text{ and } x \neq 0)$ then we have that $\chi_f([\psi]) = \frac{1}{n}(\mathbf{L} - 1)$.

We deduce from this example the following Lemma:

Lemma 1.2. *Let ψ be the ring formula whose interpretation in any field $K \supseteq k$ provides the set of K-rational points of T which lift to K-rational points of T' by a Galois covering $T' \to T$ of degree n of d-dimensional algebraic k-tori. If the field k contains all the n-th roots of unity then we have that $\chi_f([\psi]) = \frac{1}{n}(\mathbf{L} - 1)^d$.*

Proof. The morphism $T' \to T$ induces a finite index inclusion of the corresponding character group $M \subseteq M'$, and hence a map of $k-$algebras $k[M] \hookrightarrow k[M']$. By the classification theorem of finitely generated abelian groups applied to M'/M there exists a basis $\{v_1, \dots, v_d\}$ of M' and unique integers $b_1 | b_2 | \cdots | b_d$, where $|$ denotes division, such that $\{b_1 v_1, \dots, b_d v_d\}$ is a basis of M and $n = b_1 \cdots b_d$. It follows that the map of coordinate rings $K[M] \hookrightarrow K[M']$ express in coordinates as $K[z_1^{\pm b_1}, \dots, z_d^{\pm b_d}] \hookrightarrow K[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$. We deduce that the ring formula ψ is the conjunction of formulas $\psi_i : (\exists y_i)(x_i = y_i^{b_i} \text{ and } x_i \neq 0)$, for $i = 1, \dots, d$ where the variables $x_1, \dots, x_d$ are independent. Then we get that $\chi_f([\psi]) = \frac{1}{b_1 \cdots b_d}(\mathbf{L} - 1)^d$. □

Remark 1.3. Denef and Loeser defined the map χ_f by factoring it through the Grothendieck ring $K_0(\mathrm{PFF}_k)$ of ring formulas for the category of *pseudo-finite fields* containing k. See [7, 8, 5].

2. Arcs, jets spaces and motivic Poincaré series

We start this Section by recalling the definition of the space of arcs of a variety S. We assume for simplicity that S is an affine irreducible and reduced algebraic variety defined over a field k of characteristic zero.

For any integer $m \geq 0$ the functor from the category of k-algebras to the category of sets, sending a k-algebra R to the set of $R[t]/(t^{m+1})$-rational points of S is representable by a k-scheme $H_m(S)$ of finite type over k, called the m-jet scheme of S. The natural maps induced by truncation $j_m^{m+1} : H_{m+1}(S) \to H_m(S)$ are affine and hence the projective limit $H(S) := \varprojlim H_m(S)$ is a k-scheme, not necessarily of finite type, called the *arc space* of S.

In what follows we consider the schemes $H_m(S)$ and $H(S)$ with their reduced structure. We have natural morphisms $j_m : H(S) \to H_m(S)$. By an *arc* we mean a k-rational point of $H(S)$, i.e., a morphism Spec $k[[t]] \to S$. By an m-jet we mean a k-rational point of $H_m(S)$, i.e., a morphism Spec $k[t]/(t^{m+1}) \to S$. The origin of the arc (resp. of the m-jet) is the image of the closed point 0 of Spec $k[[t]]$ (resp. of Spec $k[t]/(t^{m+1})$).

If $Z \subset S$ is a closed subvariety then $H(S)_Z := j_0^{-1}(Z)$ (resp. $H_m(S)_Z := (j_0^m)^{-1}(Z)$) denotes the subscheme of $H(S)$ (resp. of $H_m(S)$) formed by arcs (resp. m-jets) in S with origin in Z.

By a Theorem of Greenberg [13], for any integer $m \geq 0$, $j_m(H(S))$ is a constructible subset of the k-variety $H_m(S)$. We can then consider the class $[j_m(H(S))] \in K_0(\mathrm{Var}_k)$. Greenberg's result implies also that there is a ring formula ψ_m over k, such that for any field K containing k, the k-rational points of $H_m(S)$ which can be lifted to K-rational points of $H(S)$ correspond to the tuples satisfying ψ_m in K. If ψ'_m is another ring formula over k with the same property then $[\psi_m] = [\psi'_m]$ in $K_0(\mathrm{Field}_k)$. The same applies for $j_m(H(S)_Z)$ if $Z \subset S$ is a closed subvariety.

Notation 2.1. We denote the class $[\psi_m]$ by $[j_m(H(S))]_f$ to avoid confusion with the class $[j_m(H(S))] \in K_0(\mathrm{Var}_k)$.

The following Poincaré series were introduced by Denef and Loeser in the papers ([6, 7]).

Definition 2.2.

(1) The *geometric motivic Poincaré series* of (S, Z) is
$$P_{\mathrm{geom}}^{(S,Z)}(T) := \sum_{m \geq 0} \chi_c([j_m(H(S)_Z)])T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]].$$

(2) The *arithmetic motivic Poincaré series* of (S, Z) is
$$P_{\mathrm{ar}}^{(S,Z)}(T) := \sum_{m \geq 0} \chi_f([j_m(H(S)_Z)]_f)T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]].$$

Remark 2.3. We have slightly modified the original definition of the geometric motivic Poincaré series, as $\sum_{m \geq 0}[j_m(H(S)_Z)]T^m \in K_0(\mathrm{Var}_k)[[T]]$ (see [6]), in order to have the geometric and arithmetic setting in the same ring. This does not affect the rationality results below.

Denef and Loeser proved that these series have a rational form:

Theorem 2.4. *(see* [6] *Theorem 1.1 and* [7] *Theorem 9.2.1) The series* $P_{\mathrm{geom}}^{(S,Z)}(T)$ *and* $P_{\mathrm{ar}}^{(S,Z)}(T)$ *belong to the subring of* $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]]$ *generated by* $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[T]$ *and the series* $(1 - \mathbf{L}^a T^b)^{-1}$*, with* $a \in \mathbf{Z}$ *and* $b > 0$*.*

The arithmetic motivic Poincaré series has interesting properties of specialization to classical arithmetic series. Let p be a prime number. The operators N_p and $N_{p,m}$ are applied to a variety V defined over the integers by $N_p(V) := \#V(\mathbf{Z}/p\mathbf{Z})$ and $N_{p,m}(V) := \#\{\pi_m(V(\mathbf{Z}_p))\}$ where $\mathbf{Z}_p$ denotes the p-adic integers, $\pi_m(V(\mathbf{Z}_p)) \subset V(\mathbf{Z}/p^{m+1}\mathbf{Z})$ is the image of $V(\mathbf{Z}_p))$ by the natural projection induced by $\mathbf{Z}_p \to \mathbf{Z}/p^{m+1}\mathbf{Z}$, and $\#$ denotes the cardinality. Suppose that the variety S is defined over the integers. The *Serre-Oesterlé* series $P_p^S(T) := \sum_{m \geq 0} N_{p,m}T^m \in \mathbf{Z}[[T]]$ of S at the prime p is a rational function of T (see [4]). Denef and Loeser proved that for $p \gg 0$ the series $P_p^S(T)$ is obtained from $P_{\mathrm{ar}}^S(T)$ by applying to each coefficient the operator N_p (see [7, 5, 8]).

Remark 2.5. These results hold in a more general setting, in particular when S is not affine as assumed here (see [6, 7]). The proof of the rationality of $P_p^S(T)$ involves the use of quantifier elimination results and p-adic integration (see [4]). The proof of the rationality of $P_{\mathrm{ar}}^S(T)$ requires also quantifier elimination results and arithmetic motivic integration (see [7, 8, 5]).

3. Affine toric varieties and monomial ideals

In this Section we introduce the basic notions and notations from toric geometry(see [9, 20, 10, 11] for proofs).

If $N \cong \mathbf{Z}^d$ is a lattice we denote by $N_{\mathbf{R}} := N \otimes \mathbf{R}$ (resp. $N_{\mathbf{Q}} := N \otimes \mathbf{Q}$) the vector space spanned by N over the field $\mathbf{R}$ (resp. over $\mathbf{Q}$). In what follows a *cone* in $N_{\mathbf{R}}$ mean a *rational convex polyhedral cone*: the set of non negative linear combinations of vectors $a_1 \dots, a_r \in N$. The cone τ is *strictly convex* if it contains no line through the origin, in that case we denote by 0 the 0-dimensional face of τ; the cone τ is *simplicial* if the primitive vectors of the 1-dimensional faces are linearly independent over $\mathbf{R}$. We denote by $\overset{\circ}{\tau}$ or by $\mathrm{int}(\tau)$ the relative interior of the cone τ.

We denote by M the dual lattice. The *dual* cone $\tau^\vee \subset M_{\mathbf{R}}$ (resp. *orthogonal* cone $\tau^\perp$) of τ is the set $\{w \in M_{\mathbf{R}} \mid \langle w, u\rangle \geq 0, \text{ (resp. } \langle w, u\rangle = 0) \ \forall u \in \tau\}$.

