UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE ESTUDIOS ESTADÍSTICOS

TESIS DOCTORAL

Valores para juegos cooperativos con dígrafos

Values for cooperative games with digraphs

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Elena del Carmen Gavilán García

Director

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Universidad Complutense de Madrid
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Tesis doctoral

Doctorado en Análisis de Datos (Data Science)

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values for cooperative games with digraphs

Memoria para optar al grado de Doctora

presentada por

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Agradecimientos

Defiende tu derecho a pensar, porque incluso pensar de manera errónea es mejor que no pensar.

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Resumen

Valores para juegos cooperativos con dígrafos

La Teoría de Juegos es una disciplina a caballo entre las matemáticas y la economía que ha alcanzado una gran relevancia en el análisis de procesos de decisión con actores interdependientes. El objetivo, al abordar desde conflictos bélicos y negociaciones comerciales hasta comportamientos sociales, es desarrollar estrategias óptimas y diseñar mecanismos para ayudar en la toma de decisiones.

La presente memoria se sitúa dentro del análisis de juegos cooperativos con utilidad transferible (o juegos TU) en los que la comunicación entre jugadores es limitada y está restringida por un grafo dirigido o dígrafo.

Se introduce un concepto novedoso de comunicación dirigida y una conectividad relacionada en grafos dirigidos que utilizamos para modelar ciertas restricciones de cooperación en juegos TU. En la literatura sobre la comunicación en redes dirigidas se pueden encontrar diferentes nociones de conectividad y, por tanto, diferentes formas en que la comunicación dirigida restringe las posibilidades de cooperación de los jugadores. En esta memoria, se introduce una noción de conectividad en dígrafos que se basa en caminos dirigidos. Asumimos que una coalición de jugadores en un juego sólo puede cooperar si estos jugadores forman un camino dirigido en un dígrafo. Definimos un juego restringido siguiendo el mismo planteamiento que Myerson para situaciones de comunicación no dirigida, y consideramos como regla de asignación el valor de Shapley de este juego restringido. Caracterizamos este regla mediante versiones ampliadas de los conocidos axiomas de eficiencia en componentes, equidad y contribuciones equilibradas.

Además, utilizando la nueva noción de conectividad y la regla de asignación introducidas definimos medidas de centralidad, eficiencia y vulnerabilidad en redes dirigidas.

En segundo lugar, utilizando el mismo concepto de comunicación dirigida, se introduce una familia de valores para juegos con restricción dada por dígrafos inspirada en el valor posicional. Para ello, definimos un juego de arcos, el cual analiza las coaliciones de arcos (dirigidos) en la generación de valor. A continuación se asigna el valor de Shapley a cada arco y se divide entre los nodos incidentes en él, permitiendo que el origen y el extremo obtengan una parte diferente de este valor. Sin embargo, la forma en que se reparte el pago del arco entre su origen y extremo es uniforme para todos los arcos. El nuevo valor propuesto, se caracteriza mediante eficiencia en la conexión y una modificación de la propiedad clásica de contribuciones equilibradas de arcos para situaciones de comunicación no dirigida que discrimina entre los roles de los nodos como origen y extremo.

Finalmente, se define el juego mixto para situaciones de comunicación dirigida, en el que los jugadores van a ser tanto los nodos como los arcos dirigidos en el dígrafo; el valor de una coalición (de jugadores y arcos) coincide con la suma de los valores (en el juego original) de las subcoaliciones máximalmente conectadas de esos jugadores en el grafo dado por esos arcos. Se propone además un valor mixto de comunicación dirigida, el cual asigna el valor de Shapley al juego mixto definido. Esta regla, a diferencia de las otras dos definidas, permiten analizar la parte del valor de la gran coalición que los jugadores deben detraer para pagar a los dueños de los arcos dirigidos cuando no se pueda asumir el dominio de estos arcos por parte de los jugadores incidentes. La regla definida se caracteriza con propiedades novedosas y razonables en este contexto.

Abstract

Values for cooperative games with digraphs

Game theory is a discipline that straddles mathematics and economics and has achieved great relevance in the analysis of decision-making processes with interdependent actors. The objective, in dealing with everything from war conflicts and commercial negotiations to social behaviour, is to develop optimal strategies and design mechanisms to aid decision-making.

This thesis is situated within the analysis of cooperative games with transferable utility (or TU games) in which communication between players is limited and constrained by a directed graph or digraph.

We introduce a novel concept of directed communication and related connectivity in directed graphs that we use to model certain cooperation constraints in TU games. In the literature on communication in directed networks one can find different notions of connectivity and thus different ways in which directed communication constrains the cooperation possibilities of the players. In this report, we introduce a notion of connectivity in digraphs that is based on directed paths. We assume that a coalition of players in a game can only cooperate if these players form a directed path in a digraph. We define a restricted game following the same approach as Myerson for undirected communication situations, and consider as an assignment rule the Shapley value of this restricted game. We characterise this rule by means of extended versions of the well-known axioms of component efficiency, fairness and balanced contributions. Furthermore, using the newly introduced notion of connectivity and the allocation rule we define measures of centrality, efficiency and vulnerability in directed networks.

Second, using the same concept of directed communication, we introduce a family of values for digraph-constrained games inspired by position value. For this, we define a set of arcs, which analyses coalitions of (directed) arcs in value generation. The Shapley value is then assigned to each arc and divided among the nodes incident on it, allowing the origin and the end to get a different share of this value. However, the way in which the arc payment is divided between its origin and end is uniform for all arcs. The proposed new value is characterised by connection efficiency and a modification of the classical property of balanced arc contributions for undirected communication situations that discriminates between the roles of nodes as origin and end.

Finally, the mixed game is defined for directed communication situations, in which the players are to be both nodes and directed arcs in the digraph; the value of a coalition (of players and arcs) coincides with the sum of the values (in the original game) of the maximally connected subcoalitions of those players in the graph given by those arcs. A mixed value of directed communication is also proposed, which assigns the Shapley value to the defined mixed game. This rule, unlike the other two defined, allows to analyse the part of the value of the grand coalition that players must subtract to pay the owners of the directed arcs when the dominance of these arcs cannot be assumed by the incident players. The defined rule is characterised by novel and reasonable properties in this context.

Contents

Chapter 1

Introduction

1.1 Introducción (en Español)

La palabra “game” tiene sus raíces en el inglés antiguo. Entonces se usaba la palabra “gamen” para referirse a alegría, diversión o entretenimiento. A lo largo del tiempo, esta palabra evolucionó y se utilizó para describir diversas formas de actividades recreativas con reglas, como competiciones y deportes. La palabra “game” ha mantenido su significado general de actividad recreativa o competición, y se ha adaptado a diferentes contextos y usos a lo largo de la historia del idioma inglés. A nivel científico, el término *juego* ha adquirido matices específicos y se contempla como cualquier modelo de conflicto en el que los actores toman decisiones y buscan maximizar sus resultados.

Se considera que el estudio de la Teoría de Juegos, con raíces desde los debates filosóficos griegos sobre la toma de decisiones, se inició a mediados del siglo XX, cuando John von Neumann y Oskar Morgenstern publican, en 1944, el libro *Theory of Games and Economic Behavior* donde se proporcionó un marco matemático sólido para el análisis de juegos y estrategias.

A lo largo de su evolución, esta disciplina ha trascendido los límites iniciales de los juegos de mesa y apuestas, expandiéndose hacia campos tan diversos como la economía, la biología, la política, la informática y la sociología. Su importancia radica en su capacidad para modelar y analizar situaciones estratégicas, desde conflictos armados hasta problemas de negociación, comportamiento empresarial, evolución biológica e interacciones sociales cotidianas.

Su relevancia y reconocimiento académico ha sido reconocido por el elevado número de premios Nobel concedidos a investigadores en Teoría de Juegos. John Nash, John Harsanyi y Reinhard Selten en 1994, recibieron el Premio Nobel en Economía "por sus análisis del equilibrio en la Teoría de Juegos no cooperativos", y en particular por el concepto de equilibrio de Nash. Robert J. Aumann y Thomas C. Schelling compartieron el Premio Nobel en Economía en 2005 "por ampliar la comprensión del conflicto y la cooperación a través de análisis basados en la Teoría de Juegos". En 2007, le fue otorgado este galardón en memoria de Alfred Nobel a Roger B. Myerson junto con Leonid Hurwicz y Eric Maskin "por establecer las bases de la teoría del diseño de los mecanismos, que determina cuándo los mercados están trabajando de manera efectiva". Unos años más tarde, Lloyd Shapley junto con Alvin E. Roth fueron laureados, en la edición de 2012, con el Premio Sveriges Riksbank en Ciencias Económicas en Memoria de Alfred Nobel, "por su trabajo en la teoría de las asignaciones estables y el diseño de mercado". En el año 2014 el francés Jean Tirole recibió también el galardón "por su análisis sobre el poder de los mercados y la regulación".

Como se ha mencionado, es un área de las Matemáticas y la Economía que se centra en el análisis de las estrategias y las interacciones de las decisiones en situaciones donde los resultados dependen de múltiples actores, conocidos como "jugadores" y permite determinar estrategias óptimas en escenarios de competencia o cooperación. Su aplicación se extiende a la comprensión de la evolución de especies, el desarrollo de políticas públicas, la toma de decisiones en empresas y la optimización de algoritmos en inteligencia artificial.

Con intención de tener una visión temporal de los diferentes hitos que han marcado la historia de esta área de conocimiento, debemos remontarnos a principios del siglo XX donde se conocen los estudios preliminares de Zermelo (1913), con el teorema que lleva su nombre y que habla de estrategias ganadoras en un juego de dos; Borel (1921), el cual introdujo la idea de estrategia mixta y Neumann (1928), con el teorema minimax. A continuación, se publica el libro anteriormente mencionado, *Theory of Games and Economic Behavior* por Neumann & Morgenstern (1944), texto innovador que creó el campo de investigación interdisciplinar de la Teoría de Juegos. Unos años después, Tucker (1950) escribe formalmente las primeras discusiones sobre el dilema del prisionero. En este mismo año, John Nash en su tesis doctoral dirigida por Tucker, detalla lo que más adelante se denominaría el Equilibrio de Nash, Nash (1950). Tres años más tarde, Shapley (1953b) publica su muy conocido trabajo en el que propone un método para asignar valores a los jugadores en un juego cooperativo, comúnmente llamado el valor de Shapley. Selten (1965) introdujo el concepto de 'perfección en subjuegos' para analizar juegos iterados o repetidos y Harsanyi (1967) exploró cómo los jugadores pueden tomar decisiones racionales en juegos con información incompleta. Poco después, Myerson (1977) desarrolló juegos donde la cooperación está restringida mediante grafos de comunicación.

En la actualidad, esta disciplina se erige como un pilar fundamental en la comprensión de la toma de decisiones y la interacción humana y sistémica, impactando directamente en múltiples aspectos de la vida moderna. Desde sus comienzos hasta su amplia influencia actual, la Teoría de Juegos continúa siendo un campo de estudio dinámico y esencial para comprender y abordar desafíos complejos en un mundo interconectado.

La teoría se encuentra diferenciada en dos grandes cuerpos: la Teoría de Juegos cooperativos y la de los no cooperativos. Por un lado, en los juegos no cooperativos cada uno de los jugadores toma decisiones estratégicas de manera independiente y sin coordinación ni negociación, únicamente con el objetivo de maximizar su propio resultado. Por otro lado, en los juegos cooperativos los jugadores colaboran y toman decisiones estratégicas de manera conjunta para obtener el máximo beneficio común, en lugar de competir entre sí. Para llevar a cabo esta cooperación, los jugadores se comunican, pudiendo alcanzar acuerdos de negociación. Aunque inicialmente la Teoría de Juegos se centró más en los juegos no cooperativos, el estudio de los juegos cooperativos ha ganado importancia y sigue siendo un campo de investigación activo.

Los juegos cooperativos en los que el pago a las coaliciones es divisible y transferible entre los jugadores reciben el nombre de juegos cooperativos con utilidad transferible (juegos-TU). Para poder conocer el reparto de ganancias de los diferentes actores en un juego cooperativo TU, se han definido a lo largo del tiempo diferentes valores o reglas de asignación. El valor de Shapley, Shapley (1953b), es una de las reglas de asignación más prominentes que se han desarrollado para juegos cooperativos. La idea principal detrás del valor de Shapley es considerar todas las posibles maneras en que los jugadores pueden unirse y formar coaliciones, y luego asignar un valor a cada jugador que coincide con su contribución marginal promedio a todas las posibles coaliciones en las que participa. Shapley (1953b) caracterizó su valor a través de tres axiomas, aditividad, simetría y propiedad soporte (*carrier*), aunque popularmente se ha difundido más la caracterización de Shubik (1962), donde se utilizan los axiomas de eficiencia, simetría, aditividad y jugador nulo. Myerson (1980) dio una nueva caracterización para este valor usando los axiomas de eficiencia y contribuciones equilibradas (*balanced contributions*). En la literatura se pueden encontrar otras caracterizaciones como las de Young (1985; 1994), Hart & Mas-Colell (1989), Chun (1989), Feltkamp (1995), Hamiache (2001), van den Brink (2002), Kongo et al. (2007), Manuel et al. (2013) y Manuel & Martín (2020).

En cuanto al cálculo y las aplicaciones del valor de Shapley, desde su origen, el número de contribuciones es significativo. Entre ellas se pueden encontrar las extensiones multilineales para simplificar el cálculo introducidas por Owen (1972). Moretti & Patrone (2008) analizaron la tranversalidad del valor de Shapley a través de diferentes aplicaciones, Castro et al. (2009) desarrollaron un método polinómico para poder estimar el valor de Shapley a través del muestreo estadístico y Skibski et al. (2014) desarrollaron un algoritmo eficiente para calcular el valor de Shapley.

En muchas situaciones de la vida real, existen restricciones en la formación de coaliciones, es decir, la capacidad para que los jugadores puedan comunicarse está limitada de alguna manera y no todas las coaliciones serán factibles. En una de las restricciones más habituales en la formación de coaliciones se asume que no todos los pares de jugadores pueden comunicarse directamente entre sí, pero también podemos encontrar restricciones de tiempo, recursos limitados o estructuras específicas de interacción. Puede haber un límite en la cantidad de mensajes que los jugadores pueden intercambiar, pueden existir canales de comunicación específicos disponibles solo para ciertos jugadores o grupos, o puede haber restricciones geográficas o espaciales que dificulten la interacción directa entre todos los jugadores.

Los juegos cooperativos con comunicación restringida tienen aplicaciones en una variedad de campos, desde la teoría de redes y la informática hasta la economía y la toma de decisiones grupales. Comprender cómo los jugadores cooperan y toman decisiones cuando la comunicación está restringida proporciona perspectivas valiosas sobre la eficacia de las estrategias de coordinación y la formación de coaliciones en entornos donde las interacciones directas no son posibles para todos los participantes.

Los primeros en hablar de estructuras coalicionales en juegos cooperativos con comunicación restringida fueron Aumann & Dreze (1974), los cuales propusieron un modelo en el que la colaboración entre jugadores no es libre, sino que solo se da entre aquellos que pertenecen a distintos conjuntos de una partición fijada *a priori*.

Una de las restricciones en la comunicación más estudiada a lo largo del tiempo es la comunicación restringida por grafos no dirigidos, problema abordado en sus inicios por Myerson (1977, 1980). Un grafo es una estructura matemática compuesta por un conjunto de nodos (vértices) y un conjunto de conexiones entre ellos (aristas). En particular, en Teoría de Juegos restringida a grafos, los nodos representan los jugadores del juego y los arcos las posibles comunicaciones simétricas entre nodos adyacentes.

En la propuesta de Myerson, solo las coaliciones que estén conectadas en el grafo son factibles y, por lo tanto, solo los jugadores que pertenecen a ellas pueden cooperar y generar el valor descrito por el juego. En la literatura, el par formado por el juego cooperativo coalicional y el grafo es llamado *situación de comunicación* o *juego restringido al grafo*.

Además, Myerson (1977) propuso aplicar el valor de Shapley (Shapley, 1953b) a un juego modificado en el que cada coalición factible (que en este contexto significa conectado) puede ganar su valor, y el valor de cualquier otra coalición es igual a la suma de los valores, en el juego original, de sus componentes conexas en el grafo. Esta solución se conoce hoy en día como el valor de Myerson.

También demostró que esta solución es la única que satisface la eficiencia en componentes (la suma de los pagos asignados a los jugadores de cada componente es igual al valor de cada componente en el juego original) y la equidad o *fairness* (romper un enlace entre dos jugadores tiene el mismo efecto en los pagos de estos dos jugadores). Más tarde, Myerson (1980) proporcionó otra axiomatización reemplazando la equidad por contribuciones equilibradas o *balanced contributions* (el efecto en un jugador de que otro se aísle es simétrico).

Debido a la relevancia del valor de Myerson, se pueden encontrar en la literatura diferentes generalizaciones, análisis y aplicaciones para este valor. Winter (1992) aplicó el enfoque propuesto por Hart & Mas-Colell (1989) relativo a la propiedad de consistencia y la función potencial para el valor de Shapley, probando que el valor de Myerson también admite una función potencial. van den Nouweland et al. (1992) extendieron el valor de Myerson para situaciones de comunicación restringida por un hipergrafo. Jackson & Wolinsky (2003) introdujeron una extensión del valor de Myerson interpretable como una regla de poder en la negociación. Algaba et al. (2001) propusieron una caracterización para el valor de Myerson en juegos cooperativos con estructuras de coalición estables. Calvo et al. (1999) propusieron, para juegos con grafos probabilísticos, una extensión del valor de Myerson en la cual cada par de nodos tiene una probabilidad de comunicación directa independiente de las de los demás pares. Haeringer (1999) generalizó el valor de Myerson incluyendo pesos y en Fernández et al. (2002) introducen algoritmos de tiempo polinómico basados en funciones generadoras para calcular el valor de Myerson. Gómez et al. (2008) definieron sobre el conjunto de todos los arcos posibles en la red una distribución de probabilidad, eliminando la condición de independencia probabilística entre los arcos. Para grafos en los que los arcos tienen una ponderación, González-Arangüena et al. (2015) extendieron el valor de Myerson y en Gómez et al. (2003) propusieron una descomposición aditiva para situaciones de comunicación con juego simétrico. Ello permitió dar una interpretación al valor de Myerson como medida de centralidad en redes descomponible en otras dos, una de comunicación y otra de intermediación. Esta descomposición se generaliza para juegos no simétricos en González-Arangüena et al. (2017). Jiménez-Losada et al. (2013) proponen un marco general para definir los valores Myerson difusos. Shan et al. (2019) una clase de extensiones eficientes del valor de Myerson. En Manuel et al. (2020) se analiza la relación del axioma de marginalismo y el valor de Myerson, proponiendo una nueva caracterización que incluye este axioma. Li & Shan (2020a) propusieron una extensión eficiente del valor de Myerson para juegos con estructura de grafo de comunicación y Manuel & Martín (2021) extendieron el valor de Myerson para situaciones en las que los jugadores tienen diferentes habilidades de regateo.

Alternativamente, en la literatura se encuentran otras reglas de asignación para situaciones de comunicación. Meessen (1988), en su tesis doctoral, introdujo el valor posicional, el cual difundieron más adelante Borm et al. (1992).

Considerando una situación de comunicación, primero definieron lo que se conoce como el juego de los arcos (*link game*), donde los *links* son los jugadores, y el valor de cada coalición de links es igual al valor de la gran coalición en el juego de Myerson para el grafo dado por esos links. E introdujeron el valor posicional que se calcula para cada jugador como la mitad de la suma de los valores de Shapley en el juego de los arcos de aquellos que inciden en él.

La caracterización que propusieron Borm et al. (1992) para este valor era válida solo para situaciones de comunicación en las que el grafo es un árbol. Una caracterización para el valor posicional definida sobre el conjunto total de situaciones de comunicación fue introducida por Slikker (2005) utilizando los axiomas de eficiencia en componentes y las contribuciones equilibradas en arcos (*balanced link contributions*). Se conocen otras caracterizaciones del valor posicional, como la propuesta por Casajus (2007) en términos del valor de Myerson de un juego restringido, conocida como la forma de agente de enlace (*link agent form*). También Gómez et al. (2004) introdujeron una aproximación unificada al cálculo del valor de Myerson y el valor posicional. En Manuel et al. (2022) propusieron una nueva caracterización para el valor posicional a través de una extensión de la clásica propiedad de marginalismo para juegos-TU.

Otro valor para situaciones de comunicación que no ha recibido tanta atención en la literatura pero que también es de interés en esta memoria es el *valor mixto*. Feltkamp & van den Nouweland (1993) introdujeron un juego TU restringido al grafo, el juego mixto o pseudo-juego, en el que los jugadores son tanto los nodos como los arcos del grafo. El valor de cada coalición (de nodos y arcos) coincide con la suma de los valores (en el juego original) de las subcoaliciones de esos jugadores máximamente conectadas en el grafo dado por esos links. Definieron el valor mixto (para jugadores y arcos) como el valor de Shapley correspondiente al juego mixto. Feltkamp & van den Nouweland (1993) caracterizaron el valor mixto en términos de eficiencia, aditividad en el juego, la propiedad del arco superfluo y el anonimato.

Recientemente, López et al. (2022) han utilizado el valor mixto para analizar el impacto de los intermediarios en una negociación. En Gavilán et al. (2023a) se extiende el valor mixto para redes ponderadas y además se proponen nuevas caracterizaciones tanto para el valor mixto como para el valor mixto ponderado utilizando modificaciones de las propiedades clásicas de eficiencia en componentes, equidad, contribuciones equilibradas y contribuciones equilibradas de arcos.

La ya mencionada versatilidad de la Teoría de Juegos alcanza para medir la centralidad de actores en redes sociales, un problema clásico en sociología. La centralidad es un concepto esquivo que trata de evaluar la importancia de un nodo en una red, teniendo en cuenta su posición y su influencia en la estructura global de la red. Bavelas (1948) introduce la idea de centralidad aplicada a redes de comunicación, especialmente en la comunicación en grupos pequeños. A partir de este trabajo, han sido muchos los esfuerzos de los investigadores para poder definir qué es la centralidad, sin llegar a ningún consenso. Por ello se han definido a lo largo de los años diferentes medidas de centralidad. Entre ellas podemos encontrar:

- Centralidad de grado (*degree centrality*) (Shaw, 1954; Nieminen, 1974), basada en el número de conexiones de un nodo.
- Centralidad por intermediación (*betweenness centrality*) (Bavelas, 1948; Freeman, 1977), basada en la cantidad de veces que un nodo se encuentra en el camino más corto entre otros nodos.
- Centralidad por proximidad (*closeness centrality*) (Beauchamp, 1965; Sabidussi, 1966), basada en la distancia promedio de un nodo a todos los demás nodos en la red.
- Centralidad de vector propio (*eigenvector centrality*) (Bonacich, 1972; 1987). Evalúa la importancia de un nodo en función de su conexión con otros nodos importantes, a través de los autovalores de la matriz de adyacencia del grafo que representa la red social.

La primera contribución juego-teórica a la centralidad en redes sociales fue propuesta por Grofman & Owen (1982). Otras contribuciones en este ámbito se encuentran en van den Brink & Borm (2002), Gómez et al. (2003), Amer et al. (2007), Aadithya et al. (2010), del Pozo et al. (2011), Michalak et al. (2013), Mazalov & Trukhina (2014), Michalak et al. (2015), Mazalov et al. (2016), Szczepański et al. (2016), Tarkowski et al. (2018), Gallardo et al. (2018), Navarro (2019) y van den Brink & Rusinowska (2023). Tarkowski et al. (2017) es una revisión interesante sobre las contribuciones a la centralidad de la red a partir de la Teoría de Juegos.

Otras medidas de gran relevancia en redes sociales que son utilizadas para evaluar la robustez de una red, son la eficiencia y la vulnerabilidad. La eficiencia en redes sociales se percibe como la capacidad de la red para conectar a los individuos, facilitar la comunicación mejorando el flujo de información y aumentar la cohesión social. Por otro lado, la vulnerabilidad de la red indica su exposición o la de sus miembros a riesgos externos o internos, como la desinformación, la pérdida de información y las interacciones perjudiciales. Latora & Marchiori (2001) introdujeron una de las más prominentes medidas de eficiencia. La Teoría de Juegos también ha sido utilizada para aportar resultados en esta dirección. Bell (2003), Jackson & Wolinsky (2003), Jackson et al. (2008), Gueye et al. (2012), Lee et al. (2018) Lindelauf et al. (2013) y Manuel & Ortega (2023) son también enfoques juego-teóricos diferentes para medir la eficiencia y la vulnerabilidad en las redes.

En la comunicación muchas veces se asume que los enlaces entre pares de jugadores son simétricos, es decir que cualquier jugador puede iniciar la comunicación. Sin embargo, frecuentemente podemos encontrar escenarios en las que la comunicación es dirigida existindo emisores y receptores y, por lo tanto, necesitamos modelos alternativos. Para modelar este tipo de problemas, en esta tesis doctoral utilizaremos situaciones de comunicación dirigida en las que los jugadores en un juego cooperativo son simultaneamente nodos en una red dirigida representada por un grafo dirigido o dígrafo. Aunque las situaciones de comunicación dirigida no son novedosas y han sido ya utilizadas en la literatura (ver van den Brink & Gilles, 1996; van den Brink, 1997 y Gonzalez-Aranguena et al., 2008) la interpretación, y por lo tanto, las propiedades deseables de las reglas de asignación, sí lo son.

De manera similar al caso de los valores de Myerson, posicional y mixto, introduciremos juegos restringidos que tengan en cuenta este tipo de limitaciones en la comunicación. En la literatura, existen diferentes nociones de conectividad en grafos dirigidos y diferentes formas en que la comunicación dirigida restringe las posibilidades de cooperación de los jugadores. En la presente memoria, se considera un nuevo tipo de conectividad en situaciones de comunicación dirigida, la cual será aplicada para la formación de coaliciones en juegos cooperativos.

Admitiremos que una coalición de jugadores solo puede cooperar si estos jugadores forman un camino dirigido en un dígrafo de comunicación.

En primer lugar, y utilizando esta idea de conexión y siguiendo los pasos de Myerson (1977) definiremos un juego (de nodos) restringido al dígrafo y propondremos como regla de asignación en la situación de comunicación dirigida el valor de Shapley de este juego. Lo caracterizamos mediante versiones extendidas de las mencionadas propiedades de eficiencia en componentes, equidad y contribuciones equilibradas. La equidad y las contribuciones equilibradas se expresan de la misma manera que en los grafos no dirigidos, solo que las aristas son arcos orientados. La eficiencia en componentes se modifica para reflejar nuestras nuevas restricciones de cooperación y recibirá el nombre de eficiencia en la conexión.

Después de introducir este modelo, la solución y las axiomatizaciones, lo aplicaremos a algunos problemas que surgen en las redes sociales. Específicamente, al tomar cualquier juego simétrico (es decir, un juego en el que todos los jugadores son idénticos en sus contribuciones) y aplicar nuestra solución a cada red dirigida, obtenemos una nueva medida de centralidad para redes dirigidas. En realidad, obtenemos una familia de medidas de centralidad que está parametrizada por el juego simétrico que se utiliza. Además, las axiomatizaciones de nuestra regla de asignación para situaciones de comunicación dirigida se pueden aplicar directamente como una axiomatización para medidas de centralidad en redes dirigidas.

Mientras que dentro de esta familia todas las medidas de centralidad satisfacen equidad y contribuciones equilibradas, cada medida difiere por el juego simétrico que se utiliza y que impacta en el axioma de eficiencia de conexión. Así, destacamos el papel del juego.

Además, analizaremos comparativamente el enfoque propuesto en esta memoria con otros modelos en los que las restricciones en las comunicaciones se establecen mediante un grafo dirigido. En particular con los juegos con estructura de permisos de Gilles et al. (1992), Gilles & Owen (1994) y van den Brink & Gilles (1996) y con los modelos de Khmelnitskaya et al. (2016) y Li & Shan (2020b).

En segundo lugar, utilizando también el concepto de conectividad introducido y, siguiendo a Meessen (1988) y Borm et al. (1992), definiremos un juego de arcos y una familia de valores posicionales para situaciones de comunicación dirigida. En este juego, se evalúa la importancia de las coaliciones de arcos (dirigidos) en la generación de valor y, a continuación, se reparte el valor de Shapley de cada arco entre sus nodos incidentes, admitiendo que el origen y extremos obtengan una proporción diferente del valor del arco. Sin embargo, la forma en que se reparte el valor del arco entre el origen y el extremo es uniforme para todos los arcos. Se caracterizan estos valores mediante la eficiencia en la conexión, antes mencionada, y una modificación de las contribuciones equilibradas de los arcos que discrimina entre los papeles de ambos extremos.

Por último, se propone una extensión del juego mixto, definido en Feltkamp & van den Nouweland (1993), para situaciones de comunicación dirigida. Se define un nuevo valor (para arcos y nodos), el valor de comunicación dirigida mixto en el que se asigna el valor de Shapley del juego mixto de arcos y nodos. Este valor se caracteriza a través de extensiones de las propiedades clásicas de eficiencia, equidad y contribuciones equilibradas, definidas ahora para situaciones dirigidas. Estas propiedades son eficiencia mixta en la conexión, equidad mixta, y contribuciones equilibradas mixtas.

Los resultados descritos en la presente memoria sobre situaciones de comunicación dirigida, así como la propuesta de extensión de los valores de Myerson, posicional y mixto para estas nuevas situaciones así como medidas de centralidad y de eficiencia y vulnerabilidad en redes dirigidas han dado lugar a las publicaciones:

- E.C. Gavilán, C.M. Manuel y R. van den Brink. (2022). A Family of Position Values for Directed Communication Situations. *Mathematics*, 10(8), 1235.

