Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 164571, 10 pages
http://dx.doi.org/10.1155/2013/164571

Research Article

Smooth Approximation of Lipschitz Functions on Finsler Manifolds

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Received 13 May 2013; Accepted 24 June 2013

Academic Editor: Miguel Martin

We study the smooth approximation of Lipschitz functions on Finsler manifolds, keeping control on the corresponding Lipschitz constants. We prove that, given a Lipschitz function $f : M \to \mathbb{R}$ defined on a connected, second countable Finsler manifold M, for each positive continuous function $\varepsilon : M \to (0, \infty)$ and each $r > 0$, there exists a C^1-smooth Lipschitz function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon(x)$, for every $x \in M$, and $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + r$. As a consequence, we derive a completeness criterium in the class of what we call quasi-reversible Finsler manifolds. Finally, considering the normed algebra $C_b^1(M)$ of all C^1 functions with bounded derivative on a complete quasi-reversible Finsler manifold M, we obtain a characterization of algebra isomorphisms $T : C_b^1(N) \to C_b^1(M)$ as composition operators. From this we obtain a variant of Myers-Nakai Theorem in the context of complete reversible Finsler manifolds.

1. Introduction

There are many geometrically significant functions on a Riemannian manifold which are typically Lipschitz but not smooth, as it is the case, for example, of distance functions. Thus it is interesting to study the regularization and smooth approximation of Lipschitz functions on Riemannian manifolds. This has been done in the classical work of Greene and Wu [1], where in particular it is proved that every Lipschitz real function on a (connected, second countable, and finite dimensional) Riemannian manifold can be approximated, in the C^0-fine topology, by smooth Lipschitz functions whose Lipschitz constants can be made arbitrarily close to the Lipschitz constant of the original function. This result has been extended in [2] to the case of infinite-dimensional Riemannian manifolds, where some interesting applications are also given. Recently, related approximation results in the setting of the so-called Banach-Finsler manifolds have been obtained in [3].

Our purpose here is to study the analogous approximation problem in the context of (finite-dimensional) Finsler manifolds, where the Finsler structure is supposed to be positively (but in general not absolutely) homogeneous. The contents of the paper are as follows. In Section 2 we collect some basic preliminary facts about Finsler manifolds. Section 3 is devoted to give a mean value inequality in this context. Next, in Section 4, we obtain our main result. Namely, we prove in Theorem 8 that every Lipschitz real function on a connected, second countable Finsler manifold can be approximated, in the C^0-fine topology, by C^1-smooth Lipschitz functions with Lipschitz constants arbitrarily close to the Lipschitz constant of the original function. This approximation result has been used in [4] in order to obtain a version of the Myers-Nakai Theorem for reversible Finsler manifolds (that is, in the case that the Finsler structure is absolutely homogeneous). In Section 5 we introduce the class of quasi-reversible Finsler manifolds, which can be described as those Finsler manifolds where distance functions are in fact Lipschitz. As a consequence of our main result, we obtain a completeness criterium for quasi-reversible Finsler manifolds, in terms of the existence of a proper C^1-smooth function with uniformly bounded derivative. In this way we extend the completeness criterium for Riemannian manifolds given by Gordon in [5]. Finally, in Section 6 we consider the normed algebra $C_b^1(M)$ of all C^1 functions with bounded derivative on a quasi-reversible Finsler manifold M, and we obtain a characterization of normed algebra isomorphisms $T : C_b^1(N) \to C_b^1(M)$ as composition operators. From this we obtain a variant of Myers-Nakai Theorem in the context of complete reversible Finsler manifolds.

2. Preliminaries

We start with the basic notion of Minskowski norm.

Definition 1. Let V be a finite-dimensional real vector space. One says that a functional $F : V \to [0,\infty)$ is a *Minkowski norm* on V if the following conditions are satisfied

(i) Positivity: $F(v) = 0$ if and only if $v = 0$.

(ii) Triangle inequality: $F(u+v) \leq F(u) + F(v)$, for every $u, v \in V$.

(iii) Positive homogeneity: $F(\lambda v) = \lambda F(v)$, for every $v \in V$ and every $\lambda > 0$.

(iv) Regularity: F is continuous on V and C^∞-smooth on $V \setminus \{0\}$.

(v) Strong convexity: for every $v \in V \setminus \{0\}$, the quadratic form g_v associated to the second derivative of the function F^2 at v, that is,

$$g_v = \frac{1}{2} d^2 \left[F^2\right](v), \tag{1}$$

is positive definite on V.

We note that conditions (i) and (ii) in the above definition are, in fact, consequence of conditions (iii)–(v) (see Theorem 1.2.2 of [6]). It is clear that every norm associated to an inner product is a Minkowski norm. Recall that, in general, a Minkowski norm needs not to be symmetric, and there are indeed very interesting examples of nonsymmetric Minkowski norms, such as, for example, Randers spaces (see [6]). We say F is symmetric or absolutely homogeneous if

$$F(\lambda v) = |\lambda| F(v), \quad \text{for every } v \in V \text{ and every } \lambda \in \mathbb{R}. \tag{2}$$

In this case, F is a norm in the usual sense.

Now the definition of Finsler manifold is as follows.

Definition 2. A *Finsler manifold* is a pair (M, F), where M is a finite-dimensional C^∞-smooth manifold and $F : TM \to [0,\infty)$ is a continuous function defined on the tangent bundle TM, satisfying

(i) F is C^∞-smooth on $TM \setminus \{0\}$.

(ii) For every $x \in M$, $F(x,\cdot) : T_xM \to [0,\infty)$ is a Minkowski norm on the tangent space T_xM.

In particular, a Riemannian manifold is a special case of Finsler manifold, where the Minkowski norm on each tangent space T_xM is given by an inner product. The Finsler structure F is said to be *reversible* if, for every $x \in M$, $F(x,\cdot)$ is symmetric. This is of course the case of Riemannian manifolds.

Now suppose that (M, F) is a connected Finsler manifold. The Finsler distance d_F on M is defined by

$$d_F(x,y) = \inf\left\{\ell_F(\sigma) : \sigma \text{ piecewise } C^1 \text{ path from } x \text{ to } y\right\}, \tag{3}$$

where the Finsler length of a piecewise C^1 path $\sigma : [a,b] \to M$ is defined as:

$$\ell_F(\sigma) = \int_a^b F\left(\sigma(t), \sigma'(t)\right) dt. \tag{4}$$

In this way we have (see Section 6.2 of [6]) that the Finsler distance d_F is the so-called an asymmetric distance on M, in the sense that it verifies

(i) $d_F(x,y) \geq 0$.

(ii) $d_F(x,y) = 0$ if and only if $x = y$.

(iii) $d_F(x,y) \leq d_F(x,z) + d_F(z,y)$, for every $x, y, z \in M$.

In general, d_F needs not to be symmetric. Nevertheless, when F is reversible the Finsler distance d_F is symmetric, and therefore (M, d_F) is a metric space in the usual sense. In general, for each $p \in M$ and $r > 0$, the forward ball of center p and radius r is defined as

$$\mathbf{B}_p^+(r) = \{x \in M : d_F(p,x) < r\}. \tag{5}$$

In the same way, the backward ball of center p and radius r is defined as

$$\mathbf{B}_p^-(r) = \{x \in M : d_F(x,p) < r\}. \tag{6}$$

Note that, as can be seen in [6], the family of forward balls and also the family of backward balls are both neighborhood basis for the topology of the manifold M.

