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On some conjuctures in singularity theory

(Sobre algunas conjeturas en teoría de singularidades)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Director

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Madrid

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Tesis Doctoral

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(Sobre algunas conjeturas en Teoría de Singularidades)

Memoria para optar al grado de Doctor en Investigación Matemática
presentada por

Leire Gorrochategui Gregorio

Bajo la dirección de

Alejandro Melle Hernández

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DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

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Sobre algunas conjeturas en Teoría de Singularidades (On some conjectures in Singularity Theory)

y dirigida por: Alejandro Melle Hernández

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And nothing else matters...

Contents

Abstract

This thesis is devoted to the study of singular points of plane curves. More specifically, we give counterexamples to some conjectures that have arisen from the investigation of plane curve singularities. In particular, we deal with two main topics:

1. The monodromy conjecture of J. Denef and F. Loeser and its generalisation by A. Némethi and W. Veys.

2. Free and nearly free plane curves with isolated singularities, and some related conjectures proposed by A. Dimca and G. Sticlaru.

The Monodromy Conjecture predicts that if s_0 is a pole of the local topological zeta function $Z_{top,0}(f, s)$ associated with the singularity defined by the germ f, then $\exp(2\pi i s_0)$ is an eigenvalue of the local monodromy of f at some complex point of $(f^{-1}(0), 0)$.

The conjecture was verified by F. Loeser for plane curves in [Loe88]. However in higher dimensions there is not so much known; there are only some partial results due to the lack of a conceptual link between the monodromy operator and the topological zeta function.

The Monodromy Conjecture was later extended for topological zeta functions associated with arbitrary differential forms by W. Veys, and afterwards A. Némethi and W. Veys introduced the set of *allowed* differential forms. They proved that for germs of plane curve singularities these allowed differential forms exist, and that this set contains the standard differential form. In this context it is natural to ask if there exists any other naturally defined (even associated canonically to the germ f) differential form which is allowed. A natural choice might be the Hessian form. A. Melle wondered whether the poles of the corresponding topological zeta function would provide eigenvalues of the monodromy, in the same way as the standard form does, given that the result holds in many examples, such as the simple singularities of type $\mathbb{A}_n$.

We show that the local topological zeta function of a germ associated with its Hessian differential form does not satisfy the Monodromy Conjecture, i.e., that the Hessian form is not an allowed differential 2-form. This result was proved by the author in [Gor18] and it is presented in Chapter 3.

We have also studied some conjectures regarding free and nearly free curves. The notion of free divisor was introduced by K. Saito [Sai80] in the study of discriminants of versal unfoldings of germs of isolated hypersurface singularities. Since then many interesting and unexpected applications to Singularity Theory and Algebraic Geometry have appeared. In this thesis we are mainly focused on complex projective plane curves and for this reason we adapt the corresponding notions and results to this setup.

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d in the polynomial ring, let $C \subset \mathbb{P}^2$ be defined by $f = 0$ and assume that C is reduced. We denote by J_f the Jacobian

ideal of f, i.e., the homogeneous ideal in S spanned by f_x, f_y, f_z, and by $M(f) = S/J_f$ the corresponding graded ring, called the Milnor algebra of f.

The study of free curves in the projective plane has a rather long tradition, being inaugurated by A. Simis in [Sim06a, Sim06b], and actively continued by several mathematicians. C is said to be a free curve if $J_f = I_f$, where I_f denotes the saturation of the ideal J_f with respect to the maximal ideal $\mathfrak{m} = (x, y, z)$.

The nearly free curves were introduced in [DS18a]. They have properties similar to the free curves, and together with the free curves may lead to a new understanding of the rational cuspidal curves, due to Conjecture 0.0.1(i) below. This class of curves forms already the subject of attention in a number of papers, see for instance [AD18, MV17].

By definition, C is a nearly free curve if the graded module $N(f) = I_f/J_f$ satisfies $N(f) \neq 0$ and the graded part $N(f)_k$ is such that $\dim_{\mathbb{C}} N(f)_k \leq 1$ for all k.

The main results of A. Dimca and G. Sticlaru in [DS17], [DS15] and [DS18a] and many series of examples motivated the following conjecture:

Conjecture 0.0.1. *[DS15]*

(i) *Any rational cuspidal curve C in the plane is either free or nearly free.*

(ii) *An irreducible plane curve C which is either free or nearly free is rational.*

In [DS18a], the authors provide some interesting results supporting the statement of Conjecture 0.0.1(i); in particular, Conjecture 0.0.1(i) holds for rational cuspidal curves of even degree [DS18a, Theorem 4.1]. They proved also that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ and for those curves whose degree is a prime power. Furthermore, using the classification given in [FLMN07a] of unicuspidal rational curves with a unique Puiseux pair, A. Dimca and G. Sticlaru proved that all of them are either free or nearly free curves, except the curves of odd degree in one of the cases of the classification. More recently, they proved that their Conjecture 0.0.1(i) is true for any rational cuspidal curve $C = V(f)$ with $\mathrm{mdr}(f) \leq 15$ (where $r = \mathrm{mdr}(f)$ is the minimal degree of a Jacobian syzygy for f), and for most curves of degree $d \leq 90$.

A. Dimca and G. Sticlaru also proposed the following conjecture:

Conjecture 0.0.2. *[DS15]*

(i) *Any free irreducible plane curve C has only singularities with at most two branches.*

(ii) *Any nearly free irreducible plane curve C has only singularities with at most three branches.*

In Chapter 5 we give some examples of irreducible free curves and nearly free curves in the complex projective plane which are not rational and thus giving counterexamples to Conjecture 0.0.1(ii). Among these counterexamples we found some examples of irreducible free and nearly free curves whose two singular points have any odd number of branches $r = 2\ell + 1$, $\ell \geq 1$, giving counterexamples to Conjectures 0.0.2(i) and 0.0.2(ii). Furthermore, an irreducible nearly free curves with just one singular point and having 4 branches, giving another counterexample to Conjecture 0.0.2(ii), is also provided. Unfortunately, our examples say nothing about the most remarkable conjecture by A. Dimca and G. Sticlaru, which predicts that every rational cuspidal plane curve is either free or nearly free.

The results contained in this thesis have been partially published in two articles, corresponding to chapters 3 and 5, respectively:

- L. Gorrochategui. Monodromy conjecture and the Hessian differential form. *Topology and its Applications*, 234 (2018), 452–456.

- E. Artal Bartolo, L. Gorrochategui, I. Luengo and A. Melle-Hernández. On some conjectures about free and nearly free divisors. *Singularities and computer algebra. Festschrift for Gert-Martin Greuel on the occasion of his 70th birthday. Based on the conference, Lambrecht (Pfalz), Germany, June 2015* (2017), 1–19.

Resumen

Esta tesis está dedicada al estudio de puntos singulares de curvas planas. En concreto, proporcionamos contraejemplos a algunas de las conjeturas que han surgido a raíz de la investigación de las singularidades de dichas curvas. En particular, tratamos dos temas principales:

1. La conjetura de la monodromía de J. Denef y F. Loeser y su generalización por A. Némethi y W. Veys.

2. Curvas planas libres y casi-libres con singularidades aisladas, y algunas conjeturas relacionadas propuestas por A. Dimca y G. Sticlaru.

La conjetura de la monodromía afirma que si s_0 es un polo de la función zeta topológica local $Z_{top,0}(f, s)$ asociada a la singularidad definida por el germen f, entonces $\exp(2\pi i s_0)$ es un autovalor de la monodromía local de f en algún punto de $(f^{-1}(0), 0)$.

La conjetura fue demostrada para curvas planas por F. Loeser en [Loe88]. Sin embargo, en dimensiones más altas no se tiene mucha más información; únicamente hay algunos resultados parciales debido a la ausencia de un vínculo conceptual entre el operador monodromía y la función zeta topológica.

La conjetura de la monodromía fue extendida más tarde para funciones zeta asociadas a formas diferenciales arbitrarias por W. Veys, y posteriormente A. Némethi y W. Veys introdujeron el conjunto de formas diferenciales *permitidas*. Demostraron que estas formas diferenciales *permitidas* existen para gérmenes de singularidades de curvas planas, y que dicho conjunto contiene a la forma diferencial estándar. En este contexto, cabe preguntarse si podría existir otra forma diferencial definida de manera natural (incluso asociada canónicamente al germen f) que esté *permitida*. Una elección natural podría ser la forma diferencial hessiana. A. Melle se preguntó si los polos de la función zeta asociada a esta forma diferencial darían lugar a autovalores de la monodromía, al igual que ocurre con la forma diferencial estándar, ya que este resultado es cierto en numerosos ejemplos, como en el caso de las singularidades simples de tipo $\mathbb{A}_n$.

Mostraremos que la función zeta topológica local de un germen asociado a su forma diferencial hessiana no cumple la conjetura de la monodromía, es decir, que la hessiana no es una forma diferencial *permitida*. Este resultado fue demostrado por la autora en [Gor18] y lo presentamos en el Capítulo 3.

También hemos estudiado algunas conjeturas relacionadas con las curvas planas libres y casi-libres. El concepto de divisor libre fue introducido por K. Saito [Sai80] en el estudio de discriminantes de deformaciones versales de gérmenes de singularidades aisladas de hipersuperficies. Desde entonces han aparecido aplicaciones interesantes e inesperadas a la Teoría de Singularidades y a la Geometría Algebraica. En esta tesis nos centraremos principalmente en las curvas proyectivas planas complejas y por esta razón se han adaptado las correspondientes notaciones y resultados a esta situación.

Consideremos el anillo de polinomios $S := \mathbb{C}[x, y, z]$ equipado con la graduación natural $S = \bigoplus_{m=0}^{\infty} S_m$ definida por polinomios homogéneos. Sea $f \in S_d$ un polinomio homogéneo de grado d en S, sea $C \subset \mathbb{P}^2$ la curva definida por $f = 0$ y supongamos que C es reducida. Denotemos por J_f al ideal jacobiano de f, esto es, al ideal homogéneo en S generado por f_x, f_y, f_z, y por $M(f) = S/J_f$ al correspondiente anillo graduado, denominado álgebra de Milnor de f.

El estudio de las curvas libres en el plano proyectivo tiene una trayectoria bastante larga, iniciada por A. Simis en [Sim06a, Sim06b], y continuada activamente por varios matemáticos. Se dice que C es una curva libre si $J_f = I_f$, donde I_f denota la saturación del ideal J_f con respecto al ideal maximal $\mathfrak{m} = (x, y, z)$.

Las curvas casi-libres se introdujeron en [DS18a]. Tienen propiedades similares a las de las curvas libres y, junto con estas últimas, pueden llevar a una nueva interpretación de las curvas racionales cuspidales, gracias a la Conjetura 0.0.3(i). Esta clase de curvas ya es objeto de atención de algunos trabajos, véanse por ejemplo [AD18] y [MV17].

Por definición, C es una curva casi-libre si el módulo graduado $N(f) = I_f/J_f$ verifica que $N(f) \neq 0$ y la parte graduada $N(f)_k$ es tal que $\dim_{\mathbb{C}} N(f)_k \leq 1$ para todo k.

Los resultados principales de A. Dimca y G. Sticlaru en [DS17], [DS15] y [DS18a] y numerosas familias de ejemplos motivaron la siguiente conjetura:

Conjetura 0.0.3. *[DS15]*

(i) *Toda curva racional cuspidal plana C es o bien libre o bien casi-libre.*

(ii) *Toda curva plana irreducible C que sea libre o casi-libre es racional.*

En [DS18a], los autores proporcionan algunos resultados interesantes que respaldan la Conjetura 0.0.3(i); en particular, la Conjetura 0.0.3(i) es cierta para curvas racionales cuspidales de grado par [DS18a, Teorema 4.1]. Probaron también que esta conjetura es cierta para las curvas C cuyo grupo fundamental $\pi_1(\mathbb{P}^2 \setminus C)$ es abeliano y para las curvas cuyo grado es una potencia de primo. Asimismo, usando la clasificación de curvas racionales unicuspidales con un único par de Puiseux que figura en [FLMN07a], A. Dimca y G. Sticlaru demostraron que todas ellas son curvas o bien libres o bien casi-libres, exceptuando las curvas de grado impar en uno de los casos de la clasificación. Recientemente han demostrado que su Conjetura 0.0.3(i) es cierta para cualquier curva racional cuspidal $C = V(f)$ con $\mathrm{mdr}(f) \leq 15$ (siendo $r = \mathrm{mdr}(f)$ el mínimo grado de una sicigia asociada al jacobiano de f), y para la gran mayoría de las curvas de grado $d \leq 90$.

A. Dimca y G. Sticlaru propusieron además la siguiente conjetura:

Conjetura 0.0.4. *[DS15]*

(i) *Toda curva plana irreducible libre C tiene singularidades con, a lo sumo, dos ramas.*

(ii) *Toda curva plana irreducible casi-libre C tiene singularidades con, a lo sumo, tres ramas.*

En el capítulo 5 damos algunos ejemplos de curvas irreducibles libres y casi-libres en el plano proyectivo complejo que no son racionales, proporcionando así contraejemplos a la Conjetura 0.0.3(ii). Entre estos contraejemplos encontramos algunos ejemplos de curvas irreducibles libres y casi-libres cuyos dos puntos singulares tienen un número impar de ramas $r = 2\ell + 1$, $\ell \geq 1$, obteniendo de esta manera contraejemplos a las Conjeturas 0.0.4(i) y 0.0.4(ii). Se presenta también una curva irreducible casi-libre con un único punto singular de cuatro ramas, dando lugar a otro contraejemplo para la Conjetura 0.0.4(ii). Desgraciadamente, nuestros ejemplos no dicen nada acerca de la conjetura más destacada de A. Dimca y G. Sticlaru, que predice que toda curva plana racional y cuspidal es libre o casi-libre.

Los resultados contenidos en esta tesis se han publicado en dos artículos, correspondientes a los capítulos 3 y 5, respectivamente:

- L. Gorrochategui. Monodromy conjecture and the Hessian differential form. *Topology and its Applications*, 234 (2018), 452–456.
- E. Artal Bartolo, L. Gorrochategui, I. Luengo and A. Melle-Hernández. On some conjectures about free and nearly free divisors. *Singularities and computer algebra. Festschrift for Gert-Martin Greuel on the occasion of his 70th birthday. Based on the conference, Lambrecht (Pfalz), Germany, June 2015* (2017), 1–19.

Introducción

El tema central de esta tesis es la Teoría de Singularidades. En concreto, proporcionamos diversos contraejemplos a algunas de las conjeturas que han surgido a raíz de la investigación de singularidades de curvas planas. En particular, tratamos dos temas principales:

1. La conjetura de la monodromía de J. Denef y F. Loeser y su generalización por parte de A. Némethi y W. Veys.

2. Curvas planas libres y casi-libres con singularidades aisladas, y algunas conjeturas relacionadas propuestas por A. Dimca y G. Sticlaru.

Este trabajo se compone de un resumen en castellano y en inglés, una introducción en castellano y la tesis completa en inglés. El cuerpo de la tesis se estructura en cinco capítulos. El primero de ellos es introductorio. El capítulo 2 contiene los conceptos y herramientas necesarios que serán utilizados a lo largo de la tesis. La mayor parte de este material es de sobra conocido por los lectores familiarizados con la Teoría de Singularidades. En los capítulos 3 y 5 presentamos los principales resultados de nuestra investigación. El capítulo 4 trata sobre las curvas planas racionales cuspidales. Finalmente sintetizamos las conclusiones del presente trabajo. A continuación, resumimos su contenido capítulo a capítulo.

En el **Capítulo 1** ofrecemos una visión general de la tesis además de la estructura de los capítulos siguientes.

El **Capítulo 2** establece un lenguaje unificado y contiene material de referencia de modo que la tesis sea lo más autocontenida posible. En las Secciones 2.1 y 2.2 revisamos brevemente los conceptos básicos requeridos para el estudio de singularidades aisladas de hipersuperficies, por lo que trabajamos con gérmenes de funciones. Haremos especial énfasis en las singularidades de curvas planas, que es un tema clásico y bien conocido de la Teoría de Singularidades. Comenzamos describiendo la resolución de singularidades en la Sección 2.2. En la Sección 2.3 presentamos otra herramienta fundamental: la fibración de Milnor. La Sección 2.4 está dedicada al estudio de la monodromía, que es uno de los componentes esenciales de la conjetura de la monodromía; en particular, introducimos la función zeta de la monodromía.

A continuación estudiamos curvas algebraicas proyectivas planas complejas. En la Sección 2.5 fijamos convenciones y recordamos las principales propiedades y los invariantes globales de dichas curvas partiendo de los invariantes locales de sus puntos singulares que ya hemos explorado en las secciones anteriores, y enunciaremos además algunas propiedades interesantes de las singularidades de la curva dual.

Los contraejemplos a las conjeturas de A. Dimca y G. Sticlaru se han construido utilizando dos técnicas fundamentales: las cubiertas de Kummer y ciertas propiedades de los sistemas lineales de dimensión 1 asociados a curvas racionales unicuspidales. Por este motivo, estos temas se han abordado en las Secciones 2.6 y 2.7, respectivamente. En la Sección 2.6 recordamos en primer lugar algunas propiedades esenciales y útiles sobre las cubiertas topológicas. En la Sección 2.7 enunciamos algunas definiciones básicas relacionadas con los sistemas lineales unidimensionales de curvas planas.

El **Capítulo 3** trata sobre la conjetura de la monodromía de J. Denef y F. Loeser, que relaciona los polos de la función zeta de Igusa, la motívica o la topológica con los autovalores de la monodromía.

Las funciones zeta se pueden asociar a varios objetos matemáticos como cuerpos, grupos, álgebras, funciones y sistemas dinámicos. Normalmente, las funciones zeta codifican información aritmética, algebraica, geométrica o topológica relevante del objeto original.

Antes de enunciar la conjetura de la monodromía, introducimos su otro componente principal, a saber, la función zeta topológica en la Sección 3.3 y su precursora, la función zeta local de Igusa, en la Sección 3.2.

En 1992 J. Denef y F. Loeser presentaron una nueva función zeta [DL92], a la que denominaron *función zeta topológica* debido a que en ella aparece la característica de Euler-Poincaré, que es un invariante topológico.

La conjetura de la monodromía predice que si s_0 es un polo de la función zeta topológica local $Z_{top,0}(f,s)$ asociada a la singularidad definida por el germen f, entonces $\exp(2\pi i s_0)$ es un autovalor de la monodromía local de f en algún punto de $(f^{-1}(0),0)$. La Sección 3.4 aporta una visión más profunda de este tema.

La conjetura de la monodromía fue demostrada para curvas planas por F. Loeser en [Loe88], inicialmente en el contexto de las funciones zeta p-ádicas de Igusa. Sin embargo, a día de hoy, solo hay algunos resultados parciales en dimensiones superiores, debido a la ausencia de un vínculo conceptual entre el operador monodromía y la función zeta topológica. De hecho, las pruebas existentes para estos casos particulares básicamente calculan los dos lados de forma independiente y comparan los resultados finales.

Con el fin de encontrar una interpretación más conceptual, la conjetura de la monodromía fue extendida más tarde para funciones zeta asociadas a formas diferenciales arbitrarias por W. Veys, y posteriormente A. Némethi y W. Veys introdujeron el conjunto de formas diferenciales *permitidas* (véanse [NV10] y [NV12]). Demostraron que estas formas diferenciales *permitidas* existen para gérmenes de singularidades de curvas planas, y que dicho conjunto contiene a la forma diferencial estándar.

Por otro lado, no es difícil observar en ejemplos concretos que no todos los autovalores del operador monodromía están a su vez inducidos por polos de la función zeta topológica, como veremos en la Sección 3.4.

La función zeta topológica fue introducida por primera vez por J. Denef y F. Loeser para la forma diferencial estándar $\omega_0 = dx_0 \wedge \ldots \wedge dx_n$. Al igual que la forma diferencial estándar está siempre *permitida*, cabe preguntarse si podría existir otra forma diferencial con su misma propiedad, y una elección razonable podría ser la forma diferencial hessiana, que depende únicamente de la función con la que se define la curva.

Por este motivo, A. Melle conjeturó y se preguntó si todos los polos de la función zeta topológica asociada a una singularidad aislada de curva plana definida por un germen analítico f y la forma diferencial hessiana $\omega_{\text{hess}(f)} := \text{hess}(f)dx \wedge dy$ siempre induciría autovalores de la monodromía, dado que este hecho ocurre en numerosos ejemplos, como en el caso de las singularidades simples de tipo $\mathbb{A}_n$. Esto se explica en mayor detalle en la Sección 3.5.

El objetivo central del Capítulo 3 es mostrar que la función zeta topológica local de un germen asociada a su forma diferencial hessiana no verifica la conjetura de la monodromía. Este resultado fue probado por la autora en [Gor18]. En la Sección 3.6 proporcionamos el ejemplo definido por el germen $(C,0) = (f^{-1}(0), 0)$, donde

$$f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6. \tag{0.0.1}$$

Sea (X,π) la resolución sumergida minimal del germen $(f^{-1}(0) \cup \operatorname{div}(\omega), 0)$. Así, π es una sucesión finita de explosiones. Denotamos a las componentes por $\tilde{E}_i, i \in I = I_e \cup I_{s^f} \cup I_{s^\omega}$, donde $\tilde{E}_i$ es:

- una componente excepcional, si $i \in I_e$;
- una componente irreducible de la transformada estricta de $f^{-1}(0)$, si $i \in I_{s^f}$;
- una componente irreducible de la transformada estricta de $\operatorname{div}(\omega)$, si $i \in I_{s^\omega}$.

Para cada $i \in I$, sean N_i y $\nu_i - 1$ las multiplicidades de $\tilde{E}_i$ en el divisor de $\pi^* f$ y $\pi^*\omega$, respectivamente. El grafo de la resolución de la singularidad en el origen $(C,0) = (f^{-1}(0), 0)$ definida por $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$ es:

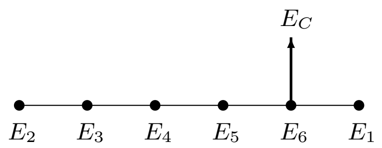


Figure 1: Grafo dual de la resolución sumergida minimal de $(f^{-1}(0), 0)$, para $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

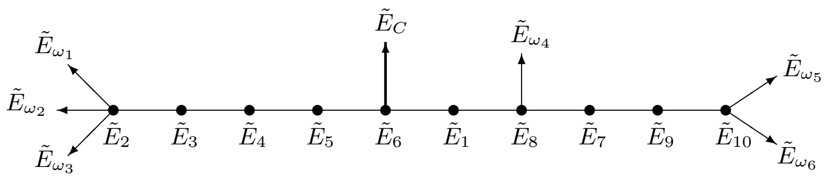
y la resolución sumergida minimal de $\left(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\mathrm{hess}(f)}\right), 0\right)$ se especifica en la Figura 2, para los conjuntos de índices:

$$I_e = \{1, \ldots, 10\},$$
$$I_{s^f} = \{C\},$$
$$I_{s^\omega_f} = \{\omega_1, \ldots, \omega_6\}.$$

Veremos que el polo de la función zeta topológica $s_0 := -13/6$, que se corresponde con la componente $\tilde{E}_2$, no induce ningún autovalor de la monodromía, pues s_0 no es una raíz del polinomio característico del germen $(C,0)$:

$$\Delta_C(t) = \frac{(t-1)(t^{30}-1)}{(t^5-1)(t^6-1)},$$

lo que significa que la conjetura de la monodromía se incumple al considerar la forma diferencial $\omega_{\mathrm{hess}(f)} = \mathrm{hess}(f) dx \wedge dy$ en vez de la estándar ω_0. En particular, la forma diferencial hessiana no es una forma *permitida*.



i	1	2	3	4	5	6	7	8	9	10	C	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
N_i	5	6	12	18	24	30	5	10	5	5	1	0	0	0	0	0	0
ν_i	9	13	22	31	40	49	13	23	16	19	1	2	2	2	2	2	2

Figure 2: Grafo dual de la resolución sumergida minimal de $\left(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\text{hess}(f)}\right), 0\right)$, junto con los datos numéricos de sus componentes, para $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

En el **Capítulo 4** resumimos los conceptos sobre curvas planas racionales cuspidales que aparecen en las próximas secciones de la tesis, en especial en el Capítulo 5. Tras esta introducción, en la Sección 4.2 recordamos algunos invariantes y propiedades que nos resultarán útiles.

La Sección 4.3 está dedicada a la clasificación de las curvas planas racionales cuspidales irreducibles en el plano proyectivo complejo en función de la acción del grupo de automorfismos $\mathrm{PGL}(3,\mathbb{C})$ en $\mathbb{P}^2(\mathbb{C})$, que es un problema abierto muy complicado a la vez que cautivador. El objetivo principal de este problema es determinar, para un d concreto, si existe una curva proyectiva plana C de grado d con un número fijo de singularidades de un tipo topológico dado. Presentamos ciertas familias de curvas racionales unicuspidales, concretamente aquellas con un solo par de Puiseux, las curvas de Tono y las curvas de Orevkov. Mencionamos además algunos problemas interesantes relacionados con las curvas racionales cuspidales.

Dada una curva plana racional unicuspidal C con cúspide $p \in C$, D. Daigle y A. Melle demostraron la existencia de un sistema lineal de dimensión 1 determinado por el par (C,p). En la Sección 4.4 revisamos un poco de teoría relacionada con este sistema lineal Λ_C, que se estudia en profundidad en [DM14].

El **Capítulo 5** está dedicado a algunas conjeturas sobre curvas planas libres y casi-libres. El concepto de divisor libre fue introducido por K. Saito [Sai80] en el estudio de discriminantes de deformaciones versales de gérmenes de singularidades aisladas de hipersuperficies. Desde entonces han aparecido aplicaciones interesantes e inesperadas a la Teoría de Singularidades y a la Geometría Algebraica. En esta tesis nos centraremos principalmente en las curvas proyectivas planas complejas y por esta razón hemos adaptado las correspondientes notaciones y resultados a esta situación.

Consideremos el anillo de polinomios $S := \mathbb{C}[x,y,z]$ equipado con la graduación natural $S = \bigoplus_{m=0}^{\infty} S_m$ definida por polinomios homogéneos. Sea $f \in S_d$ un polinomio homogéneo de grado d en S, sea $C \subset \mathbb{P}^2$ la curva definida por $f = 0$ y supongamos que C es reducida. Denotemos por J_f al ideal jacobiano de f, esto es, al ideal homogéneo en S generado por f_x, f_y, f_z, y por $M(f) = S/J_f$ al correspondiente anillo graduado, denominado álgebra de Milnor de f.

El estudio de las curvas libres en el plano proyectivo tiene una trayectoria bastante larga, iniciada por A. Simis en [Sim06a, Sim06b], y continuada activamente por varios matemáticos. Se dice que

C es una curva libre si $J_f = I_f$, donde I_f denota la saturación del ideal J_f con respecto al ideal maximal $\mathfrak{m} = (x, y, z)$.

La Sección 5.2 consiste en una selección de resultados básicos de álgebra conmutativa que serán útiles a la hora de comprender el Capítulo 5.

En las Secciones 5.3, 5.4 y 5.5 resumimos los conceptos relacionados con los divisores libres y casi-libres que se usan en el resto de la tesis, e incluimos las referencias correspondientes a los resultados mencionados.

Las curvas casi-libres se introdujeron en [DS18a]. Tienen propiedades similares a las de las curvas libres y, junto con estas últimas, pueden llevar a una nueva interpretación de las curvas racionales cuspidales, gracias a la Conjetura 0.0.3(i). Esta clase de curvas ya es objeto de atención de algunos trabajos, véanse por ejemplo [AD18] y [MV17].

Por definición, C es una curva casi-libre si el módulo graduado $N(f) = I_f/J_f$ verifica que $N(f) \neq 0$ y la parte graduada $N(f)_k$ es tal que $\dim_{\mathbb{C}} N(f)_k \leq 1$ para todo k.

Los resultados principales de A. Dimca y G. Sticlaru en [DS17], [DS15] y [DS18a] y numerosas familias de ejemplos motivaron la siguiente conjetura:

Conjetura 0.0.5. *[DS15]*

(i) *Toda curva racional cuspidal plana C es o bien libre o bien casi-libre.*

(ii) *Toda curva plana irreducible C que sea libre o casi-libre es racional.*

En [DS18a], los autores proporcionan algunos resultados interesantes que respaldan la Conjetura 0.0.3(i); en particular, la Conjetura 0.0.3(i) es cierta para curvas racionales cuspidales de grado par [DS18a, Teorema 4.1]. Para la demostración de esto último necesitan una hipótesis en las cúspides que no se cumple en general cuando el grado es impar, véase [DS18a, Teorema 4.1].

Probaron también que esta conjetura es cierta para las curvas C cuyo grupo fundamental $\pi_1(\mathbb{P}^2 \setminus C)$ es abeliano y para las curvas cuyo grado es una potencia de primo.

Recientemente, en [DS18b], A. Dimca y G. Sticlaru han demostrado que su Conjetura 0.0.3(i) es válida para toda curva racional cuspidal $C = V(f)$ tal que $\operatorname{mdr}(f) \leq 15$ y para la mayor parte de las curvas de grado $d \leq 90$.

A. Dimca y G. Sticlaru también propusieron la siguiente conjetura:

Conjetura 0.0.6. *[DS15]*

(i) *Toda curva plana irreducible libre C tiene singularidades con, a lo sumo, dos ramas.*

(ii) *Toda curva plana irreducible casi-libre C tiene singularidades con, a lo sumo, tres ramas.*

En [AGLM17] damos algunos ejemplos de curvas irreducibles libres y casi-libres en el plano proyectivo complejo que no son racionales, proporcionando así contraejemplos a la Conjetura 0.0.3(ii). Entre estos contraejemplos encontramos algunos ejemplos de curvas irreducibles libres y casi-libres cuyos dos puntos singulares tienen un número impar de ramas $r = 2\ell + 1$, $\ell \geq 1$, obteniendo de esta manera contraejemplos a las Conjeturas 0.0.4(i) y 0.0.4(ii). Además, se proporciona una curva irreducible casi-libre con un único punto singular y cuatro ramas, dando lugar a otro contraejemplo a la Conjetura 0.0.4(ii). Estos resultados se han recopilado en la Sección 5.6.

De 5.6.1 se deduce que, para todo entero impar $k \geq 1$, la curva plana irreducible C_{5k} de grado $d = 5k$ definida como

$$f_{5k} := (y^k z^k - x^{2k})^2 y^k - x^{5k} = 0$$

verifica que:

- su género geométrico es $g(C_{5k}) = \frac{(k-1)(k-2)}{2}$;
- su lugar singular consiste en dos puntos: $\mathrm{Sing}(C_{5k}) = \{p_1, p_2\}$;
- el número de ramas de C_{5k} en cada punto singular p_i es exactamente k;
- C_{5k} es una curva libre.

Así, si $k \geq 3$, C_{5k} es un contraejemplo tanto para la parte *libre* de la Conjetura 0.0.3(ii) como para la Conjetura 0.0.4(i).

De 5.6.2 también se deduce que, para todo entero impar $k \geq 1$, la curva plana irreducible C_{4k} de grado $d = 4k$ definida por

$$f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0$$

cumple que:

- su género geométrico es $g(C_{4k}) = \frac{(k-1)(k-2)}{2}$;
- su lugar singular consiste en dos puntos: $\mathrm{Sing}(C_{4k}) = \{p_1, p_2\}$;
- el número de ramas de C_{4k} en cada punto singular p_i es exactamente k;
- C_{4k} es una curva casi-libre.

Así, si $k \geq 3$, C_{4k} es un contraejemplo tanto para la parte *casi-libre* de la Conjetura 0.0.3(ii) como para la Conjetura 0.0.4(ii).

En las familias estudiadas arriba, el número de puntos singulares de sus curvas es exactamente dos. En 5.6.3, buscamos curvas con género y número de singularidades no acotado que constituyan un contraejemplo a la parte relativa a las curvas casi-libres de la Conjetura 0.0.3(ii). En particular, para todo entero impar $k \geq 3$, la curva irreducible C_{2k} de grado $d = 2k$ definida como

$$f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0$$

cumple que:

- $\mathrm{Sing}(C_{2k})$ contiene exactamente $3k$ puntos singulares de tipo $\mathbb{A}_{k-1}$;
- su género es $g(C_{2k}) = \frac{(k-1)(k-2)}{2}$;
- C_{2k} es una curva casi-libre.

Una de las herramientas fundamentales a la hora de encontrar estos ejemplos han sido las cubiertas de Kummer, un instrumento muy útil para construir curvas algebraicas complicadas a partir de otras más simples. Por esta razón hemos incluido una introducción a las cubiertas de Kummer en el Capítulo 2.

En particular, las familias de ejemplos $\{C_{5k}\}$, $\{C_{4k}\}$ y $\{C_{2k}\}$ se construyen como la transformada bajo la cubierta de Kummer π_k de la correspondiente curva racional cuspidal: la quíntica C_5 que es una curva libre, y las respectivas curvas casi-libres definidas por la cuártica C_4 y la cónica C_2.

Finalmente, en 5.6.4 se muestra una curva irreducible C_{49} de grado 49 tal que:

- C_{49} tiene un solo punto singular, que posee 4 ramas;
- su género es $g(C_{49}) = 0$, es decir, C_{49} es una curva racional;
- C_{49} es una curva casi-libre.

Esta curva C_{49} se construye como el elemento genérico del único sistema lineal de dimensión 1 asociado a una cierta curva plana racional unicuspidal de grado 49 y constituye otro contraejemplo a la Conjetura 0.0.4(ii).

Todos estos ejemplos contradicen algunas de las conjeturas propuestas por A. Dimca y G. Sticlaru en [DS15]. Sin embargo, nuestros ejemplos no dicen nada sobre la conjetura más destacada de A. Dimca y G. Sticlaru, que predice que toda curva plana racional cuspidal es o bien libre o bien casi-libre.

Chapter 1

Introduction

The main topic of this thesis is Singularity Theory. More specifically, we give counterexamples to some conjectures that have arisen from the study of plane curve singularities. In particular, we deal with two main topics:

1. The monodromy conjecture of J. Denef and F. Loeser and its generalisation by A. Némethi and W. Veys.
2. Free and nearly free plane curves with isolated singularities, and some related conjectures proposed by A. Dimca and G. Sticlaru.

This dissertation is composed of an abstract in Spanish and in English, and the complete work in English. The body of the thesis is structured in five chapters. The first of them is introductory. Chapter 2 contains the required concepts and tools that will be used throughout the body of the thesis. Most of this material is widely known for people who are familiar with Singularity Theory. In chapters 3 and 5 we present the main results of our research. Chapter 4 comprises several results on rational cuspidal plane curves. Finally, we summarise the conclusions of our work. Below, we outline the content of this memoir chapter by chapter.

In **Chapter 1** we present an overview of the thesis and the structure of the subsequent chapters.

Chapter 2 establishes a unified language and contains background information so that the thesis is as self-contained as possible. In Sections 2.1 and 2.2 we briefly review the basic concepts to locally study the isolated hypersurface singularities, so we deal with function germs. We give a special emphasis to plane curve singularities, which is a classical and very well understood topic in Singularity Theory. We start describing resolution of singularities in Section 2.2. In Section 2.3 we present another fundamental tool: the Milnor fibration. Section 2.4 is devoted to the study of the monodromy, which is one of the essential components of the Monodromy Conjecture. In particular we introduce there the zeta function of the monodromy.

We next study complex projective algebraic plane curves. In Section 2.5 we fix conventions and recall the main properties and global invariants of such curves from the local invariants of their singular points that we have explored in the previous sections and we also state some interesting features of the singularities of the dual curve.

The counterexamples to the conjectures of A. Dimca and G. Sticlaru have been constructed using two main techniques: Kummer covers and certain properties of pencils of rational unicuspidal plane curves. For this reason, these topics are covered in Sections 2.6 and 2.7, respectively. In Section

2.6 we first recall some basic and useful properties about topological coverings. In Section 2.7 we state some fundamental definitions concerning pencils of plane curves.

Chapter 3 deals with the Monodromy Conjecture of J. Denef and F. Loeser, which relates poles of the Igusa, motivic or topological zeta function to monodromy eigenvalues.

Zeta functions can be attached to several mathematical objects like fields, groups, algebras, functions and dynamical systems. Typically, zeta functions encode relevant arithmetic, algebraic, geometric or topological information about the original object.

Before stating the Monodromy Conjecture, we will introduce its other main ingredient, namely the topological zeta function in Section 3.3 and its precursor, the Igusa local zeta function, in Section 3.2.

In 1992 J. Denef and F. Loeser introduced a new zeta function [DL92], which they called the *topological zeta function* because of the topological Euler-Poincaré characteristic turning up in it.

The classical Monodromy Conjecture predicts that if s_0 is a pole of the local topological zeta function $Z_{top,0}(f, s)$ associated with the singularity defined by f, then $\exp(2\pi i s_0)$ is an eigenvalue of the local monodromy of f at some complex point of $(f^{-1}(0), 0)$. Section 3.4 gives a deeper insight into it.

The Monodromy Conjecture was verified for plane curves by F. Loeser in [Loe88], originally in the context of p-adic Igusa zeta functions. However, for the time being, there are only some partial results in higher dimensions, due to the lack of a conceptual link between the monodromy operator and the topological zeta function. Indeed, the existent proofs of the particular cases basically compute both sides independently and compare the two final results.

In order to find a more conceptual understanding, it was later extended for topological zeta functions associated with arbitrary differential forms by W. Veys, and afterwards A. Némethi and W. Veys introduced the set of *allowed* differential forms (see [NV10] and [NV12]). They proved that for germs of plane curve singularities these allowed differential forms exist, and that this set contains the standard differential form as well.

On the other hand, it is easy to observe on explicit examples that not all the eigenvalues of the monodromy operator are also induced by poles of the topological zeta function, as we will show in Section 3.4.

The topological zeta function was first introduced by J. Denef and F. Loeser for the standard differential form $\omega_0 = dx_0 \wedge \ldots \wedge dx_n$. In the same way that the standard differential form is always allowed, it is natural to wonder whether there exists any other differential form satisfying the same property, and a reasonable candidate might be the Hessian differential form, which only depends on the defining function of the curve.

For this reason, A. Melle conjectured and asked whether all the poles of the topological zeta function associated with an isolated plane curve singularity defined by an analytic germ f and the Hessian differential form $\omega_{\mathrm{hess}(f)} := \mathrm{hess}(f)dx \wedge dy$ did always induce eigenvalues of the monodromy, provided that this fact occurs in many examples, e.g., the simple singularities of type $\mathbb{A}_n$. This is explained in more detail in Section 3.5.

The main goal of Chapter 3 is to show that the local topological zeta function of a germ associated with its Hessian differential form does not satisfy the Monodromy Conjecture. This result was proved by the author in [Gor18]. In Section 3.6 we show the counterexample given by the germ

$(C, 0) = (f^{-1}(0), 0)$, where

$$f(x, y) = y^5 - 2x^3y^7 + x^4y^3 + x^6. \tag{1.0.1}$$

Let (X, π) be the minimal embedded resolution of the germ $(f^{-1}(0) \cup \mathrm{div}(\omega), 0)$. Thus, π is a finite succession of blowings-up. We denote the components by $\tilde{E}_i, i \in I = I_e \cup I_{s^f} \cup I_{s^\omega}$, where $\tilde{E}_i$ is:

- an exceptional component, if $i \in I_e$;
- an irreducible component of the strict transform of $f^{-1}(0)$, if $i \in I_{s^f}$;
- an irreducible component of the strict transform of $\mathrm{div}(\omega)$, if $i \in I_{s^\omega}$.

For each $i \in I$, let N_i and $\nu_i - 1$ be the multiplicities of $\tilde{E}_i$ in the divisor of $\pi^* f$ and $\pi^*\omega$, respectively.

The resolution graph of the singularity $(C, 0) = (f^{-1}(0), 0)$ defined by $f(x, y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$ at the origin is:

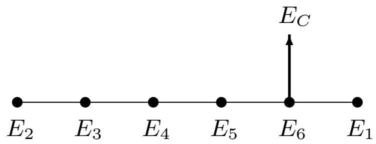


Figure 1.1: Dual graph of the minimal embedded resolution of $(f^{-1}(0), 0)$, for $f(x, y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

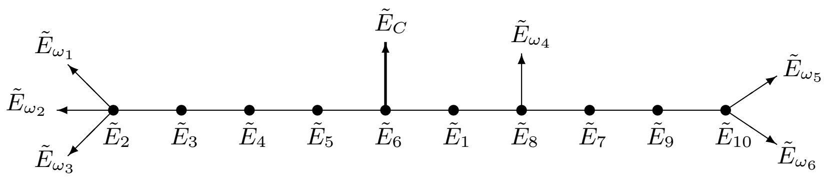
and the minimal embedded resolution of $(f^{-1}(0) \cup \mathrm{div}\left(\omega_{\mathrm{hess}(f)}\right), 0)$ is described in the figure below, for the sets of indices:

$I_e = \{1, \ldots, 10\}$,
$I_{s^f} = \{C\}$,
$I_{s^\omega_f} = \{\omega_1, \ldots, \omega_6\}$.