- E.C. Gavilán, C.M. Manuel y R. van den Brink. (2023b). Directed communication in games with directed graphs. *TOP*, 1-34.

- E.C. Gavilán, C.M. Manuel y D. Martín. (2023a). Communication in Weighted Networks: A Game Theoretic Approach. *Axioms* 12(2), 180.

- E.C. Gavilán y C.M. Manuel (2024). A Mixed Value for Directed Communication Situations. *Sometido.*

La memoria se estructura de la siguiente manera. En el Capítulo 2 se detallan los conceptos necesarios para la introducción y formalización de los resultados posteriores . En el Capítulo 3 se describe el modelo de situaciones de comunicación dirigidas, el valor-DC para esas situaciones y su caracterización. Además se describen las aplicaciones (centralidad, eficiencia y vulnerabilidad) de dicha regla de asignación y se compara con otras reglas que se encuentran definidas en la literatura. En el Capítulo 4 se introduce el juego de los arcos y se define la familia de valores posicionales para situaciones de comunicación dirigidas así como su caracterización. En el Capítulo 5 se propone un nuevo juego mixto para situaciones de comunicación dirigida, el valor de comunicación dirigida mixto y se caracteriza este. Por último, en el Capítulo 6 se exponen las conclusiones de los resultados obtenidos y se plantean futuras líneas de investigación.

1.2 Introduction (in English)

The word "game" has its roots in Old English. At that time the word "gamen" was used to refer to joy, fun or entertainment. Over time, this word evolved and came to be used to describe various forms of recreational activities with rules, such as competitions and sports. The word "game" has retained its general meaning of recreational activity or competition, and has been adapted to different contexts and uses throughout the history of the English language. At the scientific level, the term "game" has acquired specific nuances and is seen as any model of conflict in which players make decisions and seek to maximise their outcomes.

The study of Game Theory, with roots in Greek philosophical debates on decision-making, is considered to have begun in the mid-20th century, when John von Neumann and Oskar Morgenstern published the book The Theory of Games and Economic Behavior in 1944, which provided a solid mathematical framework for the analysis of games and strategies.

Throughout its evolution, this discipline has transcended the initial boundaries of table games and betting, expanding into fields as diverse as economics, biology, politics, computer science and sociology. Its importance lies in its ability to model and analyse strategic situations, from armed conflicts to negotiation problems, business behaviour, biological evolution and everyday social interactions.

Its relevance and academic recognition has been acknowledged by the large number of Nobel Prizes awarded to researchers in Game Theory. John Nash, John Harsanyi, and Reinhard Selten received the Nobel Prize in Economics in 1994 "for their pioneering analysis of equilibria in the theory of non-cooperative games," particularly for the concept of Nash equilibrium. Robert J. Aumann and Thomas C. Schelling shared the Nobel Prize in Economics in 2005 "for having enhanced our understanding of conflict and cooperation through game-theoretical analysis." In 2007, Roger B. Myerson, along with Leonid Hurwicz and Eric Maskin, was awarded this honor in memory of Alfred Nobel "for having laid the foundations of mechanism design theory, which determines when markets work effectively." A few years later, Lloyd Shapley, along with Alvin E. Roth, was awarded the 2012 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel, "for their work on the theory of stable allocations and the practice of market design."

In 2014 the frenchman Jean Tirole also received the award "for his analysis of the power of markets and regulation".

As mentioned, it is an area of Mathematics and Economics that focuses on the analysis of strategies and decision interactions in situations where outcomes depend on multiple actors, known as "players", and allows determining optimal strategies in competitive or cooperative scenarios. Its application extends to the understanding of the evolution of species, the development of public policies, decision making in companies and the optimisation of algorithms in artificial intelligence.

With the intention of having a temporal vision of the different milestones that have marked the history of this area of knowledge, we must go back to the beginning of the 20th century where we know the preliminary studies of Zermelo (1913), with the theorem that bears his name and that talks about winning strategies in a game of two; Borel (1921), who introduced the idea of mixed strategy and Neumann (1928), with the minimax theorem. This was followed by the publication of the aforementioned book, The Theory of Games and Economic Behavior by Neumann & Morgenstern (1944), a groundbreaking text that created the interdisciplinary research field of Game Theory. A few years later, Tucker (1950) formally wrote the first discussions of the prisoner's dilemma. In the same year, John Nash, in his doctoral thesis supervised by Tucker, details what would later be called the Nash Equilibrium, Nash (1950). Three years later, Shapley (1953b) published his well-known paper proposing a method for assigning values to players in a cooperative game, commonly called the Shapley value. Selten (1965) introduced the concept of 'perfection in subgames' to analyse iterated or repeated games and Harsanyi (1967) explored how players can make rational decisions in games with incomplete information. Soon after, Myerson (1977) developed games where cooperation is constrained by communication graphs.

Today, the discipline stands as a fundamental pillar in the understanding of decision making and human and systemic interaction, directly impacting multiple aspects of modern life. From its beginnings to its wide influence today, Game Theory continues to be a dynamic and essential field of study for understanding and addressing complex challenges in an interconnected world.

Game Theory is differentiated into two main bodies: the theory of cooperative games and the theory of non-cooperative games. On the one hand, in non-cooperative games, each player makes strategic decisions independently and without coordination or negotiation, solely with the aim of maximising their own outcome. On the other hand, in cooperative games, players cooperate and make strategic decisions together in order to obtain the maximum common benefit, instead of competing with each other. In order to carry out this cooperation, players communicate and can reach negotiation agreements. Although initially Game Theory focused more on non-cooperative games, the study of cooperative games has gained importance and remains an active field of research.

Cooperative games in which the payout to coalitions is divisible and transferable between players are called cooperative games with transferable utility (TU-games). In order to know the distribution of gains of the different players in a cooperative TU game, different values or allocation rules have been studied over time. Shapley value, or Shapley value, Shapley (1953b) is one of the most prominent allocation rules that have been developed for cooperative games. The main idea behind Shapley value is to consider all the possible ways in which players can join together and form coalitions, and then assign to each player his average marginal contribution (over all permutations) to all the possible coalitions in which he participates. Shapley (1953b) characterized its value through three axioms, additivity, symmetry and carrier property, although the characterization of Shubik (1962), where the axioms of efficiency, symmetry, additivity and null player are used, has become more popular. Myerson (1980) gave a new characterization for this value using the axioms of efficiency and balanced contributions. Other characterizations can be found in the literature such as those of Young (1985, 1994), Hart & Mas-Colell (1989), Chun (1989), Feltkamp (1995), Hamiache (2001), van den Brink (2002), Kongo et al. (2007), Manuel et al. (2013) and Manuel & Martín (2020).

Regarding the calculation and applications of the Shapley value, since its origin, the number of contributions is significant. Among them, we can find the multilinear extensions to simplify the calculation introduced in Owen (1972). Moretti & Patrone (2008) analysed the transversality of the Shapley value through different applications and Castro et al. (2009) developed a polynomial method to estimate the Shapley value through statistical sampling.

In many real-life situations, there are constraints on coalition formation, i.e. the ability for players to communicate is limited in some way and not all coalitions will be feasible. In one of the most common coalition formation constraints it is assumed that not all pairs of players can communicate directly with each other, but we can also find time constraints, limited resources or specific interaction structures. There may be a limit on the amount of messages players can exchange, there may be specific communication channels available only to certain players or groups, or there may be geographical or spatial restrictions that make direct interaction between all players difficult.

Cooperative games with restricted communication have applications in a variety of fields, from network theory and computer science to economics and group decision-making. Understanding how players cooperate and make decisions when communication is restricted provides valuable insights into the effectiveness of coordination strategies and coalition building in environments where direct interactions are not possible for all participants.

The first to speak of coalitional structures in cooperative games with restricted communication were Aumann & Dreze (1974), who proposed a model in which collaboration between players is not free, but only between those belonging to different sets of *a priori* fixed partition.

One of the most studied communication constraints over time is undirected graph-restricted communication, a problem first addressed by Myerson (1977, 1980). A graph is a mathematical structure composed of a set of nodes (vertices) and a set of connections between them (edges). In particular, in Game Theory restricted to graphs, the nodes represent the players in the game and the links represent the possible symmetric communications between adjacent nodes.

In Myerson's proposal, only coalitions that are connected in the network are feasible and, therefore, only the players belonging to them can cooperate and generate the value described by the game. In the literature, a pair formed by the cooperative coalitional game and the graph is called *communication situation* or *graph restricted game*.

Moreover, Myerson (1977) proposed to apply the Shapley value (Shapley, 1953b) to a modified game in which each feasible (that in this context means connected) coalition can win its value, and the value of any other coalition is equal to the sum of the values, in the original game, of its connected components in the graph. This solution is known today as the Myerson value.

He also showed that this solution is the only one that satisfies component efficiency (the sum of the payoffs assigned to the players of each component is equal to the value of each component in the original game) and *fairness* (breaking a link between two players has the same effect on the payoffs of these two players). Later, Myerson (1980) provided another axiomatization by replacing fairness with *balanced contributions* (the effect on one player of isolating another is symmetrical).

Due to the relevance of the Myerson value, different generalisations, analyses and applications for this value can be found in the literature. Winter (1992) applied the approach proposed by Hart & Mas-Colell (1989) concerning the consistency property and the potential function for the Shapley value, proving that the Myerson value also admits a potential function. van den Nouweland et al. (1992) extended the Myerson value for situations of communication restricted by a hypergraph. Jackson & Wolinsky (2003) introduced an extension of the Myerson value interpretable as a power rule in negotiation. Algaba et al. (2001) proposed a characterisation for the Myerson value in cooperative games with stable coalition structures. Calvo et al. (1999) proposed, for games with probabilistic graphs, an extension of the Myerson value in which each pair of nodes has a direct communication probability independent of those of the other pairs. Haeringer (1999) generalised the Myerson value by including weights and in Fernández et al. (2002) they introduce polynomial time algorithms based on generating functions to compute the Myerson value. Gómez et al. (2008) defined over the set of all possible arcs in the network a probability distribution, eliminating the condition of probabilistic independence between the arcs. For networks in which the arcs have a weighting, González-Arangüena et al. (2015) extended the Myerson value and Gómez et al. (2003) proposed an additive decomposition for communication situations with symmetrical play. This allowed them to give an interpretation of the Myerson value as a measure of centrality in networks that can be decomposed into two others, one for communication and the other for intermediation. This decomposition is generalised for non-symmetric games in González-Arangüena et al. (2017). Jiménez-Losada et al. (2013) propose a general framework for defining fuzzy Myerson values. Shan et al. (2019) propose a class of efficient extensions of the Myerson value. In Manuel et al. (2020) the relationship of the marginality axiom and the Myerson value is analysed, proposing a new characterisation that includes this axiom. Li & Shan (2020a) proposed an efficient extension of the Myerson value for games with communication graph structure and Manuel & Martín (2021) extended the Myerson value for situations in which players have different bargaining abilities.

Alternatively, other assignment rules for communication situations can be found in the literature. Meessen (1988), in his doctoral thesis, introduced the position value, which was later disseminated by Borm et al. (1992). Considering a communication situation, they first defined what is known as the link game, where the links are the players, and the value of each coalition of links is equal to the value of the grand coalition in Myerson's game for the graph given by those links. And they introduced the position value which is calculated for each player as half the sum of the Shapley values in the game of the arcs of those that impinge on it.

The characterization proposed by Borm et al. (1992) for this value was valid only for communication situations where the network is a tree. A characterization for the position value defined on the total set of communication situations was introduced by Slikker (2005) using the axioms of efficiency in components and balanced link contributions. Other characterisations of position value are known, such as the one proposed by Casajus (2007) in terms of the Myerson value of a restricted set, known as the link agent form. Also Gómez et al. (2004) introduced a unified approach to the computation of the Myerson value and the position value. In Manuel et al. (2022) they proposed a new characterisation for position value through an extension of the classical marginalism property for TU-games.

Another value for communication situations that has not received as much attention in the literature but is also of interest in this report is the *mixed value*. Feltkamp & van den Nouweland (1993) introduced a TU game restricted to the graph, the mixed game or pseudo-game, in which the players are both the nodes and the arcs of the graph. The value of each coalition (of nodes and arcs) coincides with the sum of the values (in the original game) of the maximally connected subcoalitions of those players in the graph given by those links. They defined the mixed value (for players and arcs) as the Shapley value corresponding to the mixed game. Feltkamp & van den Nouweland (1993) characterized the mixed value in terms of efficiency, additivity in the game, the superfluous link property and anonymity.

Recently, López et al. (2022) has been used the mixed value to analyse the impact of intermediaries in a negotiation. In Gavilán et al. (2023a), they extends the mixed value for weighted networks and furthermore propose new characterizations for both mixed value and the weighted mixed value using modifications of the classical properties of component efficiency, fairness, balanced contributions and balanced contributions of arcs.

The aforementioned versatility of game theory is enough to measure the centrality of actors in social networks, a classic problem in sociology. Centrality is an elusive concept that tries to evaluate the importance of a node in a network, taking into account its position and its influence on the overall structure of the network. Bavelas (1948) introduces the idea of centrality applied to communication networks, especially in small group communication. Since this work, there have been many efforts by researchers to define what centrality is, without reaching any consensus. For this reason, different measures of centrality have been defined over the years. Among them we can find:

- Degree centrality (Shaw, 1954; Nieminen, 1974), based on the number of connections of a node.
- Intermediation centrality (Bavelas, 1948; Freeman, 1977), based on the number of times a node is on the shortest path between other nodes.
- Closeness centrality (Beauchamp, 1965; Sabidussi, 1966), based on the average distance from a node to all other nodes in the network.
- Eigenvector centrality (Bonacich, 1972; 1987). It evaluates the importance of a node in terms of its connection to other important nodes, through the eigenvalues of the adjacency matrix of the graph representing the social network.

The first game-theoretical contribution to centrality in social networks was proposed by Grofman & Owen (1982). Other contributions in this area can be found in van den Brink & Borm (2002), Gómez et al. (2003), Amer et al. (2007), Aadithya et al. (2010), del Pozo et al. (2011), Michalak et al. (2013), Mazalov & Trukhina (2014), Michalak et al. (2015), Mazalov et al. (2016), Szczepański et al. (2016), Tarkowski et al. (2018), Gallardo et al. (2018), Navarro (2019) and van den Brink & Rusinowska (2023). Tarkowski et al. (2017) is an interesting review of contributions to network centrality from Game Theory.

In communication it is often assumed that links between pairs of players are symmetric, i.e., any player can initiate communication. However, we can often find scenarios in which communication is directed with both senders and receivers and, therefore, we need alternative models. To model such problems, in this thesis we will use directed communication situations in which the players in a cooperative game are simultaneously nodes in a directed network represented by a directed graph or digraph. Although directed communication situations are not novel and have already been used in the literature (see van den Brink & Gilles, 1996, 1996, van den Brink & Gilles; van den Brink, 1997 and Gonzalez-Aranguena et al., 2008) the interpretation, and thus the desirable properties of the assignment rules, are.

Similar to the case of Myerson, position and mixed values, we will introduce restricted games that take into account these kinds of communication constraints. In the literature, there are different notions of connectivity in directed graphs and different ways in which directed communication restricts players' possibilities for cooperation. In this PhD thesis, we consider a new type of connectivity in directed communication situations, which will be applied to the formation of coalitions in cooperative games. We will admit that a coalition of players can only cooperate if these players form a directed path in a communication digraph.

First of all, using this idea of connection and following the steps of Myerson (1977) we define a game (of players/nodes) restricted to the digraph and propose as allocation rule in the directed communication situation the Shapley value of this game. We characterize it by means of extended versions of the aforementioned properties of component efficiency, fairness and balanced contributions. Fairness and balanced contributions are expressed in the same way as in undirected graphs, only the edges are oriented arcs. Component efficiency is modified to reflect our new cooperation constraints and will be called connection efficiency.

After introducing this model, the solution and the axiomatizations, we will apply it to some problems that arise in social networks. Specifically, by taking any symmetric game (i.e., a game in which all players are identical in their contributions) and applying our solution to each directed network we obtain new centrality measures for directed networks. In fact, we obtain a family of centrality measures that is parameterised by the symmetric game that is used. Moreover, the axiomatizations of our assignment rule for directed communication situations can be directly applied as an axiomatisation for centrality measures in directed networks.

While within this family all centrality measures satisfy fairness and balanced contributions, each measure differs by the symmetric game that is used and that impacts the axiom of connection efficiency. Thus, we highlight the role of the game.

In addition, we will compare the approach proposed in this thesis with other models in which communication constraints are set by means of a directed network. In particular with the permission-structured games of Gilles et al. (1992), Gilles & Owen (1994) and van den Brink & Gilles (1996) and with the models of Khmelnitskaya et al. (2016) and Li & Shan (2020b).

Secondly, by also using the concept of connectivity introduced and, following Meessen (1988) and Borm et al. (1992), we define a game of arcs and a family of position values for directed communication situations. In this game, the importance of coalitions of (directed) arcs in generating value is assessed, and then the Shapley value of each arc is distributed among its incident nodes, admitting that the origin and ends get a different proportion of the value of the arc. However, the way in which the arc value is distributed between the origin and the end is uniform for all arcs. These values are characterised by the aforementioned connection efficiency and a modification of the balanced contributions of the arcs that discriminates between the roles of the two ends.

Finally, an extension of the mixed game, defined in Feltkamp & van den Nouweland (1993), is proposed for directed communication situations. A new value is defined (for arcs and nodes), the mixed directed communication value where the Shapley value of the mixed set of arcs and nodes is assigned. This value is characterised through extensions of the classical properties of efficiency, fairness and balanced contributions, now defined for directed situations. These properties are mixed connection efficiency, mixed fairness, and mixed balanced contributions.

The results described in this PhD thesis on directed communication situations, as well as the proposed extension of Myerson, position and mixed values for these new situations as well as measures of centrality and efficiency and vulnerability in directed networks have led to the publications:

- E.C. Gavilán, C.M. Manuel and R. van den Brink. (2022). A Family of Position Values for Directed Communication Situations. *Mathematics*, 10(8), 1235.
- E.C. Gavilán, C.M. Manuel and R. van den Brink. (2023b). Directed communication in games with directed graphs. *TOP*, 1-34.
- E.C. Gavilán, C.M. Manuel and D. Martín. (2023a). Communication in Weighted Networks: A Game Theoretic Approach. *Axioms* 12(2), 180.
- E.C. Gavilán and C.M. Manuel (2024). A Mixed Value for Directed Communication Situations. *Under review.*

The report is structured as follows. Chapter 2 details the concepts necessary for the introduction and formalisation of the subsequent results. Chapter 3 describes the model of directed communication situations, the DC-value for these situations and their characterization. Furthermore, the applications (centrality, efficiency and vulnerability) of such an allocation rule are described and compared with other rules defined in the literature. Chapter 4 introduces the arcs game and defines the family of position values for directed communication situations and their characterization. In Chapter 5 a new mixed game is proposed for directed communication situations, the value of mixed directed communication and it is characterized. Finally, in Chapter 6 the conclusions of the results obtained are presented and future lines of research are proposed.

Chapter 2

Preliminaries

In this chapter, different concepts, notions and results from game theory and graph theory are presented, which will be used throughout this manuscript. Among them, n-person cooperative games expressed in the form of characteristic function, graphs and directed graphs as well as communication situations. Different allocation rules are defined, such as the Shapley value for cooperative games and the Myerson value, the position value and the mixed value for communication situations.

2.1 Cooperative games

Formally, an n-person cooperative game in the form of a characteristic function can be defined in the following way.

Definition 2.1.1. *A cooperative game with transferable utility (or a TU-game for short) is a pair* (N, v) *in which* $N = \{1, \ldots, n\}$ *is the set of players and* $v \colon 2^N \to \mathbb{R}$*, verifying* $v(\emptyset) = 0$*, is the characteristic function.*

In this model, $v(S) \in \mathbb{R}$ is the payoff that the $s = |S|$ members of the *coalition* $S \in 2^N = \{S \mid S \subseteq N\}$ can obtain by cooperating. Similarly we will use $t = |T|$, $n = |N|$, etc. We will denote by G^N the vector space of all TU-games with player set N.

The null game $(N, \mathbf{0})$ is the game with characteristic function given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. Similarly, $(N, \mathbf{1})$ will be the game with characteristic function defined as $\mathbf{1}(S) = 1$ for all $\emptyset \neq S \subseteq N$.

Definition 2.1.2. *For each $S \subseteq N \neq \emptyset$, the unanimity game (N, u_S) is a TU-game with characteristic function given by*

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise.} \end{cases}$$

The family of TU-games $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$ is a basis in G^N known as the *unanimity games basis.*

As a consequence of being a basis, given $(N, v) \in G^N$, its characteristic function v admits the following (unique) expression

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S,$$

where the coordinates $\{\Delta_v(S)\}_{\emptyset \neq S \subseteq N}$ are known as the *Harsanyi dividends* (Harsanyi, 1959).

Given (N, v) the Harsanyi dividends, $\{\Delta_v(S)\}_{\emptyset \neq S \subseteq N}$, are obtained in terms of the payoffs of the coalitions in the following way

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

The following definitions introduce concepts that will be used extensively in this thesis.

Definition 2.1.3. *Given $(N, v) \in G^N$ and a coalition $T \subseteq N$, T is said to be a support of the game (N, v), if it holds that for any coalition $S \subseteq N$*

$$v(S) = v(S \cap T).$$

Definition 2.1.4. *Given $(N, v) \in G^N$, $i \in N$ and $S \subseteq N \backslash \{i\}$ we call marginal contribution of the player i to the coalition S in the game v, the difference:*

$$v(S \cup \{i\}) - v(S).$$

Definition 2.1.5. *We will say that $i \in N$ is a null player in (N, v) if the marginal contribution of i to any coalition S that it joins is zero, i.e.,*

$$v(S \cup \{i\}) - v(S) = 0 \text{ for all } S \subseteq N \backslash \{i\}.$$

Definition 2.1.6. *We will say that i and j are symmetric players in (N, v) if the marginal contribution to every coalition they do not both belong to is the same, i.e.,*

$$v(S \cup \{i\}) = v(S \cup \{j\}) \text{ for all } S \subseteq N \backslash \{i, j\}.$$

The previous definition establishes that two symmetric players are interchangeable in the game, at least in practical terms.

2.2 Types of games

Depending on the different properties of the characteristic function of a game, different types of cooperative games are defined.

Definition 2.2.1. *A game $(N, v) \in G^N$ is symmetric if there exist a function $f : N \cup \{0\} \to \mathbb{R}$ such that, for all $S \subseteq N$, $v(S) = f(s)$.*

In a symmetric game, any two players are symmetric, meaning the payoff of a coalition depends solely on the number of players it includes, and not on their identities. In a symmetric game, all coalitions with the same number of players have the same payoff. Therefore, all players participate in the game on equal terms and are interchangeable.

Definition 2.2.2. *A game $(N, v) \in G^N$ is superadditive if for all $S, T \subseteq N$ with $S \cap T = \emptyset$:*

$$v(S \cup T) \geq v(S) + v(T).$$

A game is superadditive if, for any pair of disjoint coalitions (i.e., coalitions that do not share any player), the sum of the game's payoff for the coalition formed by the union of those two coalitions is at least equal to the sum of the individual payoffs of each coalition. In other words, the union of disjoint coalitions cannot decrease the payoff of the grand coalition. This implies that working together always yields greater or equal benefits in terms of the payouts obtained compared to working separately.

Definition 2.2.3. *A game* $(N, v) \in G^N$ *is convex if for every* $S, T \subseteq N$*, it satisfies that*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

It means that a game is convex if when two different coalitions are joined, then the sum of the outcome from the union and the intersection is at least equal to the sum of the benefits of the joining coalitions (of the coalitions being joined), where the coalitions being not necessarily disjoint. If the inequality in the previous definition occurs in the opposite direction, the game is said to be concave. All the convex games are superadditive.

Definition 2.2.4. *A game* $(N, v) \in G^N$ *is almost-positive (Vasil'ev, 1975) if it holds that:*

$$\Delta_v(S) \geq 0 \textit{ for all } \emptyset \neq S \subseteq N.$$

All the almost-positive games are convex, and therefore supperadditive. The converse (in general) is not true.

Definition 2.2.5. *A game* $(N, v) \in G^N$ *is zero-normalized if it satisfies that:*

$$v(\{i\}) = 0 \textit{ for all } i \in N.$$

We will denote with G_0^N the subspace of G^N consisting of all zero-normalized games with player set N.

Every game (N, v) allows a zero-normalized form, (N, v_0), with

$$v_0(S) = v(S) - \sum_{i \in S} v(\{i\}) \text{ for all } S \subseteq N.$$

The zero-normalization of a game is a particular case of strategic equivalence, which is defined below.

Definition 2.2.6. *Given* (N, v) *and* $(N, w) \in G^N$*, they are said to be strategically equivalent if there exist real constants* $\lambda, \alpha_1, \alpha_2, ..., \alpha_n$*, such that for all* $S \subseteq N$ *and* $S \neq \emptyset$*,*

$$v(S) = \lambda w(S) + \sum_{i \in S} \alpha_i.$$

Definition 2.2.7. *A game* $(N, v) \in G^N$ *is simple if for all* $S \subseteq N$*, it satisfies that:*

$$v(S) = 0 \text{ or } v(S) = 1.$$

It means that a coalition $S \subseteq N$ is considered winning if $v(S) = 1$, otherwise the coalition is losing. A simple game is fully determined by its set of winning coalitions.

Definition 2.2.8. *A game* $(N, v) \in G^N$ *is called inessential or additive if for all* $S, T \subseteq N$ *with* $S \cap T = \emptyset$*, it holds that*

$$v(S \cup T) = v(S) + v(T).$$

In this kind of games, the players lose the desire to cooperate and the payoffs for them do not change by the coalitions that are formed. Given that, these games satisfy

$$v(S) = \sum_{i \in S} v(\{i\}), \text{ for all } \emptyset \neq S \subseteq N.$$

2.3 Solution concepts for cooperative games

When the players decide to cooperate in a game $(N, v) \in G^N$, it is required a way to distribute the payoff $v(N)$ among them.

Let $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}$ be a payoff distribution vector, in where for each $i = 1, 2, ..., n$, x_i represents the payoff received by player i. For any coalition $S \subseteq N$, we consider $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$.

Definition 2.3.1. *The preimputations set of a game* (N, v) *is the set of efficient distribution vectors, i.e.,*

$$PI(N, v) = \{\boldsymbol{x} = (x_1, , x_2, ..., x_n) \in \mathbb{R}^n \mid x(N) = v(N)\}.$$

When a player joins a coalition, it is sensible to think that he will not accept a lower payment than he would get individually, i.e., individual rationality.

Definition 2.3.2. *The set of preimputations of a game (N, v) is the subset of $PI(N, v)$ formed by the distribution vectors that satisfy the principle of individual rationality ,*

$$I(N, v) = \{\boldsymbol{x} = (x_1, x_2, ..., x_n) \in PI(N, v) \mid x_i \geq v(\{i\}), \text{ for all } i = 1, 2, ..., n\}.$$

Definition 2.3.3. *Given $(N, v) \in G^N$, the set of vectors of stable payment allocations, called core of (N, v) and denoted by $C(N, v)$ is defined as*

$$C(N, v) = \{\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x(N) = v(N), x(S) \geq v(S), \text{ for all } S \subseteq N\}$$

The notion of the Core as a general solution concept for a cooperative game was developed by Shapley (1953a) and Gillies (1953, 1959). Despite being a highly intuitive solution concept, it exhibits properties that may not always be desirable. Among these is the fact that it does not generally provide a unique payoff vector for each game (which, on the other hand, can be enriching sometimes) or, worse, it can be empty.

Definition 2.3.4. *An allocation rule (or a point solution) on G^N is a map $\varphi : G^N \to \mathbb{R}^n$. For each $(N, v) \in G^N$, $\varphi_i(N, v)$ represents the outcome or payoff for player $i \in N$ in the game (N, v).*

One of the more prominent allocation rules for TU games was proposed by Shapley (1953b), later called the Shapley value. It assigns to every player in a TU-game a linear combination of his marginal contributions to different coalitions.

Definition 2.3.5. *The Shapley value, Sh, is the allocation rule that, for all $(N, v) \in G^N$, and for all $i \in N$, is defined as*

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n - s - 1)!s!}{n!} [v(S \cup \{i\}) - v(S)], \quad i \in N$$

This coincides with

$$Sh_i(N, v) = \sum_{i \in S \subseteq N} \frac{\Delta_v(S)}{s}, \quad i \in N$$

in which the Shapley value of each player is determined from his respective Shapley values in the unanimity games and the Harsanyi dividends.

Shapley (1953a) characterized his value using the axioms of *symmetry*, *carrier* and *additivity*:

i) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ is symmetric if for all games $(N, v) \in G^N$ and for all permutation π of the player set N

$$\varphi_{\pi(i)}(N, v) = \varphi_i(N, \pi v) \text{ for all } i \in N,$$

where the game πv is defined as $\pi v(S) = v(\pi(S))$.

ii) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ satisfies the carrier property if for all $(N, v) \in G^N$ and for all support T of (N, v), then

$$\sum_{i \in T} \varphi_i(N, v, \gamma) = v(T).$$

iii) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ satisfies additivity if, given two games (N, v_1) and $(N, v_2) \in G^N$ then

$$\varphi(N, v_1 + v_2) = \varphi(N, v_1) + \varphi(N, v_2).$$

Due to the interest and importance of the Shapley value as a solution for TU games, different authors have proposed other characterizations for it.