If (V, F) is a Minkowski space, that is, a vector space endowed with a Minkowski norm, then the associated asymmetric distance d_F is given by

$$d_F(u,v) = F(v-u). \tag{7}$$

In this case, we will denote by $B_0(r)$ the forward ball of center $0 \in V$ and radius r, and we call it the Minkowski ball of center 0 and radius r. That is,

$$B_0(r) = \mathbf{B}_0^+(r) = \{v \in V : F(v) < r\}. \tag{8}$$

We next recall the following result by Deng and Hou (see Theorem 1.2 in [7]) concerning the exponential mapping in a Finsler manifold, which will be useful in what follows.

Theorem 3 (Deng and Hou [7]). *Let (M, F) be a connected Finsler manifold, let $x \in M$, and consider $r > 0$ such that the exponential mapping $\exp_x : B_0(r) \to \mathbf{B}_x^+(r)$ is a C^1-diffeomorphism. Then, for $u, v \in B_0(r)$ with $u \neq v$, one has that*

$$\lim_{(u,v)\to(0,0)} \frac{F(x, v-u)}{d_F\left(\exp_x(u), \exp_x(v)\right)} = 1. \tag{9}$$

A terminological remark is now in order. Suppose that X and Y are two nonempty sets, endowed with asymmetric distances d_X and d_Y, respectively. We say that a mapping $f : (X, d_X) \to (Y, d_Y)$ is Lipschitz, with constant $C > 0$ (or, briefly, C-Lipschitz) if, for every $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \le C \cdot d_X(x_1, x_2). \qquad (10)$$

As usual, we will say that $f : (X, d_X) \to (Y, d_Y)$ is C-bi-Lipschitz when f is bijective and both f and f^{-1} are C-Lipschitz mappings. With this terminology at hand and as a direct consequence of Theorem 3, we obtain the following result, describing the bi-Lipschitz behavior of the exponential mapping associated to a Finsler manifold in small balls.

Corollary 4. *Let (M, F) be a connected Finsler manifold. For each $x \in M$ and each $\varepsilon > 0$, there exists $r > 0$ such that the exponential mapping $\exp_x : B_0(r) \to \mathbf{B}_x^+(r)$ is a C^1-diffeomorphism, which is $(1+\varepsilon)$-bi-Lipschitz.*

3. Mean Value Inequality

In this section we obtain a kind of mean value inequality in the context of Finsler manifolds. If (M, F) is a connected Finsler manifold, we define the Lipschitz constant of a function $f : M \to \mathbb{R}$ as

$$\operatorname{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d_F(x, y)} : x, y \in M,\ x \neq y\right\} \in [0, \infty]. \qquad (11)$$

Of course f is Lipschitz if and only if $\operatorname{Lip}(f) < \infty$. We denote by $\operatorname{Lip}(M)$ the space of all real Lipschitz functions defined on M. If $f : M \to \mathbb{R}$ is now a C^1-smooth function, we define as usual the norm of its differential $df(x)$ at a point $x \in M$ by

$$\|df(x)\|_F = \sup\{|df(x)(v)| : v \in T_xM, F(x, v) \le 1\}. \qquad (12)$$

Next, we give the following result providing the desired mean value inequality.

Theorem 5. *Let (M, F) be a connected Finsler manifold and $f : M \to \mathbb{R}$ a C^1-smooth function. Then,*

$$\operatorname{Lip}(f) = \sup\{\|df(x)\|_F : x \in M\}. \qquad (13)$$

Thus, for every $p, q \in M$, one has that

$$|f(p) - f(q)| \le \sup\{\|df(x)\|_F : x \in M\} \cdot d_F(p, q). \qquad (14)$$

Proof. For the proof, fix a number $C \ge 0$, and we are going to see that the following conditions are equivalent:

(1) f is C-Lipschitz.

(2) $\|df(x)\|_F \le C$, for each $x \in M$.

(1) $\Rightarrow$ (2) Suppose there exists some $p \in M$ with $\|df(p)\|_F > C$. Then there is some $v \in T_pM$ such that $F(p, v) \le 1$ and $|df(p)(v)| > C$. Suppose, for example, that $df(p)(v) > C$, the other case being analogous. As it is shown [6] (see Theorem 6.3.1), we can choose $r > 0$ such that the geodesic $\gamma(t) = \exp_p(tv)$, defined for $t \in [0, r]$, minimizes the Finsler distance from point p, that is, $d_F(p, \gamma(s)) = s$, for every $s \in [0, r]$. Now define $h : [0, r] \to \mathbb{R}$ by $h(t) = f(\gamma(t))$. We then have that $h'(0) = df(p)(v) > C$, and therefore there exists some $\delta \in (0, r)$ such that

$$\frac{h(t) - h(0)}{t} > C, \quad \text{if } 0 < t \le \delta. \qquad (15)$$

Since $p = \gamma(0)$, choosing $q = \gamma(\delta)$, we obtain that

$$|f(p) - f(q)| = h(\delta) - h(0) > C \cdot \delta = C \cdot d_F(p, q) \qquad (16)$$

which contradicts the fact that f is C-Lipschitz.

(2) $\Rightarrow$ (1) Consider $x, y \in M$, and let $\varepsilon > 0$. Choose a piecewise C^1 path $\gamma : [a, b] \to M$ such that $\gamma(a) = x$, $\gamma(b) = y$, and

$$\ell_F(\gamma) \le d_F(x, y) + \varepsilon. \qquad (17)$$

Define now $h : [a, b] \to \mathbb{R}$ by $h(t) = f(\gamma(t))$. Then,

$$\begin{aligned} |f(y) - f(x)| &= \left|\int_a^b h'(t)\,dt\right| \le \int_a^b |h'(t)|\,dt \\ &= \int_a^b |df(\gamma(t))(\gamma'(t))|\,dt \\ &\le \int_a^b \|df(\gamma(t))\|_F \cdot F(\gamma(t), \gamma'(t))\,dt \\ &\le C\int_a^b F(\gamma(t), \gamma'(t))\,dt = C \cdot \ell_F(\gamma) \\ &\le C(d_F(x, y) + \varepsilon). \end{aligned} \qquad (18)$$

This shows that $|f(x) - f(y)| \le C \cdot d_F(x, y)$, for every $x, y \in M$. □

We finish this section with the following simple result giving a local characterization of Lipschitz mappings, which will be useful later.

