THE ZAREMBA PROBLEM AND THE FINITE HILBERT TRANSFORM

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ABSTRACT. We study the Zaremba problem in Lipschitz graph domains in the plane with data in weighted Lebesgue spaces. Through the use of conformal mappings, we show that the solvability of such problems (i.e. existence and uniqueness of solutions along with non-tangential maximal operator estimates) is characterized in terms of solvability results for the particular case when the domain is the upper half-plane. For the latter, we obtain characterizations in terms of mapping properties of local versions of the Hilbert transform, and in particular of the finite Hilbert transform. We illustrate our results with examples when the domain is a cone with vertex in the real line or, more generally, when the boundary of the domain is a polygonal curve with a finite number of vertices.

1. INTRODUCTION

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, consider the following mixed problem for Laplace's equation, or Zaremba problem in Ω:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = f_D & \text{on } D, \\ \nabla v \cdot \mathbf{n} = f_N & \text{on } N, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator in $\mathbb{R}^n$, $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, and D and N are disjoint open subsets of $\partial\Omega$ such that $\partial N = \partial D$ and $\overline{N} \cup \overline{D} = \partial\Omega$.

Boundary value problems closely related to (**??**) were originally motivated by experimental work due to Leopoldo Nobili at the beginning of the nineteenth century, who observed the formation of colored rings on a charged silver plate. Subsequently, Riemann [**?**] came up with a mathematical model into a mixed Dirichlet-Neumann problem for the Laplace's equation and advances in the theory were made by Weber [**?**] and Zaremba [**?**, **?**] (see Venouziou [**?**] for more details). By now there is a deep understanding of the mathematical theory for a variety of boundary value problems with mixed Dirichlet and Neumann type conditions that model physical phenomena in conductivity, heat transfer, elastic deformations, and electrostatics among other areas. See, for instance, [**?**, **?**, **?**, **?**, **?**, **?**, **?**] and references therein.

In particular, the study of the regularity of solutions of (**??**) in Lipschitz domains was motivated in Kenig [**?**, p.120, problem 3.2.15] and, since then, there have been numerous contributions regarding solvability questions of (**??**). Brown and Sykes [**?**, **?**, **?**] considered the case when Ω is a Lipschitz graph domain and, roughly speaking, D and N meet at an angle which is strictly less than π (creased domains); their results include existence and uniqueness of solutions and non-tangential maximal function estimates for the gradient of the solution in $L^p(\partial\Omega)$ for $1 < p \leq 2$ when f_N and the gradient of f_D are in $L^p(\partial\Omega)$. Other related results were obtained by I. Mitrea and M. Mitrea [**?**], Brown, Capogna and Lanzani [**?**], Brown and

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Ott [**?**, **?**], Brown, Ott and Taylor [**?**], Awala, Mitrea and Ott [**?**], Croyle and Brown [**?**], among others.

Motivated by the work in [**?**], we obtained in [**?**] an explicit expression for a parameter r_0 such that, when Ω is a Lipschitz graph domain and $1 < p < r_0$, (**??**) has unique solutions with data in $L^p(\partial\Omega)$, and their gradients satisfy non-tangential maximal function estimates; at the endpoint, we proved solvability when the data belong to the Lorentz space $L^{r_0,1}(\partial\Omega)$. For the case when the domain is the upper half-plane, we assumed the data to be in weighted spaces, and established conditions on the weights for the existence of solutions and estimates for the non-tangential maximal function of their gradients.

Our results in this article deal with the Zaremba problem (**??**) for Lipschitz graph domains Ω with data in weighted Lebesgue spaces. Through the use of conformal mappings, we show that the solvability of such problems is characterized in terms of solvability results for the particular case when $\Omega = \mathbb{R}^2_+$. For the latter, we give characterizations of solvability in terms of mapping properties of local versions of the Hilbert transform, and we show that in the particular case $D = (-1, 1)$, there is a close relation with the airfoil equation. We illustrate our results with examples where Ω is a cone with vertex in $\mathbb{R}$ or, more generally, $\partial\Omega$ is a polygonal curve with a finite number of vertices.

The organization of the article is as follows. In Section **??**, we present our main results. The solvability of the mixed problem (**??**) in the upper half-plane is treated in Section **??**; besides the proofs of the corresponding main results, this section contains characterizations of mapping properties in weighted Lebesgue spaces of local versions of the Hilbert transform along with examples of weights satisfying the hypothesis of our theorems and corollaries. In Section **??**, we deal with (**??**) in the setting of general domains and present specific examples.

2. MAIN RESULTS

Before presenting our results with more precision, we briefly summarize the setting in which we work and present some notation. We refer the reader to Sections **??** and **??** for details.

In the rest of the manuscript, Ω will always be a Lipschitz graph domain, which we next define: Let Λ be a curve in the complex plane given parametrically by $x + i\gamma(x)$ for $x \in \mathbb{R}$, where γ is a real-valued Lipschitz function with constant L, and consider the *Lipschitz graph domain*

$$\Omega = \{z \in \mathbb{C} : \text{Im}(z) > \gamma(\text{Re}(z))\}; \tag{2.1}$$

note that $\Lambda = \partial\Omega$. Given $0 < \beta < \arctan(1/L)$, define the non-tangential maximal operator $\mathcal{M}_\beta$ as

$$\mathcal{M}_\beta(F)(\xi) = \sup_{z\in\Gamma_\beta(\xi)} |F(z)|, \quad \xi \in \partial\Omega, \tag{2.2}$$

where F is a complex-valued function defined in Ω and

$$\Gamma_\beta(\xi) = \{z \in \mathbb{C} : \text{Im}(z) > \text{Im}(\xi) \text{ and } |\text{Re}(\xi) - \text{Re}(z)| < \tan(\beta)|\text{Im}(z) - \text{Im}(\xi)|\}. \tag{2.3}$$

For an open set $U \subset \partial\Omega$, the notation f_U denotes a complex-valued function defined on $\partial\Omega$ such that $f_U = 0$ on the complement of U. If F is defined on Ω and f is defined on $\partial\Omega$, the equality $F = f$ on U will be interpreted as holding almost everywhere in U in the sense of non-tangential convergence unless otherwise stated.

If $f_D \in L^1_{\text{loc}}(D)$ and $f_N \in L^1_{\text{loc}}(N)$, a function v defined on Ω is said to be *a solution of the mixed problem* (**??**) *in* Ω *with data* f_D *and* f_N if v is harmonic in Ω and the equalities $v = f_D$ on D and $\nabla v \cdot \mathbf{n} = f_N$ on N hold. This leads to the following definition regarding solvability of (**??**):

Definition 2.1. If X and Y are Banach spaces of measurable functions defined on $\partial\Omega$, we say that *the mixed problem* (**??**) *in* Ω *is solvable in* (X, Y) if there exists $1 < \beta < \arctan(1/L)$ such that for every $f_D \in L^1_{\text{loc}}(D)$ with $f_D' \in X$ and every $f_N \in L^1_{\text{loc}}(N)$ with $f_N \in X$ there exists a solution v of (**??**) in Ω with data f_D and f_N and

$$\|\mathcal{M}_\beta(\nabla v)\|_Y \lesssim \|f_D'\|_X + \|f_N\|_X,$$

where the implicit constant is independent of f_D and f_N and, $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms in X and Y, respectively. We say that *the mixed problem* (**??**) *in* Ω *is uniquely solvable in* (X, Y) if (**??**) is solvable in (X, Y) and for every $f_D \in L^1_{\text{loc}}(D)$ with $f_D' \in X$ and $f_N \in L^1_{\text{loc}}(N)$ with $f_N \in X$ there is a unique solution v of the mixed problem (**??**) in Ω with data f_D and f_N such that $\mathcal{M}_\beta(\nabla v) \in Y$. If $X = Y$ we say that (**??**) is *solvable or uniquely solvable in* X.

For $1 < p \leq \infty$ and an interval $J \subset \mathbb{R}$, the notation $A_p(J)$ refers to the class of Muckenhoupt weights in J, while $A_p(\partial\Omega)$ stands for the class of Muckenhoupt weights in $\partial\Omega$. We will denote by $L^p(J, \omega)$ and $L^p(\partial\Omega, \eta)$ the weighted Lebesgue spaces on J and $\partial\Omega$ with weights ω and η, respectively. As usual, p' denotes the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. Since Ω is simply connected, there is a conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$ such that $\Phi(\infty) = \infty$. It can be proved that Φ extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$ and $\Phi(x)$, $x \in \mathbb{R}$, is absolutely continuous when restricted to any finite interval; in particular, $\Phi'(x)$ exists for almost every $x \in \mathbb{R}$ and is locally integrable. Moreover, $\Phi'(x) \neq 0$ for almost every $x \in \mathbb{R}$, $\lim_{z \to x} \Phi'(z) = \Phi'(x)$ for almost every $x \in \mathbb{R}$ in the sense of non-tangential convergence and $|\Phi'| \in A_2(\mathbb{R})$. See Kenig [**?**, Theorems 1.1 and 1.10] for the proof of those properties and additional ones. We define $1 \leq p_\Phi \leq \infty$ such that its conjugate exponent is

$$p_\Phi' = \inf\{q \in (1, \infty) : |\Phi'| \in A_q(\mathbb{R})\}.$$

In the following, Φ will be as described above, and D and N will be open sets contained in $\partial\Omega$ as in (**??**).