A *fan* Σ is a family of strictly convex cones in $N_{\mathbf{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The relation $\theta \leq \tau$ (resp. $\theta < \tau$) denotes that θ is a face of τ (resp. $\theta \neq \tau$ is a face of τ). The *support* (resp. the *k-skeleton*) of the fan Σ is the set $|\Sigma| := \bigcup_{\tau \in \Sigma} \tau \subset N_{\mathbf{R}}$ (resp. $\Sigma^{(k)} = \{\tau \in \Sigma \mid \dim \tau = k\}$). We say that a fan Σ' is a *subdivision* of the fan Σ if both fans have the same support and if every cone of Σ' is contained in a cone of Σ. If Σ_i for $i = 1, \ldots, n$, are fans with the same support their intersection $\cap_{i=1}^n \Sigma_i := \{\cap_{i=1}^n \tau_i \mid \tau_i \in \Sigma_i\}$ is also a fan.

Notation 3.1. In this paper Λ is a sub-semigroup of finite type of a lattice M, which generates M as a group and such that the cone $\sigma^\vee = \mathbf{R}_{\geq 0}\Lambda$ is strictly convex and of dimension d. We denote by N the dual lattice of M and by $\sigma \subset N_{\mathbf{R}}$ the dual cone of $\sigma^\vee$. We denote by Z^Λ the *affine toric variety* $Z^\Lambda = \operatorname{Spec} k[\Lambda]$, where $k[\Lambda] = \{\sum_{\text{finite}} a_\lambda X^\lambda \mid a_\lambda \in k\}$ denotes the semigroup algebra of the semigroup Λ with coefficients in the field k. The semigroup Λ has a unique minimal set of generators $e_1, \ldots, e_n$ (see the proof of Chapter V, Lemma 3.5, page 155 [9]). We have an *embedding* of $Z^\Lambda \subset \mathbf{A}^n_k$ given by, $x_i := X^{e_i}$ for $i = 1, \ldots, n$.

If $\Lambda = \sigma^\vee \cap M$ then the variety Z^Λ, which we denote also by $Z_{\sigma,N}$ or by Z_σ when the lattice is clear from the context, is normal. If $\Lambda \neq \sigma^\vee \cap M$ the inclusion of semigroups $\Lambda \to \bar{\Lambda}$ defines a toric modification $Z^{\bar{\Lambda}} \to Z^\Lambda$, which is the *normalization map*.

The torus $T_N := Z^M$ is an open dense subset of Z^Λ, which acts on Z^Λ and the action extends the action of the torus on itself by multiplication. The *origin* 0 of the affine toric variety Z^Λ is the 0-dimensional orbit, defined by the maximal ideal $(X^\lambda)_{0 \neq \lambda \in \Lambda}$ of $k[\Lambda]$. There is a one to one inclusion reversing correspondence between the faces of σ and the orbit closures of the torus action on Z^Λ. If $\theta \leq \sigma$, we denote by $\operatorname{orb}_\theta^\Lambda$ the orbit corresponding to the face θ of σ. The orbit closures are of the form $Z^{\Lambda \cap \theta^\perp}$ for $\theta \leq \sigma$.

The *Newton polyhedron* of a monomial ideal corresponding to a non empty set of lattice vectors $\mathcal{I} \subset \Lambda$ is defined as the convex hull of the Minkowski sum of sets $\mathcal{I} + \sigma^\vee$. We denote this polyhedron by $\mathcal{N}(\mathcal{I})$. Notice that the vertices of $\mathcal{N}(\mathcal{I})$ are elements of $\mathcal{I}$. We denote by $\operatorname{ord}_{\mathcal{I}}$ the *support function* of the polyhedron $\mathcal{N}(\mathcal{I})$, which is defined by $\operatorname{ord}_{\mathcal{I}} : \sigma \to \mathbf{R}$, $\nu \mapsto \inf_{\omega \in \mathcal{N}(\mathcal{I})} \langle \nu, \omega \rangle$. The face of the polyhedron $\mathcal{N}(\mathcal{I})$ determined by $\nu \in \sigma$ is the set $\mathcal{F}_\nu := \{\omega \in \mathcal{N}(\mathcal{I}) \mid \langle \nu, \omega\rangle = \operatorname{ord}_{\mathcal{I}}(\nu)\}$. All faces of $\mathcal{N}(\mathcal{I})$ are of this form, the compact faces are defined by vectors $\nu \in \overset{\circ}{\sigma}$. The set $\Sigma(\mathcal{I})$ consisting of the cones $\sigma(\mathcal{F}) := \{\nu \in \sigma \mid \langle \nu, \omega \rangle = \operatorname{ord}_{\mathcal{I}}(\nu), \ \forall \omega \in \mathcal{F}\}$, for $\mathcal{F}$ running through the faces of $\mathcal{N}(\mathcal{I})$, is a fan supported on σ. Notice that if $\tau \in \Sigma(\mathcal{I})$ and if $\nu, \nu' \in \overset{\circ}{\tau}$ then $\mathcal{F}_\nu = \mathcal{F}_{\nu'}$. We denote this face of $\mathcal{N}(\mathcal{I})$ also by $\mathcal{F}_\tau$.

The affine varieties Z_τ corresponding to cones τ in a fan Σ glue up to define a *toric variety* Z_Σ. A fan Σ subdividing the cone σ defines a *toric modification* $\pi_\Sigma : Z_\Sigma \to Z_\sigma$.

If $\mathcal{I} \subset \Lambda$ defines a monomial ideal the composite $Z_{\Sigma(\mathcal{I})} \overset{\pi_{\Sigma(\mathcal{I})}}{\longrightarrow} Z_\sigma \longrightarrow Z^\Lambda$ is equal to the *normalized blowing up* of Z^Λ centered at $\mathcal{I}$ (see [17] for instance).

Definition 3.2. For $1 \leq j \leq n$ the *j-th ideal of sums of combinations* of Z^Λ is the monomial ideal $\mathcal{C}_j$ of $k[\Lambda]$ generated by X^α where α runs through:

$$\left\{ e_{i_1} + \cdots + e_{i_j} \mid \{i_1, \ldots, i_j\} \in \binom{\{1, \ldots, n\}}{j} \right\}, \tag{3}$$

where $\binom{\{1,\ldots,n\}}{j}$ denotes the set of combinations of j elements of $\{1, \ldots, n\}$, for $j = 1, \ldots, n$. We denote by Θ_j (resp. by $\operatorname{ord}_{\mathcal{C}_j}$) the dual subdivision of σ (resp. the support function) of the polyhedron $\mathcal{N}(\mathcal{C}_j)$. The maps

$$\varphi_1 \ := \ \operatorname{ord}_{\mathcal{C}_1} \quad \text{and} \quad \varphi_j \ := \ \operatorname{ord}_{\mathcal{C}_j} - \operatorname{ord}_{\mathcal{C}_{j-1}} \quad \text{for } j = 2, \ldots, n,$$

are piece-wise linear functions defined on the cone σ. If $\nu \in \sigma$ we denote by convenience $\varphi_0(\nu) := 0$ and $\varphi_{n+1}(\nu) := +\infty$.

Definition 3.3. For $1 \leq l \leq d$ the *l-th logarithmic jacobian ideal* of Z^Λ is the monomial ideal $\mathcal{J}_l$ of $k[\Lambda]$ generated by X^α where α runs through:

$$\{e_{i_1} + \cdots + e_{i_l} \mid e_{i_1} \wedge \cdots \wedge e_{i_l} \neq 0, \text{ for } 1 \leq i_1, \ldots, i_l \leq n\}. \tag{4}$$

We denote by Σ_l (resp. by $\mathrm{ord}_{\mathcal{J}_l}$) the dual subdivision of σ (resp. the support function) of the polyhedron $\mathcal{N}(\mathcal{J}_l)$. The maps

$$\phi_1 \quad := \quad \mathrm{ord}_{\mathcal{J}_1} \quad \text{and} \quad \phi_l \quad := \quad \mathrm{ord}_{\mathcal{J}_l} - \mathrm{ord}_{\mathcal{J}_{l-1}} \quad \text{for } l = 2, \ldots, d,$$

are piece-wise linear functions defined on the cone σ. If $\nu \in \sigma$ we denote by convenience $\phi_0(\nu) := 0$ and $\phi_{d+1}(\nu) := +\infty$.

We use the notation $\mathcal{J}_l$ (resp. $\mathcal{C}_j$) also for the set (4) (resp. (3)).