Proposition 6. *Let (M, F_M) and (N, F_N) be connected Finsler manifolds, with Finsler distances d_M and d_N, respectively. A mapping $f : (M, d_M) \to (N, d_N)$ is C-Lipschitz if and only if it is locally C-Lipschitz; that is, every point $p \in M$ has a neighborhood U^p such that, for every $x, y \in U^p$,*

$$d_N(f(x), f(y)) \le C \cdot d_M(x, y). \qquad (19)$$

Proof. Suppose that f is locally C-Lipschitz. Consider $x, y \in M$ and $\varepsilon > 0$. Choose a piecewise C^1 path $\gamma : [a, b] \to M$ from x to y, such that $\ell_M(\gamma) \le d_M(x, y) + \varepsilon$. Each $p \in \gamma([a, b])$ has an open neighborhood U^p where f is C-Lipschitz. Choose a partition $a = t_0 < t_1 < \cdots < t_k = b$ of

$[a,b]$ such that, for every $i = 1, \dots, k$, $\gamma([t_{i-1}, t_i])$ is contained into U^p for some $p \in \gamma([a,b])$. Then

$$\begin{aligned} d_N(f(x), f(y)) &\le \sum_{i=1}^{k} d_N(f(\gamma(t_{i-1})), f(\gamma(t_i))) \\ &\le C \cdot \sum_{i=1}^{k} d_M(\gamma(t_{i-1}), \gamma(t_i)) \\ &\le C \cdot \sum_{i=1}^{k} \ell_M(\gamma|_{[t_{i-1}, t_i]}) = C \cdot \ell_M(\gamma) \\ &\le C \cdot (d_M(x,y) + \varepsilon). \end{aligned} \tag{20}$$

In this way we obtain that f is C-Lipschitz. The converse is clear. ☐

4. Smooth Approximation of Lipschitz Functions

In this section we present our results about regularization of Lipschitz functions on Finsler manifolds. In particular, as consequence of Theorem 8 as follows, we can derive that if $f : M \to \mathbb{R}$ is a Lipschitz function defined on a connected and second-countable Finsler manifold M and $\varepsilon > 0$ is given, there exists a C^1-smooth Lipschitz function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| \le \varepsilon$, for every $x \in M$, and $\mathrm{Lip}(g) \le \mathrm{Lip}(f) + \varepsilon$. We start with the following simple Lemma, which gives a first result of smooth approximation in Minkowski spaces.

Lemma 7. *Let (V, F) be a vector space endowed with a Minkowski norm. Consider an open set $A \subset V$ and, for $\delta > 0$, denote*

$$A_\delta = \{v + w : v \in A, F(w) < \delta\} = \bigcup_{v \in A} B_v(\delta). \tag{21}$$

Suppose that $f : A_\delta \to \mathbb{R}$ is Lipschitz and let $\varepsilon > 0$. Then there exists a C^∞-smooth function $g : V \to \mathbb{R}$ such that $|f(v) - g(v)| \le \varepsilon$, for every $v \in A$, and $\mathrm{Lip}(g|_A) \le \mathrm{Lip}(f|_A)$.

Proof. By choosing a basis of V, we may assume that $V = \mathbb{R}^n$. Note that, by local compactness, it follows that the Minkowski norm F is equivalent to the usual Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$, in the sense that there exists some $R \ge 1$ such that

$$\frac{1}{R} \cdot \|v\| \le F(v) \le R \cdot \|v\| \tag{22}$$

for every $v \in \mathbb{R}^n$. Now, if $f : A_\delta \to \mathbb{R}$ is a C-Lipschitz function for the Minkowski norm F, then f is a $(C \cdot R)$-Lipschitz function for the Euclidean norm. Hence, using, for example, the well-known MacShane extension result, we can obtain a Lipschitz extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$.

Now consider a sequence $(\varphi_k)_k$ of usual C^∞-smooth mollifiers on $\mathbb{R}^n$, where each φ_k is nonnegative, $\mathrm{supp}(\varphi_k)$ is contained in the Euclidean ball $\mathbb{B}(0, 1/k)$, and $\int_{\mathbb{R}^n} \varphi_k = 1$. For each k, define $f_k : \mathbb{R}^n \to \mathbb{R}$ by

$$f_k(v) = \int_{\mathbb{R}^n} \tilde{f}(v + w)\, \varphi_k(w)\, dw. \tag{23}$$

Each f_k is C^∞-smooth, and, since $\tilde{f}$ is uniformly continuous, we have that the sequence (f_k) converges to $\tilde{f}$ uniformly on $\mathbb{R}^n$. Given $\varepsilon > 0$, choose $k > R/\delta$ and large enough so that $\|f_k - \tilde{f}\|_\infty < \varepsilon$ and define $g = f_k$. Then, if $u, v \in A$,

$$\begin{aligned} |g(u) - g(v)| &= \left| \int_{\mathbb{R}^n} \tilde{f}(u + w)\, \varphi_k(w)\, dw \right. \\ &\qquad \left. - \int_{\mathbb{R}^n} \tilde{f}(v + w)\, \varphi_k(w)\, dw \right| \\ &\le \int_{\mathbb{R}^n} \left| \tilde{f}(u + w) - \tilde{f}(v + w) \right| \cdot |\varphi_k(w)|\, dw \\ &= \int_{\mathbb{B}(0,1/k)} |f(u + w) - f(v + w)| \cdot |\varphi_k(w)|\, dw \\ &\le C \cdot F(v - u) \cdot \int_{\mathbb{R}^n} \varphi_k(w)\, dw \\ &= C \cdot d_F(u, v). \end{aligned} \tag{24}$$

Therefore, we have that $\mathrm{Lip}(g|_A) \le \mathrm{Lip}(f|_A)$, as we wanted. ☐

We next give the main result of the paper.

Theorem 8. *Let (M, F) be a connected and second countable Finsler manifold, let $f : M \to \mathbb{R}$ be a Lipschitz function, consider $\varepsilon : M \to (0, +\infty)$ a continuous function, and let $r > 0$. Then there is a C^1-smooth Lipschitz function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| \le \varepsilon(x)$, for every $x \in M$, and $\mathrm{Lip}(g) \le \mathrm{Lip}(f) + r$.*

Proof. Let us denote $C = \mathrm{Lip}(f)$. Without loss of generality we may assume that, for every $x \in M$, $\varepsilon(x) > 0$ is small enough so that $\varepsilon(x) \le r/2$ and

$$C \cdot (1 + \varepsilon(x))^2 < C + \frac{r}{2}. \tag{25}$$

Using Corollary 4, for each $x \in M$, we can choose $\delta_x > 0$ such that the exponential mapping $\exp_x$ is a C^1-diffeomorphism and $(1 + \varepsilon(x))$-bi-Lipschitz from the Minkowski ball $B_{0_x}(3\delta_x) \subset T_xM$ onto the forward ball $\mathbf{B}_x^+(3\delta_x) \subset M$, where 0_x denotes the null vector of T_xM. In addition, by the continuity of f and ε, we can also assume that $\varepsilon(y) \ge \varepsilon(x)/2$ and $|f(y) - f(x)| \le \varepsilon(x)/2$, for every $y \in \mathbf{B}_x^+(3\delta_x)$. Since M is second countable, there is a sequence (x_n) in M such that

$$M = \bigcup_{n=1}^{\infty} \mathbf{B}_{x_n}^+(\delta_n), \tag{26}$$

where we denote $\delta_n = \delta_{x_n}$. Now, for each $n \in \mathbb{N}$, we define $f_n : B_{0_{x_n}}(3\delta_n) \to \mathbb{R}$ by

$$f_n(v) = f\left(\exp_{x_n}(v)\right), \tag{27}$$

and we then have that f_n is $C \cdot (1 + \varepsilon(x_n))$-Lipschitz.