Given f_D and f_N and the corresponding mixed problem (**??**) in Ω, we will consider the following mixed problem in the upper half-plane:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = f_D \circ \Phi & \text{in } \Phi^{-1}(D), \\ \nabla u \cdot \mathbf{n} = |\Phi'| \, (f_N \circ \Phi) & \text{in } \mathbb{R} \setminus \overline{\Phi^{-1}(D)}. \end{cases} \tag{2.4}$$

Our first stated result shows a relation between the solvability of (**??**) and (**??**):

Theorem 2.2 (Solvability of (**??**) in weighted Lebesgue spaces)**.** *Let Ω be a Lipschitz graph domain in the plane, $D \subset \partial\Omega$ be open, and $1 < p < \infty$. Consider weights*

$$\eta, \rho \in A_\infty(\partial\Omega) \quad \textit{such that} \quad \omega = |\Phi'|^{1-p} \, (\eta \circ \Phi), \, \mu = |\Phi'|^{1-p} \, (\rho \circ \Phi) \in A_\infty(\mathbb{R}).$$

Then the Zaremba problem (**??**) *is (uniquely) solvable in $(L^p(\partial\Omega, \eta), L^p(\partial\Omega, \rho))$ with $N = \partial\Omega \setminus \overline{D}$ if and only if* (**??**) *is (uniquely) solvable in $(L^p(\mathbb{R}, \omega), L^p(\mathbb{R}, \mu))$.*

The condition $1 < p < p_\Phi$ is equivalent to $|\Phi'|^{1-p} \in A_p(\mathbb{R}) \subset A_\infty(\mathbb{R})$. We then have the following corollary for the case $\eta = \rho \equiv 1$ in Theorem **??**:

Corollary 2.3 (Solvability of (**??**) in $L^p(\partial\Omega)$)**.** *Let Ω be a Lipschitz graph domain in the plane, $D \subset \partial\Omega$ be open and $1 < p < p_\Phi$. Then the Zaremba problem* (**??**) *is (uniquely) solvable in $L^p(\partial\Omega)$ with $N = \partial\Omega \setminus \overline{D}$ if and only if* (**??**) *is (uniquely) solvable in $L^p(\mathbb{R}, |\Phi'|^{1-p})$.*

We next present solvability results for the Zaremba problem when Ω is the upper half-plane $\mathbb{R}^2_+$ in terms of characterizations of mapping properties for local versions of the Hilbert transform.

The Hilbert transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \frac{f(x-t)}{t}\, dt;$$

if $1 < p < \infty$, then $\mathcal{H}$ is bounded from $L^p(\mathbb{R}, \omega)$ to $L^p(\mathbb{R}, \omega)$ if and only if $\omega \in A_p(\mathbb{R})$ (see Hunt et al. [**?**]). For an open set $U \subset \mathbb{R}$, and a function h, the *Hilbert transform restricted to U* and denoted $\mathcal{H}_U$, is defined by

$$\mathcal{H}_U h = \mathcal{H}(h\chi_U)\chi_U.$$

In particular, setting $I := (-1, 1)$, $\mathcal{H}_I$ is known as the *finite Hilbert transform* and it also holds that $\mathcal{H}_I$ is bounded from $L^p(I, \omega)$ to $L^p(I, \omega)$ if and only if $\omega \in A_p(I)$ (see Astala et al. [**?**]). Analogous results hold for $\mathcal{H}_{(a,b)}$, where $\infty \leq a < b \leq \infty$. The case for (a, b) with $a, b \in \mathbb{R}$ follows from the case corresponding to $(-1, 1)$ by a change of variable. When either $a = -\infty$ or $b = \infty$, one can adapt the proof of Theorem 1.1 in Agora et al. [**?**] to reduce the problem to the case of the truncated Hardy-Littlewood maximal operator and then proceed as in the classical case $(a, b) = \mathbb{R}$.

Pioneering work concerning mapping properties of $\mathcal{H}_I$ are due to Carleman [**?**], who obtained inversion formulas for the equation $A(x)h(x) + \mathcal{H}_I h(x) = f(x)$ assuming that A and f are analytic. Tricomi [**?**, **?**] studied the case $A \equiv 0$ (known as the "airfoil equation") with $f \in L^p(I)$ for $p > 4/3$, and showed the existence of a solution $h \in L^q(I)$ with $1 < q < 4/3$. Spectral properties of $\mathcal{H}_I$ were considered by Koppelman and Pincus in [**?**] and by Rooney in [**?**]. Later on, the authors in [**?**] characterized the weights $\omega \in A_p(I)$, $1 < p < \infty$, for which $\mathcal{H}_I$ is injective, surjective, bounded from below or an isomorphism in $L^p(I, \omega)$. Theorem **??** and its Corollary **??** stated in Section **??** partially contain their results and are at the center of our characterizations of solvability results for the Zaremba problem when $\Omega = \mathbb{R}^2_+$ and $D = I$ or $D = (0, \infty)$. The reader may consult Curbera et al. [**?**] for a survey of recent developments in the study of the finite Hilbert transform and its inversion problem in other function spaces, as well as Herdman and Turi [**?**], and references therein, for applications to the theory of airfoils.

Our main result concerning the Zaremba problem in the upper half-plane and mapping properties of the Hilbert transform restricted to an open set is the following theorem:

Theorem 2.4 (Solvability of (**??**) in $\mathbb{R}^2_+$)**.** *Let $1 < p < \infty$ and $\omega \in A_p(\mathbb{R})$. Consider an open set $D \subset \mathbb{R}$ and the Zaremba problem*

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+, \\ v = f_D & \text{on } D, \\ \nabla v \cdot \mathbf{n} = f_N & \text{on } N = \mathbb{R} \setminus \overline{D}. \end{cases} \tag{2.5}$$

Then the following holds:

*(a) If the mixed problem (**??**) is solvable in $L^p(\mathbb{R}, \omega)$, then $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is surjective.*
*(b) If $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is injective and (**??**) is solvable in $L^p(\mathbb{R}, \omega)$, then $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is an isomorphism and (**??**) is uniquely solvable in $L^p(\mathbb{R}, \omega)$.*
*(c) If D is an interval (bounded or unbounded), $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is an isomorphism if and only if (**??**) is uniquely solvable in $L^p(\mathbb{R}, \omega)$.*
*(d) If D is an interval (bounded or unbounded), $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is surjective if and only if (**??**) is solvable in $L^p(\mathbb{R}, \omega)$.*

The following corollary is a consequence of Theorem **??**, along with the characterization of mapping properties of the finite Hilbert transform $\mathcal{H}_I$ proved in [**?**] (see Theorem **??**).

Corollary 2.5 (Solvability of (**??**) in $\mathbb{R}^2_+$ for $D = I$)**.** *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{R})$ and consider the Zaremba problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = f_I & \text{on } I, \\ \nabla u \cdot (0,-1) = f_{\mathbb{R}\setminus\overline{I}} & \text{on } \mathbb{R}\setminus\overline{I}. \end{cases} \tag{2.6}$$

Then (**??**) *is uniquely solvable in $L^p(\mathbb{R},\omega)$ if and only if ω satisfies*

$$\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}} \omega(x) \in A_p(I) \quad \text{or} \quad \left(\frac{1+x}{1-x}\right)^{\frac{p}{2}} \omega(x) \in A_p(I). \tag{2.7}$$

Also, (**??**) *is not uniquely solvable but solvable in $L^p(\mathbb{R},\omega)$ if and only if*

$$\left(\frac{1}{1-x^2}\right)^{\frac{p}{2}} \omega(x) \in A_p(I). \tag{2.8}$$

Specific instances of Corollary **??** for data in the unweighted space $L^p(\mathbb{R})$ are given in the following statement:

Corollary 2.6. *The following results hold:*

(a) There is no $1 < p < \infty$ such that (**??**) *is uniquely solvable in $L^p(\mathbb{R})$.*
(b) The Zaremba problem (**??**) *is solvable in $L^p(\mathbb{R})$ if and only if $1 < p < 2$.*
(c) If $2 < p < \infty$ and $\ell > 0$ with $\ell + 1 < p < 2(\ell+1)$, then (**??**) *is uniquely solvable in $(L^p(\mathbb{R}), L^p(\mathbb{R}, \min(1, |x+1|^\ell)))$ and in $(L^p(\mathbb{R}), L^p(\mathbb{R}, \min(1, |x-1|^\ell)))$.*

The following corollary is a counterpart of Corollary **??** for $D = (0,\infty)$ and follows from Theorem **??** and Corollary **??**.

Corollary 2.7 (Solvability of (**??**) in $\mathbb{R}^2_+$ for $D = (0,\infty)$)**.** *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{R})$ and consider the Zaremba problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = f_{(0,\infty)} & \text{on } (0,\infty), \\ \nabla u \cdot (0,-1) = f_{(-\infty,0)} & \text{on } (-\infty,0). \end{cases} \tag{2.9}$$

Then (**??**) *is uniquely solvable in $L^p(\mathbb{R},\omega)$ if and only if ω satisfies*

$$x^{\frac{p}{2}}\omega(x) \in A_p((0,\infty)) \quad \text{or} \quad x^{-\frac{p}{2}}\omega(x) \in A_p((0,\infty)). \tag{2.10}$$

Also, (**??**) *is not uniquely solvable but solvable in $L^p(\mathbb{R},\omega)$ if and only if*

$$\left(\frac{1+x}{\sqrt{x}}\right)^{p} \omega(x) \in A_p((0,\infty)). \tag{2.11}$$

In view of Theorem **??** and Corollaries **??** and **??**, we obtain the following results regarding the solvability if (**??**) when $D = \Phi(I)$ or $D = \Phi((0,\infty))$.