Shubik (1962) proposed a characterization for the Shapley value using the *efficiency* and *null-player* properties instead of the carrier axiom:

i) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ is efficient if for all $(N, v) \in G^N$ is verified that

$$\sum_{i \in N} \varphi_i(N, v) = v(N).$$

ii) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ satisfies the null-player property if for all $i \in N$ being a null-player in $(N, v) \in G^N$

$$\varphi_i(N, v) = 0.$$

Later, Myerson (1980) proposed a new characterization using efficiency and its well-known *balanced contributions*:

i) An allocation rule $\varphi : G^N \to \mathbb{R}^n$ satisfies the balanced contributions property if for all games $(N, v) \in G^N$ and for $i, j \in N$[1]

$$\varphi_i(N, v) - \varphi_i(N, v_{|_{N \setminus \{j\}}}) = \varphi_j(N, v) - \varphi_j(N, v_{|_{N \setminus \{i\}}}).$$

[1]Given (N, v) and $T \subseteq N$, the game $(N, v_{|_T})$ has characteristic function given by $v_{|_T}(S) = v(T \cap S)$, for all $S \subseteq N$.

To clarify several of the concepts previously introduced let us consider the following example in which we will use three different games. These three games will be used throughout the report to see how the players behave in the different situations proposed.

Example 2.3.1. *Consider* $(N, v) \in G^N$ *with* $N = \{1, 2, 3, 4\}$ *and* v_1*,* v_2 *and* v_3 *the following three different games (see Gómez et al., 2003).*

- *First of all, we consider the messages game, that calculates the number of messages that can be sent between pairs of players. The messages game is an almost-positive game which implies that it is also convex and superadditive. Moreover it is zero-normalized. In this game the characteristic function is defined by*

$$v_1(S) = \begin{cases} \frac{s(s-1)}{2} & \text{if } s \geq 2 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then, the characteristic function v_1*, in terms of the unanimity basis, is given by*

$$v_1 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

and the Shapley value for the players is

$$Sh(N, v_1) = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$$

- *Secondly, the overhead game, which considers that each coalition of size* s *has to face a unitary payment. This game is not almost positive, or convex, or zero-nomalized, but it is supperaditive. The characteristic function is defined as*

$$v_2(S) = -1 \text{ for all } S \neq \emptyset, \quad (2)$$

which in terms of the unanimity basis is given by

$$v_2 = -u_{\{1\}} - u_{\{2\}} - u_{\{3\}} - u_{\{4\}} + u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$-u_{\{1,2,3\}} - u_{\{1,2,4\}} - u_{\{1,3,4\}} - u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

and the Shapley value for the players is

$$Sh(N, v_2) = \left(\frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4}\right)$$

This game admits a zero-normalized form given by the characteristic funtion (we abuse the notation using also v_2)

$$v_2(S) = s - 1 \text{ for all } s \neq 0, \tag{3}$$

which in terms of the unanimity basis is

$$v_2 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$-u_{\{1,2,3\}} - u_{\{1,2,4\}} - u_{\{1,3,4\}} - u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

and the Shapley value for the players

$$Sh(N, v_2) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

When referring to the overhead game, we will always use its zero-normalized form in this report, because some solutions used and proposed in this report require the games to be zero-normalized.

- *Finally, the conferences game that calculates the number of conferences that can be made between players where each conference, within a coalition, is a subgroup of the coalition. Its characteristic funcition is given by*

$$v_3(S) = 2^s - s - 1, \tag{4}$$

which in terms of the unanimity basis is

$$v_3 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$+u_{\{1,2,3\}} + u_{\{1,2,4\}} + u_{\{1,3,4\}} + u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

and the Shapley value for the players is

$$Sh(N, v_3) = \left(\frac{11}{4}, \frac{11}{4}, \frac{11}{4}, \frac{11}{4}\right)$$

As can be seen, given the symmetry of the three games and the symmetry and efficiency of the Shapley value, all four players receive a quarter of the value of the grand coalition in each one of the games.

2.4 Graphs or Networks

In cooperative game theory, when modelling real situations, we may find scenarios in which communication between certain players is not feasible. One of the most studied communication restrictions for TU-games is given by graphs. In this section we present concepts related to graph theory that we will use in this report.

Definition 2.4.1. *A graph or a network is a pair (N, γ) with $N = \{1, 2, \ldots, n\}$ being the set of nodes and $\gamma \subseteq \gamma_N = \{\{i, j\} \mid i, j \in N, i \neq j\}$, which is the complete graph. Each pair $\{i, j\} \in \gamma$ is called an* edge *or* link.

We often denote a graph (N, γ) just by its set of edges γ. Γ^N denotes the set of all graphs with nodes set N.

Two nodes i and j are *directly connected* in (N, γ), if $\{i, j\} \in \gamma$. Two nodes i and j are *connected* in γ if there exists a sequence of nodes $i_1, i_2, \ldots i_k$ with $i_1 = i$, $i_k = j$ and $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \ldots, k-1$.

The graph (N, γ) is connected if all $i, j \in N$ are connected in γ. A set $\emptyset \neq S \subseteq N$ is connected in γ if $|S| = 1$ or every pair of nodes in S is connected in $(S, \gamma_{|S})$ with $\gamma_{|S} = \{\{i, j\} \in \gamma \mid i, j \in S\}$.

Definition 2.4.2. *A coalition C is a connected component in the graph (N, γ) if it is maximally connected, that is (i) C is connected in the graph and, (ii) for all $C' \subseteq N$, if $C \subsetneq C'$ then, C' is not connected.*

We will denote by N/γ the partition of N in connected components induced by (N, γ).

We will denote by S/γ the set of the connected components of S in $(S, \gamma_{|S})$. A graph (N, γ') with $\gamma' \subseteq \gamma$ is a *subgraph* of (N, γ). For $i \in N$, (N, γ_i) is the subgraph of (N, γ) of those links incident on i, i.e., $\gamma_i = \{l \in \gamma \mid i \in l\}$. And (N, γ_{-i}) is the subgraph of (N, γ) in which all links incident in i are severed, i.e., $\gamma_{-i} = \gamma \setminus \gamma_i$.

Definition 2.4.3. *A connected set, $T \subseteq N$, in (N, γ) is a minimal connection set of $S \subseteq T$ in γ, if there does not exist another $T' \subsetneq T$ with $S \subseteq T'$ and T' connected. $\mathcal{MCS}(S, N, \gamma)$ will be the family, occasionally empty, of all minimal connection sets of S in (N, γ).*

Definition 2.4.4. *Given $S \subseteq N$, (N, η) is a connection graph of S in (N, γ) if S is connected in (N, η).*

Definition 2.4.5. *A connection graph η of S in γ is said to be minimal if there is no other connection graph of S in γ strictly contained in it, i.e., if for every connection graph η' of S in γ, $\eta' \not\subset \eta$. We will denoted by $\mathcal{MCG}(S, N, \gamma)$ the family, occasionally empty, of all minimal connection graphs of S in (N, γ).*

Definition 2.4.6. *Given (N, v, γ) and $\emptyset \neq S \subseteq N$, we will say that $\{T, \eta\} \tilde{\subseteq} \{N, \gamma\}$[2] with $S \subseteq T$, is a connection set-graph of S in (N, γ), if (T, η) is a connected graph. And we will say that $\{T, \eta\}$, a connection set-graph of S in (N, γ), is minimal if for all $\{T', \eta'\}$ with $\{T', \eta'\} \tilde{\subset} \{T, \eta\}$, $\{T', \eta'\}$ is not a connection set-graph of S in (N, γ).*

Given $\emptyset \neq S \subseteq N$, we will denote $\mathcal{MCSG}(S, N, \gamma)$ the family, occasionally empty, of the minimal connection set-graphs of S in (N, γ).

2.5 Communication situations. Graph restricted games

A communication situation is an extension of TU-games, proposed by Myerson (1977), in which the players in the game have restrictions in the communication that are given by a graph.

Definition 2.5.1. *A communication situation is a triple (N, v, γ), where (N, v) is a TU-game and (N, γ) is a graph which set of nodes corresponds to the set of players in the game.*

$\mathcal{CS}^N$ will denote the set of all communication situations with player-node set N and CS_0^N will denote the subset of those elements in CS^N in which the game is zero-normalized.

[2]Given $\{S, \eta\}$ and $\{S', \eta'\}$ with $S, S' \subseteq N$ and $\eta, \eta' \subseteq \gamma$, we will denote by $\{S, \eta\} \tilde{\subseteq} \{S', \eta'\}$ the order given by $S \subseteq S'$ and $\eta \subseteq \eta'$. If $\{S, \eta\} \tilde{\subseteq} \{S', \eta'\}$ but $S \neq S'$ or $\eta \neq \eta'$ (or both), then we will write $\{S, \eta\} \tilde{\subset} \{S', \eta'\}$.

For communication situations, Myerson (1977) defined the *graph-restricted game*, as a new TU-game for each communication situation, in which the payoff of each coalition will be conditioned by the communication constraints imposed by the graph and calculated as the sum of the payoffs that the maximal connected subcoalitions can obtain.

Definition 2.5.2. *Given (N, v, γ), the graph-restricted game is the TU-game (N, v^γ), in which the characteristic function is given by:*

$$v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \text{ for all } S \subseteq N.$$

The above definition expresses the idea that only coalitions that are connected in the network can form and generate their payoff. This expression is actually called the *Myerson game.*

Gómez et al. (2003) tested how the unanimity games are transformed when moving to the graph restricted game.

Lemma 2.5.1. *Given $(N, u_S, \gamma) \in \mathcal{CS}^N$ with $\emptyset \neq S \subseteq N$, if $\mathcal{MCS}(S, N, \gamma) = \{S_1, S_2, ..., S_{r(S)}\} \neq \emptyset$ then*

$$u_S^\gamma = \mathbf{1} - \prod_{i=1}^{r(S)} [\mathbf{1} - u_{S_i}]$$

and $u_S^\gamma \equiv \mathbf{0}$ otherwise.

Example 2.5.1. *Consider the communication situations (N, v, γ) in which $N = \{1, 2, 3, 4\}$, v is one of the three games in the Example 2.3.1, and $\gamma = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}\}$. A representation of γ can be seen in Figure 2.1.*

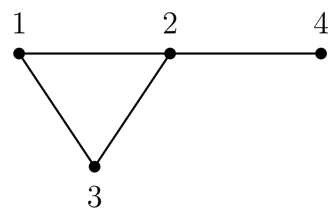

Figure 2.1

The famlily $\mathcal{MCS}(S,N,\gamma)$ in this case is

$$\mathcal{MCS}(S,N,\gamma) = \begin{cases} \{\{1,2\}\} & \text{if } S=\{1,2\} \\ \{\{1,3\}\} & \text{if } S=\{1,3\} \\ \{\{1,2,4\}\} & \text{if } S=\{1,4\} \\ \{\{2,3\}\} & \text{if } S=\{2,3\} \\ \{\{2,4\}\} & \text{if } S=\{2,4\} \\ \{\{2,3,4\}\} & \text{if } S=\{3,4\} \\ \{\{1,2,3\}\} & \text{if } S=\{1,2,3\} \\ \{\{1,2,4\}\} & \text{if } S=\{1,2,4\} \\ \{\{1,2,3,4\}\} & \text{if } S=\{1,3,4\} \\ \{\{2,3,4\}\} & \text{if } S=\{2,3,4\} \\ \{N\} & \text{if } S=\{1,2,3,4\}. \end{cases} \tag{5}$$

- *In the messages game, with communication restricted by γ, using the expression (1) and the necessary coalitions in $\mathcal{MCS}(S,N,\gamma)$, the graph restricted game in terms of the unanimity basis is given by*

$$v_1^\gamma = [\mathbf{1}-(\mathbf{1}-u_{\{1,2\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,3\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,2,4\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{2,3\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{2,4\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{2,3,4\}})]$$
$$= u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{1,2,4\}} + u_{\{2,3,4\}}.$$

- *Overhead game. Using the expression (3) and $\mathcal{MCS}(S,N,\gamma)$, the graph restricted game in terms of the unanimity basis is*

$$v_2^\gamma = [\mathbf{1}-(\mathbf{1}-u_{\{1,2\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,3\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,2,4\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{2,3\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{2,4\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{2,3,4\}})]$$
$$-[\mathbf{1}-(\mathbf{1}-u_{\{1,2,3\}})] - [\mathbf{1}-(\mathbf{1}-u_{\{1,2,4\}})] - [\mathbf{1}-(\mathbf{1}-u_{\{1,2,3,4\}})]$$
$$-[\mathbf{1}-(\mathbf{1}-u_{\{2,3,4\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,2,3,4\}})]$$
$$= u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,3\}} + u_{\{2,4\}} - u_{\{1,2,3\}}.$$

- *Conferences game. Using the expression (4) and $\mathcal{MCS}(S,N,\gamma)$, the graph restricted game in terms of the unanimity basis is*

$$v_3^\gamma = [\mathbf{1}-(\mathbf{1}-u_{\{1,2\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,3\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{1,2,4\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{2,3\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{2,4\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{2,3,4\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{1,2,3\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{1,2,4\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{1,2,3,4\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{2,3,4\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{1,2,3,4\}})]$$

$$=u_{\{1,2\}}+u_{\{1,3\}}+u_{\{2,3\}}+u_{\{2,4\}}+u_{\{1,2,3\}}+2u_{\{1,2,4\}}+2u_{\{2,3,4\}}+2u_{\{1,2,3,4\}}.$$

Following Myerson's framework on communication situations, Meessen (1988) in his Master's thesis and Borm et al. (1992) proposed an alternative to the graph restricted game of Myerson, the *link-game*. In this new game the players are the undirected edges (or links) of the graph that restricts communication, the coalitions are the different subgraphs of the graph and the worth of every coalition is determined by what the grand coalition of all players N can earn if exactly the links in that coalition are present.

Definition 2.5.3. *Given a communication situation $(N, v, \gamma) \in \mathcal{CS}_0^N$, the associated link game is the game[3] (γ, r_γ^v) with the characteristic function given by:*

$$r_\gamma^v(\eta) = v^\eta(N) = \sum_{C\in N/\eta} v(C), \text{ for all } \eta \subseteq \gamma.$$

Gómez et al. (2004) introduced the link game for each unanimity game.

Lemma 2.5.2. *Given $(N, u_S, \gamma) \in \mathcal{CS}^N$ with $\emptyset \neq S \subseteq N$, if $\mathcal{MCG}(S, N, \gamma) = \{\eta_{S_1}, \eta_{S_2}, ..., \eta_{S_{r(S)}}\} \neq \emptyset$ then*

$$r_\gamma^{u_S} = \mathbf{1} - \prod_{i=1}^{r(S)} \left(\mathbf{1} - u_{\eta_{S_i}}\right)$$

and $r_\gamma^{u_S} \equiv \mathbf{0}$ otherwise.

Example 2.5.2. *Consider the communication situations (N, v, γ) as in the Example 2.5.1. Denote $a = \{1, 2\}$, $b = \{1, 3\}$, $c = \{2, 3\}$ and $d = \{2, 4\}$ as represented in the Figure 2.2.*

[3] As can be seen, this is a slight notational deviation from the definition of game included in Definition 2.1.1 of game since the links, which are now the set of players, are not labelled 1 to n. We will admit this abuse of definition whenever we refer to the link game.

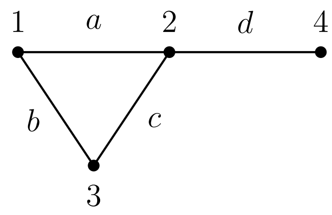

Figure 2.2

The $\mathcal{MCG}(S,N,\gamma)$ *in this case are*

$$\mathcal{MCG}(S,N,\gamma) = \begin{cases} \{\{a\},\{b,c\}\} & \textit{if } S=\{1,2\} \\ \{\{b\},\{a,c\}\} & \textit{if } S=\{1,3\} \\ \{\{a,d\},\{b,c,d\}\} & \textit{if } S=\{1,4\} \\ \{\{c\},\{a,b\}\} & \textit{if } S=\{2,3\} \\ \{\{d\}\} & \textit{if } S=\{2,4\} \\ \{\{c,d\},\{a,b,d\}\} & \textit{if } S=\{3,4\} \\ \{\{a,b\},\{a,c\},\{b,c\}\} & \textit{if } S=\{1,2,3\} \\ \{\{a,d\},\{b,c,d\}\} & \textit{if } S=\{1,2,4\} \\ \{\{b,c,d\},\{a,b,d\},\{a,c,d\}\} & \textit{if } S=\{1,3,4\} \\ \{\{c,d\},\{a,b,d\}\} & \textit{if } S=\{2,3,4\} \\ \{\{a,b,d\},\{a,c,d\},\{b,c,d\}\} & \textit{if } S=\{1,2,3,4\}. \end{cases} \quad (6)$$

- *For the messages game, in this case using (1) and* $\mathcal{MCG}(S,N,\gamma)$ *in (6), the link game in terms of the uninamity basis is*

$$r_\gamma^{v_1} = [\mathbf{1}-(\mathbf{1}-u_{\{a\}})(\mathbf{1}-u_{\{b,c\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{b\}})(\mathbf{1}-u_{\{a,c\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,d\}})(\mathbf{1}-u_{\{b,c,d\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{c\}})(\mathbf{1}-u_{\{a,b\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{d\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{c,d\}})(\mathbf{1}-u_{\{a,b,d\}})]$$

$$= u_{\{a\}} + u_{\{b\}} + u_{\{c\}} + u_{\{d\}} + u_{\{a,b\}} + u_{\{a,c\}} + u_{\{a,d\}} + u_{\{b,c\}} + u_{\{c,d\}}$$

$$+u_{\{a,b,d\}} + u_{\{b,c,d\}} - 3u_{\{a,b,c\}} - 2u_{\{a,b,c,d\}}.$$

- *For the overhead game, using (3) and* $\mathcal{MCG}(S,N,\gamma)$ *in (6), the link game in terms of the unanimity basis is*

$$r_\gamma^{v_2} = [\mathbf{1}-(\mathbf{1}-u_{\{a\}})(\mathbf{1}-u_{\{b,c\}})] + [\mathbf{1}-(\mathbf{1}-u_{\{b\}})(\mathbf{1}-u_{\{a,c\}})]$$

$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,d\}})(\mathbf{1}-u_{\{b,c,d\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{c\}})(\mathbf{1}-u_{\{a,b\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{d\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{c,d\}})(\mathbf{1}-u_{\{a,b,d\}})]$$
$$-[\mathbf{1}-(\mathbf{1}-u_{\{a,b\}})(\mathbf{1}-u_{\{a,c\}})(\mathbf{1}-u_{\{b,c\}})]-[\mathbf{1}-(\mathbf{1}-u_{\{a,d\}})(\mathbf{1}-u_{\{b,c,d\}})]$$
$$-[\mathbf{1}-(\mathbf{1}-u_{\{a,b,d\}})(\mathbf{1}-u_{\{a,c,d\}})(\mathbf{1}-u_{\{b,c,d\}})]-[\mathbf{1}-(\mathbf{1}-u_{\{c,d\}})(\mathbf{1}-u_{\{a,b,d\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,b,d\}})(\mathbf{1}-u_{\{a,c,d\}})(\mathbf{1}-u_{\{b,c,d\}})]$$
$$=u_{\{a\}}+u_{\{b\}}+u_{\{c\}}+u_{\{d\}}-u_{\{a,b,c\}}$$

- *Finally, for the conferences game, using the expression (4) and $\mathcal{MCG}(S,N,\gamma)$ in (6), the link game in terms of the unanimity basis is*

$$r_\gamma^{v_3}=[\mathbf{1}-(\mathbf{1}-u_{\{a\}})(\mathbf{1}-u_{\{b,c\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{b\}})(\mathbf{1}-u_{\{a,c\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,d\}})(\mathbf{1}-u_{\{b,c,d\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{c\}})(\mathbf{1}-u_{\{a,b\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{d\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{c,d\}})(\mathbf{1}-u_{\{a,b,d\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,b\}})(\mathbf{1}-u_{\{a,c\}})(\mathbf{1}-u_{\{b,c\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{a,d\}})(\mathbf{1}-u_{\{b,c,d\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,b,d\}})(\mathbf{1}-u_{\{a,c,d\}})(\mathbf{1}-u_{\{b,c,d\}})]+[\mathbf{1}-(\mathbf{1}-u_{\{c,d\}})(\mathbf{1}-u_{\{a,b,d\}})]$$
$$+[\mathbf{1}-(\mathbf{1}-u_{\{a,b,d\}})(\mathbf{1}-u_{\{a,c,d\}})(\mathbf{1}-u_{\{b,c,d\}})]$$
$$=u_{\{a\}}+u_{\{b\}}+u_{\{c\}}+u_{\{d\}}+2u_{\{a,b\}}+2u_{\{a,c\}}+2u_{\{a,d\}}+2u_{\{b,c\}}+2u_{\{c,d\}}$$
$$+4u_{\{a,b,d\}}+2u_{\{a,c,d\}}+4u_{\{b,c,d\}}-5u_{\{a,b,c\}}-8u_{\{a,b,c,d\}}.$$

Feltkamp & van den Nouweland (1993) proposed a new game for communication situations, pseudo-game or mixed game, in which the pseudo-players are the players in the game and the links in the network.

Definition 2.5.4. *Given a communication situation $(N,v,\gamma)\in\mathcal{CS}^N$, the mixed game $(N\cup\gamma,w_{v,\gamma})\in G^{N\cup\gamma}$ is defined as*

$$w_{v,\gamma}(\{S,\eta\})=\sum_{C\in S/\eta}v(C)\ \textit{for all}\ \{S,\eta\}\tilde{\subseteq}\{N,\gamma\}.$$

Remember that given $\{S,\eta\}$ and $\{S',\eta'\}$ with $S,S'\subseteq N$ and $\eta,\eta'\subseteq\gamma$, we will denote by $\{S,\eta\}\tilde{\subseteq}\{S',\eta'\}$ the order given by $S\subseteq S'$ and $\eta\subseteq\eta'$. If $\{S,\eta\}\tilde{\subseteq}\{S',\eta'\}$ but $S\neq S'$ or $\eta\neq\eta'$ (or both), then we will write $\{S,\eta\}\tilde{\subset}\{S',\eta'\}$.

Gavilán et al. (2023a) introduced the pseudo-games corresponding to the unanimity games.

Proposition 2.5.1. *Given* $(N, u_S, \gamma) \in \mathcal{CS}^N$ *with* $\emptyset \neq S \subseteq N$,

$$w_{u_S,\gamma} = \mathbf{1} - \prod_{\{T_i,\eta_i\} \in \mathcal{MCSG}(S,N,\gamma)} \left[\mathbf{1} - u_{\{T_i,\eta_i\}}\right]$$

if $\mathcal{MCSG}(S, N, \gamma) \neq \emptyset$, *and* $w_{u_s,\gamma} \equiv \mathbf{0}$ *(the null game), otherwise.*

Example 2.5.3. *Consider the communication situations as in the Example 2.5.2. The* $\mathcal{MCSG}(S, N, \gamma)$ *are given by*

$$\mathcal{MCSG}(S,N,\gamma) = \begin{cases} \{\{1,2,a\},\{1,2,3,b,c\}\} & \text{if } S = \{1,2\} \\ \{\{1,3,b\},\{1,2,3,a,c\}\} & \text{if } S = \{1,3\} \\ \{\{1,2,4,a,d\},\{1,2,3,4,b,c,d\}\} & \text{if } S = \{1,4\} \\ \{\{2,3,c\},\{1,2,3,a,b\}\} & \text{if } S = \{2,3\} \\ \{\{2,4,d\}\} & \text{if } S = \{2,4\} \\ \{\{2,3,4,c,d\},\{1,2,3,4,a,b,d\}\} & \text{if } S = \{3,4\} \\ \{\{1,2,3,a,b\},\{1,2,3,a,c\},\{1,2,3,b,c\}\} & \text{if } S = \{1,2,3\} \\ \{\{1,2,4,a,d\},\{1,2,3,4,b,c,d\}\} & \text{if } S = \{1,2,4\} \\ \{\{1,2,3,4,a,b,d\},\{1,2,3,4,a,c,d\}, & \text{if } S = \{1,3,4\} \\ \quad \{1,2,3,4,b,c,d\}\} & \\ \{\{2,3,4,c,d\},\{1,2,3,4,a,b,d\}\} & \text{if } S = \{2,3,4\} \\ \{\{1,2,3,4,a,b,d\},\{1,2,3,4,a,c,d\}, & \text{if } S = N \\ \quad \{1,2,3,4,b,c,d\}\} & \end{cases} \tag{7}$$

Then, taking into account the expression (7)

- *For the messages game, using the expression of the original game in (1), the mixed game in terms of the unanimity basis is*

$$\begin{aligned} w_{v_1,\gamma} &= [\mathbf{1} - (\mathbf{1} - u_{\{1,2,a\}})(\mathbf{1} - u_{\{1,2,3,b,c\}})] + [\mathbf{1} - (\mathbf{1} - u_{\{1,3,b\}})(\mathbf{1} - u_{\{1,2,3,a,c\}})] \\ &+ [\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,d\}})(\mathbf{1} - u_{\{1,2,3,4,b,c,d\}})] + [\mathbf{1} - (\mathbf{1} - u_{\{2,3,c\}})(\mathbf{1} - u_{\{1,2,3,a,b\}})] \\ &+ [\mathbf{1} - (\mathbf{1} - u_{\{2,4,d\}})] + [\mathbf{1} - (\mathbf{1} - u_{\{2,3,4,c,d\}})(\mathbf{1} - u_{\{1,2,3,4,a,b,d\}})] \\ &= u_{\{1,2,a\}} + u_{\{1,3,b\}} + u_{\{2,3,c\}} + u_{\{2,4,d\}} + u_{\{1,2,3,a,b\}} + u_{\{1,2,3,a,c\}} + u_{\{1,2,3,b,c\}} + u_{\{1,2,4,a,d\}} \\ &+ u_{\{2,3,4,c,d\}} + u_{\{1,2,3,4,a,b,d\}} + u_{\{1,2,3,4,b,c,d\}} - 3u_{\{1,2,3,a,b,c\}} - 2u_{\{1,2,3,4,a,b,c,d\}} \end{aligned}$$

  As can be seen, the level of complexity in the calculation of the mixed game is high and we limit ourselves in this case to the message game.

2.6 Allocation rules for communication situations

In this section we include the more relevant values for the communication situations existing in the literature: the Myerson value, the position value and the mixed value.

Definition 2.6.1. *An allocation rule for communication situations is a map* $\varphi : \mathcal{CS}^N \to \mathbb{R}^n$ *where* $\varphi_i(N, v, \gamma)$ *represents the payoff obtained by player* i *in the communication situation* (N, v, γ)*.*

As mentioned above, Shapley value is highly relevant as a point solution for cooperative games. So much so that the Myerson value, the position value and the mixed value are calculated from it.

Myerson (1977) proposed to apply the Shapley value to the graph-restricted game, yielding the allocation rule μ. Nowadays, this rule is known as the *Myerson value.*

Definition 2.6.2. *The Myerson value,* μ*, is the allocation rule for communication situations defined as:*

$$\mu(N, v, \gamma) = Sh(N, v^{\gamma}) \text{ for all } (N, v, \gamma) \in \mathcal{CS}^N.$$

Three properties strongly associated to the Myerson value are given in the following definitions.

Definition 2.6.3. *An allocation rule* φ *on* $\mathcal{CS}^N$ *satisfies component efficiency (Myerson, 1977) if, for all* $(N, v, \gamma) \in \mathcal{CS}^N$ *and all* $C \in N/\gamma$

$$\sum_{i \in C} \varphi_i(N, v, \gamma) = v(C).$$

It means that in every component, the sum of the payoffs assigned to the players in that component equals its worth in the original game.

Definition 2.6.4. *An allocation rule* φ *on* $\mathcal{CS}^N$ *satisfies fairness (Myerson, 1977) if, for all* $(N, v, \gamma) \in \mathcal{CS}^N$ *and every* $l = \{i, j\} \in \gamma$

$$\varphi_i(N, v, \gamma) - \varphi_i(N, v, \gamma \setminus \{l\}) = \varphi_j(N, v, \gamma) - \varphi_j(N, v, \gamma \setminus \{l\}).$$

When two players break a link among them the effect on the payoffs of these two players is the same.

Definition 2.6.5. *An allocation rule φ on $\mathcal{CS}^N$ satisfies balanced contributions (Myerson, 1980) if, for all $(N, v, \gamma) \in \mathcal{CS}^N$ and all $i, j \in N$,*

$$\varphi_i(N, v, \gamma) - \varphi_i(N, v, \gamma_{-j}) = \varphi_j(N, v, \gamma) - \varphi_j(N, v, \gamma_{-i}).$$

The effect of isolating a player, in the sense that all its links are broken, on the payoff of another player is the same as the effect the other way around.

Myerson (1977) showed that μ is the unique allocation rule on communication situations that satisfies component efficiency and fairness.

Later, Myerson (1980) showed that the value μ is the unique allocation rule for communication situations satisfying component efficiency and balanced contributions.

Example 2.6.1. *Consider the communication situations (N, v, γ) as in Example 2.5.1. Using the definitions of the three different games, v_1^γ, v_2^γ and v_3^γ, the Myerson values payoffs for the players are*

$$\mu(N, v_1, \gamma) = Sh(N, v_1^\gamma) = \left(\frac{8}{6}, \frac{13}{6}, \frac{8}{6}, \frac{7}{6}\right)$$

$$\mu(N, v_2, \gamma) = Sh(N, v_2^\gamma) = \left(\frac{4}{6}, \frac{7}{6}, \frac{4}{6}, \frac{3}{6}\right)$$

$$\mu(N, v_3, \gamma) = Sh(N, v_3^\gamma) = \left(\frac{15}{6}, \frac{22}{6}, \frac{15}{6}, \frac{14}{6}\right)$$

Meessen (1988) and Borm et al. (1992) proposed another allocation rule, π, that assigns to a player i in a communication situation half of the sum of the Shapley values of the links (in the link game) incident on it. This value is called *the position value.*

Definition 2.6.6. *The position value, π, is the allocation rule for communication situations defined as:*

$$\pi_i(N, v, \gamma) = \frac{1}{2} \sum_{l \in \gamma_i} Sh_l(\gamma, r_\gamma^v), \text{ for all } i \in N,$$

where, for $i \in N$ and $(N, \gamma) \in \Gamma^N$, $\gamma_i = \{l \in \gamma \mid i \in l\}$ is the set of links incident on i.