Next, we are going to construct a partition of unity subordinated to the covering $\{\mathbf{B}^+_{x_n}(2\delta_n)\}_{n\in\mathbb{N}}$ of M, estimating the Lipschitz constant of the respective functions. Thus, for each $n \in \mathbb{N}$, let $\theta_n : \mathbb{R} \to [0,1]$ be a C^∞-smooth function such that

$$\theta_n(x) = \begin{cases} 1 & \text{if } x \in (-\infty, \delta_n], \\ 0 & \text{if } x \in [2\delta_n, +\infty), \end{cases} \tag{28}$$

and define $\varphi_n : M \to [0,1]$ by

$$\varphi_n(x) = \begin{cases} \theta_n\left(F\left(x_n, \exp^{-1}_{x_n}(x)\right)\right) & \text{if } x \in \mathbf{B}^+_{x_n}(3\delta_n), \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

It is clear that each φ_n is C^1-smooth and Lipschitz. Furthermore, $\varphi_n = 1$ on the forward ball $\mathbf{B}^+_{x_n}(\delta_n)$ and $\varphi_n = 0$ on $M \setminus \mathbf{B}^+_{x_n}(2\delta_n)$. Now we define the functions $\psi_n : M \to [0,1]$ by setting $\psi_1 = \varphi_1$ and, for $n \geq 2$,

$$\psi_n = \varphi_n \cdot \prod_{j<n}\left(1 - \varphi_j\right). \tag{30}$$

Then, it is easy to check that, for every $n \in \mathbb{N}$,

(i) ψ_n is C^1-smooth and C_n-Lipschitz, where $C_n = \sum_{j\leq n} \operatorname{Lip}(\varphi_j)$.

(ii) $\operatorname{supp}(\psi_n) \subset \operatorname{supp}(\varphi_n) \subset \mathbf{B}^+_{x_n}(2\delta_n)$.

(iii) $\psi_m = 0$ on $\mathbf{B}^+_{x_n}(\delta_n)$, whenever $m > n$.

Thus, $\{\psi_n\}$ is the desired partition of unity. Indeed, for each $x \in M$, let $n = n(x)$ be the first integer such that $x \in \mathbf{B}^+_{x_n}(\delta_n)$. Then $\varphi_n(x) = 1$ and $\psi_m(\mathbf{B}^+_{x_n}(\delta_n)) = 0$, for $m > n$. Therefore, the family $\{\operatorname{supp}(\psi_m)\}_m$ is locally finite. In addition $\sum_m \psi_m(x) = 1$, since

$$\begin{aligned} \sum_m \psi_m(x) &= \psi_1(x) + \psi_2(x) + \cdots + \psi_n(x) \\ &= \psi_1(x) + \psi_2(x) + \cdots + \psi_{n-1}(x) \\ &\quad + \varphi_n(x) \prod_{j<n}\left(1 - \varphi_j\right)(x) \\ &= \psi_1(x) + \psi_2(x) + \cdots + \varphi_{n-1}(x) \\ &\quad \times \prod_{j<n-1}\left(1 - \varphi_j\right)(x) + \prod_{j<n}\left(1 - \varphi_j\right)(x) \\ &= \psi_1(x) + \psi_2(x) + \cdots + \varphi_{n-2}(x) \\ &\quad \times \prod_{j<n-2}\left(1 - \varphi_j\right)(x) + \prod_{j<n-1}\left(1 - \varphi_j\right)(x) \\ &= \cdots = \varphi_1(x) + \left(1 - \varphi_1(x)\right) = 1. \end{aligned} \tag{31}$$

Now using Lemma 7 we can find, for each n, a C^∞-smooth function $g_n : T_{x_n}M \to \mathbb{R}$ such that

$$\left|g_n(v) - f_n(v)\right| \leq \frac{\varepsilon(x_n)}{2^{n+2}\left(C_n + 1\right)} \tag{32}$$

for every $v \in B_{0_{x_n}}(2\delta_n)$, and

$$\operatorname{Lip}\left(g_n|_{B_{0_{x_n}}(2\delta_n)}\right) \leq \operatorname{Lip}\left(f_n|_{B_{0_{x_n}}(2\delta_n)}\right) \leq C \cdot \left(1 + \varepsilon(x_n)\right). \tag{33}$$

Thus, we define the approximation function $g : M \to \mathbb{R}$ by

$$g(x) = \sum_n \psi_n(x) \cdot g_n\left(\exp^{-1}_{x_n}(x)\right) \tag{34}$$

for each $x \in M$. Note that, since the exponential mapping $\exp_{x_n}$ is a C^1-diffeomorphism from $B_{0_{x_n}}(3\delta_n)$ onto $\mathbf{B}^+_{x_n}(3\delta_n)$, the expression $\psi_n(x) \cdot g_n(\exp^{-1}_{x_n}(x))$ is well defined for $x \in \mathbf{B}^+_{x_n}(3\delta_n)$ and it is C^1-smooth on $\mathbf{B}^+_{x_n}(3\delta_n)$. On the other hand, if $x \notin \mathbf{B}^+_{x_n}(2\delta_n) \supset \operatorname{supp}(\psi_n)$, then $\psi_n(x) = 0$. Thus, for every $x \notin \mathbf{B}^+_{x_n}(3\delta_n)$, we may assume that $\psi_n(x) \cdot g_n(\exp^{-1}_{x_n}(x))$ is zero. With this convention, and taking into account that $\{\psi_n\}$ is a C^1-smooth partition of unity, we obtain that g is well defined and C^1-smooth on M.

We are going to see that g is also Lipschitz and that g and $\operatorname{Lip}(g)$ approximate to f and $\operatorname{Lip}(f)$, respectively. Fix $x \in M$, and consider again $n = n(x)$ the first integer such that $x \in \mathbf{B}^+_{x_n}(\delta_n)$. To simplify, denote $v_m = \exp^{-1}_{x_m}(x) \in T_{x_m}M$, for all m. Then we have

$$\begin{aligned} \left|g(x) - f(x)\right| &= \left|\sum_{m\leq n} \psi_m(x) \cdot g_m\left(\exp^{-1}_{x_m}(x)\right) - f(x)\right| \\ &= \left|\sum_{m\leq n} \psi_m(x) \cdot \left(g_m(v_m) - f(x)\right)\right| \\ &= \left|\sum_{m\leq n} \psi_m(x) \cdot \left(g_m(v_m) - f_m(v_m)\right)\right| \\ &\leq \sum_{m\leq n} \psi_m(x) \frac{\varepsilon(x_m)}{2^{m+2}\left(C_m + 1\right)} \\ &\leq \sum_{m\leq n} \psi_m(x) \frac{\varepsilon(x_m)}{2} \\ &\leq \sum_{m\leq n} \psi_m(x)\, \varepsilon(x) \leq \varepsilon(x). \end{aligned} \tag{35}$$

Finally, let us check that g is $(C + r)$-Lipschitz, and hence $\operatorname{Lip}(g) \leq C + r$. By Proposition 6, it will suffice to see that g is locally $(C + r)$-Lipschitz. Fix $z \in M$ and, as before, consider $n = n(z)$ the first integer such that $z \in \mathbf{B}^+_{x_n}(\delta_n)$. We are going to see that g is $(C + r)$-Lipschitz on the open set $U^z = \mathbf{B}^+_z(\delta_z) \cap \mathbf{B}^-_z(\delta_z)$, where

$$\delta_z = \frac{1}{2} \min\left\{\delta_1, \ldots, \delta_n, \delta_n - d_F(x_n, z)\right\}. \tag{36}$$

For each $x, y \in M$, denote

$$P_{x,y} = \left\{m \in \{1, \dots, n\} : \mathbf{B}^+_{x_m}(2\delta_m) \cap \{x, y\} \neq \emptyset\right\}. \quad (37)$$

Then, whenever $x, y \in U^z$, the following holds:

(1) if $m \in \{1, \dots, n\}$ and $y \in \mathbf{B}^+_{x_m}(2\delta_m)$, then $x \in \mathbf{B}^+_{x_m}(3\delta_m)$. Indeed, if $x, y \in U^z$, then $d_F(y, x) < 2\delta_z \leq \delta_m$. Thus if $y \in \mathbf{B}^+_{x_m}(2\delta_m)$ we have that

$$d_F(x_m, x) \leq d_F(x_m, y) + d_F(y, x) < 2\delta_m + \delta_m = 3\delta_m, \quad (38)$$

and therefore $x \in \mathbf{B}^+_{x_m}(3\delta_m)$.