Corollary 2.8 (Solvability of (**??**) in weighted Lebesgue spaces for $D = \Phi(I)$)**.** *Let Ω be a Lipschitz graph domain in the plane and $D = \Phi(I)$. For $1 < p < \infty$, let η be a weight in $A_\infty(\partial\Omega)$ such that $|\Phi'|^{1-p}(\eta\circ\Phi)$ is in $A_\infty(\mathbb{R})$. Then* (**??**) *is uniquely solvable in $L^p(\partial\Omega,\eta)$ if and only if*

$$\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}} |\Phi'(x)|^{1-p}(\eta\circ\Phi)(x) \in A_p(I) \quad \text{or} \quad \left(\frac{1+x}{1-x}\right)^{\frac{p}{2}} |\Phi'(x)|^{1-p}(\eta\circ\Phi)(x) \in A_p(I). \tag{2.12}$$

Also, (**??**) *is not uniquely solvable but solvable in* $L^p(\partial\Omega,\eta)$ *if and only if*

$$\left(\frac{1}{1-x^2}\right)^{\frac{p}{2}} |\Phi'(x)|^{1-p}\,(\eta\circ\Phi)(x)\in A_p(I). \tag{2.13}$$

Corollary 2.9 (Solvability of (**??**) in weighted Lebesgue spaces for $D=\Phi((0,\infty))$)**.** *Let* Ω *be a Lipschitz graph domain in the plane and* $D=\Phi((0,\infty))$*. For* $1<p<\infty$*, let* η *be a weight in* $A_\infty(\partial\Omega)$ *such that* $|\Phi'|^{1-p}\,(\eta\circ\Phi)$ *is in* $A_\infty(\mathbb{R})$*. Then* (**??**) *is uniquely solvable in* $L^p(\partial\Omega,\eta)$ *if and only if*

$$x^{\frac{p}{2}}|\Phi'(x)|^{1-p}\,(\eta\circ\Phi)(x)\in A_p((0,\infty)) \quad or \quad x^{-\frac{p}{2}}|\Phi'(x)|^{1-p}\,(\eta\circ\Phi)(x)\in A_p((0,\infty)). \tag{2.14}$$

Also, (**??**) *is not uniquely solvable but solvable in* $L^p(\partial\Omega,\eta)$ *if and only if*

$$\left(\frac{1+x}{\sqrt{x}}\right)^{p} |\Phi'(x)|^{1-p}\,(\eta\circ\Phi)(x)\in A_p((0,\infty)). \tag{2.15}$$

3. Solvability of the mixed problem (??) in the upper half-plane

The main goal of this section is to present the proof of one of our main results, Theorem **??**. We first introduce needed tools and lemmas on classes of weights, characterizations of mapping properties for $\mathcal{H}_I$ and $\mathcal{H}_{(0,\infty)}$, and some facts about the Neumann problem in the upper half-plane. The proof of Theorem **??** is contained in Section **??** along with examples of weights satisfying the conditions of Corollaries **??** and **??**, and the statement of a general instance of Corollary **??**.

3.1. **Classes of weights and the finite Hilbert trasnsform.**

In this section, we define the Muckenhoupt classes of weights and present some known useful results about them and mapping characterizations of $\mathcal{H}_I$ and $\mathcal{H}_{(0,\infty)}$.

For an open interval $J\subset\mathbb{R}$, a weight ω on J is a non-negative locally integrable function defined in J. We will use the weighted Lebesgue space $L^p(J,\omega)$, where $1\le p\le\infty$, of all measurable functions $f:J\to\mathbb{C}$ such that

$$\|f\|_{L^p(J,\omega)}=\left(\int_J |f(x)|^p\omega(x)\,dx\right)^{\frac{1}{p}}<\infty,$$

with the corresponding changes for $p=\infty$. When $\omega\equiv 1$, we use the notation $L^p(J)$ instead of $L^p(J,\omega)$.

Given $1<p<\infty$, we will consider weights ω in the Muckenhoupt class $A_p(J)$ defined by

$$\|\omega\|_{A_p(J)}=\sup_{K\subset J}\left(\frac{1}{|K|}\int_K\omega(x)dx\right)\left(\frac{1}{|K|}\int_K\omega(x)^{1-p'}dx\right)^{p-1}<\infty, \tag{3.1}$$

where the supremum is taken over all intervals $K\subset J$, and $A_\infty(J)=\cup_{p>1}A_p(J)$. We recall that, for $1<p<\infty$, $\omega\in A_p(J)$ if and only if $\omega^{1-p'}\in A_{p'}(J)$, $A_p(J)\subset A_q(J)$ if $p<q$ and, if $\omega\in A_p(J)$ with $p>1$ then $\omega\in A_{p-\varepsilon}(J)$ for some $\varepsilon>0$.

It is well known that $|x|^s\in A_p(\mathbb{R})$ if and only if $-1<s<p-1$, and

$$\prod_{k=1}^m |x-x_k|^{s_k}\in A_p(\mathbb{R}) \quad \text{iff} \quad -1<s_k<p-1 \text{ for } k=1,\cdots m, \text{ and } -1<\sum_{k=1}^m s_k<p-1. \tag{3.2}$$

Also, we will frequently use that if $x_1<\cdots<x_m$ belong to a bounded interval J, then

$$\prod_{k=1}^m |x-x_k|^{s_k}\in A_p(J) \quad \text{iff} \quad -1<s_k<p-1 \text{ for } k=1,\cdots,m. \tag{3.3}$$

3.1.1. *Characterizations of mapping properties of $\mathcal{H}_I$ and $\mathcal{H}_{(0,\infty)}$ in weighted Lebesgue spaces.* Define $\tau : I \to (0,\infty)$ by $\tau(x) = \frac{1-x}{1+x}$; note that τ is a homeomorphism from I onto $(0,\infty)$ and $\tau^{-1}(x) = \frac{1-x}{1+x}$ for $x \in (0,\infty)$.

We have the following theorem from [**?**] and a corollary.

Theorem 3.1. *Let* $1 < p < \infty$ *and* $\omega \in A_p(I)$.

(a) [**?**, Theorem 3.1]*:* $\mathcal{H}_I : L^p(I,\omega) \to L^p(I,\omega)$ *is an isomorphism if and only if*

$$\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}} \omega(x) \in A_p(I) \quad \textit{or} \quad \left(\frac{1+x}{1-x}\right)^{\frac{p}{2}} \omega(x) \in A_p(I).$$

(b) [**?**, Theorem 3.2, Part (i)]*:* $\mathcal{H}_I : L^p(I,\omega) \to L^p(I,\omega)$ *is surjective and not injective if and only if and only if*

$$\left(\frac{1}{1-x^2}\right)^{\frac{p}{2}} \omega(x) \in A_p(I).$$

In that case, ker$(\mathcal{H}_I) =$ *span* $\left\{\frac{1}{\sqrt{1-x^2}}\right\}$.

We next state a corollary based on [**?**, Remark 3.5]; we include its proof for completeness.

Given a weight ω on $(0,\infty)$ or on I define

$$\omega^*(x) = \left(\frac{1+x}{\sqrt{2}}\right)^{p-2} \omega\left(\frac{1-x}{1+x}\right)$$

for $x \in I$ or $x \in (0,\infty)$, respectively. Note that $(\omega^*)^* = \omega$, the mapping $\mathcal{T} : L^p((0,\infty),\omega) \to L^p(I,\omega^*)$ given by

$$\mathcal{T}f(x) = \frac{\sqrt{2}}{1+x} f\left(\frac{1-x}{1+x}\right), \quad x \in I,$$

is an isometric isomorphism and $\mathcal{T}^{-1}$ has the same formula as $\mathcal{T}$. It follows that

$$\mathcal{H}_{(0,\infty)} f = -\mathcal{T}^{-1}\mathcal{H}_I\mathcal{T}f. \tag{3.4}$$

Corollary 3.2. *Let* $1 < p < \infty$ *and* $\omega \in A_p((0,\infty))$.

(a) $\mathcal{H}_{(0,\infty)} : L^p((0,\infty),\omega) \to L^p((0,\infty),\omega)$ *is an isomorphism if and only if*

$$x^{\frac{p}{2}}\,\omega(x) \in A_p((0,\infty)) \quad \textit{or} \quad x^{-\frac{p}{2}}\,\omega(x) \in A_p((0,\infty)).$$

(b) $\mathcal{H}_{(0,\infty)} : L^p((0,\infty),\omega) \to L^p((0,\infty),\omega)$ *is surjective and not injective if and only if*

$$\left(\frac{1+x}{\sqrt{x}}\right)^{p} \omega(x) \in A_p((0,\infty)).$$

In that case, ker$(\mathcal{H}_{(0,\infty)}) =$ *span* $\left\{\frac{1}{\sqrt{x}}\right\}$.

Proof. The identity (**??**) gives that $\mathcal{H}_{(0,\infty)}$ is bounded on $L^p((0,\infty),\omega)$ if and only if $\mathcal{H}_I$ is bounded on $L^p(I,\omega^*)$; therefore,

$$\omega \in A_p((0,\infty)) \quad \text{if and only if} \quad \omega^* \in A_p(I). \tag{3.5}$$

Also, we obtain from (**??**) that

$$\mathcal{H}_{(0,\infty)} \;\text{ is injective in }\; L^p((0,\infty),\omega) \text{ if and only if } \mathcal{H}_I \;\text{ is an injective in }\; L^p(I,\omega^*), \tag{3.6}$$

$$\mathcal{H}_{(0,\infty)} \;\text{ is surjective in }\; L^p((0,\infty),\omega) \text{ if and only if } \mathcal{H}_I \;\text{ is surjective in }\; L^p(I,\omega^*). \tag{3.7}$$

Proof of Item (??): Let $\omega \in A_p((0,\infty))$; then $\omega^* \in A_p(I)$ by (??). It then follows from (??), (??) and Item (??) in Theorem ?? that $\mathcal{H}_{(0,\infty)}$ is an isomorphism in $L^p((0,\infty),\omega)$ if and only if

$$\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}}\omega^*(x) \in A_p(I) \quad \text{or} \quad \left(\frac{1+x}{1-x}\right)^{\frac{p}{2}}\omega^*(x) \in A_p(I),$$

which, by (??), is equivalent to

$$\left(\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}}\omega^*(x)\right)^* \in A_p((0,\infty)) \quad \text{or} \quad \left(\left(\frac{1+x}{1-x}\right)^{\frac{p}{2}}\omega^*(x)\right)^* \in A_p((0,\infty)).$$

Since $\left(\left(\frac{1-x}{1+x}\right)^{\frac{p}{2}}\omega^*(x)\right)^* = x^{\frac{p}{2}}\,\omega(x)$ and $\left(\left(\frac{1+x}{1-x}\right)^{\frac{p}{2}}\omega^*(x)\right)^* = x^{-\frac{p}{2}}\,\omega(x)$ the desired result follows.