Lemma 3.4. *If $\nu \in \overset{\circ}{\sigma}$ and if $(p_1, \ldots, p_n)$ is a permutation of $(1, \ldots, n)$ such that*

$$\langle \nu, e_{p_1} \rangle \leq \cdots \leq \langle \nu, e_{p_n} \rangle.$$

then $\mathrm{ord}_{\mathcal{C}_j}(\nu) = \langle \nu, \sum_{r=1}^{j} e_{p_r} \rangle$ *and* $\varphi_j(\nu) = \langle \nu, e_{p_j} \rangle$ *for* $1 \leq j \leq n$*. Moreover, the following holds*

$$0 = \varphi_0(\nu) \leq \varphi_1(\nu) \leq \cdots \leq \varphi_n(\nu) \quad \text{and} \quad 0 = \phi_0(\nu) \leq \phi_1(\nu) \leq \cdots \leq \phi_d(\nu).$$

Proof. The first assertion follows by induction on $j \in \{1, \ldots, n\}$.
See Lemma 5.3 [2] for the second sequence of inequalities. □

Proposition 3.5. *The Newton polyhedra of the ideals $\mathcal{C}_j$, $j = 1, \ldots, n$, determine and are determined by the minimal system of generators of the semigroup Λ.*

Proof. The Newton polyhedron $\mathcal{N}(\mathcal{C}_j)$ determines and it is determined by its support function $\mathrm{ord}_{\mathcal{C}_j}$, for $j = 1, \ldots, n$. By Lemma 3.4 and the definitions if θ is a d dimensional cone of the fan $\cap_{r=1}^{n} \Theta_r$ there exists a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ such that $\varphi_j(\nu) = \langle \nu, e_{i_j} \rangle$ for $j = 1, \ldots, n$ and all $\nu \in \overset{\circ}{\theta}$. Thus, the functions φ_j, $j = 1, \ldots, n$, , or equivalently, $\mathrm{ord}_{\mathcal{C}_j}$, $j = 1, \ldots, n$, determine the vectors $e_1, \ldots, e_n$. □

4. Arcs and jets on a toric singularity

Let Λ be a semigroup as in Notation 3.1. If R is a k-algebra, a R-rational point of Z^Λ is a homomorphism of semigroups $(\Lambda, +) \to (R, \cdot)$, where $(R, \cdot)$ denotes the semigroup R for the multiplication. In particular, the closed points are obtained for $R = k$. An arc h on the affine toric variety Z^Λ is given by a semigroup homomorphism $(\Lambda, +) \to (k[[t]], \cdot)$. An arc in the torus T_N is defined by a semigroup homomorphisms $\Lambda \to k[[t]]^*$, where $k[[t]]^*$ denotes the group of units of the ring $k[[t]]$.

Notation 4.1. We denote the set of arcs $H(Z^\Lambda)_0$ of Z^Λ with origin at the distinguished point 0 of Z^Λ simply by H_Λ, and by H_Λ^* the set consisting of those arcs of H_Λ with generic point in the torus T_N.

Notice that $h \in H_\Lambda^*$ if and only if for all $u \in \Lambda$ the formal power series $X^u \circ h \in k[[t]]$ is non-zero. Any arc $h \in H_\Lambda^*$ defines two group homomorphisms $\nu_h : M \to \mathbf{Z}$ and $\omega_h : M \to k[[t]]^*$ by: $X^m \circ h = t^{\nu_h(m)} \omega_h(m)$. If $m \in \Lambda$ then $\nu_h(m) > 0$ hence ν_h belongs to $\overset{\circ}{\sigma} \cap N$. Notice that ω_h defines an arc in the torus, i.e., $\omega_h \in H(T_N)$.

Remark 4.2. The space of arcs in the torus acts on the arc space of a toric variety (see [15, 16]).

Lemma 4.3. (see Theorem 4.1 of [15], Lemma 5.6 of [16], and Proposition 3.3 [17]). *The map $\overset{\circ}{\sigma} \cap N \times H(T_N) \to H_\Lambda^*$ which applies a pair (ν, ω) to the arc h defined by $X^u \circ h = t^{\langle \nu, u \rangle} \omega(u)$, for $u \in \Lambda$, is a one to one correspondence. The sets $H_{\Lambda,\nu}^* := \{h \in H_\Lambda^* \mid \nu_h = \nu\}$ for $\nu \in \overset{\circ}{\sigma} \cap N$ are orbits for the action of H_{T_N} on H_Λ^* and we have that $H_\Lambda^* = \bigsqcup_{\nu \in \overset{\circ}{\sigma} \cap N} H_{\Lambda,\nu}^*$.*

Remark 4.4. We often denote the set H_Λ^* (resp. the orbit $H_{\Lambda,\nu}^*$) by H^* (resp. by H_ν^*) if Λ is clear from the context.

An arc $h \in H_\Lambda$ has its generic point η contained in exactly one orbit of the torus action on Z^Λ. If $h(\eta) \in \mathrm{orb}_\theta^\Lambda$, for some $\theta \leq \sigma$, then h factors through the orbit closure $Z^{\Lambda\cap\theta^\perp}$ and $h \in H^*_{\Lambda\cap\theta^\perp}$, i.e., h is an arc through $(Z^{\Lambda\cap\theta^\perp}, 0)$ with generic point in the torus $\mathrm{orb}_\theta^\Lambda$. We can apply Lemma 4.3 to describe the set $H^*_{\Lambda\cap\theta^\perp}$, just replacing the semigroup Λ by $\Lambda\cap\theta^\perp$ (see [2]). In particular, if $\theta = 0$ then $h \in H^*_\Lambda$; if $\theta = \sigma$ then $\Lambda \cap \theta^\perp = 0$ and h is the constant arc at the distinguished point $0 \in Z^\Lambda$. We have a partition $H_\Lambda = \bigsqcup_{\theta\leq\sigma} H^*_{\Lambda\cap\theta^\perp}$.

5. The image of the class of the formula defining $j_s(H^*_\nu)$

Definition 5.1. We associate to $(\nu, s) \in (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$ the sets

$$M^s_\nu := \mathrm{span}_{\mathbf{Z}}\{e_i \mid \langle \nu, e_i\rangle \leq s, i = 1, \dots, n\}, \quad \text{and} \quad \ell^s_\nu := \mathrm{span}_{\mathbf{Q}}\{e_i \mid \langle \nu, e_i\rangle \leq s, i = 1, \dots, n\}.$$

We denote by $l(\nu, s)$ the dimension of the $\mathbf{Q}$-vector space ℓ^s_ν. The integer $l(\nu, s)$ is also the rank of the lattice M^s_ν. We denote by $q(\nu, s)$ the index of the lattice extension $M^s_\nu \subset \ell^s_\nu \cap M$.

Proposition 5.2. *If $(\nu, s) \in \overset{\circ}{\sigma} \times \mathbf{Z}_{>0}$, $l(\nu, s) > 0$, and if the field k contains all the $q(\nu, s)$-th roots of unity then we have*

$$\chi_f([j_s(H^*_\nu)]_f) = \frac{1}{q(\nu, s)} (\mathbf{L} - 1)^{l(\nu,s)} \times \mathbf{L}^{sl(\nu,s) - \mathrm{ord}_{\mathcal{J}_{l(\nu,s)}}(\nu)}.$$

If $l(\nu, s) = 0$ then we have $\chi_f([j_s(H^*_\nu)]_f) = 1$.

Proof. If $h \in H^*_\nu$ the equality $\mathrm{ord}_t(X^{e_i} \circ h) = \langle \nu, e_i\rangle$ holds for $1 \leq i \leq n$. By Definition 5.1 those vectors e_i such that $j_s(X^{e_i} \circ h) \neq 0$ span the $\mathbf{Q}$-vector space ℓ^s_ν since $\langle \nu, e_i\rangle \leq s$. If $l(\nu, s) = 0$ this vector space is empty, the jet space $j_s(H^*_\nu)$ consists of the constant 0-jet and the conclusion follows easily from the definitions.

Suppose then that $l := l(\nu, s) > 0$. If $h \in H^*_\nu$ then it is given by n series of the form

$$X^{e_i} \circ h = t^{\langle \nu, e_i\rangle} c(e_i)\big(1 + \sum_{m\geq 1} u_m(e_i) t^m\big),\ i = 1, \dots, n.$$

We have that the s-jet $j_s(X^{e_i} \circ h)$ is different from zero if and only if $\langle \nu, e_i\rangle \leq s$.

By Lemma 5.7 of [2] there exist integers $1 \leq k_1, \dots, k_l \leq n$ such that $\phi_i(\nu) = \langle \nu, e_{k_i}\rangle \leq s$, for $i = 1, \dots, l$, $\ell^s_\nu = \mathrm{span}_{\mathbf{Q}}\{e_{k_1}, \dots, e_{k_l}\}$ and $\mathrm{ord}_{\mathcal{J}_l}(\nu) = \sum_{i=1}^l \langle \nu, e_{k_i}\rangle$.

By Section 6 of [2] if h is the universal family of arcs parametrizing H^*_ν, the terms $\{u_m(e_{k_i}) \mid i = 1, \dots, l, m \geq 1\}$ are algebraically independent over $\mathbf{Q}$ and the terms $\{c(e_i)^{\pm 1} \mid i = 1, \dots, n\}$ generate a k-algebra isomorphic to $k[M]$ by the isomorphism which maps $c(e_i) \mapsto X^{e_i}$.