Slikker (2005) characterized the position value in terms of two properties, *component efficiency* and *balanced link contributions.*

Definition 2.6.7. *An allocation rule φ on $\mathcal{CS}^N$ satisfies balanced link contributions (Slikker, 2005) if, for all $(N, v, \gamma) \in \mathcal{CS}^N$ and all $i, j \in N$,*

$$\sum_{l \in \gamma_i} [\varphi_j(N, v, \gamma) - \varphi_j(N, v, \gamma \setminus \{l\})] = \sum_{l \in \gamma_j} [\varphi_i(N, v, \gamma) - \varphi_i(N, v, \gamma \setminus \{l\})],$$

where $(N, \gamma \setminus \{l\})$ is the subgraph of (N, γ) obtained when the relation l is broken.

This property says that the sum of the effects of breaking each individual link of a player on the payoff of another player is the same as the effect the other way around. The following example shows the position value for the communication situation discussed above.

Example 2.6.2. *Consider the communication situations (N, v, γ) as in the Example 2.5.2. Using the expressions of the three different games, $r_\gamma^{v_1}$, $r_\gamma^{v_2}$ and $r_\gamma^{v_3}$, the position values are given by*

$$Sh(N, r^\gamma_{v_1}) = \left(\frac{8}{6}, \frac{7}{6}, \frac{8}{6}, \frac{13}{6}\right) \text{ and } \pi(N, v_1, \gamma) = \left(\frac{15}{12}, \frac{29}{12}, \frac{15}{12}, \frac{13}{12}\right)$$

$$Sh(N, r^\gamma_{v_2}) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1\right) \text{ and } \pi(N, v_2, \gamma) = \left(\frac{2}{3}, \frac{7}{6}, \frac{2}{3}, \frac{1}{2}\right)$$

$$Sh(N, r^\gamma_{v_3}) = \left(\frac{7}{3}, \frac{6}{3}, \frac{7}{3}, \frac{13}{3}\right) \text{ and } \pi(N, v_3, \gamma) = \left(\frac{13}{6}, \frac{27}{6}, \frac{13}{6}, \frac{13}{6}\right).$$

Feltkamp & van den Nouweland (1993) proposed the mixed value, ρ, that assigns in a communication situation, for every player $i \in N$ and $l \in \gamma$ the Shapley value in the mixed game.

Definition 2.6.8. *The mixed value, ρ, is the allocation rule for communication situations defined as*

$$\rho(N, v, \gamma) = Sh(N \cup \gamma, w_{v,\gamma}).$$

Feltkamp & van den Nouweland (1993) showed that the mixed value is the unique allocation rule for communication situations satisfiying efficiency, additivity in the game, the superfluous link property and the anonymity property.

Definition 2.6.9. *A mixed allocation rule, φ, defined on $\mathcal{CS}^N$ satisfies mixed component efficiency if, for each $(N, v, \gamma) \in \mathcal{CS}^N$ and each $C \in N/\gamma$*

$$\sum_{i \in C} \varphi_i(N, v, \gamma) + \sum_{l \in \gamma_{|C}} \varphi_l(N, v, \gamma) = v(N).$$

An allocation rule φ, for nodes and links, satisfies additivity if the allocation in a communication situation in which the game is the sum of TU-games (*ceteris paribus*) coincides with the sum of the allocations in the respective communication situations. Formally,

Definition 2.6.10. *A rule φ on CS^N is additive if for $(N, v + v', \gamma)$, (N, v, γ), $(N, v', \gamma) \in CS^N$, $\varphi(N, v + v', \gamma) = \varphi(N, v, \gamma) + \varphi(N, v', \gamma)$.*

Definition 2.6.11. *An allocation rule φ, for nodes and links, satisfies the superfluous link property if the allocation does not change when eliminating links that are null players in the pseudogame.*

Definition 2.6.12. *A rule φ on CS^N, for players and links, satisfies anonymity if the allocation only depends on the number of non-isolated players and links.*

Later, Gavilán et al. (2023a) obtain three different characterizations for the mixed value using mixed component efficiency, mixed fairness, balanced contributions, mixed balanced contributions and balanced link contributions.

The mixed fairness says that when two players break a link among them the effect on the payoffs of these two players is the same and coincides with the payoff of this link.

Definition 2.6.13. *A mixed allocation rule on $\mathcal{CS}^N$, φ, satisfies mixed fairness, if for every $(N, v, \gamma) \in \mathcal{CS}^N$ and every directly connected pair of players $i, j \in N$, with $l = \{i, j\}$,*

$$\varphi_i(N, v, \gamma) - \varphi_i(N, v, \gamma \setminus \{l\}) = \varphi_j(N, v, \gamma) - \varphi_j(N, v, \gamma \setminus \{l\}) = \varphi_l(N, v, \gamma).$$

The mixed balanced contributions property says that the effect of isolating a player, in the sense that all its links are broken, on the payoff of a link is the same as the effect in the player of severing that link.

Definition 2.6.14. *A mixed allocation rule, φ, on $\mathcal{CS}^N$ satisfies the mixed balanced contributions property, if for $i \in N$ and $l \in \gamma$ it holds that*

$$\varphi_l(N, v, \gamma) - \varphi_l(N, v, \gamma_{-i}) = \varphi_i(N, v, \gamma) - \varphi_i(N, v, \gamma \setminus \{l\})$$

Gavilán et al. (2023a) shows that

i) The mixed value is the unique mixed allocation rule on $\mathcal{CS}^N$ satisfying mixed component efficiency and mixed fairness.

ii) The mixed value is the unique mixed allocation rule on $\mathcal{CS}^N$ satisfying mixed component efficiency, balanced contributions and mixed balanced contributions.

iii) The mixed value is the unique mixed allocation rule on $\mathcal{CS}^N$ satisfying mixed component efficiency, balanced link contributions and mixed balanced contributions.

Example 2.6.3. *Consider the communication situation (N, v, γ) as in Example 2.5.3. Using the expression of the mixed game $w_{v_1,\gamma}$, the mixed values for the set of players and links are given by*

$$\rho(N, v_1, \gamma) = Sh(N \cup \gamma, w_{v_1,\gamma}) = \Big(\frac{421}{420}, \frac{645}{420}, \frac{421}{420}, \frac{323}{420}, \frac{137}{420}, \frac{113}{420}, \frac{137}{420}, \frac{323}{420}\Big)$$

2.7 Directed graphs or digraphs

This section introduces different definitions and notations related to directed graphs or digraphs, which will be useful for the understanding of the results presented in this manuscript.

Definition 2.7.1. *A directed graph or a digraph is a pair (N, D) where $N = \{1, 2, \ldots, n\}$ is a (finite) set of nodes and $D \subseteq N \times N$ is a binary relation on N. Each directed edge or arc $(i, j) \in D$ is an ordered pair of nodes called endpoints of the arc where i is the tail and j is the head. We say that an arc is from its tail to its head.*

Also, if $(i, j) \in D$, then i is called a predecessor of j, and j a successor of i. A *loop* in a digraph is an arc in which its endpoints are equal. Multiple arcs are those with identical tails and identical heads. We will assume the digraph to be

i) Irreflexive, i.e., with no loops

ii) Simple, i.e., with no multiple arcs

When there is no ambiguity with respect to N, we will simply identify the digraph with its set of arcs D. $\mathcal{D}^N$ denotes the set of all irreflexive, simple digraphs with nodes set N.

Definition 2.7.2. *Given $(N, D) \in \mathcal{D}^N$ and $i \in N$, the* outdegree, *$d_i^O(N, D) = \Big|\{j \in N \mid (i, j) \in D\}\Big|$ (respectively the* indegree, *$d_i^I(N, D) = \Big|\{j \in N \mid (j, i) \in D\}\Big|$) is the number of arcs with i as the tail (respectively, i as the head). Then, $d_i(N, D) = d_i^O(N, D) + d_i^I(N, D)$ is the* degree *of node i in (N, D).*

Definition 2.7.3. *The total out-degree, respectively, the total in-degree, in the digraph (N, D), will be denoted by $d^O(D) = \sum\limits_{i \in N} d_i^O(D)$, respectively, $d^I(D) = \sum\limits_{i \in N} d_i^I(D)$. It is easy to see that $d^O(D) = d^I(D) = |D|$, as every edge has one tail and one head.*

Definition 2.7.4. *The relative out-degree of node i in (N, D), denoted $rd_i^O(D)$, is defined as*

$$rd_i^O(D) = \frac{d_i^O(D)}{\sum\limits_{j \in N} d_j^O(D)} = \frac{d_i^O(D)}{d^O(D)}.$$

Similarly,

$$rd_i^I(D) = \frac{d_i^I(D)}{\sum\limits_{j \in N} d_j^I(D)} = \frac{d_i^I(D)}{d^I(D)}$$

denotes the relative in-degree of node $i \in N$ in the digraph (N, D).

Definition 2.7.5. *For $(N, D) \in \mathcal{D}^N$, a graph (N, D') with $D' \subseteq D$, is called a subdigraph of (N, D). For each $i \in N$, the subdigraph (N, D_{-i}) of (N, D) is obtained when severing all arcs incident with i, i.e. $D_{-i} = \{(j, k) \in D \mid i \notin \{j, k\}\}$.*

Moreover, the restriction of (N,D) to $\emptyset \neq S \subseteq N$ is the directed graph $(S, D_{|S})$ in which $D_{|S} = \{(i,j) \in D \mid i,j \in S\}$. For $L \subseteq D$, we will use the abuse of notation $\{L\}$ to indicate the nodes corresponding to the arcs in L, i.e., $\{L\} = \{i \in N \mid \text{there is } j \in N \text{ with } (i,j) \in L \text{ or } (j,i) \in L\}$. Notice that the restriction $(\{L\}, D_{|\{L\}})$ coincides with $(\{L\}, L)$.

Definition 2.7.6. *Given a digraph (N,D), a (directed) path from i to j is a sequence of distinct nodes $P = (i_1, ..., i_t)$ with $i_1 = i$, $i_t = j$ and $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, t-1$. We assume that (i), $i \in N$, is a path.*

For convenience we will sometimes identify P with its set of arcs $\{(i_1, i_2), \ldots, (i_{t-1}, i_t)\}$. We also will use the abuse of notation $\{P\}$ to denote the set of nodes of a path P, and thus $\{P\} = \{i_1, \ldots, i_t\}$ for $P = (i_1, \ldots, i_t)$.

Definition 2.7.7. *Given two paths $P = (i_1, ..., i_t)$ and $Q = (j_1, ..., j_r)$ in (N,D), we say that P is a* subpath *of Q, denoted by $P \tilde{\subseteq} Q$, if for each $k = 1, ..., t-1$ there exists $l = 1, ..., r-1$ such that $j_l = i_k$ and $j_{l+1} = i_{k+1}$.*

A path P in (N,D) is *maximal* if it is maximal for the defined partial order $\tilde{\subseteq}$, i.e., if there is no other path $P' \neq P$ such that $P \tilde{\subseteq} P'$.

Definition 2.7.8. *Given a digraph (N,D), we will denote by $\mathcal{P}(N,D)$ the set of all maximal paths of (N,D).*

Definition 2.7.9. *Given a digraph (N,D) and $\emptyset \neq S \subseteq N$, we say that a path P in (N,D) is a* connection path *of S in (N,D) if $S \subseteq \{P\}$. We say that a path P is a* minimal connection path *of S in (N,D) if P is a connection path of S and there does not exist another connection path $P' \neq P$ of S such that $P' \tilde{\subseteq} P$.*

By $\mathcal{MCP}(S,N,D)$ we will denote the family (occasionally empty) of all minimal connection paths of S in (N,D).

Definition 2.7.10. *A Hamiltonian path (or traceable path) is a path in which $t = n$, where n is the cardinality of N, i.e., it is a path that visits each node exactly once. Thus, if P is a Hamiltonian path, then $\{P\} = N$. A Hamiltonian path in a digraph is necessarily maximal.*

Definition 2.7.11. *Given a digraph $(N,D) \in \mathcal{D}^N$, its underlying (undirected) graph $(N, \gamma_D) \in \Gamma^N$ is obtained by replacing all directed arcs with corresponding undirected links, i.e. $\gamma_D = \{\{i,j\} \mid (i,j) \in D\}$.*

A digraph (N, D) is *weakly connected* if its underlying graph (N, γ_D) is connected. A weak component of (N, D) is a component of (N, γ_D).

Definition 2.7.12. *A digraph $(N, D) \in \mathcal{D}^N$, is strongly connected or strong if for every $i, j \in N$ there is a directed path P starting at i and ending at j.*

A set of nodes C is a *strongly connected component* of a given digraph (N, D) if $(C, D_{|_C})$ is strongly connected, and for $C' \supset C$, $(C', D_{|_{C'}})$ is not strong. Notice that every strongly connected digraph is weakly connected.

We illustrate these notions with examples.

Example 2.7.1. *Consider three different digraphs, $(N, D^1), (N, D^2)$ and (N, D^3) in which $N = \{1, 2, 3, 4\}$. See Fig. 2.3.*

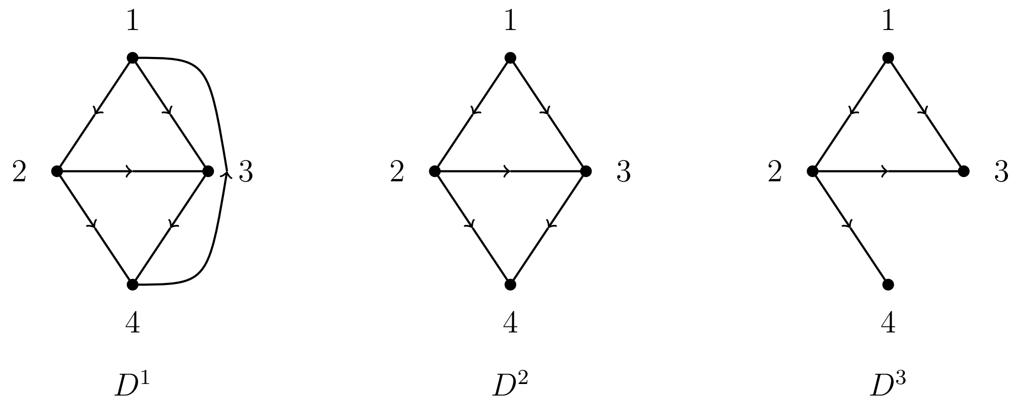

Figure 2.3: Digraphs (N, D^1),(N, D^2) and (N, D^3) Example 2.7.1

- *The digraph (N, D^1), with $D^1 = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (4, 1)\}$, is strongly connected, and $((1, 2), (2, 3), (3, 4))$ is a Hamiltonian path.*
- *The digraph (N, D^2), with $D^2 = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$, is weakly connected, but not strongly connected, and $((1, 2), (2, 3), (3, 4))$ is a Hamiltonian path. This is also a connection path of $S = \{1, 2, 4\}$, but it is not a minimal connection path of S. The path $((1, 2), (2, 4))$ is a minimal connection path of $S = \{1, 2, 4\}$.*
- *The digraph (N, D^3), with $D^3 = \{(1, 2), (1, 3), (2, 3), (2, 4)\}$, is weakly connected, but not strongly connected. It does not have a Hamiltonian path.*

The path $((1,2),(2,4))$ *is a maximal path in* (N, D)*. It is also a minimal connection path of* $S = \{1, 2, 4\}$*.*

Example 2.7.2. *Consider the digraph* $(N, D) \in \mathcal{D}^N$ *with* $N = \{1, 2, 3, 4\}$*, and* $D = \{a = (1, 2), b = (2, 4), c = (1, 3), d = (3, 4)\}$*; see Figure 2.4.*

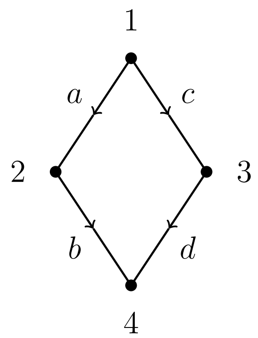

Figure 2.4: The digraph (N, D) of Example 2.7.2.

In this case, the set of maximal paths in (N, D) *is*

$$\mathcal{P}(N, D) = \{(1, 2, 4), (1, 3, 4)\}$$

and in $(S, D_{|S})$ *for* $S = \{1, 2, 3\}$ *is*

$$\mathcal{P}(S, D_{|S}) = \{(1, 2), (1, 3)\}.$$

There is no Hamiltonian path in (N, D)*.*

Moreover,

$$\mathcal{MCP}(\{1, 4\}, N, D) = \{(1, 2, 4), (1, 3, 4)\}$$
$$\mathcal{MCP}(\{2, 3\}, N, D) = \emptyset$$
$$\mathcal{MCP}(\{1, 2\}, N, D) = \{(1, 2)\}.$$

Chapter 3

Directed communication situations. The DC-value

As mentioned above, in this chapter we introduce a new digraph restricted game and a new allocation rule for directed communication situations, in which the players in a TU-game have their possibilities restricted to a directed network that is represented as a directed graph. Although this model is not new, the interpretation given in this report, and therefore the desirable properties of the assignment rules, are.

For this interpretation, a new concept of connectivity is defined which expresses that only coalitions that form a directed path in a directed communication situation can cooperate. We say that a path in a directed graph is a 'connecting path' of a coalition if all players in this coalition belong to the path.

At it is said, a new restricted game is defined, the *digraph restricted game* in which, following Myerson (1977), the payoff of a coalition will be equal to the 'sum' of the payoffs of its maximally connected subcoalitions (paths).

For directed communication situations, a new allocation rule, the Directed Communication value, DC-value for short, is proposed. It is obtained applying the Shapley value to the players in the digraph restricted game. Also we obtain two characterizations through the properties of connection efficiency, which is new, fairness and balanced contributions that are defined adapting the classical Myerson properties to this setting.

The chapter ends with a section of applications of the proposed value. In particular, we show that the value can be used as a centrality measure for directed networks. It can be decomposed as a sum of two other measures, one of them measuring connection centrality and the other, betweenness centrality. And thus, this decomposition gives a vector character to the centrality. Additionally, the first of these two measures can be decomposed in the in-connection centrality and the out-connection centrality, and respectively, the second one in the in-betweennes centrality and the out-betweennes centrality.

Also, efficiency and vulnerability measures are defined based on the proposed value, and a relationship is established between them and centrality in directed networks.

Finally, the proposed value is compared with other measures for directed graphs existing in the literature.

In the following definition we formalized those situations where cooperation among players in a TU-game is limited because of restricted directed communication possibilities.

Definition 3.0.1. *A directed communication situation is a triple* (N, v, D) *in which* (N, v) *is a TU-game and* (N, D) *is a directed graph, the nodes in the digraph being the players in the game.*

The set of all directed communication situations with player set N will be denoted by $\mathcal{DCS}^N$.

The model of a directed communication situation is mathematically identical to other models of a game with an order (or digraph) on the player set (see Introduction), but because of the interpretation of the directed graph as a directed communication network, we refer to it as a directed communication situation. In the following definition, we introduce a modification of the original game to take into account the restrictions in the communication given by the digraph. In this modified game, we assume that the worth of each coalition is the "sum" of the payoffs of its (path) maximally connected subcoalitions.[1]

[1]This definition mimics (in some sense) the one introduced by Myerson (1977) for the graph-restricted game, where the connectedness of subcoalitions in graphs is replaced by the path-connectedness in digraphs.

Definition 3.0.2. *A subcoalition $T \subseteq S$ is a (path) maximally connected subcoalition of S in directed graph (N, D) if there is a maximal path P in $(S, D_{|S})$ with $T = \{P\}$.*

For $\emptyset \neq S \subseteq N$, we denote the family of the maximal paths in $(S, D_{|S})$ by $\mathcal{P}(S, D_{|S})$.

To clarify the previous definition, let us consider the following example.

Example 3.0.1. *Consider the directed communication situation (N, v, D) with N=$\{1, 2, 3, 4\}$, and $D = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$. See Fig 3.1.*

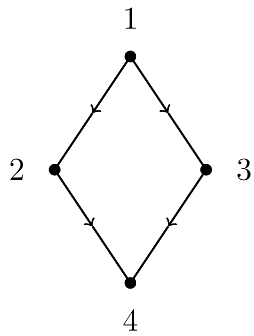

Figure 3.1

In this case,

$$\mathcal{P}(S, D_{|S}) = \begin{cases} \{(i)\}, & \text{if } S = \{i\}, i = 1, 2, 3, 4, \\ \{(1, 2)\}, & \text{if } S = \{1, 2\}, \\ \{(1, 3)\}, & \text{if } S = \{1, 3\}, \\ \{(1), (4)\} & \text{if } S = \{1, 4\}, \\ \{(2), (3)\} & \text{if } S = \{2, 3\}, \\ \{(2, 4)\}, & \text{if } S = \{2, 4\}, \\ \{(3, 4)\}, & \text{if } S = \{3, 4\}, \\ \{(1, 2), (1, 3)\}, & \text{if } S = \{1, 2, 3\}, \\ \{(1, 2, 4)\}, & \text{if } S = \{1, 2, 4\}, \\ \{(1, 3, 4)\}, & \text{if } S = \{1, 3, 4\}, \\ \{(2, 4), (3, 4)\}, & \text{if } S = \{2, 3, 4\}, \\ \{(1, 2, 4), (1, 3, 4)\}, & \text{if } S = N. \end{cases}$$

The $\mathcal{MCP}$ for the digraph in Fig 3.1 is given

$$\mathcal{MCP}(S,N,D) = \begin{cases} \{(i)\} & \textit{if } S=\{i\} \\ \{(1,2)\} & \textit{if } S=\{1,2\} \\ \{(1,3)\} & \textit{if } S=\{1,3\} \\ \{(1,2,4),(1,3,4)\} & \textit{if } S=\{1,4\} \\ \{(2,4)\} & \textit{if } S=\{2,4\} \\ \{(3,4)\} & \textit{if } S=\{3,4\} \\ \{(1,2,4)\} & \textit{if } S=\{1,2,4\} \\ \{(1,3,4)\} & \textit{if } S=\{1,3,4\} \\ \emptyset & \textit{otherwise.} \end{cases} \tag{8}$$

As can be seen, for the different coalitions S that can be formed with the set of nodes N in this directed communication situation, the $\mathcal{MCP}$ may contain one path as in the case $S = \{1,2\}$, several as in the case of coalition $S = \{1,4\}$ or none as in the case of $S = \{2,3\}$ because there is no directed path among the coalition members.

Connectedness being related to directed (connection) paths occurs in, for example, supply chain management, attribution models and vaccination policy. In supply chains, value can be created when a manufacturer (source) is connected to a retailer (sink) by a sequence of intermediaries (wholesalers, shipping companies etc.). In the other direction, to dampen the bullwhip effect[2] efficiency gains can be reached when the agents on a supply chain share information. In marketing attribution, advertisers use ‘attractiveness’ of various advertisement strategies, including online advertising, to decide in which form of advertisement they will invest. Typically, the attractiveness of one form of advertisement depends on its position in the advertisement channel and the rate to which a customer visiting a path containing multiple ads from the same advertiser (channel) eventually results in a conversion. Attribution models assess the ‘value’ of each ad on the path leading to conversion. Regarding the third example mentioned above, vaccination was one of the main strategies to ‘beat’ the Covid19 pandemic. To increase the vaccination grade, it is essential to get information about the virus and vaccination policy to members of the society. Problematic was that some parts of

[2]The bullwhip effect occurs when information about demand becomes less precise when moving up the supply chain from retailer to manufacturer.

society was difficult to reach, for example because of language problems or distrust in government organizations. To get information from the government to those parts of society a chain of intermediary social clubs, doctors etc, helped in passing the information from government to the people.

3.1 The digraph restricted game

Notice that the maximal paths might have players/nodes in common, who will contribute only once in the coalition, and therefore, we must use the classical inclusion-exclusion principle to respect that coalitions cannot obtain more under the restrictions than in the original game. This brings us to the following definition of the modified game, taking account of the cooperation restrictions.

Definition 3.1.1. *Given $(N, v, D) \in \mathcal{DCS}^N$, the digraph restricted game is defined as the TU-game (N, v^D) with characteristic function given by:*

$$v^D(S) = \sum_{i=1}^{r(S)} v(\{P_i^S\}) - \sum_{i=1}^{r(S)-1} \sum_{j=i+1}^{r(S)} v(\{P_i^S\} \cap \{P_j^S\})$$

$$+ \sum_{i=1}^{r(S)-2} \sum_{j=i+1}^{r(S)-1} \sum_{k=j+1}^{r(S)} v(\{P_i^S\} \cap \{P_j^S\} \cap \{P_k^S\}) + \cdots +$$

$$+(-1)^{r(S)-1} v(\{P_1^S\} \cap \ldots \cap \{P_{r(S)}^S\}),$$

where for $\emptyset \neq S \subseteq N$, $\mathcal{P}(S, D_{|S}) = \{P_1^S, \cdots, P_{r(S)}^S\}$ is the family of the maximal paths in $(S, D_{|S})$, and $v^D(\emptyset) = 0$.

Example 3.1.1. *Consider the directed communication situation (N, v, D) of Example 3.0.1. Using the (path) maximally connected sets and the $\mathcal{MCP}(S, N, D)$ given in Example 3.0.1, the characteristic function is given by*

$$v^D(S) = \begin{cases} v(\{i\}), & \text{if } S = \{i\}, i = 1, 2, 3, 4, \\ v(\{1\}) + v(\{4\}) & \text{if } S = \{1, 4\}, \\ v(\{2\}) + v(\{3\}) & \text{if } S = \{2, 3\}, \\ v(\{1, 2\}) + v(\{1, 3\}) - v(\{1\}), & \text{if } S = \{1, 2, 3\}, \\ v(\{2, 4\}) + v(\{3, 4\}) - v(\{4\}), & \text{if } S = \{2, 3, 4\}, \\ v(\{1, 2, 4\}) + v(\{1, 3, 4\}) - v(\{1, 4\}), & \text{if } S = N \\ v(S) & \text{otherwise.} \end{cases}$$

Notice that in determining the worth of the grand coalition N, we use the worth of coalition $\{1, 4\}$ although $\{1, 4\}$ is not connected. Since worth is generated by directed paths, players 1 and 4 cooperate in the coalition/path $\{1, 2, 4\}$, as well as in $\{1, 3, 4\}$. To avoid that the worth generated by cooperation of 1 and 4 would be double counted, we subtract the worth of $\{1, 4\}$.

Remark 3.1.1. *Given $(N, v, D) \in \mathcal{DCS}^N$, the characteristic functions v^D and v coincide if for all $S \subseteq N$, the digraph restricted to S, $(S, D_{|S})$, has a Hamiltonian path. As an example, the digraphs (N, D) with $N = \{1, 2, 3\}$ and $D = \{(1, 2), (1, 3), (2, 3)\}$ and (N, D') with $N = \{1, 2, 3, 4\}$ and $D' = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (4, 1)\}$ satisfy such a condition (see Fig. 3.2). For the digraph of Example 3.0.1, the digraphs restricted to the coalitions $S \in \{\{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ do not have a Hamiltonian path.*

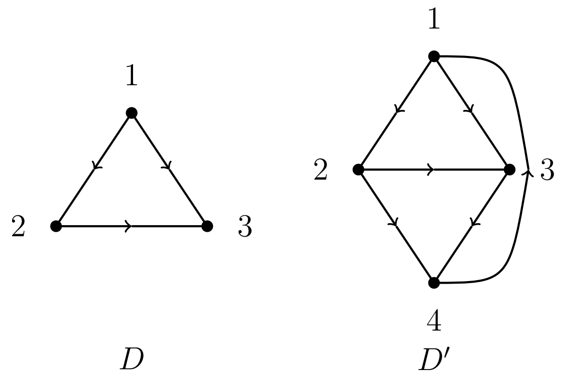

Figure 3.2

Remark 3.1.2. *Given (N, v, D), the characteristic functions v^D and v^{γ_D} coincide if for all $S \subseteq N$, there exists a Hamiltonian path in each element of $S/(\gamma_D)_{|S}$, i.e., in each component of S. As an example, the digraph (N, D) with $N = \{1, 2, 3\}$ and $D = \{(1, 2), (2, 3)\}$ satisfies such a condition. Specifically, every line-digraph, i.e. rooted tree with a single sink, satisfies this condition.*

In the following proposition, we obtain the expression of the digraph restricted game for each unanimity game.

Proposition 3.1.1. *Given* $(N, u_S, D) \in \mathcal{DCS}^N$ *with* $\emptyset \neq S \subseteq N$*, if* $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$*, then*

$$u_S^D = 1 - \prod_{i=1}^{t(S)} (1 - u_{\{Q_i^S\}}),$$

and $u_S^D \equiv \mathbf{0}$ *otherwise.*

Proof: Suposse that $\emptyset \neq S \subseteq N$ and $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$. Let $T \subseteq N$. For $T = \emptyset$, trivially $u_S^D(\emptyset) = 0 = \left[1 - \prod_{i=1}^{t(S)} (1 - u_{\{Q_i^S\}}(\emptyset))\right]$, since $u_{\{Q\}}(\emptyset) = 0$ for all $Q \in \mathcal{MCP}(S, N, D)$. For $T \neq \emptyset$, let $\mathcal{P}(T, D_{|_T}) = \{P_1^T, \cdots, P_{r(T)}^T\}$ be the family of the maximal paths in $(T, D_{|_T})$. Then, by the definition of the digraph restricted game,

$$u_S^D(T) = \sum_{i=1}^{r(T)} u_S(\{P_i^T\}) - \sum_{i=1}^{r(T)-1} \sum_{j=i+1}^{r(T)} u_S(\{P_i^T\} \cap \{P_j^T\})$$

$$+ \sum_{i=1}^{r(T)-2} \sum_{j=i+1}^{r(T)-1} \sum_{k=j+1}^{r(T)} u_S(\{P_i^T\} \cap \{P_j^T\} \cap \{P_k^T\}) + (-1)^{r(T)-1} u_S(\{P_1^T\} \cap \cdots \cap \{P_{r(T)}^T\}).$$

We consider the following two cases with respect to $r'(T)$, the cardinality of the subset of $\mathcal{P}(T, D_{|_T})$ with nodes set containing S, i.e., we assume there are $r'(T)$ maximal paths in $(T, D_{|_T})$ which nodes set contains S.