We will show that the pole of the topological zeta function $s_0 := -13/6$, which corresponds to the component $\tilde{E}_2$, does not induce a monodromy eigenvalue, since s_0 is not a root of the characteristic polynomial of the germ $(C, 0)$:

$$\Delta_C(t) = \frac{(t-1)(t^{30}-1)}{(t^5-1)(t^6-1)},$$

and this means that the Monodromy Conjecture fails when we consider the differential form $\omega_{\mathrm{hess}(f)} = \mathrm{hess}(f)dx \wedge dy$ instead of the standard ω_0. In particular, the Hessian differential form is not allowed.



i	1	2	3	4	5	6	7	8	9	10	C	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
N_i	5	6	12	18	24	30	5	10	5	5	1	0	0	0	0	0	0
ν_i	9	13	22	31	40	49	13	23	16	19	1	2	2	2	2	2	2

Figure 1.2: Dual graph of the minimal embedded resolution of $(f^{-1}(0) \cup \text{div}\left(\omega_{\text{hess}(f)}\right), 0)$, along with the numerical data of its components, for $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

In **Chapter 4** we summarise the concepts of rational cuspidal plane curves that appear in the forthcoming parts of the thesis, especially in Chapter 5. After their introduction, in Section 4.2 we recall some useful invariants and properties.

Section 4.3 is devoted to the classification of irreducible projective plane rational cuspidal curves in the complex projective plane, up to the action of the automorphism group $\text{PGL}(3,\mathbb{C})$ on $\mathbb{P}^2(\mathbb{C})$, which is a very difficult and interesting open problem. The main goal of this problem is to determine, for a given d, whether there exists a projective plane curve C of degree d having a fixed number of singularities of given topological type. We introduce certain families of rational unicuspidal curves, namely those having one Puiseux pair, Tono's curves and Orevkov's curves. We also mention some intringuing problems related to rational cuspidal curves.

Given a unicuspidal rational plane curve C with cusp $p \in C$, D. Daigle and A. Melle proved the existence of a pencil determined by the pair (C, p). In Section 4.4 we review some theory related to this pencil Λ_C, which is extensively studied in [DM14].

Chapter 5 is dedicated to some conjectures on free and nearly free plane curves. The notion of free divisor was introduced by K. Saito [Sai80] in the study of discriminants of versal unfoldings of germs of isolated hypersurface singularities. It was originally associated with hyperplane arrangement theory. Since then many interesting and unexpected applications to Singularity Theory and Algebraic Geometry have appeared. In this thesis we are mainly focused on complex projective plane curves and for this reason we adapt the corresponding notions and results to this setup.

Let $S := \mathbb{C}[x,y,z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a squarefree homogeneous polynomial of degree d in the polynomial ring, and let $C \subset \mathbb{P}^2$ be the reduced curve defined by $f = 0$. We denote by J_f the Jacobian ideal of f, i.e., the homogeneous ideal in S spanned by f_x, f_y, f_z, and by $M(f) = S/J_f$ the corresponding graded ring, called the Milnor algebra of f.

The study of free curves in the projective plane has a rather long tradition, being inaugurated by A. Simis in [Sim06a, Sim06b], and actively continued by several mathematicians (see the article by A. Dimca and G. Sticlaru [DS19] and the references given there). C is a free curve if $J_f = I_f$, where I_f denotes the saturation of the ideal J_f with respect to the maximal ideal $\mathfrak{m} = (x,y,z)$.

Section 5.2 consists in a selection of basic commutative algebra results that are useful for the understanding of Chapter 5.

In Sections 5.3, 5.4 and 5.5 we give an overview of the concepts related to free and nearly free divisors used in the rest of the thesis and provide references for all the results mentioned therein.

The nearly free curves were introduced in [DS18a, Dim17]. They have properties similar to the free curves, and together with the free curves may lead to a new understanding of the rational cuspidal curves, due to Conjecture (i) below. This class of curves forms already the subject of attention in a number of papers, see for instance [AD18] and [MV17].

By definition, C is a nearly free curve if the graded module $N(f) = I_f/J_f$ satisfies $N(f) \neq 0$ and the graded part $N(f)_k$ is such that $\dim_{\mathbb{C}} N(f)_k \leq 1$ for all k.

The main results of A. Dimca and G. Sticlaru in [DS17], [DS15] and [DS18a] and many series of examples motivate the following conjecture:

Conjecture 1.0.7. *[DS15]*

(i) *Any rational cuspidal curve C in the plane is either free or nearly free.*

(ii) *An irreducible plane curve C which is either free or nearly free is rational.*

In [DS18a], the authors provide some interesting results supporting the statement of Conjecture 0.0.1(i); in particular, Conjecture 0.0.1(i) holds for rational cuspidal curves of even degree [DS18a, Theorem 4.1]. They need a topological assumption on the cusps which is not fulfilled all the time when the degree is odd, see [DS18a, Theorem 4.1].

They proved also that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ and for those curves whose degree is a prime power, see [DS18a, Corollary 4.2] and the discussion in [AD15].

Furthermore, using the classification given in [FLMN07a] of unicuspidal rational curves with a unique Puiseux pair, A. Dimca and G. Sticlaru proved in [DS18a, Corollary 4.5] that all of them are either free or nearly free curves, except the curves of odd degree in one of the cases of the classification.

More recently, in [DS18b], A. Dimca and G. Sticlaru proved that their Conjecture 0.0.1(i) is true for any rational cuspidal curve $C = V(f)$ with $\mathrm{mdr}(f) \leq 15$ and for most curves of degree $d \leq 90$.

A. Dimca and G. Sticlaru also proposed the following conjecture:

Conjecture 1.0.8. *[DS15]*

(i) *Any free irreducible plane curve C has only singularities with at most two branches.*

(ii) *Any nearly free irreducible plane curve C has only singularities with at most three branches.*

In [AGLM17] we gave some examples of irreducible free curves and nearly free curves in the complex projective plane which are not rational and thus giving counterexamples to Conjecture 0.0.1(ii). Among these counterexamples we found some examples of irreducible free and nearly free curves whose two singular points have any odd number of branches $r = 2\ell + 1$, $\ell \geq 1$, giving counterexamples to Conjectures 0.0.2(i) and 0.0.2(ii). Furthermore, an irreducible nearly free curve with just one singular point and having 4 branches, giving another counterexample to Conjecture 0.0.2(ii), was also provided. These results are gathered in Section 5.6.

From 5.6.1 it can be deduced that, for every odd integer $k \geq 1$, the irreducible plane curve C_{5k} of degree $d = 5k$ defined by

$$f_{5k} := (y^k z^k - x^{2k})^2 y^k - x^{5k} = 0$$

satisfies:

- its geometric genus is $g(C_{5k}) = \frac{(k-1)(k-2)}{2}$;
- its singular locus consists of two points, say $\mathrm{Sing}(C_{5k}) = \{p_1, p_2\}$;
- the number of branches of C_{5k} at each p_i is exactly k;
- C_{5k} is a free curve.

Hence, for $k \geq 3$, C_{5k} is a counterexample to both the free part of Conjecture 0.0.1(ii) and Conjecture 0.0.2(i).

From 5.6.2 it can also be inferred that, for any odd integer $k \geq 1$, the irreducible plane curve C_{4k} of degree $d = 4k$ defined by

$$f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0$$

satisfies:

- its geometric genus is $g(C_{4k}) = \frac{(k-1)(k-2)}{2}$;
- its singular locus consists of two points: $\mathrm{Sing}(C_{4k}) = \{p_1, p_2\}$;
- the number of branches of C_{4k} at each p_i is exactly k;
- C_{4k} is a nearly free curve.

Thus, for $k \geq 3$, C_{4k} is a counterexample to both the nearly-free part of Conjecture 0.0.1(ii) and Conjecture 0.0.2(ii) too.

In the familes studied above the number of singular points of the curves is exactly two. In 5.6.3, we are looking for curves with unbounded genus and number of singularities which give a counterexample to the part regarding nearly free curves of Conjecture 0.0.1(ii). In particular, for every odd integer $k \geq 3$, the irreducible curve C_{2k} of degree $d = 2k$ defined by

$$f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0$$

satisfies:

- $\mathrm{Sing}(C_{2k})$ contains exactly $3k$ singular points of type $\mathbb{A}_{k-1}$;
- its genus is $g(C_{2k}) = \frac{(k-1)(k-2)}{2}$;
- C_{2k} is a nearly free curve.

One of the main tools to find such examples is the use of Kummer covers, which are a very useful tool in order to construct complicated algebraic curves starting from simple ones. For this reason, we have included an introduction to Kummer covers in Chapter 2.

In particular, these familes of examples $\{C_{5k}\}$, $\{C_{4k}\}$ and $\{C_{2k}\}$ are constructed as the pullback under the Kummer cover π_k of the corresponding rational cuspidal curves: the quintic C_5 which is a free curve, and the corresponding nearly free divisors defined by the quartic C_4 and the conic C_2.

Finally, in 5.6.4, an irreducible curve C_{49} of degree 49 is given. This curve satisfies:

- C_{49} has just one singular point which has 4 branches;
- its genus is $g(C_{49}) = 0$, i.e., C_{49} is a rational curve;
- C_{49} is a nearly free curve.

This curve is constructed as a generic element of the unique pencil associated with a certain rational unicuspidal plane curve of degree 49 and it provides another counterexample to Conjecture 0.0.2(ii).

All these examples contradict some of the conjectures proposed by A. Dimca and G. Sticlaru in [DS15]. Our examples say nothing about the most remarkable conjecture by A. Dimca and G. Sticlaru, which predicts that every rational cuspidal plane curve is either free or nearly free.

Chapter 2

Prerequisites

In this chapter we settle the foundations for the subsequent chapters. We will recall some basic definitions, notations and results which will be used throughout the rest of the thesis in order to be as self-contained as possible.

2.1 Germs of isolated hypersurface singularities

We start with the definition of one of the main objects of this thesis: a germ of isolated hypersurface singularity (see [GLS07]).

Let $U \subset \mathbb{C}^{n+1}$ be an open subset, and let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function on U such that $f(0) = 0$. This means that f can be expanded as a convergent power series, i.e., $f \in \mathbb{C}\{x_0, \dots, x_n\}$.

A germ of a holomorphic function at $x \in \mathbb{C}^{n+1}$ is the equivalence class of a holomorphic function f defined in an open neighbourhood of x, where two functions, defined in open neighbourhoods of x, are equivalent if they coincide in some (usually smaller) common neighbourhood of x.

Let us denote by V the hypersurface $f^{-1}(0)$. We denote by

- $Crit(f) = \operatorname{Sing}(f) := \left\{ x \in U : \frac{\partial f}{\partial x_0}(x) = \dots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$

  the set of *critical* or *singular points* of f, and

- $\operatorname{Sing}(V) := \left\{ x \in U : f(x) = \frac{\partial f}{\partial x_0}(x) = \dots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$

  the set of *singular points* of V.

A point $x \in U$ is called an *isolated critical point* of f if there exists a neighbourhood W of x such that $(Crit(f) \cap W) \setminus \{x\} = \emptyset$. It is called an *isolated singular point* of V if $x \in V$ and $(\operatorname{Sing}(V) \cap W) \setminus \{x\} = \emptyset$. Then we say also that the germ $(V, x) \subset (\mathbb{C}^{n+1}, x)$ is an *isolated hypersurface singularity.*

We will mainly focus on isolated plane curve singularities ($n = 1$).

2.2 Resolution of singularities

A useful instrument for studying the topology of a singularity is its resolution. Many topological and analytical invariants of a singularity, such as its multiplicity or the characteristic polynomial of its classical monodromy operator can be expressed in terms of the topological characteristics of the divisors which are glued in during the resolution of the singularity.

2.2.1 Resolution of hypersurface singularities

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function defining an isolated hypersurface singularity $(V, 0)$. We define an *embedded resolution* of $(V, 0)$ as follows (see [AGV88]):

Definition 2.2.1 (Embedded resolution of a hypersurface)**.** An *embedded resolution* (X, π) for the hypersurface $(V, 0)$ consists in a non-singular variety X and a proper analytic map $\pi : \big(X, \pi^{-1}(0)\big) \to (\mathbb{C}^{n+1}, 0)$ such that:

1. The restriction $\pi\mid_{X\setminus\pi^{-1}(0)}: X \setminus \pi^{-1}(0) \to \mathbb{C}^{n+1} \setminus \{0\}$ is an analytic isomorphism.
2. The subspace $E := \pi^{-1}(0)$ of the space X, called the *exceptional divisor*, is the union of non-singular n-dimensional divisors in X which are in general position.
3. The total transform of V, $\pi^{-1}(V)$, is a divisor with normal crossings in X. This means that in a neighbourhood of any point of $\pi^{-1}(V)$ there exists a local system of coordinates $x_0, \dots, x_n$ such that the lifting $f \circ \pi$ of the function f to the space X of the map π has the form $x_0^{k_0} \cdot \ldots \cdot x_n^{k_n} \cdot u$, where u is a unit in $\mathbb{C}\{x_0, \dots, x_n\}$.

Hironaka proved that for a variety of arbitrary dimension over any field of characteristic zero, its embedded resolution of singularities does always exist [Hir64].

2.2.2 Resolution of plane curve singularities

In the following we describe how to resolve plane curve singularities via the blowing up of a point p in a smooth surface S. This is a purely local process in which the point p is replaced by a projective line $E \simeq \mathbb{P}^1$. As a result, curves which previously met at p get separated or, at least, their intersection multiplicity decreases. Anyway, the singularities of curves at p become simpler after blowing up. It turns out that by successively blowing up points, we can resolve a reduced plane curve singularity, that is, we can transform it into a smooth germ. The results of this subsection can be found in the books of de Jong and Pfister [dP00], Wall [Wal04] and Greuel, Lossen and Shustin [GLS07].

Let (C, p) be a germ of plane curve, so one can find local coordinates such that $p = (0, 0)$ and such that (C, p) is defined by the local equation $f(x, y) = 0$ for a certain $f \in \mathbb{C}\{x, y\}$. Then, the order of f is called the *multiplicity* of C at p, and it is denoted by $m_p(C)$. The curve C is smooth at p if and only if $m_p(C) = 1$.

Although blowing up may be defined in much more general situations, in this thesis we will only use the simplest form of it. We start from a point p on a smooth analytic surface S, and we will construct a new surface T and a map $\pi : T \to S$, called the *blowing up of S with centre p*, such that:

- $\pi^{-1}(p) \subset T$ is a curve E.

- π gives a bijection, indeed an analytic isomorphism, from $T \setminus E$ to $S \setminus \{p\}$.
- The points on E correspond to the different directions in S at p.

The construction of the blowing up of a smooth surface S, while conveniently expressed in terms of local coordinates, is not dependent on them.

It is interesting to explore the geometry of the curve E which appears on blowing up, and its relation to other curves which arise when we repeat the process.

Indeed, if $\pi : T \to S$ is the blowing up of S with centre the point $p \in S$, then $E = \pi^{-1}(p)$ is the exceptional divisor of the blowing up and C is a curve in S not passing through p, then C corresponds to the unique curve $\pi^{-1}(C)$ in T. However, if C is a curve through p, then $\pi^{-1}(C)$ is called the *total transform* of C, and it contains the exceptional curve E. The closure of $\pi^{-1}(C) \setminus E$ in the Zariski topology is called the *strict transform* of C.

From the viewpoint of calculations, let us explain the blowing-up process in coordinates (see [Wal04]). For this aim, let us introduce local coordinates (x, y) in a neighbourhood U of a point $p \in S$. Assume that the projective line $\mathbb{P}^1(\mathbb{C})$ has coordinates $(\xi : \eta)$. Recall that $\mathbb{P}^1(\mathbb{C})$ is the union of two affine coordinate charts: U_0, where $\xi \neq 0$ and we can take η/ξ as coordinate, and U_1, where $\eta \neq 0$ and we can take ξ/η as coordinate.

Then define T to be the subspace of points in the product $S \times \mathbb{P}^1(\mathbb{C})$ satisfying the equation $x\eta = y\xi$. The projection of $S \times \mathbb{P}^1(\mathbb{C})$ to S defines a map $\pi : T \to S$. Any point $(x, y) \neq p$ determines a unique pair $(\xi : \eta)$ such that $(\xi : \eta) = (x : y)$, hence a unique point $\pi^{-1}(x, y)$, while corresponding to the point p we have the entire projective line $\mathbb{P}^1(\mathbb{C})$, so that $\pi^{-1}(p)$ is a curve E (the exceptional divisor) isomorphic to $\mathbb{P}^1(\mathbb{C})$.

On the part of T where $\xi \neq 0$ we write Y for η/ξ, and the equation $x\eta = y\xi$ then simplifies to $y = xY$, showing that this part of T can be identified with $\mathbb{C}^2$ by taking the coordinates (x, Y). Similarly, on the part of T where $\eta \neq 0$ we write X for ξ/η; the equation $x\eta = y\xi$ simplifies to $x = Xy$, and we identify this part of T with $\mathbb{C}^2$ using the coordinates (X, y). The existence of these local coordinates exhibits the fact that the blow up of $\mathbb{P}^2(\mathbb{C})$ is another non-singular surface. Note in particular that the preimage E of p is isomorphic to $\mathbb{P}^1(\mathbb{C})$, and is given in the first chart by $x = 0$ (with coordinate Y) and in the second by $y = 0$ (with coordinate X).

Recall that a collection of curves in a smooth surface is said to have *normal crossings* if:

- each curve is smooth;
- no three meet in a point;
- any intersection of two of them is transverse.

The procedure to be followed in order to resolve a curve singularity is simple: whenever there is a singular point or one point where the normal crossing condition fails, choose one such point, and blow it up. A resolution obtained by following this procedure is called minimal (as opposed to one where additional unnecessary blowings up are performed). The order in which we blow points up does not affect the result, since if two points in a surface T are both to be blown up, blowing up one of the points does not change what happens in a neighbourhood of the other.

We shall denote the ambient space after the i-th blowing up by X_i, the corresponding exceptional line by E_i and the strict transform by C_i, where $C_i \subset X_i$.

Let us denote by p_i the point corresponding to $p \in C$ on the curve C_i. The points p_i are called *infinitely near points* of p. If we denote by m_i the multiplicity of C_i at p_i, then $m_{i-1} \geq m_i$, and at some point it drops strictly. This implies that there is a natural number k such that $m_k = 1$ and therefore C_k is smooth, which means that the singularity is resolved after a finite number k of blowings up. We denote $m_p := m_0$. If $\bigcup_{i=1}^k E_i$ intersects C_k not transversally, we blow up again. If not, the process stops. Let us denote by $\pi = \pi_1 \circ \ldots \circ \pi_k$ the composition of all the blowings up and $E = \bigcup_{i=1}^k E_i$ the exceptional divisor.

Definition 2.2.2 (Embedded resolution of a plane curve)**.** With the notation above, the composition

$$X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{C}^2 \tag{2.2.1}$$

is called *standard embedded resolution* of the singularity of the irreducible curve $C \subset \mathbb{C}^2$ at $(0,0)$ if either:

- $C_{i-1} \subset X_{i-1}$ still has a singular point and π_i is the blowing up of X_{i-1} with centre the singular point, or
- C_{i-1} is smooth but the intersection with E_{i-1} in X_{i-1} is not transversal and π_i is the blowing up of X_{i-1} with centre the intersection point of E_{i-1} and C_{i-1}.

and if, furthermore, C_k is smooth and intersects $E = \bigcup_{i=1}^k E_i$ transversally.

Definition 2.2.3 (Minimal resolution of singularities)**.** The *minimal resolution* of singularities of C is the shortest sequence $X_\ell \xrightarrow{\pi_\ell} X_{\ell-1} \xrightarrow{\pi_{\ell-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{C}^2$ such that the strict transform of C on X_ℓ is a non-singular curve.

Definition 2.2.4. (Multiplicity sequence) We call $\bar{m}_p = (m_0, \ldots, m_{k-1})$ the *multiplicity sequence* of the singularity $p \in C$. Whenever there are ℓ_i subsequent identical terms m_i in the sequence, we compress the notation by writing $\bar{m}_p = (m_0, m_1, \ldots, (m_i)_{\ell_i}, \ldots, 1)$. Additionally, we usually omit the ending 1's in the sequence. Furthermore, if we need to specify the singular point p in the multiplicity sequence we write $\bar{m}_p = (m_0^p, \ldots, m_{k-1}^p)$.

Definition 2.2.5 (Delta invariant)**.** We define the *delta invariant* δ_p of a point p of C by

$$\delta_p(C) = \sum \frac{m_q(m_q - 1)}{2}, \tag{2.2.2}$$

where the sum is taken over all infinitely near points q lying over p, including p.

We will put the useful information of the resolution into a diagram, which we call the *dual graph*. It is obtained as follows: one associates a vertex to each exceptional component in the embedded resolution (represented by a dot) and to each component of the strict transform of $f^{-1}(0)$ (represented by an arrow and known as *branch*); one also associates to each intersection an edge, connecting the corresponding vertices. The fact that E_i has numerical data (N_i, ν_i) is denoted by $E_i(N_i, \nu_i)$ [LSV06] (see Section 3.2).

As an example, we present the resolution graph and the dual graph associated with the plane curve singularity $(C, 0) = (f^{-1}(0), 0)$, where $f(x, y) = (x^2 + y^3)(x^2y^2 + x^6 + y^6)$ [A'C75], along with the corresponding numerical data, in Figure 2.1:

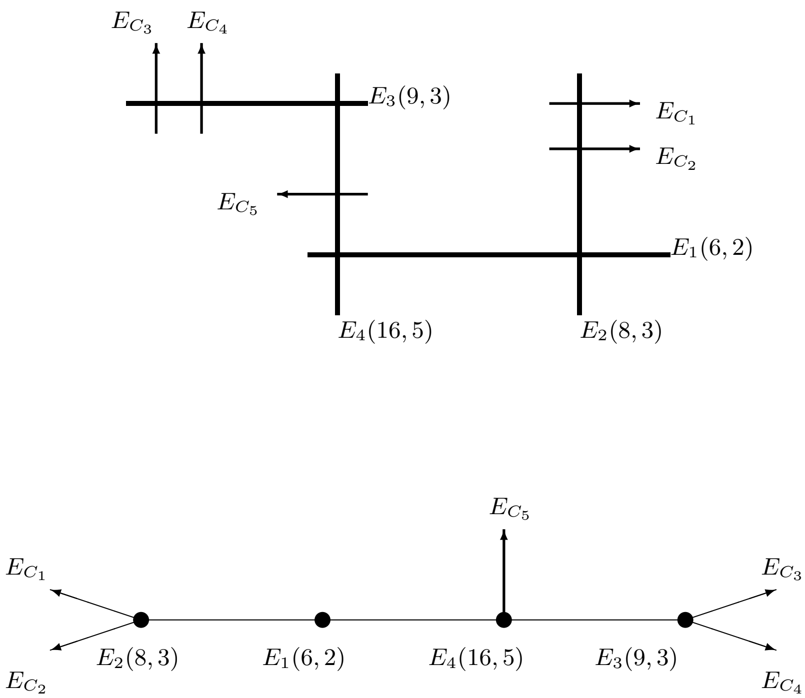


Figure 2.1: Resolution graph and dual graph of the minimal embedded resolution of $(f^{-1}(0), 0)$, for $f(x, y) = (x^2 + y^3)(x^2y^2 + x^6 + y^6)$.

2.3 Milnor fibration

The Milnor fibration is a fundamental tool in the study of the topology of hypersurfaces. The ideas and results explained in this section were developed by J. Milnor in the book [Mil68].

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function, and denote by $(V, 0)$ the hypersurface $(f^{-1}(0), 0)$.

Milnor fibration provides a framework to study the topology of $(V, 0)$, especially in the case in which we are interested, that is, when f is not smooth at 0. There are, in fact, two equivalent fibrations which, in the literature, are called the *Milnor fibration* of the function germ f or of the hypersurface singularity $(V, 0)$: one is defined on small spheres and the other one inside small open balls. Let us introduce both fibrations for the case in which $(V, 0)$ has an isolated singularity at $0 \in \mathbb{C}^{n+1}$.

For this aim, let $\epsilon > 0$ be small enough such that the closed ball $\mathbb{B}_\epsilon \subset \mathbb{C}^{n+1}$ of radius ϵ around the origin intersects the fibre $f^{-1}(0)$ transversally. Take $\delta \in \mathbb{R}$ such that $0 < \delta \ll \epsilon$ verifying that for any t in the disc $\mathbb{D}_\delta \subset \mathbb{C}$ of radius δ around the origin, the fibre $f^{-1}(t)$ intersects the ball $\mathbb{B}_\epsilon$ transversally. Let $\mathbb{D}^*_\delta$ denote the punctured disc $\{z \in \mathbb{C} : 0 < |z| < \delta\}$ and let $\mathbb{S}^{2n+1}_\epsilon = \partial\mathbb{B}_\epsilon$ be the boundary of $\mathbb{B}_\epsilon$ and let $K = V \cap \mathbb{S}^{2n+1}_\epsilon$ be the corresponding link (notice that $K = \emptyset$ for $n = 0$). The following results are due to J. Milnor:

Theorem 2.3.1 (Milnor)**.** *[Mil68] The map*

$$\varphi : \mathbb{S}^{2n+1}_{\epsilon} \setminus K \to \mathbb{S}^1, \qquad \varphi(x) = \frac{f(x)}{|f(x)|} \tag{2.3.1}$$

is a smooth locally trivial fibration.

From a historical point of view, this theorem is motivated by the classical study of fibred knots. Indeed, when f has an isolated singularity at the origin, the link K is a smooth manifold, and any fibre $F_a := \varphi^{-1}(a)$ for $a \in \mathbb{S}^1$ is a smooth open manifold whose closure $\overline{F_a}$ coincides with the union $F_a \cup K$.

The second fibration can be described as follows:

Theorem 2.3.2 (Milnor)**.** *[Mil68]*

1. *The map*
$$\overline{\psi} : \mathbb{B}_{\epsilon} \cap f^{-1}(\mathbb{D}^*_{\delta}) \to \mathbb{D}^*_{\delta}, \qquad \overline{\psi}(x) = f(x) \tag{2.3.2}$$
is a topological locally trivial fibration.

2. *The map*
$$\psi : \mathbb{B}^{\circ}_{\epsilon} \cap f^{-1}(\mathbb{D}^*_{\delta}) \to \mathbb{D}^*_{\delta}, \qquad \psi(x) = f(x), \tag{2.3.3}$$
where $\mathbb{B}^{\circ}_{\epsilon} \subset \mathbb{C}^{n+1}$ is the open ball of radius ϵ around the origin, is a smooth locally trivial fibration.

Moreover:

- The fibrations φ and ψ are fibre diffeomorphic equivalent.
- The fibrations ψ and $\overline{\psi}$ are fibre homotopy equivalent.

These theorems motivate the following definition:

Definition 2.3.3 (Milnor fibration)**.** Any of the (equivalent) fibrations φ, ψ or $\overline{\psi}$ is called the *Milnor fibration* of the function germ f (or the singularity $(V, 0)$). Any of the corresponding fibres $\varphi^{-1}(a)$ $(a \in \mathbb{S}^1)$, $\psi^{-1}(a)$ or $\overline{\psi}^{-1}(a)$ $(a \in \mathbb{D}^*_{\delta})$ is called the *Milnor fibre* of the function germ f (or the singularity $(V, 0)$).

Remark 2.3.4. The Milnor fibration associated with a hypersurface singularity $(V, 0)$ does not depend on the choice of an equation $f = 0$ for $(V, 0)$.

In [Mil68] J. Milnor proved that the Milnor fibre is a CW-complex of dimension at most n. In the case in which f has an isolated singularity at the origin, he also proved that the Milnor fibre is homotopy equivalent to a bouquet of n-spheres.

The number of spheres is called the *Milnor number* μ, which can be computed from the equation f as shown in (2.3.4). For a proof of the fact that both definitions agree, see [Bri70].

For further details regarding the Milnor fibration, the reader is referred to the books by J. Milnor [Mil68] and A. Dimca [Dim92].

2.3.1 The Milnor number and the Tjurina number

In this subsection we present two useful invariants: the Milnor number and the Tjurina number (see [dP00] and [GLS07]). We start introducing the notions of analytic and topological types in the context of isolated hypersurface singularities.

Definition 2.3.5 (Analytical and topological invariants)**.** Let $(X, z) \subset (\mathbb{C}^{n+1}, z)$ and $(Y, w) \subset (\mathbb{C}^{n+1}, w)$ be two germs of isolated hypersurface singularities. Then:

- (X, z) and (Y, w) (or any defining power series) are said to be *analytically equivalent* (or *contact equivalent*) if there exists a local analytic isomorphism $(\mathbb{C}^{n+1}, z) \to (\mathbb{C}^{n+1}, w)$ mapping (X, z) to (Y, w). The corresponding equivalence classes are called *analytic types.*
- (X, z) and (Y, w) (or any defining power series) are said to be *topologically equivalent* if there exists a homeomorphism $(\mathbb{C}^{n+1}, z) \to (\mathbb{C}^{n+1}, w)$ mapping (X, z) to (Y, w). The corresponding equivalence classes are called *topological types.*
- A number (or a set, or a group, ...) associated with a singularity is called an *analytic, respectively topological, invariant* if it does not change its value within an analytic, respectively topological, equivalence class.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function.

Definition 2.3.6. The *Jacobian ideal* of f is defined as $J(f) := J_f := \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$.

The *Milnor algebra* of f by definition is the $\mathbb{C}-$algebra $M(f) := \frac{\mathbb{C}\{x_0,\dots,x_n\}}{J(f)}$.

The *Milnor number* of f, denoted by $\mu(f^{-1}(0), 0)$ is the $\mathbb{C}-$vector space dimension of the Milnor algebra:

$$\mu(f^{-1}(0), 0) = \mu(V, 0) := \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x_0, \dots, x_n\}}{J(f)} \right). \tag{2.3.4}$$

The Milnor number is a topological invariant of the hypersurface singularity $(V, 0)$.

Another important invariant of $(V, 0)$ is the Tjurina number, which is an analytical invariant:

Definition 2.3.7. The *Tjurina algebra* of f is the $\mathbb{C}-$algebra $\frac{\mathbb{C}\{x_0,\dots,x_n\}}{(f,J(f))}$.

The *Tjurina number* of f, denoted by $\tau(f^{-1}(0), 0)$ is the $\mathbb{C}-$vector space dimension of the Tjurina algebra:

$$\tau(f^{-1}(0), 0) = \tau(V, 0) := \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\{x_0, \dots, x_n\}}{(f, J(f))} \right). \tag{2.3.5}$$

The Milnor and the Tjurina algebra and, in particular, their dimensions, play an important role in the study of isolated hypersurface singularities ([Loo84]). Indeed:

- $(V, 0)$ has an isolated singularity if and only if $\mu(V, 0) < \infty$.
- $(V, 0)$ has an isolated singularity if and only if $\tau(V, 0) < \infty$.

Notice that, with the definitions of the Milnor and Tjurina numbers given above, it is clear that $\mu(V,0) \geq \tau(V,0)$.

Furthermore, the Milnor number of a plane curve germ (C,p) is given by

$$\mu(C,0) = 2\delta_0(C) + r_0(C) - 1, \tag{2.3.6}$$

where $r_0(C)$ is the number of branches of C at the origin and $\delta_0(C)$ denotes the delta invariant.

2.4 Monodromy

The word *monodromy* comes from the greek word $\mu o \nu o - \delta \rho o \mu \psi$ and means something like 'uniformly running' or 'uniquely running'. It was first used by Riemann, and it arose in keeping track of the solutions of the hypergeometric differential equation going once around a singular point on a closed path. Since then, monodromy groups have played a substantial role in many areas of mathematics (see the survey by Ebeling [Ebe06] and the references given there).

In this thesis we focus our attention on the classical local geometric monodromy in singularity theory. More precisely we focus on the monodromy operator of an isolated hypersurface singularity. The study of this operator started in 1967 with the proof of the well-known monodromy theorem (see Theorem 2.4.2). We are interested in the monodromy since it encodes a lot of information about the topology of the singularity.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function defining a hypersurface $(V,0)$ with an isolated point at the origin, and let $\epsilon, \delta \in \mathbb{R}, 0 < \delta \ll \epsilon$ be small enough such that the mapping $f|_{f^{-1}(\mathbb{D}^*_\delta)\cap\mathbb{B}_\epsilon} \colon f^{-1}(\mathbb{D}^*_\delta) \cap \mathbb{B}_\epsilon \to \mathbb{D}^*_\delta$ defines a Milnor fibration.

As we have stated before, a fibre $F_t = f^{-1}(t)$ (for $t \in \mathbb{D}^*_\delta$) of this bundle is a Milnor fibre and has the homotopy type of a bouquet of μ n-spheres, where μ denotes the Milnor number. Its only interesting homology group is $H_n(F_t, \mathbb{C})$. Indeed, since $(V,0)$ defines an isolated singularity, the only two non-zero $H_q(F_\delta, \mathbb{C})$ are for $q = 0$ and $q = n$. In particular, $H_n(F_t, \mathbb{C})$ has dimension μ.

Parallel translation along the loop

$$\gamma : [0,1] \to \mathbb{D}_\delta \qquad \gamma(s) = \delta \exp(2\pi i s)$$

yields a well-defined (up to isotopy) diffeomorphism $h : F_\delta \to F_\delta$ called the *geometric monodromy* of the singularity.

The map h is therefore obtained by moving the Milnor fibre in a specific way through the fibre bundle $f^{-1}(\mathbb{D}^*_\delta)$ along the path γ that circles once around the origin in the complex plane; this explains the word *monodromy*.

Definition 2.4.1 (Monodromy operator)**.** The induced homomorphism

$$h_* : H_n(F_\delta, \mathbb{C}) \to H_n(F_\delta, \mathbb{C})$$

is called the *complex algebraic monodromy* of f at $0 \in \mathbb{C}^{n+1}$.

This operator is also sometimes called the *Picard-Lefschetz monodromy operator* (or simply *monodromy operator*) since the consideration of this operator goes back to E. Picard and S. Lefschetz.

The characteristic polynomial $\Delta(t)$ of the monodromy operator is

$$\Delta(t) = \det\left(t \cdot \mathrm{id}_* - h_*; H_n(F_\delta, \mathbb{C})\right), \tag{2.4.1}$$

where id_* denotes the identical transformation of the homology group $H_n(F_\delta, \mathbb{C})$. The roots of the characteristic polynomial $\Delta(t)$ are the eigenvalues of the monodromy operator h_* of the singularity (see [AGV88]).

Theorem 2.4.2 (Monodromy Theorem)**.** *With the notation above, the following holds:*

(a) *The eigenvalues of h_* are roots of unity.*

(b) *The size of the blocks in the Jordan normal form of h_* is at most $(n+1) \times (n+1)$.*

(c) *If $(V, 0)$ is non-smooth, then the size of the Jordan blocks for the eigenvalue 1 is at most $n \times n$.*

2.4.1 Zeta Function of the Monodromy

In this subsection we introduce one of the zeta functions involved in the Monodromy Conjecture: the zeta function of the monodromy.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a non-constant analytic function defining a hypersurface $(V, 0)$ with an isolated point at the origin, and let $\epsilon, \delta \in \mathbb{R}, 0 < \delta \ll \epsilon$ small enough such that the mapping $f|_{f^{-1}(\mathbb{D}^*_\delta)\cap\mathbb{B}_\epsilon}$: $f^{-1}(\mathbb{D}^*_\delta) \cap \mathbb{B}_\epsilon \to \mathbb{D}^*_\delta$ defines a Milnor fibration. Let $F_\delta = f^{-1}(\delta)$ be the corresponding Milnor fibre and let $h_* : H_n(F_\delta, \mathbb{C}) \to H_n(F_\delta, \mathbb{C})$ be the monodromy operator. As mentioned before, since the origin is an isolated singularity of $f^{-1}(0)$, then $H_q(F_\delta, \mathbb{C}) = 0$ for $q \neq 0, n$.

Sometimes instead of the characteristic polynomial of the singularity it is more convenient to use what is called the zeta function of the monodromy transformation h_* of the singularity. The zeta function of the monodromy usually gives more beautiful solutions and, furthermore, it is defined also for non-isolated singularities, whilst the characteristic polynomial becomes practically meaningless. The definition of the zeta function of the monodromy takes into account the zeroth homology of the Milnor fibre $H_0(F_\delta, \mathbb{C})$:

Definition 2.4.3 (Zeta function of the monodromy)**.** The *zeta function of the monodromy* at 0 is the rational function:

$$\zeta_{f,0}(t) := \prod_{q \geq 0} \left\{\det\left(id_* - t h_*; H_q(F_\delta, \mathbb{C})\right)\right\}^{(-1)^{q+1}} \tag{2.4.2}$$

in the complex variable t.

The zeta function of the monodromy thus encodes the monodromy eigenvalues of f. In particular, for an isolated singularity the knowledge of $\zeta_{f,0}(t)$ and of $\Delta(t)$ are equivalent. Indeed, since $H_n(F_\delta, \mathbb{C})$ has dimension μ, where μ is the Milnor number of f:

$$\zeta_{f,0}(t) = \frac{1}{1-t}\left(t^\mu \Delta\left(\frac{1}{t}\right)\right)^{(-1)^{n+1}} \tag{2.4.3}$$

or, equivalently:

$$\Delta(t) = t^{\mu}\left(\frac{t-1}{t}\zeta_{f,0}\left(\frac{1}{t}\right)\right)^{(-1)^{n+1}}. \tag{2.4.4}$$

In [A'C75], N. A' Campo proved a relatively easy formula for the zeta function of the monodromy of f in terms of an embedded resolution of singularities of $f^{-1}(0)$ and its associated numerical data.

Theorem 2.4.4 (A' Campo)**.** *[A'C75] Let $\pi : (X, E) \to (\mathbb{C}^{n+1}, 0)$ be an embedded resolution of $(V, 0)$. Then:*

1. *The zeta function of the monodromy of f at 0 is given by*

$$\zeta_{f,0}(t) = \prod_{i\in S}(1-t^{N_i})^{-\chi(E_i^\circ\cap\pi^{-1}(0))}, \tag{2.4.5}$$

where $E_i, i \in S$, denote the irreducible components of $\pi^{-1}(f^{-1}(0))$, the integers N_i are the multiplicities of E_i in the divisor of $\pi^* f$, and the sets E_i° are defined as $E_i^\circ := E_i \setminus \left(\bigcup_{j\in S\setminus\{i\}} E_j\right)$.

2. *For the isolated singularity $(V, 0)$, the characteristic polynomial of f is*

$$\Delta(t) = \left(\frac{1}{t-1}\prod_{i\in S}(t^{N_i}-1)^{\chi(E_i^\circ\cap\pi^{-1}(0))}\right)^{(-1)^n}. \tag{2.4.6}$$

Additionally,

$$\mu = \dim H_n(F_\delta, \mathbb{C}) = (-1)^n\left[-1+\sum_{i\in S}N_i\chi(E_i^\circ\cap\pi^{-1}(0))\right]. \tag{2.4.7}$$

This formula is often used in practice to compute monodromy eigenvalues.

2.5 Projective plane curves

The previous discussion is concentrated on what happens in a small neighbourhood of the origin in the plane $\mathbb{C}^2$. However, sometimes we wish to think of curves in the large, and then it is more convenient to work in the projective plane.

Let $[x : y : z]$ denote the coordinates of a point in $\mathbb{P}^2(\mathbb{C})$. Let $F(x, y, z) \in \mathbb{C}[x, y, z]$ be a homogeneous reduced polynomial, and let $V(F)$ denote the zero set of F. Then $C = V(F) \subset \mathbb{P}^2(\mathbb{C})$ is called a *plane algebraic curve.* If F is a polynomial of degree d, we say that the curve C has degree d.

2.5.1 Properties and invariants

Now that we have introduced some local properties of plane curves and their singular points, we next want to study complex projective algebraic plane curves. We will define global invariants of such curves from the local invariants of their singular points that we have explored earlier. We refer to the books of Wall [Wal04] and Brieskorn and Knörrer [BK86] for further details.

For curves in the projective plane, the most basic invariant is the degree of the defining equation. This gives a qualitative bound for the possible complexities of the curve, and also of its singularities.

Since we are now studying reduced curves C rather than just germs at a point, we have to introduce the following notations in order to specify the singular point that we are dealing with (we will omit C from the notation if it is clear from the context):

- $m_p(C)$ for the multiplicity of (the germ of) C at the point $p \in C$;
- $\mu_p(C)$ for the Milnor number of C at the point p;
- $r_p(C)$ for the number of branches of C at p;
- $\delta_p(C)$ for the delta invariant of C at p.

We also write $\mu(C) := \sum_{p\in \mathrm{Sing}(C)} \mu_p$, where $\mathrm{Sing}(C)$ is the set of all singular points of C.