(2) For every $m \in P_{x,y}$ we have that $x, y \in \mathbf{B}^+_{x_m}(3\delta_m)$. That is clear from the above. In particular, if $m \in P_{x,y}$ then $v_m = \exp^{-1}_{x_m}(x)$ and $w_m = \exp^{-1}_{x_m}(y)$ are well defined, and, using (33) and the fact that $\exp^{-1}_{x_m} : \mathbf{B}^+_{x_m}(3\delta_m) \to B_{0_{x_m}}(3\delta_m)$ is $(1 + \varepsilon(x_m))$-Lipschitz, we see that

$$\begin{aligned} |g_m(v_m) - g_m(w_m)| &\leq C \cdot (1 + \varepsilon(x_m)) \\ &\quad \times d_F\left(\exp^{-1}_{x_m}(x), \exp^{-1}_{x_m}(y)\right) \\ &\leq C \cdot (1 + \varepsilon(x_m))^2 \, d_F(x, y). \end{aligned} \quad (39)$$

(3) If $m \in \mathbb{N} \setminus P_{x,y}$ then $\psi_m(x) = 0 = \psi_m(y)$. That follows, since $\operatorname{supp}(\psi_m) \subset \mathbf{B}^+_{x_m}(2\delta_m)$ and $\operatorname{supp}(\psi_\ell) \cap \mathbf{B}^+_{x_n}(\delta_n) = \emptyset$, for every $l > n$.

As a consequence of the above we have, for every $x, y \in U^z$, and using the notation $v_m = \exp^{-1}_{x_m}(x)$ and $w_m = \exp^{-1}_{x_m}(y)$, the following:

(i) $g(x) = \sum_{m \in P_{x,y}} \psi_m(x) \cdot g_m(v_m)$,

(ii) $g(y) = \sum_{m \in P_{x,y}} \psi_m(y) \cdot g_m(w_m)$,

(iii) $\sum_{m \in P_{x,y}} \psi_m(x) = 1 = \sum_{m \in P_{x,y}} \psi_m(y)$,

(iv) $|g_m(v_m) - g_m(w_m)| \leq C(1 + \varepsilon(x_m))^2 d_F(x, y)$, whenever $m \in P_{x,y}$.

Therefore, since

$$\sum_{m \in P_{x,y}} (\psi_m(x) - \psi_m(y)) \cdot f(x) = 0, \quad (40)$$

we deduce that

$$\begin{aligned} g(x) - g(y) &= \sum_{m \in P_{x,y}} \psi_m(x) \cdot g_m(v_m) \\ &\quad - \sum_{m \in P_{x,y}} \psi_m(y) \cdot g_m(w_m) \\ &= \sum_{m \in P_{x,y}} (\psi_m(x) - \psi_m(y)) \cdot (g_m(v_m) - f(x)) \\ &\quad + \sum_{m \in P_{x,y}} \psi_m(y) \cdot (g_m(v_m) - g_m(w_m)). \end{aligned} \quad (41)$$

Finally, using (25), (32), and the fact that ψ_m is C_m-Lipschitz, we then have that

$$\begin{aligned} |g(x) - g(y)| &\leq \sum_{m \in P_{x,y}} |\psi_m(x) - \psi_m(y)| \\ &\quad \cdot |g_m(v_m) - f(x)| \\ &\quad + \sum_{m \in P_{x,y}} \psi_m(y) \cdot |g_m(v_m) - g_m(w_m)| \\ &\leq \sum_{m \leq n} \frac{C_m \cdot \varepsilon(x_m)}{(C_m + 1) 2^{m+2}} d_F(x, y) \\ &\quad + \sum_{m \leq n} \psi_m(y) \left(C + \frac{r}{2}\right) d_F(x, y) \\ &\leq \sum_{m \leq n} \frac{\varepsilon(z)}{2^{m+1}} d_F(x, y) + \left(C + \frac{r}{2}\right) d_F(x, y) \\ &\leq (C + r) \cdot d_F(x, y) \end{aligned} \quad (42)$$

since $\sum_{m \leq n} (\varepsilon(z)/2^{m+1}) \leq \varepsilon(z) \leq r/2$. This shows that g is locally $(C + r)$-Lipschitz and we finish the proof. □

Remark 9. In general, the exponential mapping in a Finsler manifold is only C^1-smooth. According to a result of Akbar-Zadeh in [8] (see also [6], page 127), the exponential mapping is C^2-smooth if and only if it is C^∞-smooth, and this property characterizes a special class of Finsler manifolds, called manifolds of Berwald type. Thus, if (M, F) is a connected, second countable manifold of Berwald type, the same proof above gives that the approximating function g in Theorem 8 can be chosen to be C^∞-smooth.

5. Quasi-Reversible Manifolds and a Completeness Criterium

In this section, as an application of the approximation result given in the above section, we obtain a completeness criterium for the class of manifolds that we call quasi-reversible. These are defined as follows.

Definition 10. A Finsler manifold (M, F) is said to be *quasi-reversible* if there exists some $C \geq 1$ such that

$$F(x, -v) \leq C \cdot F(x, v), \quad \text{for every } (x, v) \in TM. \quad (43)$$

It is clear that every reversible Finsler manifold is quasi-reversible. In fact, a Finsler manifold is reversible if and only if it is quasi-reversible for $C = 1$. On the other hand, a remarkable class of quasi-reversible (not necessarily reversible) manifolds are those manifolds of Berwald type. Indeed, we can deduce this, using a result due to Ichijyō [9] (see also [6], page 258) saying that if M is a manifold of Berwald type, then all its tangent spaces $(T_x M, F(x, \cdot))$, for every $x \in M$, are linearly isometric to each other.

We next give a useful characterization of connected quasi-reversible manifolds.

Theorem 11. *Let (M, F) be a connected Finsler manifold, and let $C \geq 1$. The following conditions are equivalent:*

(1) *$F(x, -v) \leq C \cdot F(x, v)$, for every $(x, v) \in TM$.*

(2) *$(1/C) \cdot d_F(y, x) \leq d_F(x, y) \leq C \cdot d_F(y, x)$, for every $x, y \in M$.*

(3) *For all $p \in M$, the forward distance function $\Phi_p = d_F(p, \cdot)$ is C-Lipschitz.*

(4) *For all $p \in M$, the backward distance function $\Psi_p = d_F(\cdot, p)$ is C-Lipschitz.*

Proof. (1) ⇒ (2) Let $x, y \in M$ and consider $\gamma : [0, 1] \to M$ a piecewise C^1 path from y to x. Then the reverse path $\tilde{\gamma} : [0, 1] \to M$ given by $\tilde{\gamma}(t) = \gamma(1 - t)$ is a piecewise C^1 path from x to y. Now,

$$\begin{aligned} \ell_F(\tilde{\gamma}) &= \int_0^1 F\left(\tilde{\gamma}(t), \tilde{\gamma}'(t)\right) dt \\ &= \int_0^1 F\left(\gamma(1-t), -\gamma'(1-t)\right) dt \\ &= \int_0^1 F\left(\gamma(s), -\gamma'(s)\right) ds \\ &\leq C \cdot \int_0^1 F\left(\gamma(s), \gamma'(s)\right) ds \\ &= C \cdot \ell_F(\gamma). \end{aligned} \tag{44}$$

This implies that $d_F(x, y) \leq C \cdot d_F(y, x)$. Interchanging the roles of x and y, we obtain the reverse inequality.