By the proof of Theorem 7.1 [2] a formula defining $j_s(H^*_\nu)$ is the conjunction of two formulas ψ_1 and ψ_2 with independent sets of variables. The first formula ψ_1 is a finite sequence of polynomial equalities with rational coefficients expressing the terms $u_r(e_i)$ appearing in $j_s(X^{e_i} \circ h)$, for $1 \leq r \leq s - \langle \nu, e_i\rangle$, in terms of the variables $\{u_r(e_{k_i}) \mid 1 \leq i \leq l,\ 1 \leq r \leq s - \langle \nu, e_{k_i}\rangle\}$. We deduce that $\chi_f([\psi_1]) = \mathbf{L}^{sl - \mathrm{ord}_{\mathcal{J}_l}(\nu)}$. The second formula comes from the effect on the initial coefficients $c(e_i)$ for $e_i \in \ell^s_\nu$, of the operation taking the s-jet of an arc. This operation is described by taking the image by the map

$$\Psi : T' := \mathrm{Spec}\, k[c(e_i)^{\pm 1}]_{e_i \in \ell^s_\nu} \to T := \mathrm{Spec}\, k[c(e_i)^{\pm 1}]_{\langle \nu, e_i\rangle \leq s},$$

of the point determined by $h \in H^*_\nu$. The map Ψ is the unramified covering of l-dimensional algebraic tori determined by the inclusion $M^s_\nu \subset \ell^s_\nu \cap M$ of index $q(\nu, s)$ of rank $l(\nu, s)$ lattices. Thus the second formula is equivalent to $\psi_2 : (\exists y \in T')(\Psi(y) = x, \text{ and } x \in T)$, hence by Lemma 1.2 we get that $\chi_f([\psi_2]) = \frac{1}{q(\nu,s)} (\mathbf{L} - 1)^l$. □

6. Sequences of convex piece-wise linear functions and fans

Let $\sigma \subset N_{\mathbf{R}}$ be a rational convex polyhedral cone of dimension $d = \dim N_{\mathbf{R}}$. Consider a sequence of piece-wise linear continuous functions

$$h_p : \sigma \to \mathbf{R},\ \text{for } 1 \leq p \leq m,$$

such that $h_p(\sigma \cap N) \subset \mathbf{Z}$, and

$$0 \leq h_1(\nu) \leq \cdots \leq h_m(\nu) \quad \forall \nu \in \sigma. \tag{5}$$

By convenience we set $h_0(\nu) = 0$ and $h_{m+1}(\nu) = +\infty$. We denote by Ξ_0 the fan consisting on the faces of σ and by Ξ_p the coarser fan such that the restriction of h_p to η is linear for any cone $\eta \in \Xi_p$ for $1 \leq p \leq m$. In addition we assume that for any cone $\eta \in \Xi_{p-1}$ the restriction $h_{p|\eta}$ is *upper convex*, that is $h_p(\nu) + h_p(\nu') \leq h_p(\nu + \nu')$ for all $\nu, \nu' \in \eta$.

Notation 6.1. For $0 \leq p \leq m$ and for $\eta \in \cap_{r=1}^{p} \Xi_r$ we set

$$\eta(\underline{h}, p) := \{(\nu, s) \in N_{\mathbf{R}} \times \mathbf{R}_{\geq 0} \mid \nu \in \overset{\circ}{\sigma} \cap \overset{\circ}{\eta},\ h_p(\nu) \leq s < h_{p+1}(\nu)\}.$$

Lemma 6.2. *The closure $\bar{\eta}(\underline{h}, p)$ of the set $\eta(\underline{h}, p)$ is a convex polyhedral cone which is rational for the lattice $N \times \mathbf{Z}$.*

Proof. If $\eta \in \cap_{r=0}^{p} \Xi_r$ then the restriction $h_{j|\eta} \colon \eta \to \mathbf{R}$ is linear if $j = p$ and upper convex if $j = p + 1$. It follows that $\bar{\eta}(\underline{h}, p)$ is a convex polyhedral cone, rational for the lattice $N \times \mathbf{Z}$ since h_p and h_{p+1} take integral values on N. □

Notation 6.3. For $0 \leq p \leq m$ and $\eta \in \Xi_p$ we define the following sets:

(i) $A(\underline{h}, p) := \{(\nu, s) \in N \times \mathbf{Z} \mid \nu \in \overset{\circ}{\sigma},\ h_p(\nu) \leq s < h_{p+1}(\nu)\}$.

(ii) $A(\underline{h}, p, \eta) := \{(\nu, s) \in N \times \mathbf{Z} \mid \nu \in \overset{\circ}{\sigma} \cap \overset{\circ}{\eta},\ h_p(\nu) \leq s < h_{p+1}(\nu)\}$.

Remark 6.4. We have partitions

$$(\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{\geq 0} = \bigsqcup_{p=0}^{m} A(\underline{h}, p) \quad \text{and} \quad A(\underline{h}, p) = \bigsqcup_{\eta \in \cap_{r=0}^{p} \Xi_r} A(\underline{h}, p, \eta).$$

7. Refinements of partitions

We apply the procedure of Section 6 to both sequences $\underline{\phi} = (\phi_1, \ldots, \phi_d)$ and $\underline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ (see Lemma 3.4).

Remark 7.1. Notice that the sequence of fans associated to $\underline{\phi}$ (resp. $\underline{\varphi}$) is $\cap_{r=0}^{i} \Sigma_r$, $i = 0, \ldots, d$ (resp. $\cap_{r=0}^{i} \Theta_r$, $i = 0, \ldots, n$), where for convenience we denote by Σ_0 or by Θ_0 the fan consisting of the faces of the cone σ.

Lemma 7.2. *If $A(\underline{\varphi}, j, \theta) \neq \emptyset$ for some $1 \leq j \leq n$ and $\theta \in \cap_{r=1}^{j} \Theta_r$ (cf. Notation 6.3) then the restriction of the functions $(\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0} \to \mathbf{Z}_{\geq 0}$ given by*

$$(\nu, s) \mapsto l(\nu, s), \quad \text{and} \quad (\nu, s) \mapsto q(\nu, s),$$

to the set $A(\underline{\varphi}, j, \theta)$ are constant functions. We denote their values on the set $A(\underline{\varphi}, j, \theta)$ by $l(j, \theta)$ and $q(j, \theta)$ respectively.

Proof. By elementary properties of Minkowski sums every vector in the relative interior of θ, for $\theta \in \cap_{r=1}^{j} \Theta_r$, defines the same face $\mathcal{F}_{r,\theta}$ of the polyhedron $\mathcal{N}(\mathcal{C}_r)$ for $1 \leq r \leq j$. Suppose that $\nu, \nu' \in \overset{\circ}{\theta}$ and $(p_1, \ldots, p_n)$ and $(p'_1, \ldots, p'_n)$ are two permutations of $(1, \ldots, n)$ such that the inequalities

$$\langle \nu, e_{p_1} \rangle \leq \cdots \leq \langle \nu, e_{p_n} \rangle \text{ and } \langle \nu', e_{p'_1} \rangle \leq \cdots \leq \langle \nu', e_{p'_n} \rangle, \tag{6}$$

hold. We prove first that

$$\langle \nu, e_{p'_1} \rangle \leq \cdots \leq \langle \nu, e_{p'_j} \rangle. \tag{7}$$

By definition, for any $1 \leq r \leq j$ we have that $\mathrm{ord}_{\mathcal{C}_r}(\nu) = \langle \nu, u_r \rangle$ for any $u_r \in \mathcal{F}_{r,\theta}$. We get from Lemma 3.4 that the vectors $u_r := e_{p_1} + \cdots + e_{p_r}$ and $u'_r := e_{p'_1} + \cdots + e_{p'_r}$ belong to $\mathcal{F}_{r,\theta}$ for $1 \leq r \leq j$. This implies (7).

If $(\nu, s) \in A(\underline{\varphi}, j, \theta)$ then by Lemma 3.4 we obtain that $\varphi_j(\nu) = \langle \nu, e_{p_j} \rangle \leq s < \varphi_{j+1}(\nu)$. We deduce that if (ν, s) and (ν', s') belong to $A(\underline{\varphi}, j, \theta)$ then

$$\{e_i \mid 1 \leq i \leq n,\ \langle \nu, e_i \rangle \leq s\} = \{e_i \mid 1 \leq i \leq n,\ \langle \nu', e_i \rangle \leq s'\} = \{e_{p_1}, \ldots, e_{p_j}\}. \tag{8}$$

Since (8) spans the lattice M_ν^s and the vector space ℓ_ν^s we get that the sublattices $\ell_\nu^s \cap M$ and M_ν^s are independent of the choice of (ν, s) in $A(\underline{\varphi}, j, \theta)$. This implies the constancy of the functions l and q on $A(\underline{\varphi}, j, \theta)$. $\square$

Remark 7.3. If $1 \leq l \leq d$ and if $\tau \in \cap_{r=1}^{l} \Sigma_r$ we denoted in [2] the set $A(\underline{\phi}, l)$ (resp. $A(\underline{\phi}, l, \tau)$) by A_l (resp. by $A_{l,\tau}$). The map $l(\nu, s)$ is also constant on the sets of the form $A(\underline{\phi}, l, \tau)$ for $\tau \in \cap_{i=0}^{l} \Sigma_i$, see Lemma 5.7 [2].