Let χ denote the Euler characteristic with compact support. Then:

Theorem 2.5.1. *[Wal04] If C is a reduced curve of degree d in $\mathbb{P}^2(\mathbb{C})$, then*

$$\chi(C) = 3d - d^2 + \mu(C). \tag{2.5.1}$$

Proposition 2.5.2. *[Wal04] A non-singular curve C of degree d is connected and has genus*

$$g(C) = \frac{1}{2}(d-1)(d-2). \tag{2.5.2}$$

For the next property, recall that the normalisation $n : \tilde{C} \to C$ of the curve C is defined by resolving the singularities of C. When C is a curve in the projective space, the normalisation $\tilde{C}$ itself carries important information. For any resolution, the strict transform of C is a smooth curve $\tilde{C}$, and we have a projection $n : \tilde{C} \to C$, which is (up to isomorphism) independent of the resolution. This projection, or the curve $\tilde{C}$, is called the *normalisation* of C. As far as the topology is concerned, the map n is bijective except at singular points, and over a singular point of C with r branches there are just r points of $\tilde{C}$.

Lemma 2.5.3. *[BK86] Let C be a plane algebraic curve and let $n : \tilde{C} \to C$ be the resolution of singularities. Then the connected components of C correspond bijectively to the irreducible components of C.*

Theorem 2.5.4. *[Wal04] The normalisation $\tilde{C}$ of a reduced plane curve C of degree d has Euler characteristic*

$$\chi(\tilde{C}) = 3d - d^2 + \sum_{p \in \mathrm{Sing}(C)} (\mu_p + r_p - 1). \tag{2.5.3}$$

In particular, if C is irreducible, then $\tilde{C}$ is connected, of genus

$$g(\tilde{C}) = \frac{1}{2}\left[(d-1)(d-2) - \sum_{p \in \mathrm{Sing}(C)} (\mu_p + r_p - 1)\right]. \tag{2.5.4}$$

Theorem 2.5.5 (Bézout's theorem)**.** *[BK86] For plane algebraic curves C and D of degrees* $\deg C$ *and* $\deg D$ *which do not have any common component, we have that*

$$\sum_{p \in C \cap D} (C.D)_p = \deg C \cdot \deg D. \tag{2.5.5}$$

In particular, for the tangent line T_pC of the curve C at a point $p \in C$ we have that

$$i(C)_p \leq \deg C, \tag{2.5.6}$$

where $i(C)_p := (C.T_pC)_p$ is defined as the local intersection number of C with its tangent line at the point p.

2.5.2 The dual curve

Let C be an algebraic curve in $\mathbb{P}^2(\mathbb{C})$. At each non-singular point $p \in C$ there is a tangent line to C, denoted by L_p, which determines a point (which we denote by the same symbol) in the dual projective space $\mathbb{P}^2(\mathbb{C})^\vee$. Consider the locus of these points as p varies: its closure is another algebraic curve $C^\vee$, called the *dual curve* of C. The singularities of C and $C^\vee$ are related in the following way:

Theorem 2.5.6. *[Wal04, Theorem 7.4.1] Let C be a curve with a unique tangent at the point $p \in C$. Then*

$$i(C)_p = i(C^\vee)_p = m_p(C) + m_{p^\vee}(C^\vee). \tag{2.5.7}$$

Finally, the following property will be useful:

Theorem 2.5.7. *[Wal04, Theorem 7.4.6] There is an isomorphism between the trees of infinitely near points for C and $C^\vee$, which preserves the multiplicity of each branch at each infinitely near point (except at p itself), and is such that a point $O_s^\vee$ is proximate to $O_0^\vee$ if and only if O_s belongs to the tangent line L to C at O_0; and dually a point O_s is proximate to O_0 if and only if $O_s^\vee$ belongs to the tangent line $L^\vee$ to $C^\vee$ at $O_0^\vee$.*

For further details, see the book by Wall [Wal04].

2.6 Coverings and Kummer covers

Kummer covers are a very useful tool in order to construct complicated algebraic curves starting from simple ones. In order to introduce Kummer covers and their properties, we start by recalling some definitions and interesting properties of topological coverings. The definitions and results below appear in much more detail in the books by Hatcher [Hat02], Khovanskii [Kho13], Manetti [Man15], Massey [Mas81] and Szamuely [Sza09].

Definition 2.6.1 (Covering)**.** Continuous maps $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ from topological spaces Y_1 and Y_2, respectively, to a topological space X are called *left equivalent* if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $f_1 = f_2 \circ h$.

A topological space Y together with a projection $f : Y \to X$ to a topological space X is called a *covering with fibre* D over X (where D is a discrete set) if for each point $c \in X$ there exists an open neighborhood U such that the projection map of $U \times D$ onto the first factor is left equivalent to the map $f : Y_U \to U$, where $Y_U = f^{-1}(U)$.

A triple $f : (Y, y) \to (X, x)$ consisting of spaces with marked points (X, x) and (Y, y) and a map f is called a *covering with marked points* if $f : Y \to X$ is a covering and $f(y) = x$. A covering $f : (Y, y) \to (X, x)$ induces the homomorphism $f_* : \pi_1(Y, y) \to \pi_1(X, x)$ defined by $f_*([\alpha]) = [f \circ \alpha]$.

If $f : (Y, y) \to (X, x)$ is a covering space, then the cardinality of the set $f^{-1}(x)$ is locally constant over X. Hence if X is connected, then $|f^{-1}(x)|$ is constant as x ranges over all of X. It is called the *number of sheets* of the covering.

The following theorem holds for every connected, locally connected and locally simply connected topological space (X, x).

Theorem 2.6.2 (Classification of coverings with marked points)**.** *[Kho13]*

1. *For every subgroup G of the fundamental group of the space X there exist a connected space (Y, y) and a covering over (X, x) by the covering space (Y, y) such that the image of the fundamental group of the space (Y, y) coincides with the subgroup G.*
2. *Two coverings over (X, x) by connected covering spaces (Y, y_1) and (Y, y_2) are equivalent if the images of the fundamental group of these spaces in the fundamental group of (X, x) coincide.*

The previous theorem shows that coverings with marked points over a topological space X with a marked point x considered up to left equivalence are classified by subgroups G of the fundamental group $\pi_1(X, x)$. Let us discuss the correspondence between coverings with marked points and subgroups of the fundamental group. Let $f : (Y, y) \to (X, x)$ be a covering that corresponds to the subgroup $G \subset \pi_1(X, x)$ and let $F = f^{-1}(x)$ denote the fibre over the point x.

Lemma 2.6.3. *[Kho13] The fibre F is in bijective correspondence with the right cosets of the group $\pi_1(X, x)$ modulo the subgroup G. If a right coset h corresponds to a point c of the fibre F, then the group hGh^{-1} corresponds to the covering $f : (Y, c) \to (X, x)$ with marked point c.*

2.6.1 Covering transformations and monodromy

Lemma 2.6.4. *[Kho13] For each curve $\gamma : [0, 1] \to X$, where $\gamma(0) = x$ and for each point $y \in Y$ that is projected to x, i.e., $f(y) = x$, there exists a unique curve $\tilde{\gamma} : [0, 1] \to Y$ such that $\tilde{\gamma}(0) = y$ and $f \circ \tilde{\gamma} = \gamma$.*

Consider a covering $f : Y \to X$. A homeomorphism $h : Y \to Y$ is called a *deck transformation* or *covering transformation* of this covering if $f = f \circ h$. Deck transformations form a group, $\mathrm{Deck}(f)$. A covering is called *normal* or *Galois* if its group of deck transformations acts transitively on each fibre of the covering.

Lemma 2.6.5. *[Kho13] A covering is normal if and only if it corresponds to a normal subgroup H of the fundamental group $\pi_1(X, x)$. For this normal subgroup, the group of deck transformations is isomorphic to the quotient group $\pi_1(X, x)/H$.*

The fundamental group $\pi_1(X, x)$ acts on the fibre $f^{-1}(x)$ of the covering $f : (Y, y) \to (X, x)$. We now define this action.

Let γ be a path in the space X that originates and terminates at the point x. For every point $c \in f^{-1}(x)$, let $\tilde{\gamma}_c$ denote the lift of the path γ to Y such that $\tilde{\gamma}(0) = c$. The map $S_\gamma : f^{-1}(x) \to f^{-1}(x)$ that takes the point c to the point $\tilde{\gamma}_c(1) \in f^{-1}(x)$ belongs to the group $\mathrm{Bij}(f^{-1}(x))$ of bijections from the set $f^{-1}(x)$ to itself. The map S_γ depends only on the homotopy class of the path γ, that is, on the element of the fundamental group $\pi_1(X, x)$ represented by the path γ.

With this notation, the action of the fundamental group $\pi_1(X, x)$ on the fibre $f^{-1}(x)$ is defined as:

$$\begin{array}{ccc} \pi_1(X,x) \times f^{-1}(x) & \to & f^{-1}(x) \\ ([\gamma], y) & \mapsto & S_\gamma(y) \end{array}$$

The homomorphism

$$Mon(f) : \begin{array}{ccc} \pi_1(X,x) & \to & \mathrm{Bij}(f^{-1}(x)) \\ [\gamma] & \mapsto & S_\gamma \end{array}$$

is called the *monodromy homomorphism*, and the image of the fundamental group in the group $\mathrm{Bij}(f^{-1}(x))$ is called the *monodromy group* of the covering $f : (Y, y) \to (X, x)$.

Theorem 2.6.6. *[Żołądek02] Let $f : (Y, y) \to (X, x)$ be a covering map. We have the isomorphism*

$$\mathrm{Mon}(f) \cong \frac{\pi_1(X,x)}{\bigcap_{j=1}^n \mathrm{Stab}(b_j)},$$

where $f^{-1}(x) = \{b_1, \ldots, b_n\}$ and $\mathrm{Stab}(b_j)$ is the stabiliser of b_j, and

$$\mathrm{Deck}(f) \cong \frac{\mathrm{Norm}(f_*(\pi_1(Y,y))}{f_*(\pi_1(Y,y))},$$

where $\mathrm{Norm}(f_*(\pi_1(Y, y))$ is the normaliser of $f_*(\pi_1(Y, y))$ in $\pi_1(X, x)$.

In particular, the equality $\mathrm{Deck}(f) = \mathrm{Mon}(f)$ holds if and only if $f_*(\pi_1(Y, b))$ is a normal subgroup of $\pi_1(X, x)$ or, equivalently, if f is a Galois covering.

If $f : (Y, y) \to (X, x)$ is a Galois covering, we denote $\mathrm{Gal}(f) := \mathrm{Deck}(f) = \mathrm{Mon}(f)$.

Remark 2.6.7. Theorem 2.6.6 above also holds when the fibre $f^{-1}(x)$ is not a finite set.

2.6.2 The induced covering space over a subspace

Let $f : (Y, y) \to (X, x)$ be a covering space of (X, x), let $A \subset X$ be a subspace of X which is connected and locally arcwise connected, and let B be an arc component of $f^{-1}(A)$. Then $\tilde{f} := f|_B : B \to A$ is a covering space of A. We are interested in studying under which conditions is $f^{-1}(A)$ connected (see [Mas81]).

Let $b \in B$ be such that $b \in f^{-1}(x)$, and let $i : A \to X$ denote the inclusion map, as shown in the diagram below.

$$\begin{array}{ccc} \pi_1(B,b) & \xrightarrow{\tilde{f}_*} & \pi_1(A,x) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \pi_1(Y,b) & \xrightarrow{f_*} & \pi_1(X,x) \end{array}$$

Then

$$\tilde{f}_*(\pi_1(B,b)) = i_*^{-1}(f_*(\pi_1(Y,b))).$$

Proposition 2.6.8. *[Mas81] Under the above hypotheses, $f^{-1}(A)$ is connected (i.e., $B = f^{-1}(A)$) if and only if the subgroup $i_*(\pi_1(A,x))$ meets every coset of the subgroup $f_*(\pi_1(Y,b))$.*

We are going to present the consequences of this result for the case of normal or Galois coverings. Assume that $f : (Y, y) \to (X, x)$ is a normal covering of X. Then $\tilde{f} : (B, b) \to (A, x)$ is a normal covering of A. Therefore, if $f_*(\pi_1(Y,b))$ is a normal subgroup of $\pi_1(X,x)$ then $i_*^{-1}(f_*(\pi_1(Y,b)))$ is a normal subgroup of $\pi_1(A,x)$.

Note that the group of automorphisms $\mathrm{Deck}(\tilde{f})$ may be considered a subgroup of the group of automorphisms $\mathrm{Deck}(Y)$. In this case, $f^{-1}(A)$ is connected if and only if the homomorphism of quotient groups

$$\frac{\pi_1(A,x)}{\tilde{f}_*(\pi_1(B,b))} \to \frac{\pi_1(X,x)}{f_*(\pi_1(Y,b))} \cong \mathrm{Mon}(f)$$

induced by i_* is an epimorphism (it is always a monomorphism).

Additionally, the number of connected components of $f^{-1}(A)$ is given by the index of the subgroup $\frac{\pi_1(A,x)}{\tilde{f}_*(\pi_1(B,b))}$ in $\frac{\pi_1(X,x)}{f_*(\pi_1(Y,b))}$, i.e.,

$$\text{Number of connected components of } f^{-1}(A) = |\,\mathrm{Mon}(f) : \mathrm{Im}(\tilde{p})|. \qquad (2.6.1)$$

$$\begin{array}{ccccc} \pi_1(B,b) & \xrightarrow{\tilde{f}_*} & \pi_1(A,x) & \xrightarrow{\tilde{p}} & \frac{\pi_1(A,x)}{\tilde{f}_*(\pi_1(B,b))} \\ \downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow \\ \pi_1(Y,b) & \xrightarrow{f_*} & \pi_1(X,x) & \xrightarrow{p} & \frac{\pi_1(X,x)}{f_*(\pi_1(Y,b))} \end{array}$$

2.6.3 Branched coverings

Definition 2.6.9 (Branched covering)**.** [Cog11] Let M be an $m-$dimensional (connected) complex manifold. A *branched covering* of M is an m-dimensional irreducible normal complex space Y together with a surjective holomorphic map $\pi : Y \to M$ such that:

- every fibre of π is discrete in Y;
- the set $R_\pi := \{y \in Y | \pi^* : \mathcal{O}_{\pi(y),M} \to \mathcal{O}_{y,Y}$ is not an isomorphism$\}$, called the *ramification locus*, and $B_\pi = \pi(R_\pi)$, called the *branched locus*, are hypersurfaces of Y and M, respectively;
- the map $\pi| : Y \setminus \pi^{-1}(B_\pi) \to M \setminus B_\pi$ is an unramified (topological) covering;
- for any $q \in M$ there is a connected open neighborhood $W^q \subset M$ such that for every connected component U of $\pi^{-1}(W)$:
    1. $\pi^{-1}(q) \cap U$ has only one element, and
    2. $\pi|_U : U \to W$ is surjective and proper.

A branched cover $\pi : Y \to M$ will be called *Galois*, if $\pi_*(\pi_1(Y \setminus \pi^{-1}(B_\pi)))$ is a normal subgroup of $\pi_1(M \setminus B_\pi)$.

In order to introduce the key concept of meridian, let M be a complex manifold, B' an irreducible component of a hypersurface $B \subset M$, and $b \in B'$ a smooth point on B. By definition, this means that there exists an open neighborhood U of b in M and a holomorphic function f on U such that $B \cap U = \{z \in U : f(z) = 0\}$. By the Implicit Function Theorem, there exists a change of coordinates such that U can be chosen to be $V \times B$, where V is a polydisk and $B \cap U = V \times \{0\}$. Hence the point $b \in B \cap U$ will have coordinates $b = (b_0, 0)$. Let $\gamma_b = \{b_0\} \times \{\exp(2\pi i\lambda)\}$ be a closed path centred at $\tilde{b} = (b_0, 1)$.

Definition 2.6.10 (Meridian)**.** [Cog11] Under the above conditions, a closed path in $\pi_1(M \setminus B, q_0)$ is called a *meridian* of B if there is a representative γ in its homotopy class that can be written as

$$\gamma = \alpha\gamma_b\alpha^{-1},$$

where $\alpha \in \pi_1(M \setminus B, q_0, \tilde{b})$ for a certain $b \in B$ as above.

Proposition 2.6.11. *[Cog11] Any two meridians, say $\gamma_1, \gamma_2 \in \pi_1(M \setminus B, q_0)$ of the same irreducible component B are conjugated, that is, $\gamma_2 = \omega\gamma_1\omega^{-1}$ for a certain $\omega \in \pi_1(M \setminus B, q_0)$. Moreover, the conjugacy class of a meridian coincides with the set of homotopy classes of meridians around the same irreducible component.*

Proposition 2.6.12. *[Don11] Let Y and M be connected Riemann surfaces and $\pi : Y \to M$ a non-constant holomorphic map. For each point y in Y, there is a unique integer $e_y \geq 1$ such that we can find charts around y in Y and $\pi(y)$ in M in which π is represented by the map $z \mapsto z^{e_y}$.*

The integer $e_y \geq 1$ is called the *ramification index* of $y \in Y$.

Proposition 2.6.13 (Riemann-Hurwitz formula)**.** *[Sza09] Let $f : Y \to M$ be a holomorphic map of compact Riemann surfaces having degree d as a branched cover. The Euler characteristics $\chi(M)$ and $\chi(Y)$ of M and Y are related by the formula*

$$\chi(Y) = d \cdot \chi(M) - \sum_y (e_y - 1), \tag{2.6.2}$$

where the sum is over the branch points of f and e_y is the ramification index at the branch point $y \in Y$.

Equivalently,

$$2g(Y) - 2 = d \cdot (2g(M) - 2) + \sum_y (e_y - 1). \tag{2.6.3}$$

2.6.4 Kummer covers

Kummer covers are a very useful tool in order to construct complicated algebraic curves starting from simple ones.

Given $k \in \mathbb{N}, k \geq 1$, a *Kummer cover* is a map $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by

$$\pi_k([x : y : z]) := [x^k : y^k : z^k]. \tag{2.6.4}$$

Since the fundamental group $\pi_1(\mathbb{P}^2 \setminus V(xyz)) \cong \mathbb{Z} \times \mathbb{Z}$ is abelian, Kummer covers are finite Galois unramified covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\mathrm{Gal}(\pi_k) \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$. Therefore, many topological properties of the new curves can be obtained: Alexander polynomial, fundamental group, characteristic varieties and so on (see [Art94, AC98, Cog99, Ulu01, Hir92, CK12, ACO14, Lin12] for papers using these techniques).

Example 2.6.14. In [Ulu01], A. M. Uludağ constructed new examples of Zariski pairs using former ones and Kummer covers. He also used the same techniques to construct infinite families of curves with finite non-abelian fundamental groups.

Example 2.6.15. In [Cog99, Hir92], the Kummer covers allow to construct curves with *many cusps* and extremal properties for their Alexander invariants. These ideas are pushed further in [CK12], where the authors find Zariski triples of curves of degree 12 with 32 ordinary cusps (distinguished by their Alexander polynomial).

Example 2.6.16. Within the same ideas N. Lindner [Lin12] constructed an example of a cuspidal curve C' of degree 12 with 30 cusps and Alexander polynomial $t^2 - t + 1$. To this end, he started with a sextic C_0 with 6 cusps, admitting a toric decomposition. He pulled back C_0 under a Kummer map $\pi_2 : \mathbb{P}^2 \to \mathbb{P}^2$ ramified above three inflectional tangents of C_0. Since the sextic is of torus type, then same holds for the pullback. N. Lindner showed that the Mordell-Weil lattice has rank 2 and that the Mordell-Weil group contains $A_2(2)$.

A systematic study of Kummer covers of projective plane curves has been done by E. Artal, J. I. Cogolludo and J. Ortigas in [ACO14, §5]. Some of the most relevant results that appear in this survey are collected below.

Let C be a (reduced) projective curve of degree d of equation $F_d(x, y, z) = 0$ and let $C_k := \pi_k^{-1}(C)$ be its transform by the Kummer cover π_k, $k \geq 1$.

Note that C_k is a projective curve of degree dk of equation $F_d(x^k, y^k, z^k) = 0$. Another obvious remark is that if C is reducible so is C_k. The converse is not true as we will see in section 5.6, see Theorems 5.6.1, 5.6.2 and 5.6.3.

Definition 2.6.17. [ACO14] Let $p := [x_0 : y_0 : z_0]$ be a point of $\mathbb{P}^2$.

- We say that p is a point of *type* $(\mathbb{C}^*)^2$ (or simply of *type* 2) if $x_0 y_0 z_0 \neq 0$.

- If $x_0 = 0$ but $y_0 z_0 \neq 0$ the point is said to be of *type* $\mathbb{C}^*_x$ (types $\mathbb{C}^*_y$ and $\mathbb{C}^*_z$ are defined accordingly). Such points will also be referred to as *type* 1 points. The corresponding line (either $L_X := \{x = 0\}$, $L_Y := \{y = 0\}$, or $L_Z := \{z = 0\}$) where a type-1 point lies on will be referred to as their *axis*.

- The remaining points $p_x := [1 : 0 : 0]$, $p_y := [0 : 1 : 0]$ and $p_z := [0 : 0 : 1]$ will be called *vertices* (or type 0 points) and their axes are the two lines (either L_X, L_Y, or L_Z) they lie on.

Remark 2.6.18. The map π_k is a finite surjective morphism of degree k^2.

Remark 2.6.19. [ACO14] Note that a point of type ℓ ($\ell = 0, 1, 2$) in $\mathbb{P}^2$ has exactly k^ℓ preimages under π_k. It is also clear that the local type of C_k at any two points on the same fibre are analytically equivalent.

Since the cardinality of the fibres drops at $V(xyz)$, this set is called the *ramification locus* of π_k. Outside the ramification locus, i.e., when restricted to $\mathbb{P}^2 \setminus V(xyz)$ on both sides, $\pi_k : \mathbb{P}^2 \setminus V(xyz) \to \mathbb{P}^2 \setminus V(xyz)$ is a covering map of degree k^2 with respect to the Euclidean topology. Furthermore, the corresponding field extension is a Kummer extension with Galois group $(\mathbb{Z}/k\mathbb{Z})^2$. For this reason, the map π_k is called a Kummer cover.

Let Δ be the set of points where C intersects $V(xyz)$ with multiplicity at least two.

Proposition 2.6.20. *[Lin12] Let $C = V(F_d)$ be a complex projective curve of degree d. Suppose that C intersects $V(xyz)$ in smooth points only and that C does not contain any of the points p_x, p_y, p_z. Then the following relation between the singular loci* $\operatorname{Sing}(C)$ *and* $\operatorname{Sing}(C_k)$ *holds:*

$$\operatorname{Sing}(C_k) = \pi_k^{-1}(\operatorname{Sing}(C) \cup \Delta). \tag{2.6.5}$$

Lemma 2.6.21. *[ACO14] Let $p \in \mathbb{P}^2$ be a point of type ℓ and $q \in \pi_k^{-1}(p)$. Then there exist local coordinates (u_0, v_0) and (u_1, v_1) centred at q and p, respectively, such that:*

1. *If $\ell = 2$, then $(u_1, v_1) = \pi_k(u_0, v_0) = (u_0, v_0)$.*
2. *If $\ell = 1$, then $(u_1, v_1) = \pi_k(u_0, v_0) = (u_0^k, v_0)$, where $u_0 = 0$ and $u_1 = 0$ are the local equations (at q and p, respectively) of the axes containing the points.*
3. *If $\ell = 0$, then $(u_1, v_1) = \pi_k(u_0, v_0) = (u_0^k, v_0^k)$, where $u_i = 0$ and $v_i = 0$ are the local equations of the axes containing p and q.*

The singularities of C_k are described in the next proposition.

Proposition 2.6.22. *[ACO14] Let $p \in \mathbb{P}^2$ be a point of type ℓ and $q \in \pi_k^{-1}(p)$. One has the following:*

(1) *If $\ell = 2$, then (C, p) and (C_k, q) are analytically isomorphic.*

(2) *If $\ell = 1$, then (C_k, q) is a singular point of type 1 if and only if $m > 1$, where $m := (C \cdot L)_p$ and L is the axis of p.*

(3) *If $\ell = 0$, then (C_k, q) is a singular point.*

Remark 2.6.23. Using Proposition 2.6.22 (1), if $\mathrm{Sing}(C) \subset \{xyz = 0\}$ then $\mathrm{Sing}(C_k) \subset \{xyz = 0\}$.

Example 2.6.24. [ACO14] In some cases, we can be more explicit about the singularity type of (C_k, q). If p is of type 1, (C, p) is smooth and $m := (C \cdot L)_p$, then (C_k, q) has the same topological type as $u_0^k - v_0^m = 0$. In particular, if $m = 2$, then (C_k, q) is of type $\mathbb{A}_{k-1}$.

Note that it is always possible to find a change of coordinates such that $\Delta = \emptyset$. For example, if C is a cuspidal curve of degree d with n cusps, and $\Delta = \emptyset$, then 2.6.20 and 2.6.22 state that $\pi_k^{-1}(C)$ is a cuspidal curve of degree kd with nk^2 cusps. So the Kummer cover π_k enables to produce cuspidal curves with a high number of cusps. Using 2.6.24, if $\Delta \neq \emptyset$ and the intersection with $V(xyz)$ is sufficiently nice, then one can put even more cusps into $\pi_k^{-1}(C)$.

In order to describe better the singular points of type 0 and 1 of C_k we will introduce some notation. Let $p \in \mathbb{P}^2$ be a point of type $\ell = 0, 1$ and $q \in \pi_k^{-1}(p)$ a singular point of C_k. Denote by μ_p (resp. μ_q) the Milnor number of C at p (resp. C_k at q). Since $\ell = 0, 1$, then p and q belong to either exactly one or two axes. If p and q belong to an axis L, then $m_p^L := (C \cdot L)_p$ (analogous notation for q). More specific details about singular points of types 0 and 1 can be described as follows.

Proposition 2.6.25. *[ACO14] Under the above conditions and notation one has the following properties:*

(1) *For $\ell = 1$, p belongs to a unique axis (L), and:*

   (a) $\mu_q = k\mu_p + (m_p^L - 1)(k - 1)$.

   (b) If (C, p) is locally irreducible and $r := \gcd(k, m_p^L)$, then (C, q) has r irreducible components which are analytically isomorphic to each other.

(2) *For $\ell = 0$, p belongs to exactly two axes (L_1 and L_2), and:*

   (a) $\mu_q = k^2(\mu_p - 1) + k(k - 1)(m_p^{L_1} + m_p^{L_2}) + 1$.

   (b) If (C, p) is locally irreducible and $r := \gcd(k, m_p^{L_1}, m_p^{L_2})$, then (C, q) has kr irreducible components which are analytically isomorphic to each other.

2.7 Pencils of plane curves

Our last counterexample to the conjectures of A. Dimca and G. Sticlaru is constructed as a general element of the unique pencil associated with a certain rational unicuspidal plane curve. In this section we recall some basic results on pencils of plane curves (see [BY14] and [GH78] for further details).

A *pencil of plane curves* in $\mathbb{P}^2$ is a line in the projective space of homogeneous polynomials of $\mathbb{C}[x, y, z]$ of some fixed degree d. In other words, a pencil of plane curves is a linear system in $\mathbb{P}^2$ of dimension 1.

Any two distinct plane curves of the same degree generate a pencil, and conversely a pencil is determined by any two of its curves C_1 and C_2.

An arbitrary curve C in the pencil (called a *fibre*) is defined by

$$C = aC_1 + bC_2,$$

where $[a:b] \in \mathbb{P}^1$.

Every two fibres in a pencil $\Lambda = \left\{aC_1 + bC_2 : [a:b] \in \mathbb{P}^1\right\}$ intersect in the same set of points, namely

$$\mathrm{Bs}(\Lambda) = C_1 \cap C_2,$$

called the *base* of the pencil. If fibres do not have a common component (called a *fixed component*), then the base is a finite set of points.

We say that a *generic element* of the pencil Λ has a property P if the set of elements in Λ that do not have the property P is contained in a subvariety of strictly smaller dimension, i.e., the set of elements that do not have that property is finite.

A relevant property about linear systems is the following:

Theorem 2.7.1 (Bertini's Theorem)**.** *[GH78] On $\mathbb{P}^n$, if a linear system has no fixed components, then a generic element has no singular points away from the base locus of the system.*

2.7.1 Resolving base points of pencils of curves

Let Λ be a pencil of plane curves $\{\Gamma_t : t \in \mathbb{P}^1\}$ in $\mathbb{P}^2$ which has no fixed components, and let $p \in \mathbb{P}^2$ be a base point of the pencil (so that p lies in the intersection of all curves in the pencil). According to [MW01], we write

$$m_p(\Lambda) := \min\{m_p(\Gamma_t) : t \in \mathbb{P}^1\}.$$

Then $m_p(\Lambda) = m_p(\Gamma_t)$ for all but finitely many t.

Let $\pi : X_1 \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at p and E the exceptional divisor of π. Then the *total transform* of Λ, $\{\pi^{-1}(\Gamma_t) : t \in \mathbb{P}^1\}$, is a linear system with $m_p(\Lambda)E$ as a fixed component. Removing this component gives the *strict transform* $\Lambda_1 = \{\pi^{-1}(\Gamma_t) - m_p(\Lambda)E : t \in \mathbb{P}^1\}$ of Λ, which has no fixed components.

Let $q \in X_r$ be an infinitely near point of p obtained by a finite sequence of blowings-up $\pi_i : X_{i+1} \to X_i$, where $X_0 = \mathbb{P}^2$. Set $m_q(\Lambda) := m_q(\Lambda_r)$, where Λ_r is the strict transform of Λ on X_r. We say that q is a *base point* of the pencil Λ if $m_q(\Lambda) > 0$.

Inductively blow up at a base point of the pencil, take the strict transform of the pencil and continue. Since for two members of the pencil with multiplicity $m_p(\Lambda)$ at p, blowing up p reduces the total intersection number there by $m_p(\Lambda)^2$, the blow up reduces the total intersection number of these two, and hence of any two members of the pencil by $m_p(\Lambda)^2$. As the original intersection number is finite, we may continue till no base points remain.

In the end, we have a smooth projective surface S with a well defined morphism $\Psi_\Lambda : S \to \mathbb{P}^1$ given by the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow{\ \pi\ } & S \\ {\scriptstyle \Phi_\Lambda}\downarrow & \swarrow {\scriptstyle \Psi_\Lambda} & \\ \mathbb{P}^1 & & \end{array} \tag{2.7.1}$$

whose fibres $S_t = \Psi_\Lambda^{-1}(t)$ project to the curves Γ_t of the original pencil. π is the birational morphism given by the resolution of the pencil and $\Phi_\Lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is the rational map defined by Λ.

Dicriticals

Let us consider the exceptional locus $\mathcal{E} = exc(\pi) \subset S$ of π. A curve $E \subset S$ is *vertical* if $\Psi_\Lambda(E)$ is a point, so that it lies in a fibre, and it is called *horizontal* otherwise (see [DM14]).

The horizontal curves included in $\mathcal{E}$ are called the *dicriticals* of diagram (2.7.1).

If $E \subseteq \mathcal{E}$ is a dicritical of (2.7.1) then the composition $E \hookrightarrow S \xrightarrow{\Psi_\Lambda} \mathbb{P}^1$ is a surjective morphism $f_E : \mathbb{P}^1 \to \mathbb{P}^1$; the positive integer $\deg(f_E)$ is called the *degree of the dicritical* E.

Suppose that diagram (2.7.1) has $s \geq 0$ dicriticals, of degrees $d_1, \dots, d_s$ respectively. Then the number s and the unordered s-tuple $[d_1, \dots, d_s]$ are uniquely determined by Λ, i.e., they are independent of the choice of a diagram (2.7.1) which resolves the points of indeterminacy of Φ_Λ. So it makes sense to speak of the *number of dicriticals* of Λ and of the degrees of these dicriticals.

Chapter 3

Monodromy Conjecture for the Hessian differential form

3.1 Introduction

This chapter deals with the Monodromy Conjecture of J. Denef and F. Loeser, which relates poles of the Igusa, motivic or topological zeta function to monodromy eigenvalues. The conjecture was verified by F. Loeser for plane curves. However in higher dimensions there is not so much known. The conjecture was later extended for topological zeta functions associated with arbitrary differential forms by W. Veys, and afterwards A. Némethi and W. Veys introduced the set of *allowed* differential forms. They proved that for germs of plane curve singularities these allowed differential forms exist, and that this set contains the standard differential form as well.

In this context it is natural to ask if there exists any other naturally defined (even associated canonically to the germ f) differential form which is allowed, apart from the standard differential form. A natural choice might be the Hessian form. A. Melle asked whether the poles of the corresponding topological zeta function would provide eigenvalues of the monodromy, in the same way as the standard form does.

In this chapter we show that the local topological zeta function of a germ associated with its Hessian differential form does not satisfy the Monodromy Conjecture, and this fact implies that the Hessian form is not an allowed differential 2-form. This result was proved by the author in [Gor18].

3.2 The p-adic Igusa zeta function

We have already presented the zeta function of the monodromy in Section 2.4.1. Before stating the Monodromy Conjecture, we will introduce its other main ingredient, namely the topological zeta function and its precursor, the Igusa local zeta function.

Zeta functions can be attached to several mathematical objects like fields, groups, algebras, functions and dynamical systems. Typically, zeta functions encode relevant arithmetic, algebraic, geometric or topological information about the original object [CCM$^+$12].

Since the 19^{th} century many zeta functions have been defined and studied, such as the Riemann zeta function, the Weil zeta functions and the Igusa zeta function. More concretely, local zeta functions were introduced by A. Weil in the 1960s and have been extensively studied by J.-I. Igusa, J. Denef and F. Loeser, among others. More recently, using ideas of motivic integration due to M. Kontsevich, a generalisation of these functions, called motivic zeta functions, was introduced by J. Denef and F. Loeser. We will explore some aspects of the topological zeta function and the zeta function of the monodromy, which seem to be related, as the Monodromy Conjecture predicts.

Let us begin with the definition of the p-adic Igusa zeta function. For more details see, for instance, [Bor13] and [Vey01a].

The p-adic Igusa zeta function was introduced by A. Weil in 1965 and it was first studied by J.-I. Igusa. Such a p-adic Igusa zeta function associated with a function germ is closely related to the following number theoretic problem:

Let $f(x)$ be a polynomial over $\mathbb{Z}$ in the variables $x_0, \ldots, x_n$. For any $d \in \mathbb{N} \setminus \{0\}$ we want to count the number of solutions $M(d)$ (in $\mathbb{Z}/d\mathbb{Z}$) of the congruence $f(x) \equiv 0 \mod d$. Thanks to the Chinese Remainder Theorem this problem is simplified to the case that d is a prime power. Indeed, if $d = p_1^{k_1} \ldots p_r^{k_r}$, is the prime factorisation of m, then the theorem states that

$$\mathbb{Z}/d\mathbb{Z} \cong \left(\mathbb{Z}/p_1^{k_1}\mathbb{Z}\right) \times \ldots \times \left(\mathbb{Z}/p_r^{k_r}\mathbb{Z}\right) \tag{3.2.1}$$

and therefore $M(d) = M\left(p_1^{k_1}\right) \cdot \ldots \cdot M\left(p_r^{k_r}\right)$.

Henceforth fix a prime number p and denote by M_i the number of solutions (in $\mathbb{Z}/p^i\mathbb{Z}$) of $f(x) \equiv 0 \mod p^i$. As two series are equal if and only if all their coefficients coincide, a series is a good tool to encode discrete information, so these numbers can be arranged in a Poincaré series:

$$P(t) := \sum_{i=0}^{\infty} M_i \left(p^{-(n+1)}t\right)^i. \tag{3.2.2}$$

Since the coefficients $\frac{M_i}{p^{i(n+1)}}$ are bounded by one, the series above defines a holomorphic function P on the open unit disk in the complex plane. In 1966 Z. I. Borevich and I. R. Shafarevich conjectured that the formal power series P was in fact a rational function and thus allowed a meromorphic continuation to $\mathbb{C}$. This fact was first proved by J.-I. Igusa in 1974 and later by J. Denef, who used a completely different approach (see [Bor13] and the references therein). In both proofs they expressed $P(t)$ as an integral over the p-adic integers.

Notice that since $\mathbb{Z}/p^i\mathbb{Z} \cong \mathbb{Z}_p/p^i\mathbb{Z}_p$ we can consider f as a polynomial $f \in \mathbb{Z}_p[x_0, \ldots, x_n]$.

Let $|\cdot|$ denote the standard absolute value on the field $\mathbb{Q}_p$ of p-adic numbers and $|dx|$ the Haar measure on $\mathbb{Q}_p^{n+1}$ normalised in such a way that such that $\mathbb{Z}_p^{n+1}$ has measure 1. Then the function

$$Z(s) = Z_p(s, f) := \int_{\mathbb{Z}_p^{n+1}} |f(x)|^s |dx| \tag{3.2.3}$$

is defined for $s \in \mathbb{C}$ with $Re(s) > 0$ and can be meromorphically continued to $\mathbb{C}$; it is now known as Igusa local zeta function of f.

There is an exact relation between $Z(s)$ and $P(t)$, namely

$$Z(s) = (1 - p^s)P(p^{-s}) + p^s. \tag{3.2.4}$$

J.-I. Igusa proved that $Z(s)$ is a rational function of p^{-s} using embedded resolution of singularities and this fact implies that $P(t)$ is rational in the variable t.

The rationality result allows to extend Z and P as meromorphic functions to the whole complex plane, and this way it introduces the notion of poles. Hereafter we shall use Z and P to denote the meromorphic continuation to $\mathbb{C}$ of the original functions Z and P, respectively.

Igusa zeta function can be generalised as follows: let $a \in \mathbb{Q}_p$ and denote the p-adic order of a by $ord_p a \in \mathbb{Z} \cup \{\infty\}$. We write $|a| = p^{-ord_p a}$ for the p-adic norm of a and $ac\ a = |a|\ a$ for its angular component. Let $\varkappa : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ be a (multiplicative) character of $\mathbb{Z}_p^\times$, the group of units of $\mathbb{Z}_p$. To f and $\varkappa$ one associates a more general Igusa local zeta function $Z(s)$, which is the meromorphic continuation to $\mathbb{C}$ of

$$Z(s) = Z_p(s, f, \varkappa) := \int_{\mathbb{Z}_p^{n+1}} \varkappa\,(ac\ f(x))\,|f(x)|^s |dx|. \tag{3.2.5}$$

This function is still rational in p^{-s} (see [Vey01a]).

The study of the poles of Igusa local zeta function of f is interesting since the poles of $Z(s)$ determine the poles of $P(t)$, which in turn describe the behaviour of the numbers M_i when $i \gg 0$.

A useful tool to study $Z(s)$ and especially its poles is the embedded resolution of singularities of $f^{-1}(0)$, considered as an algebraic set in the affine space $\mathbb{A}^{n+1}$. Such a resolution provides in a natural way a complete list of candidate poles.

To this end, fix an embedded resolution (with normal crossings) $\pi : X \to \mathbb{A}^{n+1}$ of $f^{-1}(0)$. We denote by E_i, for $i \in S$, the (reduced) irreducible components of $\pi^{-1}\left(f^{-1}(0)\right)$, and by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor of respectively $f \circ \pi$ and $\pi^*(dx_0 \wedge \ldots \wedge dx_n)$, the pullback of the volume form $dx_0 \wedge \ldots \wedge dx_n$. The pairs (N_i, ν_i) are called the *numerical data* of the resolution (X, π).

These numerical data appear in fact very naturally in the context of p-adic Igusa local zeta functions. The idea is to compute the defining integral of $Z(s)$ on X instead of $\mathbb{A}^{n+1}$, exploiting the normal-crossings property.

Roughly, we can find in a neighbourhood of any point $q \in X$ local coordinates $y_0, \ldots, y_n$ such that any E_i passing through q is locally described by the vanishing of one coordinate, say y_i, and then locally $f \circ \pi = u \cdot \prod_{i\in I} y_i^{N_i}$ and $\pi^*(dx_0 \wedge \ldots \wedge dx_n) = v \cdot \prod_{i\in I} y_i^{\nu_i - 1} dy_0 \wedge \ldots \wedge dy_n$, where q belongs exactly to E_i, $i \in I$, and $u, v \in \mathbb{C}\{y_0, \ldots, y_n\}$ are units. This observation is the starting point of the theorem below and, more generally, of the definition of the topological zeta function:

Theorem 3.2.1. *[Vey01a, Proposition 5.6] Let d denote the order of the character $\varkappa$. Then:*

1. *The real poles of $Z(s)$ are part of the set $\left\{-\frac{\nu_i}{N_i} : i \in S\right\}$.*
2. *Suppose that $\varkappa$ is trivial on $1 + p\mathbb{Z}_p$ (this is the relevant case). Then for almost all p we have that*

$$Z(s) = p^{-(n+1)} \sum_{\substack{I \subset S \\ \forall i \in I : d | N_i}} c_I^\varkappa \prod_{i \in I} \frac{p-1}{p^{\nu_i + sN_i} - 1}, \tag{3.2.6}$$

where the $c_I^\varkappa$ are constants depending on p. In particular, denoting $E_I^\circ := \left(\bigcap_{i\in I} E_i\right) \setminus \left(\bigcup_{j \notin I} E_j\right)$ then one has that $c_I^\varkappa = 0$ if $E_I^\circ = \emptyset$.