Proof of Item (??): If $\mathcal{H}_{(0,\infty)}$ is surjective and not injective in $L^p((0,\infty),\omega)$, then, by (??), (??), Item (??) in Theorem ?? and (??), we have

$$\left(\left(\frac{1}{1-x^2}\right)^{\frac{p}{2}}\omega^*(x)\right)^* \in A_p((0,\infty)).$$

Noting that $\left(\left(\frac{1}{1-x^2}\right)^{\frac{p}{2}}\omega^*(x)\right)^* = \left(\frac{1+x}{\sqrt{x}}\right)^p \omega(x)$, we obtain the condition in Item (??). Conversely, if $\left(\frac{1+x}{\sqrt{x}}\right)^p \omega(x) \in A_p((0,\infty))$ holds, retracing the previous steps gives that $\mathcal{H}_{(0,\infty)}$ is surjective and not injective. Regarding the ker of $\mathcal{H}_{(0,\infty)}$, using (??), that $\mathcal{T}$ is an isomorphism and Part (??) in Theorem ??, we have

$$\mathcal{H}_{(0,\infty)}f = 0 \iff \mathcal{H}_I(\mathcal{T}f) = 0 \iff \mathcal{T}f(x) = c\,\frac{\chi_I(x)}{\sqrt{1-x^2}} \iff f(x) = \frac{c}{\sqrt{2}}\,\frac{\chi_{(0,\infty)}(x)}{\sqrt{x}}.$$

□

3.2. **The Neumann problem in the upper half-plane.** In this section we gather some facts about the Neumann problem in the upper half-plane.

Consider the classical Neumann boundary value problem in $\mathbb{R}^2_+$:

$$\Delta u = 0 \text{ on } \mathbb{R}^2_+ \qquad \text{and} \qquad \nabla u \cdot (0,-1) = f \text{ on } \mathbb{R}. \tag{3.8}$$

For $f : \mathbb{R} \to \mathbb{C}$ define

$$u_f(x,y) := -\frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{\sqrt{(x-t)^2+y^2}}{1+|t|}\right) f(t)\,dt, \quad (x,y) \in \mathbb{R}^2_+. \tag{3.9}$$

We note that the integral on the right-hand side of (??) is absolutely convergent for all f satisfying $\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\,dx < \infty$. In particular, u_f is well defined and absolutely convergent for all $f \in L^p(\mathbb{R},\omega)$ if $\omega \in A_p(\mathbb{R})$.

The Neumann problem (??) in the upper half-plane is uniquely solvable in $L^p(\mathbb{R},\omega)$ for $\omega \in A_p(\mathbb{R})$ (see [?, Theorem 1.3] and [?, Theorem 2.5]); more precisely, the following result holds.

Theorem 3.3. *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{R})$ and $0 < \beta < \pi/2$. If $f \in L^p(\mathbb{R},\omega)$, then u_f is harmonic in $\mathbb{R}^2_+$, $\nabla u \cdot (0,-1) = f$ on $\mathbb{R}$ and*

$$\|\mathcal{M}_\beta(\nabla u_f)\|_{L^p(\mathbb{R},\omega)} \lesssim \|f\|_{L^p(\mathbb{R},\omega)},$$

where the implicit constant is independent of f. Moreover, if u is a solution of (??) with datum f and $\mathcal{M}_\beta(\nabla u) \in L^p(\mathbb{R},\omega)$, then there exists a constant C such that $u = u_f + C$.

The next lemma deals with the boundary values of u_f (see [?, Lemma 3.4]).

Lemma 3.4. *Let $f \in L^p(\mathbb{R}, \omega)$ with $\omega \in A_p(\mathbb{R})$. The function $x \to \int_{\mathbb{R}} \left| \log\left(\frac{|x-t|}{1+|t|}\right) f(t) \right| dt$ is locally integrable in $\mathbb{R}$. Moreover, the function given by*

$$\mathcal{B}f(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \log\left(\frac{|x-t|}{1+|t|}\right) f(t)\, dt \tag{3.10}$$

satisfies $(\mathcal{B}f)' = -\mathcal{H}f$ in the sense of distributions and $u_f = \mathcal{B}f$ on $\mathbb{R}$.

3.3. **The proof of Theorem ??, examples and corollaries.** In this section we prove Theorem **??**. We also give examples of weights that satisfy the conditions (**??**) or (**??**) of Corollary **??** and use them to prove Corollary **??**; examples of weights that verify (**??**) or (**??**) of Corollary **??** are also presented. Finally, we state the general version of Corollary **??** when D in (**??**) is an arbitrary bounded interval.

The following lemma will be used in the proof of Theorem **??** (see, for instance, [**?**, Lemma 3.1]):

Lemma 3.5. *Let X and Y be Banach spaces and $T : X \to Y$ be a surjective bounded linear operator. Then for every $y \in Y$, there exists $x_y \in X$ such that $T(x_y) = y$ and $\|x_y\|_X \lesssim \|y\|_Y$.*

3.3.1. *Proof of Theorem* **??**. We start with the proof of Part (**??**); that is, we prove that if (**??**) is solvable in $L^p(\mathbb{R}, \omega)$, then $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is surjective. For $F_D \in L^p(D, \omega)$, let $f_D(x) = -\int_0^x F_D(y)\, dy$ (which is a well-defined continuous function since F_D is locally integrable); we have $f_D' = -F_D$. Consider (**??**) with data f_D and $f_N = 0$ and let u be a solution with $\mathcal{M}_\beta(\nabla u) \in L^p(\mathbb{R}, \omega)$. Since u is harmonic in $\mathbb{R}^2_+$ and $\omega \in A_p(\mathbb{R})$, we have that $\nabla u \cdot (0, -1)$ has non-tangential limit in $L^p(\mathbb{R}, \omega)$. Therefore, there exists p_D in $L^p(D, \omega)$ such that $\nabla u \cdot (0, -1) = p_D$ on $\mathbb{R}$; this is, u is a solution of the Neumann problem (**??**) with datum p_D. By the uniqueness result stated in Theorem **??** and by Lemma **??**, we conclude that $\mathcal{H}_D p_D = -f_D' = F_D$.

We next prove Part (**??**). Assume $\mathcal{H}_D : L^p(D, \omega) \to L^p(D, \omega)$ is injective and (**??**) is solvable in $L^p(D, \omega)$. Then $\mathcal{H}_D$ is surjective by Part (**??**), and, since $\omega \in A_p(\mathbb{R})$, $\mathcal{H}_D$ is continuous; this gives that $\mathcal{H}_D$ is an isomorphism. To prove uniqueness, suppose u_1 and u_2 are solutions of the mixed problem (**??**) with data f_N and f_D' in $L^p(\mathbb{R}, \omega)$ and $\mathcal{M}_\beta(\nabla u_1)$ and $\mathcal{M}_\beta(\nabla u_2)$ in $L^p(\mathbb{R}, \omega)$. Since u_1 and u_2 are harmonic in $\mathbb{R}^2_+$ and $\omega \in A_p(\mathbb{R})$, we have that $\nabla u_1 \cdot (0, -1)$ and $\nabla u_2 \cdot (0, -1)$ have non-tangential limits in $L^p(\mathbb{R}, \omega)$. Therefore, there exist p_D^1 and p_D^2 in $L^p(D, \omega)$ such that $\nabla u_1 \cdot (0, -1) = f_N + p_D^1$ and $\nabla u_2 \cdot (0, -1) = f_N + p_D^2$ on $\mathbb{R}$. Then u_1 and u_2 are solutions of the Neumann problem (**??**) with data $f_N + p_D^1 \in L^p(\mathbb{R}, \omega)$ and $f_N + p_D^2 \in L^p(\mathbb{R}, \omega)$, respectively, and by the uniqueness result in Theorem **??** and by Lemma **??** we must have $f_D'(x) = -\mathcal{H}(f_N + p_D^1)(x)$ and $f_D'(x) = -\mathcal{H}(f_N + p_D^2)(x)$ almost everywhere in D. This implies that $\mathcal{H}_D(p_D^1 - p_D^2) = 0$ almost everywhere in D. The injectivity assumption of $\mathcal{H}_D$ then gives $p_D^1 = p_D^2$ almost everywhere in D. Applying again the uniqueness result in Theorem **??**, we then conclude that there exists a constant C such that $u_1 = u_2 + C$ almost everywhere on $\mathbb{R}$. Since u_1 and u_2 are both equal to f_D on D, we have $C = 0$.