By Remark 6.4 we have two partitions:

$$(9) \qquad (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0} = \bigsqcup_{j=0}^{n} \bigsqcup_{\theta \in \cap_{r=0}^{j} \Theta_r} A(\underline{\varphi}, j, \theta) \ \text{ and } \ (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0} = \bigsqcup_{l=0}^{d} \bigsqcup_{\eta \in \cap_{i=0}^{l} \Sigma_i} A(\underline{\phi}, l, \eta),$$

associated to the sequences $\underline{\varphi}$ and $\underline{\phi}$.

Proposition 7.4. *If $\theta(\underline{\varphi}, j) \neq \emptyset$ for some $1 \leq j \leq n$ and $\theta \in \cap_{r=1}^{j} \Theta_r$ then there exists a unique cone $\tau \in \cap_{r=1}^{l(j,\theta)} \Sigma_r$ such that $\theta \subset \tau$ and*

$$(10) \qquad A(\underline{\varphi}, j, \theta) \subset A(\underline{\phi}, l(j, \theta), \tau).$$

Proof. Given (ν, s) and (ν', s') in $A(\underline{\varphi}, j, \theta) \subset (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$, we deduce from (9) that there exist cones $\tau \in \cap_{i=0}^{l} \Sigma_i$ and $\tau' \in \cap_{i=0}^{l'} \Sigma_i$ for integers $0 \leq l, l' \leq d$ such that $(\nu, s) \in A(\underline{\phi}, l, \tau)$ and $(\nu', s') \in A(\underline{\phi}, l', \tau')$. By Lemma 5.7 [2] we have that $l = l(\nu, s)$ and $l' = l(\nu', s')$, and then $l = l'$ by (8). Notice then that $l = l(j, \theta)$ by definition in Lemma 7.2.

Let $(p_1, \dots, p_n)$ and $(p'_1, \dots, p'_n)$ be two permutations of $(1, \dots, n)$ such that (6) holds. Then we can apply the method given in Proposition 5.1 [2] to determine the value of $\mathrm{ord}_{\mathcal{J}_i}(\nu)$, $1 \leq i \leq l(\nu, s)$. Moreover, it is enough to apply this on the set (8) instead of on $\{e_1, \dots, e_n\}$. We deduce from (7) that ν and ν' define the same face of $\mathcal{N}(\mathcal{J}_i)$ for $1 \leq i \leq l(\nu, s)$. This is equivalent to the equality $\tau = \tau'$. We have proven (10) and, as a consequence, the inclusion $\theta \subset \tau$ holds. $\square$

Definition 7.5. (see Definition 8.1 and Remark 8.6 [2]). We consider the equivalence relation $\sim$ defined on the set $(\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$ by:

$$(\nu, s) \sim (\nu', s') \quad \Leftrightarrow \quad s = s',\, \ell_\nu^s = \ell_{\nu'}^s \text{ and } \nu_{|\ell_\nu^s} = \nu'_{|\ell_{\nu'}^s}.$$

Lemma 7.6. *The set $A(\underline{\varphi}, j, \theta)$ is union of equivalence classes by the relation $\sim$ of Definition 7.5, for $1 \leq j \leq n$ and $\theta \in \cap_{r=1}^{j} \Theta_r$. Moreover we have that*

$$(11) \qquad A(\underline{\phi}, l, \tau)/_\sim = \bigsqcup_{\theta \in \cap_{r=1}^{j} \Theta_r,\, l(j,\theta) = l}^{\theta \subset \tau} A(\underline{\varphi}, j, \theta)/_\sim.$$

Proof. By (9) and Proposition 7.4 it follows that $A(\underline{\phi}, l, \tau) = \bigsqcup_{\theta \in \cap_{r=1}^{j} \Theta_r,\, l(j,\theta) = l}^{\theta \subset \tau} A(\underline{\varphi}, j, \theta)$. If (ν, s) belongs to $A(\underline{\varphi}, j, \theta)$ and $(\nu, s) \sim (\nu', s)$ then (8) holds. The vectors ν and ν' define the same face of $\mathcal{N}(\mathcal{C}_r)$ for $1 \leq r \leq j$, and therefore $\nu' \in \mathrm{int}\theta$. Since $\varphi_j(\nu') \leq s < \varphi_{j+1}(\nu')$ we conclude that $(\nu', s) \in A(\underline{\varphi}, j, \theta)$. $\square$

8. The structure of the series $P_{\mathrm{ar}}^{(Z^\Lambda, 0)}(T)$

We consider the following auxiliary Poincaré series:

$$(12) \qquad P_{\mathrm{ar}}(\Lambda) := \sum_{s \geq 0} \chi_f([j_s(H_\Lambda) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_s(H_{\Lambda \cap \theta^\perp})]_f) T^s \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]].$$

Notice that the Poincaré series $P_{\mathrm{ar}}(\Lambda)$ measures the class of the formula defining the set of jets of arcs with origin in 0 which are not jets of arcs factoring through proper orbit closures of the toric variety Z^Λ.

Proposition 8.1. *We have that* $P_{\mathrm{ar}}^{(Z^\Lambda, 0)}(T) = \sum_{\theta \leq \sigma} P_{\mathrm{ar}}(\Lambda \cap \theta^\perp)$.

It follows from Proposition 8.1 that in order to describe the motivic series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ it is enough to describe the form of the auxiliary series $P_{\mathrm{ar}}(\Lambda)$ for any semigroup Λ.

Remark 8.2. In the normal case the equality $j_m(H_\Lambda) = j_m(H_\Lambda^*)$ holds for all $m \geq 0$, see [18], but this property fails in general.

We recall the following definition from [2].

Definition 8.3. (see [2] Definition 8.9) If $1 \leq l \leq d$ the set $\mathcal{D}_l$ is the subset of cones $\tau \in \bigcap_{i=1}^{l} \Sigma_i$ such that the face $\mathcal{F}_\tau$ of $\mathcal{N}(\mathcal{J}_l)$ is contained in the interior of $\sigma^\vee$. We denote by $|\mathcal{D}_l|$ the set $\cup_{\tau \in \mathcal{D}_l} \tau$.

Proposition 8.4. *Let us fix an integer $s_0 \geq 1$. The set $j_{s_0}(H_\Lambda^*) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_{s_0}(H_{\Lambda \cap \theta^\perp})$ expresses as a finite disjoint union of locally closed subsets, as follows:*

$$j_{s_0}(H_\Lambda^*) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_{s_0}(H_{\Lambda \cap \theta^\perp}) = \bigsqcup_{j=1}^{n} \bigsqcup_{\theta \in \cap_{r=1}^{j} \Theta_r}^{\theta \subset |\mathcal{D}_{l(j,\theta)}|} \bigsqcup_{[(\nu, s_0)] \in A(\underline{\varphi}, j, \theta)/\sim} j_{s_0}(H_{\Lambda,\nu}^*). \tag{13}$$

Proof. This partition follows from the partition given in Proposition 8.11 [2] by using Formula (11) (see Remark 7.3). □

If $s_0 \geq 1$ the coefficient of T^{s_0} in the auxiliary series $P(\Lambda)$ is obtained by applying the map χ_f to the class of the formula defining (13). Then we determine this class by using Proposition 5.2.

We introduce the following auxiliary series for $\theta \in \cap_{r=1}^{j} \Theta_r$:

$$P_{\underline{\varphi},j,\theta}(\Lambda) := (\mathbf{L} - 1)^{l(j,\theta)} \sum_{s \geq 1} \sum_{[(\nu,s)] \in A(\underline{\varphi},j,\theta)/\sim} \mathbf{L}^{l(j,\theta)s - \mathrm{ord}_{\mathcal{J}_{l(j,\theta)}}(\nu)} T^s. \tag{14}$$

We deduce the following Proposition from Proposition 8.4 and Formula (14).

Proposition 8.5. *We have that*

$$P_{\mathrm{ar}}(\Lambda) = \sum_{j=1}^{n} \sum_{\theta \in \cap_{r=1}^{j} \Theta_r}^{\theta \subset |\mathcal{D}_{l(j,\theta)}|} \frac{1}{q(j,\theta)} P_{\underline{\varphi},j,\theta}(\Lambda). \tag{15}$$

9. The rational form of some generating series

In this Section we fix an integer $1 \leq j \leq n$ and a cone $\theta \in \cap_{r=1}^{j} \Theta_r$ such that $A(\underline{\varphi}, j, \theta) \neq \emptyset$. For simplicity we denote by l the integer $l(j,\theta)$ defined in Lemma 7.2 and by τ the unique element of the fan $\cap_{r=1}^{l} \Sigma_r$ such that (10) holds.

Since $\theta \subset \tau \subset \cap_{r=1}^{l} \Sigma_r$ the restriction of ϕ_r to θ is a linear function of the form

$$(\phi_r)_{|\theta}(\nu) = \langle \nu, e_{i_r} \rangle, \text{ for } r = 1, \dots, l$$

where $\{i_1, \dots, i_l\} \subset \{1, \dots, n\}$.