The poles of $Z(s)$ occur in the Monodromy Conjecture, which was initially formulated by J.-I. Igusa in 1988 as follows (see, for instance, [Vey01a]):

Conjecture 3.2.2 (Monodromy Conjecture of Igusa)**.** *For all but finitely many p we have that if s_0 is a pole of $Z(s)$, then* $\exp(2\pi i Re(s_0))$ *is an eigenvalue of the local monodromy of f at some complex point of $f^{-1}(0)$.*

This conjecture relates arithmetical properties of the polynomial f to geometrical properties of f (considered as a polynomial over $\mathbb{C}$), since the local monodromy is a geometrical invariant of f.

3.3 The topological zeta function

Some related zeta functions were created after the Igusa zeta function. In 1992 J. Denef and F. Loeser introduced a new zeta function [DL92], which they called the *topological zeta function* because of the topological Euler-Poincaré characteristic turning up in it [LSV06]. The topological zeta function can be understood as a limit of p-adic Igusa zeta functions.

Indeed, taking heuristically the limit when p tends to 1 in the formula in (3.2.6) yields

$$\sum_{\substack{I\subset S\\ \forall i\in I: d|N_i}} \chi(E_I^\circ)\prod_{i\in I}\frac{1}{\nu_i+sN_i}, \tag{3.3.1}$$

where $\chi(\cdot)$ denotes the Euler-Poincaré characteristic.

Remark that by the cohomological interpretation of the $c_I^{\varkappa}$ it is at least plausible that their 'limit' is $\chi(E_I^\circ)$.

J. Denef and F. Loeser defined the topological zeta function $Z_{top}^{(d)}(s,f)$ associated with a germ of analytic function $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ and $d\in\mathbb{N}\setminus\{0\}$ as the rational function (3.3.1) in the variable s. We denote $Z_{top}(s,f):=Z_{top}^{(1)}(s,f)$. More precisely, we define the topological zeta function as follows:

Definition 3.3.1 (Topological zeta function)**.** Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of a non-constant analytic function. Let $E_i, i\in S$, be the (reduced) irreducible components of $\pi^{-1}\left(f^{-1}(0)\right)$, and N_i and ν_i-1 the multiplicities of E_i in the divisor of respectively $f\circ\pi$ and $\pi^*(dx_0\wedge\ldots\wedge dx_n)$, the pullback of the volume form $dx_0\wedge\ldots\wedge dx_n$. The *global topological zeta function* of f is the rational function

$$Z_{top}(f,s):=\sum_{I\subset S}\chi(E_I^\circ)\prod_{i\in I}\frac{1}{\nu_i+sN_i}. \tag{3.3.2}$$

The *local topological zeta function* of f is the rational function

$$Z_{top,0}(f,s):=\sum_{I\subset S}\chi(E_I^\circ\cap\pi^{-1}(0))\prod_{i\in I}\frac{1}{\nu_i+sN_i}, \tag{3.3.3}$$

where s is a variable.

It is clear that all real poles of $Z(s)$ are part of the set $\left\{-\frac{\nu_i}{N_i}: i\in S\right\}$. For this reason, the negative rational numbers $-\frac{\nu_i}{N_i}$ are called the *candidate poles.*

The topological zeta function is by definition a rational function. However, it is not an intrinsic invariant like Igusa zeta function since it is defined in terms of an embedded resolution. In order to guarantee that it is an invariant, the fact that the topological zeta function does not depend on the embedded resolution (X, π) by which it is defined must be shown. J. Denef and F. Loeser originally proved this by expressing $Z_{top}(s, f)$ in an exact way as a limit of Igusa local zeta functions. They proved that every embedded resolution gives rise to the same function, so the topological zeta function is a well-defined singularity invariant (see [LSV06]).

In spite of its name, some years later E. Artal, Pi. Cassou-Noguès, I. Luengo and A. Melle pointed out in [ACLM02b] that the topological zeta function is not a topological invariant as one might expect: they found two singularities $(V_1, 0)$ and $(V_2, 0)$ with the same topological type but whose local topological zeta functions are not equal. Thus, the *topological* term refers to the fact that the topological zeta functions only depend on topological properites of the resolution.

In 1998 J. Denef and F. Loeser introduced another related zeta function: the *motivic zeta function* [DL98], which can be interpreted as a geometrisation of the p-adic zeta function. The motivic zeta function is an intrinsic invariant and, in fact, it is a finer invariant than the other two zeta functions we have defined so far; in particular, it specialises to both the topological zeta function and the various p-adic Igusa zeta functions (for almost all p). This implies that proving something for the motivic zeta function has consequences for the others.

J. Denef and F. Loeser formulated analogous versions of the Monodromy Conjecture of J.-I. Igusa for the topological and the motivic zeta functions.

3.4 The Monodromy Conjecture

The Monodromy Conjecture relates poles of the Igusa, motivic or topological zeta function to monodromy eigenvalues. In this thesis we will focus on the topological zeta function.

The classical Monodromy Conjecture predicts that if s_0 is a pole of the local topological zeta function $Z_{top,0}(f, s)$ associated with the singularity defined by f, then $\exp(2\pi i s_0)$ is an eigenvalue of the local monodromy of f at some complex point of $f^{-1}(0)$.

The Monodromy Conjecture was verified for plane curves by F. Loeser in [Loe88], originally in the context of p-adic Igusa zeta functions. However, for the time being, there are only some partial results in higher dimensions, due to the lack of a conceptual link between the monodromy operator and the topological zeta function, see [ACLM02a], [ACLM05], [BV16], [BMT11], [Cau16], [GL14], [LV09], [LV11] and [Vey93]. Indeed, the existent proofs of the particular cases basically compute both sides independently and compare the two final results.

It is easy to see on explicit examples that not all the eigenvalues of the monodromy operator are also induced by poles of the topological zeta function, as we will show in Example 3.5.1.

The topological zeta function was first introduced by J. Denef and F. Loeser for the standard differential form $\omega_0 = dx_0 \wedge \ldots \wedge dx_n$. However, in order to find a more conceptual understanding of it, A. Némethi and W. Veys, in [NV10] and [NV12], gave a new focus to the Monodromy Conjecture, by introducing other more general local topological zeta functions associated with the original germ f and differential forms besides the standard ω_0. In the literature similar generalisations were already present (see [NV12]); however they were subject to the restriction $\text{Supp}\,(\text{div}(\omega)) \subset f^{-1}(0)$. In [NV12] this condition is released, so that arbitrary differential forms ω may be considered. Hence, different sets of numbers $\nu_i, i \in S$, are generated, since the ν_i were originally associated with the differential form $dx_0 \wedge \ldots \wedge dx_n$.

Let us introduce these new topological local zeta functions. Again, they are defined in terms of an embedded resolution $\pi : (X, \tilde{E}) \to (\mathbb{C}^{n+1}, 0)$ of $f^{-1}(0) \cup \mathrm{div}(\omega)$, where ω denotes a differential $n+1$-form. We denote by $\tilde{E}_i$ the irreducible components of $\pi^{-1}(f^{-1}(0) \cup \mathrm{div}(\omega))$.

Definition 3.4.1 (Topological zeta function (general version))**.** [Vey01b] The *local topological zeta function* of the pair (f, ω) at $0 \in \mathbb{C}^{n+1}$ is the rational function

$$Z_{top,0}(f, \omega; s) := \sum_{I \subset S} \chi(\tilde{E}_I^{\circ} \cap \pi^{-1}(0)) \prod_{i \in I} \frac{1}{\nu_i + sN_i}. \tag{3.4.1}$$

The *global topological zeta function* of the pair (f, ω) is the rational function

$$Z_{top}(f, \omega; s) := \sum_{I \subset S} \chi(\tilde{E}_I^{\circ}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}, \tag{3.4.2}$$

where s is a variable.

These definitions extend naturally the definition of J. Denef and F. Loeser. Moreover, since they do not depend on any particular resolution, the local (resp. global) topological zeta function $Z_{top,0}(f, \omega; s)$ (resp. $Z_{top}(f, \omega; s)$) are well-defined invariants of the pair (f, ω) (see [Vey07]).

The poles of these zeta functions are rational numbers of the form $\frac{\nu_i}{N_i}, i \in S$, where N_i and ν_i are obtained from the multiplicities of $\tilde{E}_i$ in the divisor of $\pi^* f$ and $\pi^* \omega$, respectively.

It is not difficult to see that the Monodromy Conjecture is in general false when considering an arbitrary analytic differential form ω. For this reason, A. Némethi and W. Veys studied under which conditions the corresponding new set of poles still induce eigenvalues of the monodromy and which eigenvalues can be found in this way. Before that, W. Veys proved the following theorem:

Theorem 3.4.2. *[Vey07] Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-zero polynomial function or germ.*

1. *If λ is a monodromy eigenvalue of f at 0, then there exists a differential $(n+1)$-form ω and a point x in a neigbourhood of 0 such that $Z_{top,x}(f, \omega; s)$ has a pole s_0 satisfying $\exp(2\pi i s_0) = \lambda$.*
2. *Suppose that $f^{-1}(0)$ has an isolated singularity at 0, and let λ be a monodromy eigenvalue of f at 0. Then we can take 0 itself as point x in the previous statement.*

Notice that the new zeta functions $Z_{top,0}(f, \omega; s)$ obtained this way may in general have other poles that do not induce monodromy eigenvalues of f. In [Vey07], Veys also gave the following example for plane curve singularities:

Example 3.4.3. Let $f(x, y) = y^p - x^q$ on $(\mathbb{C}^2, 0)$ with $2 \leq p < q$ and $\gcd(p, q) = 1$.

Take $\omega_{ij} := x^{i-1}y^{j-1}dx \wedge dy$ for $1 \leq i \leq q-1$ and $1 \leq j \leq p-1$.

- If s_0 is a pole of $Z_{top,0}(f, \omega_{ij}; s)$ for some ω_{ij}, then $\exp(2\pi i s_0)$ is a monodromy eigenvalue of f at 0.
- If λ is a monodromy eigenvalue of f at 0, then there is a form ω_{ij} and a pole s_0 of $Z_{top,0}(f, \omega_{ij}; s)$ such that $\exp(2\pi i s_0) = \lambda$.

In this setup, A. Némethi and W. Veys predicted in [NV12] the existence of a set of *allowed* differential forms such that the conditions below hold:

1. For every allowed ω and every pole s_0 of $Z_{top,0}(f,\omega;s)$, $\exp(2\pi i s_0)$ is a local monodromy eigenvalue of f.
2. The standard form $\omega_0 = dx_0 \wedge \ldots \wedge dx_n$ is allowed.
3. Every local monodromy eigenvalue λ of f is obtained as a pole of $Z_{top,0}(f,\omega;s)$ for some allowed ω.

Notice that (1) and (2) combined imply the classical Monodromy Conjecture.

3.5 The Hessian differential form and the Monodromy Conjecture in dimension two

Henceforth, assume that $n = 1$ and let $f \in \mathbb{C}\{x,y\}$. In [NV10] and [NV12], A. Némethi and W. Veys actually proved the existence of allowed forms for germs in $(\mathbb{C}^2, 0)$.

Let (X,π) be the minimal embedded resolution of the germ $(f^{-1}(0) \cup \mathrm{div}(\omega), 0)$. Thus, π is a finite succession of blowings-up. We denote the components by $\tilde{E}_i, i \in I = I_e \cup I_{s^f} \cup I_{s^\omega}$, where $\tilde{E}_i$ is:

- an exceptional component, if $i \in I_e$; these components are ordered as created;
- an irreducible component of the strict transform of $f^{-1}(0)$, if $i \in I_{s^f}$;
- an irreducible component of the strict transform of $\mathrm{div}(\omega)$, if $i \in I_{s^\omega}$.

Notice that a component could be simultaneously a component of strict transform of the function and strict transform of the form.

For each $i \in I$, let N_i and $\nu_i - 1$ be the multiplicities of $\tilde{E}_i$ in the divisor of $\pi^* f$ and $\pi^*\omega$, respectively. In this case $\nu_i \geq 1$ and, if f is reduced, then $(N_i, \nu_i) = (1,1)$ for $i \in I_{s^f}$. The defining expression for $Z_{top,0}(f,\omega;s)$ reduces to

$$Z_{top,0}(f,\omega;s) = \sum_{j \in I_e} \frac{\chi(\tilde{E}_j^\circ)}{\nu_j + sN_j} + \sum_{\{i,j\} \subset I} \frac{\chi(\tilde{E}_i \cap \tilde{E}_j)}{(\nu_i + sN_i)(\nu_j + sN_j)}. \tag{3.5.1}$$

Since $\tilde{E}_i$ is homeomorphic to a 2-sphere and we obtain $\tilde{E}_i^\circ$ by making r_i punctures, where r_i denotes the number of intersection points of the exceptional component $\tilde{E}_i^\circ$ with other $\tilde{E}_j$, for $j \in I$, then $\chi(\tilde{E}_i^\circ) = 2 - r_i$.

Given a germ $f \in \mathbb{C}\{x,y\}$ one can associate to it the corresponding Hessian holomorphic function $\mathrm{hess}(f)$, defined as the determinant of the matrix of the second derivatives $\mathrm{Hess}(f)$, namely

$$\mathrm{hess}(f) := f_{xx} f_{yy} - f_{xy}^2 \in \mathbb{C}\{x,y\}.$$

Let us introduce the Hessian differential 2-form:

$$\omega_{\mathrm{hess}(f)} := \mathrm{hess}(f) dx \wedge dy.$$

In the same way that the standard differential form is always allowed, it is natural to wonder whether there exists any other differential form satisfying the same property, and a reasonable candidate might be the Hessian differential form, which only depends on the defining function of the curve.

The interest of this differential form resides in the fact that the rank of the Hessian matrix at a singular point $p \in f^{-1}(0)$ is an invariant of the contact class (see [GLS07]). Recall that two polynomials $f, g \in \mathbb{C}[x, y]$ are said to be contact equivalent if there exists an automorphism ϕ of $\mathbb{C}\{x, y\}$ and a unit $u \in (\mathbb{C}\{x, y\})^*$ such that $f = u \circ \phi(g)$.

For this reason, A. Melle conjectured and asked whether all the poles of the topological zeta function associated with an irreducible plane curve singularity defined by f and the Hessian differential form $\omega_{\text{hess}(f)} := \text{hess}(f) dx \wedge dy$ did always induce eigenvalues of the monodromy, provided that this fact occurs in many examples.

Example 3.5.1. An easy example is the singularity at the origin of the cusp

$$f(x, y) = y^2 - x^3$$

whose resolution graph is shown in Figure 3.1:

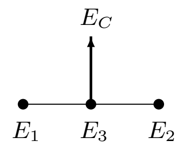


Figure 3.1: Dual graph of the embedded resolution of $(f^{-1}(0), 0)$, for the cusp $f(x, y) = y^2 - x^3$.

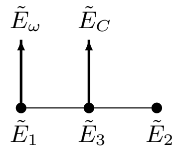
The Hessian polynomial corresponding to f is

$$\text{hess}(f) = -12x$$

and the minimal embedded resolution of $(f^{-1}(0) \cup \text{div}\left(\omega_{\text{hess}(f)}\right), 0)$ is described in Figure 3.2, where the following notation is used:
$I_e = \{1, 2, 3\}$,
$I_{s^f} = \{C\}$,
$I_{s^\omega_f} = \{\omega\}$.

Thus, the topological zeta function associated with f and $\omega_{\text{hess}(f)}$ is

$$\begin{aligned} Z_{top,0}(f, \omega; s) &= \sum_{j \in I_e} \frac{\chi(\tilde{E}_j^\circ)}{\nu_j + sN_j} + \sum_{\{i,j\} \subset I} \frac{\chi(\tilde{E}_i \cap \tilde{E}_j)}{(\nu_i + sN_i)(\nu_j + sN_j)} \\ &= \frac{1}{4 + 3s} + \frac{-1}{7 + 6s} + \frac{1}{(3 + 2s)(2 + 0s)} + \frac{1}{(3 + 2s)(7 + 6s)} \\ &\quad + \frac{1}{(4 + 3s)(7 + 6s)} + \frac{1}{(7 + 6s)(1 + s)} \\ &= \frac{-7}{2(7 + 6s)} + \frac{1}{(1 + s)}, \end{aligned}$$



i	1	2	3	C	ω
N_i	2	3	6	1	0
ν_i	3	4	7	1	2

Figure 3.2: Dual graph of the minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\mathrm{hess}(f)}\right), 0)$, along with the numerical data of its components, for $f(x, y) = y^2 - x^3$.

whose actual poles come from the components $\tilde{E}_3$ and $\tilde{E}_C$.

Applying A'Campo's formula, one obtains the characteristic polynomial of the monodromy:

$$\Delta(t) = \frac{(t-1)(t^6-1)}{(t^2-1)(t^3-1)} = t^2 - t + 1. \tag{3.5.2}$$

The pole of the topological zeta function $s_0 := -7/6$, which corresponds to the component $\tilde{E}_3$, induces a monodromy eigenvalue, since $\Delta(\exp(2\pi i s_0)) = 0$. The pole $s_0 := -1$ corresponds to the characteristic polynomial of the monodromy operator acting on $H_0(F_\delta, \mathbb{C})$, namely $\Delta_0(t) = t - 1$ and therefore the Monodromy Conjecture holds.

However, as mentioned in Section 3.4, not all the eigenvalues of the monodromy operator correspond to poles of the topological zeta function. Indeed, in this example, the characteristic polynomial of the monodromy has two roots:

- $t_1 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$, which corresponds to the pole of the zeta function $s_0 := -7/6$, since $t_1 = \exp(2\pi i s_0)$;
- $t_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, which is not induced by any pole of the topological zeta function.

There are many other examples for which the Monodromy Conjecture is also true for the Hessian differential form. For instance, it holds for the singularity defined by $f(x, y) = (x^2+y^3)(x^2y^2+x^6+y^6)$ (see the resolution graph of this singularity in Figure 2.1).

Example 3.5.2. The Monodromy Conjecture associated with the Hessian differential form holds for the simple singularities of type $\mathbb{A}_n$, for $n \geq 2$. These singularities are defined by the local equation

$$f(x, y) = y^2 - x^{n+1} = 0.$$

The Hessian polynomial corresponding to f is

$$\mathrm{hess}(f) = -2n(n+1)x^{n-1}.$$

The resolution graph of $V(f)$ depends on the parity of n:

- If $n = 2k$, then the singularity has one branch and the resolution graph is shown in Figure 3.3:

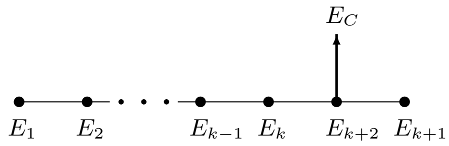


Figure 3.3: Dual graph of the minimal embedded resolution of $(f^{-1}(0), 0)$, along with the numerical data of its components, for $f(x, y) = y^2 - x^{2k+1}$.

- If $n = 2k+1$, then the singularity has two smooth branches and the resolution graph is shown in Figure 3.4:

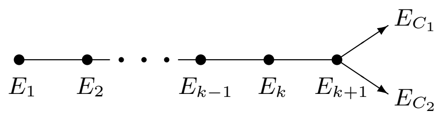
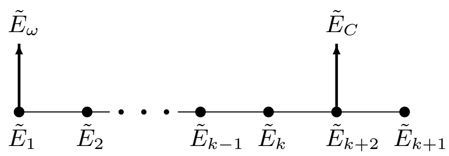


Figure 3.4: Dual graph of the minimal embedded resolution of $(f^{-1}(0), 0)$, along with the numerical data of its components, for $f(x, y) = y^2 - x^{2k+2}$.

Let us analyse both cases:

- If $n = 2k$:

  The minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\text{hess}(f)}\right), 0)$ is described in Figure 3.5, where the same notation as in the previous example is used:



i	$1, 2, \ldots, k$	$k+1$	$k+2$	C	ω
N_i	$2i$	$2k+1$	$4k+2$	1	0
ν_i	$i+2$	$k+3$	$2k+5$	1	2

Figure 3.5: Dual graph of the minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\text{hess}(f)}\right), 0)$, along with the numerical data of its components, for $f(x, y) = y^2 - x^{2k+1}$.

Thus, the topological zeta function associated with f and $\omega_{\text{hess}(f)}$ is

$$
\begin{aligned}
Z_{top,0}(f,\omega;s) &= \sum_{j\in I_e}\frac{\chi(\tilde{E}_j^\circ)}{\nu_j+sN_j}+\sum_{\{i,j\}\subset I}\frac{\chi(\tilde{E}_i\cap\tilde{E}_j)}{(\nu_i+sN_i)(\nu_j+sN_j)}\\
&= \frac{1}{(k+3)+(2k+1)s}+\frac{-1}{(2k+5)+(4k+2)s}+\frac{1}{2(3+2s)}\\
&\quad+\sum_{m=1}^{k-1}\frac{1}{[(m+2)+2ms][(m+3)+2(m+1)s]}\\
&\quad+\frac{1}{[(k+2)+2ks][(2k+5)+(4k+2)s]}\\
&\quad+\frac{1}{[(2k+5)+(4k+2)s][(k+3)+(2k+1)s]}\\
&\quad+\frac{1}{[(2k+5)+(4k+2)s][1+s]}\\
&= \frac{1}{(k+3)+(2k+1)s}+\frac{-1}{(2k+5)+(4k+2)s}+\frac{1}{2(3+2s)}\\
&\quad+\sum_{m=1}^{k-1}\left[-\frac{m}{2}\cdot\frac{1}{(m+2)+2ms}+\frac{m+1}{2}\cdot\frac{1}{(m+3)+2(m+1)s}\right]\\
&\quad-\frac{k}{2}\cdot\frac{1}{(k+2)+2ks}+\frac{2k+1}{2}\cdot\frac{1}{(2k+5)+(4k+2)s}\\
&\quad+\frac{2}{(2k+5)+(4k+2)s}-\frac{1}{(k+3)+(2k+1)s}\\
&\quad+\frac{4k+2}{2k-3}\cdot\frac{1}{(2k+5)+(4k+2)s}-\frac{1}{2k-3}\cdot\frac{1}{1+s}\\
&= \frac{4k^2+8k-5}{4k-6}\cdot\frac{1}{(2k+5)+(4k+2)s}-\frac{1}{2k-3}\cdot\frac{1}{1+s},
\end{aligned}
$$

whose actual poles $-\frac{2k+5}{4k+2}$ and -1 come from the components $\tilde{E}_{k+2}$ and $\tilde{E}_C$, respectively.

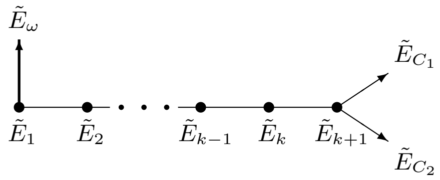
Applying A'Campo's formula, one obtains the characteristic polynomial of the monodromy:

$$\Delta(t)=\frac{(t-1)(t^{4k+2}-1)}{(t^2-1)(t^{2k+1}-1)}=\frac{t^{2k+1}+1}{t+1}. \tag{3.5.3}$$

The pole of the topological zeta function $s_0 := -\frac{2k+5}{4k+2}$, which corresponds to the component $\tilde{E}_{k+2}$, induces a monodromy eigenvalue, since

$$\left[\exp\left(-2\cdot\frac{2k+5}{4k+2}\pi i\right)\right]^{2k+1}+1=\exp\left(-(2k+5)\pi i\right)+1=0\quad\forall k\in\mathbb{N}. \tag{3.5.4}$$

The pole $s_0 := -1$ corresponds to the characteristic polynomial of the monodromy operator acting on $H_0(F_\delta,\mathbb{C})$, namely $\Delta_0(t)=t-1$ and the Monodromy Conjecture holds.



i	$1,2,\ldots,k+1$	C	ω
N_i	$2i$	1	0
ν_i	$i+2$	1	2

Figure 3.6: Dual graph of the minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\operatorname{hess}(f)}\right), 0)$, along with the numerical data of its components, for $f(x,y) = y^2 - x^{2k+2}$.

- If $n = 2k+1$:

  The minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\operatorname{hess}(f)}\right), 0)$ is described in Figure 3.6.

  The topological zeta function associated with f and $\omega_{\operatorname{hess}(f)}$ is

$$
\begin{aligned}
Z_{top,0}(f,\omega;s) &= \sum_{j \in I_e} \frac{\chi(\tilde{E}_j^\circ)}{\nu_j + sN_j} + \sum_{\{i,j\} \subset I} \frac{\chi(\tilde{E}_i \cap \tilde{E}_j)}{(\nu_i + sN_i)(\nu_j + sN_j)} \\
&= \frac{-1}{(k+3)+(2k+2)s} + \frac{1}{2(3+2s)} \\
&\quad + \sum_{m=1}^{k} \frac{1}{[(m+2)+2ms][(m+3)+2(m+1)s]} \\
&\quad + \frac{2}{[(k+3)+(2k+2)s][1+s]} \\
&= \frac{-1}{(k+3)+(2k+2)s} + \frac{1}{2(3+2s)} \\
&\quad + \sum_{m=1}^{k} \left[-\frac{m}{2} \cdot \frac{1}{(m+2)+2ms} + \frac{m+1}{2} \cdot \frac{1}{(m+3)+2(m+1)s} \right] \\
&\quad + \frac{4k+4}{k-1} \cdot \frac{1}{(k+3)+(2k+2)s} - \frac{2}{k-1} \cdot \frac{1}{1+s} \\
&= \frac{4k^2+8k-5}{4k-6} \cdot \frac{1}{(k+3)+(2k+2)s} - \frac{2}{k-1} \cdot \frac{1}{1+s}
\end{aligned}
$$

whose actual poles come from the components $\tilde{E}_{k+1}$ and $\tilde{E}_C$.

Applying A'Campo's formula, one obtains the characteristic polynomial of the monodromy:

$$
\Delta(t) = \frac{(t-1)(t^{2k+2}-1)}{(t^2-1)} = \frac{t^{2k+2}-1}{t+1}. \tag{3.5.5}
$$

The pole of the topological zeta function $s_0 := -\frac{k+3}{2k+2}$, which corresponds to the component

$\tilde{E}_{k+1}$, induces a monodromy eigenvalue, since

$$\left[\exp\left(-2\cdot\frac{k+3}{2k+2}\pi i\right)\right]^{2k+2} - 1 = \exp\left(-2(k+3)\pi i\right) - 1 = 0 \quad \forall k\in\mathbb{N}. \tag{3.5.6}$$

The pole $s_0 := -1$ corresponds to the characteristic polynomial of the monodromy operator acting on $H_0(F_\delta, \mathbb{C})$, namely $\Delta_0(t) = t-1$ and the Monodromy Conjecture also holds.

3.6 The Hessian differential form is not allowed

However, the conjecture turned out to be false, as the following counterexample reveals.

Let us consider the irreducible curve defined by the germ

$$f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6 \tag{3.6.1}$$

and let us focus on the singularity at the origin, whose multiplicity sequence is $[5, 1_5]$.

The resolution graph of this curve is shown in Figure 3.7:

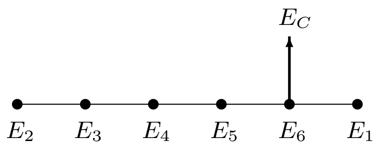


Figure 3.7: Dual graph of the minimal embedded resolution of $(f^{-1}(0), 0)$, for $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

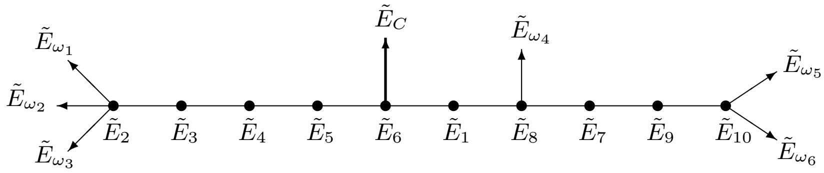
The Hessian polynomial corresponding to f is

$$\operatorname{hess}(f) = -12xy(63x^3y^{11} + 6x^4y^7 + 210x^6y^4 + 20y^9 + 6x^5y^3 - 15x^7 - 20xy^5 - 50x^3y^2) \tag{3.6.2}$$

and the minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\operatorname{hess}(f)}\right), 0)$ is described in Figure 5.4, for the sets of indices:

$I_e = \{1, \ldots, 10\}$,
$I_{s^f} = \{C\}$,
$I_{s^\omega_f} = \{\omega_1, \ldots, \omega_6\}$.

Thus, the topological zeta function associated with f and $\omega_{\operatorname{hess}(f)}$ is



i	1	2	3	4	5	6	7	8	9	10	C	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
N_i	5	6	12	18	24	30	5	10	5	5	1	0	0	0	0	0	0
ν_i	9	13	22	31	40	49	13	23	16	19	1	2	2	2	2	2	2

Figure 3.8: Dual graph of the minimal embedded resolution of $(f^{-1}(0) \cup \operatorname{div}\left(\omega_{\mathrm{hess}(f)}\right), 0)$, along with the numerical data of its components, for $f(x,y) = y^5 - 2x^3y^7 + x^4y^3 + x^6$.

$$
\begin{aligned}
Z_{top,0}(f,\omega;s) &= \frac{-2}{13+6s} + \frac{-1}{49+30s} + \frac{-1}{23+10s} + \frac{-1}{19+5s} + \frac{1}{(9+5s)(49+30s)} \\
&+ \frac{1}{(9+5s)(23+10s)} + \frac{1}{(13+6s)(22+12s)} + \frac{3}{2(13+6s)} \\
&+ \frac{1}{(22+12s)(31+18s)} + \frac{1}{(31+18s)(40+24s)} + \frac{1}{(40+24s)(49+30s)} \\
&+ \frac{1}{(49+30s)(1+s)} + \frac{1}{(13+5s)(23+10s)} + \frac{1}{(13+5s)(16+5s)} \\
&+ \frac{1}{2(23+10s)} + \frac{1}{(16+5s)(19+5s)} + \frac{2}{2(19+5s)} \\
&= -\frac{1500s^4 + 11560s^3 + 30513s^2 + 31576s + 9849}{(13+6s)(49+30s)(23+10s)(19+5s)(1+s)}
\end{aligned}
$$

whose actual poles come from the components $\tilde{E}_2, \tilde{E}_6, \tilde{E}_8, \tilde{E}_{10}$ and $\tilde{E}_C$.

Applying A'Campo's formula, one obtains the characteristic polynomial of the monodromy:

$$\Delta(t) = \frac{(t-1)(t^{30}-1)}{(t^5-1)(t^6-1)}. \tag{3.6.3}$$

The pole of the topological zeta function $s_0 := -13/6$, which corresponds to the component $\tilde{E}_2$, does not induce a monodromy eigenvalue, since

$$\Delta(\exp(2\pi i s_0)) = \frac{5}{2}(1 - \sqrt{3}i) \neq 0. \tag{3.6.4}$$

Therefore, the Monodromy Conjecture fails when we consider the differential form $\omega_{\mathrm{hess}(f)} = \mathrm{hess}(f) dx \wedge dy$ instead of the standard ω_0. In particular, the Hessian differential form is not allowed.

Chapter 4

Rational cuspidal plane curves

4.1 Rational cuspidal plane curves

Let $F(x, y, z) \in \mathbb{C}[x, y, z]$ be a homogeneous irreducible polynomial, and let $V(F)$ denote the zero set of F. Then $C = V(F) \subset \mathbb{P}^2$ is a *plane algebraic curve*. If F is a polynomial of degree d, we say that the curve C has degree d.

A curve C is *rational* if it is birationally equivalent to $\mathbb{P}^1$ and hence admits a parametrisation.

Given C, we assign to each point $p \in C$ its *multiplicity* m_p. Let us assume that $p = [0 : 0 : 1]$ (otherwise we move p to $[0 : 0 : 1]$ using a linear change of coordinates). We write

$$f(x, y) := F(x, y, 1) = f_m(x, y) + f_{m+1}(x, y) + \ldots + f_d(x, y), \tag{4.1.1}$$

where $f_i(x, y)$ denotes a homogeneous polynomial in the variables x and y of degree i. Then the multiplicity m_p is the order of f, i.e., $m_p := m$.

The *tangent* to a curve C at a point $p = [x_0 : y_0 : z_0]$ is denoted by T_pC. If p is a smooth point, then $m_p = 1$ and there exists a unique tangent T_pC to C at p.

Otherwise, if p is a singular point, then

$$f_m(x, y) = \prod_{i=1}^{m} L_i(x, y),$$

where $L_i(x, y)$ are linear polynomials, not necessarily distinct. Let us define the lines $T_i := V(L_i)$. Then $V(f_m)$ is a union of r lines T_i through p, where $1 \leq r \leq m$:

$$V(f_m) = \bigcup_{i=1}^{r} T_i.$$

The r lines T_i are called the *tangents* to C at p. In the particular case in which $r = 1$ and C has only one branch through p, the singular point p is called a *cusp*.

If the set of singular points of C only consists of cusps, we say that the curve is *cuspidal*.

A *rational cuspidal curve* is therefore a plane algebraic curve which is birational to $\mathbb{P}^1$ and is such that all its singularities are cusps.

The results below can be found in the survey by T. K. Moe [Moe15] and the articles by J. Fernández de Bobadilla, I. Luengo, A. Melle and A. Némethi [FLMN06, FLMN07a, FLMN07b].

4.2 Invariants and properties of rational cuspidal plane curves

In this section we summarise some important definitions and properties of rational cuspidal curves. We start with some notation.

Let C be a rational cuspidal plane curve with k cusps: p_1, $p_2, \ldots$, p_k. Then the curve C can be completely described by the multiplicity sequences of its cusps. We write $[\bar{m}_{p_1}; \ldots; \bar{m}_{p_k}]$ and call this sequence the *cuspidal configuration* of the curve.

There are many relevant results regarding the multiplicity sequence of a point. For instance, if $\bar{m}_p = (m_0, m_1, \ldots, m_s)$ is the multiplicity sequence of a point p, then

$$m_0 \geq m_1 \geq \ldots \geq m_s = 1. \tag{4.2.1}$$

Furthermore:

Proposition 4.2.1. *[FZ96, Proposition 1.2] Let $\bar{m}_p = (m_0, m_1, \ldots, m_s)$ be the multiplicity sequence of a cusp p. Then:*

- *For each $i = 1, \ldots, s$ there exists $\ell \geq 0$ such that*

  $$m_{i-1} = m_i + \ldots + m_{i+\ell},$$

  where $m_i = m_{i+1} = \ldots = m_{i+\ell-1}$.

- *The number of ending 1's in the multiplicity sequence equals the smallest $m_i > 1$.*

In order to introduce the next property of rational cuspidal plane curves, let us provisionally change the convention and define the multiplicity sequences to be infinite. To this end, for the multiplicity sequence $\bar{m}_p = (m_0, m_1, \ldots, m_s)$ we set $m_\nu = 1$ for all $\nu \geq s$. Thus, the sequence $(1, 1, \ldots)$ denotes the multiplicity sequence of a smooth germ.

Proposition 4.2.2. *[FZ96, Lemma 1.4] Let (C, p) be an irreducible germ of a curve, and let p have multiplicity sequence $\bar{m}_p = (m_0, m_1, \ldots)$. Then there exists a germ of a smooth curve (Γ, p) through p with local intersection number with the curve C given by $(C.\Gamma)_p = l$ if and only if l satisfies the condition*

$$l = m_0 + m_1 + \ldots + m_t$$

for some $t \geq 1$ with $m_0 = m_1 = \ldots = m_{t-1}$.

In particular,

$$i(C)_p := (C.T_pC)_p = \sum_{i=0}^{a} m_i = a \cdot m_0 + m_a \tag{4.2.2}$$

for some $a \geq 1$ (see [Moe15]).

For a cusp p with multiplicity sequence $\bar{m}_p = (m_0, m_1, \ldots, m_s)$, its Milnor number is:

$$\mu_p(C) = \sum_{i=0}^{s} m_i(m_i - 1) \tag{4.2.3}$$

and its delta invariant is

$$\delta_p(C) = \frac{\mu_p + r_p - 1}{2} = \sum_{i=0}^{s} \frac{m_i(m_i - 1)}{2}. \tag{4.2.4}$$

Furthermore, since a rational curve has genus 0, and the genus of a curve C of degree d can be computed as

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{p \in \mathrm{Sing}(C)} \delta_p, \tag{4.2.5}$$

then for a rational cuspidal curve with k cusps we have that

$$\frac{(d-1)(d-2)}{2} = \sum_{i=1}^{k} \delta_{p_i} = \sum_{i=1}^{k} \sum_{j=0}^{s_{p_i}} \frac{m_j^{p_i}(m_j^{p_i} - 1)}{2}, \tag{4.2.6}$$

where s_{p_i} is the number of blowings-up necessary to resolve the singularity p_i and $m_j^{p_i}$ is the multiplicity of p_i after j blowings-up.

Finally, let $\pi : V \to \mathbb{P}^2$ be the minimal embedded resolution of $C \subset \mathbb{P}^2$. Let $\tilde{C} \subset V$ be the strict transform of C. One of the integers which plays a special role in the classification problem is the *self-intersection* of $\tilde{C}$ in V [FLMN07b]. It equals

$$\tilde{C}^2 = d^2 - \sum_{p \in \mathrm{Sing}(C)} \sum_{j=0}^{s_p} (m_j^p)^2. \tag{4.2.7}$$

4.3 The classification problem

The classification of irreducible projective plane curves in the complex projective plane $\mathbb{P}^2(\mathbb{C})$, up to the action of the automorphism group $\mathrm{PGL}(3, \mathbb{C})$ on $\mathbb{P}^2(\mathbb{C})$, is a very difficult and interesting open problem.

4.3.1 The logarithmic Kodaira dimension

Let C be an irreducible curve in the complex projective plane. One of the main invariants of such curves is the logarithmic Kodaira dimension $\bar{\kappa}(\mathbb{P}^2 \setminus C)$, introduced by Sh. Iitaka in [Iit70].

Let us consider a smooth algebraic variety S and a complete smooth variety $V \supset S$ such that $D := V \setminus S$ is a simple normal crossing divisor. Let K_V be a canonical divisor of V and take the divisor $K_V + D$. Then:

- either there exists a natural number a_0 such that the linear system $|a_0(K_V + D)|$ is not empty, or

- for every natural number n we have that $|n(K_V + D)|$ is the empty set.

In the first case, the dimension of the cohomology group $H^0\big(na_0(K_V + D)\big)$ behaves like n^k when $n \gg 0$. The *logarithmic Kodaira dimension* of S is $\bar{\kappa}(S) := k$ and it satisfies $0 \leq \bar{\kappa} \leq \dim S$. In the second case we say that the logarithmic Kodaira dimension of S is $\bar{\kappa}(S) = -\infty$.

Hence, since S is a surface, its logarithmic Kodaira dimension verifies that $\bar{\kappa}(S) \in \{-\infty, 0, 1, 2\}$.

4.3.2 The classification problem

The main goal of the classification problem is to determine, for a given d, whether there exists a projective plane curve C of degree d having a fixed number of singularities of given topological type. The logarithmic Kodaira dimension $\bar{\kappa}$ of the open surface $\mathbb{P}^2 \setminus C$ helps. For example, the classification of rational cuspidal projective plane curves C with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ has been concluded, and is the following:

- $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ if and only if $\tilde{C}^2 \geq -1$, where $\tilde{C}$ denotes the strict transform of C via its minimal embedded resolution. In this case, the curve C has only one cusp; all these curves have been classified by M. Miyanishi and T. Sugie. This family also contains the Abhyankar-Moh-Suzuki (AMS) curves.

- $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 0$ cannot occur by a result of Sh. Tsunoda, i.e., there exist no rational cuspidal plane curves with $\bar{\kappa} = 0$. Thus, $\bar{\kappa}(\mathbb{P}^2 \setminus C) \geq 1$ if and only if $\tilde{C}^2 \leq -2$.

- If $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$, the curve C has at most two cusps, and these curves are classified in both cases by K. Tono. We will describe these curves in 4.3.4.

- $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$: in this case, the classification problem remains open. The only known unicuspidal rational plane curves are the families described by S. Orevkov, that we will introduce in 4.3.5. I. Wakabayashi showed in [Wak78] that if a rational curve has at least three cusps, then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$.

4.3.3 Rational unicuspidal curves with one Puiseux pair

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a local holomorphic function which defines a cusp in p, i.e., f is irreducible in $\mathbb{C}\{x, y\}$. Then it has a *local parametrisation*, i.e., there exists $x(t), y(t) \in \mathbb{C}\{t\}$ such that $f(x(t), y(t)) \equiv 0$ and $t \mapsto (x(t), y(t))$ is a bijection for $|t|$ small enough. Up to local homeomorphism, we can assume that the parametrisation has the form

$$x(t) = t^m, \quad y(t) = t^{n_1} + \cdots + t^{n_r} \tag{4.3.1}$$

with $1 < m < n_1 < \cdots < n_r$, and $\gcd(m, n_1, n_2, \ldots, n_k)$ does not divide n_{k+1} for $0 \leq k \leq r-1$. If $r = 1$, we say that the singularity has *one Puiseux pair*. More precisely, we say that a local plane curve singularity $\{f(x, y) = 0\}$ has one Puiseux pair (a, b) if after a local homeomorphism the singularity can be parametrised by

$$x(t) = t^a, \quad y(t) = t^b, \tag{4.3.2}$$

where $\gcd(a, b) = 1$.

This implies that a curve with a unique Puiseux pair (a, b) has the same topology as the cusp $x^a + y^b = 0$.