We next prove Part (**??**). Let D be an interval. Assume first that (**??**) is uniquely solvable in $L^p(\mathbb{R}, \omega)$. By Part (**??**), $\mathcal{H}_D$ is surjective. To prove that $\mathcal{H}_D$ is injective, assume $p_D \in L^p(D, \omega)$ is such that $\mathcal{H}_D p_D = 0$. Let $u = u_{p_D}$ be the solution of the Neumann problem (**??**) with data p_D given by (**??**). By Lemma **??**, we have that $0 = -\mathcal{H}_D p_D = \partial_x u$ almost everywhere on D; since D is an interval, u converges non-tangentially to a constant c on D, and we can assume $c = 0$. Then u is also a solution of (**??**) with $f_D = f_N = 0$ and $\mathcal{M}_\beta(\nabla u) \in L^p(\mathbb{R}, \omega)$; uniqueness of solutions of (**??**) gives that $u \equiv 0$ and therefore $p_D = 0$. As a consequence, $\mathcal{H}_D$ is injective. Since $\mathcal{H}_D$ is bounded in $L^p(D, \omega)$ ($\omega \in A_p(\mathbb{R})$) and bijective, we have that $\mathcal{H}_D$ is an isomorphism from $L^p(D, \omega)$ onto $L^p(D, \omega)$.

Conversely, suppose that $\mathcal{H}_D$ is an isomorphism from $L^p(D,\omega)$ onto $L^p(D,\omega)$. By Part (**??**), it is enough to prove that (**??**) is solvable in $L^p(\mathbb{R},\omega)$. Let f_D and f_N be such that $f_D' \in L^p(\mathbb{R},\omega)$ and $f_N \in L^p(\mathbb{R},\omega)$. Since $\mathcal{H}_D$ is an isomorphism, there exists $h_D \in L^p(D,\omega)$ such that

$$\mathcal{H}_D h_D = (-f_D' - \mathcal{H} f_N)\chi_D \quad \text{a.e.} \tag{3.11}$$

and

$$\|h_D\|_{L^p(D,\omega)} \lesssim \|f_D' + \mathcal{H} f_N\|_{L^p(D,\omega)} \lesssim \|f_D'\|_{L^p(\mathbb{R},\omega)} + \|f_N\|_{L^p(\mathbb{R},\omega)}\,. \tag{3.12}$$

We will next proceed as in [**?**, Theorem 1.5] to show that if $h = f_N + h_D$, with h_D as obtained above and u_h is as in (**??**), then there exists a constant C such that $u_h + C$ is a solution of (**??**) satisfying the required estimates. We have, by (**??**), that $\mathcal{H}h = -f_D'$ on D and, by Lemma **??**, $\mathcal{H}h = -(\mathcal{B}h)'$ on D, both equalities in the sense of distributions. As a consequence, $(\mathcal{B}h)' = f_D'$ on D, and since D is an interval, there exists a constant C such that $f_D = \mathcal{B}h + C$ almost everywhere on D. By Theorem **??**, $u_h + C$ is harmonic in $\mathbb{R}^2_+$ and $\nabla(u_h + C)\cdot(0,-1) = h$ on $\mathbb{R}$; in particular, the latter implies that we have $\nabla(u_h + C)\cdot(0,-1) = f_N$ on N. Moreover,

$$\|\mathcal{M}_\beta(\nabla(u_h + C))\|_{L^p(\mathbb{R},\omega)} \lesssim \|h\|_{L^p(\mathbb{R},\omega)} \le \|h_D\|_{L^p(D,\omega)} + \|f_N\|_{L^p(\mathbb{R},\omega)}; \tag{3.13}$$

this inequality and (**??**) lead to

$$\|\mathcal{M}_\beta(\nabla(u_h + C))\|_{L^p(\mathbb{R},\omega)} \lesssim \|f_D'\|_{L^p(\mathbb{R},\omega)} + \|f_N\|_{L^p(\mathbb{R},\omega)}.$$

Finally, Lemma **??** gives that $u_h = \mathcal{B}h$ on $\mathbb{R}$ and therefore $u_h + C = f_D$ on D.

Regarding Part (**??**); let D be an interval. If (**??**) is solvable in $L^p(\mathbb{R},\omega)$, then Part (**??**) gives that $\mathcal{H}_D$ is surjective. For the converse, we proceed as in Part (**??**), noting that (**??**) and (**??**) hold by Lemma **??**.

□

3.3.2. *Examples of weights in* $A_p(\mathbb{R})$ *satisfying the conditions* (**??**) *or* (**??**) *and proof of Corollary* **??**. We next give necessary and sufficient conditions on values $\ell, s \in \mathbb{R}$ for weights of the form $\omega(x) = |x+1|^\ell |x-1|^s$ to be in $A_p(\mathbb{R})$ and to satisfy the hypothesis (**??**) or (**??**) of Corollary **??**. Using (**??**) and (**??**), we obtain the following equivalencies:

- The weight $|x+1|^\ell |x-1|^s$ belongs to $A_p(\mathbb{R})$ and satisfies (**??**) if and only if

$$\begin{aligned} &s > -1,\ s+\ell > -1, \text{ and } \max\{1, 2(s+1), \ell+1, s+\ell+1\} < p < 2(\ell+1) \quad \text{or} \\ &\ell > -1,\ s+\ell > -1, \text{ and } \max\{1, 2(\ell+1), s+1, s+\ell+1\} < p < 2(s+1). \end{aligned} \tag{3.14}$$

- The weight $|x+1|^\ell |x-1|^s$ belongs to $A_p(\mathbb{R})$ and satisfies (**??**) if and only if

$$s+\ell > -1 \text{ and } \max\{1, s+1, \ell+1, s+\ell+1\} < p < \min\{2(s+1), 2(\ell+1)\}. \tag{3.15}$$

The proof of Corollary **??** follows using the examples of weights given above.

Proof of Corollary **??**. For Items (**??**) and (**??**), take $s = \ell = 0$. Since (**??**) does not hold, we have that (**??**) is not uniquely solvable in $L^p(\mathbb{R})$. Since (**??**) holds if and only if $1 < p < 2$ and (**??**) is not uniquely solvable in $L^p(\mathbb{R})$ for any $1 < p < 2$, the claim in Item (**??**) follows.

Regarding Item (**??**), (**??**) gives that (**??**) is uniquely solvable in $(L^p(\mathbb{R},\omega), L^p(\mathbb{R},\omega))$ with $\omega(x) = \min(1, |x \pm 1|^\ell)$. The claim then follows since $\omega \lesssim 1$ on $\mathbb{R}$.

□

3.3.3. *Examples of weights in $A_p(\mathbb{R})$ satisfying the conditions* (**??**) *or* (**??**). Regarding power weights of the form

$$w(x) := \prod_{j=1}^{m} |x + x_j|^{\ell_j} |x|^s, \quad s, x_j \in \mathbb{R}, x_j \neq 0, j = 1, \cdots, m,$$

we have the following with $\ell = \ell_1 + \cdots, \ell_m$:

- $w \in A_p(\mathbb{R})$ and w satisfies (**??**) if and only if

$$\begin{aligned} &\ell + s > -1,\ s > -1,\ \ell_j > -1\,\forall j \text{ s.t. } x_j < 0 \text{ and} \\ &p > \max\{1, 2(s+1), 2(\ell + s + 1), \max_{j:x_j<0}\{\ell_j + 1\}\} \\ &\text{or} \\ &\ell_j > -1\,\forall j \text{ s.t. } x_j < 0 \text{ and} \\ &\max\{1, s+1, s+\ell+1, \max_{j:x_j<0}\{\ell_j+1\}\} < p < 2\min\{s+1, \ell+s+1\}. \end{aligned} \tag{3.16}$$

- $w \in A_p(\mathbb{R})$ and w satisfies (**??**) if and only if

$$\begin{aligned} &\ell + s > -1,\ \ell_j > -1\,\forall j \text{ s.t. } x_j < 0 \text{ and} \\ &\max\{1, s+1, 2(s+\ell+1), \max_{j:x_j<0}\{\ell_j+1\}\} < p < 2(s+1)\}. \end{aligned} \tag{3.17}$$

3.3.4. *The general case of Corollary* **??**. Here we state the result for the solvability of the Zaremba problem (**??**) with $D = (a, b)$

Corollary 3.6. *Let $1 < p < \infty$, $-\infty < a < b < \infty$, $\omega \in A_p(\mathbb{R})$ and consider the Zaremba problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = f_{(a,b)} & \text{on } (a,b), \\ \nabla u \cdot (0,-1) = f_{\mathbb{R}\setminus\overline{(a,b)}} & \text{on } \mathbb{R} \setminus \overline{(a,b)}. \end{cases} \tag{3.18}$$

Then (**??**) *is uniquely solvable in $L^p(\mathbb{R}, \omega)$ if and only if ω satisfies*

$$\left(\frac{b-x}{x-a}\right)^{\frac{p}{2}} \omega(x) \in A_p((a,b)) \quad \text{or} \quad \left(\frac{x-a}{b-x}\right)^{\frac{p}{2}} \omega(x) \in A_p((a,b)). \tag{3.19}$$

Also, (**??**) *is not uniquely solvable but solvable in $L^p(\mathbb{R}, \omega)$ if and only if*

$$\left(\frac{(b-a)^2}{(b-a)^2 - (2x - (b+a))^2}\right)^{\frac{p}{2}} \omega(x) \in A_p((a,b)). \tag{3.20}$$

This corollary follows through a change of variable via the transformation $T_{a,b} : I \to (a, b)$ given by $T_{a,b}(x) = \frac{b-a}{2}x + \frac{b+a}{2}$, and using a proof similar to that of Corollary **??** with the following definitions for ω^* and $\mathcal{T}$: For $\omega \in A_p((a, b))$ or $\omega \in A_p(I)$ define, respectively,

$$\omega^*(x) = \mu \circ T_{a,b} \text{ for } x \in I \quad \text{and} \quad \omega^*(x) = \mu \circ T_{a,b}^{-1} \text{ for } x \in (a,b).$$

The mapping $\mathcal{T} : L^p((a, b), \omega) \to L^p(I, \omega^*)$ is given by $\mathcal{T}f(x) = \left(\frac{b-a}{2}\right)^{\frac{1}{p}} f \circ T_{a,b}$.