Case (i) Suppose $1 \leq r'(T)$. Then,

$$u_S^D(T) = \binom{r'(T)}{1} - \binom{r'(T)}{2} + \cdots + (-1)^{r'(T)-1} \binom{r'(T)}{r'(T)}$$

$$= -\binom{r'(T)}{0} + \binom{r'(T)}{1} - \binom{r'(T)}{2} + \cdots + (-1)^{r'(T)} \binom{r'(T)}{r'(T)} + \binom{r'(T)}{0}$$

$$= (1-1)^{r'(T)} + \binom{r'(T)}{0} = 1$$

where the first equality follows since $u_S(\{P_1^T\} \cap \cdots \cap \{P_k^T\})$, $1 \leq k \leq r(T)$, equals 1 if the set of nodes S belongs to every path in $\{P_1^T\} \cap \cdots \cap \{P_k^T\}$, and equals 0 otherwise, and the third equality follows from the binomial formula.

Case (ii) If, on the other hand, $r'(T) = 0$, then $u_S^D(T) = 0$.

Thus, we can conclude that

$$u_S^D(T) = \begin{cases} 1 & \text{if there exists at least one path in } P \in \mathcal{P}(T, D_{|_T}) \text{ with } S \subseteq \{P\} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $\left[1 - \prod_{i=1}^{t(S)} (1 - u_{\{Q_i^S\}}(T))\right] = 1$ if there is $i = 1, ..., t(S)$ such that $(1 - u_{\{Q_i^S\}}(T)) = 0$, or equivalently $u_{\{Q_i^S\}}(T) = 1$, i.e., if there is a path with nodes set containing S and contained in T.

Finally, if for $\emptyset \neq S \subseteq N$, $\mathcal{MCP}(S, N, D) = \emptyset$, by the definition of u_S^D, $u_S^D(T) = 0$ for all $T \subseteq N$, as the nodes in the elements of $\mathcal{P}(T, D_{|T})$ cannot contain S since S has no minimal connection paths in (N, D). This completes the proof. □

Example 3.1.2. *Consider the directed communication situation (N, v, D) in which (N, D) is as in the Example 3.0.1 and v is one of the three characteristic functions v_1, v_2 and v_3 that we are usimng in this report.*

- *For the messages game, we recall the expression of the game in terms of the unanimity basis,*

$$v_1 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

Particularizing the expression of v^D obtained in Example 3.1.1, we have

$$v_1^D(S) = \begin{cases} 0 & \text{if } S = \emptyset, s = 1, S = \{1,4\} \text{ or } S = \{2,3\} \\ 1 & \text{if } S = \{1,2\},\ S = \{1,3\},\ S = \{2,4\} \text{ or } S = \{3,4\} \\ 2 & \text{if } S = \{1,2,3\} \text{ or } S = \{2,3,4\} \\ 3 & \text{if } S = \{1,2,4\} \text{ or } S = \{1,3,4\} \\ 5 & \text{if } S = N. \end{cases}$$

which in terms of the unanimity basis is

$$v_1^D(S) = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,4\}} + u_{\{3,4\}} + u_{\{1,2,4\}} + u_{\{1,3,4\}} - u_N. \qquad (9)$$

This result can be obtained from the expression of the game u_S^D obtained in Proposition (3.1.1). Indeed, $u_{\{1,2\}}^D = u_{\{1,2\}}$, $u_{\{1,3\}}^D = u_{\{1,3\}}$, $u_{\{1,4\}}^D = u_{\{1,2,4\}} + u_{\{1,3,4\}} - u_{\{1,2,3,4\}}$, $u_{\{2,4\}}^D = u_{\{2,4\}}$ and $u_{\{3,4\}}^D = u_{\{3,4\}}$.

And thus

$$v_1^D = \sum_{i<j} u^D_{\{i,j\}} = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,4\}} + u_{\{3,4\}} + u_{\{1,2,4\}} + u_{\{1,3,4\}} - u_N,$$

which coincides with the expression in (9).

- *In the overhead game case, the expression of the game in terms of the unanimity basis is*

$$v_2 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$-u_{\{1,2,3\}} - u_{\{1,2,4\}} - u_{\{1,3,4\}} - u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

The expression of v^D *is*

$$v_2^D(S) = \begin{cases} 0 & \text{if } S = \emptyset, s = 1, S = \{1,4\}, S = \{2,3\} \\ 1 & \text{if } S = \{1,2\},\ S = \{1,3\},\ S = \{2,4\} \text{ or } S = \{3,4\} \\ 2 & \text{if } S = \{1,2,4\},\ S = \{1,3,4\}, \\ & S = \{1,2,3\} \text{ or } S = \{2,3,4\} \\ 3 & \text{if } S = N. \end{cases}$$

which in terms of the unanimity basis is

$$v_2^D(S) = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,4\}} + u_{\{3,4\}} + u_{\{1,2,4\}} + u_{\{1,3,4\}} - u_N. \quad (10)$$

- *In the conferences game case, the expression of the game in terms of the unanimity basis is*

$$v_3 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$+u_{\{1,2,3\}} + u_{\{1,2,4\}} + u_{\{1,3,4\}} + u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

The expression of v^D *is*

$$v_2^D(S) = \begin{cases} 0 & \text{if } S = \emptyset, s = 1, S = \{1,4\}, S = \{2,3\} \\ 1 & \text{if } S = \{1,2\},\ S = \{1,3\},\ S = \{2,4\} \text{ or } S = \{3,4\} \\ 2 & \text{if } S = \{1,2,3\} \text{ or } S = \{2,3,4\} \\ 4 & \text{if } S = \{1,2,4\} \text{ or } S = \{1,3,4\} \\ 7 & \text{if } S = N. \end{cases}$$

which in terms of the unanimity basis is

$$v_3^D(S) = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,4\}} + u_{\{3,4\}} + 2u_{\{1,2,4\}} + 2u_{\{1,3,4\}} - u_N. \quad (11)$$

3.2 A value for directed communication situations

First of all, we introduce the definition of a value or allocation rule on $\mathcal{DCS}^N$. As usual, an allocation rule assigns a payoff vector to every, in this case, directed communication situation.

Definition 3.2.1. *A value or an allocation rule on $\mathcal{DCS}^N$ is a function $\varphi : \mathcal{DCS}^N \to \mathbb{R}^n$. For $(N, v, D) \in \mathcal{DCS}^N$, $\varphi_i(N, v, D)$ represents the outcome for player i, $i \in N$, in the directed communication situation (N, v, D).*

Following Myerson (1977) for undirected communication situations, we define an allocation rule that assigns to every directed communication situation, the Shapley value of the corresponding digraph restricted game.

Definition 3.2.2. *The Directed Communication-value (or DC-value for short) μ^d is given by $\mu^d(N, v, D) = Sh(N, v^D)$, for all $(N, v, D) \in \mathcal{DCS}^N$, where v^D is given by Definition 3.1.1.*

Remark 3.2.1. *Taking Remark 3.1.1 into account, given $(N, v, D) \in \mathcal{DCS}^N$, $\mu^d(N, v, D) = \mu(N, v, \gamma_D)$=Sh(N,v) if for all $S \subseteq N$, the digraph restricted to S, $(S, D_{|S})$, has a Hamiltonian path. Moreover, representing $\gamma \in \Gamma^N$ as $D^\gamma = \{(i, j), (j, i) \mid \{i, j\} \in \gamma\}$, it is clear that D^γ has a Hamiltonian path if γ is connected, and thus $\mu^d(N, v, D^\gamma) = \mu(N, v, \gamma)$ in that case.*

The digraph restricted game for a directed communication situation $(N, v, D) \in \mathcal{DCS}^N$ coincides with the graph restricted game of the communication situation $(N, v, \gamma_D) \in \mathcal{CS}^N$ if in (N, v, D) each ”component” has a Hamiltonian path.

Example 3.2.1. *Consider the directed communication situations (N, v, D) in which (N, D) is as in the Example 3.0.1 and v is one of the three games used in 3.1.2. From the expressions for the digraphs restricted games obtained, the DC-values for the players are given by*

- *For the messages game using (9),*

$$\mu^d(N, v_1, D) = Sh(N, v_1^D)$$

$$= \left(2 \times \frac{1}{2} + 2 \times \frac{1}{3} - \frac{1}{4}, 2 \times \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, 2 \times \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, 2 \times \frac{1}{2} + 2 \times \frac{1}{3} - \frac{1}{4}\right)$$

$$= (\frac{17}{12}, \frac{13}{12}, \frac{13}{12}, \frac{17}{12}).$$

- *For the overhead game using (10)*

$$\mu^d(N, v_2, D) = Sh(N, v_2^D) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}).$$

- *For the conferences game using (11)*

$$\mu^d(N, v_3, D) = Sh(N, v_3^D) = (\frac{25}{12}, \frac{17}{12}, \frac{17}{12}, \frac{25}{12}).$$

3.3 Characterizations of the DC-value

In this section, we characterize the DC-value with modifications of properties as component efficiency, fairness and balanced contributions for directed networks. First, connection efficiency expresses that worth is generated by (maximal) paths. This occurs, for example, in marketing attribution or supply chains, where worth is generated when, through a sequence of advertisements, respectively a sequence of intermediary retailers and other agents on the supply chain, a conversion takes place when a consumer (sink of the path) buys a product that is produced by a producer (source of the path). Another situation where this occurs is in communications (e.g., sending messages) when the only thing that matters is whether a message from a sender reaches the intended receiver, possibly through a chain of intermediaries. As we will see in next section this notion of connection efficiency can be used to introduce game theoretical measures of efficiency and vulnerability for directed networks.

Definition 3.3.1. *An allocation rule φ on $\mathcal{DCS}^N$ satisfies connection efficiency if, for all $(N, v, D) \in \mathcal{DCS}^N$ and all $C \in N/\gamma_D$, $\sum_{i \in C} \varphi_i(N, v, D) = v^D(C)$.*

Connection efficiency reflects our new cooperation restrictions and requires that the sum of the payoffs of the players in a component is equal to the ”sum” of the worths of the (path) maximally connected subcoalitions in this component. This emphasizes our idea of connectedness that is needed to generate worth. Notice that the axiom of connection efficiency explicitly uses the worth $v^D(C)$

of components C. In terms of the original game v and digraph D, this means that the total payoff in every component is equal to the worth generated by all directed paths in D in the original game v.

Fairness and balanced contributions are expressed the same as in Myerson (1977, 1980) (see Preliminaries), but for digraphs, requiring that the payoffs of two nodes on a directed arc change by the same amount when the arc is deleted.

Definition 3.3.2. *An allocation rule φ on $\mathcal{DCS}^N$ satisfies fairness if, for all $(N, v, D) \in \mathcal{DCS}^N$ and every $e = (i, j) \in D$, $\varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{e\}) = \varphi_j(N, v, D) - \varphi_j(N, v, D \setminus \{e\})$.*

Notice that fairness is more or less stated the same way as Myerson's fairness for undirected communication situations (as mentioned in the preliminaries), just replacing undirected graphs and undirected edges by directed networks and arcs.

Finally, balanced contributions requires that the effect of isolation of a player on the payoff of another player, is equal to the effect the other way around, that is the effect on the payoff of this player of the isolation of the other player.

Definition 3.3.3. *An allocation rule φ on $\mathcal{DCS}^N$ satisfies balanced contributions if, for all $(N, v, D) \in \mathcal{DCS}^N$ and all $i, j \in N$, $\varphi_i(N, v, D) - \varphi_i(N, v, D_{-j}) = \varphi_j(N, v, D) - \varphi_j(N, v, D_{-i})$.*

In the following propositions, we prove that the DC-value μ^d is the unique connection efficient rule that satisfies either fairness or balanced contributions. Notice that this mimics other results in the literature on undirected graph games, starting with the results of Myerson (1977) and Myerson (1980), that characterize a value by an efficiency and a fairness or balanced contributions property. Since uniqueness follows similar as in Myerson (1977) or Myerson (1980), we first prove that μ^d satisfies the three previous properties.

Proposition 3.3.1. *The DC-value, μ^d, satisfies connection efficiency.*

Proof: Let $(N, v, D) \in \mathcal{DCS}^N$. Suppose that $C \in N/\gamma_D$ is a weak component in (N, D). Then,

$$\sum_{i \in C} \mu_i^d(N, v, D) = \sum_{i \in C} Sh_i(N, v^D) = \sum_{i \in C} Sh_i(C, (v^D)_{|C}) = (v^D)_{|C}(C) = v^D(C),$$

the first equality holding by the definition of μ^d, the second one as the Shapley value of a player in (N, v^D) only depends on the component to which he belongs[3], the third one by the efficiency of the Shapley value, and the fourth one by the definition of the restriction of a game to a coalition. □

Proposition 3.3.2. *The DC-value, μ^d, satisfies fairness.*

Proof: As μ^d is clearly linear in the game, it is sufficient to prove the result for directed communication situations of the form (N, u_S, D) with $\emptyset \neq S \subseteq N$. Suppose that $i, j \in N$ are such that $(i, j) \in D$. If $\mathcal{MCP}(S, N, D) = \emptyset$, then $\mu_k^d(N, u_S, D) = \mu_k^d(N, u_S, D \backslash \{(i, j)\}) = 0$, for $k = i, j$, by connection efficiency, and thus the result is proved in this case.

If, on the other hand, $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ then, for $k = i, j$, $\mu_k^d(N, u_S, D) = Sh_k[N, 1 - \prod_{l=1}^{t(S)}(1 - u_{\{Q_l^S\}})]$, by Proposition 3.1.1.

Since $\mathcal{MCP}(S, N, D \backslash \{(i, j)\}) \subseteq \mathcal{MCP}(S, N, D)$, suppose, without loss of generality, that $\mathcal{MCP}(S, N, D \backslash \{(i, j)\}) = \{Q_1^S, \cdots, Q_{t'(S)}^S\}$ with $t'(S) \leq t(S)$. The case $t'(S) = t(S)$ is trivial. Therefore, suppose that $t'(S) < t(S)$, i.e., $Q_{t'(S)}^S, \cdots, Q_{t(S)}^S$ are the paths in $\mathcal{MCP}(S, N, D)$ to which the arc (i, j) belongs.

Then, for $k = i, j$,

$$
\begin{aligned}
&\mu_k^d(N, u_S, D) - \mu_k^d(N, u_S, D \backslash \{(i, j)\}) \\
&= Sh_k\Big[N, 1 - \prod_{l=1}^{t(S)}(1 - u_{\{Q_l^S\}})\Big] - Sh_k\Big[N, 1 - \prod_{l=1}^{t'(S)}(1 - u_{\{Q_l^S\}})\Big] \\
&= Sh_k\Big[N, \prod_{l=1}^{t'(S)}(1 - u_{\{Q_l^S\}}) - \prod_{l=1}^{t(S)}(1 - u_{\{Q_l^S\}})\Big] \\
&= Sh_k\Big[N, \prod_{l=1}^{t'(S)}(1 - u_{\{Q_l^S\}})\Big(1 - \prod_{l=t'(S)+1}^{t(S)}(1 - u_{\{Q_l^S\}})\Big)\Big], \qquad (12)
\end{aligned}
$$

where the second equality follows from linearity of the Shapley value. The characteristic function $1 - \prod_{l=t'(S)+1}^{t(S)}(1 - u_{\{Q_l^S\}})$ can be written as a linear combination of unanimity games as follows

[3]This follows from the Shapley value satisfying marginality or strong monotonicity as discussed in Young (1985), and the fact that changes in the directed graph outside the component of a player, does not affect the marginal contributions of that player in the digraph restricted game v^D.

$$\sum_{l=t'(S)+1}^{t(S)} u_{\{Q_l^S\}} - \sum_{l=t'(S)+1}^{t(S)-1} \sum_{m=l+1}^{t(S)} u_{\{Q_l^S\} \cup \{Q_m^S\}}$$
$$+ \cdots + (-1)^{t(S)-t'(S)-1} u_{\{Q_{t'(S)+1}^S\} \cup \cdots \cup \{Q_{t(S)}^S\}}. \tag{13}$$

Notice that players i and j belong to every coalition in $\{\{Q_{t'(S)}^S\}, \cdots, \{Q_{t(S)}^S\}\}$, and thus both players belong to every unanimity coalition in this expression. As a consequence, $\prod_{l=1}^{t(S)} (1 - u_{\{Q_l^S\}}) \Big(1 - \prod_{l=t'(S)+1}^{t(S)} (1 - u_{\{Q_l^S\}})\Big)$ is also a linear combination of unanimity games u_T with $i, j \in T$. Given the symmetry of the Shapley value, we have that

$$Sh_i\Big[N, \prod_{l=1}^{t'(S)} (1 - u_{\{Q_l^S\}}) \Big(1 - \prod_{l=t'(S)+1}^{t(S)} (1 - u_{\{Q_l^S\}})\Big)\Big]$$
$$= Sh_j\Big[N, \prod_{l=1}^{t'(S)} (1 - u_{\{Q_l^S\}}) \Big(1 - \prod_{l=t'(S)+1}^{t(S)} (1 - u_{\{Q_l^S\}})\Big)\Big],$$

and thus the result follows with (12). □

Proposition 3.3.3. *The DC-value, μ^d, satisfies balanced contributions.*

Proof: Again, as μ^d is linear in the game, it is sufficient to prove that μ^d satisfies the property for directed communication situations of the form (N, u_S, D) with $\emptyset \neq S \subseteq N$.

Suppose $i, j \in N$. If $\mathcal{MCP}(S, N, D) = \emptyset$, then $u_S^D = u_S^{D \setminus D_i} = u_S^{D \setminus D_j} \equiv \mathbf{0}$, and thus $\mu_i^d(N, u_S, D) = \mu_j^d(N, u_S, D) = \mu_i^d(N, u_S, D_{-j}) = \mu_j^d(N, u_S, D_{-i}) = 0$ by connection efficiency, and then, the result is proved in this case.

Consider, then, the case in which $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ and (without loss of generality) $\mathcal{MCP}(S, N, D \setminus D_j) = \{Q_1^S, \cdots, Q_{t'(S)}^S\}$ with $t'(S) \leq t(S)$. Again (similar as in the proof of Proposition 3.3.2) the case $t'(S) = t(S)$ is trivial. Therefore, suppose that $t'(S) < t(S)$.

Then, in a parallel way to the proof of the previous proposition,

$$\mu_i^d(N, u_S, D) - \mu_i^d(N, u_S, D_{-j})$$
$$= Sh_i\Big[N, \prod_{l=1}^{t'(S)} (1 - u_{\{Q_l^S\}}) \Big(1 - \prod_{l=t'(S)+1}^{t(S)} (1 - u_{\{Q_l^S\}})\Big)\Big],$$

but now the sets in $\{Q_1^S, \cdots, Q_{t'(S)}^S\}$ are the minimal connection paths in $D \backslash D_j$, and thus is the Shapley value of player i in a linear combination of unanimity games u_T with $j \in T$ (and not necessarily $i \in T$). Similar as in the proof of Proposition 3.3.2, this expression can be written in a form as (13), but with coalitions $\{Q_l^S\}$ containing j (but not necessarily containing i).

On the other hand, $\mu_j^d(N, u_S, D) - \mu_j^d(N, u_S, D_{-i})$ is the Shapley value of player j in a linear combination of unanimity games u_T with $i \in T$ (but not necessarily $j \in T$).

By the null player property of the Shapley value, in both expressions, we can ignore the unanimity games of coalitions not containing player i, respectively player j, and thus both $\mu_i^d(N, u_S, D) - \mu_i^d(N, u_S, D_{-j})$ as well as $\mu_j^d(N, u_S, D) - \mu_j^d(N, u_S, D_{-i})$ can be expressed as sum of unanimity games of coalitions containing both i and j, and thus, by the symmetry of the Shapley value,

$$\mu_i^d(N, u_S, D) - \mu_i^d(N, u_S, D_{-j}) = \mu_j^d(N, u_S, D) - \mu_j^d(N, u_S, D_{-i}),$$

which completes the proof. □

In the next two theorems we obtain two characterizations of μ^d.

Theorem 3.3.1. *The DC-value μ^d is the unique allocation rule on $\mathcal{DCS}^N$ satisfying connection efficiency and fairness.*

Proof: It is already proved that μ^d satisfies connection efficiency and fairness, see Propositions 3.3.1 and 3.3.2. Reciprocally, suppose that φ is an allocation rule on $\mathcal{DCS}^N$ satisfying connection efficiency and fairness. We will prove that $\varphi(N, v, D)$ is uniquely determined for all $(N, v, D) \in \mathcal{DCS}^N$, by induction on the cardinality of D.

If $|D| = 0$, then clearly every singleton $\{i\}$, $i \in N$, is a weak component, and thus by connection efficiency, $\varphi_i(N, v, D) = v^D(\{i\}) = v(\{i\}) = \mu_i^d(N, v, D)$.

Proceeding by induction, suppose that the result holds for $(N, v, D') \in \mathcal{DCS}^N$ such that $|D'| \leq k$ and consider (N, v, D) with $|D| = k + 1$. Let $C \in N/\gamma_D$ be a weak component of (N, D), and let (C, T_C) be a spanning tree in $(C, (\gamma_D)_{|C})$.

Then, for every $\{i, j\} \in T_C$, fairness implies that

$$\varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\}))\})$$

$$= \varphi_j(N, v, D) - \varphi_j(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\})\}), \tag{14}$$

where $t(e)$ denotes the tail of arc $e \in D$, while $h(e)$ denotes the head of arc $e \in D$.

Notice that $|T_C| = |C| - 1$. Moreover, for every weak component $C \in N/\gamma_D$, connection efficiency implies that

$$\sum_{i \in C} \varphi_i(N, v, D) = v^D(C). \tag{15}$$

Since $\varphi_i(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\})\})$ and $\varphi_j(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\})\})$ are uniquely determined by the induction hypothesis, we have $(|C| - 1) + 1 = |C|$ linear equations (14) and (15) that are clearly independent. Thus, the numbers (payoffs) $\varphi_i(N, v, D)$, $i \in C$, are uniquely determined.

Since we have shown that μ^d satisfies connection efficiency and fairness, it must hold that $\varphi(N, v, D) = \mu^d(N, v, D)$ for all $(N, v, D) \in \mathcal{DCS}^N$, which completes the proof. □

Looking at the two axioms in Theorem 3.3.1, it is interesting to observe that fairness has a symmetric flavour in the sense that it seems not to take account of the orientation of the arcs. Although in the model the arcs are oriented, this axiom expresses that both players on an arc benefit equally from building the arc. So, in the axiomatization, the impact of the arc orientation fully comes from connection efficiency which, although it is an axiom requiring an efficiency for the total payoff in a component, takes account of the orientation of the arcs in determining what a component can earn.

Theorem 3.3.2. *The DC-value is the unique allocation rule on $\mathcal{DCS}^N$ satisfying connection efficiency and balanced contributions.*

Proof: It is already proved that μ^d satisfies connection efficiency and balanced contributions, see Propositions 3.3.1 and 3.3.3. Reciprocally, suppose that φ is an allocation rule in $\mathcal{DCS}^N$ satisfying connection efficiency and balanced contributions. Consider $(N, v, D) \in \mathcal{DCS}^N$ and $i \in N$. We will prove that $\varphi_i(N, v, D) = \mu_i^d(N, v, D)$ by induction on the cardinality of D. If $|D| = 0$, then i is an isolated node in the digraph and by connection efficiency, $\varphi_i(N, v, D) = v^d(\{i\}) = v(\{i\}) = \mu_i^d(N, v, D)$.

Proceeding by induction, suppose that $\varphi(N, v, D')$ is uniquely determined for directed communication situations (N, v, D') such that $|D'| \leq k$ and consider

(N, v, D) with $|D| = k+1$. Let C_i be the weak component of (N, D) to which i belongs. If $|C_i| = 1$, then connection efficiency implies that $\varphi_i(N, v, D) = v^D(\{i\}) = \mu_i^d(N, v, D)$. Next, suppose $|C_i| > 1$, and consider $j \in C_i$, $j \neq i$. As φ satisfies the balanced contributions property,

$$\varphi_i(N, v, D) - \varphi_i(N, v, D \backslash D_j) = \varphi_j(N, v, D) - \varphi_j(N, v, D \backslash D_i),$$

or equivalently,

$$\varphi_i(N, v, D) - \varphi_j(N, v, D) = \varphi_i(N, v, D \backslash D_j) - \varphi_j(N, v, D \backslash D_i). \qquad (16)$$

Since $|D \backslash D_i| \leq k$ and $|D \backslash D_j| \leq k$, by the induction hypothesis, $\varphi_i(N, v, D \backslash D_j) - \varphi_j(N, v, D \backslash D_i)$ is uniquely determined., Moreover, for the weak component C_i, connection efficiency implies that

$$\sum_{j \in C_i} \varphi_j(N, v, D) = v^D(C_i). \qquad (17)$$

Since the $(|C_i| - 1) + 1 = |C_i|$ linear equations (16) and (17) are independent, the numbers (payoffs) $\varphi_j(N, v, D)$ for $j \in C_i$, are uniquely determined.

Since we have shown that μ^d satisfies connection efficiency and balanced contributions, it must hold that $\varphi(N, v, D) = \mu^d(N, v, D)$ for all $(N, v, D) \in \mathcal{DCS}^N$, which completes the proof. □

Remark 3.3.1. *We remark that, instead of the sum of the payoffs of the players in a component being equal to the "sum" of the worths of the (path) maximally connected subcoalitions in this component (i.e.* $v^D(C)$*), we could use other definitions of efficiency. For example, we could require the sum of the payoffs in a component to be equal to the maximum worth of any connection path in the coalition, or the maximum sum of the connection paths over every partition of the coalition (and thus every player contributing to at most one path). From the proofs of Theorems 3.3.1 and 3.3.2, it is easy to see that combining such an alternative efficiency with either fairness or balanced contributions also gives a unique allocation rule. What version of efficiency is most appropriate depends on the application one has in mind. We have considered connection efficiency as defined in Definition 3.3.1 since it seems appropriate for the applications mentioned in the paragraph after Example 3.0.1.*

This also shows the 'power' of Myerson's fairness and balanced contributions axioms in models where efficiency is less obvious than in the undirected communication situations. Although, contrary to undirected communication situations, *efficiency in directed communication situations is not obvious, fairness and balanced contributions stated in their original form (but in a different, more general, model) give $c-1$ linear independent equations for every component with c players in every induction step of the proof for every component which, with an efficiency condition, give unique payoffs. In particular, for directed communication situations this gives the possibility to express the desired efficiency requirement, to get a unique allocation rule that satisfies fairness or balanced contributions.*

3.4 Applications to directed network centrality

In this section, we discuss several applications of the new allocation rule for directed communication situations to centrality, efficiency and vulnerability in directed networks.

3.4.1 Connection and betweenness centrality

The defined DC-value can be used to introduce game-theoretical centrality measures for directed networks. A (directed) network centrality measure p is a mapping that assigns to every directed network (N, D) an n-dimensional vector measuring the centrality of nodes in some way. In the literature there are many concepts of centrality and corresponding centrality measures. Social networks theorists have developed relevant centrality measures as degree (Shaw, 1954; Nieminen, 1974), closeness (Beauchamp, 1965; Sabidussi, 1966), betweeness (Bavelas, 1948; Freeman, 1977) and eigenvector centrality (Bonacich, 1972; 1987).

Suppose that (N, v, D) is a directed communication situation in which (N, v) is a symmetric game. Following Gómez et al. (2003), for all $i \in N$, since the Shapley value assigns equal payoff $\frac{v(N)}{n}$ to every player in a symmetric game, the difference

$$\mu_i^d(N, v, D) - Sh_i(N, v) = \mu_i^d(N, v, D) - \frac{v(N)}{n}$$

can be viewed as the advantages (or disadvantages) that the (directed) communications present in the network (and the restrictions) provide to player i. Adding $\frac{v(N)}{n}$ to the measure of each player-node, we can use $\mu_i^d(N, v, D)$ as a centrality measure. This gives rise to the following family of centrality measures for directed networks.

Definition 3.4.1. *Given a symmetric game* (N, v)*, the centrality measure* κ_v^d *is defined as* $\kappa_v^d(N, D) = \mu^d(N, v, D)$ *for every directed network* (N, D)*.*

Remark 3.4.1. *Definition 3.4.1 defines a family of centrality measures for directed networks that is parameterized by a symmetric game. This implies that each centrality measure in this family is determined by the symmetric game that is used. The appropriate game depends on what kind of centrality one wants to measure. The game reflects (economic or social) interests that motivate the interactions among the players/nodes. But, since it is a symmetric game, differences between the nodes is fully determined by the differences in network positions.*

Since in this section we will only use symmetric games, we denote the class of directed communication situations where the game (N, v) is symmetric by $\mathcal{DCSS}^N$.