A. Némethi, J. Fernández de Bobadilla, I. Luengo and A. Melle gave in [FLMN07a] a complete topological classification of those cases when C is a rational unicuspidal curve with one Puiseux pair. We denote by $\{\varphi_j\}_{j\geq 0}$ the Fibonacci numbers: $\varphi_0 = 0, \varphi_1 = 1$ and $\varphi_{j+2} = \varphi_{j+1} + \varphi_j$ for $j \geq 0$.

Theorem 4.3.1. *[FLMN07a, Theorem 1.1] The Puiseux pair (a, b) can be realised by a unicuspidal rational plane curve of degree d if and only if (d, a, b) appears in the list below:*

1. $(a, b) = (d - 1, d)$;
2. $(a, b) = (d/2, 2d - 1)$, *where d is even;*
3. $(a, b) = (\varphi_{j-2}^2, \varphi_j^2)$ *and* $d = \varphi_{j-1}^2 + 1 = \varphi_{j-2}\varphi_j$, *where j is odd and* ≥ 5;
4. $(a, b) = (\varphi_{j-2}, \varphi_{j+2})$ *and* $d = \varphi_j$, *where j is odd and* ≥ 5;
5. $(a, b) = (\varphi_4, \varphi_8 + 1) = (3, 22)$ *and* $d = \varphi_6 = 8$;
6. $(a, b) = (2\varphi_4, 2\varphi_8 + 1) = (6, 43)$ *and* $d = 2\varphi_6 = 16$.

All these cases are realisable: see [FLMN07a] for further details.

4.3.4 Tono's curves

In this thesis we will be particularly interested in the curves of K. Tono, since one of these curves will be used in order to determine a counterexample to one of the conjectures of A. Dimca and G. Sticlaru, namely that any nearly free irreducible plane curve C has only singularities with at most three branches.

Let us recall a classification result of K. Tono [Ton01]. Assume that C is a unicuspidal rational plane curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$. Then C is projectively equivalent to one of the following curves C':

- *Type I*: C' is given by the equation:

$$((f_1^s y + \sum_{i=2}^{s+1} a_i f_1^{s+1-i} x^{ia-a+1})^a - f_1^{as+1})/x^{a-1} = 0, \tag{4.3.3}$$

  where $f_1 = x^{a-1}z + y^a$, $a \geq 3$, $s \geq 1$, $a_2, \ldots, a_{s+1} \in \mathbb{C}$ with $a_{s+1} \neq 0$.

  In this case, $d = a^2 s + 1$, and the multiplicity sequence of (C', p) is

$$\bar{m}_p = [(a^2 - a)s, (sa)_{2a-1}, a_{2s}]. \tag{4.3.4}$$

- *Type IIa*: C' is given by

$$((yf_2^n + x^{2n+1})^{4n+1} - f_3^{2n+1})/f_2^n = 0, \tag{4.3.5}$$

where $f_2 = xz - y^2$, $f_3 = f_2^{2n}z + 2x^{2n}yf_2^n + x^{4n+1}$ and $n \geq 2$.

In this case, $d = 8n^2 + 4n + 1$ and the multiplicity sequence of (C', p) is

$$\bar{m}_p = [(n(4n+1))_4, (4n+1)_{2n}, 3n+1, n_3]. \tag{4.3.6}$$

- *Type IIb*: C' is given by

$$((f_3^{2s-1}(f_2^n y + x^{2n+1}) + \sum_{i=1}^{s} a_i f_3^{2(s-i)} f_2^{i(4n+1)-n})^{4n+1} - f_3^{2((4n+1)s-n)})/f_2^n = 0, \tag{4.3.7}$$

where $n \geq 2$, $s \geq 1$, $a_1, \ldots, a_s \in \mathbb{C}$ with $a_s \neq 0$.

The degree of C' is $d = 2(4n+1)^2 s - 4n(2n+1)$. Set $a^* := 4n+1$ and $s^* := 4s - 1$. The multiplicity sequence for $s = 1$ is

$$\bar{m}_p = [(3na^*)_4, (3a^*)_{2n}, (a^*)_3, 3n+1, n_3], \tag{4.3.8}$$

otherwise it is

$$\bar{m}_p = [(s^*a^*n)_4, (s^*a^*)_{2n}, (sa^*)_3, (s-1)a^*, (a^*)_{2(s-1)}, 3n+1, n_3]. \tag{4.3.9}$$

4.3.5 Orevkov's curves

In [Ore02], S. Orevkov gave a proof of the existence of certain unicuspidal rational curves with logarithmic Kodaira dimension 2. That proof explains a way of constructing explicitly series of such curves by applying a product of Cremona transformations to simple, non-singular algebraic curves of low degree.

Theorem 4.3.2. *[Ore02, Theorem C] For any* $j \geq 0$, $\;j \not\equiv 2 \mod 4$, *there exists a rational cuspidal curve* C_j *of degree* $d_j = \varphi_{j+2}$ *which has a single cusp* p_0 *of multiplicity* $m_j = \varphi_j$. *In particular:*

- *If* j *is odd, then* $\bar{\kappa}(\mathbb{P}^2 \setminus C_j) = -\infty$ *and the cusp of* C_j *has one Puiseux pair* (m_j, n_j), *where* $n_j = \varphi_{j+4}$.

- *If* j *is even (and therefore divisible by 4), then* $\bar{\kappa}(\mathbb{P}^2 \setminus C_j) = 2$. *In particular:*
  - *The cusp of* C_j *for* $j \geq 8$ *has two Puiseux pairs:* $(\varphi_j, \varphi_{j+4})$ *and* $(3, 1)$.
  - *The cusp of* C_4 *has degree* 8 *and one Puisuex pair:* $(\varphi_4, \varphi_8 + 1) = (3, 22)$.

- *For any* $j > 0$, $\;j \equiv 0 \mod 4$ *there exists a rational cuspidal curve* C_j^* *of degree* $d_j^* = 2\varphi_{j+2}$ *which has a single cusp of multiplicity* $m_j^* = 2\varphi_j$. *In this case,* $\bar{\kappa}(\mathbb{P}^2 \setminus C_j) = 2$. *Additionally:*

– *The cusp of* C_j^* *for* $j \geq 8$ *has two Puiseux pairs:* $(2\varphi_j, 2\varphi_{j+4})$ *and* $(6, 1)$.
– *The cusp of* C_4^* *has degree* 16 *and one Puiseux pair:* $(2\varphi_4, 2\varphi_8 + 1) = (6, 43)$.

The multiplicity sequence of C_j at the cusp p_0 is

$$\bar{m}_p = (\varphi_j, S_j, S_{j-4}, \ldots, S_r), \tag{4.3.10}$$

where $j = 4k + r$ for $r = 3, 4, 5$, $k \geq 0$ and S_i denotes the subsequence $(\varphi_i)_5, \varphi_i - \varphi_{i-4}$.

In the case of C_j^*, all the multiplicities are multiplied by 2, i.e.,

$$\bar{m}_p^* = (2\varphi_j, 2S_j, 2S_{j-4}, \ldots, 2S_r). \tag{4.3.11}$$

For instance, the curve C_4 has degree 8 and multiplicity sequence

$$(\varphi_4, (\varphi_4)_5, \varphi_4 - \varphi_0) = (3_7).$$

4.3.6 Problems related to rational cuspidal plane curves

The classification problem is not only interesting for its own sake; it is also connected with crucial properties, problems and conjectures in the theory of open surfaces. Let us mention some of them that are related to rational cuspidal curves:

- The open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic (i.e., $H_i(\mathbb{P}^2 \setminus C; \mathbb{Q}) = 0, \forall i > 0$) if and only if C is a rational cuspidal curve [FZ94].

- The *rigidity conjecture* of H. Flenner and M. Zaidenberg says that any $\mathbb{Q}$-acyclic affine surface Y with logarithmic Kodaira dimension $\bar{\kappa}(Y) = 2$ must be rigid (e.g., if C has at least three cusps then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$). This conjecture for $Y = \mathbb{P}^2 \setminus C$ would imply the projective rigidity of the curve C in the sense that every equisingular deformation of C in $\mathbb{P}^2$ would be projectively equivalent to C [FZ94].

- The Coolidge-Nagata problem predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line. This long-standing conjecture has been proved only recently by M. Koras and K. Palka in [KP17].

- Another problem is the determination of the maximal number of cusps among all the rational cuspidal plane curves. This number is expected to be small (the maximal known is four). K. Tono proved that this maximal number is strictly less than nine [Ton05] and the strongest bound of 6 has been recently settled by K. Palka [Pal19].

There are more open problems connected to the classification of rational projective curves. For further reading about the subject, see for example [FLMN07a] and [FLMN07b].

4.3.7 The rational cuspidal curve $[(3_4);(3);(3);(3)]$ does not exist

The only currently known rational cuspidal curve with four cusps is the quintic curve with cuspidal configuration $[(2_3);(2);(2);(2)]$. On the other hand, there exists a quintic unicuspidal rational curve with multiplicity sequence (2_6). Analogously, starting from the Orevkov's unicuspidal curve C_4 of degree 8, whose multiplicity sequence is (3_7), we wondered whether the rational cuspidal curve with four cusps p_1, p_2, p_3, p_4 and cuspidal configuration $[(3_4);(3);(3);(3)]$ could exist.

Hence, let us assume that there exists a curve C_* with cuspidal configuration $[(3_4);(3);(3);(3)]$. Then C has degree 8, since

$$(d-1)(d-2) = \sum_{i=1}^{4}\sum_{j=1}^{r_i} m_j^{p_i}(m_j^{p_i}-1) = 4\cdot 3\cdot 2 + 3\cdot 3\cdot 2 = 42. \tag{4.3.12}$$

Furthermore, the degree of its dual curve $C_*^\vee$ is

$$d^\vee = 2d-2-\sum_{i=1}^{4}(m_{p_i}-1) = 16-2-4\cdot 2 = 6. \tag{4.3.13}$$

Now, let $p_* \in C_*$ be the cusp whose multiplicity sequence is (3_4). Then:

- By the formula 4.2.2 we know that $i(C_*)_{p_*} = 3a+3$ for a suitable $a \geq 1$ and, by Bézout's Theorem, $i(C_*)_{p_*} \leq 8$, so that $i(C_*)_{p_*} = 6$.

- By Theorem 2.5.6,
$$i(C_*)_{p_*} = i(C_*^\vee)_{p_*^\vee} = m(C_*) + m(C_*^\vee),$$
which implies that $m(C_*^\vee) = 3$.

By Theorem 2.5.7, $p_*^\vee$ has multiplicity (3_4).

Furthermore, since C_* is rational, its dual curve $C_*^\vee$ is also rational and satisfies the identity

$$(d^\vee-1)(d^\vee-2) = \sum_{p\in Sing C_*^\vee}\ \sum_{q \text{ inf. near } p} m_q(m_q-1) = 20. \tag{4.3.14}$$

But on the other hand we have that for the cusp $p_*^\vee$ the sum

$$(d^\vee-1)(d^\vee-2) = \sum_{q \text{ inf. near } p_*^\vee} m_q(m_q-1) = 3\cdot 2\cdot 4 = 24 > 20 \tag{4.3.15}$$

which is inconsistent with the equation 4.3.14. This proves that the curve C_* cannot exist.

4.4 Pencil associated with unicuspidal rational plane curves

Let C be a unicuspidal rational plane curve and let $p \in C$ be its cusp. We denote the unicuspidal plane curve by (C,p) and we call p the *distinguished point of C*.

We are particularly interested in a pencil on $\mathbb{P}^2$ determined by the pair (C,p), namely the unique pencil Λ_C on $\mathbb{P}^2$ satisfying:

- $C \in \Lambda_C$;
- $\mathrm{Bs}(\Lambda_C) = \{p\}$.

The existence of this pencil was proved by D. Daigle and A. Melle and it is extensively studied in their article [DM14]. We are going to describe this pencil in more detail.

Let $\Gamma = \Gamma_{(C,p)} \subset \mathbb{N}$ denote the semigroup of (C, p), i.e.,

$$\Gamma_{(C,p)} := \{(C.D)_p : D \text{ is an effective divisor such that } C \not\subset \mathrm{Supp}(D)\}.$$

Proposition 4.4.1. *[DM14] Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree d and with distinguished point p. For each pair $(l, j) \in \mathbb{N}^2$ such that $l > 0$ and $j \leq ld$, let $X_{l,j}(C)$ be the set of effective divisors D of $\mathbb{P}^2$ such that $\deg(D) = l$ and $(C.D)_p \geq j$. Then the following properties hold:*

1. *$X_{l,j}(C)$ is a linear system on $\mathbb{P}^2$ for all l, j, and $\dim X_{l,j}(C) \geq 1$ whenever $l \geq d$.*
2. *For each $j \in \mathbb{N}$ such that $j \leq d^2$, the dimension of the linear system $X_{d,j}(C)$ is equal to the cardinality of the set $[j, d^2] \cap \Gamma$, where $\Gamma = \Gamma_{(C,P)}$.*
   In particular, for each integer j such that $(d-1)(d-2) \leq j \leq d^2$, $\dim X_{d,j}(C) = d^2 - j + 1$. Consequently, $X_{d,d^2}(C)$ is a pencil and $X_{d,d^2-1}(C)$ is a net (recall that a net *is a linear system of dimension 2).*

Remark 4.4.2. $C \in X_{d,j}(C)$ for all j, because $(C.C) = \infty > j$.

Definition 4.4.3. Let (C, p) be a unicuspidal rational plane curve of degree d. We define

$$\Lambda_C = X_d(C) = X_{d,d^2}(C), \tag{4.4.1}$$

where $d = \deg(C)$.

By Proposition 4.4.1, Λ_C is a pencil on $\mathbb{P}^2$. The definition of $X_{d,d^2}(C)$ and Bézout's Theorem yield the following explicit description of Λ_C:

$$\Lambda_C = \{C\} \cup \left\{D \in \mathrm{Div}(\mathbb{P}^2) : D \geq 0,\ \deg(D) = \deg(C) \text{ and } C \cap \mathrm{Supp}(D) = \{p\}\right\}. \tag{4.4.2}$$

The pencil Λ_C can also be characterised as follows:

Corollary 4.4.4. *[DM14] Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point p. Then Λ_C is the unique pencil on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{p\}$.*

Note that the linear system Λ_C is primitive (i.e., its generic member is irreducible and reduced), because C is irreducible and reduced and it is an element of Λ_C.

Let S be a rational non-singular projective surface.

Definition 4.4.5. [DM14] We say that a linear system $\mathbb{L}$ on S is *rational* if $\dim \mathbb{L} \geq 1$ and the generic member of $\mathbb{L}$ is an irreducible rational curve.

For the following definitions, consider sequences

$$S = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} S_n \tag{4.4.3}$$

where, for each $i = 1, \ldots, n$, $\pi_i : S_i \to S_{i-1}$ is the blowing-up of the non-singular projective surface S_{i-1} at a point $p_i \in S_{i-1}$.

1. Given a curve $C \subset S$, consider the minimal resolution of singularities $X \to S$ of C, let $\tilde{C}$ be the strict transform of C on X, and let $\tilde{\nu}(C) = \tilde{C}^2$ denote the self-intersection number of $\tilde{C}$ in X. When $\tilde{\nu}(C) \geq 0$ (resp. $\tilde{\nu}(C) > 0$), we say that C is *of non-negative type* (resp. *of positive type*).

2. We also consider the embedded resolution of singularities $Y \to S$ of C, and define $\tilde{\nu}_{emb}(C)$ to be the self-intersection number of the strict transform of C on Y.

   Clearly, $\tilde{\nu}_{emb}(C) \leq \tilde{\nu}(C)$.

3. We say that the sequence (4.4.3) is a *chain* if $\pi_{i-1}(p_i) = p_{i-1}$ for all i such that $2 \leq i \leq n$.

4. A curve $C \subset S$ is *uniresolvable* if there exists a chain (4.4.3) with the property that the strict transform of C on S_n is a non-singular curve.

Given a unicuspidal rational curve $C \subset \mathbb{P}^2$ we consider the pencil Λ_C, and ask when this pencil is rational (in the sense of Definition 4.4.5).

Theorem 4.4.6. *[DM14] For a unicuspidal rational curve $C \subset \mathbb{P}^2$, the following are equivalent:*

1. *C is of non-negative type.*
2. *Λ_C is rational.*

Moreover, if these conditions hold then Λ_C is uniresolvable.

D. Daigle and A. Melle proved the following useful and interesting result:

Theorem 4.4.7. *[DM14] Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point p and let Λ_C be the unique pencil on $\mathbb{P}^2$ such that $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{p\}$. If C is of non-negative type then Λ_C has either 1 or 2 dicriticals, and at least one of them has degree 1.*

Chapter 5

On some conjectures about Free and Nearly Free curves

5.1 Introduction

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d in the polynomial ring, let $C \subset \mathbb{P}^2$ be defined by $f = 0$ and assume that C is reduced. We denote by J_f or $J(f)$ the Jacobian ideal of f, i.e., the homogeneous ideal in S spanned by f_x, f_y, f_z, and by $M(f) := S/J_f$ the corresponding graded ring, called the Milnor algebra of f.

The study of free curves in the projective plane has a rather long tradition, being inaugurated by A. Simis in [Sim06a, Sim06b], and actively continued by several mathematicians (see the article by A. Dimca and G. Sticlaru [DS19] and the references given there). We say that C is a free curve if $J_f = I_f$, where I_f denotes the saturation of the ideal J_f with respect to the maximal ideal $\mathfrak{m} = (x, y, z)$.

The nearly free curves were introduced in [DS15]. They have properties similar to the free curves, and together with the free curves may lead to a new understanding of the rational cuspidal curves, due to Conjecture 5.1.1(i) below. This class of curves forms already the subject of attention in a number of papers, see for instance [AD18, MV17].

By definition, C is a nearly free curve if the graded module $N(f) = \bigoplus_{m=0}^{\infty} N_k = I_f/J_f$ satisfies $N(f) \neq 0$ and the graded part $N(f)_k$ is such that $\dim_{\mathbb{C}} N(f)_k \leq 1$ for all k.

In addition, a reduced curve $C = V(f)$ is nearly free if and only if

$$\tau(C) = (d-1)^2 - r(d-r-1) - 1, \tag{5.1.1}$$

where $r = \mathrm{mdr}(f)$ is the minimal degree of a Jacobian syzygy for f, see [DS18a].

The main results of A. Dimca and G. Sticlaru in [DS17], [DS15] and [DS15] and many series of examples motivate the following conjecture:

Conjecture 5.1.1. *[DS15]*

(i) *Any rational cuspidal curve C in the plane is either free or nearly free.*

(ii) *An irreducible plane curve C which is either free or nearly free is rational.*

In [DS18a], the authors provide some interesting results supporting the statement of Conjecture 5.1.1(i); in particular, Conjecture 5.1.1(i) holds for rational cuspidal curves of even degree [DS18a, Theorem 3.1]. They need a topological assumption on the cusps which is not fulfilled all the time when the degree is odd, see [DS18a, Theorem 3.1].

They proved also that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ and for those curves whose degree is a prime power, see [DS18a, Corollary 3.2] and the discussion in [AD15].

Furthermore, using the classification given in [FLMN07a] of unicuspidal rational curves with a unique Puiseux pair, A. Dimca and G. Sticlaru proved in [DS18a, Corollary 3.5] that all of them are either free or nearly free curves, except the curves of odd degree in one of the cases of the classification.

More recently, in [DS18b], A. Dimca and G. Sticlaru proved that their Conjecture 5.1.1(i) is true for any rational cuspidal curve $C = V(f)$ with $\operatorname{mdr}(f) \leq 15$, and for most curves of degree $d \leq 90$.

Regarding Conjecture 5.1.1(ii), note that reducible nearly free curves may have irreducible components which are not rational, see [DS15, Example 2.8]. For example, a smooth cubic with three tangents at aligned inflection points is nearly free (note that the condition of alignment can be removed, at least in some examples computed using `Singular` [DGPS19]).

For free curves, examples can be found using [Val15, Theorem 2.7] e.g. $C = V(f)$, where

$$f = (x^3 - y^3)(y^3 - z^3)(x^3 - z^3)(ax^3 + by^3 + cz^3)$$

for generic $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$. The conjectures in [Val15] give some candidate examples of lower degree; it is possible to check that $D = V(g)$, for

$$g = (y^2 z - x^3)(y^2 z - x^3 - z^3)$$

is free (also compued with `Singular` [DGPS19]). Furthermore, A. Dimca and G. Sticlaru proposed the following conjecture:

Conjecture 5.1.2. *[DS15]*

(i) *Any free irreducible plane curve C has only singularities with at most two branches.*

(ii) *Any nearly free irreducible plane curve C has only singularities with at most three branches.*

In [AGLM17] we gave some examples of irreducible free curves and nearly free curves in the complex projective plane which are not rational and thus giving counterexamples to Conjecture 5.1.1(ii). Among these counterexamples we found some examples of irreducible free and nearly free curves whose two singular points have any odd number of branches $r = 2\ell + 1$, $\ell \geq 1$, giving counterexamples to Conjectures 0.0.2(i) and 0.0.2(ii). Furthermore, an irreducible nearly free curve with just one singular point and having 4 branches, giving another counterexample to Conjecture 5.1.2(ii), was also provided.

5.2 Various prerequisites

5.2.1 Commutative algebra

We introduce some basic results of commutative algebra. The following definitions and results can be found in [GP07] in much more detail.

Throughout this section, let A be a ring, $I, J \subset A$ ideals, and let $\mathbb{K}$ be a field.

Definition 5.2.1 (Saturation of an ideal)**.** The *ideal quotient* of I by J is defined as:

$$I : J := \{a \in A | aJ \subset I\}.$$

The *saturation of I with respect to J* is

$$I : J^\infty := \{a \in A | \exists n \text{ such that } aJ^n \subset I\}.$$

The *saturation* of the homogeneous ideal $I \subset A[x_0, \ldots, x_n]$ is the saturation of I with respect to the maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$:

$$\mathrm{sat}(I) := I : \mathfrak{m} = \{f \in A | \exists r \geq 0 \text{ such that } \mathfrak{m}^r f \subset I\}.$$

I is called *saturated* if $\mathrm{sat}(I) = I$.

Definition 5.2.2 (Free module)**.** Let A be a ring and let M be an $A-$module. M is called *free* if $M \cong \bigoplus_{\lambda \in \Lambda} A$. The cardinality of the index set Λ is called the *rank* of M.

Definition 5.2.3 (Graded ring)**.** A *graded ring A* is a ring together with a direct sum decomposition $A = \bigoplus_{\nu \geq 0} A_\nu$, where the A_ν are abelian groups satisfying

$$A_\nu A_\mu \subset A_{\nu+\mu} \ \forall \nu, \mu \geq 0.$$

Definition 5.2.4 (Graded module)**.** Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded ring. An $A-$module M together with a direct sum decomposition $M = \bigoplus_{\mu \in \mathbb{Z}} M_\mu$ into abelian groups is called a *graded $A-$module* if

$$A_\nu M_\mu \subset M_{\nu+\mu} \ \forall \nu \geq 0, \mu \in \mathbb{Z}.$$

The elements from M_ν are called *homogeneous of degree ν*.

Notation 5.2.5. Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded $S-$module with d-th graded component M_d. For any graded module M we denote by $M(a)$ the module M shifted by a so that $M(a)_d = M_{a+d}$. For example, we will denote the free $S-$module of rank 1 generated by an element of degree a by $S(-a)$.

Definition 5.2.6 (Graded ideal)**.** A submodule $N \subset M$ generated by homogeneous elements is called a *graded* or *homogeneous* submodule. A graded submodule of a graded ring is called a *graded ideal* or *homogeneous ideal*.

Remark 5.2.7. In this case, N is graded with the induced grading, that is:

$$N = \bigoplus_{\nu \in \mathbb{Z}} (M_\nu \cap N).$$

Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded ring, and let $I \subset A$ be a homogeneous ideal. Then the quotient A/I has an induced structure as a graded ring:

$$A/I = \bigoplus_{\nu \geq 0} (A_\nu + I)/I \cong \bigoplus_{\nu \geq 0} A_\nu/(I \cap A_\nu).$$

Definition 5.2.8 (Free resolution)**.** Let A be a ring and M a finitely generated $A-$module. A *free resolution* of M is an exact sequence

$$\ldots \to F_{k+1} \xrightarrow{\varphi_{k+1}} F_k \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

with finitely generated free $A-$modules F_i for $i \geq 0$.

If $(A, \mathfrak{m})$ is a local ring, then a free resolution as above is called *minimal* if $\varphi_k(F_k) \subset \mathfrak{m}F_{k-1}$ for $k \geq 1$. In this case, $b_k(M) := \text{rank}(F_k)$, for $k \geq 0$, is called the *k-th Betti number* of M.

In this thesis we will be especially interested in resolutions of modules over the $\mathbb{C}-$algebra $A[x_0, \ldots, x_n]$.

Theorem 5.2.9. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated $A-$module. Then, the following holds:*

1. *M has a minimal free resolution.*
2. *The rank of F_k in a minimal free resolution is independent of the resolution.*

Furthermore, if M has a minimal resolution of finite length n

$$0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$$

and if

$$0 \to G_m \to G_{m-1} \to \ldots \to G_0 \to M \to 0$$

is any free resolution, then $m \geq n$.

Definition 5.2.10 (Homogeneous free resolution)**.** Let $\mathbb{K}$ be a field, A a graded $\mathbb{K}-$algebra and M a graded $A-$module. A *homogeneous free resolution* of M is a resolution

$$\ldots \to F_{k+1} \xrightarrow{\varphi_{k+1}} F_k \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

such that:

1. The F_k are finitely generated free $A-$modules:
$$F_k = \bigoplus_{j \in \mathbb{Z}} A(-j)^{b_{j-k},k}.$$
2. The φ_k are homogeneous maps of degree 0.

Such a resolution is called *minimal* if $\varphi_k(F_k) \subset \mathfrak{m}F_{k-1}$ for $k \geq 1$, where $\mathfrak{m}$ is the ideal generated by all elements of positive degree. The numbers $b_{j,k} := b_{j,k}(M)$ are called the *graded Betti numbers* of M and $b_k(M) := \sum_j b_{j,k}(M)$ is called the *k-th Betti number* of M.

Definition 5.2.11 (Syzygy)**.** Let A be a ring. A *syzygy* or *relation* between k elements $f_1, \ldots, f_k$ of an $A-$module M is a k-tuple $(g_1, \ldots, g_k) \in A_k$ satisfying

$$\sum_{i=1}^{k} g_i f_i = 0. \tag{5.2.1}$$

The set of all syzygies between $f_1, \ldots, f_k$ is a submodule of A^k. Indeed, it is the kernel of the ring homomorphism

$$\begin{array}{rcl} \varphi : F_1 := \bigoplus_{i=1}^{k} A\epsilon_i & \longrightarrow & M \\ \epsilon_i & \mapsto & f_i \end{array}$$

where $\{\epsilon_1, \ldots, \epsilon_k\}$ denotes the canonical basis of A^k. φ surjects onto the $A-$module $I := (f_1, \ldots, f_k)_A$ and

$$\mathrm{syz}(I) := \mathrm{syz}(f_1, \ldots, f_k) := \ker(\varphi)$$

is called the *module of syzygies* of I with respect to the generators $f_1, \ldots, f_k$.

Remark 5.2.12. If A is a graded ring and $\{f_1, \ldots, f_k\}$ and $\{g_1, \ldots, g_k\}$ are minimal sets of homogeneous generators of I, then

$$\mathrm{syz}(f_1, \ldots, f_k) \cong \mathrm{syz}(g_1, \ldots, g_k)$$

are isomorphic as graded $A-$modules.

Hence $\mathrm{syz}(I)$ is well-defined up to graded isomorphism.

Let $\mathbb{K}$ be a field and consider the polynomial ring $S_r := \mathbb{K}[x_0, \ldots, x_r]$.

Theorem 5.2.13 (Hilbert's Syzygy Theorem, [Eis05])**.** *Any finitely generated graded S_r-module M has a finite graded free resolution*

$$0 \to F_m \xrightarrow{\varphi_m} F_{m-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0.$$

Moreover, we may take $m \leq r+1$, where $r+1$ is the number of variables in S_r.

Remark 5.2.14. If M is a finitely generated graded S_r-module, then the projective dimension of M is equal to the length of its minimal free resolution.

The results below are valid for more general classes of rings and ideals. Since we are interested in codimension 2 ideals $I \subset \tilde{S} := \mathbb{K}[x, y, z]$ minimally generated by three homogeneous polynomials of the same degree, we will state our results according to this setup.

Lemma 5.2.15. *[ScOT14] Let $I \subset \tilde{S} = \mathbb{K}[x, y, z]$ be an ideal of codimension 2 generated by three forms of degree d. Then the following conditions are equivalent:*

1. *There exist two distinct minimal generating syzygies of degrees r_1 and r_2 such that $r_1 + r_2 \geq d + 1$.*

2. *I is not a perfect ideal (i.e., the ring $\tilde{S}/I$ is not Cohen-Macaulay).*

3. *For any two distinct minimal generating syzygies of degrees r_1 and r_2, one has $r_1 + r_2 \geq d + 1$.*

The last general properties are the following, see [DS19, Lin12].

Lemma 5.2.16. *[DS19, Lemma 2.1] Let I be a homogeneous ideal in $\tilde{S}$ of codimension 2. Then the projective dimension* $\operatorname{pd}\tilde{S}/I$ *of the graded $\tilde{S}-$module $\tilde{S}/I$ is either 2 or 3. More precisely,* $\operatorname{pd}\tilde{S}/I = 2$ *if and only if the ideal I is saturated.*

Lemma 5.2.17. *[Lin12, Lemma 4.3] Let I be a homogeneous ideal of $\tilde{S}$ of codimension 2. Then the following are equivalent:*

1. $\operatorname{pd}\tilde{S}/I = 2$.
2. *$\tilde{S}/I$ is a Cohen-Macaulay module over $\tilde{S}$.*
3. *(x, y, z) is not an associated prime of $\tilde{S}/I$.*
4. *I is a saturated ideal.*

5.2.2 Further definitions

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d. Let C be the plane curve in $\mathbb{P}^2$ defined by $f = 0$ and assume that C is reduced, i.e., f is squarefree. We denote by J_f the Jacobian ideal of f. Let $M(f) = S/J_f$ be the corresponding graded ring, called the Jacobian (or Milnor) algebra of f. Notice that if C is a singular curve, then J_f has codimension 2.

The study of such Milnor algebras is related to the singularities of the corresponding projective curve $C = V(f)$. We will follow the definitions, notations and ideas of A. Dimca and G. Sticlaru.

Definition 5.2.18. Consider the graded $S-$submodule

$$\mathrm{AR}(f) = \{(a, b, c) \in S^3 \mid af_x + bf_y + cf_z = 0\} \subset S^3$$

of *all relations* involving the partial derivatives of f, and denote by $\mathrm{AR}(f)_m$ its homogeneous part of degree m.

The *minimal degree of a Jacobian relation* for f is the integer $\mathrm{mdr}(f)$ defined to be the smallest integer $m \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0, \quad (a, b, c) \in S_m^3 \setminus (0, 0, 0). \tag{5.2.2}$$

When $\mathrm{mdr}(f) = 0$, then C is a union of lines passing through one point, a situation easy to analyse. For this reason, we will assume from now on that $\mathrm{mdr}(f) \geq 1$.

Let $I_f := \mathrm{sat}(J_f)$ be the saturation of the ideal J_f with respect to the maximal ideal $\mathfrak{m} = (x, y, z)$ in S and let us define the Jacobian module of f to be $N(f) := I_f/J_f$, which is the local cohomology group $I_f/J_f = H_0^{\mathfrak{m}}(M(f))$.

Notation 5.2.19. We set the following notation:

- $\mathrm{ar}(f)_k := \dim \mathrm{AR}(f)_k$,
- $m(f)_k := \dim M(f)_k$,
- $n(f)_k := \dim N(f)_k$ for any integer k,

- $\nu(C) := \max\{n(f)_k : k \geq 0\}$.

Let us give the definition of some invariants associated with the Milnor algebra $M(f)$.

Definition 5.2.20. Let $C = V(f)$ be a plane curve of degree d with isolated singularities.
(i) the *coincidence threshold* is defined as

$$\mathrm{ct}(f) := \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree d such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

(ii) the *stability threshold* is defined as

$$\mathrm{st}(f) := \min\{q : \ \dim M(f)_k = \tau(C) \text{ for all } k \geq q\}.$$

Note that for $j < d - 1$ one has the following equality:

$$\mathrm{AR}(f)_j = H^2(K^*(f))_{j+2}, \tag{5.2.3}$$

where $K^*(f)$ is the Koszul complex of f_x, f_y, f_z with the natural grading.

It is known that one has

$$\mathrm{ct}(f) = \mathrm{mdr}(f) + d - 2. \tag{5.2.4}$$

Finally, let $T = 3(d - 2)$ denote the degree of the socle of the Gorenstein ring $M(f_s)$.

5.3 Free curves

Let Y be a reduced divisor on a smooth algebraic variety X over the complex field $\mathbb{C}$. According to [Sai80], Y is called a *free divisor* on X if the $\mathcal{O}_X-$module

$$\mathrm{Der}_X(-\log Y) := \{\theta \in \mathrm{Der}(X) \,|\, \theta(\mathcal{O}_X(-Y)) \subseteq \mathcal{O}_X(-Y)\} \tag{5.3.1}$$

is free, where $\mathcal{O}_X$ denotes the sheaf of regular functions on X.

A special case is that of a non-smooth reduced divisor $Y = V(f) \subset \mathbb{P}^2$ on the projective plane, where f is a squarefree homogeneous form of degree d in $S = \mathbb{C}[x, y, z]$. Let $\mathrm{Der}(S)$ be the $S-$module of derivations on S, and let

$$D_f := \{\theta \in \mathrm{Der}(S) : \theta(f) \in \langle f \rangle\}$$

be the $S-$module of logarithmic derivations on f. Y is called a *free divisor* if and only if D_f is a free $S-$module. Since $df = xf_x + yf_y + zf_z$, one has that $\theta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is in D_f. In fact,

$$D_f = \theta_E S \oplus D_f^0, \tag{5.3.2}$$

where D_f^0 is an $S-$submodule of D_f whose elements are in one-to-one correspondence with the syzygies of J_f. We have that Y is free if and only if J_f is a perfect ideal.

The notion of *free divisor* was originally associated with hyperplane arrangement theory. When Y is a hyperplane arrangement in a vector space of finite dimension, free divisors received a great deal of interest, see [ScOT14] and the references therein. When X is not necessarily the projective or the affine space, [DSS$^+$13] gives some insights into the freeness of divisors on X in this general setup. In the case of divisors on the projective space or affine space, some important results were obtained, especially when $X = \mathbb{P}^2$. The advantage of working with divisors on $\mathbb{P}^n$ is that Saito's Criterion ([Sai80]) translates into the following: the divisor Y is free if and only if the gradient ideal of Y is a perfect ideal of codimension 2.

On $\mathbb{P}^2$, because this gradient ideal is generated by three homogeneous polynomials of the same degree, ideals of this type have been studied extensively (see [ScOT14]).

We use the definition of freeness given by A. Dimca in [Dim17]. Recall that:

Definition 5.3.1 (Free divisor)**.** [Dim17] The curve $C = V(f)$ is a *free divisor* if the following equivalent conditions hold:

1. $N(f) = 0$, i.e., the Jacobian ideal is saturated ($\nu(C) = 0$).
2. The minimal resolution of the Milnor algebra $M(f)$ has the following form:
$$0 \to S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \to S^3(-d+1) \xrightarrow{(f_x, f_y, f_z)} S \tag{5.3.3}$$
for some positive integers d_1, d_2.
3. The graded $S-$module $\mathrm{AR}(f)$ is free of rank 2, i.e., there is an isomorphism
$$\mathrm{AR}(f) = S(-d_1) \oplus S(-d_2) \tag{5.3.4}$$
for some positive integers d_1 and d_2.

Definition 5.3.2. With the notation above, we refer to the pair (d_1, d_2) as the *exponents* of the free curve C, where $d_1 \leq d_2$.

These exponents satisfy the relations:

$$d_1 + d_2 = d - 1; \tag{5.3.5}$$

$$\tau(C) = (d-1)^2 - d_1 d_2, \tag{5.3.6}$$

where $\tau(C)$ is the *total Tjurina number* of C.

It is interesting to note that the freeness of the plane curve C can be characterised in terms of these invariants, as we will show in Section 5.5.1.

Theorem 5.3.3. *[DS17, Theorem 2.5] Let $C = V(f)$ be a free curve of degree d and total Tjurina number $\tau(C)$, which is not a pencil of lines. Let d_1 and d_2 with $d_1 \leq d_2$ be the degrees of two homogeneous generators of the free graded $S-$module* $\mathrm{AR}(f)$*. Then, the following holds:*

(i) The degrees d_1 and d_2 are the roots of the equation
$$t^2 - (d-1)t + (d-1)^2 - \tau(C) = 0. \tag{5.3.7}$$
In particular, $d = d_1 + d_2 + 1$ and $\tau(C) = (d-1)^2 - d_1 d_2$ and hence the pairs $(d, \tau(C))$ and (d_1, d_2) determine each other.

(ii) $\mathrm{mdr}(f) = d_1$, $\mathrm{ct}(f) = d + d_1 - 2$ *and* $\mathrm{st}(f) = d + d_2 - 3$.

(iii) $\mathrm{ct}(f) \leq d + j \leq \mathrm{st}(f)$ *if and only if* $d_1 - 2 \leq j \leq d_2 - 3$, *and for such j's one has*

$$m(f)_{d+j} = m(f_s)_{d+j} + \binom{j - d_1 + 3}{2}. \tag{5.3.8}$$

In particular, one has

$$\tau(C) = m(f_s)_{d+d_2-3} + \binom{d_2 - d_1}{2}. \tag{5.3.9}$$

(iv) Let $U = \mathbb{P}^2 \setminus C$. *Then the Euler number* $\chi(U)$ *of* U *is given by*

$$\chi(U) = \tau(C) - \mu(C) + (d_1 - 1)(d_2 - 1), \tag{5.3.10}$$

where $\mu(C)$ *is the total Milnor number of* C. *In particular, if* C *is irreducible one has that* $\chi(U) \geq 1$ *and* $d_1 > 1$.

5.4 Nearly free curves

We have seen that for a reduced curve $C = V(f)$ the existence of a resolution (5.3.3) is equivalent to the vanishing of the $S-$graded module $N(f) = I_f/J_f$, and the curves satisfying these equivalent properties are called free.

A. Dimca and G. Sticlaru introduced a more subtle notion for a curve to be nearly free, see [DS18a], imposing the condition that the Jacobian module $N(f)$ is non-zero but as small as possible.

Definition 5.4.1 (Nearly free divisor)**.** [Dim17] The curve $C = V(f)$ is a *nearly free divisor* if the following equivalent conditions hold:

1. $N(f) \neq 0$ and $n(f)_k \leq 1$ for any k ($\nu(C) = 1$).
2. The Milnor algebra $M(f)$ has a minimal resolution of the form:

$$0 \to S(-d-d_2) \to S(-d-d_1+1) \oplus S^2(-d-d_2+1) \to S^3(-d+1) \xrightarrow{(f_x, f_y, f_z)} S \tag{5.4.1}$$

   for some integers $1 \leq d_1 \leq d_2$, called the *exponents of* C.
3. There are 3 syzygies ρ_1, ρ_2, ρ_3 of degrees d_1, $d_2 = d_3 = d - d_1$ which form a minimal system of generators for the first syzygy module $\mathrm{AR}(f)$.

If $C = V(f)$ is nearly free, then the exponents $d_1 \leq d_2$ satisfy:

$$d_1 + d_2 = d; \tag{5.4.2}$$

$$\tau(C) = (d-1)^2 - d_1(d_2 - 1) - 1. \tag{5.4.3}$$

Remark 5.4.2. Note that for both a free and a nearly free curve $C = V(f)$, it is clear that $\mathrm{mdr}(f) = d_1$.

The first natural question is whether such nearly free curves exist. The following examples show that the answer is positive.

Example 5.4.3. [DS15] In degree $d = 3$, consider a conic plus a secant line, e.g. $C = V(f)$, for $f = x^3 + xyz$. Then C is nearly free with the resolution for $M(f)$ of the form

$$0 \to S(-5) \to S(-3) \oplus S(-4)^2 \to S(-2)^3 \to S,$$

and hence $(d_1, d_2, d_3) = (1, 2, 2)$.

Example 5.4.4. [DS15] On the other hand, the nodal cubic $C = V(f)$, where $f = xyz + x^3 + y^3$, is not nearly free, since $\dim N(f)_1 = 2$.

Example 5.4.5. [DS15] Let $C = V(f)$, for $f = xyz(x + y + z)$, be the union of 4 lines in general position. Then C is a nearly free curve with the resolution for $M(f)$ given by

$$0 \to S(-6) \to S(-5)^3 \to S(-3)^3 \to S,$$

with $(d_1, d_2, d_3) = (2, 2, 2)$ and $\dim N(f)_3 = 1$, and $N(f)_k = 0$ for other k's.