4. The mixed problem (??) for general domains

The main goals of this section are to prove Theorem **??** and to present specific examples corresponding to domains Ω such as a cone and, more generally, a Schwarz-Christoffel Lipschitz domain. We will use the results in Section **??**.

4.1. **Preliminaries.** Recall that given a weight η on $\partial\Omega$ (i.e. a non-negative locally integrable function in $\partial\Omega$ with respect to arc-length) and $1 < p < \infty$, $L^p(\partial\Omega, \eta)$ is the space of measurable functions in $\partial\Omega$ that are p-integrable with respect to $\eta(\xi)\, d\xi$, where $d\xi$ indicates arc-length measure; if $\eta \equiv 1$, we write $L^p(\partial\Omega)$. The definition of the Muckenhoupt class on $\partial\Omega$, denoted $A_p(\partial\Omega)$, is analogous to (**??**) with the supremum taken over all intervals contained in $\partial\Omega$ ($J \subset \partial\Omega$ is an interval if $b(J)$ is an interval in $\mathbb{R}$, where $b : \partial\Omega \to \mathbb{R}$ is defined by $b(x+i\gamma(x)) = x$ for $x \in \mathbb{R}$); set $A_\infty(\partial\Omega) = \cup_{p>1} A_p(\partial\Omega)$. We note that if $\eta \in A_\infty(\partial\Omega)$ if and only if $|\Phi'|\,(\eta \circ \Phi) \in A_\infty(\mathbb{R})$ (see [**?**, Lemma 1.16]).

Given $0 < p < \infty$, $0 < \beta < \arctan(1/L)$ and $\mathcal{M}_\beta$ as in (**??**), we define the Hardy space $H^p(\Omega, \eta)$ as

$$H^p(\Omega, \eta) := \{h : \Omega \to \mathbb{C} : h \text{ is analytic in } \Omega \text{ and } \|\mathcal{M}_\beta(h)\|_{L^p(\partial\Omega,\eta)} < \infty\},$$

setting $\|h\|_{H^p(\Omega,\eta)} = \|\mathcal{M}_\beta(h)\|_{L^p(\partial\Omega,\eta)}$. This definition is independent of β and any other chosen value gives an equivalent norm.

Given $f : \partial\Omega \longrightarrow \mathbb{R}$ and $\xi = \nu(x) = x + i\gamma(x) \in \partial\Omega$, we define $f'(\xi)$ by the condition

$$f'(\xi)(1 + i\gamma'(x)) = (f \circ \nu)'(x),$$

whenever $f \circ \nu$ and γ are differentiable at x. We say that f is differentiable (in a weak sense) over the curve $\partial\Omega$ if this holds for almost every $x \in \mathbb{R}$.

We end this section with the statements of lemmas to be used in the proof of Theorem **??**. The domain Ω and the conformal map Φ are as above.

Lemma 4.1 (Proposition 1.1 in Jerison-Kenig [**?**] and Lemma 1.13 in [**?**]). *There exist β, β_1 and $\beta_2 > 0$ such that for any $x \in \mathbb{R}$*

$$\Gamma_{\beta_1}(\Phi(x)) \subset \Phi(\Gamma_\beta(x)) \subset \Gamma_{\beta_2}(\Phi(x)).$$

Lemma 4.2 (Theorem 2.8 in [**?**]). *Let $0 < p < \infty$ and $\eta \in A_\infty(\partial\Omega)$. Then $h \in H^p(\Omega, \eta)$ if and only if $h \circ \Phi \in H^p(\mathbb{R}^2_+, |\Phi'|\,(\eta \circ \Phi))$, with equivalence of norms.*

Lemma 4.3 (Corollary 4.15 in Ballesta-Yagüe [**?**]). *Let $1 < p < \infty$ and $\eta \in A_\infty(\partial\Omega)$ such that $|\Phi'|^{1-p}\,(\eta \circ \Phi) \in A_\infty(\mathbb{R})$. Then*

$$h \in H^p\left(\mathbb{R}^2_+, |\Phi'|^{1-p}\,(\eta \circ \Phi)\right) \iff h \cdot \frac{1}{\Phi'} \in H^p\left(\mathbb{R}^2_+, |\Phi'|\,(\eta \circ \Phi)\right)$$

with equivalent norms.

4.2. **Proof of Theorem ??.** Consider $f_D \in L^1_{\text{loc}}(D)$ with $f'_D \in L^p(\partial\Omega, \eta)$ and $f_N \in L^1_{\text{loc}}(N)$ with $f_N \in L^p(\partial\Omega, \eta)$, and suppose that (**??**) is solvable in $(L^p(\mathbb{R}, \omega), L^p(\mathbb{R}, \mu))$. A change of variables shows

$$\|(f_D \circ \Phi)'\|_{L^p(\mathbb{R},\omega)} = \|f'_D\|_{L^p(\partial\Omega,\eta)} \quad \text{and} \quad \||\Phi'|\,(f_N \circ \Phi)\|_{L^p(\mathbb{R},\omega)} = \|f_N\|_{L^p(\partial\Omega,\eta)}. \tag{4.1}$$

We see that a solution v of the Zaremba problem for Ω corresponding to the sets D and N with data f_D and f_N is given by $u \circ \Phi^{-1}$, where u is a solution of (**??**). Indeed, v is harmonic in Ω; $v = f_D$ on D in view of Lemma **??** and the fact that $v \circ \Phi = u = f_D \circ \Phi$ on $\Phi^{-1}(D)$; and $\nabla v \cdot \mathbf{n} = f_N$ on N as seen by using the same ideas as in [**?**, (4.3)].

We next prove that

$$\|\mathcal{M}_\beta(\nabla v)\|_{L^p(\partial\Omega,\rho)} \lesssim \|f'_D\|_{L^p(\partial\Omega,\eta)} + \|f_N\|_{L^p(\partial\Omega,\eta)}. \tag{4.2}$$

Define

$$F(y_1, y_2) = \frac{\partial u}{\partial_{y_1}}(y_1, y_2) - i\frac{\partial u}{\partial_{y_2}}(y_1, y_2) \quad \text{and} \quad G(z_1, z_2) = \frac{\partial v}{\partial_{z_1}}(z_1, z_2) - i\frac{\partial v}{\partial_{z_2}}(z_1, z_2).$$

Since u and v are harmonic, the Cauchy-Riemann equations imply that F and G are analytic in $\mathbb{R}^2_+$ and Ω, respectively. Moreover, since

$$\|\mathcal{M}_\beta(\nabla u)\|_{L^p(\mathbb{R},\mu)} \lesssim \|(f_D \circ \Phi)'\|_{L^p(\mathbb{R},\omega)} + \||\Phi'|\,(f_N \circ \Phi)\|_{L^p(\mathbb{R},\omega)}, \tag{4.3}$$

we have that $F \in H^p(\mathbb{R}^2_+, \mu)$ and therefore $F/\Phi' \in H^p(\mathbb{R}^2_+, |\Phi'|\,(\rho \circ \Phi))$, with equivalent norms, by Lemma **??**. Noting that $G = (F \circ \Phi^{-1})(\Phi^{-1})'$, Lemma **??** then gives that $G \in H^p(\Omega, \rho)$, with equivalent norms. This fact, (**??**) and (**??**) give the desired estimate (**??**).

Conversely, suppose (**??**) is solvable in $(L^p(\partial\Omega, \eta), L^p(\partial\Omega, \rho))$. Then the Zaremba problem (**??**) is solvable in $(L^p(\mathbb{R}, \omega), L^p(\mathbb{R}, \mu))$, with the solutions related through $u = v \circ \Phi$, where v represents the solution for the Zaremba problem (**??**). This follows from reasonings similar to the ones given above. The equivalence for uniqueness follows from the fact that v is a solution of (**??**) if and only if $v \circ \Phi$ is a solution of (**??**). □

4.3. **Applications to the cone and related domains.** In this section we consider Theorem **??** when Ω is a cone with vertex at zero or, more generally, a Schwarz-Christoffel Lipschitz domain.

4.3.1. Ω *is a cone with vertex at zero and aperture* $\alpha\pi$ *with* $0 < \alpha < 2$. We will present examples of open sets D and Lebesgue spaces $L^p(\partial\Omega, \eta)$ illustrating situations where the Zaremba problem (**??**) is solvable, uniquely solvable or not solvable in $L^p(\partial\Omega, \eta)$. We will consider $D = \Phi(I)$ or $D = \Phi((0, \infty))$. For all these examples we have $\Phi(z) = e^{i\theta}\,(z - x_0)^\alpha$ for some fixed $x_0, \theta \in \mathbb{R}$ and all $z \in \mathbb{R}^2_+$.

We start with the case $\eta \equiv 1$; then $|\Phi'(x)|^{1-p} = \alpha^{1-p}|x - x_0|^{(\alpha-1)(1-p)}$ for $x \in \mathbb{R}$ and we must assume $1 < p < p_\Phi$, so that $|\Phi'|^{1-p} \in A_p(\mathbb{R})$. Note that $p_\Phi = \infty$ if $0 < \alpha \leq 1$ and $p_\Phi = \frac{\alpha}{\alpha-1}$ if $1 < \alpha \leq 2$.

$D = \Phi(I)$ and $\eta \equiv 1$ on $\partial\Omega$. The examples presented in this case follow from Corollary **??** with $\eta \equiv 1$.

D is a segment with an endpoint at the vertex of the cone. Let $\Phi(z) = e^{i\theta}(z + 1)^\alpha$. Then, $D = \Phi(I)$ is a segment on the right ray of Ω with an endpoint at zero, as shown in Figure **??**.