(2) ⇒ (1) Let $x \in M$. From Theorem 3 we have that if $v \neq 0$ then

$$\begin{aligned} &\lim_{t \to 0^+} \frac{F(x, tv - t(-v))}{d_F(\exp_x(t(-v)), \exp_x(tv))} \\ &\quad = 1 = \lim_{t \to 0^+} \frac{F(x, t(-v) - tv)}{d_F(\exp_x(tv), \exp_x(t(-v)))}. \end{aligned} \tag{45}$$

That is,

$$\begin{aligned} &\lim_{t \to 0^+} \frac{F(x, 2tv)}{d_F(\exp_x(-tv), \exp_x(tv))} \\ &\quad = 1 = \lim_{t \to 0^+} \frac{F(x, 2t(-v))}{d_F(\exp_x(tv), \exp_x(-tv))}. \end{aligned} \tag{46}$$

Thus we obtain that

$$\begin{aligned} F(x, -v) &= \lim_{t \to 0^+} \frac{d_F(\exp_x(tv), \exp_x(-tv))}{2t} \\ &\leq C \lim_{t \to 0^+} \frac{d_F(\exp_x(-tv), \exp_x(tv))}{2t} = C \cdot F(x, v). \end{aligned} \tag{47}$$

(2) ⇒ (3) For every $p, x, y \in M$ we have, from the triangle inequality, that

$$\begin{aligned} d_F(p, x) - d_F(p, y) &\leq d_F(y, x) \\ d_F(p, y) - d_F(p, x) &\leq d_F(x, y). \end{aligned} \tag{48}$$

By the hypothesis (2), we follow at once that

$$\left| d_F(p, x) - d_F(p, y) \right| \leq C \cdot d_F(x, y). \tag{49}$$

This means that the function $\Phi_p = d_F(p, \cdot)$ is C-Lipschitz.

(3) ⇒ (2) For every $p, x, y \in M$ we have that, in particular,

$$d_F(p, x) - d_F(p, y) \leq C \cdot d_F(x, y). \tag{50}$$

Choosing $p = y$ we have that

$$d_F(y, x) \leq C \cdot d_F(x, y). \tag{51}$$

Reversing the roles of x and y, we also have that $d_F(x, y) \geq (1/C) \cdot d_F(y, x)$.

(4) ⇔ (2) This can be seen as before. □

Note that, by choosing $C = 1$ in the above result, we can deduce at once the following characterization of connected reversible manifolds.

Corollary 12. *Let (M, F) be a connected Finsler manifold. The following conditions are equivalent:*

(1) *$F(x, -v) = F(x, v)$, for every $(x, v) \in TM$,*

(2) *$d_F(y, x) = d_F(x, y)$, for every $x, y \in M$.*

As an application of Theorem 8, we are going to obtain a completeness criterium in the context of quasi-reversible manifolds. This will extend the corresponding result by Gordon [5] for Riemannian manifolds. First recall that a sequence (x_n) in a Finsler manifold (M, F) is said to be forward Cauchy (resp., backward Cauchy) if, for every $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that, if $n_0 \leq n \leq m$, then $d_F(x_n, x_m) < \varepsilon$, (resp., $d_F(x_m, x_n) < \varepsilon$). We say then that (M, F) is forward complete (resp., backward complete) if every forward Cauchy sequence is convergent (resp., every backward Cauchy sequence is convergent). It is clear that, for quasi-reversible manifolds, forward and backward completeness are equivalent. On the other hand, recall that a continuous function $f : M \to \mathbb{R}$ is said to be proper if, for every compact set $K \subset \mathbb{R}$, its preimage $f^{-1}(K)$ is compact.

Theorem 13. *Let (M, F) be a connected, second countable, and quasi-reversible Finsler manifold. The following conditions are equivalent:*

(1) *(M, F) is forward complete.*

(2) *There exists a proper Lipschitz function $f : M \to \mathbb{R}$.*

(3) *There exists a proper C^1-smooth function $g : M \to \mathbb{R}$ whose differential is uniformly bounded in norm.*

Proof. (1) ⇒ (2) Fix $p \in M$ and consider the forward distance function $\varphi_p = d_F(p, \cdot)$. The Hopf-Rinow Theorem (see Theorem 6.6.1 in [6]) gives that (M, F) is forward complete if and only if every closed and (forward) bounded subset of M is compact. This implies that φ_p is a proper function. Furthermore, by Theorem 11, we have that φ_p is Lipschitz.

(2) $\Rightarrow$ (3) Suppose that the proper function $f : M \to \mathbb{R}$ is C-Lipschitz, and fix some $\varepsilon > 0$. By Theorem 8, there exists a C^1-smooth function $g : M \to \mathbb{R}$ such that

(i) $|f(x) - g(x)| \leq \varepsilon$, for every $x \in M$.

(ii) $\mathrm{Lip}(g) \leq C + \varepsilon$.

It is easy to check that g is a proper function, since f is so. On the other hand, by Theorem 5, we have that $\|dg(x)\|_F \leq C+\varepsilon$, for every $x \in M$.

(3) $\Rightarrow$ (1) Let (x_n) be a forward Cauchy sequence in M. Since $g : M \to \mathbb{R}$ is Lipschitz, then $\{g(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and therefore $\{g(x_n)\}$ converges to some point z in $\mathbb{R}$. Thus $K = \{z\} \cup \{g(x_n) : n \in \mathbb{N}\}$ is a compact subset of $\mathbb{R}$. Now (x_n) is contained in $g^{-1}(K)$, which is compact since g is proper. Then (x_n) is convergent in M. □

6. Algebras of Differentiable Functions on Finsler Manifolds

The classical Myers-Nakai Theorem asserts that the Riemannian structure of a Riemannian manifold M is determined by the natural normed algebra structure on the space $C_b^1(M)$ of all bounded C^1 functions on M which have bounded derivative (or, equivalently, which are Lipschitz on M with respect to the geodesic distance). This was proved by Myers [10] in the case that M is compact, and later on by Nakai [11] in the general case. More recently, analogous results have been obtained in the case of infinite-dimensional Riemannian manifolds (see [12]) and the case of Banach-Finsler manifolds (see [13]). Our aim in this section is to obtain a description of algebra isomorphisms between spaces of type $C_b^1(M)$ in the setting of quasi-reversible Finsler manifolds. From this we will obtain a variant of Myers-Nakai Theorem in the context of reversible Finsler manifolds.