Remark 5.4.6. In [DS18a] is shown that to construct a resolution (5.4.1) for a given polynomial f, the following ingredients are required:

(i) the integer $b := d_2 - d + 2$;

(ii) three syzygies $r_i = (a_i, b_i, c_i) \in S^3_{d_i}$, $i = 1, 2, 3$, for (f_x, f_y, f_z), i.e., such that

$$a_i f_x + b_i f_y + c_i f_z = 0,$$

necessary to construct the morphism

$$\bigoplus_{i=1}^{3} S(-d_i - (d-1)) \to S^3(-d+1), \quad (u_1, u_2, u_3) \mapsto u_1 r_1 + u_2 r_2 + u_3 r_3; \tag{5.4.4}$$

(iii) one relation $R = (v_1, v_2, v_3) \in \bigoplus_{i=1}^{3} S(-d_i - (d-1))_{b+2(d-1)}$ among r_1, r_2, r_3, i.e., such that $v_1 r_1 + v_2 r_2 + v_3 r_3 = 0$, necessary to construct the morphism

$$S(-b - 2(d-1)) \to \bigoplus_{i=1}^{3} S(-d_i - (d-1)) \tag{5.4.5}$$

by the formula $w \mapsto wR$. Note that $v_i \in S_{b-d_i+d-1}$.

Now we state some results on nearly free curves.

Theorem 5.4.7. *[DS18a] Suppose the curve $C = V(f)$ has a minimal resolution for $M(f)$ as in* (5.4.1) *with $d_1 \le d_2 \le d_3$. Then one has that:*

(i)

$$d_1 + d_2 \ge d, \tag{5.4.6}$$

$$b = \sum_{i=1}^{3} d_i - 2(d-1), \tag{5.4.7}$$

$$\tau(C) = (d-1)\sum_{i=1}^{3} d_i - \sum_{i<j} d_i d_j. \tag{5.4.8}$$

Moreover, $\mathrm{mdr}(f) = d_1$, $\mathrm{ct}(f) = d_1 + d - 2$ *and* $\mathrm{st}(f) = b + 2d - 4$.

(ii) Suppose in addition that the curve $C = V(f)$ is nearly free. Then one has the following:

$$d_1 + d_2 = d, \quad d_2 = d_3, \tag{5.4.9}$$

$$b = d_2 - d + 2, \tag{5.4.10}$$

$$\tau(C) = (d-1)^2 - d_1(d_2 - 1) - 1. \tag{5.4.11}$$

Moreover, $\mathrm{st}(f) = d_2 + d - 2$ *and* $\mathrm{ct}(f) + \mathrm{st}(f) = T + 2 = 3d - 4$.

5.5 Free and nearly free divisors and rational cuspidal plane curves

5.5.1 Characterisation of free and nearly free reduced plane curves

Denote by $\tau(C)$ the global Tjurina number of the curve C, which is defined as the sum of the Tjurina numbers of the singular points of C. A. Dimca provides in [Dim17] a characterisation of free and nearly free reduced plane curves C of degree d.

As it is recalled in [Dim17], the following theorem is a corollary of [dW99, Theorem 3.2] by A. A. du Plessis and C. T. C. Wall.

Theorem 5.5.1. *[Dim17, Theorem 1.1] For a positive integer r, define the two integers:*

$$\tau(r)_{min} := (d-1)(d-r-1) \tag{5.5.1}$$

and

$$\tau(r)_{max} := (d-1)(d-r-1) + r^2. \tag{5.5.2}$$

If $r = \mathrm{mdr}(f) < d/2$, then one has that

$$\tau(r)_{min} \leq \tau(C) \leq \tau(r)_{max}. \tag{5.5.3}$$

Moreover, if d is even and $r = d/2$, then

$$\tau(r)_{min} \leq \tau(C) \leq \tau(r)_{max} - 1. \tag{5.5.4}$$

Corollary 5.5.2. *[Dim17, Corollary 1.2] If* $r = \mathrm{mdr}(f) < d/2$*, then one has that*

$$\tau(C) = \tau(C)_{max}$$

if and only if $C = V(f)$ *is a free curve.*

Theorem 5.5.3. *[Dim17, Theorem 1.3] If* $r = \mathrm{mdr}(f) \leq d/2$*, then one has that*

$$\tau(C) = \tau(C)_{max} - 1$$

if and only if $C = V(f)$ *is a nearly free curve.*

A. Dimca and G. Sticlaru also proposed the following conjecture:

Conjecture 5.5.4. *A plane curve* $C = V(f)$ *is free if and only if*

$$\mathrm{ct}(f) + \mathrm{st}(f) = T.$$

5.5.2 Rational cuspidal plane curves

The notion of nearly free curve, introduced by A. Dimca and G. Sticlaru in [DS18a], motivated the study of the rational cuspidal plane curves in this context.

In [DS17] A. Dimca and G. Sticlaru investigated a stronger property for cuspidal curves, namely they searched among them the free divisors. This can be explained by the fact that the number of known examples of irreducible free curves seems to be very limited. For instance, they proposed the following conjecture:

Conjecture 5.5.5. *[DS17] An irreducible plane curve of degree* $d \geq 2$ *which is a free divisor is a rational curve.*

They also proved the property below:

Corollary 5.5.6. *[DS17] If* C *is a free irreducible curve, then* C *is rational cuspidal if and only if*

$$(d_1 - 1)(d_2 - 1) = \mu(C) - \tau(C) + 1.$$

In particular, a free rational cuspidal plane curve cannot have only weighted homogeneous singularities unless $d_1 = d_2 = 2$ *(and hence* $d = 5$*).*

Next we present an infinite series of irreducible free curves obtained from the classification of rational cuspidal curves.

Theorem 5.5.7. *[DS17, Theorem 4.6] The rational cuspidal curve*

$$C_{2k+1} : f_{2k+1} = (y^{k-1}z + x^k)^2 y - x^{2k+1} = 0$$

of type $(2k+1, 2k-1)$ *has two cusps of type* $(2k+1, 2k-1)$ *and respectively* $(2k+1, 2)$ *and it is free for any* $k \geq 2$*. The corresponding Jacobian ideal* $J_{f_{2k+1}}$ *is of linear type if and only if* $k = 2$*.*

Moreover $\tau(C_{2k+1}) = 3k^2$*,* $\mu(C_{2k+1}) = 2k(2k-1)$ *and* $d_1 = d_2 = k$*.*

Proof. To prove that we have free divisors for any k we proceed as follows. We look at the syzygies among the partial derivatives f_x, f_y, f_z and find that we have two such syzygies in degree k, namely:

$$(s_1): \quad a_x f_x + a_y f_y + a_z f_z = 0,$$

where:
$a_x = 2x^k + 2y^{k-1}z$,
$a_y = (4k+2)x^k - 4kx^{k-1}y - (8k^2-2)y^{k-1}z$,
$a_z = 4k(k-1)x^{k-1}z + (8k^3 - 4k^2 - 2k + 1)y^{k-2}z^2$, and

$$(s_2): \quad b_x f_x + b_y f_y + b_z f_z = 0,$$

where:
$b_x = 0$,
$b_y = -2y^k$, and
$b_z = x^k + (2k-1)y^{k-1}z$.

It is clear that (s_1) and (s_2) are linearly independent as $b_x = 0$ and $a_x \neq 0$. Then we apply Lemma 1.1 and Proposition 1.8 in [ScOT14], exactly as in the proof of Proposition 2.2 in [ScOT14].

The Milnor number $\mu(C_{2k+1})$ is computed using Corollary 5.5.6 and this implies that the largest cusp of C_{2k+1} has type $(2k+1, 2k-1)$. Indeed, the other cusp is described in Proposition 3.2 (ii)[DS17] and we know that it has type $(2k+1, 2)$, i.e., it is an $\mathbb{A}_{2k}$-singularity with Milnor number $2k$.

□

We also present some other results by A. Dimca and G. Sticlaru regarding families of rational cuspidal curves which are either free or nearly free.

Theorem 5.5.8. *[DS18a, Theorem 3.1] Let $C = V(f)$ be a rational cuspidal curve of degree d. Assume that either:*

1. *d is even, or*
2. *d is odd and for any singularity x of C, the order of any eigenvalue λ_x of the local monodromy operator h_x is not d.*

Then C is either a free or a nearly free curve.

As a corollary, A. Dimca and G. Sticlaru proved the following:

Corollary 5.5.9. *[DS18a, Corollary 3.2] Let $C = V(f)$ be a rational cuspidal curve of degree d such that*

1. *either $d = p^k$ is a prime power, or*
2. *$\pi_1(U)$ is abelian, where $U = \mathbb{P}^2 \setminus C$.*

Then C is either a free or a nearly free curve.

Corollary 5.5.10. *[DS18a, Corollary 3.5] A unicuspidal rational curve with a unique Puiseux pair not of the type (3) of the Puiseux pairs realisable by a unicsupidal rational plane curve of degree d with $d = \varphi_{j-2}\varphi_j$ odd (see Theorem 4.3.1) is either free or nearly free.*

Assume from now on that d is odd, and let

$$d = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m} \tag{5.5.5}$$

be the prime decomposition of d. We assume also that $m \geq 2$, the case $m = 1$ being settled in the first statement of [DS18a, Corollary 3.2] (see 5.5.9 above). By changing the order of the p_j's if necessary, we can and do assume that $p_1^{k_1} > p_j^{k_j}$, for any $2 \leq j \leq m$. Set $e_1 = d/p_1^{k_1}$. Then:

Theorem 5.5.11. *[DS18b] Let $C = V(f)$ be a rational cuspidal curve of odd degree $d = 2d' + 1$. Then $\mathrm{mdr}(f) \leq d'$ and if equality holds, then C is either free or nearly free.*

Theorem 5.5.12. *[DS18b] Let $C = V(f)$ be a rational cuspidal curve of degree $d = 2d' + 1$ (an odd number as in (5.5.5)). If*

$$\mathrm{mdr}(f) \leq r_0 := \frac{d - e_1}{2},$$

then C is either free or nearly free. In particular, the following holds:

i) If $d = 3p^k$, with p a prime number, then C is either free or nearly free.

ii) If $d = 5p^k$, with p a prime number, $p^k > 3$, then C is either free or nearly free, unless $\mathrm{mdr}(f) = d' - 1$.

Corollary 5.5.13. *[DS18b] A rational cuspidal curve $C = V(f)$ of degree d is either free or nearly free, if one of the following holds:*

1. $\mathrm{mdr}(f) \leq 15$, *or*
2. $d \leq 90$, *unless we are in one of the following situations:*

   i) $d = 35$ *and* $\mathrm{mdr}(f) = 16$;
   ii) $d = 45$ *and* $\mathrm{mdr}(f) = 21$;
   iii) $d = 55$ *and* $\mathrm{mdr}(f) = 26$;
   iv) $d = 63$ *and* $\mathrm{mdr}(f) \in \{29, 30\}$;
   v) $d = 65$ *and* $\mathrm{mdr}(f) = 31$;
   vi) $d = 77$ *and* $\mathrm{mdr}(f) \in \{36, 37\}$;
   vii) $d = 85$ *and* $\mathrm{mdr}(f) = 41$.

In the excluded situations, the results in [DS18b] do not allow to conclude.

Example 5.5.14. Up to projective transformation, there are two quintic curves with two singular points of type $\mathbb{A}_4$ and $\mathbb{E}_8$. For these curves, the Milnor number is $\mu = \mu_{\mathbb{A}_4} + \mu_{\mathbb{E}_8} = 4 + 8 = 12$.

One of these quintic curves is the curve C_5 that will be presented in the next Theorem 5.6.1: $C_5 = V(f_5)$, where $f_5 = (yz - x^2)^2 y - x^5$; this curve is free (see Theorem 5.5.7) and satisfies $d_1 = d_2 = 4$ and $\tau(C_5) = 12$ (see 5.3.6).

There is another one, $D_5 = V(g_5)$, where $g_5 = y^3 z^2 - x^5$ (the contact of the tangent line to the $\mathbb{A}_4$-point distinguishes both curves).

The curve D_5 is nearly free, since $\mathrm{syz}(J_{g_5})$ is generated by $R_1 := (0, 2y, -3z)$, $R_2 := (3y^2z^2, 5x^4, 0)$ and $R_3 := (2y^3z, 0, 5x^4)$, so that $\mathrm{mdr}(g_5) = 1$, $d_1 = 1$ and $d_2 = 4$.

These syzygies satisfy the relation $-5x^4R_1 + 2yR_2 - 3zR_3 = 0$. Furthermore, since D_5 is nearly free, then $\tau(D_5) = 12$ according to equation 5.4.3.

The pair (C_5,D_5) is a kind of counterexample to Terao's conjecture, which asserts that freeness is a combinatorial invariant of an arrangement, [OT92, Conjecture 4.138] for irreducible divisors (with constant Tjurina number), compare with [ScOT09].

5.6 Counterexamples to the conjectures of A. Dimca and G. Sticlaru

5.6.1 Irreducible free curves with many branches and high genus

Let us consider the quintic curve C_5, see Figure 5.1, defined by

$$f_5 := (yz - x^2)^2y - x^5 = 0.$$

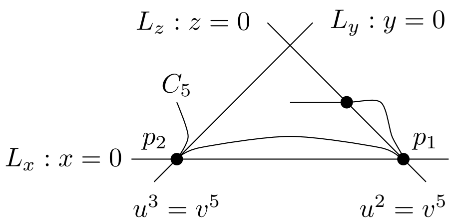


Figure 5.1: Curve C_5

It has two singular points, namely $p_1 = [0 : 1 : 0]$, of type $\mathbb{A}_4$, and $p_2 = [0 : 0 : 1]$, of type $\mathbb{E}_8$. Therefore, it is a rational and cuspidal plane curve. Furthermore, C_5 is free, as shown in Theorem 5.5.7.

Now consider the Kummer cover $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_k([x : y : z]) = [x^k : y^k : z^k]$ and its Kummer transform C_{5k}, defined by

$$f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k} = 0.$$

Theorem 5.6.1. *For any $k \geq 1$, the curve C_{5k} of degree $d = 5k$ defined by $C_{5k} = V(f_{5k})$, where*

$$f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k}, \tag{5.6.1}$$

verifies the following properties:

(1) $\mathrm{Sing}(C_{5k}) = \{q_1 = [0 : 1 : 0], q_2 = [0 : 0 : 1]\}$. *The number of branches of C_{5k} at q_2 is k, and at q_1 it equals k (if k is odd) or $2k$ (if k is even).*

(2) *C_{5k} is a free curve with exponents $d_1 = 2k$ and $d_2 = 3k - 1$ and $\tau(C_{5k}) = 19k^2 - 8k + 1$.*

(3) *C_{5k} has two irreducible components of genus $\frac{(k-2)^2}{4}$ if k is even, and it is irreducible of genus $\frac{(k-1)(k-2)}{2}$ otherwise.*

Proof. Part (1) is an easy consequence of Lemma 2.6.21, Proposition 2.6.22 and Proposition 2.6.25. The singularities $\mathrm{Sing}(C_5) = \{p_1, p_2\}$ are of type 0 in the sense of the Kummer cover π_k and C_5 has no singularities outside the intersection points of the axes. Moreover C_5 intersects the line L_z transversally at a smooth point $p_3 = [1 : 1 : 0]$ of type 1. Then by Proposition 2.6.22(2) and by Remark 2.6.23, the singularities of C_{5k} are exactly $q_1 = \pi_k^{-1}(p_1)$ and $q_2 = \pi_k^{-1}(p_2)$.

Since p_1 and p_2 are of type 0 we deduce the structure of C_{5k} at these points using Proposition 2.6.25(2)(b). At p_1 one has $(C_5, L_z)_{p_1} = 4$, $(C_5, L_x)_{p_1} = 2$ and $r_1 = \gcd(k, 2, 4) = \gcd(k, 2)$. If k is odd, then $r_1 = 1$ and so the number of branches of C_{5k} at q_1 is equal to k. Otherwise, $r_1 = 2$ and the number of branches of C_{5k} at q_1 is equal to $2k$. In the same way, in order to study the number of branches at p_2, we compute the intersection numbers $(C_5, L_x)_{p_2} = 3$ and $(C_5, L_y)_{p_2} = 5$, so $r_2 = \gcd(3, 5) = 1$ for all k, and therefore the number of branches of C_{5k} at q_2 is equal to k.

In order to prove (2), we develop the ideas of [DS17, Theorem 4.6] (see Theorem 5.5.7). Let us study first the syzygies of the free curve C_5.

Let us denote by J the Jacobian ideal of f_5, and let J_x be the ideal generated by $x{f_5}_x$, ${f_5}_y$ and ${f_5}_z$. In the same way, we consider the ideals J_y, J_z, J_{xy}, J_{xz}, J_{yz}, J_{xyz}:

- $J_y = ({f_5}_x, y{f_5}_y, {f_5}_z)$,
- $J_z = ({f_5}_x, {f_5}_y, z{f_5}_z)$,
- $J_{xy} = (x{f_5}_x, y{f_5}_y, {f_5}_z)$,
- $J_{xz} = (x{f_5}_x, {f_5}_y, z{f_5}_z)$,
- $J_{yz} = ({f_5}_x, y{f_5}_y, z{f_5}_z)$,
- $J_{xyz} = (x{f_5}_x, y{f_5}_y, z{f_5}_z)$.

Note that the chain rule implies that:

- ${f_{5k}}_x = kx^{k-1}{f_5}_x(x^k, y^k, z^k)$,
- ${f_{5k}}_y = ky^{k-1}{f_5}_y(x^k, y^k, z^k)$,
- ${f_{5k}}_z = kz^{k-1}{f_5}_z(x^k, y^k, z^k)$.

Let $S_k := \mathbb{C}[x^k, y^k, z^k]$. We have a decomposition

$$S = \bigoplus_{i,j,l \in \{0,\ldots,k-1\}} x^i y^j z^l S_k. \tag{5.6.2}$$

By construction, ${f_{5k}}_x \in x^{k-1}S_k$, ${f_{5k}}_y \in y^{k-1}S_k$ and ${f_{5k}}_z \in z^{k-1}S_k$. Hence, in order to compute the syzygies (a, b, c) among the partial derivatives of f_{5k}, we need to characterise the triples (a, b, c) such that each entry belongs to a factor of the decomposition (5.6.2).

Let us assume that:
$a \in x^{i_a} y^{j_a} z^{l_a} S_k$,

$b \in x^{i_b}y^{j_b}z^{l_b}S_k$,
$c \in x^{i_c}y^{j_c}z^{l_c}S_k$,

where $0 \leq i_a, j_a, l_a, i_b, j_b, l_b, i_c, j_c, l_c \leq k-1$.

We deduce that

$$i_x + k - 1 \equiv i_y \equiv i_z \bmod k \Longrightarrow i = i_y = i_z \text{ and } i_x = \begin{cases} i+1 & \text{if } i < k-1 \\ 0 & \text{if } i = k-1 \end{cases}$$

(analogous relations hold for the other indices). Let us explain these relationships in more detail:

For this aim, we define the polynomial

$$\begin{aligned} H(x,y,z) := & x^{i_a}y^{j_a}z^{l_a}\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k) \\ & + x^{i_b}y^{j_b}z^{l_b}\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k) \\ & + x^{i_c}y^{j_c}z^{l_c}\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k). \end{aligned}$$

Since (a,b,c) is a syzygy of the ideal $J(f_{5k})$, then $H(x,y,z) \equiv 0$. In particular, $H(x,1,1) = 0$:

$$\begin{aligned} H(x,1,1) = & x^{i_a}\alpha(x^k,1,1)kx^{k-1}f_{5_x}(x^k,1,1) \\ & + x^{i_b}\beta(x^k,1,1)ky^{k-1}f_{5_y}(x^k,1,1) \\ & + x^{i_c}\gamma(x^k,1,1)kz^{k-1}f_{5_z}(x^k,1,1) \quad = 0. \end{aligned}$$

Therefore:

$$\begin{aligned} & i_a + kd_{\alpha_x} + (k-1) + 4k = i_b + kd_{\beta_x} + (k-1) + 4k = i_c + kd_{\gamma_x} + (k-1) + 4k \Rightarrow \\ & i_a + (k-1) + 4k \equiv i_b + (k-1) \equiv i_c + (k-1) \mod k, \end{aligned}$$

where d_{α_x} (respectively d_{β_x} and d_{γ_x}) is the degree of the polynomial α (resp. β, γ) in the variable x.

Consequently,

$$i_a + k - 1 \equiv i_b \equiv i_c \bmod k \Rightarrow i := i_b = i_c.$$

- If $i = k-1$, then:

$$i_a + k - 1 \equiv k - 1 \bmod k \Rightarrow i_a \equiv 0 \bmod k \Rightarrow i_a = 0.$$

- Otherwise, if $i < k-1$, then:

$$i_a + k - 1 \equiv i \mod k \Rightarrow i_a - 1 \equiv i \mod k \Rightarrow i_a \equiv i + 1 \mod k \Rightarrow i_a = i + 1.$$

Analogously, since $H(1,y,1) = 0$:

$$j_a \equiv j_b + k - 1 \equiv j_c \bmod k \Longrightarrow j := j_a = j_c \text{ and } j_b = \begin{cases} j+1 & \text{if } j < k-1 \\ 0 & \text{if } j = k-1. \end{cases}$$

Finally, $H(1,1,z)=0$ implies that:

$$l_a \equiv l_b \equiv l_c + k - 1 \bmod k \Longrightarrow l := l_a = l_b \text{ and } l_c = \begin{cases} l+1 & \text{if } l < k-1 \\ 0 & \text{if } l = k-1. \end{cases}$$

Therefore:

$a = x^{i_a} y^j z^l \alpha(x^k, y^k, z^k),$
$b = x^i y^{j_b} z^l \beta(x^k, y^k, z^k),$
$c = x^i y^j z^{l_c} \gamma(x^k, y^k, z^k).$

and we distinguish eight cases:

Case 1:
$i = j = l = k-1 \Longrightarrow i_a = j_b = l_c = 0;$

$a = y^{k-1} z^{k-1} \alpha(x^k, y^k, z^k),$
$b = x^{k-1} z^{k-1} \beta(x^k, y^k, z^k),$
$c = x^{k-1} y^{k-1} \gamma(x^k, y^k, z^k).$

(a,b,c) is a syzygy among the partial derivatives of f_{5k}, so

$$\begin{aligned} & y^{k-1}z^{k-1}\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k) + x^{k-1}z^{k-1}\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k) \\ & + x^{k-1}y^{k-1}\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k) = 0 \Longrightarrow \\ & kx^{k-1}y^{k-1}z^{k-1}[\alpha(x^k,y^k,z^k)f_{5_x}(x^k,y^k,z^k) + \beta(x^k,y^k,z^k)f_{5_y}(x^k,y^k,z^k) \\ & + \gamma(x^k,y^k,z^k)f_{5_z}(x^k,y^k,z^k)] = 0, \end{aligned}$$

and this implies that (α,β,γ) is a syzygy of the generators of the ideal J.

$\operatorname{syz}(J)$ is generated by $R_1^J = (0, 2y^2, x^2 - 3yz)$ and $R_2^J = (2x^2 - 2yz, 10x^2 - 8xy + 30yz, 8xz - 45z^2)$.

Hence (a,b,c) is a combination of $\big(0, 2x^{k-1}z^{k-1}y^{2k}, x^{k-1}y^{k-1}(x^{2k} - 3y^k z^k)\big)$ and $\big(y^{k-1}z^{k-1}(2x^{2k} - 2y^k z^k), x^{k-1}z^{k-1}(10x^{2k} - 8x^k y^k + 30y^k z^k), x^{k-1}y^{k-1}(8x^k z^k - 45z^{2k})\big)$.

Taking out common factors we get syzygies of degrees $2k$ and $3k-1$:

$S_1^{J,k} = \big(0, 2z^{k-1}y^{k+1}, x^{2k} - 3y^k z^k\big);$
$S_2^{J,k} = \big(y^{k-1}(2x^{2k} - 2y^k z^k), x^{k-1}(10x^{2k} - 8x^k y^k + 30y^k z^k), x^{k-1}y^{k-1}(8x^k z - 45z^{k+1})\big).$

Case 2:
$i < k-1; j = l = k-1 \Longrightarrow i_a = i+1; j_b = l_c = 0;$

$a = x^{i+1} y^{k-1} z^{k-1} \alpha(x^k, y^k, z^k),$
$b = x^i z^{k-1} \beta(x^k, y^k, z^k),$
$c = x^i y^{k-1} \gamma(x^k, y^k, z^k).$

(a,b,c) is a syzygy among the partial derivatives of f_{5k}, so

$$x^{i+1}y^{k-1}z^{k-1}\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k)+x^iz^{k-1}\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k)$$
$$+x^iy^{k-1}\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k)=0\Longrightarrow$$
$$kx^iy^{k-1}z^{k-1}[x^k\alpha(x^k,y^k,z^k)f_{5_x}(x^k,y^k,z^k)+\beta(x^k,y^k,z^k)f_{5_y}(x^k,y^k,z^k)$$
$$+\gamma(x^k,y^k,z^k)f_{5_z}(x^k,y^k,z^k)]=0$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_x.

The following generators of $\text{syz}(J_x)$ are obtained using `Singular` [DGPS19]:

$R_1^{J_x} = (0, 2y^2, x^2 - 3yz)$,
$R_2^{J_x} = (2x^2 - 2yz, 10x^3 - 8x^2y + 30xyz - 16y^2z, -45xz^2 + 24yz^2)$.

We consider the system of generators of $\text{syz}(J_x)$ given by $R_1^{J_x}$ and $8zR_1^{J_x} + R_2^{J_x}$.

Hence (a, b, c) is a combination of $\big(0, 2x^iz^{k-1}y^{2k}, x^iy^{k-1}(x^{2k} - 3y^kz^k)\big)$ and

$$(x^{i+1}y^{k-1}z^{k-1}(2x^{2k}-2y^kz^k), x^iz^{k-1}(10x^{3k}-8x^{2k}y^k+30x^ky^kz^k),$$
$$x^iy^{k-1}(-45x^kz^{2k}+8z^kx^{2k})).$$

Taking out common factors we get syzygies of degrees $2k$ and $3k-1$:

$$S_1^{J_x,k} = \big(0, 2z^{k-1}y^{k+1}, x^{2k} - 3y^kz^k\big);$$

$$S_2^{J_x,k} = (y^{k-1}(2x^{2k}-2y^kz^k), 10x^{3k-1}-8x^{2k-1}y^k+30x^{k-1}y^kz^k,$$
$$y^{k-1}(-45x^{k-1}z^{k+1}+8zx^{2k-1})).$$

Case 3:
$j < k-1;\ = l = k-1 \Longrightarrow j_b = j+1; i_a = l_c = 0;$

$a = y^jz^{k-1}\alpha(x^k, y^k, z^k)$,
$b = x^{k-1}y^{j+1}z^{k-1}\beta(x^k, y^k, z^k)$,
$c = x^{k-1}y^j\gamma(x^k, y^k, z^k)$.

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$y^jz^{k-1}\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k)+x^{k-1}y^{j+1}z^{k-1}\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k)$$
$$+x^{k-1}y^j\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k)=0\Longrightarrow$$
$$kx^{k-1}y^jz^{k-1}[\alpha(x^k,y^k,z^k)f_{5_x}(x^k,y^k,z^k)+y^k\beta(x^k,y^k,z^k)f_{5_y}(x^k,y^k,z^k)$$
$$+\gamma(x^k,y^k,z^k)f_{5_z}(x^k,y^k,z^k)]=0$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_y.

$\text{syz}(J_y)$ is generated by $R_1^{J_y} = (0, 2y, x^2 - 3yz)$ and
$R_2^{J_y} = (2x^2y - 2y^2z, 10x^2 - 8xy + 30yz, 8xyz - 45yz^2)$.

Hence (a, b, c) is a combination of $\big(0, 2x^{k-1}y^{j+1}z^{k-1}y^k, x^{k-1}y^j(x^{2k} - 3y^kz^k)\big)$ and $\big(y^jz^{k-1}(2x^{2k}y^k - 2y^{2k}z^k), x^{k-1}y^{j+1}z^{k-1}(10x^{2k} - 8x^ky^k + 30y^kz^k), x^{k-1}y^j(8x^ky^kz^k - 45y^kz^{2k})\big)$.

Taking out common factors we get syzygies of degrees $2k$ and $3k - 1$:

$$S_1^{J_y,k} = \big(0, 2yz^{k-1}y^k, x^{2k} - 3y^kz^k\big);$$
$$S_2^{J_y,k} = \big(2x^{2k}y^{k-1} - 2y^{2k-1}z^k, x^{k-1}(10x^{2k} - 8x^ky^k + 30y^kz^k), x^{k-1}(8x^ky^{k-1}z - 45y^{k-1}z^{k+1})\big).$$

Case 4:
$l < k - 1; i = j = k - 1 \Longrightarrow l_c = l + 1; i_a = j_b = 0;$

$a = y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^{k-1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^{k-1}z^{l+1}\gamma(x^k, y^k, z^k).$

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$\begin{aligned}&y^{k-1}z^l\alpha(x^k, y^k, z^k)kx^{k-1}f_{5_x}(x^k, y^k, z^k) + x^{k-1}z^l\beta(x^k, y^k, z^k)ky^{k-1}f_{5_y}(x^k, y^k, z^k)\\ &+ x^{k-1}y^{k-1}z^{l+1}\gamma(x^k, y^k, z^k)kz^{k-1}f_{5_z}(x^k, y^k, z^k) = 0 \Longrightarrow\\ &kx^{k-1}y^{k-1}z^l[\alpha(x^k, y^k, z^k)f_{5_x}(x^k, y^k, z^k) + \beta(x^k, y^k, z^k)f_{5_y}(x^k, y^k, z^k)\\ &+ z^k\gamma(x^k, y^k, z^k)f_{5_z}(x^k, y^k, z^k)] = 0\end{aligned}$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_z.

syz(J_z) is generated by $R_1^{J_z} = (0, 2y^2z, x^2 - 3yz)$ and $R_2^{J_z} = (2x^2 - 2yz, 10x^2 - 8xy + 30yz, 8x - 45z)$.

Hence (a, b, c) is a combination of $\big(0, 2x^{k-1}z^ly^{2k}z^k, x^{k-1}y^{k-1}z^{l+1}(x^{2k} - 3y^kz^k)\big)$ and $\big(y^{k-1}z^l(2x^{2k} - 2y^kz^k), x^{k-1}z^l(10x^{2k} - 8x^ky^k + 30y^kz^k), x^{k-1}y^{k-1}z^{l+1}(8x^k - 45z^k)\big)$.

Taking out common factors we get syzygies of degrees $2k$ and $3k - 1$:

$$S_1^{J_z,k} = \big(0, 2y^{k+1}z^{k-1}, x^{2k} - 3y^kz^k\big);$$
$$S_2^{J_z,k} = \big(y^{k-1}(2x^{2k} - 2y^kz^k), x^{k-1}(10x^{2k} - 8x^ky^k + 30y^kz^k), x^{k-1}y^{k-1}z(8x^k - 45z^k)\big).$$

Case 5:
$i, j < k - 1; l = k - 1 \Longrightarrow i_a = i + 1; j_b = j + 1; l_c = 0;$

$a = x^{i+1}y^jz^{k-1}\alpha(x^k, y^k, z^k),$
$b = x^iy^{j+1}z^{k-1}\beta(x^k, y^k, z^k),$
$c = x^iy^j\gamma(x^k, y^k, z^k).$

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$\begin{aligned}&x^{i+1}y^jz^{k-1}\alpha(x^k, y^k, z^k)kx^{k-1}f_{5_x}(x^k, y^k, z^k) + x^iy^{j+1}z^{k-1}\beta(x^k, y^k, z^k)ky^{k-1}f_{5_y}(x^k, y^k, z^k)\\ &+ x^iy^j\gamma(x^k, y^k, z^k)kz^{k-1}f_{5_z}(x^k, y^k, z^k) = 0 \Longrightarrow\\ &kx^iy^jz^{k-1}[x^k\alpha(x^k, y^k, z^k)f_{5_x}(x^k, y^k, z^k) + y^k\beta(x^k, y^k, z^k)f_{5_y}(x^k, y^k, z^k)\\ &+ \gamma(x^k, y^k, z^k)f_{5_z}(x^k, y^k, z^k)] = 0\end{aligned}$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_{xy}.

The following generators of $\text{syz}(J_{xy})$ are obtained using `Singular` [DGPS19]:

$$R_1^{J_{xy}} = (0, 2y, x^2 - 3yz),$$
$$R_2^{J_{xy}} = (-2x^2y + 2y^2z, -10x^3 + 8x^2y - 30xyz + 16y^2z, 45xyz^2 - 24y^2z^2).$$

We consider the system of generators of $\text{syz}(J_{xy})$ given by $R_1^{J_{xy}}$ and $R_2^{J_{xy}} - 8yzR_1^{J_{xy}}$.

Hence (a, b, c) is a combination of $\left(0, 2x^iy^{j+1}z^{k-1}y^k, x^iy^j(x^{2k} - 3y^kz^k)\right)$ and

$$(x^{i+1}y^jz^{k-1}(-2x^{2k}y^k + 2y^{2k}z^k), x^iy^{j+1}z^{k-1}(-10x^{3k} + 8x^{2k}y^k - 30x^ky^kz^k),$$
$$x^iy^j(45x^ky^kz^{2k} + 8x^{2k}y^kz^k)).$$

Taking out common factors we get syzygies of degrees $2k$ and $3k - 1$:

$$S_1^{J_{xy},k} = \left(0, 2yz^{k-1}y^k, x^{2k} - 3y^kz^k\right);$$

$$S_2^{J_{xy},k} = (-2x^{2k}y^{k-1} + 2y^{2k-1}z^k, -10x^{3k-1} + 8x^{2k-1}y^k - 30x^{k-1}y^kz^k,$$
$$45x^{k-1}y^{k-1}z^{k+1} + 8x^{2k-1}y^{k-1}z).$$

Case 6:
$i, l < k - 1; j = k - 1 \Longrightarrow i_a = i + 1; l_c = l + 1; j_b = 0;$

$a = x^{i+1}y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^iz^l\beta(x^k, y^k, z^k),$
$c = x^iy^{k-1}z^{l+1}\gamma(x^k, y^k, z^k).$

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$x^{i+1}y^{k-1}z^l\alpha(x^k, y^k, z^k)kx^{k-1}f_{5_x}(x^k, y^k, z^k) + x^iz^l\beta(x^k, y^k, z^k)ky^{k-1}f_{5_y}(x^k, y^k, z^k)$$
$$+ x^iy^{k-1}z^{l+1}\gamma(x^k, y^k, z^k)kz^{k-1}f_{5_z}(x^k, y^k, z^k) = 0 \Longrightarrow$$
$$kx^iy^{k-1}z^l[x^k\alpha(x^k, y^k, z^k)f_{5_x}(x^k, y^k, z^k) + \beta(x^k, y^k, z^k)f_{5_y}(x^k, y^k, z^k)$$
$$+ z^k\gamma(x^k, y^k, z^k)f_{5_z}(x^k, y^k, z^k)] = 0$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_{xz}.

The following generators of $\text{syz}(J_{xz})$ are obtained using `Singular` [DGPS19]:

$$R_1^{J_{xz}} = (0, 2y^2z, x^2 - 3yz),$$
$$R_2^{J_{xz}} = (2x^2 - 2yz, 10x^3 - 8x^2y + 30xyz + 8y^2z, 12x^2 - 45xz - 12yz).$$

We consider the system of generators of $\text{syz}(J_{xz})$ given by $R_1^{J_{xz}}$ and $R_2^{J_{xz}} - 4R_1^{J_{xz}}$.

Hence (a, b, c) is a combination of $\left(0, 2x^iz^ly^{2k}z^k, x^iy^{k-1}z^{l+1}(x^{2k} - 3y^kz^k)\right)$ and

$$(x^{i+1}y^{k-1}z^l(2x^{2k}-2y^kz^k), x^iz^l(10x^{3k}-8x^{2k}y^k+30x^ky^kz^k),$$
$$x^iy^{k-1}z^{l+1}(12x^{2k}-45x^kz^k)).$$

Taking out common factors we get syzygies of degrees $2k$ and $3k-1$:

$$S_1^{J_{xz},k} = \left(0, 2y^{k+1}z^{k-1}, x^{2k}-3y^kz^k\right);$$

$$S_2^{J_{xz},k} = (y^{k-1}(2x^{2k}-2y^kz^k), 10x^{3k-1}-8x^{2k-1}y^k+30x^{k-1}y^kz^k,$$
$$y^{k-1}z(12x^{2k-1}-45x^{k-1}z^k)).$$

Case 7:
$j, l < k-1; i = k-1 \Longrightarrow j_b = j+1; l_c = l+1; i_a = 0;$

$a = y^jz^l\alpha(x^k, y^k, z^k),$
$b = x^{k-1}y^{j+1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^jz^{l+1}\gamma(x^k, y^k, z^k).$

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$y^jz^l\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k)+x^iy^{j+1}z^l\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k)$$
$$+x^iy^jz^{l+1}\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k)=0 \Longrightarrow$$
$$kx^{k-1}y^jz^l[\alpha(x^k,y^k,z^k)f_{5_x}(x^k,y^k,z^k)+y^k\beta(x^k,y^k,z^k)f_{5_y}(x^k,y^k,z^k)$$
$$+z^k\gamma(x^k,y^k,z^k)f_{5_z}(x^k,y^k,z^k)]=0$$

and this implies that (α, β, γ) is a syzygy of the generators of the ideal J_{yz}.

$\text{syz}(J_{yz})$ is generated by $R_1^{J_{yz}} = (0, 2yz, x^2-3yz)$ and
$R_2^{J_{yz}} = (2x^2y-2y^2z, 10x^2-8xy+30yz, 8xy-45yz)$.

Hence (a, b, c) is a combination of $\left(0, 2x^{k-1}y^{j+1}z^ly^kz^k, x^{k-1}y^jz^{l+1}(x^{2k}-3y^kz^k)\right)$ and
$\left(y^jz^l(2x^{2k}y^k-2y^{2k}z^k), x^{k-1}y^{j+1}z^l(10x^{2k}-8x^ky^k+30y^kz^k), x^{k-1}y^jz^{l+1}(8x^ky^k-45y^kz^k)\right)$.

Taking out common factors we get syzygies of degrees $2k$ and $3k-1$:

$$S_1^{J_{yz},k} = \left(0, 2y^{k+1}z^{k-1}, x^{2k}-3y^kz^k\right);$$
$$S_2^{J_{yz},k} = \left(2x^{2k}y^{k-1}-2y^{2k-1}z^k, x^{k-1}(10x^{2k}-8x^ky^k+30y^kz^k), x^{k-1}z(8x^ky^{k-1}-45y^{k-1}z^k)\right).$$

Case 8:
$i, j, l < k-1 \Longrightarrow i_a = i+1; j_b = j+1; l_c = l+1;$

$a = x^{i+1}y^jz^l\alpha(x^k, y^k, z^k),$
$b = x^iy^{j+1}z^l\beta(x^k, y^k, z^k),$
$c = x^iy^jz^{l+1}\gamma(x^k, y^k, z^k).$

(a, b, c) is a syzygy among the partial derivatives of f_{5k}, so

$$
\begin{aligned}
&x^{i+1}y^jz^l\alpha(x^k,y^k,z^k)kx^{k-1}f_{5_x}(x^k,y^k,z^k)+x^iy^{j+1}z^l\beta(x^k,y^k,z^k)ky^{k-1}f_{5_y}(x^k,y^k,z^k)\\
&+x^iy^jz^{l+1}\gamma(x^k,y^k,z^k)kz^{k-1}f_{5_z}(x^k,y^k,z^k)=0\Longrightarrow\\
&kx^iy^jz^l[x^k\alpha(x^k,y^k,z^k)f_{5_x}(x^k,y^k,z^k)+y^k\beta(x^k,y^k,z^k)f_{5_y}(x^k,y^k,z^k)\\
&+z^k\gamma(x^k,y^k,z^k)f_{5_z}(x^k,y^k,z^k)]=0
\end{aligned}
$$

and this implies that (α,β,γ) is a syzygy of the generators of the ideal J_{xyz}.

The following generators of $\mathrm{syz}(J_{xyz})$ are obtained using `Singular` [DGPS19]:

$$
\begin{aligned}
R_1^{J_{xyz}}&=(0,2yz,x^2-3yz),\\
R_2^{J_{xyz}}&=(2x^2y-2y^2z,10x^3-8x^2y+30xyz-16y^2z,-45xyz+24y^2z).
\end{aligned}
$$

We consider the system of generators of $\mathrm{syz}(J_{xyz})$ given by $R_1^{J_{xyz}}$ and $R_2^{J_{xyz}}+8yR_1^{J_{xyz}}$.