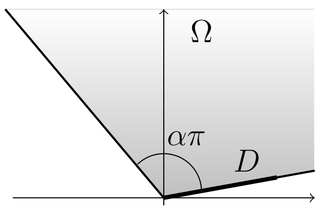

FIGURE 1. D is a segment with an endpoint at the vertex of the cone

(a) *Uniquely solvable* in $L^p(\partial\Omega)$ if and only if α and p are as follows:
   (i) $0 < \alpha \leq 1/2$ and $p > 2$,
   (ii) $1/2 < \alpha < 1$ and $2 < p < \frac{2\alpha}{2\alpha-1}$,
   (iii) $1 < \alpha < 2$ and $\frac{2\alpha}{2\alpha-1} < p < 2$.

   In order to conclude we need to see that one of the conditions in (**??**) hold with $s = 0$ and $\ell = (\alpha - 1)(1 - p)$ under the assumptions in Items (**??**), (**??**) and (**??**); this would show that (**??**) is satisfied with $\eta \equiv 1$. The first condition in (**??**) is equivalent to

$$\max\{2, (\alpha - 1)(1 - p) + 1\} < p < 2(\alpha - 1)(1 - p) + 2,$$

which in turn is equivalent to Item (**??**) if $0 < \alpha \le 1/2$ and to Item (**??**) if $1/2 < \alpha < 1$. The second condition in (**??**) is equivalent to

$$(\alpha - 1)(1 - p) > -1 \quad \text{and} \quad \max\{1, 2(\alpha - 1)(1 - p) + 2\} < p < 2,$$

which is equivalent to Item (**??**).

(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega)$ if and only if α and p are in the following ranges:
  (i) $0 < \alpha \le 1$ and $1 < p < 2$,
  (ii) $1 < \alpha < 2$ and $1 < p < \frac{2\alpha}{2\alpha-1}$.
  This follows from checking the conditions in (**??**) for $s = 0$ and $\ell = (\alpha - 1)(1 - p)$.
(c) *Not solvable* in $L^p(\partial\Omega)$ for the following values of p and α as neither (**??**) nor (**??**) are satisfied for $\eta \equiv 1$:
  (i) $0 < \alpha \le 1/2$ and $p = 2$,
  (ii) $1/2 < \alpha \le 1$ with $p = 2$ or $\frac{2\alpha}{2\alpha-1} \le p < \infty$,
  (iii) $1 < \alpha < 2$ with $p = \frac{2\alpha}{2\alpha-1}$ or $2 \le p < p_\Phi$.

D is a segment away from the vertex of the cone. Let $\Phi(z) = e^{i\theta}(z + 2)^\alpha$. Then, $D = \Phi(I)$ is a segment away from the vertex of the cone as shown in Figure **??**.

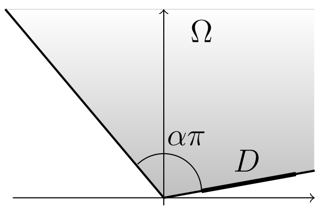

FIGURE 2. D is a segment away from the vertex of the cone

(a) *Not uniquely solvable* in $L^p(\partial\Omega)$ for $1 < p < p_\Phi$ since $|\Phi'|^{1-p} \sim 1$ in I gives that (**??**) is not satisfied with $\eta \equiv 1$.
(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega)$ for $1 < p < 2$ noting that (**??**) is satisfied with $\eta \equiv 1$.
(c) *Not solvable* in $L^p(\partial\Omega)$ for $2 \le p < p_\Phi$ as neither (**??**) nor (**??**) are satisfied for $\eta \equiv 1$.

D is a bounded polygonal curve that "bends" at the vertex of the cone. Let $\Phi(z) = e^{i\theta} z^\alpha$. Then, $D = \Phi(I)$ is as shown in Figure **??**.

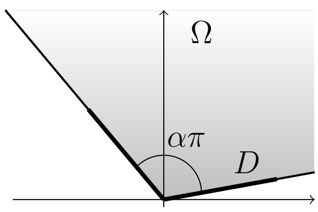

FIGURE 3. D is a bounded polygonal curve that "bends" at the vertex of the cone

(a) *Not uniquely solvable in* $L^p(\partial\Omega)$ for $1 < p < p_\Phi$ as (**??**) does not hold if $\eta \equiv 1$.
(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega)$ for $1 < p < 2$ since condition (**??**) is satisfied with $\eta \equiv 1$.
(c) *Not solvable in* $L^p(\partial\Omega)$ for $2 \le p < p_\Phi$ since neither (**??**) nor (**??**) are true with $\eta \equiv 1$.

$D = \Phi((0,\infty))$ and $\eta \equiv 1$ on $\partial\Omega$. The examples presented in this case follow from Corollary **??** with $\eta \equiv 1$.

D is a ray of the cone. Let $\Phi(z) = e^{i\theta} z^{\alpha}$. Then, $D = \Phi((0,\infty))$ is the right ray of the cone as shown in Figure **??**.

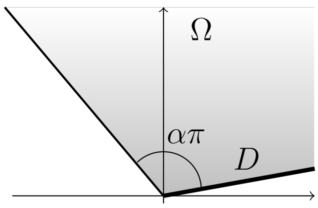

FIGURE 4. D is a ray of the cone

(a) *Uniquely solvable* in $L^p(\partial\Omega)$ for the following ranges of p and α :
  (i) $0 < \alpha \le \frac{1}{2}$ and $1 < p < \infty$,
  (ii) $\frac{1}{2} < \alpha \le 1$ and $1 < p < \infty$ such that $p \ne \frac{2\alpha}{2\alpha-1}$,
  (iii) $1 < \alpha < 2$ and $1 < p < p_\Phi$ such that $p \ne \frac{2\alpha}{2\alpha-1}$.
  This follows from using (**??**) with $m = 1$, $\ell = \ell_1 = 0$ and $s = (\alpha-1)(1-p)$ to show that (**??**) with $\eta \equiv 1$ is satisfied for those values of p and α. The solvability results stated above add uniqueness to those in [**?**, Corollary 5.1]; see also [**?**, Theorem 5] for a closely related statement.
(b) *Not solvable in* $L^p(\partial\Omega)$ for $\frac{1}{2} < \alpha < 2$ and $p = \frac{2\alpha}{2\alpha-1}$. This follows from (**??**) with $m = 1$, $\ell = \ell_1 = 0$ and $s = (\alpha-1)(1-p)$ to show that (**??**) with $\eta \equiv 1$ is not satisfied.

D is a half-line away from the vertex of the cone. Let $\Phi(z) = e^{i\theta}(z+1)^{\alpha}$. Then, $D = \Phi((0,\infty))$ is a half-line away from the vertex of the cone as shown in Figure **??**.

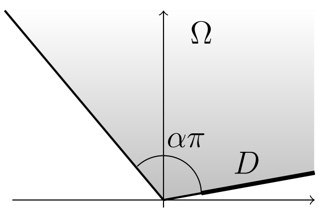

FIGURE 5. D is a half-line away from the vertex of the cone

(a) *Uniquely solvable* $L^p(\partial\Omega)$ if and only if
  (i) $0 < \alpha \le \frac{1}{2}$ with $1 < p < 2$,
  (ii) $\frac{1}{2} < \alpha \le 1$ with $1 < p < 2$ or $p > \frac{2\alpha}{2\alpha-1}$,
  (iii) $1 < \alpha < 2$ with $1 < p < \frac{2\alpha}{2\alpha-1}$ or $2 < p < p_\Phi$.
  This follows from (**??**) with $m = 1$, $\ell = \ell_1 = (\alpha-1)(1-p)$ and $s = 0$ to show that (**??**) with $\eta \equiv 1$ is satisfied if and only if the parameters are as above.
(b) *Solvable but not uniquely solvable in* $L^p(\partial\Omega)$ if and only if $1 < \alpha < 2$ and $\frac{2\alpha}{2\alpha-1} < p < 2$. This follows from (**??**) with $m = 1$, $\ell = \ell_1 = (\alpha-1)(1-p)$ and $s = 0$ to show that (**??**) with $\eta \equiv 1$ is satisfied for such values of p and α.
(c) *Not solvable in* $L^p(\partial\Omega)$ for the following ranges of p and α :
  (a) $0 < \alpha \le \frac{1}{2}$ with $p \ge 2$,
  (b) $\frac{1}{2} < \alpha \le 1$ with $2 \le p \le \frac{2\alpha}{2\alpha-1}$,
  (c) $1 < \alpha < 2$ with $p = \frac{2\alpha}{2\alpha-1}$ or $p = 2$.

Reasoning as in the example corresponding to Figure **??**, we have the same exact solvability results for the following case:

D is an unbounded polygonal curve that "bends" at the vertex of the cone. Let $\Phi(z) = e^{i\theta}(z-1)^\alpha$. Then, $D = \Phi((0,\infty))$ is as shown in Figure **??**.

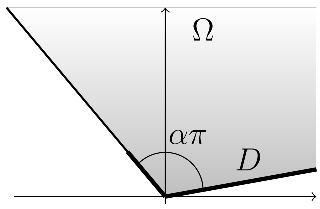

FIGURE 6. D is an unbounded polygonal curve that "bends" at the vertex of the cone

We next consider cases where η is a power weight.