We can decompose, for each $i \in N$, the characteristic function v in $v_i + v_{-i}$ with

$$v_i = \sum_{S:i \in S} \Delta_v(S) u_S$$

and $v_{-i} = v - v_i$. The game (N, v_i) represents the productivity of i via the characteristic function v, whereas in the game (N, v_{-i}), i is a null player. Notice that (N, v_i) and (N, v_{-i}) are not symmetric games, but for different $i, j \in N$, the games (N, v_i) and (N, v_j) are isomorphic (similar for (N, v_{-i}) and (N, v_{-j})). Then, given the obvious linearity (in the game) of the defined centrality measure, we have, for each $(N, v, D) \in \mathcal{DCSS}^N$, and each $i \in N$,

$$\kappa_{v,i}^d(N, D) = \mu_i^d(N, v_i, D) + \mu_i^d(N, v_{-i}, D).$$

The game (N, v) being symmetric, $\mu_i^d(N, v_i, D)$ represents player $i's$ productivity in the characteristic function that is preserved by the digraph. Since the productivity of all players in the unrestricted game is the same, $\mu_i^d(N, v_i, D)$ can be viewed as the part of $i's$ centrality due to his communication possibilities in the graph. Even though $\mu_i^d(N, v_i, D)$ itself is not a centrality measure, it defines part of the centrality of the nodes in a directed communication network.

On the other hand, as i is a null player in (N, v_{-i}), it is obvious that $\mu_i^d(N, v_{-i}, D)$ exclusively depends on the ability of i to intermediate between others, improving their communication and, thus, it can be viewed as a contribution to its betweenness centrality. This motivates the following definitions.

Definition 3.4.2. *Given a symmetric game (N, v), we define the connection centrality of i in directed network (N, D) as*

$$\kappa_{v,i}^{d,C}(N, D) = \mu_i^d(N, v_i, D).$$

Definition 3.4.3. *Given a symmetric game (N, v), we define the betweenness centrality of i in directed network (N, D) as*

$$\kappa_{v,i}^{d,B}(N, D) = \mu_i^d(N, v_{-i}, D).$$

A nice feature of Myerson (1977)'s characterization of his value by component efficiency and fairness, is that it also gives uniqueness on smaller classes of games, specifically even on classes that contain only one game. As can be seen from the proofs of Theorem 3.3.1 and 3.3.2, the same holds for the characterizations given by these results. Specifically, considering the class that consists of a single symmetric game (N, v), we can obtain the following characterization of the centrality measure κ_v^d.

Definition 3.4.4. *A centrality measure p for directed networks satisfies fairness if, for every directed network (N, D) and every $e = (i, j) \in D$, $p_i(N, D) - p_i(N, D \setminus \{e\}) = p_j(N, D) - p_j(N, D \setminus \{e\})$.*

Definition 3.4.5. *A centrality measure p for directed networks satisfies balanced contributions if, for every directed network (N, D) and all $i, j \in N$, $p_i(N, D) - p_i(N, D_{-j}) = p_j(N, D) - p_j(N, D_{-i})$.*

Notice that the above two axioms do not depend on the specific symmetric game that is used. These axioms are satisfied by every centrality measure κ_v^d defined in this section. A specific measure is obtained by applying connection efficiency with respect to a specific symmetric game.

Definition 3.4.6. *Let (N, v) be a symmetric game. A centrality measure p for directed networks satisfies connection efficiency (with respect to v) if, for every directed network (N, D) and all $C \in N/\gamma_D$, $\sum_{i \in C} p_i(N, D) = v^D(C)$.*

Theorem 3.4.1. *Let (N, v) be a symmetric game.*

(i) κ_v^d is the unique centrality measure for directed networks satisfying connection efficiency (with respect to (N, v)) and fairness.

(ii) κ_v^d is the unique centrality measure for directed networks satisfying connection efficiency (with respect to (N, v)) and balanced contributions.

Since this theorem is an application of Theorems 3.3.1 and 3.3.2 on a smaller class of games, the measure κ_v^d satisfying the axioms is a corollary from Theorems 3.3.1 and 3.3.2. Uniqueness is not a corollary, but can be proved in the same way as Theorems 3.3.1 and 3.3.2, and the proof is therefore omitted.

Since all centrality measures in our family satisfy fairness and balanced contributions, it can be seen from these characterizations that the choice of the symmetric game v determines the specific centrality measure by connection efficiency. For example, when it is about connecting pairs of nodes, then the messages game seems appropriate.

3.4.2 Out- and in-centrality

By linearity of the allocation rule μ^d, for any symmetric game (N, v), the centrality measure κ_v^d can be decomposed into separate 'centralities' related to the possibility to generate connection in any subset or coalition of nodes $S \subseteq N$. This also makes it possible to specifically take account of nodes' possibilities to send or receive messages in their communication by distinguishing their out- and in-degree. We can do this both for connection as well as for betweenness. We define the following new measures for directed networks using unanimity games. They are extended by linearity. Recall that $d_i^O(N, D)$ and $d_i(N, D)$ are the out-degree and the degree of node $i \in N$ in the digraph (N, D) respectively.

Definition 3.4.7. *Consider a symmetric game (N, v). The out-connection centrality (with respect to (N, v)) of i in (N, D) is defined as,*

$$\kappa_{v,i}^{d,C_O}(N, D) = \sum_{S \subseteq N} \Delta_{v_i}(S) \kappa_{u_S,i}^{d,C_O}(N, D)$$

where, denoting $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ for every $S \subseteq N$, $\kappa_{u_S,i}^{d,C_O}(N, D)$

is given by

$$\kappa^{d,C_O}_{u_S,i}(N,D) = \sum_{j=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}}) \frac{d_i^O(N, Q_j^S)}{d_i(N, Q_j^S)}$$
$$- \sum_{j=1}^{t(S)-1} \sum_{k=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}\cup\{Q_k^S\}}) \frac{d_i^O(N, Q_j^S \cup Q_k^S)}{d_i(N, Q_j^S \cup Q_k^S)}$$
$$+ \cdots + (-1)^{t(S)-1} Sh_i \left[N, u_{\cup_{j=1}^{t(S)}\{Q_j^S\}}\right] \frac{d_i^O(N, \cup_{j=1}^{t(S)} Q_j^S)}{d_i(N, \cup_{j=1}^{t(S)} Q_j^S)}.$$

Notice that $\kappa^{d,C_O}_{u_S}$ is not a centrality measure in our family, since u_S is not a symmetric game. But this is used to define the centrality measure κ^{d,C_O}_v which is in our family. A similar remark applies to the following definitions. Recall that $d_i^I(N,D)$ and $d_i(N,D)$ are the in-degree and the degree of node $i \in N$ in the digraph (N,D) respectively.

Definition 3.4.8. *Consider a symmetric game (N,v). The in-connection centrality (with respect to (N,v)) of i in (N,D) is defined as,*

$$\kappa^{d,C_I}_{v,i}(N,D) = \sum_{S\subseteq N} \Delta_{v_i}(S) \kappa^{d,C_I}_{u_S,i}(N,D)$$

where, denoting $\mathcal{MCP}(S,N,D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ for every $S \subseteq N$, $\kappa^{d,C_I}_{u_S,i}(N,D)$ is given by

$$\kappa^{d,C_I}_{u_S,i}(N,D) = \sum_{j=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}}) \frac{d_i^I(N, Q_j^S)}{d_i(N, Q_j^S)}$$
$$- \sum_{j=1}^{t(S)-1} \sum_{k=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}\cup\{Q_k^S\}}) \frac{d_i^I(N, Q_j^S \cup Q_k^S)}{d_i(N, Q_j^S \cup Q_k^S)}$$
$$+ \cdots + (-1)^{t(S)-1} Sh_i \left[N, u_{\cup_{j=1}^{t(S)}\{Q_j^S\}}\right] \frac{d_i^I(N, \cup_{j=1}^{t(S)} Q_j^S)}{d_i(N, \cup_{j=1}^{t(S)} Q_j^S)}.$$

Definition 3.4.9. *Consider a symmetric game (N,v). The out-betweenness centrality (with respect to (N,v)) of i in (N,D) is defined as,*

$$\kappa^{d,B_O}_{v,i}(N,D) = \sum_{S\subseteq N} \Delta_{v_{-i}}(S) \kappa^{d,B_O}_{u_S,i}(N,D)$$

where, denoting $\mathcal{MCP}(S,N,D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ for every $S \subseteq N$, $\kappa^{d,B_O}_{u_S,i}(N,D)$ is given by

$$\kappa^{d,B_O}_{u_S,i}(N,D) = \sum_{j=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}}) \frac{d_i^O(N, Q_j^S)}{d_i(N, Q_j^S)}$$

$$- \sum_{j=1}^{t(S)-1} \sum_{k=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\} \cup \{Q_k^S\}}) \frac{d_i^O(N, Q_j^S \cup Q_k^S)}{d_i(N, Q_j^S \cup Q_k^S)}$$

$$+ \cdots + (-1)^{t(S)-1} Sh_i \left[N, u_{\cup_{j=1}^{t(S)} \{Q_j^S\}} \right] \frac{d_i^O(N, \cup_{j=1}^{t(S)} Q_j^S)}{d_i(N, \cup_{j=1}^{t(S)} Q_j^S)}.$$

Definition 3.4.10. *Consider a symmetric game (N, v). The in-betweenness centrality (with respect to (N, v)) of i in (N, D) is defined as,*

$$\kappa_{v,i}^{d,B_I}(N, D) = \sum_{S \subseteq N} \Delta_{v_{-i}}(S) \kappa_{u_S,i}^{d,B_I}(N, D)$$

where, denoting $\mathcal{MCP}(S, N, D) = \{Q_1^S, \cdots, Q_{t(S)}^S\} \neq \emptyset$ for every $S \subseteq N$, $\kappa_{u_S,i}^{d,B_I}(N, D)$ is given by

$$\kappa_{u_S,i}^{d,B_I}(N, D) = \sum_{j=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\}}) \frac{d_i^I(N, Q_j^S)}{d_i(N, Q_j^S)}$$

$$- \sum_{j=1}^{t(S)-1} \sum_{k=1}^{t(S)} Sh_i(N, u_{\{Q_j^S\} \cup \{Q_k^S\}}) \frac{d_i^I(N, Q_j^S \cup Q_k^S)}{d_i(N, Q_j^S \cup Q_k^S)}$$

$$+ \cdots + (-1)^{t(S)-1} Sh_i \left[N, u_{\cup_{j=1}^{t(S)} \{Q_j^S\}} \right] \frac{d_i^I(N, \cup_{j=1}^{t(S)} Q_j^S)}{d_i(N, \cup_{j=1}^{t(S)} Q_j^S)}.$$

Notice the differences between these four definitions. On one hand, the two connection centralities ($\kappa_{v,i}^{d,C_O}$ and $\kappa_{v,i}^{d,C_I}$) use the game (N, v_i), while the two betweeness centralities ($\kappa_{v,i}^{d,B_O}$ and $\kappa_{v,i}^{d,B_I}$) use the game (N, v_{-i}). On the other hand, the two 'out' centralities ($\kappa_{v,i}^{d,C_O}$ and $\kappa_{v,i}^{d,B_O}$) use the outdegree d_i^O, while the two 'in' centralities ($\kappa_{v,i}^{d,C_I}$ and $\kappa_{v,i}^{d,B_I}$) use the indegree d_i^I. To clarify these ideas of distinguishing the centrality in a digraph into four components based on out- and in-connection, and out- and in-betweenness centralities, let us consider the following example.

Example 3.4.1. *Consider again the directed communication situation (N, v, D) in Example 3.1.2 and $D = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$.*

Using the messages game, and the expression in terms of the unanimity basis, for player 1 we have

$$\kappa_{v,1}^{d,C_O}(N, D) = \kappa_{u_{\{1,2\}},1}^{d,C_O}(N, D) + \kappa_{u_{\{1,3\}},1}^{d,C_O}(N, D)$$

$$+ \kappa_{u_{\{1,4\}},1}^{d,C_O}(N, D) = Sh_1(N, u_{\{1,2\}}) \frac{d_1^O(N, (1, 2))}{d_1(N, (1, 2))} + Sh_1(N, u_{\{1,3\}}) \frac{d_1^O(N, (1, 3))}{d_1(N, (1, 3))}$$

$$+Sh_1(N,u_{\{1,2,4\}})\frac{d_1^O(N,(1,2,4))}{d_1(N,(1,2,4))}+Sh_1(N,u_{\{1,3,4\}})\frac{d_1^O(N,(1,3,4))}{d_1(N,(1,3,4))}$$

$$-Sh_1(N,u_{\{1,2,3,4\}})\frac{d_1^O(N,(1,2,3,4))}{d_1(N,(1,2,3,4))}=\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{3}\cdot\frac{1}{1}+\frac{1}{3}\cdot\frac{1}{1}-\frac{1}{4}\cdot\frac{1}{1}=\frac{17}{12},$$

and thus, as can be expected, all the centrality of player 1 is out-connection centrality. He does not intermediate between the communication of the others and he is tail in all the arcs he is involved in.

The out-connection centrality of player 2 is

$$\kappa_{v,2}^{d,C_O}(N,D)=\kappa_{u_{\{1,2\}},2}^{d,C_O}(N,D)+\kappa_{u_{\{2,3\}},2}^{d,C_O}(N,D)$$

$$+\kappa_{u_{\{2,4\}},2}^{d,C_O}(N,D)=Sh_2(N,u_{\{1,2\}})\frac{d_2^O(N,(1,2))}{d_2(N,(1,2))}+Sh_2(N,u_{\{2,4\}})\frac{d_2^O(N,(2,4))}{d_2(N,(2,4))}$$

$$=\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{2},$$

and the in-connection centrality of player 2 is

$$\kappa_{v,2}^{d,C_I}(N,D)=\kappa_{u_{\{1,2\}},2}^{d,C_I}(N,D)+\kappa_{u_{\{2,3\}},2}^{d,C_I}(N,D)$$

$$+\kappa_{u_{\{2,4\}}2}^{d,C_I}(N,D)=Sh_2(N,u_{\{1,2\}})\frac{d_2^I(N,(1,2))}{d_2(N,(1,2))}+Sh_2(N,u_{\{2,4\}})\frac{d_2^I(N,(2,4))}{d_2(N,(2,4))}$$

$$=\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{2}\cdot 0=\frac{1}{2}.$$

The out-betweenness of player 2 is

$$\kappa_{v,2}^{d,B_O}(N,D)=\kappa_{u_{\{1,3\}},2}^{d,B_O}(N,D)+\kappa_{u_{\{1,4\}},2}^{d,B_O}(N,D)$$

$$+\kappa_{u_{\{3,4\}},2}^{d,B_O}(N,D)=Sh_2(N,u_{\{1,3\}})\frac{d_2^O(N,(1,3))}{d_2(N,(1,3))}+Sh_2(N,u_{\{1,2,4\}})\frac{d_2^O(N,(1,2,4))}{d_2(N,(1,2,4))}$$

$$+Sh_2(N,u_{\{1,3,4\}})\frac{d_2^O(N,(1,3,4))}{d_2(N,(1,3,4))}-Sh_2(N,u_{\{1,2,3,4\}})\frac{d_2^O(N,(1,2,4)\cup(1,3,4))}{d_2(N,(1,2,4)\cup(1,3,4))}$$

$$+Sh_2(N,u_{\{3,4\}})\frac{d_2^O(N,(3,4))}{d_2(N,(3,4))}=0\cdot 0+\frac{1}{3}\cdot\frac{1}{2}+0\cdot 0-\frac{1}{4}\cdot\frac{1}{2}+0\cdot 0=\frac{1}{24},$$

and the in-betweenness centrality of player 2 is

$$\kappa_{v,2}^{d,B_I}(N,D)=\kappa_{u_{\{1,3\}},2}^{d,B_I}(N,D)+\kappa_{u_{\{1,4\}},2}^{d,B_I}(N,D)$$

$$+\kappa_{u_{\{3,4\}},2}^{d,B_I}(N,D)=Sh_2(N,u_{\{1,3\}})\frac{d_2^I(N,(1,3))}{d_2(N,(1,3))}+Sh_2(N,u_{\{1,2,4\}})\frac{d_2^I(N,(1,2,4))}{d_2(N,(1,2,4))}$$

$$+Sh_2(N, u_{\{1,3,4\}})\frac{d_2^I(N,(1,3,4))}{d_2(N,(1,3,4))} - Sh_2(N, u_{\{1,2,3,4\}})\frac{d_2^I(N,(1,2,4)\cup(1,3,4))}{d_2(N,(1,2,4)\cup(1,3,4))}$$

$$+Sh_2(N, u_{\{3,4\}})\frac{d_2^I(N,(3,4))}{d_2(N,(3,4))} = 0\cdot 0 + \frac{1}{3}\cdot\frac{1}{2} + 0\cdot 0 - \frac{1}{4}\cdot\frac{1}{2} + 0\cdot 0 = \frac{1}{24},$$

and thus, the total centrality of player 2, which is $\frac{13}{12}$, can be obtained by adding $\frac{1}{2}, \frac{1}{2}, \frac{1}{24}$ and $\frac{1}{24}$ that are, respectively, his out-connection centrality, his in-connection centrality, his out-betweenness centrality and his in-betweenness centrality.

Remark 3.4.2. *Comparing the obtained results for centrality of nodes in Example 3.4.1 with the ones obtained using the measure, κ, defined (for undirected graphs) in Gómez et al. (2003) we have $\kappa(N, v, \gamma_D) = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ in which the part corresponding to communication centrality is $(\frac{17}{12}, \frac{17}{12}, \frac{17}{12}, \frac{17}{12})$, whereas, the betweenness centrality of nodes is $(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$. In such a situation the symmetry in the game and in the graph gives the same figures for all players. Then, they have the same ability to communicate and to intermediate in the communication of messages of the others. Let us recall that for the directed communication situation in the Example 3.4.1, given the directed edges, player 1 and 4 can not intermediate among the others. For the nodes intermediating the proportion of his centrality due to intermediation is $\frac{1}{13}$ in (N, v, D) and $\frac{1}{17}$ in (N, v, γ_D).*

3.4.3 An application to efficiency and vulnerability in directed networks

Finally, in this section we illustrate how the obtained DC-value μ^d can be used to define game theoretical measures of efficiency and vulnerability for directed networks, in which players have (economic or social) interests given by a TU-game. As mentioned in the previous subsection, given a directed communication situation (N, v, D), and $i \in N$, v_i gives the productivity of player i via the characteristic function whereas in v_{-i} player i is null. Then, in a connected directed network, $\sum_{i\in N} \mu_i^d(N, v_i, D)$ can be seen as the part of the total worth, $v(N)$, of the grand coalition that players can preserve given the existing communications in the network and thus,

$$\mathcal{E}(N, v, D) = \frac{\sum_{i\in N} \mu_i^d(N, v_i, D)}{v(N)} = \frac{\sum_{i\in N} \kappa_{v,i}^{d,C}(N, D)}{v(N)}$$

can be used as a measure of the efficiency of the directed network to preserve the interests of the players.

Similarly,

$$\mathcal{V}(N,v,D) = \frac{\sum_{i\in N} \mu_i^d(N, v_{-i}, D)}{v(N)} = \frac{\sum_{i\in N} \kappa_{v,i}^{d,B}(N,D)}{v(N)} = \frac{v^D(N)}{v(N)} - \mathcal{E}(N,v,D)$$

is a vulnerability network measure that represents the part of the total worth $v(N)$ corresponding to intermediation costs. These costs must be payed by coalitions to unproductive players that permit members in the coalition to be connected. The vulnerability of the network increases with these intermediation costs because the members of the coalitions are more dependent on other members.

We illustrate these measures with the following example.

Example 3.4.2. *Let (N,v,T) be a directed communication situation in which v is the messages game and T is an oriented tree (Harary & Sumner, 1980). Then, for each $i \in N$, $v_i = \sum_{j:j\neq i} u_{\{i,j\}}$. As T is an oriented tree for $j \neq i$, $\mathcal{MCP}(\{i,j\},N,D) = \{Q^{\{i,j\}}\}$ or is equal to the emptyset. Let us denote the geodesic distance between i and j by $d(i,j)$, i.e., the number of arcs in $Q^{\{i,j\}}$ if such arcs exist, and $d(i,j)=\infty$, otherwise. Then,*

$$\mu_i^d(N, v_i, T) = \sum_{j:j\neq i} Sh_i(N, u_{\{Q^{\{i,j\}}\}}) = \sum_{j:j\neq i} \frac{1}{d(i,j)+1}$$

As a consequence,

$$\mathcal{E}(N,v,T) = \frac{2}{n(n-1)} \sum_{i<j} \frac{1}{d(i,j)+1} = \frac{1}{n(n-1)} \sum_{i\neq j} \frac{1}{d(i,j)+1},$$

which is very similar to the expression for the average efficiency of Latora & Marchiori (2001). Indeed the only difference is that in our proposal the measure is defined in terms of the distance geodesic plus 1 whereas the average effciency is measure in terms of geodesic.

Example 3.4.3. *For the communication situation (N,v,D) of Example 4.1 (and*

Example 3.4.1) we have [4]

$$\mathcal{E}(N,v,D) = \frac{\sum_{i\in N} \mu_i^d(N,v_i,D)}{v(N)} = \frac{\sum_{i\in N} \kappa_{v,i}^{d,C}(N,D)}{v(N)} = \frac{\frac{17}{12}+1+1+\frac{17}{12}}{6} = \frac{29}{36}$$

$$\mathcal{V}(N,v,D) = \frac{\sum_{i\in N} \mu_i^d(N,v_{-i},D)}{v(N)} = \frac{\sum_{i\in N} \kappa_{v,i}^{d,B}(N,D)}{v(N)} = \frac{0+\frac{1}{12}+\frac{1}{12}+0}{6} = \frac{1}{36}.$$

Five of the six total possible messages can be sent using this (directed) network. Only the communication between 1 and 4 is vulnerable because of players 2 and 3 must be rewarded with $\frac{1}{12}$ *each. Then, the network efficiency is* $\frac{5}{6} - \frac{2\cdot\frac{1}{12}}{6} = \frac{29}{36} = 0.806$*, measuring the fraction of the total worth (6) that players can retain given the restrictions in the connection, and after paying the intermediation fees. The proportion corresponding to these intermediation fees can be seen as a vulnerability measure.*

3.5 The defined value versus other allocation rules

In this section, we illustrate our defined value μ^d and compare it with some other values from the literature. To obtain some numerical comparisons, we will use the examples in Khmelnitskaya et al. (2016) that are also used in Li & Shan (2020b).

Consider the directed communication situations (N,v,D_1), (N,v,D_2) and (N,v,D_3) with $N=\{1,2,3,4,5\}$, $v(S)=s^2$ for all $S\subseteq N$ and

$$D_1 = \{(1,2),(3,5),(4,5)\}$$

$$D_2 = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$$

$$D_3 = \{(1,2),(2,3),(3,1),(3,4),(4,1),(4,5)\},$$

see Fig 3.3.

[4]The contribution to the network efficiency (centrality) is equal for players 2 and 3 and it is also equal for 1 and 4 (they differ in the in-and out-communication centrality)

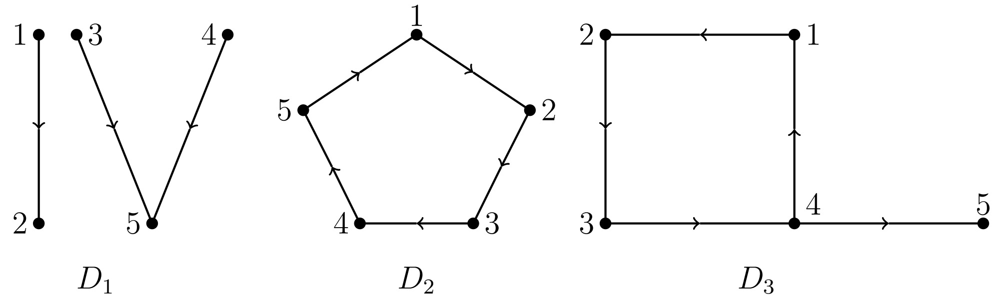

Figure 3.3: The examples of Khmelnitskaya et al. (2016)

As $v = \sum_{i=1}^{5} u_{\{i\}} + 2 \sum_{S \subseteq N, s=2} u_S$, each player can generate a dividend equal to 1 on his own, and each couple generates a dividend of 2 if both players can communicate.

The digraph restricted game defined in this paper, for these directed communication situations, is given by[5]

$$v^{D_1} = \sum_{i=1}^{5} u_{\{i\}} + 2u_{\{1,2\}} + 2u_{\{3,5\}} + 2u_{\{4,5\}},$$

$$\begin{aligned} v^{D_2} = \sum_{i=1}^{5} u_{\{i\}} &+ 2u_{\{1,2\}} + 2(u_{\{1,2,3\}} + u_{\{1,3,4,5\}} - u_{\{1,2,3,4,5\}}) + 2(u_{\{1,2,3,4\}} + u_{\{1,4,5\}} \\ &- u_{\{1,2,3,4,5\}}) + 2u_{\{1,5\}} + 2u_{\{2,3\}} + 2(u_{\{2,3,4\}} + u_{\{1,2,4,5\}} - u_{\{1,2,3,4,5\}}) + 2(u_{\{1,2,5\}} \\ &+ u_{\{2,3,4,5\}} - u_{\{1,2,3,4,5\}}) + 2u_{\{3,4\}} + 2(u_{\{3,4,5\}} + u_{\{1,2,3,5\}} - u_{\{1,2,3,4,5\}}) + 2u_{\{4,5\}}, \end{aligned}$$

and, finally

$$\begin{aligned} v^{D_3} = \sum_{i=1}^{5} u_{\{i\}} &+ 2u_{\{1,2\}} + 2u_{\{1,3\}} + 2u_{\{1,4\}} + 2u_{\{1,2,3,4,5\}} + 2u_{\{2,3\}} \\ &+ 2(u_{\{2,3,4\}} + u_{\{1,2,4\}} - u_{\{1,2,3,4\}}) + 2u_{\{2,3,4,5\}} + 2u_{\{3,4\}} + 2u_{\{3,4,5\}} + 2u_{\{4,5\}}. \end{aligned}$$

In game v^{D_3}, for example, the coalition $S = \{3,4,5\}$ obtains $v^{D_3}(S) = 9$ given that all the subcoalitions $\{3,4\}, \{3,5\}$ and $\{4,5\}$ can communicate (players 3 and 5 using 4 as intermediary). However, coalition $S = \{1,4,5\}$ only obtains

[5] Notice that these games are not zero-normalized.

$v^{D_3}(S) = 7$ as the connection between players 1 and 5 is unfeasible, since there is no path within S connecting these three players. Regarding our value, we have,

$$\mu^d(N, v, D_1) = Sh(N, v^{D_1}) = (2, 2, 2, 2, 3)$$

$$\mu^d(N, v, D_2) = Sh(N, v^{D_2}) = (5, 5, 5, 5, 5)$$

and

$$\mu^d(N, v, D_3) = Sh(N, v^{D_3}) = (\frac{137}{30}, \frac{142}{30}, \frac{172}{30}, \frac{192}{30}, \frac{107}{30})$$

Connection efficiency of μ^d in these examples says that in (N, v, D_1) player 1 and 2 can communicate and, then, they preserve the 4 units of $v(\{1, 2\})$. On the other hand, in the coalition $\{3, 4, 5\}$ (the other weak component) only the couples $\{3, 5\}$ and $\{4, 5\}$ can communicate, as there is no directed path connecting 3 and 4. Therefore, this coalition loses 2 units, preserving only 7 units of the initial 9.

In (N, v, D_2) and (N, v, D_3) all couples can communicate and thus, connection efficieny requires the usual efficiency in these two cases.

Next, we compare our value μ^d with some other values from the literature for this example.

3.5.1 Myerson value for undirected communication situation

Although the Myerson value is not defined for games with digraphs but for games with graphs, we can compare the obtained figures with the corresponding Myerson values for the (undirected) communication situations (N, v, γ_{D_1}), (N, v, γ_{D_2}) and (N, v, γ_{D_3}). We have

$$\mu(N, v, \gamma_{D_1}) = (2, 2, \frac{8}{3}, \frac{8}{3}, \frac{11}{3})$$

$$\mu(N, v, \gamma_{D_2}) = (5, 5, 5, 5, 5)$$

and

$$\mu(N, v, \gamma_{D_3}) = (\frac{296}{60}, \frac{266}{60}, \frac{296}{60}, \frac{406}{60}, \frac{236}{60})$$

As the Myerson value is component efficient, the only case in which players cannot obtain $v(N)$ is (N, v, γ_{D_1}): $\sum_{i \in N} \mu_i(N, v, \gamma_{D_1}) = v(\{1, 2\}) + v(\{3, 4, 5\}) <$

$v(N)$, as players in the coalition $\{1,2\}$ cannot communicate with players in $\{3,4,5\}$.

As mentioned, in (N, v, γ_{D_2}) and in (N, v, γ_{D_3}) all couples can communicate. Nevertheless, $\mu^d \neq \mu$. In our model this occurs as the communication induced by a digraph needs a higher level of intermediation, which generates a different allocation between players. As an example, in (N, γ_{D_3}) the connection between players 1 and 5 only needs the intermediation of player 4, whereas in (N, D_3) it also needs the collaboration of 2 and 3. The corresponding dividend (equal to 2) is divided among $1, 4$ and 5 in the case of the Myerson value but among players $1, 2, 3, 4$ and 5 in our proposal. The difference between both allocation rules is due to the intermediation costs.