Hence (a,b,c) is a combination of $\big(0,2x^iy^{j+1}z^ly^kz^k,x^iy^jz^{l+1}(x^{2k}-3y^kz^k)\big)$ and

$$
\begin{aligned}
&(x^{i+1}y^jz^l(2x^{2k}y^k-2y^{2k}z^k),x^iy^{j+1}z^l(10x^{3k}-8x^{2k}y^k+30x^ky^kz^k),\\
&x^iy^jz^{l+1}(-45x^ky^kz^k+8y^kx^{2k})).
\end{aligned}
$$

Taking out common factors we get syzygies of degrees $2k$ and $3k-1$:

$$
S_1^{J_{xyz},k}=\big(0,2y^{k+1}z^{k-1},x^{2k}-3y^kz^k\big)\,;
$$

$$
\begin{aligned}
S_2^{J_{xyz},k}=\;&(2x^{2k}y^{k-1}-2y^{2k-1}z^k,10x^{3k-1}-8x^{2k-1}y^k+30x^{k-1}y^kz^k,\\
&z(-45x^{k-1}y^{k-1}z^k+8y^{k-1}x^{2k-1})).
\end{aligned}
$$

In view of these results, it is clear that $\mathrm{mdr}(f_{5k})=2k$, and a system of generators of $\mathrm{AR}(f_{5k})$ is:

$$
\begin{aligned}
S_1^{J,k}&=\big(0,2z^{k-1}y^{k+1},x^{2k}-3y^kz^k\big);\\
S_2^{J,k}&=\big(y^{k-1}(2x^{2k}-2y^kz^k),x^{k-1}(10x^{2k}-8x^ky^k+30y^kz^k),x^{k-1}y^{k-1}(8x^kz-45z^{k+1})\big),
\end{aligned}
$$

which are linearly independent.

We conclude that C_{5k} is free with $d_1=\mathrm{mdr}(f_{5k})=2k$ and $d_2=d-1-d_1=5k-1-2k=3k-1$. By equation (5.3.6), $\tau(C_{5k})=(5k-1)^2-2k(3k-1)=19k^2-8k+1$ for all k.

In order to prove (3), we study the branched cover $\tilde{\pi}_k:\tilde{C}_{5k}\to\tilde{C}_5$ between the normalisations of the curves C_{5k} and C_5, respectively. The monodromy of this map as an unramified cover of $\mathbb{P}^2\setminus\{xyz=0\}$ is determined by an epimorphism

$$
H_1(\mathbb{P}^2\setminus\{xyz=0\};\mathbb{Z})\to\mathbb{Z}/k\mathbb{Z}\times\mathbb{Z}/k\mathbb{Z}=:G_k \tag{5.6.3}
$$

such that the meridians of the lines are sent to a_x,a_y,a_z, a system of generators of G_k such that $a_x+a_y+a_z=0$. Notice that, since $\pi_1(\mathbb{P}^2\setminus V(xyz))\cong\mathbb{Z}\times\mathbb{Z}$ is abelian, we can identify $\pi_1(\mathbb{P}^2\setminus V(xyz))\cong H_1(\mathbb{P}^2\setminus V(xyz),\mathbb{Z})$.

We have the map

$$\begin{array}{ccccccc}
\pi_1(\mathbb{P}^2\setminus V(xyz)) & \xrightarrow{(\pi_k)_*} & \pi_1(\mathbb{P}^2\setminus V(xyz)) & \xrightarrow{p} & \mathrm{Gal}(\pi_k) & \cong & G_k := \mathbb{Z}/k\mathbb{Z}\times\mathbb{Z}/k\mathbb{Z} \\
[\gamma_x] & \mapsto & [\gamma_x^k] & \mapsto & a_x & \leftrightarrow & (1,0) \\
[\gamma_y] & \mapsto & [\gamma_y^k] & \mapsto & a_y & \leftrightarrow & (0,1) \\
[\gamma_z] & \mapsto & [\gamma_z^k] & \mapsto & a_z & \leftrightarrow & (-1,-1)
\end{array}$$

where the meridians γ_L of the lines satisfy $\gamma_x\gamma_y\gamma_z = 1$.

Since the singularities of C_5 are locally irreducible, then C_5 and $\tilde{C}_5$ are homeomorphic. Hence $\tilde{C}_5\setminus\{xyz=0\}$ is isomorphic to $\mathbb{P}^1\setminus\{\text{three points}\}$, so that $\pi_1(\tilde{C}_5\setminus V(xyz)) \cong \pi_1(\mathbb{P}^1\setminus\{p_1,p_2,p_3\}) \cong \mathbb{Z}*\mathbb{Z}$.

The covering $\tilde{\pi}_k$ is determined by the monodromy map

$$H_1(\tilde{C}_5\setminus\{xyz=0\};\mathbb{Z}) \to G_k$$

obtained by composing with the map defined by the inclusion.

The image of a meridian corresponding to a point P in the axes is given by

$$m_P^{L_x}a_x + m_P^{L_y}a_y + m_P^{L_z}a_z.$$

More specifically,

$$\begin{array}{ccccc}
\pi_1(\tilde{C}_5\setminus V(xyz)) & \xrightarrow{i_*} & H_1(\mathbb{P}^2\setminus V(xyz),\mathbb{Z}) & \xrightarrow{p} & \mathrm{Gal}(\pi_k) \\
[\alpha_P] & \mapsto & [m_P^{L_x}\gamma_x + m_P^{L_y}\gamma_y + m_P^{L_z}\gamma_z] & \mapsto & m_P^{L_x}a_x + m_P^{L_y}a_y + m_P^{L_z}a_z \\
[\alpha_{p_1}] & \mapsto & [2\gamma_x + 4\gamma_z] & \mapsto & 2a_x + 4a_z \\
[\alpha_{p_2}] & \mapsto & [3\gamma_x + 5\gamma_y] & \mapsto & 3a_x + 5a_y \\
[\alpha_{p_3}] & \mapsto & [\gamma_z] & \mapsto & a_z
\end{array}$$

Hence, we obtain a_z (the smooth point), $3a_x+5a_y$ (the $\mathbb{E}_8$-point) and $2a_x+4a_z$ (the $\mathbb{A}_4$-point). In terms of the basis a_y, a_z they read as $a_z, 2a_y-3a_z, -2a_y+2a_z$, i.e., the monodromy group is generated by $2a_y, a_z$. If k is even ($k=2\ell$), the monodromy group is of index 2 in G_k, and hence $\tilde{C}_{5k}$ has two connected components (recall 2.6.1):

$$f_{5k} = (x^{5\ell} - x^{4\ell}y^{\ell} + y^{3\ell}z^{2\ell})(-x^{5\ell} - x^{4\ell}y^{\ell} + y^{3\ell}z^{2\ell}),$$

whereas it is G_k when k is odd, so that $\tilde{C}_{5k}$ is connected. These properties give us the statement about the number of irreducible components.

The genus can be computed using the singularities of C_{5k} or via Riemann-Hurwitz's formula. The Euler characteristic of $\tilde{C}_{5k}$ is:

$$\chi(\tilde{C}_{5k}) = 3\cdot 5k - (5k)^2 + \sum_{q\in\mathrm{Sing}(C_{5k})}(\mu_q + r_q - 1),$$

where the Milnor number of $q\in\mathrm{Sing}(C_{5k})$ is $\mu_q = k^2(\mu_p - 1) + k(k-1)(m_p^{L_1} + m_p^{L_2}) + 1$ and the number of branches of C_{5k} at q is $r_q = k\cdot\gcd(k, m_p^{L_1}, m_p^{L_2})$ (recall Proposition 2.6.25(2)).

We have that $\mu_{q_1} = 9k^2 - 6k + 1$, $\mu_{q_2} = 15k^2 - 8k + 1$, since $\mu_{p_1} = 4$ and $\mu_{p_2} = 8$. The number of branches at the singular points of C_{5k} is $r_{q_1} = k$, if k is odd and $r_{q_1} = 2k$, if k is even, and $r_{q_2} = k$.

Therefore, $\chi(\tilde{C}_{5k}) = -k^2 + 2k + r_{q_1}$. Hence, for k odd,

$$\chi(\tilde{C}_{5k}) = -k^2 + 3k \Longrightarrow g(\tilde{C}_{5k}) = \frac{(k-1)(k-2)}{2},$$

and for k even,

$$\chi(\tilde{C}_{5k}) = -k^2 + 4k \Longrightarrow g(\tilde{C}^i_{5k}) = \frac{2 - \frac{\chi(\tilde{C}_{5k})}{2}}{2} = \frac{(k-2)^2}{4}.$$

where $\tilde{C}_{5k} = \tilde{C}^1_{5k} \cup \tilde{C}^2_{5k}$. $\square$

So, for odd $k \geq 3$, the curve C_{5k} is an irreducible free curve of positive genus whose singularities have k branches each. This is a counterexample to both the free part of Conjecture 5.1.1(ii) and Conjecture 5.1.2(i).

5.6.2 Irreducible nearly free curves with many branches and high genus

The quartic curve C_4, defined by

$$f_4 := (yz - x^2)^2 - x^3 y = 0,$$

has two singular points, namely $p_1 = [0:1:0]$, of type $\mathbb{A}_2$, and $p_2 = [0:0:1]$, of type $\mathbb{A}_4$. Therefore C_4 is rational and cuspidal. We will consider the Kummer transform C_{4k} of the curve C_4, defined by

$$f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0.$$

Theorem 5.6.2. *For any $k \geq 1$, the curve C_{4k} of degree $d = 4k$ defined by $C_{4k} = V(f_{4k})$, where*

$$f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k \tag{5.6.4}$$

verifies the following properties:

(1) $\mathrm{Sing}(C_{4k}) = \{q_1 = [0:1:0], q_2 = [0:0:1]\}$. *The number of branches of C_{4k} at q_2 is k, and at q_1 it equals k (if k is odd) or $2k$ (if k is even).*

(2) *C_{4k} is a nearly free curve with exponents $d_1 = d_2 = d_3 = 2k$, and $\tau(C_{4k}) = 6k(2k-1)$.*

(3) *C_{4k} has two irreducible components of genus $\frac{(k-2)^2}{4}$ if k is even, and it is irreducible of genus $\frac{(k-1)(k-2)}{2}$ otherwise.*

Proof. The singularities $\mathrm{Sing}(C_4) = \{p_1, p_2\}$ are of type 0 in the sense of the Kummer cover π_k and C_4 has no singularities outside the intersection points of the axes. Moreover, C_4 intersects the line L_z transversally at a smooth point $p_3 = [1:1:0]$ of type 1. Then by Proposition 2.6.22 (2) and by Remark 2.6.23, singularities of C_{4k} are exactly $q_1 = \pi_k^{-1}(p_1)$ and $q_2 = \pi_k^{-1}(p_2)$.

To prove Part (1) it is enough to find the number of branches of C_{4k} at these singular points using Proposition 2.6.25 (2) (b). At p_1 one has $(C_4, L_z)_{p_1} = 3$, $(C_4, L_x)_{p_1} = 2$ and $r_1 = \gcd(k, 2, 3) = 1$

for all k, so that the number of branches of C_{4k} at q_1 is equal to k. In the same way, at p_2, the intersection $(C_4, L_x)_{p_2} = 2$, $(C_4, L_y)_{p_2} = 4$ and $r_2 = \gcd(k, 2, 4) = \gcd(k, 2)$. If k is odd, $r_2 = 1$ and the number of branches of C_{4k} at q_2 is equal to k. Otherwise, $r_2 = 2$ and the number of branches of C_{4k} at q_2 is equal to $2k$.

The proof of Part (2) follows the same guidelines as in Theorem 5.6.1. With the notations of that proof, a system of generators for the syzygies of J (the Jacobian ideal of f_4) is given by:

$$\begin{aligned} R_1^J &:= (y(3x-4z),\, 3y(4x-3y),\, z(9y-20x))\,, \\ R_2^J &:= \left(-x(x+2z),\, -4x^2+3xy+10yz,\, -z(3x+10z)\right), \\ R_3^J &:= \left(xy,\, -3y^2,\, 2x^2+3yz\right). \end{aligned} \tag{5.6.5}$$

These syzygies satisfy the relation $xR_1^J + 3yR_2^J + 10zR_3^J = 0$. Thus, by Remark 5.4.6, C_4 is a nearly free curve with exponents $d_1 = d_2 = d_3 = 2$.

For the ideal J_z, we have a similar situation. For the other ideals, their syzygy space is free of rank 2. Using these results it is not hard to prove that the syzygies of the Jacobian ideal of f_{4k} are generated by

$$\begin{aligned} R_{k,1} &:= \left(y^k(3x^k-4z^k),\, 3x^{k-1}y(4x^k-3y^k),\, x^{k-1}z(9y^k-20x^k)\right), \\ R_{k,2} &:= \left(-xy^{k-1}(x^k+2z^k),\, -4x^{2k}+3x^ky^k+10y^kz^k,\, -y^{k-1}z(3x^k+10z^k)\right), \\ R_{k,3} &:= \left(xy^kz^{k-1},\, -3y^{k+1}z^{k-1},\, 2x^{2k}+3y^kz^k\right). \end{aligned} \tag{5.6.6}$$

The details of this proof are given below.

As in the proof of the previous theorem, in order to compute the syzygies (a, b, c) among the partial derivatives of f_{4k}, we need to characterise the triples (a, b, c) such that each entry belongs to a factor of the decomposition (5.6.2).

Recall that:

- $f_{4k_x} = kx^{k-1}f_{4_x}(x^k, y^k, z^k)$,
- $f_{4k_y} = ky^{k-1}f_{4_y}(x^k, y^k, z^k)$,
- $f_{4k_z} = kz^{k-1}f_{4_z}(x^k, y^k, z^k)$.

and let us assume that:

$a = x^{i_a}y^jz^l\alpha(x^k, y^k, z^k)$,
$b = x^iy^{j_b}z^l\beta(x^k, y^k, z^k)$,
$c = x^iy^jz^{l_c}\gamma(x^k, y^k, z^k)$.

We distinguish eight cases:

Case 1:
$i = j = l = k-1 \Longrightarrow i_a = j_b = l_c = 0$;

$a = y^{k-1}z^{k-1}\alpha(x^k, y^k, z^k)$,
$b = x^{k-1}z^{k-1}\beta(x^k, y^k, z^k)$,
$c = x^{k-1}y^{k-1}\gamma(x^k, y^k, z^k)$.

(α, β, γ) is a syzygy of the generators of the ideal J.

syz(J) is generated by $R_1^J = (3xy - 4yz, 12xy - 9y^2, -20xz + 9yz)$,
$R_2^J = (-x^2 + 2xz, -4x^2 + 3xy + 10yz, -3xz - 10z^2)$ and $R_3^J = (xy, -3y^2, 2x^2 + 3yz)$.

Hence (a, b, c) is a combination of

$\big(y^{k-1}z^{k-1}(3x^ky^k - 4y^kz^k), x^{k-1}z^{k-1}(12x^ky^k - 9y^{2k}), x^{k-1}y^{k-1}(-20x^kz^k + 9y^kz^k)\big)$,

$(y^{k-1}z^{k-1}(-x^{2k} + 2x^kz^k), x^{k-1}z^{k-1}(-4x^{2k} + 3x^ky^k + 10y^kz^k), x^{k-1}y^{k-1}(-3x^kz^k - 10z^{2k}))$

and $\big(y^{k-1}z^{k-1}x^ky^k, -3x^{k-1}z^{k-1}y^{2k}, x^{k-1}y^{k-1}(2x^{2k} + 3y^kz^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J,k} = \big(3x^ky^k - 4y^kz^k, x^{k-1}(12x^ky - 9y^{k+1}), x^{k-1}(-20x^kz + 9y^kz)\big);$$
$$S_2^{J,k} = (y^{k-1}(-x^{k+1} + 2xz^k), -4x^{2k} + 3x^ky^k + 10y^kz^k, y^{k-1}(-3x^kz - 10z^{k+1}));$$
$$S_3^{J,k} = \big(-3z^{k-1}xy^k, z^{k-1}y^{k+1}, 2x^{2k} + 3y^kz^k\big).$$

Case 2:
$i < k-1; j = l = k-1 \Longrightarrow i_a = i+1; j_b = l_c = 0;$

$a = x^{i+1}y^{k-1}z^{k-1}\alpha(x^k, y^k, z^k)$,
$b = x^iz^{k-1}\beta(x^k, y^k, z^k)$,
$c = x^iy^{k-1}\gamma(x^k, y^k, z^k)$.

(α, β, γ) is a syzygy of the generators of the ideal J_x.

syz(J_x) is generated by $R_1^{J_x} = (x+2z, 4x^2-3xy-10yz, 3xz+10z^2)$ and $R_2^{J_x} = (y, -3y^2, 2x^2+3yz)$.

Hence (a, b, c) is a combination of
$\big(x^{i+1}y^{k-1}z^{k-1}(x^k + 2z^k), x^iz^{k-1}(4x^{2k} - 3x^ky^k - 10y^kz^k), x^iy^{k-1}(3x^kz^k + 10z^{2k})\big)$ and
$\big(x^{i+1}y^{k-1}z^{k-1}y^k, -3x^iz^{k-1}y^{2k}, x^iy^{k-1}(2x^{2k} + 3y^kz^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_x,k} = \big(xy^{k-1}(x^k + 2z^k), 4x^{2k} - 3x^ky^k - 10y^kz^k, y^{k-1}3x^kz + 10z^{k+1}\big);$$
$$S_2^{J_x,k} = \big(xz^{k-1}y^k, -3z^{k-1}y^{k+1}, 2x^{2k} + 3y^kz^k\big).$$

Case 3:
$j < k-1;\ = l = k-1 \Longrightarrow j_b = j+1; i_a = l_c = 0;$

$a = y^jz^{k-1}\alpha(x^k, y^k, z^k)$,
$b = x^{k-1}y^{j+1}z^{k-1}\beta(x^k, y^k, z^k)$,
$c = x^{k-1}y^j\gamma(x^k, y^k, z^k)$.

(α, β, γ) is a syzygy of the generators of the ideal J_y.

syz(J_y) is generated by $R_1^{J_y} = (3xy - 4yz, 12x - 9y, -20xz + 9yz)$ and $R_2^{J_y} = (xy, -3y, 2x^2 + 3yz)$.

Hence (a, b, c) is a combination of

$\big(y^j z^{k-1}(3x^k y^k - 4y^k z^k), x^{k-1}y^{j+1}z^{k-1}(12x^k - 9y^k), x^{k-1}y^j(-20x^k z^k + 9y^k z^k)\big)$ and
$\big(y^j z^{k-1}x^k y^k, -3x^{k-1}y^{j+1}z^{k-1}y^k, x^{k-1}y^j(2x^{2k} + 3y^k z^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_y,k} = \big(3x^k y^k - 4y^k z^k, x^{k-1}y(12x^k - 9y^k), x^{k-1}(-20x^k z + 9y^k z)\big);$$
$$S_2^{J_y,k} = \big(z^{k-1}xy^k, -3yz^{k-1}y^k, 2x^{2k} + 3y^k z^k\big).$$

Case 4:
$l < k-1; i = j = k-1 \Longrightarrow l_c = l+1; i_a = j_b = 0;$

$a = y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^{k-1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^{k-1}z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_z.

The following generators of syz(J_z) are obtained using `Singular` [DGPS19]:

$$R_1^{J_z} = (3xy - 4yz, 12xy - 9y^2, -20x + 9y),$$
$$R_2^{J_z} = (4x^2 + 3xy + 8xz - 4yz, 16x^2 - 9y^2 - 40yz, -8x + 9y + 40z),$$
$$R_3^{J_z} = (xyz, -3y^2 z, 2x^2 + 3yz).$$

We consider the system of generators of syz(J_z) given by $R_1^{J_z}$, $R_2^{J_z} - R_1^{J_z}$ and $R_3^{J_z}$.

Hence (a, b, c) is a combination of
$\big(y^{k-1}z^l(3x^k y^k - 4y^k z^k), x^{k-1}z^l(12x^k y^k - 9y^{2k}), x^{k-1}y^{k-1}z^{l+1}(-20x^k + 9y^k)\big)$,
$\big(y^{k-1}z^l(4x^{2k} + 8x^k z^k), x^{k-1}z^l(16x^{2k} - 12x^k y^k - 40y^k z^k), x^{k-1}y^{k-1}z^{l+1}(12x^k + 40z^k)\big)$ and
$\big(y^{k-1}z^l x^k y^k z^k, -3x^{k-1}z^l y^{2k} z^k, x^{k-1}y^{k-1}z^{l+1}(2x^{2k} + 3y^k z^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_z,k} = \big(3x^k y^k - 4y^k z^k, x^{k-1}(12x^k y - 9y^{k+1}), x^{k-1}z(-20x^k + 9y^k)\big);$$
$$S_2^{J_z,k} = \big(y^{k-1}(4x^{k+1} + 8xz^k), 16x^{2k} - 12x^k y^k - 40y^k z^k, y^{k-1}z(12x^k + 40z^k)\big);$$
$$S_3^{J_z,k} = \big(xy^k z^{k-1}, -3y^{k+1}z^{k-1}, 2x^{2k} + 3y^k z^k\big).$$

Case 5:
$i, j < k-1; l = k-1 \Longrightarrow i_a = i+1; j_b = j+1; l_c = 0;$

$a = x^{i+1}y^j z^{k-1}\alpha(x^k, y^k, z^k),$
$b = x^i y^{j+1}z^{k-1}\beta(x^k, y^k, z^k),$
$c = x^i y^j\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{xy}.

syz(J_{xy}) is generated by $R_1^{J_{xy}} = (y, -3y, 2x^2 + 3yz)$ and
$R_2^{J_{xy}} = (xy + 2yz, 4x^2 - 3xy - 10yz, 3xyz + 10yz^2)$.

Hence (a, b, c) is a combination of $\left(x^{i+1}y^jz^{k-1}y^k, -3x^iy^{j+1}z^{k-1}y^k, x^iy^j(2x^{2k}+3y^kz^k)\right)$ and

$$(x^{i+1}y^jz^{k-1}(x^ky^k+2y^kz^k), x^iy^{j+1}z^{k-1}(4x^{2k}-3x^ky^k-10y^kz^k),$$
$$x^iy^j(3x^ky^kz^k+10y^kz^{2k})).$$

Taking out common factors we get syzygies of degrees $2k$:

$$S_1^{J_{xy},k} = \left(xz^{k-1}y^k, -3yz^{k-1}y^k, 2x^{2k}+3y^kz^k\right);$$

$$S_2^{J_{xy},k} = (x(x^ky^{k-1}+2y^{k-1}z^k), 4x^{2k}-3x^ky^k-10y^kz^k,$$
$$3x^ky^{k-1}z+10y^{k-1}z^{k+1}).$$

Case 6:
$i, l < k-1; j = k-1 \Longrightarrow i_a = i+1; l_c = l+1; j_b = 0;$

$a = x^{i+1}y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^iz^l\beta(x^k, y^k, z^k),$
$c = x^iy^{k-1}z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{xz}.

$\mathrm{syz}(J_{xz})$ is generated by $R_1^{J_{xz}} = (x+2z, 4x^2-3xy-10yz, 3x+10z)$ and
$R_2^{J_{xz}} = (-yz, 3y^2z, -2x^2-3yz)$.

Hence (a, b, c) is a combination of
$\left(x^{i+1}y^{k-1}z^l(x^k+2z^k), x^iz^l(4x^{2k}-3x^ky^k-10y^kz^k), x^iy^{k-1}z^{l+1}(3x^k+10z^k)\right)$ and
$\left(-x^{i+1}y^{k-1}z^ly^kz^k, 3x^iz^ly^{2k}z^k, x^iy^{k-1}z^{l+1}(-2x^{2k}-3y^kz^k)\right)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_{xz},k} = \left(xy^{k-1}(x^k+2z^k), 4x^{2k}-3x^ky^k-10y^kz^k, y^{k-1}z(3x^k+10z^k)\right);$$
$$S_2^{J_{xz},k} = \left(-xy^kz^{k-1}, 3y^{k+1}z^{k-1}, -2x^{2k}-3y^kz^k\right).$$

Case 7:
$j, l < k-1; i = k-1 \Longrightarrow j_b = j+1; l_c = l+1; i_a = 0;$

$a = y^jz^l\alpha(x^k, y^k, z^k),$
$b = x^{k-1}y^{j+1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^jz^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{yz}.

The following generators of $\mathrm{syz}(J_{yz})$ are obtained using `Singular` [DGPS19]:

$$R_1^{J_{yz}} = (3xy-4yz, 12x-9y, -20x+9y),$$
$$R_2^{J_{yz}} = (2yz^2, -6xz, 3x^2+10xz).$$

We consider the system of generators of $\mathrm{syz}(J_{yz})$ given by $R_1^{J_{yz}}$ and $zR_1^{J_{yz}} + 2R_2^{J_{yz}}$.

Hence (a,b,c) is a combination of
$\big(y^jz^l(3x^ky^k-4y^kz^k), x^{k-1}y^{j+1}z^l(12x^k-9y^k), x^{k-1}y^jz^{l+1}(-20x^k+9y^k)\big)$ and
$\big(3y^jz^lx^ky^kz^k, -9x^{k-1}y^{j+1}z^ly^kz^k, x^{k-1}y^jz^{l+1}(6x^{2k}+9y^kz^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_{yz},k} = \big(3x^ky^k-4y^kz^k, x^{k-1}y(12x^k-9y^k), x^{k-1}z(-20x^k+9y^k)\big);$$
$$S_2^{J_{yz},k} = \big(3xy^kz^{k-1}, -9y^{k+1}z^{k-1}, 6x^{2k}+9y^kz^k\big).$$

Case 8:
$i,j,l<k-1 \Longrightarrow i_a=i+1; j_b=j+1; l_c=l+1;$

$a = x^{i+1}y^jz^l\alpha(x^k,y^k,z^k)$,
$b = x^iy^{j+1}z^l\beta(x^k,y^k,z^k)$,
$c = x^iy^jz^{l+1}\gamma(x^k,y^k,z^k)$.

(α,β,γ) is a syzygy of the generators of the ideal J_{xyz}.

The following generators of $\operatorname{syz}(J_{xyz})$ are obtained using `Singular` [DGPS19]:

$$R_1^{J_{xyz}} = (xy+2yz, 4x^2-3xy-10yz, 3xy+10yz),$$
$$R_2^{J_{xyz}} = (-xy, -4x^2+3xy+4yz, 4x^2-3xy-4yz).$$

We consider the system of generators of $\operatorname{syz}(J_{xyz})$ given by $R_1^{J_{xyz}}$ and $R_1^{J_{xyz}}+R_2^{J_{xyz}}$.

Hence (a,b,c) is a combination of
$\big(x^{i+1}y^jz^l(x^ky^k+2y^kz^k), x^iy^{j+1}z^l(4x^{2k}-3x^ky^k-10y^kz^k), x^iy^jz^{l+1}(3x^ky^k+10y^kz^k)\big)$ and
$\big(2x^{i+1}y^jz^ly^kz^k, -6x^iy^{j+1}z^ly^kz^k, x^iy^jz^{l+1}(4x^{2k}+6y^kz^k)\big)$.

Taking out common factors we get syzygies of degree $2k$:

$$S_1^{J_{xyz},k} = \big(x(x^ky^{k-1}+2y^{k-1}z^k), 4x^{2k}-3x^ky^k-10y^kz^k, z(3x^ky^{k-1}+10y^{k-1}z^k)\big);$$
$$S_2^{J_{xyz},k} = \big(2xy^kz^{k-1}, -6y^{k+1}z^{k-1}, 4x^{2k}+6y^kz^k\big).$$

The syzygies of the Jacobian ideal $J_{f_{4k}}$ given in 5.6.6 satisfy the relation $xR_{k,1}+3yR_{k,2}+10zR_{k,3}=0$ and, by Remark 5.4.6, C_{4k} is a nearly free curve with exponents $d_1=d_2=d_3=2k$. Finally, by equation (5.4.3), we have that $\tau(C_{4k})=6k(2k-1)$.

The proof of part (3) follows the same ideas as the previous theorem.
We have the monodromy map:

$$\begin{array}{ccccc}
\pi_1(\tilde{C}_4\setminus V(xyz)) & \xrightarrow{i_*} & H_1(\mathbb{P}^2\setminus V(xyz),\mathbb{Z}) & \xrightarrow{\ \ p\ \ } & \mathrm{Gal}(\pi_k)\\
[\alpha_P] & \mapsto & [m_P^{L_x}\gamma_x+m_P^{L_y}\gamma_y+m_P^{L_z}\gamma_z] & \mapsto & m_P^{L_x}a_x+m_P^{L_y}a_y+m_P^{L_z}a_z\\
[\alpha_{p_1}] & \mapsto & [2\gamma_x+3\gamma_z] & \mapsto & 2a_x+3a_z\\
[\alpha_{p_2}] & \mapsto & [2\gamma_x+4\gamma_y] & \mapsto & 2a_x+4a_y\\
[\alpha_{p_3}] & \mapsto & [\gamma_z] & \mapsto & a_z
\end{array}$$

Hence, we obtain a_z (the smooth point), $2a_x+4a_y$ (the $\mathbb{A}_4$-point) and $2a_x+3a_z$ (the $\mathbb{A}_2$-point). Therefore, the monodromy group is generated by $2a_y, a_z$. If k is even ($k=2\ell$), the monodromy group is of index 2 in G_k, and hence $\tilde{C}_{4k}$ has two connected components (see 2.6.1):

$$f_{4k} = (-x^{4\ell} + x^{3\ell}y^{\ell} + y^{2\ell}z^{2\ell}) \cdot (-x^{4\ell} - x^{3\ell}y^{\ell} + y^{2\ell}z^{2\ell}),$$

whereas the monodromy group is G_k when k is odd and thus $\tilde{C}_{4k}$ is connected.

The Euler characteristic of $\tilde{C}_{4k}$ is:

$$\chi(\tilde{C}_{4k}) = 3 \cdot 4k - (4k)^2 + \sum_{q \in \mathrm{Sing}(C_{4k})} (\mu_q + r_q - 1),$$

where the Milnor number of $q \in \mathrm{Sing}(C_{4k})$ is $\mu_q = k^2(\mu_p - 1) + k(k-1)(m_p^{L_1} + m_p^{L_2}) + 1$ and the number of branches of C_{4k} at q is $r_q = k \cdot \gcd(k, m_p^{L_1}, m_p^{L_2})$.

We have that $\mu_{q_1} = 6k^2 - 5k + 1$, $\mu_{q_2} = 9k^2 - 6k + 1$, since $\mu_{p_1} = 2$ and $\mu_{p_2} = 4$. The number of branches at the singular points of C_{4k} is $r_{q_1} = k$, if k is odd and $r_{q_1} = 2k$, if k is even, and $r_{q_2} = k$. Therefore, $\chi(\tilde{C}_{4k}) = -k^2 + 2k + r_{q_1}$. Hence, for k odd,

$$\chi(\tilde{C}_{4k}) = -k^2 + 3k \Longrightarrow g(\tilde{C}_{4k}) = \frac{(k-1)(k-2)}{2},$$

and for k even,

$$\chi(\tilde{C}_{4k}) = -k^2 + 4k \Longrightarrow g(\tilde{C}^i_{4k}) = \frac{2 - \frac{\chi(\tilde{C}_{4k})}{2}}{2} = \frac{(k-2)^2}{4}.$$

where $\tilde{C}_{4k} = \tilde{C}^1_{4k} \cup \tilde{C}^2_{4k}$. □

So, for odd $k \geq 3$, the curve C_{4k} is an irreducible nearly free curve of positive genus whose singularities have k branches each. This is a counterexample to both the nearly-free part of Conjecture 5.1.1(ii) and Conjecture 5.1.2(ii).

5.6.3 Positive genus nearly-free curves with many singularities

Let us consider the conic C_2 given by

$$f_2 = x^2 + y^2 + z^2 - 2(xy + xz + yz) = 0.$$

This conic is tangent to the three axes and it is very useful to produce interesting curves using Kummer covers.

Theorem 5.6.3. *For any $k \geq 2$, the curve C_{2k} of degree $d = 2k$ defined by $C_{2k} = V(f_{2k})$, where*

$$f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k), \tag{5.6.7}$$

verifies the following properties:

(1) *$\mathrm{Sing}(C_{2k})$ are $3k$ singular points of type $\mathbb{A}_{k-1}$.*

(2) *C_{2k} is a nearly free curve with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.*

(3) *C_{2k} is irreducible of genus $\frac{(k-1)(k-2)}{2}$ if k is odd and it has four irreducible smooth components of degree $\frac{k}{2}$ if k is even.*

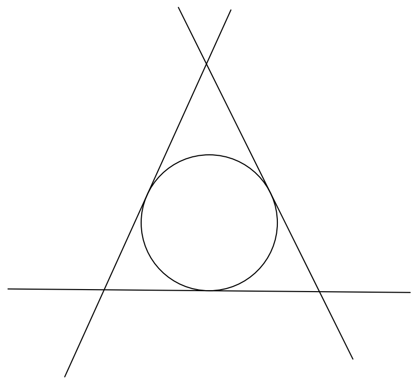


Figure 5.2: Conic C_2.

Proof. In order to prove (1) it is enough to take into account that C_2 is non-singular and by Remark 2.6.23 the singularities of C_{2k} verify $\operatorname{Sing}(C_{2k}) \subset \{xyz = 0\}$. Moreover C_2 is tangent to the three axes at three points of type 1, namely $p_1 = [0 : 1 : 1], p_2 = [1 : 0 : 1]$ and $p_3 = [1 : 1 : 0]$, such that $(C_2, L_x)_{p_1} = (C_2, L_y)_{p_2} = (C_2, L_z)_{p_3} = 2$. For $i = 1, 2, 3$, the points p_i are of type 1 and, as pointed out in Remark 2.6.19, all the k preimages of each p_i under π_k are analytically equivalent. Furthermore, by Example 2.6.24, over each p_i one has k singular points of type $\mathbb{A}_{k-1}$.

To prove (3) we proceed as in the proof of Theorem 5.6.1; the main difference is that π_2 has no ramification over C_2 and in fact C_4 is the union of four lines in general position: their preimages. If $k = 2\ell$, since $\pi_k = \pi_\ell \circ \pi_2$, each irreducible component is a smooth Fermat curve.

In this case, we have the monodromy map:

$$\begin{array}{ccccc}
\pi_1(\tilde{C}_2 \setminus V(xyz)) & \xrightarrow{i_*} & H_1(\mathbb{P}^2 \setminus V(xyz), \mathbb{Z}) & \xrightarrow{\quad p \quad} & \operatorname{Gal}(\pi_k) \\
{[\alpha_P]} & \mapsto & [m_P^{L_x}\gamma_x + m_P^{L_y}\gamma_y + m_P^{L_z}\gamma_z] & \mapsto & m_P^{L_x}a_x + m_P^{L_y}a_y + m_P^{L_z}a_z \\
{[\alpha_{p_1}]} & \mapsto & [2\gamma_x] & \mapsto & 2a_x \\
{[\alpha_{p_2}]} & \mapsto & [2\gamma_y] & \mapsto & 2a_y \\
{[\alpha_{p_3}]} & \mapsto & [2\gamma_z] & \mapsto & 2a_z
\end{array}$$

The monodromy group is generated by $2a_y, 2a_z$. Thus, if k is even ($k = 2\ell$), the monodromy group is of index 4 in G_k, and hence $\tilde{C}_{2k}$ has four connected components (see 2.6.1):

$$f_{2k} = (x^\ell + y^\ell + z^\ell) \cdot (x^\ell + y^\ell - z^\ell) \cdot (-x^\ell + y^\ell - z^\ell) \cdot (-x^\ell + y^\ell + z^\ell),$$

whereas the monodromy group is G_k when k is odd and thus $\tilde{C}_{2k}$ is connected.

The Euler characteristic of $\tilde{C}_{2k}$ is:

$$\chi(\tilde{C}_{2k}) = 3 \cdot 2k - (2k)^2 + \sum_{q \in \operatorname{Sing}(C_{2k})} (\mu_q + r_q - 1),$$

where the Milnor number of $q \in \operatorname{Sing}(C_{2k})$ is $\mu_q = k\mu_p + (m_p^L - 1)(k - 1)$ and the number of branches of C_{2k} at q is $r_q = \gcd(k, m_p^L)$ (recall Proposition 2.6.25(1)).

We have that $\mu_{q_i} = k - 1$ and the number of branches at the singular points of C_{2k} is $r_q = 1$ if k is odd and $r_q = 2$ if k is even.

Therefore, $\chi(\tilde{C}_{2k}) = -k^2 + 3k \cdot r_q$. Hence, for k odd

$$\chi(\tilde{C}_{2k}) = -k^2 + 3k \Longrightarrow g(\tilde{C}_{2k}) = \frac{(k-1)(k-2)}{2}$$

and for k even,

$$\chi(\tilde{C}_{2k}) = -k^2 + 6k \Longrightarrow g(\tilde{C}^i_{2k}) = \frac{2 - \frac{\chi(\tilde{C}_{2k})}{4}}{2} = \frac{(k-2)(k-4)}{4},$$

where $\tilde{C}_{2k} = \bigcup_{j=1}^4 \tilde{C}^j_{2k}$.

Let us study (2). A system of generators for the syzygies of J (Jacobian ideal of f_2) is given by:

$$\begin{aligned} R^J_1 &:= (y - z,\ y,\ -z)\,, \\ R^J_2 &:= (-x,\ z - x,\ z)\,, \\ R^J_3 &:= (x,\ -y,\ x - y)\,. \end{aligned} \tag{5.6.8}$$

These syzygies satisfy the relation $xR^J_1 + yR^J_2 + zR^J_3 = 0$. The other ideals have free 2-rank syzygy modules. A computation similar to the one performed in the proof of Theorem 5.6.2 gives a system of generators for the syzygy module of the Jacobian ideal of f_{2k}.

Recall that:

- $f_{2k_x} = kx^{k-1} f_{2_x}(x^k, y^k, z^k);,$
- $f_{2k_y} = ky^{k-1} f_{2_y}(x^k, y^k, z^k);,$
- $f_{2k_z} = kz^{k-1} f_{2_z}(x^k, y^k, z^k).$

Again, as in the proof of the previous theorem, in order to calculate the syzygies (a, b, c) among the partial derivatives of f_{2k}, we need to characterise the triples (a, b, c) such that each entry belongs to a factor of the decomposition (5.6.2).

Let us assume that:

$a = x^{i_a} y^j z^l \alpha(x^k, y^k, z^k),$
$b = x^i y^{j_b} z^l \beta(x^k, y^k, z^k),$
$c = x^i y^j z^{l_c} \gamma(x^k, y^k, z^k).$

and we distinguish eight cases:

Case 1:
$i = j = l = k - 1 \Longrightarrow i_a = j_b = l_c = 0;$

$a = y^{k-1} z^{k-1} \alpha(x^k, y^k, z^k),$
$b = x^{k-1} z^{k-1} \beta(x^k, y^k, z^k),$
$c = x^{k-1} y^{k-1} \gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J.

The following generators of $\operatorname{syz}(J)$ are obtained using `Singular` [DGPS19]:

$R_1^J = (y-z, y, -z),$
$R_2^J = (x, x-z, -z),$
$R_3^J = (x+y-z, 0, x-y-z).$

We consider the system of generators of syz(J) given by R_1^J, R_2^J and $R_3^J - R_1^J$.

Hence (a,b,c) is a combination of $\big(y^{k-1}z^{k-1}(y^k-z^k), x^{k-1}z^{k-1}y^k, -x^{k-1}y^{k-1}z^k\big)$, $\big(y^{k-1}z^{k-1}x^k, x^{k-1}z^{k-1}(x^k-z^k), -x^{k-1}y^{k-1}z^k\big)$ and $\big(y^{k-1}z^{k-1}x^k, -x^{k-1}z^{k-1}y^k, x^{k-1}y^{k-1}(x^k-y^k)\big)$.

Taking out common factors we get syzygies of degree k:

$S_1^{J,k} = \big(y^k - z^k, x^{k-1}y, -x^{k-1}z\big);$
$S_2^{J,k} = \big(y^{k-1}x, x^k - z^k, -y^{k-1}z\big);$
$S_3^{J,k} = \big(z^{k-1}x, -z^{k-1}y, x^k - y^k\big).$

Case 2:
$i < k-1; j = l = k-1 \Longrightarrow i_a = i+1; j_b = l_c = 0;$

$a = x^{i+1}y^{k-1}z^{k-1}\alpha(x^k, y^k, z^k),$
$b = x^i z^{k-1}\beta(x^k, y^k, z^k),$
$c = x^i y^{k-1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_x.

The following generators of syz(J_x) are obtained using `Singular` [DGPS19]:

$R_1^{J_x} = (1, x-z, -z),$
$R_2^{J_x} = (0, -x-y+z, x-y+z).$

We consider the system of generators of syz(J_x) given by $R_1^{J_x}$ and $R_1^{J_x} + R_2^{J_x}$.

Hence (a,b,c) is a combination of $\big(x^{i+1}y^{k-1}z^{k-1}, x^i z^{k-1}(x^k - z^k), -x^i y^{k-1}z^k\big)$ and $\big(x^{i+1}y^{k-1}z^{k-1}, -x^i z^{k-1}y^k, x^i y^{k-1}(x^k - y^k)\big)$.

Taking out common factors we get syzygies of degree k:

$S_1^{J_x,k} = \big(xy^{k-1}, x^k - z^k, -y^{k-1}z\big);$
$S_2^{J_x,k} = \big(xz^{k-1}, -z^{k-1}y, x^k - y^k\big).$

Case 3:
$j < k-1;\ = l = k-1 \Longrightarrow j_b = j+1; i_a = l_c = 0;$

$a = y^j z^{k-1}\alpha(x^k, y^k, z^k),$
$b = x^{k-1}y^{j+1}z^{k-1}\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^j\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_y.

syz(J_y) is generated by $R_1^{J_y} = (y-z, 1, -z)$ and $R_2^{J_y} = (x+y-z, 0, x-y-z)$.