$D = \Phi(I)$ and $\eta(\xi) = |\xi|^\beta$ for $\xi \in \partial\Omega$ and some $\beta > -1$. We present solvability results for the mixed problem (**??**) in $L^p(\partial\Omega, \eta)$ with Ω as in Figures **??**, **??** and **??**, which follow from Corollary **??** and recover previously stated results with data in $L^p(\partial\Omega)$. Recall that in all those cases $\Phi(z) = e^{i\theta}(z - x_0)^\alpha$. Since $\eta \in A_\infty(\partial\Omega)$ if and only if $|\Phi'|(\eta \circ \Phi) \in A_\infty(\mathbb{R})$, we must have $\beta > -1$. Moreover, $|\Phi'|^{1-p}(\eta \circ \Phi)$ is in $A_\infty(\mathbb{R})$ if and only if $\beta > p - 1 - \frac{p}{\alpha}$. We must then assume

$$\beta > -1 \text{ if } \alpha \le 1 \quad \text{or} \quad \beta > p - 1 - \frac{p}{\alpha} \text{ if } 1 < \alpha < 2. \tag{4.4}$$

Observe that the second condition in β is equivalent to $1 < p < \frac{\alpha(\beta+1)}{\alpha-1}$ and define

$$p_{\Phi,\beta} = \infty \text{ if } 0 < \alpha \le 1 \quad \text{and} \quad p_{\Phi,\beta} = \frac{\alpha(\beta+1)}{\alpha - 1} \text{ if } 1 < \alpha \le 2.$$

In particular, we obtain $p_{\Phi,0} = p_\Phi$.

Ω is as in Figure **??** and β is as in (**??**):

(a) *Uniquely solvable* in $L^p(\partial\Omega, \eta)$ if and only if α and p are as follows:
  (i) $0 < \alpha \le \frac{1}{2}$ and $p > \max\{2, \frac{2\alpha(\beta+1)}{2\alpha+1}\}$,
  (ii) $1/2 < \alpha \le 1$ with $\max\{2, \frac{2\alpha(\beta+1)}{2\alpha+1}\} < p < \frac{2\alpha(\beta+1)}{2\alpha-1}$ or $\max\{1, \frac{2\alpha(\beta+1)}{2\alpha-1}\} < p < 2$,
  (iii) $1 < \alpha < 2$ with $\max\{2, \frac{2\alpha(\beta+1)}{2\alpha+1}\} < p < \frac{2\alpha(\beta+1)}{2\alpha-1}$ or $\max\{1, \frac{2\alpha(\beta+1)}{2\alpha-1}\} < p < \min\{2, p_{\Phi,\beta}\}$.

(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega, \eta)$ if and only if α and p are as follows:
  (i) $0 < \alpha \le 1/2$ and $\max\{1, \frac{2\alpha(\beta+1)}{2\alpha+1}\} < p < 2$,
  (ii) $1/2 < \alpha < 2$ and $\max\{1, \frac{2\alpha(\beta+1)}{2\alpha+1}\} < p < \min\{2, \frac{2\alpha(\beta+1)}{2\alpha-1}\}$.

Ω is as in Figure **??** and β is as in (**??**): In this case we have $|\Phi'|^{1-p}(\eta \circ \Phi) \sim 1$ in I. For $0 < \alpha < 2$, the following holds:

(a) *Not uniquely solvable* in $L^p(\partial\Omega, \eta)$ for $1 < p < p_{\Phi,\beta}$.

(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega, \eta)$ for $1 < p < \min\{2, p_{\Phi,\beta}\}$.

Ω is as in Figure **??** and β is as in (**??**):

(a) *Not uniquely solvable in* $L^p(\partial\Omega, \eta)$ for $1 < p < p_{\Phi,\beta}$ and $0 < \alpha < 2$.

(b) *Solvable but not uniquely solvable* in $L^p(\partial\Omega, \eta)$ as follows:
  (i) $0 < \alpha \le 1$ and $\max\{1, \beta + 1\} < p < 2$,
  (ii) $1 < \alpha < 2$ and $\max 1, \beta + 1 < p < \min\{2, p_{\Phi,\beta}\}$.

4.3.2. *Ω is a Schwarz-Christoffel Lipschitz domain.* In this section, we present examples of Theorem **??** in domains Ω of which the cone is a particular case: $\partial\Omega$ is a polygonal curve with a finite number of vertices $w_1, \cdots, w_m$ in $\mathbb{C}$ and interior angles $\alpha_1\pi, \cdots, \alpha_m\pi$, respectively.

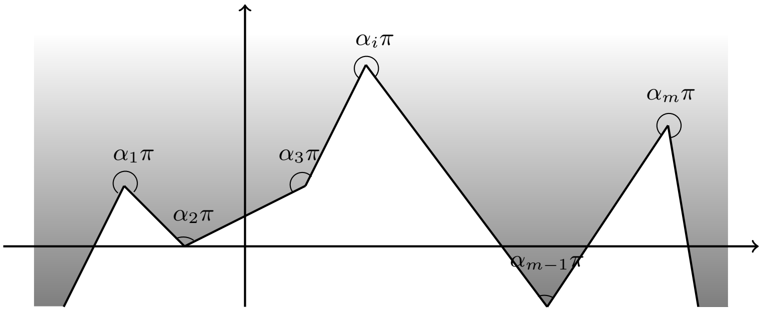

FIGURE 7. Unbounded polygon with vertices $w_1, \ldots, w_m \in \mathbb{C}$

We have the following result regarding conformal mappings from $\mathbb{R}^2_+$ onto a domain Ω as the one shown in Figure **??**:

Theorem 4.4. *(Theorem 2.1 in Driscoll and Trefethen* [**?**]*) Suppose Ω is a polygon as shown in Figure* **??** *with vertices $w_1, \ldots, w_m$ in $\mathbb{C}$ and interior angles $\alpha_1\pi, \ldots \alpha_m\pi$ in counterclockwise order. If Φ is a conformal mapping from $\mathbb{R}^2_+$ onto Ω with $\Phi(\infty) = \infty$ then*

$$\Phi(z) = A + B \int_{[z_0,z]} (\xi - x_1)^{\alpha_1 - 1} \cdots (\xi - x_m)^{\alpha_m - 1}\, d\xi, \tag{4.5}$$

where $A, B \in \mathbb{C}$, z_0 is a suitably chosen point in $\mathbb{R}^2_+$ or its boundary, $[z_0, z]$ denotes the straight line segment from z_0 to z, $x_1, \ldots, x_m$ are real numbers such that $x_1 < \cdots < x_m$ and $\Phi(x_k) = w_k$ for $k = 1, \ldots, m$.

A mapping of the form given on the right-hand side of (**??**) is called a *Schwarz-Christoffel transformation.*

We next give examples on the solvability of the Zaremba problem in a Schwarz-Christoffel domain: *Ω has two vertices and D is the segment joining them.*

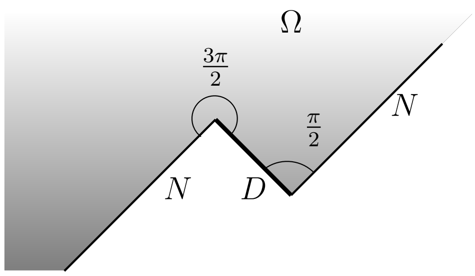

FIGURE 8. $m = 2$, $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = \frac{1}{2}$

Consider Ω a polygon with vertices $w_1, w_2 \in \mathbb{C}$ such that $x_1 = -1$, $x_2 = 1$, $A = 0$ and $B = 1$ in the formula (**??**). Set $D = \Phi(I) = [w_1, w_2]$. Note that in order for Ω to be a Lipschitz graph domain, we must have $\alpha_1 > 1$ and $\alpha_2 < 1$ or, $\alpha_1 < 1$ and $\alpha_2 > 1$ (See Figure **??**). We have

$$|\Phi'(x)| = |x + 1|^{\alpha_1 - 1} |x - 1|^{\alpha_2 - 1}, \quad x \in \mathbb{R},$$

and

$$|\Phi'(x)|^{1-p} = |x+1|^{(\alpha_1-1)(1-p)}|x-1|^{(\alpha_2-1)(1-p)}, \quad x \in \mathbb{R}.$$

Setting $\ell = (\alpha_1 - 1)(1-p)$ and $s = (\alpha_2 - 1)(1-p)$, and using (**??**), (**??**) and Corollary **??**, it is possible to determine values of α_1, α_2 and p to determine solvability results of the Zaremba problem in this setting. To illustrate, we give an specific example taking $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{1}{2}$. We have $p_\Phi = 3$ ([**?**, Lemma 5.2]); assume $1 < p < 3$.

(a) *Uniquely solvable in* $L^p(\partial\Omega)$ if and only if $3/2 < p < 3$. This follows from the second condition in (**??**) to show that the second condition in (**??**) holds with $\eta \equiv 1$ (the first condition never holds).

(b) *Not uniquely solvable but solvable in* $L^p(\partial\Omega)$ if and only if $1 < p < 3/2$. This follows by checking that (**??**) (but not (**??**)) is satisfied.

Remark 4.5. As pointed out in Section **??**, when Ω is a Lipschitz graph domain with D and N meeting at an angle different from π (creased domains), the Zaremba problem is uniquely solvable in $L^2(\partial\Omega)$ (see [**?**]). In the examples corresponding to Figures **??** and **??**, the sets D and N meet at an angle different from π at all points of ∂D $(= \partial N)$; as shown, the Zaremba problem in those settings is uniquely solvable in $L^2(\partial\Omega)$ as expected. In all other examples that we presented, D and N meet at an angle equal to π at least at one point of ∂D, and the corresponding Zaremba problem is not uniquely solvable in $L^2(\partial\Omega)$. This raises the question of whether the condition for a creased domain that N and D meet at an angle different from π at all points of their (common) boundary is necessary for the Zaremba problem to be uniquely solvable in $L^2(\partial\Omega)$. Note that the latter is not the case for $L^p(\partial\Omega)$ with $p \neq 2$; see, for instance, [**?**, Corollary 1.6], which states that the Zaremba problem with $\Omega = \mathbb{R}^2_+$ and $D = (0, \infty)$ is uniquely solvable in $L^p(\mathbb{R})$ with $p \neq 2$.

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