3.5.2 Permission values

In Gilles et al. (1992) and Gilles & Owen (1994), the digraph in the model (N, v, D) is interpreted as a permission structure that restricts the cooperation possibilities because some players need permission from other players before they are allowed to cooperate in a coalition. In the conjunctive approach, every player i needs permission from all its predecessors being the players j such that $(j, i) \in D$.[6] This yields the following conjunctive restricted games v_c^D for (N, v, D_1), (N, v, D_2) and (N, v, D_3):

$$\begin{aligned} v_c^{D_1} &= u_{\{1\}} + u_{\{1,2\}} + u_{\{3\}} + u_{\{4\}} + u_{\{3,4,5\}} \\ &+2u_{\{1,2\}} + 2u_{\{1,3\}} + 2u_{\{1,4\}} + 2u_{\{1,3,4,5\}} + 2u_{\{1,2,3\}} \\ &+2u_{\{1,2,4\}} + 2u_{\{1,2,3,4,5\}} + 2u_{\{3,4\}} + 4u_{\{3,4,5\}}, \\ v_c^{D_2} &= \left(5 + 2\binom{5}{2}\right) u_{\{1,2,3,4,5\}} = 25u_{\{1,2,3,4,5\}}, \end{aligned}$$

and,

$$v_c^{D_3} = 4u_{\{1,2,3,4\}} + u_{\{1,2,3,4,5\}} + 2\binom{4}{2} u_{\{1,2,3,4\}} + 4 \cdot 2u_{\{1,2,3,4,5\}}$$

[6] Although games with a permission structure are defined only for asymmetric digraphs, the definition can directly be extended to all digraphs. In that case, when $(i, j), (j, i) \in D$ then players i and j veto each other in the sense that one cannot cooperate in a coalition without the other.

$$= 16u_{\{1,2,3,4\}} + 9u_{\{1,2,3,4,5\}}.$$

This results in the following conjunctive permission value payoff vectors:

$$\varphi^c(N,v,D^1) = \frac{1}{30}(202,97,187,187,77),\ \varphi^c(N,v,D^2) = (5,5,5,5,5),$$

$$\text{and } \varphi^c(N,v,D^3) = \frac{1}{5}(29,29,29,29,9).$$

Notice that, digraph (N,D_2) having a Hamiltonian cycle, and v being a symmetric game, results in an equal payoff $\frac{v(N)}{n}$ for all players according to the conjunctive permission value, as is also the case in our value μ^d. The conjunctive permission value yields this equal allocation for any game, while our value might assign different payoffs to the players in case the game is not symmetric.

On the other hand, in the disjunctive approach for acyclic permission structures, every player needs permission from at least one of its predecessors (if it has any). Since D^1 is an acyclic digraph, we compute the disjunctive restricted game and disjunctive permission value for this digraph, yielding restricted game

$$v_d^{D_1} = u_{\{1\}} + u_{\{1,2\}} + u_{\{3\}} + u_{\{4\}} + (u_{\{3,5\}} + u_{\{4,5\}} - u_{\{3,4,5\}})$$

$$+2u_{\{1,2\}} + 2u_{\{1,3\}} + 2u_{\{1,4\}} + 2(u_{\{1,3,5\}} + u_{\{1,4,5\}} - u_{\{1,3,4,5\}})$$

$$+2u_{\{1,2,3\}} + 2u_{\{1,2,4\}} + 2(u_{\{1,2,3,5\}} + u_{\{1,2,4,5\}} - u_{\{1,2,3,4,5\}})$$

$$+2u_{\{3,4\}} + 2u_{\{3,5\}} + 2u_{\{4,5\}},$$

giving the disjunctive permission value payoff vector:

$$\varphi^d(N,v,D^1) = \frac{1}{30}(218,103,153,153,123).$$

In the permission approach to games with a digraph, cooperation is restricted by players needing permission from other players before being allowed to cooperate. Since the grand coalition contains all players that can generate worth as well as give permission, the two permission values are efficient, meaning that the total unrestricted worth $v(N)$ of the grand coalition will be allocated. How this is allocated over the players depends on the cooperation restrictions. The disjunctive permission value satisfies a weaker fairness axiom, where we only delete arcs such that the head/successor has at least one more ingoing arc, i.e. the payoff of the two players i, j on an oriented arc (i, j) change by the same amount if the deleted arc (i, j) satisfies $\#\{h \in N \mid (h, i) \in D\} \geq 2$.

The conjunctive permission value satisfies the alternative fairness axiom where the deletion of an arc has the same effect on the payoff of the head/successor of the arc and any other predecessor of this head. Notice that this also fits with the difference in interpretation of conjuctive and disjunctive permission. Every arc (i, j) in the conjunctive approach restricts the cooperation possibilities of the head j. But, in case a player j is the head of at least one arc, adding more arcs with j as head increases the disjunctive cooperation possibilites of j, as is also the case in the directed communication situations considered in this paper. This is also reflected in the weak fairness that these values have in common.

3.5.3 The value of Li & Shan (2020b)

Comparing our value with the Myerson value for digraphs introduced in Li & Shan (2020b), the more important difference is the type of efficiency used (which depends on the type of connection that is considered) as both rules satisfy fairness and balanced contributions. Their strong component efficiency is a property that assumes a high connection requirement for generating worth which, usually, implies a low level of cooperation. As a consequence, a lot of payoff of the original coalitions can be lost. Specifically this occurs in directed communication situations (N, v, D_1) and in (N, v, D_3) where Li & Shan (2020b)'s Myerson value for digraphs assigns, respectively, $(1, 1, 1, 1, 1)$ and $(9/2, 9/2, 9/2, 5/2, 1)$ (see Li & Shan (2020b)). From the total initial worth of 25, the players can only retain a payoff of 5 in the first case and a payoff of 17 in the second case. For (N, v, D_2), our value coincides with Li & Shan (2020b)'s Myerson value for digraphs. This follows since the presence of a Hamiltonian cycle and the game being symmetric results in payoffs $v(N)/n$ for each player in both values, as is also the case in the conjunctive permission value above.[7]

If the game is not symmetrical, the allocation according to our rule and Li &

[7] The restricted game introduced by Li & Shan (2020b) in the example of this section is $\sum_{i=1}^{n} u_{\{i\}} + (v(N) - n)u_N$ in such a case. In our proposal elementary considerations of symmetry permit us to conclude that players must be equally rewarded.

Shan (2020b)'s Myerson value can differ, as is shown in the following example.

Example 3.5.1. *Consider* (N, v, D) *with* $N = \{1, 2, 3\}$ $v = u_{\{1,2\}} + u_{1,3} - u_{\{1,2,3\}}$ *and* $D = \{(1, 2), (2, 3), (3, 1)\}$ *as in Fig 3.4.*

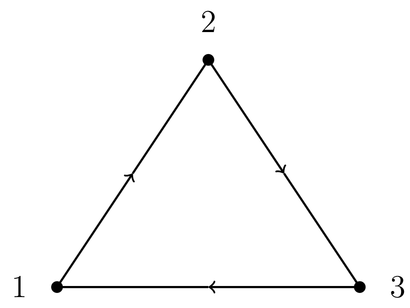

Figure 3.4

Then Li & Shan (2020b)'s Myerson value for digraphs gives $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *whereas* $\mu^d(N, v, D) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

3.6 Conclusions

In this chapter we have introduced a new concept of directed communication and a related connectedness in directed graphs, and applied this to model certain cooperation restrictions in cooperative games. This notion of connectedness is based on directed paths and fits well with the modelling of supply chain management, attribution models and vaccination policy, for example. And thus, we have assumed that a coalition of players in a game can only cooperate if these players form a directed path in a directed communication graph. We have defined a restricted game following the same approach as Myerson for undirected communication situations, and considered the allocation rule that applies the Shapley value to this restricted game. This value, the DC-value, has been characterized using a new property, the connection efficiency, and extended versions of the well-known fairness and balanced contributions axioms. Moreover, using the new notion of connectedness, we apply this allocation rule to define network centrality, efficiency and vulnerability measures for directed networks.

Chapter 4

A family of position values for directed communication situations

This chapter introduces, following the idea of connectivity in directed communication situation defined in the previous chapter, a new game called the arc game for directed communication situations. Then, a family of position values are defined for this game. They depend on a parameter α, which is related to the different assignment to players-head or tail in an arc. They are characterized by connection efficiency and different balanced contributions properties introduced for this new setting.

4.1 An arc game for directed communication situations

As mentioned in Chapter 2, Meessen (1988) and Borm et al. (1992) introduced a link game for communication situations, with a zero-normalized game, in which the links are the players and the worth of each coalition of links is determined by the worth of the coalition of all players in the digraph-restricted game corresponding to this set of links. Following their ideas, we define an arc game for directed communication situations, with a zero-normalized game, based on the restricted game v^D defined in the previous chapter (see Definition 3.1.1).

Definition 4.1.1. *Given* (N, v, D) *in* $\mathcal{DCS}_0$*, the arc game is defined as the TU game* (D, r_D^v) *with characteristic function given by:*

$$r_D^v(L) = v^L(N) = v^L(\{L\}) = \sum_{i=1}^{r(\{L\})} v(\{P_i^{\{L\}}\}) - \sum_{i=1}^{r(\{L\})-1} \sum_{j=i}^{r(\{L\})} v(\{P_i^{\{L\}}\} \cap \{P_j^{\{L\}}\})$$

$$+ \sum_{i=1}^{r(\{L\})-2} \sum_{j=i}^{r(\{L\})-1} \sum_{k=j+1}^{r(\{L\})} v(\{P_i^{\{L\}}\} \cap \{P_j^{\{L\}}\} \cap \{P_k^{\{L\}}\})$$

$$+ \cdots + (-1)^{r(\{L\})-1} v(\{P_1^{\{L\}}\} \cap \{P_2^{\{L\}}\} \cap \cdots \cap \{P_{r(\{L\})}^{\{L\}}\}) \text{ , for all } L \subseteq D,$$

with $\mathcal{P}(\{L\}, L) = \{P_1^{\{L\}} \cdots P_{r(\{L\})}^{\{L\}}\}$ *being the set of all maximal paths in the digraph* $(\{L\}, L)$*.*

To clarify the previous definition[1], let us consider the next example.

Example 4.1.1. *Consider* $(N, v, D) \in \mathcal{DCS}_0$ *with* $N = \{1, 2, 3, 4\}$*,* $D = \{a = (1, 2), b = (2, 4), c = (1, 3), d = (3, 4)\}$ *as in Example 2.7.2, and* (N, v) *a zero-normalized game.*

The characteristic function r_D^v *is given by:*

$$r_D^v(L) = \begin{cases} v(\{1,2\}) + v(\{1,3\}) - v(\{1\}), & \text{if } L = \{a, c\}, \\ v(\{1,2\}) + v(\{3,4\}), & \text{if } L = \{a, d\}, \\ v(\{2,4\}) + v(\{1,3\}), & \text{if } L = \{b, c\}, \\ v(\{2,4\}) + v(\{3,4\}) - v(\{4\}), & \text{if } L = \{b, d\}, \\ v(\{1,2,4\}) + v(\{1,3\}) - v(\{1\}), & \text{if } L = \{a, b, c\}, \\ v(\{1,2,4\}) + v(\{3,4\}) - v(\{4\}), & \text{if } L = \{a, b, d\}, \\ v(\{1,2\}) + v(\{1,3,4\}) - v(\{1\}), & \text{if } L = \{a, c, d\}, \\ v(\{2,4\}) + v(\{1,3,4\}) - v(\{4\}), & \text{if } L = \{b, c, d\}, \\ v(\{1,2,4\}) + v(\{1,3,4\}) - v(\{1\}) - v(\{4\}), & \text{if } L = \{a, b, c, d\}, \\ v(\{L\}), & \text{otherwise .} \end{cases}$$

[1] The need for the original game to be zero-normalized is clear, otherwise $r_D^v(\emptyset) = v^{\emptyset}(N) = \sum_{i \in N} v(\{i\})$ could be non-zero.

In the following proposition, we give an expression for the arc game associated with a zero-normalized unanimity game.

Proposition 4.1.1. *Given* $(N, u_S, D) \in \mathcal{DCS}_0^N$ *with* $S \subseteq N$, $s \geq 2$, *the characteristic function* $r_D^{u_S}$ *is given by*

$$r_D^{u_S} = \mathbf{1} - \prod_{i=1}^{t(S)} (\mathbf{1} - u_{Q_i^S}) \text{ if } \mathcal{MCP}(S, N, D) = \{Q_1^S \cdots Q_{t(S)}^S\} \neq \emptyset,$$

and $r_D^{u_S} \equiv \mathbf{0}$, *the null game, otherwise.*

We denote by $(N, \mathbf{1}) \in G^N$ *the game with characteristic function given by* $\mathbf{1}(S) = 1$ *for all* $\emptyset \neq S \subseteq N$ *and* $\mathbf{1}(\emptyset) = 0$. *Similarly, we denote by* $(N, \mathbf{0}) \in G^N$ *the game with characteristic function given by* $\mathbf{0}(S) = 0$ *for all* $S \subseteq N$.

Proof: The result for $L = \emptyset$ is trivial. Consider $\emptyset \neq L \subseteq D$. Using Definition 4.1.1 of the arc game,

$$r_D^{u_S}(L) = u_S^L(\{L\}) = \sum_{i=1}^{r(\{L\})} u_S(\{P_i^{\{L\}}\}) - \sum_{i=1}^{r(\{L\})-1} \sum_{j=i}^{r(\{L\})} u_S(\{P_i^{\{L\}}\} \cap \{P_j^{\{L\}}\})$$

$$+ \sum_{i=1}^{r(\{L\})-2} \sum_{j=i}^{r(\{L\})-1} \sum_{k=j+1}^{r(\{L\})} u_S(\{P_i^{\{L\}}\} \cap \{P_j^{\{L\}}\} \cap \{P_k^{\{L\}}\})$$

$$+ \cdots + (-1)^{r(\{L\})-1} u_S(\{P_1^{\{L\}}\} \cap \{P_2^{\{L\}}\} \cap \cdots \cap \{P_{r(\{L\})}^{\{L\}}\}),$$

with $\mathcal{P}(\{L\}, L) = \{P_1^{\{L\}} \cdots P_{r(\{L\})}^{\{L\}}\}$ the set of all maximal paths in the digraph $(\{L\}, L)$. Let $r'(\{L\}) \leq r(\{L\})$ be the cardinality of the subset of $\mathcal{P}(\{L\}, L)$ with node set $\{L\}$ containing S. If $r'(\{L\}) = 0$, then $r_D^{u_S}(L)$ is clearly the null game, $\mathcal{MCP}(S, N, D) = \emptyset$, and thus the statement holds in this case.

If $r'(\{L\}) \geq 1$, then we have

$$u_S^L(\{L\}) = \binom{r'(\{L\})}{1} - \binom{r'(\{L\})}{2} + \cdots + (-1)^{r'(\{L\})-1} \binom{r'(\{L\})}{r'(\{L\})}$$

$$= -\binom{r'(\{L\})}{0} + \binom{r'(\{L\})}{1} - \binom{r'(\{L\}))}{2} + \cdots + (-1)^{r'(\{L\})-1} \binom{r'(\{L\})}{r'(\{L\})} + \binom{r'(\{L\})}{0}$$

$$= -\left[\binom{r'(\{L\})}{0} - \binom{r'(\{L\})}{1} + \binom{r'(\{L\}))}{2} + \cdots + (-1)^{r'(\{L\})} \binom{r'(\{L\})}{r'(\{L\})} \right]$$

$$+\binom{r'(\{L\})}{0} = -(1-1)^{r'(\{L\})} + \binom{r'(\{L\}))}{0} = 1.$$

On the other hand, for $\emptyset \neq L \subseteq D$, $\prod_{i=1}^{t(S)}(\mathbf{1}-u_{Q_i}^S)(L) = 0$, and thus $[\mathbf{1}-\prod_{i=1}^{t(S)}(\mathbf{1}-u_{Q_i}^S)](L) = 1$ if there is at least one path contained in L whose node set contains S. Thus, the result is proven. □

4.2 A family of position values for directed communication situations

In this section, we introduce a family of allocation rules for directed communication situations based on the idea of the position value.

Definition 4.2.1. *An allocation rule on $\mathcal{DCS}_0^N$ is a function $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ that assigns to each $i \in N$ in a directed communication situation $(N, v, D) \in \mathcal{DCS}_0^N$ his reward $\varphi_i(N, v, D)$.*

Next, we define a class of allocation rules that is based on the idea behind the position value, using the arc game associated with a directed communication situation; see Definition 4.1.1. Whereas the position value for undirected communication situations shares the Shapley value payoff of every link in the link game equally between the two players on the link, in the case of directed communication, it is not obvious why the Shapley value payoff of every arc should be shared equally between the two nodes on the arc. The head and tail of an arc are clearly in an asymmetric position, and therefore discrimination in the payoff allocation seems plausible. In the definition below, we allow any split of the payoff of an edge between the head and the tail, but require a uniform sharing across all arcs. Recall from the preliminaries that $D_i^O = \{(i,j) \mid (i,j) \in D\}$ and $D_i^I = \{(j,i) \mid (j,i) \in D\}$.

Definition 4.2.2. *Let $\alpha \in [0,1]$. The value π^α is defined, for every $(N, v, D) \in \mathcal{DCS}_0^N$, as:*

$$\pi_i^\alpha(N,v,D) = \alpha \sum_{a \in D_i^I} Sh_a(D, r_D^v) + (1-\alpha) \sum_{a \in D_i^O} Sh_a(D, r_D^v) \text{ for } i = 1, \ldots, n.$$

The family of values $\{\pi^\alpha \mid \alpha \in [0,1]\}$ is called the family of position values.

Notice that, in the definition above, we can give the arc payoff fully to the head (if $\alpha = 1$), fully to the tail (if $\alpha = 0$) or allow an equal sharing between head and tail (if $\alpha = \frac{1}{2}$). However, we use the same means of splitting on every arc. We illustrate this allocation rule with an example.

Example 4.2.1. *Consider $(N, v, D) \in \mathcal{DCS}_0^N$ with $N = \{1,2,3,4\}$ and $D = \{a = (1,2), b = (2,4), c = (1,3), d = (3,4)\}$ and v the three games, as in the Example 4.1.1.*

(a) Let v be the messages game described in (1). As

$$v_1 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}},$$

we have, using Proposition 4.1.1, that

$$r_D^{v_1} = (u_{\{a\}}) + (u_{\{c\}}) + (u_{\{a,b\}} + u_{\{c,d\}} - u_{\{a,b,c,d\}}) + (u_{\{b\}}) + (u_{\{d\}})$$
$$= u_{\{a\}} + u_{\{b\}} + u_{\{c\}} + u_{\{d\}} + u_{\{a,b\}} + u_{\{c,d\}} - u_{\{a,b,c,d\}}.$$

Notice that $r_D^{u_{\{2,3\}}} \equiv \mathbf{0}$ as there is no path connecting 2 and 3.

Thus, $Sh(D, r_D^{v_1}) = (\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4})$ and

$$\pi_1^\alpha(N, v_1, D) = (1-\alpha)(\frac{5}{4} + \frac{5}{4}) = \frac{5(1-\alpha)}{2}$$
$$\pi_2^\alpha(N, v_1, D) = \alpha\frac{5}{4} + (1-\alpha)\frac{5}{4} = \frac{5}{4}$$
$$\pi_3^\alpha(N, v_1, D) = \alpha\frac{5}{4} + (1-\alpha)\frac{5}{4} = \frac{5}{4}$$
$$\pi_4^\alpha(N, v_1, D) = \alpha(\frac{5}{4} + \frac{5}{4}) = \frac{5\alpha}{2}.$$

We emphasize the following intuitive behavior of these allocation rules in this example:

(i) Given that all players are symmetrical in the game, and that players 2 and 3 are also symmetrical in the digraph, it is not surprising that the payoff is equal for both of them and it does not depend on α because the payoff lost (increased) being the tail is compensated by the payoff increased (lost) being the head.

(ii) *The payoff for 1 is greater than the payoff for 4 when* $\alpha < \frac{1}{2}$*, illustrating that, in this case, the tail is better paid. Reciprocally, for* $\alpha > \frac{1}{2}$.

(iii) *The sum of the payoffs is 5 as only 5 of the 6 bilateral connections are feasible given the digraph. Notice that connection of 2 and 3 is not possible.*

(b) *Consider the overhead game with characteristic function given in (3). Remember the expression of game in terms of the unanimity basis given by*

$$v_2 = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}}$$

$$-u_{\{1,2,3\}} - u_{\{1,2,4\}} - u_{\{1,3,4\}} - u_{\{2,3,4\}} + u_{\{1,2,3,4\}}$$

and using the Proposition 4.1.1, the arc game is

$$r_D^{v_2} = u_{\{a\}} + u_{\{c\}} + (u_{\{a,b\}} + u_{\{c,d\}} - u_{\{a,b,c,d\}}) + u_{\{b\}} + u_{\{d\}} - u_{\{a,b\}} - u_{\{c,d\}}$$

$$= u_{\{a\}} + u_{\{b\}} + u_{\{c\}} + u_{\{d\}} - u_{\{a,b,c,d\}},$$

and $Sh(N, r_D^{v_2}) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$

For this game,

$$\pi_1^\alpha(N, v_2, D) = (1-\alpha)\left(\frac{3}{4} + \frac{3}{4}\right) = \frac{3(1-\alpha)}{2}$$

$$\pi_2^\alpha(N, v_2, D) = \alpha\frac{3}{4} + (1-\alpha)\frac{3}{4} = \frac{3}{4}$$

$$\pi_3^\alpha(N, v_2, D) = \alpha\frac{3}{4} + (1-\alpha)\frac{3}{4} = \frac{3}{4}$$

$$\pi_4^\alpha(N, v_2, D) = \alpha\left(\frac{3}{4} + \frac{3}{4}\right) = \frac{3\alpha}{2}$$

(c) *Consider the conference game with characteristic function given in (4).*

We have, using Proposition 4.1.1, that

$$r_D^{v_3} = u_{\{a\}} + u_{\{c\}} + (u_{\{a,b\}} + u_{\{c,d\}} - u_{\{a,b,c,d\}}) + u_{\{b\}} + u_{\{d\}} + u_{\{a,b\}} + u_{\{c,d\}}$$

$$= u_{\{a\}} + u_{\{b\}} + u_{\{c\}} + u_{\{d\}} + 2u_{\{a,b\}} + 2u_{\{c,d\}} - u_{\{a,b,c,d\}},$$

and $Sh(D, r_D^{v_3}) = (\frac{7}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4})$*. In this case,*

$$\pi_1^\alpha(N, v_3, D) = (1-\alpha)\left(\frac{7}{4} + \frac{7}{4}\right) = \frac{7(1-\alpha)}{2}$$

$$\pi_2^{\alpha}(N, v_3, D) = \alpha\frac{7}{4} + (1-\alpha)\frac{7}{4} = \frac{7}{4}$$
$$\pi_3^{\alpha}(N, v_3, D) = \alpha\frac{7}{4} + (1-\alpha)\frac{7}{4} = \frac{7}{4}$$
$$\pi_4^{\alpha}(N, v_3, D) = \alpha\left(\frac{7}{4} + \frac{7}{4}\right) = \frac{7\alpha}{2}.$$

In the following, we obtain some useful results relating the particular (extreme) values π^0 and π^1 to the relative out-degree and the relative in-degree, respectively.

Proposition 4.2.1. *Let $(N, v, D) \in \mathcal{DCS}_0^N$. Then, for $i \in N$,*

$$\pi_i^0(N, v, D) = \sum_{\emptyset \neq A \subseteq D} \Delta_{r_D^v}(A) rd_i^O(A)$$

where, for each $\emptyset \neq A \subseteq D$, $\Delta_{r_D^v}(A)$ is the Harsanyi dividend of the coalition (of directed edges) A in the arc game r_D^v, and $rd_i^O(A)$ is the relative out-degree of node i in the directed graph (N, A).

Proof: The game (N, r_D^v) admits the following expression in terms of the Harsanyi dividends[2]:

$$r_D^v = \sum_{A \subseteq D} \Delta_{r_D^v}(A) u_A.$$

Then, using the definition of $\pi_i^0(N, v, D)$ (see Definition 4.2.2), for $(N, v, D) \in \mathcal{DCS}_0^N$ and $i \in N$, we have

$$\pi_i^0(N, v, D) = \sum_{a \in D_i^O} Sh_a(D, r_D^v) = \sum_{a \in D_i^O} Sh_a\left[D, \sum_{A \subseteq D} \Delta_{r_D^v}(A) u_A\right]$$
$$= \sum_{a \in D_i^O} \sum_{A \subseteq D} \Delta_{r_D^v}(A) Sh_a(D, u_A), \tag{18}$$

the last equality holding because of linearity of the Shapley value.

Rearranging the terms in (18), we obtain

$$\sum_{a \in D_i^O} \sum_{A \subseteq D} \Delta_{r_D^v}(A) Sh_a(D, u_A) = \sum_{A \subseteq D} \Delta_{r_D^v}(A) \sum_{a \in D_i^O} Sh_a(D, u_A)$$

[2] In the following, in order to lighten the notation, we will exclude the inequality corresponding to the empty set, since dividends are only defined for non-empty sets.

$$= \sum_{A \subseteq D} \Delta_{r_D^v}(A) \sum_{a \in D_i^O \cap A} Sh_a(D, u_A) = \sum_{A \subseteq D} \Delta_{r_D^v}(A) \frac{|D_i^O \cap A|}{|A|}$$

$$= \sum_{A \subseteq D} \Delta_{r_D^v}(A) \frac{d_i^O(A)}{|A|} = \sum_{A \subseteq D} \Delta_{r_D^v}(A) \frac{d_i^O(A)}{d^O(A)} = \sum_{A \subseteq D} \Delta_{r_D^v}(A) rd_i^O(A),$$

where the second equality follows since $a \in D \setminus A$ implies that $Sh_a(D, u_A) = 0$, the fifth equality holding as the out-degree in (N, A) equals the number of edges in A (each edge has one tail and one head), and the last equality holding as $\frac{d_i^O(A)}{d^O(A)}$, by definition, is the relative out-degree of node i in the digraph (N, A), $rd_i^O(A)$. The result is proven. □

The proof of the following proposition follows similar lines as the previous one and is therefore omitted.

Proposition 4.2.2. *Let $(N, v, D) \in \mathcal{DCS}_0^N$ and $i \in N$. Then,*

$$\pi_i^1(N, v, D) = \sum_{A \subseteq D} \Delta_{r_D^v}(A) rd_i^I(A)$$

where, for each $A \subseteq D$, $\Delta_{r_D^v}(A)$ is the Harsanyi dividend of $A \subseteq D$, and $rd_i^I(A)$ is the relative in-degree of i in (N, A).

Since, by definition, for all $\alpha \in [0, 1]$

$$\pi_i^\alpha(N, v, D) = \alpha\ \pi_i^1(N, v, D) + (1 - \alpha)\ \pi_i^0(N, v, D),$$

(i.e., π^α is a convex combination of π^0 and π^1) the following corollary is a direct consequence of the previous propositions.

Corollary 4.2.1. *For each $\alpha \in (0, 1)$, $(N, v, D) \in \mathcal{DCS}_0^N$ and $i \in N$,*

$$\pi_i^\alpha(N, v, D) = \sum_{A \subseteq D} \Delta_{r_D^v}(A)[\alpha\ rd_i^I(A) + (1 - \alpha)\ rd_i^O(A)].$$

We illustrate this result with an example.

Example 4.2.2. *Consider $\pi_1^\alpha(N, v, D)$ in Example 4.2.1.*

(a) If v is the messages game, then, by Corollary 4.2.1, we have:

$$\pi_1^\alpha(N, v_1, D) = \Delta_{r_D^{v_1}}(\{a\}) \left[\alpha\ rd_1^I(\{a\}) + (1 - \alpha)\ rd_1^O(\{a\})\right]$$

$$+\Delta_{r_D^{v_1}}(\{b\})\left[\alpha\ rd_1^I(\{b\})+(1-\alpha)\ rd_1^O(\{b\})\right]$$
$$+\Delta_{r_D^{v_1}}(\{c\})\left[\alpha\ rd_1^I(\{c\})+(1-\alpha)\ rd_1^O(\{c\})\right]$$
$$+\Delta_{r_D^{v_1}}(\{d\})\left[\alpha\ rd_1^I(\{d\})+(1-\alpha)\ rd_1^O(\{d\})\right]$$
$$+\Delta_{r_D^{v_1}}(\{a,b\})\left[\alpha\ rd_1^I(\{a,b\})+(1-\alpha)\ rd_1^O(\{a,b\})\right]$$
$$+\Delta_{r_D^{v_1}}(\{c,d\})\left[\alpha\ rd_1^I(\{c,d\})+(1-\alpha)\ rd_1^O(\{c,d\})\right]$$
$$+\Delta_{r_D^{v_1}}(\{a,b,c,d\})\left[\alpha\ rd_1^I(\{a,b,c,d\})+\ (1-\alpha)\ rd_1^O(\{a,b,c,d\})\right]$$
$$=(1-\alpha)+0+(1-\alpha)+0+\frac{(1-\alpha)}{2}+\frac{(1-\alpha)}{2}-\frac{2(1-\alpha)}{4}=\frac{5(1-\alpha)}{2}.$$

Similarly,

$$\pi_2^\alpha(N,v_1,D)=\alpha+(1-\alpha)+\left(\frac{\alpha}{2}+\frac{(1-\alpha)}{2}\right)-\left(\frac{\alpha}{4}+\frac{(1-\alpha)}{4}\right)=\frac{5}{4}$$

$$\pi_3^\alpha(N,v_1,D)=\alpha+(1-\alpha)+\left(\frac{\alpha}{2}+\frac{(1-\alpha)}{2}\right)-\left(\frac{\alpha}{4}+\frac{(1-\alpha)}{4}\right)=\frac{5}{4}$$

$$\pi_4^\alpha(N,v_1,D)=0+\alpha+0+\alpha+\frac{\alpha}{2}+\frac{\alpha}{2}-\frac{2\alpha}{4}=\frac{5\alpha}{2}.$$

Notice that these outcomes coincide with those in Example 4.2.1 (a).