Consider the lattice homomorphisms

$$\mu \colon N \times \mathbf{Z} \longrightarrow \mathbf{Z}^{l+1}, \quad (\nu, s) \mapsto (\langle \nu, e_{i_1} \rangle, \dots, \langle \nu, e_{i_l} \rangle, s)$$

and

$$\pi = (\pi_1, \pi_2) \colon \mathbf{Z}^{l+1} \longrightarrow \mathbf{Z}^2, \quad (a_1, \dots, a_{l+1}) \mapsto (l a_{l+1} - a_1 - \dots - a_l, a_{l+1}).$$

We set $\xi = \pi \circ \mu$.

Remark 9.1. The homomorphisms π, μ and ξ were also considered in [2]. Since θ is contained in τ we get that the kernels of μ and ξ intersect the cone θ only at the origin. Similarly by Formula (10) the inclusion $\xi(A(\underline{\varphi}, j, \theta)) \subset \mathbf{Z}_{\geq 0}^2 \setminus \{(0,0)\}$ holds. See [2] Section 9.

If $j \neq n$ the *lower boundary* of the cone θ is the set $\partial_- \theta := \{(\nu, s) \mid \nu \in \theta, \ s = \varphi_j(\nu)\}$. Notice that $\partial_- \theta$ is a cone since $\theta \in \cap_{r=1}^{j} \Theta_j$ and then the function φ_j is linear on θ. The *upper boundary* is the set $\partial_+ \theta(\underline{\varphi}, j) := \{(\nu, s) \mid \nu \in \theta, \ s = \varphi_{j+1}(\nu) \neq \varphi_j(\nu)\}$. If $j = n$ then $l = d$ and $\varphi_{n+1}(\nu) = +\infty$ and the upper boundary is the union of cones spanned by $(0,1) \in N_{\mathbf{R}} \times \mathbf{R}$ and the proper faces of the cone $\partial_- \theta(\underline{\varphi}, j)$. The edges of the cone $\theta(\underline{\varphi}, j)$ are edges of $\partial_- \theta(\underline{\varphi}, j) \cup \partial_+ \theta(\underline{\varphi}, j)$.

Notation 9.2. If $\rho \subset \tau$ is a one-dimensional cone rational for the lattice N we denote by ν_ρ the primitive integral vector on ρ, that is, the generator of the semigroup $\rho \cap N$.

Remark 9.3. The primitive integral vectors for the lattice $N \times \mathbf{Z}$ on the edges of the cone θ are

$$(\nu_\rho, \varphi_j(\nu_\rho)) \quad \text{for} \quad \rho \leq \theta,\, \dim \rho = 1$$

together with

$$\begin{cases} (\nu_\rho, \varphi_{j+1}(\nu_\rho)) \ \text{ for } \rho \in \Theta_{j+1},\ \rho \subset \theta,\ \dim \rho = 1 \ \text{ and } \varphi_j(\nu) \neq \varphi_{j+1}(\nu) & \text{if} \quad j \neq n \\ (0,1) & \text{if} \quad j = n. \end{cases}$$

Then notice that

$$\xi(\nu, s) = \begin{cases} (l\varphi_j(\nu_\rho) - \mathrm{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_j(\nu_\rho)) & \text{if} \quad (\nu, s) = (\nu_\rho, \varphi_j(\nu_\rho)) \\ (l\varphi_{j+1}(\nu_\rho) - \mathrm{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_{j+1}(\nu_\rho)) & \text{if} \quad (\nu, s) = (\nu_\rho, \varphi_{j+1}(\nu_\rho)) \\ (d, 1) & \text{if} \quad (\nu, s) = (0, 1). \end{cases} \tag{16}$$

Definition 9.4. Suppose that $A(\underline{\varphi}, j, \theta) \neq \emptyset$. We denote by $B_{\underline{\varphi},j,\theta}(\Lambda)$ the finite set:

$$\{(l\varphi_j(\nu_\rho) - \mathrm{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_j(\nu_\rho)) \mid \rho \leq \theta, \mathrm{dim}\rho = 1\}$$
$$\cup \begin{cases} \{(l\varphi_{j+1}(\nu_\rho) - \mathrm{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_{j+1}(\nu_\rho)) \mid \rho \in \Theta^{(1)}_{j+1}, \rho \subset \theta\} & \text{if} \quad j \neq n, \\ \\ \{(d, 1)\} & \text{if} \quad j = n. \end{cases}$$

Definition 9.5. If $A \subset \mathbf{Z}^{l+1}$ is a set we denote by $F_A(x) := \sum_{a \in A} x^a$ the *generating function* of A (see Section 12 of [2]) .

Proposition 9.6. *We have the following equality:*

$$P_{\underline{\varphi},j,\theta}(\Lambda) = (\mathbf{L} - 1)^{l(j,\theta)} \sum_{a \in \mu(A(\underline{\varphi},j,\theta))} \mathbf{L}^{\pi_1(a)} T^{\pi_2(a)} \in \mathbf{Z}[\mathbf{L}][[T]]. \tag{17}$$

There exists a polynomial $R_{\underline{\varphi},j,\theta} \in \mathbf{Z}[\mathbf{L}, T]$ such that $P_{\underline{\varphi},j,\theta}(\Lambda)$ has the rational form:

$$P_{\underline{\varphi},j,\theta}(\Lambda) = R_{\underline{\varphi},j,\theta} \prod_{(a,b) \in B_{\underline{\varphi},j,\theta}(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}.$$

Proof. The map μ defines a bijection

$$A(\underline{\varphi}, j, \theta)/_\sim \longrightarrow \mu(A(\underline{\varphi}, j, \theta)), \quad [(\nu, s)] \mapsto \mu(\nu, s).$$

(see Lemma 9.3 [2] and Lemma 7.6). Then the equality (17) follows from the definitions.

We denote by $\pi_* : k[[\mathbf{Z}^{l+1}]] \to k[[\mathbf{L}, T]]$ the monomial transformation defined by $\pi_*(x^a) := \mathbf{L}^{\pi_1(a)} T^{\pi_2(a)}$ for $a \in \mathbf{Z}^{l+1}$. Then we get that

$$P_{\underline{\varphi},j,\theta}(\Lambda) = (\mathbf{L} - 1)^{l(j,\theta)} \pi_*(F_{\mu(A_{\underline{\varphi},j,\theta})}(x)).$$

We apply the Theorem 12.4 of [2]. We obtain that the denominator of a rational form of $F_{\mu(A_{\underline{\varphi},j,\theta})}(x)$ consists of products of terms $1 - x^{\mu(b)}$ for b running through the primitive integral vectors in the edges of the closure of the cone $\theta(\underline{\varphi}, j)$. Then the result follows by Remark 9.3 and Definition 9.4. □

10. Main results

We summarize the main results of the paper.

Definition 10.1.

(i) If $0 \leq \eta < \sigma$ then $B_{\mathrm{ar}}(\Lambda \cap \eta^\perp)$ is the finite subset of $\mathbf{Z}^2_{\geq 0}$ given by Definition 9.4 when we replace Λ by the semigroup $\Lambda \cap \eta^\perp$. If $\eta = \sigma$ we set $B_{\mathrm{ar}}(\Lambda \cap \sigma^\perp) := \{(0, 1)\}$. We define the finite sets:

$$B_{\mathrm{ar}}(\Lambda) := \bigcup_{\theta \in \cap_{r=1}^j \Theta_r,\, \theta \subset |\mathcal{D}_{l(j,\theta)}|}^{1 \leq j \leq n} B_{\underline{\varphi},j,\theta}(\Lambda) \quad \text{and} \quad B_{\mathrm{ar},\Lambda} := \bigcup_{0 \leq \eta \leq \sigma} B_{\mathrm{ar}}(\Lambda \cap \eta^\perp).$$

(ii) We define the integer

$$q(\Lambda) := \mathrm{lcm}\{q(j,\theta) \mid \theta \in \cap_{r=1}^{j}\Theta_r,\ \theta \subseteq |D_{l(j,\theta)}|, 1 \leq j \leq n\}. \tag{18}$$

If $\eta < \sigma$ then $q(\Lambda \cap \eta^{\perp})$ is the number obtained by replacing Λ by $\Lambda \cap \eta^{\perp}$ in (18). We set $q(\Lambda \cap \sigma^{\perp}) := 1$. We define also the integer

$$q_{\Lambda} := \mathrm{lcm}\{q(\Lambda \cap \eta^{\perp}) \mid \eta \leq \sigma\}.$$

Theorem 10.2. *Suppose that the field k contains all $q(\Lambda)$-th roots of unity. Then there exists a polynomial $Q_{\mathrm{ar}}(\Lambda) \in \mathbf{Z}[\mathbf{L}, T]$ such that*

$$P_{\mathrm{ar}}(\Lambda) = \frac{1}{q(\Lambda)} Q_{\mathrm{ar}}(\Lambda) \prod_{(a,b)\in B_{\mathrm{ar}}(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}.$$

Proof. This follows from Propositions 8.5 and 9.6. □

Notation 10.3. If $\eta < \sigma$ then the polynomial $Q_{\mathrm{ar}}(\Lambda \cap \eta^{\perp})$ is obtained from Theorem 10.2 by replacing Λ by the semigroup $\Lambda \cap \eta^{\perp}$. We set $Q_{\mathrm{ar}}(\Lambda \cap \sigma^{\perp}) := 1$.