We consider the system of generators of $\mathrm{syz}(J_y)$ given by $R_1^{J_y}$ and $R_2^{J_y} - R_1^{J_y}$.

Hence (a,b,c) is a combination of $\big(y^j z^{k-1}(y^k - z^k), x^{k-1}y^{j+1}z^{k-1}, -x^{k-1}y^j z^k\big)$ and $\big(y^j z^{k-1}x^k, -x^{k-1}y^{j+1}z^{k-1}, x^{k-1}y^j(x^k - y^k)\big)$.

Taking out common factors we get syzygies of degree k:

$S_1^{J_y,k} = \big(y^k - z^k, x^{k-1}y, -x^{k-1}z\big)$;
$S_2^{J_y,k} = \big(z^{k-1}x, -yz^{k-1}, x^k - y^k\big)$.

Case 4:
$l < k-1; i = j = k-1 \Longrightarrow l_c = l+1; i_a = j_b = 0;$

$a = y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^{k-1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^{k-1}z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_z.

The following generators of $\mathrm{syz}(J_z)$ are obtained using `Singular` [DGPS19]:

$R_1^{J_z} = (y - z, y, -1),$
$R_2^{J_z} = (x - y + z, x - y - z, 0).$

We consider the system of generators of $\mathrm{syz}(J_z)$ given by $R_1^{J_z}$ and $R_1^{J_z} + R_2^{J_z}$.

Hence (a,b,c) is a combination of $\big(y^{k-1}z^l(y^k - z^k), x^{k-1}z^l y^k, -x^{k-1}y^{k-1}z^{l+1}\big)$ and $\big(y^{k-1}z^l x^k, x^{k-1}z^l(x^k - z^k), -x^{k-1}y^{k-1}z^{l+1}\big)$.

Taking out common factors we get syzygies of degree k:

$S_1^{J_z,k} = \big(y^k - z^k, x^{k-1}y, -x^{k-1}z\big)$;
$S_2^{J_z,k} = \big(y^{k-1}x, x^k - z^k, -y^{k-1}z\big)$.

Case 5:
$i, j < k-1; l = k-1 \Longrightarrow i_a = i+1; j_b = j+1; l_c = 0;$

$a = x^{i+1}y^j z^{k-1}\alpha(x^k, y^k, z^k),$
$b = x^i y^{j+1}z^{k-1}\beta(x^k, y^k, z^k),$
$c = x^i y^j\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{xy}.

$\mathrm{syz}(J_{xy})$ is generated by $R_1^{J_{xy}} = (1, -1, x - y)$ and $R_2^{J_{xy}} = (-y, -x + z, yz)$.

Hence (a,b,c) is a combination of $\big(x^{i+1}y^j z^{k-1}, -x^i y^{j+1}z^{k-1}, x^i y^j(x^k - y^k)\big)$ and $\big(-x^{i+1}y^j z^{k-1}y^k, -x^i y^{j+1}z^{k-1}(-x^k + z^k), x^i y^j y^k z^k\big)$.

Taking out common factors we get syzygies of degree k:

$S_1^{J_{xy},k} = \big(xz^{k-1}, -yz^{k-1}, x^k - y^k\big)$;

$$S_2^{J_{xy},k} = \left(-xy^{k-1}, x^k - z^k, y^{k-1}z\right).$$

Case 6:
$i, l < k-1; j = k-1 \Longrightarrow i_a = i+1; l_c = l+1; j_b = 0;$

$a = x^{i+1}y^{k-1}z^l\alpha(x^k, y^k, z^k),$
$b = x^i z^l \beta(x^k, y^k, z^k),$
$c = x^i y^{k-1} z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{xz}.

$\mathrm{syz}(J_{xz})$ is generated by $R_1^{J_{xz}} = (1, x-z, -1)$ and $R_2^{J_{xz}} = (-z, yz, -x+y)$.

Hence (a, b, c) is a combination of $\left(x^{i+1}y^{k-1}z^l, x^i z^l(x^k - z^k), -x^i y^{k-1}z^{l+1}\right)$ and $\left(-x^{i+1}y^{k-1}z^l z^k, x^i z^l y^k z^k, x^i y^{k-1} z^{l+1}(-x^k + y^k)\right)$.

Taking out common factors we get syzygies of degree k:

$$S_1^{J_{xz},k} = \left(xy^{k-1}, x^k - z^k, -y^{k-1}z\right);$$
$$S_2^{J_{xz},k} = \left(-xz^{k-1}, yz^{k-1}, -x^k + y^k\right).$$

Case 7:
$j, l < k-1; i = k-1 \Longrightarrow j_b = j+1; l_c = l+1; i_a = 0;$

$a = y^j z^l \alpha(x^k, y^k, z^k),$
$b = x^{k-1}y^{j+1}z^l\beta(x^k, y^k, z^k),$
$c = x^{k-1}y^j z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{yz}.

$\mathrm{syz}(J_{yz})$ is generated by $R_1^{J_{yz}} = (y-z, 1, -1)$ and $R_2^{J_{yz}} = (xz, -z, x-y)$.

Hence (a, b, c) is a combination of $\left(y^j z^l(y^k - z^k), x^{k-1}y^{j+1}z^l, -x^{k-1}y^j z^{l+1}\right)$ and $\left(y^j z^l x^k z^k, -x^{k-1}y^{j+1}z^l z^k, x^{k-1}y^j z^{l+1}(x^k - y^k)\right)$.

Taking out common factors we get syzygies of degree k:

$$S_1^{J_{yz},k} = \left(y^k - z^k, x^{k-1}y, -x^{k-1}z\right);$$
$$S_2^{J_{yz},k} = \left(xz^{k-1}, -yz^{k-1}, x^k - y^k\right).$$

Case 8:
$i, j, l < k-1 \Longrightarrow i_a = i+1; j_b = j+1; l_c = l+1;$

$a = x^{i+1}y^j z^l \alpha(x^k, y^k, z^k),$
$b = x^i y^{j+1} z^l \beta(x^k, y^k, z^k),$
$c = x^i y^j z^{l+1}\gamma(x^k, y^k, z^k).$

(α, β, γ) is a syzygy of the generators of the ideal J_{xyz}.

$\mathrm{syz}(J_{xyz})$ is generated by $R_1^{J_{xyz}} = (y, x-z, -y)$ and $R_2^{J_{xyz}} = (z, -z, x-y)$.

Hence (a,b,c) is a combination of $\left(x^{i+1}y^jz^ly^k, x^iy^{j+1}z^l(x^k-z^k), -x^iy^jz^{l+1}y^k\right)$ and $\left(x^{i+1}y^jz^lz^k, -x^iy^{j+1}z^lz^k, -x^iy^jz^{l+1}(x^k-y^k)\right)$.

Taking out common factors we get syzygies of degree k:

$$S_1^{J_{xyz},k} = \left(xy^{k-1}, x^k - z^k, -zy^{k-1}\right);$$
$$S_2^{J_{xyz},k} = \left(xz^{k-1}, -yz^{k-1}, -x^k + y^k\right).$$

We conclude that the syzygies below constitute a system of generators of $\mathrm{AR}(f_{2k})$:

$$\begin{aligned} R_{k,1} &:= \left(y^k - z^k,\, x^{k-1}y,\, -x^{k-1}z\right), \\ R_{k,2} &:= \left(-xy^{k-1},\, z^k - x^k,\, y^{k-1}z\right), \\ R_{k,3} &:= \left(xz^{k-1},\, -yz^{k-1},\, x^k - y^k\right). \end{aligned}$$

These syzygies satisfy the relation $xR_{k,1} + yR_{k,2} + zR_{k,3} = 0$ and therefore, by Remark 5.4.6, C_{2k} is a nearly free curve with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.

□

These curves, for $k \geq 3$ odd, are of positive genus and give a counterexample to the nearly-free part of Conjecture 5.1.1(ii) (with unbounded genus and number of singularities).

5.6.4 Rational nearly free plane curves with four branches

In this final section we are going to show that it is possible to construct a rational nearly free curve with singular points with more than three branches, i.e., we do not need high genus curves.

Given a curve $C \subset \mathbb{P}^2$, let $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the minimal, (not the "embedded" minimal) resolution of singularities of C; let $\widetilde{C} \subset \widetilde{\mathbb{P}^2}$ be the strict transform of C, and let $\tilde{\nu}(C) = \widetilde{C} \cdot \widetilde{C}$ denote the self-intersection number of $\widetilde{C}$ on $\widetilde{\mathbb{P}^2}$.

Recall that a *unicuspidal rational curve* is a pair (C,p), where C is a curve and $p \in C$ satisfies $C \setminus \{p\} \cong \mathbb{A}^1$. We call p the *distinguished point* of C. Given a unicuspidal rational curve $C \subset \mathbb{P}^2$ with singular point p, D. Daigle and A. Melle were interested in [DM12, DM14] in the unique pencil Λ_C on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{p\}$, where $\mathrm{Bs}(\Lambda_C)$ denotes the base locus of Λ_C on $\mathbb{P}^2$.

Let $\pi_m : \widetilde{\mathbb{P}^2_m} \to \mathbb{P}^2$ be the minimal resolution of the base points of the pencil. By Bertini theorem, the singularities of the general member C_{gen} of Λ_C are contained in $\mathrm{Bs}(\Lambda_C) = \{p\}$.

For a unicuspidal rational curve $C \subset \mathbb{P}^2$, we know from [DM14, Theorem 4.1] that the general member of Λ_C is a rational curve if and only if $\tilde{\nu}(C) \geq 0$. In such a case:

1. the general element C_{gen} of Λ_C is such that the weighted cluster of infinitely near points of C_{gen} and C are equal (see [DM12, Proposition 2.7]).

2. Λ_C has either 1 or 2 dicriticals, and at least one of them has degree 1.

In view of these results, it is worth noting that all currently known unicuspidal rational curves $C \subset \mathbb{P}^2$ satisfy $\tilde{\nu}(C) \geq 0$.

Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree d with distinguished point p. In [DM14, Proposition 1] it is proved that Λ_C is in fact the set of effective divisors D of $\mathbb{P}^2$ such that $\deg(D) = d$ and $i_p(C, D) \geq d^2$ (recall Definition 4.4.3). The curve $C \in \Lambda_C$ because $i_p(C, C) = \infty > d^2$.

The main idea here is to take the general member C_{gen} of the pencil Λ_C for a non-negative curve, i.e., a curve C such that $\tilde{\nu}(C) \geq 0$. Doing this one gets a rational curve C_{gen} whose unique singular point is $\text{Sing}(C_{gen}) = \{p\}$ and the number of branches of C_{gen} at p is nothing else than the sum of the degrees of the dicriticals.

Our next example starts with a curve C_{49} with $\bar{\kappa}(\mathbb{P}^2 \setminus C_{49}) = 1$. Then we take the pencil $\Lambda_{C_{49}}$, and finally its general member $C_{49,gen}$ has degree 49 and it is rational nearly free with just one singular point which has 4 branches.

The curve C_{49} is a unicuspidal curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C_{49}) = 1$ of type I according to Tono's classification (see 4.3.4) and it is defined by

$$((f_4^s y + \sum_{i=2}^{s+1} a_i f_4^{s+1-i} x^{ia-a+1})^a - f_4^{as+1})/x^{a-1} = 0, \tag{5.6.9}$$

where $f_4 = x^{4-1}z + y^4$, $a = 4$, $s = 3$, $a_2 = \ldots = a_s \in \mathbb{C}$ and $a_{s+1} \in \mathbb{C} \setminus \{0\}$. We can take, for instance, $a_2 = \ldots = a_s = 0 \in \mathbb{C}$ and $a_{s+1} = 1$. Let us fix $C_{49} = V(f_{49})$, where

$$f_{49} = \frac{(f_4^3 y + x^{13})^4 - f_4^{13}}{x^3}. \tag{5.6.10}$$

In this case, $d = a^2 s + 1 = 49$, and the multiplicity sequence of (C_{49}, p) of the singular point $p := [0 : 0 : 1]$ is $[36, 12_7, 4_6]$. This curve is non-negative with $\tilde{\nu}(C_{49}) = 1$.

We consider the rational curves C_4 defined by $f_4 = 0$ and C_{13} defined by $f_{13} : (f_4)^3 y + x^{13} = 0$. Since $C_{49} \cap C_4 = C_{49} \cap C_{13} = \{p\}$, it is clear that $i_p(C_{49}, C_4) = 4 \cdot 49$ and $i_p(C_{49}, C_{13}) = 13 \cdot 49$. Thus the curve $C_{13} C_4^{s(a-1)}$ belongs to the pencil $\Lambda_{C_{49}}$ if $\deg(f_{13} f_4^{s(a-1)}) = 49$, i.e., if $s(a-1) = 9$.

Let us take the generic curve of the pencil $C_{49,gen}$ defined by

$$f_{49,gen} := f_{49} + 13 f_{13} f_4^9 = 0.$$

This curve is irreducible, rational (see 4.4.6) and $\text{Sing}(C_{49,gen}) = \{p\}$. Furthermore, the number of branches of $C_{49,gen}$ at p is 4, as shown in the resolution graph of the curve given by $f_{49} + 13 f_{13} f_4^9 = 0$ (see Figure 5.3).

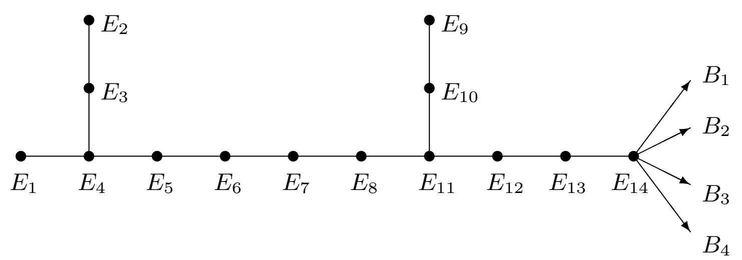


Figure 5.3: Dual graph of the minimal embedded resolution of the singularities of $C_{49,gen}$.

The curve $C_{49,gen}$ is nearly free. A minimal resolution (5.4.1) for $f_{49,gen}$ is determined by three syzygies of degrees $d_1 = 24$ and $d_2 = d_3 = 25$, so that $\mathrm{mdr}(f_{49,gen}) = 24$. It can be computed using `Singular` [DGPS19].

To construct a minimal resolution of the module of syzygies of the Jacobian ideal of $f_{49,gen}$ (5.4.1) we proceed as before (recall Remark 5.4.6):

(i) Looking at the syzygies among the partial derivatives $f_{49,gen_x}, f_{49,gen_y}, f_{49,gen_z}$ and find that we have three syzygies, one of them of degree $d_1 = 24$, and the other two with the same degree $d_2 = d - d_1 = 25$ and $d_3 = d_2 = 25$, namely:

$$(s_1): \quad A_x f_{49,gen_x} + A_y f_{49,gen_y} + A_z f_{49,gen_z} = 0,$$

where

$$\begin{aligned}
A_x \;=\;& 47628x^{23}y + 35672x^{20}y^4 - 78479037x^{19}y^5 + 9390654x^{16}y^8 - 223236x^{13}y^{11} \\
&+24588x^{10}y^{14} - 168x^7y^{17} + 6406452x^6y^{18} - 2778867x^3y^{21} + 17550y^{24} \\
&+35672x^{23}z - 78479037x^{22}yz + 18781308x^{19}y^4z - 446472x^{16}y^7z \\
&+73764x^{13}y^{10}z - 672x^{10}y^{13}z + 25625808x^9y^{14}z - 13894335x^6y^{17}z \\
&+328536x^3y^{20}z + 9390654x^{22}z^2 - 223236x^{19}y^3z + 73764x^{16}y^6z^2 \\
&-1008x^{13}y^9z^2 + 38438712x^{12}y^{10}z^2 - 27788670x^9y^{13}z^2 + 1379430x^6y^{16}z^2 \\
&+24588x^{19}y^2z^3 - 672x^{16}y^5z^3 + 25625808x^{15}y^6z^3 - 27788670x^{12}y^9z^3 \\
&+2583360x^9y^{12}z^3 - 168x^{19}yz^4 + 6406452x^{18}y^2z^4 - 13894335x^{15}y^5z \\
&+2495610x^{12}y^8z^4 - 2778867x^{18}yz^5 + 1221480x^{15}y^4z^5 + 240786x^{18}z^6,
\end{aligned}$$

$$\begin{aligned}
A_y \;=\;& 468796x^{23}y - 198744x^{20}y^4 - 142688x^{17}y^7 + 313916148x^{16}y^8 \\
&-37562616x^{13}y^{11} + 1031940x^{10}y^{14} - 107016x^7y^{17} - 26159679x^3y^{21} \\
&+11179350y^{24} - 8232x^{23}z + 142688x^{20}y^3z + 313916148x^{19}y^4z \\
&-75125232x^{16}y^7z + 2253420x^{13}y^{10}z - 324096x^{10}y^{13}z + 8064x^7y^{16}z \\
&-124391943x^6y^{17}z + 58260384x^3y^{20}z + 842400y^{23}z - 37562616x^{19}y^3z^2 \\
&+1411020x^{16}y^6z^2 - 330192x^{13}y^9z^2 + 32256x^{10}y^{12}z^2 - 235970982x^9y^{13}z^2 \\
&+123611670x^6y^{16}z^2 + 4212000x^3y^{19}z^2 + 189540x^{19}y^2z^3 - 116160x^{16}y^5z^3 \\
&+48384x^{13}y^8z^3 - 223158078x^{12}y^9z^3 + 135429840x^9y^{12}z^3 - 8424000x^6y^{15}z^3 \\
&-3048x^{19}yz^4 + 32256x^{16}y^4z^4 + 105172587x^{15}y^5z^4 + 79533090x^{12}y^8z^4 \\
&-8424000x^9y^{11}z^4 + 8064x^{19}z^5 - 19753227x^{18}yz^5 + 22997520x^{15}y^4z^5 \\
&-4212000x^{12}y^7z^5 + 2363634x^{18}z^6 - 842400x^{15}y^3z^6,
\end{aligned}$$

and

$$\begin{aligned}
A_z \;=\;& -16848x^{14}y^{10} - 1872x^{11}y^{13} - 2912x^8y^{16} + 6406452x^7y^{17} - 766584x^4y^{20} \\
&-16848xy^{23} - 33696x^{17}y^6z - 5616x^{14}y^9z - 11648x^{11}y^{12}z + 25625808x^{10}y^{13}z \\
&-3832920x^7y^{16}z - 84240x^4y^{19}z - 16848x^{20}y^2z^2 - 5616x^{17}y^5z - 17472x^{14}y^8z^2 \\
&+38438712x^{13}y^9z^2 - 7665840x^{10}y^{12}z^2 - 168480x^7y^{15}z^2 - 1872x^{20}yz^3 \\
&-11648x^{17}y^4z^3 + 25625808x^{16}y^5z^3 - 7665840x^{13}y^8z^3 - 168480x^{10}y^{11}z^3 \\
&-2912x^{20}z^4 + 6406452x^{19}yz^4 - 3832920x^{16}y^4z^4 - 84240x^{13}y^7z^4 - 766584x^{19}z^5 \\
&-16848x^{16}y^3z^5;
\end{aligned}$$

$$(s_2): \quad B_x f_{49,gen_x} + B_y f_{49,gen_y} + B_z f_{49,gen_z} = 0,$$

where

$$\begin{aligned}
B_x = \; & 134152200x^{25} + 15433236x^{22}y^3 + 1192464x^{19}y^6 + 1020227481x^{18}y^7 \\
& +313916148x^{15}y^{10} + 126103068x^{12}y^{13} + 756756x^9y^{16} - 5616x^6y^{19} \\
& -83283876x^5y^{20} + 533871x^2y^{23} + 1001952x^{22}y^2z + 1020227481x^{21}y^3z \\
& -142688x^{19}y^5z + 941748444x^{18}y^6z + 337844520x^{15}y^9z - 11742588x^{12}y^{12}z \\
& -120816x^9y^{15}z - 333135504x^8y^{16}z + 672x^6y^{18}z - 22956453x^5y^{19}z \\
& +8213400x^2y^{22}z - 142688x^{22}yz^2 + 627832296x^{21}y^2z^2 + 297379836x^{18}y^5z^2 \\
& -40661244x^{15}y^8z^2 - 328752x^{12}y^{11}z^2 - 499703256x^{11}y^{12}z^2 + 2688x^9y^{14}z^2 \\
& -97164522x^8y^{15}z^2 + 41067000x^5y^{18}z^2 - 892944x^2y^{21}z^2 + 85638384x^{21}yz^3 \\
& -43067700x^{18}y^4z^3 - 317520x^{15}y^7z^3 - 333135504x^{14}y^8z^3 + 4032x^{12}y^{10}z^3 \\
& -148416138x^{11}y^{11}z^3 + 82134000x^8y^{14}z^3 - 4464720x^5y^{17}z^3 - 14905800x^{21}z^4 \\
& -103968x^{18}y^3z^4 - 83283876x^{17}y^4z^4 + 2688x^{15}y^6z^4 - 99833877x^{14}y^7z^4 \\
& +82134000x^{11}y^{10}z^4 - 8929440x^8y^{13}z^4 + 672x^{18}y^2z^5 - 25091937x^{17}y^3z^5 \\
& +41067000x^{14}y^6z^5 - 8929440x^{11}y^9z^5 + 8213400x^{17}y^2z^6 - 4464720x^{14}y^5z^6 \\
& -892944x^{17}yz^7,
\end{aligned}$$

$$\begin{aligned}
B_y = \; & -537503148x^{22}y^3 - 62008128x^{19}y^6 - 4769856x^{16}y^9 - 4080909924x^{15}y^{10} \\
& -1255664592x^{12}y^{13} - 505306620x^9y^{16} - 3577392x^6y^{19} + 340075827x^2y^{23} \\
& -550368x^{22}y^2z - 3974880x^{19}y^5z - 4080909924x^{18}y^6z + 570752x^{16}y^8z \\
& -3766993776x^{15}y^9z - 1320951996x^{12}y^{12}z + 47790288x^9y^{15}z \\
& +697632x^6y^{18}z + 1617095259x^5y^{19}z + 183651624x^2y^{22}z + 32928x^{22}yz^2 \\
& +570752x^{19}y^4z^2 - 2511329184x^{18}y^5z^2 - 1092217932x^{15}y^8z^2 \\
& +168204816x^{12}y^{11}z^2 + 2374656x^9y^{14}z^2 + 3067622766x^8y^{15}z^2 \\
& -32256x^6y^{17}z^2 + 892632312x^5y^{18}z^2 + 24311664x^2y^{21}z^2 \\
& -242806356x^{18}y^4z^3 + 178729200x^{15}y^7z^3 + 2938176x^{12}y^{10}z^3 \\
& +2901055014x^{11}y^{11}z^3 - 129024x^9y^{13}z^3 + 1734013008x^8y^{14}z^3 \\
& +121558320x^5y^{17}z^3 - 3369600x^2y^{20}z^3 + 33766200x^{21}z^4 \\
& +61892064x^{18}y^3z^4 + 1542912x^{15}y^6z^4 + 1367243631x^{14}y^7z^4 - 193536x^{12}y^9z^4 \\
& +1682761392x^{11}y^{10}z^4 + 243116640x^8y^{13}z^4 - 16848000x^5y^{16}z^4 \\
& +281760x^{18}y^2z^5 + 256791951x^{17}y^3z^5 - 129024x^{15}y^5z^5 + 815754888x^{14}y^6z^5 \\
& +243116640x^{11}y^9z^5 + 33696000x^8y^{12}z^5 - 32256x^{18}yz^6 + 158025816x^{17}y^2z^6 \\
& +121558320x^{14}y^5z^6 - 33696000x^{11}y^8z^6 + 24311664x^{17}yz^7 - 16848000x^{14}y^4z^7 \\
& -3369600x^{17}z^8,
\end{aligned}$$

and

$$\begin{aligned} B_z &= -10732176x^{13}y^{12} - 1192464x^{10}y^{15} - 97344x^7y^{18} - 83283876x^6y^{19} \\ &\quad -25625808x^3y^{22} - 10732176y^25 - 32415552x^{16}y^8z - 3510000x^{13}y^{11}z \\ &\quad -381888x^{10}y^{14}z - 333135504x^9y^{15}z + 11648x^7y^{17}z - 153754848x^6y^{18}z \\ &\quad -61545744x^3y^{21}z + 1284192y^{24}z - 32634576x^{19}y^4z^2 - 3442608x^{16}y^7z^2 \\ &\quad -561600x^{13}y^{10}z^2 - 499703256x^{12}y^{11}z^2 + 46592x^{10}y^{13}z^2 - 358761312x^9y^{14}z^2 \\ &\quad -146746080x^6y^{17}z^2 + 7637760x^3y^{20}z^2 - 10951200x^{22}z^3 - 1125072x^{19}y^3z^3 \\ &\quad -366912x^{16}y^6z^3 - 333135504x^{15}y^7z^3 + 69888x^{13}y^9z^3 - 410012928x^{12}y^{10}z^3 \\ &\quad -186170400x^9y^{13}z^3 + 18925920x^6y^{16}z^3 - 89856x^{19}y^2z^4 - 83283876x^18y^3z^4 \\ &\quad +46592x^{16}y^5z^4 - 230632272x^{15}y^6z^4 - 132509520x^{12}y^9z4 + 25009920x^9y^{12}z^4 \\ &\quad +11648x^{19}yz^5 - 51251616x^{18}y^2z^5 - 50156496x^{15}y^5z^5 + 18588960x^{12}y^8z^5 \\ &\quad -7884864x^{18}yz^6 + 7368192x^{15}y^4z^6 + 1216800x^{18}z^7; \end{aligned}$$

and

$$(s_3): \quad C_xf_{49,gen_x} + C_yf_{49,gen_y} + C_zf_{49,gen_z} = 0,$$

where

$$\begin{aligned} C_x &= 176627556x^{22}y^3 - 459102366450x^{21}y^4 - 2476656x^{19}y^6 + 52816391901x^{18}y^7 \\ &\quad -1957910292x^{15}y^{10} - 118066572x^{12}y^{13} - 2554524x^9y^{16} + 37477744200x^8y^{17} \\ &\quad +11664x^6y^{19} - 16083397746x^5y^{20} + 101558691x^2y^{23} - 459102366450x^{24}z \\ &\quad -4953312x^{22}y^2z + 52816391901x^{21}y^3z - 1854944x^{19}y^5z + 165089340x^{18}y^6z \\ &\quad -992751552x^{15}y^9z - 189830700x^{12}y^{12}z + 149910976800x^{11}y^{13}z - 1231920x^9y^15z \\ &\quad -76105446534x^8y^{16}z + 8736x^6y^{18}z + 174657951x^5y^{19}z - 5736744x^2y^{22}z \\ &\quad -1854944x^{22}yz^2 + 2122999632x^{21}y^2z^2 - 1631303388x^{18}y^5z^2 - 565773228x^{15}y^8z^2 \\ &\quad +224866465200x^{14}y^9z^2 - 3765744x^{12}y^{11}z^2 - 143587808676x^{11}y^{12}z^2 \\ &\quad +34944x^9y^{14}z^2 - 316955106x^8y^{15}z^2 - 28683720x^5y^{18}z^2 - 11608272x^2y^{21}z^2 \\ &\quad -756618408x^{21}yz^3 - 572272452x^{18}y^4z^3 + 149910976800x^{17}y^5z^3 - 3789072x^{15}y^7z^3 \\ &\quad -134964724284x^{14}y^8z^3 + 52416x^{12}y^{10}z^3 - 983226114x^{11}y^{11}z^3 - 57367440x^8y^{14}z^3 \\ &\quad -58041360x^5y^{17}z^3 - 193775400x^{21}z^4 + 37477744200x^{20}yz^4 - 1266912x^{18}y^3z^4 \\ &\quad -63170819946x^{17}y^4z^4 + 34944x^{15}y^6z^4 - 824748561x^{14}y^7z^4 - 57367440x^{11}y^{10}z^4 \\ &\quad -116082720x^8y^{13}z^4 - 11771855550x^{20}z^5 + 8736x^{18}y^2z^5 - 231576813x^{17}y^3z^5 \\ &\quad -28683720x^{14}y^6z^5 - 116082720x^{11}y^9z^5 - 5736744x^{17}y^2z^6 - 58041360x^{14}y^5z^6 \\ &\quad -11608272x^{17}yz^7, \end{aligned}$$

$$
\begin{aligned}
C_y \;=\;& 402456600x^{25} - 46299708x^{22}y^{3} - 705938688x^{19}y^{6} + 1836409465800x^{18}y^{7} \\
&+9906624x^{16}y^{9} - 211265567604x^{15}y^{10} + 423439380x^{9}y^{16} + 7429968x^{6}y^{19} \\
&-153034122150x^{5}y^{20} + 64692886167x^{2}y^{23} - 3005856x^{22}y^{2}z \\
&+1836409465800x^{21}y^{3}z + 20241312x^{19}y^{5}z - 211265567604x^{18}y^{6}z \\
&+7419776x^{16}y^{8}z + 2074704840x^{15}y^{9}z + 3835226772x^{12}y^{12}z + 784618128x^{9}y^{15}z \\
&-727692866550x^{8}y^{16}z + 5004960x^{6}y^{18}z + 319152888639x^{5}y^{19}z \\
&+1902551976x^{2}y^{22}z + 428064x^{22}yz^{2} + 7419776x^{1}9y^{4}z^{2} - 5461253928x^{18}y^{5}z^{2} \\
&+6444852804x^{15}y^{8}z^{2} + 2353079376x^{12}y^{11}z^{2} - 1380430244700x^{11}y^{12}z^{2} \\
&+14613504x^{9}y^{14}z^{2} + 629682692886x^{8}y^{15}z^{2} - 419328x^{6}y^{17}z^{2} \\
&+9179624376x^{5}y^{18}z^{2} - 108517968x^{2}y^{21}z^{2} + 1108809000x^{21}yz^{3} \\
&+3077782812x^{18}y^{4}z^{3} + 2382024240x^{15}y^{7}z^{3} - 1305474756300x^{14}y^{8}z^{3} \\
&+13810752x^{12}y^{10}z^{3} + 621059608494x^{11}y^{11}z^{3} - 1677312x^{9}y^{13}z^{3} \\
&+17692977744x^{8}y^{14}z^{3} - 542589840x^{5}y^{17}z^{3} - 43804800x^{2}y^{20}z^{3} \\
&+44717400x^{21}z^{4} + 806133024x^{18}y^{3}z^{4} - 615259633950x^{17}y^{4}z^{4} + 3800832x^{15}y^{6}z^{4} \\
&+306218262051x^{14}y^{7}z^{4} - 2515968x^{12}y^{9}z^{4} + 17026706736x^{11}y^{10}z^{4} \\
&-1085179680x^{8}y^{13}z^{4} - 219024000x^{5}y^{16}z^{4} - 115556377950x^{20}z^{5} - 401376x^{18}y^{2}z^{5} \\
&+60381343971x^{17}y^{3}z^{5} - 1677312x^{15}y^{5}z^{5} + 8180217864x^{14}y^{6}z^{5} \\
&-1085179680x^{11}y^{9}z^{5} - 438048000x^{8}y^{12}z^{5} - 419328x^{18}yz^{6} + 1569416472x^{17}y^{2}z^{6} \\
&-542589840x^{14}y^{5}z^{6} - 438048000x^{11}y^{8}z^{6} - 108517968x^{17}yz^{7} - 219024000x^{14}y^{4}z^{7} \\
&-43804800x^{17}z^{8},
\end{aligned}
$$

and

$$
\begin{aligned}
C_z \;=\;& -98560800x^{16}y^{9} + 11338704x^{13}y^{12} - 14558544x1^{10}y^{15} + 37477744200x^{9}y^{16} \\
&+202176x^{7}y^{18} - 4311542196x^{6}y^{19} - 45337968x^{3}y^{22} + 22289904y^{25} \\
&-197121600x^{19}y^{5}z + 22677408x^{16}y^{8}z - 42799536x^{13}y^{11}z \\
&+149910976800x^{12}y^{12}z + 906048x^{10}y^{14}z - 17246168784x^{9}y^{15}z + 151424x^{7}y^{17}z \\
&-559825344x^{6}y^{18}z + 162263088x^{3}y^{21}z + 16694496y^{24}z - 98560800x^{22}yz^{2} \\
&+11338704x^{19}y^{4}z^{2} - 41923440x^{16}y^{7}z^{2} + 224866465200x^{15}y^{8}z^{2} \\
&+1505088x^{13}y^{10}z^{2} - 25869253176x^{12}y^{11}z^{2} + 605696x^{10}y^{13}z^{2} \\
&-1785921696x^{9}y^{14}z^{2} + 476966880x^{6}y^{17}z^{2} + 99290880x^{3}y^{20}z^{2} \\
&-13682448x^{19}y^{3}z^{3} + 149910976800x^{18}y^{4}z^{3} + 1100736x^{16}y^{6}z^{3} \\
&-17246168784x^{15}y^{7}z^{3} + 908544x^{13}y^{9}z^{3} - 2452192704x^{12}y^{10}z^{3} \\
&+731034720x^{9}y^{13}z^{3} + 246036960x^{6}y^{16}z^{3} + 37477744200x^{21}z^{4} + 299520x^{19}y^{2}z^{4} \\
&-4311542196x^{18}y^{3}z^{4} + 605696x^{16}y^{5}z^{4} - 1559231856x^{15}y^{6}z^{4} + \\
&619585200x^{12}y^{9}z^{4} + 325128960x^{9}y^{12}z^{4} + 151424x^{19}yz^{5} - 378473472x^{18}y^{2}z^{5} \\
&+276357744x^{15}y^{5}z^{5} + 241656480x^{12}y^{8}z^{5} + 50813568x^{18}yz^{6} + 95786496x^{15}y^{4}z^{6} \\
&+15818400x^{18}z^{7}.
\end{aligned}
$$

(ii) Then we need one relation $R = (v_1, v_2, v_3) \in \bigoplus_{i=1}^{3} S(-d_i - (d-1))_{b+2(d-1)}$ among s_1, s_2, s_3, i.e., $v_1 s_1 + v_2 s_2 + v_3 s_3 = 0$, necessary to construct the morphism

$$S(-b-2(d-1)) \to \bigoplus_{i=1}^{3} S(-d_i-(d-1))$$

by the formula $w \mapsto wR$, where $b = d_2 - d + 2$.

Note that $v_1 \in S_2$ and $v_i \in S_{b-d_i+d-1} = S_1$, $i = 2, 3$. The computations yield a relation R between these syzygies of multidegree $(2, 1, 1)$, namely $R = (684450x^2, -243y - 182z, -117y + 14z)$. Then $C_{49,gen}$ is a rational nearly free curve.

Remark 5.6.4. Let us note that a direct computation using `Singular` [DGPS19] of the Tjurina number of the singular point of the curve fails, but the nearly-free condition makes the computation possible via Theorem 5.5.3: $\tau(C_{49,gen}) = (49-1)(49-24-1) + 24^2 - 1 = 1727$ which is the result in `Singular` using characteristic $p = 1666666649$.

Conclusions

We give a summary of the main conclusions of this thesis, which have already been presented and discussed in more detail in each chapter.

As we mentioned in the introduction, we have investigated two central subjects regarding Singularity Theory of plane curves:

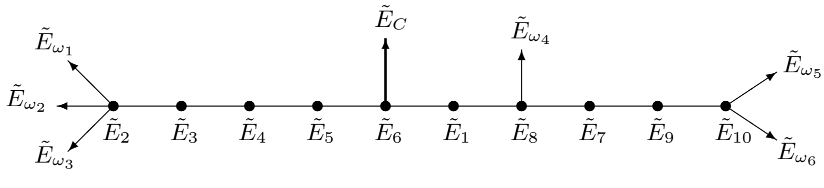
- The monodromy conjecture of J. Denef and F. Loeser and its generalisation by A. Némethi and W. Veys.
- Free and nearly free plane curves with isolated singularities, and some related conjectures proposed by A. Dimca and G. Sticlaru.

We have studied the extension of the topological zeta function using the Hessian differential form, following the ideas of A. Némethi and W. Veys, and we have found a counterexample which proves that the Monodromy Conjecture does not hold when we consider the Hessian differential form instead of the standard one.

Our counterexample is given by the germ of plane curve $(C,0)=(f^{-1}(0),0)$ defined by

$$f(x,y)=y^5-2x^3y^7+x^4y^3+x^6.$$

The minimal embedded resolution of $\left(f^{-1}(0)\cup \operatorname{div}\left(\omega_{\mathrm{hess}(f)}\right),0\right)$ is described in the figure below (the notation is specified in Chapter 3):



i	1	2	3	4	5	6	7	8	9	10	C	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
N_i	5	6	12	18	24	30	5	10	5	5	1	0	0	0	0	0	0
ν_i	9	13	22	31	40	49	13	23	16	19	1	2	2	2	2	2	2

Figure 5.4: Dual graph of the minimal embedded resolution of $\left(f^{-1}(0)\cup \operatorname{div}\left(\omega_{\mathrm{hess}(f)}\right),0\right)$, along with the numerical data of its components, for $f(x,y)=y^5-2x^3y^7+x^4y^3+x^6$.

We have shown that the pole of the topological zeta function $s_0 := -13/6$, corresponding to the component $\tilde{E}_2$, does not induce a monodromy eigenvalue, since s_0 is not a root of the characteristic polynomial of the germ $(C, 0)$:

$$\Delta_C(t) = \frac{(t-1)(t^{30}-1)}{(t^5-1)(t^6-1)},$$

and this implies that the Monodromy Conjecture fails when we consider the differential form $\omega_{\mathrm{hess}(f)} = \mathrm{hess}(f) dx \wedge dy$ instead of the standard ω_0. In particular, the Hessian differential form is not allowed.

Regarding the study of free and nearly free divisors, we have recalled the conjectures proposed by A. Dimca and G. Sticlaru in [DS17], [DS15] and [DS18a], namely:

1. Any rational cuspidal curve C in the plane is either free or nearly free.
2. An irreducible plane curve C which is either free or nearly free is rational.
3. Any free irreducible plane curve C has only singularities with at most two branches.
4. Any nearly free irreducible plane curve C has only singularities with at most three branches.

We have provided counterexamples to the last three conjectures:

- For every odd integer $k \geq 1$, the irreducible plane curve C_{5k} of degree $d = 5k$ defined by

  $$f_{5k} := (y^k z^k - x^{2k})^2 y^k - x^{5k} = 0$$

  is such that its geometric genus is $g(C_{5k}) = \frac{(k-1)(k-2)}{2}$, its singular locus consists of two points, say $\mathrm{Sing}(C_{5k}) = \{p_1, p_2\}$, the number of branches of C_{5k} at each p_i is exactly k, and C_{5k} is a free curve.

  Hence, if $k \geq 3$, C_{5k} is a counterexample to both the free part of Conjecture 2 and of Conjecture 3.

- For any odd integer $k \geq 1$, the irreducible plane curve C_{4k} of degree $d = 4k$ defined by

  $$f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0$$

  is such that its geometric genus is $g(C_{4k}) = \frac{(k-1)(k-2)}{2}$, its singular locus consists of two points: $\mathrm{Sing}(C_{4k}) = \{p_1, p_2\}$, the number of branches of C_{4k} at each p_i is exactly k and C_{4k} is a nearly free curve.

  Thus, if $k \geq 3$, C_{4k} is a counterexample to both the nearly-free part of Conjecture 2 and Conjecture 4 too.

- In the familes studied above the number of singular points of the curves is exactly two. We have also found curves with unbounded genus and number of singularities which give a counterexample to the part regarding nearly free curves of Conjecture 2. In particular, for every odd integer $k \geq 3$, the irreducible curve C_{2k} of degree $d = 2k$ defined by

  $$f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0$$

is such that $\mathrm{Sing}(C_{2k})$ contains exactly $3k$ singular points of type $\mathbb{A}_{k-1}$, its genus is $g(C_{2k}) = \frac{(k-1)(k-2)}{2}$ and C_{2k} is a nearly free curve.

In particular, these familes of examples $\{C_{5k}\}$, $\{C_{4k}\}$ and $\{C_{2k}\}$ are constructed as the pull-back under the Kummer cover π_k of the corresponding rational cuspidal curves: the quintic C_5 which is a free curve, and the corresponding nearly free divisors defined by the quartic C_4 and the conic C_2.

- Finally, an irreducible curve C_{49} of degree 49 is given. This curve has just one singular point which has 4 branches, its genus is $g(C_{49}) = 0$, i.e., C_{49} is a rational curve and it is a nearly free curve.

  The curve C_{49} is constructed as a generic element of the unique pencil associated with a certain rational unicuspidal plane curve of degree 49 and it provides another counterexample to Conjecture 4.

All these examples contradict some of the conjectures proposed by A. Dimca and G. Sticlaru in [DS15]. Unfortunately, our examples say nothing about the most remarkable conjecture by A. Dimca and G. Sticlaru, which predicts that every rational cuspidal plane curve is either free or nearly free.

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