Now let (M, F) be a Finsler manifold, and let $C_b^1(M)$ denote the space of all real bounded C^1-smooth functions defined on M whose derivative has uniformly bounded norm. We endow $C_b^1(M)$ with the natural norm:

$$\|f\|_{C_b^1} = \max\left\{\sup_{x\in M}|f(x)|, \sup_{x\in M}\|df(x)\|_F\right\}. \tag{52}$$

Endowed with this norm, $C_b^1(M)$ is a complete normed algebra. Note that $\|\cdot\|_{C_b^1}$ is not submultiplicative, but it satisfies that $\|fg\|_{C_b^1} \leq 2\|f\|_{C_b^1} \cdot \|g\|_{C_b^1}$.

Next we are going to recall the definition of the structure space associated to $C_b^1(M)$. This construction is standard, but we give some details for the reader's convenience. Let $\mathfrak{M}(M)$ denote the set of all nonzero, multiplicative, continuous linear forms $\varphi : C_b^1(M) \to \mathbb{R}$.

Claim. Each $\varphi \in \mathfrak{M}(M)$ satisfies that $\|\varphi\| = 1$, and furthermore φ is positive, that is, $\varphi(f) \geq 0$ whenever $f \geq 0$.

Proof. Since φ is multiplicative and nonzero, it is clear that $\varphi(1) = 1$. Now we are going to see that, for every $f \in C_b^1(M)$, we have that $\varphi(f)$ belongs to the closure of $f(M)$. Indeed, if $\alpha = \varphi(f)$ is not in the closure of $f(M)$, then $(f-\alpha)^2 \geq \varepsilon$ for some $\varepsilon > 0$. Thus $1/(f-\alpha)^2 \in C_b^1(M)$ and

$$\begin{aligned} 1 &= \varphi\left((f-\alpha)^2 \cdot \frac{1}{(f-\alpha)^2}\right) \\ &= \varphi\left((f-\alpha)^2\right) \cdot \varphi\left(\frac{1}{(f-\alpha)^2}\right). \end{aligned} \tag{53}$$

But we have that $\varphi((f-\alpha)^2) = 0$, which is a contradiction. From this we obtain that φ is positive. We also obtain that $|\varphi(f)| \leq \sup_{x\in M}|f(x)|$ for every $f \in C_b^1(M)$, so we deduce that $\|\varphi\| = 1$.

We endow $\mathfrak{M}(M)$ with the weak* topology it inherits from the dual space $C_b^1(M)^*$. Since $\mathfrak{M}(M)$ is a weak*-closed subset of the unit ball, we see that $\mathfrak{M}(M)$ is a compact space. Now we consider the embedding $\delta : M \to \mathfrak{M}(M)$ given by $\delta(x) = \delta_x$, where $\delta_x(f) = f(x)$ for every $f \in C_b^1(M)$ and $x \in M$. Note that every C^1-smooth function $f : M \to \mathbb{R}$ with compact support belongs to $C_b^1(M)$, and thus, in particular, $C_b^1(M)$ separates points and closed sets of M. From this it is not difficult to deduce that δ is a topological embedding, that is, a net (x_ι) in M converges to x if and only if the net of evaluations (δ_{x_ι}) converges to δ_x in the weak* topology. On the other hand, the set of evaluations $\delta(M)$ is dense in $\mathfrak{M}(M)$. Indeed, let $\varphi \in \mathfrak{M}(M)$, and consider a weak* basic neighborhood of φ of the form

$$W = \left\{\psi \in \mathfrak{M}(M) : \left|\psi(f_j) - \varphi(f_j)\right| < \varepsilon, \text{ for } j = 1, \ldots, m\right\}, \tag{54}$$

where $f_1, \ldots, f_m \in C_b^1(M)$ and $\varepsilon > 0$. Then there is some $x \in M$ such that $\delta_x \in W$, since otherwise the function $g = \sum_{j=1}^m (f_j - \varphi(f_j))^2 \in C_b^1(M)$ would satisfy $g \geq \varepsilon^2$ and $\varphi(g) = 0$, and this is impossible since φ is positive. In this way we see that $\mathfrak{M}(M)$ is a compactification of M. □

In what follows, we concentrate on the case of complete, quasi-reversible Finsler manifolds. Our next lemma will provide a topological characterization of point evaluations inside the structure space.

Lemma 14. *Let (M, F) be a connected, second countable, (forward) complete, quasi-reversible Finsler manifold, and let $\varphi \in \mathfrak{M}(M)$. The following conditions are equivalent:*

(1) *φ has a countable neighborhood basis in $\mathfrak{M}(M)$.*

(2) *There exists some $x \in M$ such that $\varphi = \delta_x$.*

Proof. (1) $\Rightarrow$ (2) Since M is (forward) complete, by Theorem 13, there exists a proper C^1-smooth function $g : M \to \mathbb{R}$ whose differential is uniformly bounded in norm. Suppose now that $\varphi \in \mathfrak{M}(M) \setminus \delta(M)$ has a countable neighborhood basis in $\mathfrak{M}(M)$. Since $\delta(M)$ is dense in $\mathfrak{M}(M)$, there is a sequence (x_n) in M such that (δ_{x_n}) converges to φ. Since $\varphi \notin \delta(M)$, we see that (x_n) has no convergent subsequence in M. Since g is proper, we deduce that $\lim_{n\to\infty}|g(x_n)| = +\infty$. Then there is a subsequence (x_{n_k}) such that $|g(x_{n_{k+1}})| > 1 + |g(x_{n_k})|$ for every k. Now we can choose a C^1 function $\theta : \mathbb{R} \to [0,1]$, with bounded derivative, such that $\theta(g(x_{n_{2k+1}})) = 1$ and $\theta(g(x_{n_{2k}})) = 0$ for every k. Then the function $f = \theta \circ g$ belongs to $C_b^1(M)$, but the sequence $(\delta_{x_{n_k}}(f))$ is not convergent, which is a contradiction.

(2) ⇒ (1) Conversely, if $\varphi = \delta_x$ for some $x \in M$, consider a countable neighborhood basis (V_n) of x in M. Then the family of closures $\{\mathrm{cl}_{\mathfrak{M}(M)} V_n\}$ is easily seen to be a countable neighborhood basis of δ_x in $\mathfrak{M}(M)$ as required. □

The following Lemma shows the metric properties of the embedding $\delta : M \to \mathfrak{M}(M)$.

Lemma 15. *Let (M, F) be a connected, second countable (forward) complete, quasi-reversible Finsler manifold, with constant C. Then, for each $x, y \in M$, we have that*

$$\frac{1}{C}\min\left\{1, d_F(x,y)\right\} \le \left\|\delta_x - \delta_y\right\| \le d_F(x,y). \tag{55}$$

Proof. Recall that

$$\left\|\delta_x - \delta_y\right\| = \sup\left\{\left|f(x) - f(y)\right| : f \in C_b^1(M); \|f\|_{C_b^1} \le 1\right\}. \tag{56}$$

Thus by the mean value inequality contained in Theorem 5 we deduce at once that $\|\delta_x - \delta_y\| \le d_F(x,y)$. For the other inequality, suppose that $x \neq y$, and consider the function $\Phi : M \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{C}\min\left\{1, d_F(x,u)\right\}. \tag{57}$$