Corollary 10.4. *If the field k contains all q_{Λ}-th roots of unity then there exists a polynomial $Q_{\mathrm{ar},\Lambda} \in \mathbf{Z}[\mathbf{L}, T]$ such that such that*

$$P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T) = \frac{1}{q_{\Lambda}} Q_{\mathrm{ar},\Lambda} \prod_{(a,b)\in B_{\mathrm{ar},\Lambda}} (1 - \mathbf{L}^a T^b)^{-1}.$$

Moreover, we have the equality:

$$P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T) = \sum_{\eta\leq\sigma} \frac{1}{q(\Lambda\cap\eta^{\perp})} Q_{\mathrm{ar}}(\Lambda\cap\eta^{\perp}) \prod_{(a,b)\in B_{\mathrm{ar}}(\Lambda\cap\eta^{\perp})} (1 - \mathbf{L}^a T^b)^{-1}. \tag{19}$$

Proof. The result follows by Theorem 10.2 and Proposition 8.1. □

We can compare at this moment the series $P_{\mathrm{geom}}^{(Z,0)}(T)$ and $P_{\mathrm{ar}}^{(Z,0)}(T)$ (see Definition 2.2). In [2] we introduced the series

$$P_{\mathrm{geom}}(\Lambda) := \sum_{s\geq 0} \chi_c([j_s(H_{\Lambda}^{*}) \setminus \bigcup_{0\neq\theta\leq\sigma} j_s(H_{\Lambda\cap\theta^{\perp}})]) T^s \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}[[T]], \tag{20}$$

and we proved that

$$P_{\mathrm{geom}}^{(Z^{\Lambda},0)}(T) = \sum_{\theta\leq\sigma} P_{\mathrm{geom}}(\Lambda\cap\theta^{\perp}).$$

Proposition 10.5. *If the field k contains all $q(\Lambda)$-th roots of unity, then*

$$P_{\mathrm{ar}}(\Lambda) - P_{\mathrm{geom}}(\Lambda) = \sum_{j=1}^{n} \sum_{\theta\in\cap_{r=1}^{j}\Theta_r}^{\theta\subset|\mathcal{D}_{l(j,\theta)}|} (1 - \frac{1}{q(j,\theta)})\ R_{\underline{\varphi},j,\theta} \prod_{(a,b)\in B_{\underline{\varphi},j,\theta}(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}.$$

Proof. This follows from Proposition 9.6, Formula (20), Theorem 10.2, and the results in [2] for $P_{\mathrm{geom}}(\Lambda)$. □

Corollary 10.6. *If for every integer $1 \leq l \leq d$, and any vertex v of the Newton polyhedra $\mathcal{N}(\mathcal{J}_l)$ there exists a subset $I_v \subset \{1, \ldots, n\}$ of l elements such that $v = \sum_{i\in I_v} e_i$ and the vectors $e_i, i \in I_v$, form part of a basis of M then the series $P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T)$ and $P_{\mathrm{geom}}^{(Z^{\Lambda},0)}(T)$ coincide .*

Proof. This condition implies that $q(\nu, s) = 1$ for every $(\nu, s) \in (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$. By Proposition 10.5 we get that $P_{\mathrm{ar}}(\Lambda) = P_{\mathrm{geom}}(\Lambda)$. Now for any face $\eta \leq \sigma$ the vertices of the Newton polyhedra of the logarithmic jacobian ideals of $\Lambda \cap \eta^{\perp}$ are also vertices of the logarithmic jacobian ideals of Λ. The hypothesis implies that $\Lambda\cap\eta^{\perp}$ spans the lattice $M\cap\eta^{\perp}$ and then also that $P_{\mathrm{ar}}(\Lambda\cap\eta^{\perp}) = P_{\mathrm{geom}}(\Lambda\cap\eta^{\perp})$. □

Remark 10.7. Corollary 10.6 is a generalization of Nicaise condition in the case of normal toric varieties (see Theorem 1 [18]).

Remark 10.8. The coordinates of the vectors in the set $B_{\underline{\varphi},j,\theta}(\Lambda)$ can be described geometrically in terms of the ideals $\mathcal{C}_j$ and $\mathcal{J}_l$, for $l = l(j,\theta)$. Let $\pi_j : Z_j \to Z^\Lambda$ be the composite of the normalization of Z^Λ with the toric modification defined by the subdivision $\cap_{r=1}^{j}\Theta_r$ of σ. The modification π_j is the minimal toric modification which factors through the normalized blowing up with center C_r, for $r = 1, \dots, j$. If ρ is an edge of θ the orbit closure E_ρ of the orbit associated to ρ on Z_j has codimension 1. We denote by v_ρ the divisorial valuation defined by E_ρ. It verifies that $v_\rho(X^m) = \langle \nu_\rho, m\rangle$, for $m \in M$. The pull back $\pi_j^*(\mathcal{I})$ of a monomial ideal $\mathcal{I}$ of Z^Λ is a sheaf of monomial ideals on Z_j. The ideals $\pi_j^*(\mathcal{C}_r)$, $r = 1, \dots, j$ are locally principal on Z_j. Then we get the following identities:

$$\varphi_j(\nu_\rho) = v_\rho(\pi_j^*(\mathcal{C}_j)) - v_\rho(\pi_j^*(\mathcal{C}_{j-1})), \varphi_{j+1}(\nu_\rho) = v_\rho(\pi_j^*(\mathcal{C}_{j+1})) - v_\rho(\pi_j^*(\mathcal{C}_j)), \mathrm{ord}_{\mathcal{J}_l}(\nu_\rho) = v_\rho(\pi_j^*(\mathcal{J}_l)).$$

Compare with the geometrical description of the set of candidate poles of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$, see [2].

11. The normal case

In the normal case, when the semigroup Λ is saturated, i.e., $\Lambda = \sigma^\vee \cap M$ we describe the motivic arithmetic series in a simpler way by using that $j^s(H_\Lambda) = j_s(H_\Lambda^*)$ (see [18]).

Notation 11.1.

(i) $\mathcal{A} = \bigsqcup_{l=1}^{d} \bigsqcup_{\tau \in \cap_{r=1}^{l}\Sigma_r} A(\underline{\phi}, l, \tau)/\sim$.

(ii) For $s_0 \geq 0$ we set $\mathcal{A}_{s_0} = \{[(\nu, s)] \in \mathcal{A} \mid s = s_0\}$.

Remark 11.2. The set $\mathcal{A}_s$ is finite (see Remark 8.2 in [2]). By (9) and Lemma 7.6 we deduce that $\mathcal{A} = \bigsqcup_{j=1}^{n} \bigsqcup_{\theta \in \cap_{r=1}^{j}\Theta_r} A(\underline{\varphi}, j, \theta)/\sim$.

Proposition 11.3. *Let us fix an integer $s_0 \geq 1$. The set $j_{s_0}(H^*)$ expresses as a finite disjoint union of locally closed subsets as $j_{s_0}(H^*) = \bigsqcup_{[(\nu,s)]\in\mathcal{A}_{s_0}} j_{s_0}(H_\nu^*)$. We deduce that $\chi_f([j_s(H^*)]_f) = \sum_{[(\nu,s)]\in\mathcal{A}_s} \chi_f([j_s(H_\nu^*)]_f)$.*

Proof. The first assertion follows by applying the method of Proposition 8.11 of [2]. The second assertion is a consequence of the first and Proposition 5.2. □

Theorem 11.4. *If Z^Λ is normal then we have*

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)} = \sum_{j=1}^{n} \sum_{\theta \in \cap_{r=1}^{j}\Theta_r} \frac{1}{q(j,\theta)} R_{\underline{\varphi},j,\theta} \prod_{(a,b)\in B_{\underline{\varphi},j,\theta}(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}.$$

Proof. It is consequence of Proposition 11.3, Remark 11.2 and Proposition 9.6. □

Corollary 11.5. *Suppose that the affine toric variety Z^Λ is normal. If $\theta \leq \sigma$ we denote by $\sigma_\theta^\vee$ the image of the cone $\sigma^\vee$ in $(M_\theta)_{\mathbf{R}}$, where $M_\theta := M/\theta^\perp \cap M$ and by $\Lambda(\theta)$ the saturated semigroup $\Lambda(\theta) := (\sigma_\theta^\vee \cap M_\theta) \times \mathbf{Z}_{\geq 0}^{\mathrm{codim}\theta}$. With this notation we have*

$$P_{\mathrm{ar}}^{Z^\Lambda}(T) = \sum_{\theta\leq\sigma} (\mathbf{L} - 1)^{\mathrm{codim}\theta} P_{\mathrm{ar}}^{(Z^{\Lambda(\theta)},0)}(T).$$

Proof. The proof follows by the same arguments as in Corollary 4.11 [2]. □

12. Examples

12.1. **The case of monomial curves.** Let $\Lambda \subset \mathbf{Z}_{\geq 0}$ be a semigroup with minimal system of generators $e_1 < e_2 < \cdots < e_n$ such that $\gcd\{e_1, \dots, e_n\} = 1$. In this case we have that $\overset{\circ}{\sigma} \cap N = \mathbf{Z}_{>0}$. If $q_i := \gcd\{e_1, \dots, e_i\}$ then we obtain that:

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = \frac{1}{1-T} + \frac{\mathbf{L}-1}{1-\mathbf{L}T}\left(\frac{1}{q_1}\frac{T^{e_1}}{1-T^{e_1}} + \sum_{i=2}^{n} \frac{q_{i-1}-q_i}{q_{i-1}q_i} \frac{\mathbf{L}^{e_i-e_1}T^{e_i}}{1-\mathbf{L}^{e_i-e_1}T^{e_i}}\right). \tag{21}$$

This follows from the results of this paper taking the following observations into account:

- We have the equality $j_s(H) = j_s(H^*)$.