It is clear that $0 \le \Phi \le 1/C \le 1$, and from Theorem 11 we have that $\mathrm{Lip}(\Phi) \le 1$. Now given $0 < \varepsilon < (1/2C)\min\{1, d_F(x,y)\}$, by Theorem 8, there exists a C^1-smooth function $f : M \to \mathbb{R}$ such that $|f(u) - \Phi(u)| \le \varepsilon$, for every $u \in M$, and $\mathrm{Lip}(f) \le 1 + \varepsilon$. Thus $\|f\|_{C_b^1} \le 1 + \varepsilon$. Now we consider $\hat{f} = (1/(1+\varepsilon))f$, and we have that $\|\hat{f}\|_{C_b^1} \le 1$. Furthermore,

$$\begin{aligned}\left|\hat{f}(x) - \hat{f}(y)\right| &= \frac{1}{1+\varepsilon}\left|f(x) - f(y)\right| \\ &\ge \frac{1}{1+\varepsilon}\left(\left|\Phi(x) - \Phi(y)\right| - 2\varepsilon\right) \\ &= \frac{1}{1+\varepsilon}\,\frac{1}{C}\min\left\{1, d_F(x,y)\right\} - \frac{2\varepsilon}{1+\varepsilon},\end{aligned} \tag{58}$$

and the result follows. □

Recall that, if M and N are Finsler manifolds, a mapping $T : C_b^1(N) \to C_b^1(M)$ is said to be a normed algebra isomorphism provided T is a bicontinuous linear bijection such that $T(f \cdot g) = T(f) \cdot T(g)$ for every $f, g \in C_b^1(N)$. Now we give the main result in this section, which provides a characterization of such normed algebra isomorphisms.

Theorem 16. *Let (M, F_M) and (N, F_N) be second countable, connected, (forward) complete, quasi-reversible Finsler manifolds, with constants C_M and C_N, respectively. For a mapping $T : C_b^1(N) \to C_b^1(M)$, the following are equivalent:*

1. *T is a normed algebra isomorphism.*
2. *There exists a C^1-diffeomorphism, $h : M \to N$, which is bi-Lipschitz for the respective Finsler distances, such that $T(f) = f \circ h$, for every $f \in C_b^1(N)$.*

Moreover, in this case, the bi-Lipschitz constant of h can be chosen to be

$$\max\left\{C_N \cdot \|T\|, C_M \cdot \left\|T^{-1}\right\|\right\}. \tag{59}$$

Proof. (1) ⇒ (2) Suppose that $T : C_b^1(N) \to C_b^1(M)$ is an isomorphism of normed algebras. Consider the transpose map $T^* : C_b^1(M)^* \to C_b^1(N)^*$, defined by $T^*(\varphi) = \varphi \circ T$ for every $\varphi \in C_b^1(M)^*$. Since T is multiplicative and T^* is weak*-to-weak* bicontinuous, we see that the restriction of T^* defines a homeomorphism from $\mathfrak{M}(M)$ onto $\mathfrak{M}(N)$. Consider now the natural embeddings $\delta_M : M \to \mathfrak{M}(M)$ and $\delta_N : N \to \mathfrak{M}(N)$. By Lemma 14 we deduce that $T^*(\delta_M(M)) = \delta_N(N)$, so that the restriction of T^* defines a homeomorphism from $\delta_M(M)$ onto $\delta_N(N)$. Thus we can define $h := (\delta_N)^{-1} \circ T^* \circ \delta_M : M \to N$, which is a homeomorphism from M onto N. Furthermore, we have that, for every $x \in M$ and $f \in C_b^1(N)$,

$$\begin{aligned}T(f)(x) &= \delta_x(T(f)) = T^*(\delta_x)(f) \\ &= \delta_{h(x)}(f) = f(h(x)),\end{aligned} \tag{60}$$

that is, $T(f) = f \circ h$. In particular, note that $f \circ h$ is C^1-smooth for every C^1-smooth function $f : N \to \mathbb{R}$ with compact support. From this it is easily deduced that h is C^1-smooth, and the same can be said about h^{-1}, so that h is a C^1-diffeomorphism.

Now we are going to see that $h : M \to N$ is bi-Lipschitz for the respective Finsler distances d_M and d_N. Using Proposition 6, it will suffice to prove that h and h^{-1} are locally Lipschitz. Given $p \in M$, consider the open neighborhood

$$\begin{aligned}U^p = {} & \mathbf{B}_p^+\left(\frac{1}{2}\right) \cap \mathbf{B}_p^-\left(\frac{1}{2}\right) \\ & \cap h^{-1}\left(\mathbf{B}_{h(p)}^+\left(\frac{1}{2}\right)\right) \cap h^{-1}\left(\mathbf{B}_{h(p)}^-\left(\frac{1}{2}\right)\right).\end{aligned} \tag{61}$$

If $x, y \in U^p$ we have that $d_M(x,y) < 1$ and $d_N(h(x), h(y)) < 1$, so by Lemma 15 and taking into account that $\|T^*\| = \|T\|$, we obtain that

$$\begin{aligned}\frac{1}{C_N} d_N(h(x), h(y)) &\le \left\|\delta_{h(x)} - \delta_{h(y)}\right\| \\ &= \left\|T^*\left(\delta_x - \delta_y\right)\right\| \le \|T\| \cdot \left\|\delta_x - \delta_y\right\| \\ &\le \|T\|\, d_M(x,y).\end{aligned} \tag{62}$$

Therefore, we have that h is $C_N \cdot \|T\|$-Lipschitz. In the same way, h^{-1} is $C_M \cdot \|T^{-1}\|$-Lipschitz and the result follows.

(2) ⇒ (1) Taking into account Theorem 5, we have that $C_b^1(M)$ is the set of all C^1-smooth functions $f : M \to \mathbb{R}$ which are Lipschitz, and the same holds for $C_b^1(N)$, so this implication is clear. □

We are now ready to deduce from our previous results a version of the classical Myers-Nakai Theorem in the context of reversible Finsler manifolds. Recall that a mapping $h : (M, F_M) \to (N, F_N)$ between Finsler manifolds is said to be a Finsler isometry if h is a C^1 diffeomorphism which preserves the Finsler structure, that is, for every $x \in M$ and every $v \in T_x M$,

$$F_M(x, v) = F_N(h(x), dh(x)(v)). \tag{63}$$

Extending the classical result by Myers and Steenrod [14] about Riemannian manifolds, Deng and Hou have proved in ([7], Theorem 2.2) that, if (M, F_M) and (N, F_N) are connected Finsler manifolds, a mapping $h : M \to N$ is a Finsler isometry if and only if h is bijective and preserves the corresponding Finsler distances d_M and d_N, that is, for every $x, y \in M$,

$$d_M(x, y) = d_N(h(x), h(y)). \tag{64}$$

Now combining the above result with Theorem 16 we obtain at once the following theorem, which has been also obtained in [4] (see Theorem 13 there) but with a different proof.

Theorem 17. *Let M and N be second countable, connected, reversible, and complete Finsler manifolds. Then M and N are equivalent as Finsler manifolds if and only if $C_b^1(M)$ and $C_b^1(N)$ are isometric normed algebras. Moreover, every normed algebra isometry $T : C_b^1(N) \to C_b^1(M)$ is of the form $T(f) = f \circ h$, where $h : M \to N$ is a Finsler isometry.*

Acknowledgments

M. I. Garrido and J. A. Jaramillo have been supported in part by D.G.I. (Spain) Grant MTM2009-07848. Y. C. Rangel has been associated to the Project 014-CT-2012 (CDCHT-UCLA) (Venezuela).

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