- If $\nu, \nu' \in \mathbf{Z}_{>0}$ verify that $j_s(H^*_\nu)$, $j_s(H^*_{\nu'}) \neq \{0\}$ then the equality $j_s(H^*_\nu) = j_s(H^*_{\nu'})$ implies that $\nu = \nu'$.
- If $\nu \in \mathbf{Z}_{>0}$ verifies that $\nu e_i \leq s < \nu e_{i+1}$ then $q(\nu, s) = q_i$.

Then, setting $e_{d+1} := \infty$, we get the following equality which implies (21):

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = \frac{1}{1-T} + \sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \sum_{s=\nu e_i}^{\nu e_{i+1}-1} (\mathbf{L}-1)\frac{1}{q_i}\mathbf{L}^{s-\nu e_1} T^s.$$

Remark 12.1. The inequalities $q_1 \geq q_2 \geq \cdots \geq q_n = 1$ are not always strict. For instance if Λ is generated by $e_1 = 8, e_2 = 18, e_3 = 20$ and $e_4 = 21$ then we get $q_1 = 8$, $q_2 = q_3 = 2$, $q_4 = 1$. It follows from (21) that the term $1 - \mathbf{L}^{12}T^{20}$ is not a factor of the denominator of the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$. Notice that if Λ' is the semigroup generated by e_1, e_2 and e_4 then we obtain from (21) that $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = P_{\mathrm{ar}}^{(Z^{\Lambda'},0)}(T)$ while the semigroups Λ and Λ' are not isomorphic. In contrast with this behavior, the motivic series $P_{\mathrm{ar}}^{(C,0)}(T)$ of a plane branch $(C,0)$ determines the semigroup of the branch $(C,0)$ (see [7]).

12.2. **An example of non-normal toric surface.** Consider the semigroup Λ generated by the vectors $e_1 = (5,0), e_2 = (0,2), e_3 = (0,3)$ and $e_4 = (6,2)$. The cone σ is $\mathbf{R}^2_{\geq 0}$ and the lattice M is equal to $\mathbf{Z}^2$. We have the semigroups $\Lambda \cap \eta_1^\perp = (5,0)\mathbf{Z}_{>0}$, and $\Lambda \cap \eta_2^\perp = (0,2)\mathbf{Z}_{>0} + (0,3)\mathbf{Z}_{>0}$, where η_1 and η_2 are the one-dimensional faces of σ. By the case of monomial curves we get that:

$$P_{\mathrm{ar}}(\Lambda \cap \eta_1^\perp) = \tfrac{\mathbf{L}-1}{1-\mathbf{L}T}\tfrac{T}{1-T} \text{ and } P_{\mathrm{ar}}(\Lambda \cap \eta_2^\perp) = \tfrac{\mathbf{L}-1}{2(1-\mathbf{L}T)}\left(\tfrac{T^2}{1-T^2} + \tfrac{\mathbf{L}T^3}{1-\mathbf{L}T^3}\right).$$

The subdivisions associated with the ideals $\mathcal{C}_r$, $r = 1, \ldots, 4$ are indicated in Figure 1

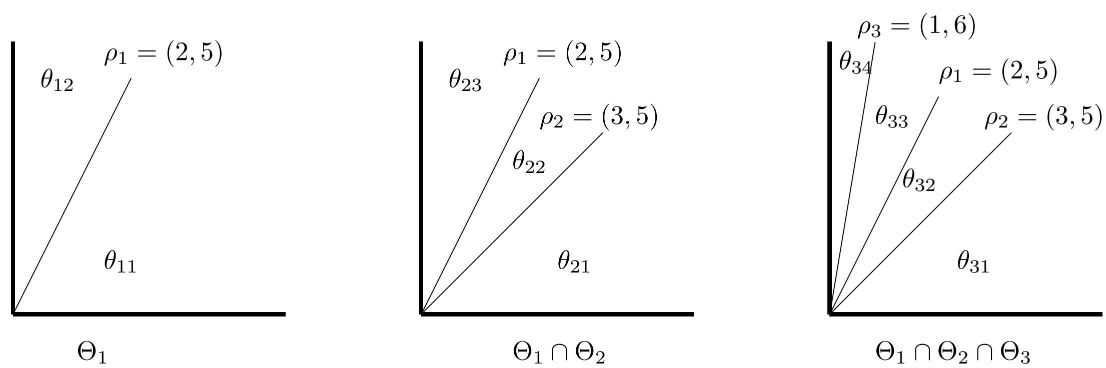

FIGURE 1. The subdivisions Θ_1, $\Theta_1 \cap \Theta_2$ and $\Theta_1 \cap \Theta_2 \cap \Theta_3$

In the following table we give the different values of $q(j,\theta)$ and $l(j,\theta)$, for θ in the subdivisions of Figure 1 and j such that $A(\underline{\varphi}, j, \theta) \neq \emptyset$. We exclude from this table the cones in $\theta \in \cap_{r=1}^4 \Theta_r$ for $j = 4$ since in this case $q(4,\theta) = 1$ and $l(4,\theta) = 2$.

$j=1$	$q(1,\theta_{11}) = 2$	$q(1,\theta_{12}) = 5$				
	$l(1,\theta_{11}) = 1$	$l(1,\theta_{12}) = 1$				
$j=2$	$q(2,\theta_{21}) = 1$	$q(2,\theta_{22}) = 10$	$q(2,\theta_{23}) = 10$	$q(2,\rho_1) = 10$		
	$l(2,\theta_{21}) = 1$	$l(2,\theta_{22}) = 2$	$l(2,\theta_{23}) = 2$	$l(2,\rho_1) = 2$		
$j=3$	$q(3,\theta_{31}) = 5$	$q(3,\theta_{32}) = 5$	$q(3,\theta_{33}) = 5$	$q(3,\theta_{34}) = 2$	$q(3,\rho_1) = 5$	$q(3,\rho_2) = 5$
	$l(3,\theta_{31}) = 2$	$l(3,\theta_{32}) = 5$	$l(3,\theta_{33}) = 2$	$l(3,\theta_{34}) = 2$	$l(3,\rho_1) = 2$	$l(3,\rho_2) = 2$

Notice that we have $A(\underline{\varphi}, 1, \rho_1) = A(\underline{\varphi}, 2, \rho_2) = A(\underline{\varphi}, 3, \rho_3) = \emptyset$. In the following table we have filled in the cases corresponding to the pairs $(a,b) \in B_{\mathrm{ar}}(\Lambda)$, $(a,b) \neq (2,1)$ in terms of the rays appearing in the subdivisions of Figure 1:

$(a,b) \in B_{\mathrm{ar}}(\Lambda)$	$\nu_{\rho_1} = (2,5)$	$\nu_{\rho_2} = (3,5)$	$\nu_{\rho_3} = (1,6)$	$\nu_{\sigma^\vee \cap \eta_1^\perp} = (1,0)$	$\nu_{\sigma^\vee \cap \eta_2^\perp} = (0,1)$
$(2\varphi_2 - \mathrm{ord}_{\mathcal{J}_2}, \varphi_2)$	(0,10)	(5,15)			(2,2)
$(2\varphi_3 - \mathrm{ord}_{\mathcal{J}_2}, \varphi_3)$	(10,15)	(5,15)	(19,18)	(5,5)	(2,2)
$(2\varphi_4 - \mathrm{ord}_{\mathcal{J}_2}, \varphi_4)$	(24,22)	(31,28)	(19,18)	(7,6)	(4,3)

It follows that $B_{\mathrm{ar},\Lambda} = B_{\mathrm{ar}}(\Lambda) \cup \{(1,3),(0,2),(1,1),(0,1)\}$. We have computed the sum of the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ with the methods of [2]. We have obtained an irredundant representation of the form $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = R(\mathbf{L},T)\prod_{(a,b)\in B}(1-\mathbf{L}^a T^b)^{-1}$ with $R(\mathbf{L},T) \in \mathbf{Q}[\mathbf{L},T]$ and where $B = B_{\mathrm{ar},\Lambda} \setminus \{(24,22),(31,28)\}$.

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