(b) If v is the overhead game, y Corollary 4.2.1, we have:

$$\pi_1^\alpha(N,v_2,D)=(1-\alpha)+0+(1-\alpha)+0-\frac{2(1-\alpha)}{4}=\frac{3(1-\alpha)}{2}$$

$$\pi_2^\alpha(N,v_2,D)=\alpha+(1-\alpha)+0+0-\left(\frac{\alpha}{4}+\frac{(1-\alpha)}{4}\right)=\frac{3}{4}$$

$$\pi_3^\alpha(N,v_2,D)=0+0+\alpha+(1-\alpha)+0-\left(\frac{\alpha}{4}+\frac{(1-\alpha)}{=}\frac{3}{4}\right.$$

$$\pi_4^\alpha(N,v_2,D)=0+\alpha+0+\alpha-\frac{2\alpha}{4}=\frac{3\alpha}{2}$$

Notice that these outcomes coincide with those in Example 4.2.1 (b).

(c) If v is the conference game, then, by Corollary 4.2.1, we have:

$$\pi_1^\alpha(N,v_3,D)=(1-\alpha)+0+(1-\alpha)+0+\frac{2(1-\alpha)}{2}+\frac{2(1-\alpha)}{2}-\frac{2(1-\alpha)}{4}=\frac{7(1-\alpha)}{2}$$

$$\pi_2^\alpha(N,v_3,D)=\alpha+(1-\alpha)+0+0+\frac{2(\alpha+(1-\alpha))}{2}+0-\frac{\alpha+(1-\alpha)}{4}=\frac{7}{4}$$

$$\pi_3^\alpha(N, v_3, D) = 0 + 0 + \alpha + (1-\alpha) + 0 + \frac{2(\alpha + (1-\alpha))}{2} - \frac{\alpha + (1-\alpha)}{4} = \frac{7}{4}$$
$$\pi_4^\alpha(N, v_3, D) = 0 + \alpha + 0 + \alpha + \frac{2\alpha}{2} + \frac{2\alpha}{2} - \frac{2\alpha}{4} = \frac{7\alpha}{2}$$

Notice that these outcomes coincide with those in Example 4.2.1 (c).

4.3 Characterization of the defined position values

In this section, we characterize the family of values defined in Definition 4.2.2 in terms of two properties, connection efficiency and α-balanced arc contributions, which are defined as follows.

Given a digraph $(N, D) \in \mathcal{D}^N$, its underlying (undirected) graph $(N, \gamma_D) \in \Gamma^N$ is obtained by replacing all directed edges with corresponding undirected links, i.e., $\gamma_D = \{\{i, j\} \mid (i, j) \in D\}$.

Definition 4.3.1. *An allocation rule $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ satisfies connection efficiency if, for all $(N, v, D) \in \mathcal{DCS}_0^N$ and all $C \in N/\gamma_D$,*

$$\sum_{i \in C} \varphi_i(N, v, D) = v^D(C).$$

As mentioned in Chapter 3, the use of connection efficiency is motivated in situations where worth is generated by (maximal) paths. This is a very useful concept of efficiency in, for example, marketing attribution or supply chains. In these applications, worth is generated when, through a sequence of advertisements—respectively, a sequence of intermediary retailers and other agents (such as transportation companies) on the supply chain—a conversion takes place when a consumer (sink of the path) buys a product that is produced by a producer (source of the path). Another situation where this occurs is in communications (e.g., sending messages), when the only aspect that matters is whether a message from a sender reaches the intended receiver, possibly through a chain of intermediaries. In recent years, this occurred, for example, in COVID-19 vaccination policy, where governments tried to reach to people who lived somewhat isolated through intermediaries such as doctors and social workers.

Proposition 4.3.1. *Let* $\alpha \in [0,1]$*. The allocation rule* $\pi^\alpha : \mathcal{DCS}_0^N \to \mathbb{R}^n$ *satisfies connection efficiency.*

Proof: Let $(N,v,D) \in \mathcal{DCS}_0^N$ and $C \in N/\gamma_D$. Then, using Corollary 4.2.1,

$$\begin{aligned}
\sum_{i\in C} \pi_i^\alpha(N,v,D) &= \sum_{i\in C}\sum_{A\subseteq D} \Delta_{r_D^v}(A)\left[\alpha\; rd_i^I(A) + (1-\alpha)\; rd_i^O(A)\right] \\
&= \sum_{i\in C}\sum_{A\subseteq D|_C} \Delta_{r_D^v}(A)\left[\alpha\; rd_i^I(A) + (1-\alpha)\; rd_i^O(A)\right] \\
&= \sum_{A\subseteq D|_C} \Delta_{r_D^v}(A) \sum_{i\in C}\left[\alpha\; rd_i^I(A) + (1-\alpha)\; rd_i^O(A)\right] \\
&= \sum_{A\subseteq D|_C} \Delta_{r_D^v}(A)\left[\alpha \sum_{i\in C} rd_i^I(A) + (1-\alpha)\sum_{i\in C} rd_i^O(A)\right] \\
&= \sum_{A\subseteq D|_C} \Delta_{r_D^v}(A)\left[\alpha + (1-\alpha)\right] = \sum_{A\subseteq D|_C} \Delta_{r_D^v}(A) = r_D^v(D|_C)
\end{aligned}$$

where $D|_C = \{(k,l) \in D \mid k,l \in C\}$, the second equality holding because the dividend of a coalition in the game (D, r_D^v) is zero if the coalition contains arcs from different components, and the fifth equality holding because the sum of the relative (in- and out-) degrees in a set of arcs that all belong to the same component is equal to one.

Finally, $r_D^v(D|_C) = v^D(C)$ by the definition of r_D^v, and, thus, the result is proven. □

Before defining an extension of the balanced link contribution property for directed communication situations, we first define two special cases focusing on the nodes' out-arcs, respectively, the in-arcs. First, balanced out-arc contributions requires that the sum of the effects of breaking each outgoing arc of a player on the payoff of another player is the same as the effect the other way around.

Definition 4.3.2. *An allocation rule* $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ *satisfies balanced out-arc contributions if, for all* $(N,v,D) \in \mathcal{DCS}_0^N$ *and all* $i,j \in N$*,*

$$\sum_{a\in D_j^O} [\varphi_i(N,v,D) - \varphi_i(N,v,D\backslash\{a\})] = \sum_{a\in D_i^O} [\varphi_j(N,v,D) - \varphi_j(N,v,D\backslash\{a\})].$$

The extreme position value where $\alpha = 0$ satisfies balanced out-arc contributions.

Proposition 4.3.2. *The allocation rule $\pi^0 : \mathcal{DCS}_0^N \to \mathbb{R}^n$ satisfies balanced out-arc contributions.*

Proof: Let $(N, v, D) \in \mathcal{DCS}_0^N$ and $i, j, \in N$. Using Proposition 4.2.1, we have that $\pi_i^0(N, v, D) = \sum\limits_{A \subseteq D} \Delta_{r_D^v}(A) r d_i^O(A)$ and similarly for j. Then,

$$\sum_{a \in D_j^O} \left[\pi_i^0(N, v, D) - \pi_i^0(N, v, D \backslash \{a\})\right]$$

$$= \sum_{a \in D_j^O} \left[\sum_{A \subseteq D} \Delta_{r_D^v}(A) r d_i^O(A) - \sum_{A \subseteq D \backslash \{a\}} \Delta_{r_D^v}(A) r d_i^O(A)\right]$$
$$= \sum_{a \in D_j^O} \sum_{A \subseteq D, a \in A} \Delta_{r_D^v}(A) r d_i^O(A) = \sum_{A \subseteq D, a \in A} \Delta_{r_D^v}(A) d_j^O(A) r d_i^O(A)$$
$$= \sum_{A \subseteq D, a \in A} \Delta_{r_D^v}(A) \frac{d_j^O(A) d_i^O(A)}{d^O(A)}.$$

As this last expression is symmetric in i and j, it coincides with

$$\sum_{a \in D_i^O} \left[\pi_j^0(N, v, D) - \pi_j^0(N, v, D \backslash \{a\}\right]$$

and, thus, the result is proven. □

Similarly, we can define balanced in-arc contributions, and have the next proposition for $\alpha = 1$. The proof mimics the previous one and, therefore, it is omitted.

Definition 4.3.3. *An allocation rule $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ satisfies balanced in-arc contributions if, for all $(N, v, D) \in \mathcal{DCS}_0^N$ and all $i, j \in N$,*

$$\sum_{a \in D_j^I} [\varphi_i(N, v, D) - \varphi_i(N, v, D \backslash \{a\})] = \sum_{a \in D_i^I} [\varphi_j(N, v, D) - \varphi_j(N, v, D \backslash \{a\})].$$

Proposition 4.3.3. *The allocation rule $\pi^1 : \mathcal{DCS}_0^N \to \mathbb{R}^n$ satisfies balanced in-arc contributions.*

Which type of balanced arc contributions is appropriate depends on the application that one considers. In some cases, such as sharing information to dampen the bullwhip effect in a supply chain, it seems that the heads (which are closer to the retailer) should receive higher weight. However, in other cases, such as channels in marketing attribution, it is less clear how the weight between heads and tails must be allocated since the tails are closer to the origin of the marketing channel, but the heads are closer to the point of conversion. To allow a compromise between the effect on heads and tails, next, we state a balanced arc contribution property for any $\alpha \in [0,1]$ with a balanced out-arc, and balanced in-arc contributions as two extreme cases.

Definition 4.3.4. *Let $\alpha \in [0,1]$. An allocation rule $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ satisfies the α-balanced arc contributions property if, for all $(N,v,D) \in \mathcal{DCS}_0^N$ and all $i,j \in N$,*

$$\begin{aligned}\alpha \sum_{a \in D_j^I} [\varphi_i(N,v,D) - \varphi_i(N,v,D\backslash\{a\})] \\ +(1-\alpha) \sum_{a \in D_j^O} [\varphi_i(N,v,D) - \varphi_i(N,v,D\backslash\{a\})] \\ = \alpha \sum_{a \in D_i^I} [\varphi_j(N,v,D) - \varphi_j(N,v,D\backslash\{a\})] \\ +(1-\alpha) \sum_{a \in D_i^O} [\varphi_j(N,v,D) - \varphi_j(N,v,D\backslash\{a\})].\end{aligned}$$

Before exploring the implications of these properties, in the following lemma, we first state a property relating the rules π^0 and π^1. This property is a kind of *cross*-balanced arc contribution, in the sense that the sum of the differences in π^0 that a player experiences when another player breaks the edges in which he is head is equal to the differences experienced in π^1 for the second player when the other breaks the edges in which he is tail.

Lemma 4.3.1. *Let $(N,v,D) \in \mathcal{DCS}_0^N$ and $i,j \in N$. Then,*

$$\sum_{a \in D_j^I} \left[\pi_i^0(N,v,D) - \pi_i^0(N,v,D\backslash\{a\})\right] = \sum_{a \in D_i^O} \left[\pi_j^1(N,v,D) - \pi_j^1(N,v,D\backslash\{a\})\right].$$

Proof: For $(N,v,D) \in \mathcal{DCS}_0^N$ and $i,j \in N$,

$$\sum_{a \in D_j^I} \left[\pi_i^0(N,v,D) - \pi_i^0(N,v,D\backslash\{a\}\right]$$

$$= \sum_{a \in D_j^I} \left[\sum_{A \subseteq D} \Delta_{r_D^v}(A) r d_i^O(A) - \sum_{A \subseteq D\backslash\{a\}} \Delta_{r_D^v}(A) r d_i^O(A)\right]$$

$$= \sum_{a \in D_j^I} \sum_{A \subseteq D, a \in A} \Delta_{r_D^v}(A) r d_i^O(A) = \sum_{A \subseteq D} \Delta_{r_D^v}(A) d_j^I(A) r d_i^O(A)$$

$$= \sum_{A \subseteq D} \Delta_{r_D^v}(A) \frac{d_j^I(A) d_i^O(A)}{d^O(A)}$$

where the first equality follows from Proposition 4.2.1.

Similarly, we can obtain, for $(N, v, D) \in \mathcal{DCS}_0^N$ and $i, j \in N$, that

$$\sum_{a \in D_i^O} \left[\pi_j^I(N, v, D) - \pi_j^I(N, v, D \backslash \{a\}\right]$$

$$= \sum_{a \in D_i^O} \left[\sum_{A \subseteq D} \Delta_{r_D^v}(A) r d_j^I(A) - \sum_{A \subseteq D \backslash \{a\}} \Delta_{r_D^v}(A) r d_j^I(A) \right]$$

$$= \sum_{a \in D_i^O} \sum_{A \subseteq D, a \in A} \Delta_{r_D^v}(A) r d_j^I(A) = \sum_{A \subseteq D} \Delta_{r_D^v}(A) d_i^O(A) r d_j^I(A)$$

$$= \sum_{A \subseteq D} \Delta_{r_D^v}(A) \frac{d_i^O(A) d_j^I(A)}{d^I(A)},$$

showing the result since $d^O(A) = d^I(A)$. □

Using this lemma and Propositions 4.3.2 and 4.3.3, we have the following proposition.

Proposition 4.3.4. *Let* $\alpha \in [0, 1]$. *The allocation rule* $\pi^\alpha : \mathcal{DCS}_0^N \to \mathbb{R}^n$, *satisfies the* α*-balanced arc contributions property.*

Proof: Given $(N, v, D) \in \mathcal{DCS}_0^N$ and $i, j \in N$,

$$\alpha \sum_{a \in D_j^I} [\pi_i^\alpha(N, v, D) - \pi_i^\alpha(N, v, D \backslash \{a\})]$$

$$+(1-\alpha) \sum_{a \in D_j^O} [\pi_i^\alpha(N, v, D) - \pi_i^\alpha(N, v, D \backslash \{a\})]$$

$$= \alpha \sum_{a \in D_j^I} [\alpha \pi_i^1(N, v, D) + (1-\alpha)\pi_i^0(N, v, D)$$

$$-\alpha \pi_i^1(N, v, D \backslash \{a\}) - (1-\alpha)\pi_i^0(N, v, D \backslash \{a\})]$$

$$+(1-\alpha) \sum_{a \in D_j^O} [\alpha \pi_i^1(N, v, D) + (1-\alpha)\pi_i^0(N, v, D)$$

$$
\begin{aligned}
&-\alpha\pi_i^1(N,v,D\backslash\{a\}) - (1-\alpha)\pi_i^0(N,v,D\backslash\{a\})] \\
&= \alpha^2 \sum_{a\in D_j^I} \left[\pi_i^1(N,v,D) - \pi_i^1(N,v,D\backslash\{a\})\right] \\
&+(1-\alpha)^2 \sum_{a\in D_j^O} \left[\pi_i^0(N,v,D) - \pi_i^0(N,v,D\backslash\{a\})\right] \\
&+\alpha(1-\alpha) \sum_{a\in D_j^I} \left[\pi_i^0(N,v,D) - \pi_i^0(N,v,D\backslash\{a\})\right] \\
&+(1-\alpha)\alpha \sum_{a\in D_j^O} \left[\pi_i^1(N,v,D) - \pi_i^1(N,v,D\backslash\{a\})\right],
\end{aligned}
$$

where the first equality follows from the definition of π^α. Taking into account that π^0 and π^1 satisfy balanced out-arc contributions and balanced in-arc contributions, respectively, and using Lemma 4.3.1, the last expression coincides with

$$
\begin{aligned}
&\alpha^2 \sum_{a\in D_i^O} \left[\pi_j^1(N,v,D) - \pi_j^1(N,v,D\backslash\{a\})\right] \\
&+(1-\alpha)^2 \sum_{a\in D_i^I} \left[\pi_j^0(N,v,D) - \pi_j^0(N,v,D\backslash\{a\})\right] \\
&+\alpha(1-\alpha) \sum_{a\in D_j^I} \left[\pi_i^1(N,v,D) - \pi_i^1(N,v,D\backslash\{a\})\right] \\
&+(1-\alpha)\alpha \sum_{a\in D_j^O} \left[\pi_i^0(N,v,D) - \pi_i^0(N,v,D\backslash\{a\})\right].
\end{aligned}
$$

A similar calculation as above shows that the last expression is equal to

$$
\begin{aligned}
&\alpha \sum_{a\in D_i^O} \left[\pi_j^\alpha(N,v,D) - \pi_j^\alpha(N,v,D\backslash\{a\})\right] \\
&+(1-\alpha) \sum_{a\in D_i^I} \left[\pi_j^\alpha(N,v,D) - \pi_j^\alpha(N,v,D\backslash\{a\})\right]
\end{aligned}
$$

and thus π^α satisfies α-balanced arc contributions, for $\alpha \in [0,1]$. □

Finally, we can characterize the allocation rules π^α, $\alpha \in [0,1]$, by connection efficiency and the corresponding α-balanced arc contributions property.

Theorem 4.3.1. *Let $\alpha \in [0,1]$. The allocation rule $\pi^\alpha : \mathcal{DCS}_0^N \to \mathbb{R}^n$ is the unique allocation rule satisfying connection efficiency and α-balanced arc contributions.*

Proof: It is already proven that π^α satisfies connection efficiency and α-balanced arc contributions; see Propositions 4.3.1 and 4.3.4. Therefore, it is sufficient to show the uniqueness of an allocation rule satisfying the two properties. Let $\varphi : \mathcal{DCS}_0^N \to \mathbb{R}^n$ be an allocation rule satisfying these two properties. We will prove that $\varphi(N, v, D)$ is uniquely determined for all $(N, v, D) \in \mathcal{DCS}_0^N$ by induction on $|D|$, the cardinality of D. The proof follows similar steps as in Slikker (2005).

If $|D| = 0$, uniqueness is trivial by connection efficiency. Proceeding by induction, suppose that uniqueness holds for (N, v, D) with $|D| \leq k$ and consider (N, v, D) such that $|D| = k + 1$. Let $C \in N/\gamma_D$ and suppose, without loss of generality, that $C = \{1, 2, \ldots, c\}$. If $c = |C| = 1$, uniqueness holds using connection efficiency. Thus, let us consider the case in which $c = |C| > 1$. Take any $j \in C\backslash\{1\}$. Applying the α-balanced arc contributions property to players–nodes 1 and j, we have

$$\begin{aligned}
&\alpha \sum_{a \in D_j^I} [\varphi_1(N, v, D) - \varphi_1(N, v, D\backslash\{a\})] \\
&+(1-\alpha) \sum_{a \in D_j^O} [\varphi_1(N, v, D) - \varphi_1(N, v, D\backslash\{a\})] \\
&= \alpha \sum_{a \in D_1^I} [\varphi_j(N, v, D) - \varphi_j(N, v, D\backslash\{a\})] \\
&+(1-\alpha) \sum_{a \in D_1^O} [\varphi_j(N, v, D) - \varphi_j(N, v, D\backslash\{a\})]
\end{aligned}$$

or, alternatively, by rearranging the terms,

$$\begin{aligned}
&\alpha \sum_{a \in D_j^I} \varphi_1(N, v, D) + (1-\alpha) \sum_{a \in D_j^O} \varphi_1(N, v, D) - \alpha \sum_{a \in D_1^I} \varphi_j(N, v, D) \\
&-(1-\alpha) \sum_{a \in D_1^O} \varphi_j(N, v, D) \\
&= \alpha \sum_{a \in D_j^I} \varphi_1(N, v, D\backslash\{a\}) \\
&+(1-\alpha) \sum_{a \in D_j^O} \varphi_1(N, v, D\backslash\{a\}) - \alpha \sum_{a \in D_1^I} \varphi_j(N, v, D\backslash\{a\}) \\
&-(1-\alpha) \sum_{a \in D_1^O} \varphi_j(N, v, D\backslash\{a\}).
\end{aligned}$$

Notice that the left-hand side can be written as

$$\alpha d_j^I(D)\varphi_1(N,v,D) + (1-\alpha)d_j^O(D)\varphi_1(N,v,D)$$

$$-\alpha d_1^I(D)\varphi_j(N,v,D) - (1-\alpha)d_1^O(D)\varphi_j(N,v,D),$$

and the right-hand side is determined by the induction hypothesis.

Since this holds for every $j \in C \setminus \{1\}$, we have $c-1$ linear independent equations in the c unknown payoffs $\varphi_i(N,v,D), i \in C$. Moreover, connection efficiency gives the equation

$$\sum_{i=1}^{c} \varphi_i(N,v,D) = v^D(C).$$

Thus, we have c linear independent equations in the c unkown payoffs $\varphi_i(N,v,D)$, $i \in C$, which thus are uniquely determined.

(It is straightforward to prove that the determinant of the coefficient matrix is equal to $\sum_{i=1}^{c}\left(\alpha d_i^I(D) + (1-\alpha)d_i^O(D)\right)\left[-\alpha d_1^I(D) - (1-\alpha)d_1^O(D)\right]^{c-2} = -\sum_{i=1}^{c} d_i^O(D)\left[\alpha d_1^I(D) + (1-\alpha)d_1^O(D)\right]^{c-2} = -|D_{|C}|\left[\alpha d_1^I(D) + (1-\alpha)d_1^O(D)\right]^{c-2} \neq 0$, for all $\alpha \in (0,1)$.

If $\alpha = 0$, $-|D_{|C}|d_1^O(D)^{c-2}$ is also different from zero when node 1 is such that $d_1^O(D) > 0$. If $d_1^O(D) = 0$. This can be shown by taking another node with positive out-degree in C.

If $\alpha = 1$, $-|D_{|C}|d_1^I(D)^{c-2}$ is also different from zero when node 1 is such that $d_1^I(D) > 0$. If $d_1^I(D) = 0$. This can be shown by taking another node with positive in-degree in C.)

Since π^α satisfies the two properties, it must be that $\varphi(N,v,D) = \pi^\alpha(N,v,D)$.

□

4.4 Conclusions

In this chapter, we have introduced a family of position values for directed communication situations, based on the idea of the position value for (undirected) communication situations. To do this, in a manner consistent with the definition of the digraph restricted game introduced in Chapter 3, we have introduced an arc game. The different defined position values differs from the proportion of the Shapley value of each arc that they assign to head and tail nodes. Moreover, we have characterized each position value in our family by the connection efficiency introduced in the Chapter 3 and a corresponding version of balanced arc contributions weighing out- and in-arcs in a different way, but uniform across arcs. Our family of position values can be expressed in terms of the (relative) out- or in-degree. This suggests, as a future research, the possibility of replacing this relative degrees by other power or centrality measures for directed graphs and also other weights determined exogenously by, for example, bargaining, political, military, etc., power can be taken into consideration. Moreover, other types of axioms can be considered, such as monotonicity axioms related to adding/deleting arcs, or related to changes in contributions in the game.

Chapter 5

A mixed value for directed communication situations

In this chapter, following Feltkamp & van den Nouweland (1993), a pseudo-game is introduced for directed communication situations. The set of players of this restricted game includes both nodes and arcs. Furthermore, a new assignment rule is defined, the mixed directed communication value for which we obtain two characterizations.

5.1 The mixed game or pseudo-game for directed communication situations

Using the type of connection that we have been defending throughout this thesis, we can define a mixed game that takes into account the payoff of coalitions not only of players (as in the digraph restricted game) nor only of arcs (as in the arc game) but the payoff of coalitions formed simultaneously by players and arcs.

Definition 5.1.1. *Given (N, v, D) in $\mathcal{DCS}^N$, the mixed game or pseudogame for directed communication situations is defined as the TU-game $(N \cup D, w_{v,D})$, in which the pseudoplayers are either nodes (players) or arcs in the digraph.*

Its characteristic function is given by:

$$w_{v,D}(S,L) = v^L(S) = \sum_{i=1}^{r(S,L)} v(\{P_i^{S,L}\}) - \sum_{i=1}^{r(S,L)-1} \sum_{j=i}^{r(S,L)} v(\{P_i^{S,L}\} \cap \{P_j^{S,L}\})$$

$$+ \sum_{i=1}^{r(S,L)-2} \sum_{j=i}^{r(S,L)-1} \sum_{k=j+1}^{r(S,L)} v(\{P_i^{S,L}\} \cap \{P_j^{S,L}\} \cap \{P_k^{S,L}\})$$

$$+ \cdots + (-1)^{r(S,L)-1} v(\{P_1^{S,L}\} \cap \{P_2^{S,L}\} \cap \cdots \cap \{P_{r(L,S)}^{S,L}\}) \text{ , for all } \{S,L\} \tilde{\subseteq} \{N,D\}$$

with $\mathcal{P}(S, L_{|S}) = \{P_1^{S,L} \cdots P_{r(S,L)}^{S,L}\}$ *being the set of all maximal paths in the digraph* $(S, L_{|S})$*.*

In the following proposition we obtain the expression of the pseudogames for the unanimity games.

Proposition 5.1.1. *For* $(N, u_S, D) \in \mathcal{DCS}^N$*, with* (N, u_S) *the unanimity game of* S *in* G^N*, we have* [1]*:*

$$w_{u_S,D} = \mathbf{1} - \prod_{i=1}^{t(S,D)} \left[\mathbf{1} - u_{\{\{Q_i^S\},Q_i^S\}}\right] \tag{19}$$

if $\mathcal{MCP}(S,N,D) = \{Q_1^S \cdots Q_{t(S,D))}^S\} \neq \emptyset$ *and* $w_{u_s,D} \equiv \mathbf{0}$ *(the null game), otherwise* [2]*.*

Proof: Suppose $\mathcal{MCP}(S,N,D) \neq \emptyset$. The other case is trivial.

For $\{T,\eta\} \tilde{\subseteq} \{N,D\}$, we have that

$$w_{u_S,D}(\{T,\eta\}) = u_s^\eta(T) = \sum_{C \in T/\eta} u_s(C)$$

$$= \begin{cases} 1 & \text{if there is } C \in T/\eta \text{ such that } S \subseteq C \\ 0 & \text{otherwise} \end{cases}$$

[1] We will denote $(N \cup \gamma, u_{\{S,\eta\}})$ if $\{\emptyset,\emptyset\} \neq \{S,\eta\} \tilde{\subseteq} \{N,\gamma\}$ for the games of the unanimity basis in $G^{N\cup\gamma}$. The characteristic function is given by $u_{\{S,\eta\}}(\{T,\delta\}) = \begin{cases} 1 & \text{if } \{S,\eta\} \tilde{\subseteq} \{T,\delta\} \\ 0 & \text{otherwise} \end{cases}$ for all $S \subseteq N$ and $\delta \subseteq \gamma$.

[2] Let us recall that $\{Q_i^S\}$ is a subset of N and Q_i^S is a subset of D for all $Q_i^S \in \mathcal{MCP}(S,N,D)$.

$$= \begin{cases} 1 & \text{if } S \text{ is connected in } (T, \eta) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, using the right hand side of (19)

$$\left\{ \mathbf{1} - \prod_{i=1}^{t(S,D)} \left[\mathbf{1} - u_{\{\{Q_i^S\}, Q_i^S\}} \right] \right\} (\{T, \eta\})$$

$$= \begin{cases} 1 & \text{if there is } Q_i^S \in \mathcal{MCP}(S, N, D) \text{ such that } \{\{Q_i^S\}, Q_i^S\} \tilde{\subseteq} \{T, \eta\} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } S \text{ is connected in } \{T, \eta\} \\ 0 & \text{otherwise} \end{cases}$$

And thus the result is proved. □

Corollary 5.1.1. *For* $(N, u_S, D) \in \mathcal{DCS}^N$*, with* (N, u_S) *the unanimity game of* S *in* G^N*, and* $\mathcal{MCP}(S, N, D) = \left\{Q_1^S, Q_2^S, ..., Q_{t(S,D)}^S\right\}$*, we have*

$$w_{u_S,D} = \sum_{i=1}^{t(S,D)} u_{\{\{Q_i^S\}, Q_i^S\}} - \sum_{i=1}^{t(S,D)-1} \sum_{j=i+1}^{t(S,D)} u_{\{\{Q_i^S\} \cup \{Q_j^S\}, Q_i^S \cup, Q_j^S\}}$$

$$+ ... + (-1)^{t(S,D)-1} u_{\left\{\cup_{i=1}^{t(S,D)} \{Q_i^S\}, \cup_{i=1}^{t(S,D)} Q_i^S\right\}}.$$

Example 5.1.1. *Consider the directed communication situation as in Example 4.1.1. Let us recall the digraph that we have been using throughout this report, (see Fig. 5.1). Now, the players in the pseudogame are the nodes and the directed arcs. The characteristic function* $w_{v,D}(S, L)$ *is given by:*

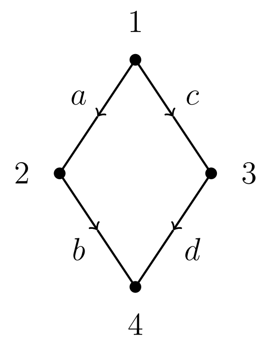

Figure 5.1: The digraph (N, D)

$$
\begin{array}{cc}
v(\{i\}) & \textit{if } S=\{i\} \textit{ and } \forall L\subseteq D, \\
v(\{1,2\}) & \textit{if } S=\{1,2\} \textit{ and } \{a\}\subseteq L, \\
v(\{1,3\}) & \textit{if } S=\{1,3\} \textit{ and } \{c\}\subseteq L, \\
v(\{2,4\}) & \textit{if } S=\{2,4\} \textit{ and } \{b\}\subseteq L, \\
v(\{3,4\}) & \textit{if } S=\{3,4\} \textit{ and } \{d\}\subseteq L, \\
v(\{1\})+v(\{4\}) & \textit{if } S=\{1,4\} \textit{ and } \forall L\subseteq D, \\
v(\{2\})+v(\{3\}) & \textit{if } S=\{2,3\} \textit{ and } \forall L\subseteq D, \\
v(\{1,2\})+v(\{3\}) & \textit{if } S=\{1,2,3\} \textit{ and } \{a\}\subseteq L,\{c\}\not\subseteq L \\
v(\{1,3\})+v(\{2\}) & \textit{if } S=\{1,2,3\} \textit{ and } \{c\}\subseteq L,\{a\}\not\subseteq L \\
v(\{1,2\})+v(\{1,3\})-v(\{1\}) & \textit{if } S=\{1,2,3\} \textit{ and } \{a,c\}\subseteq L, \\
v(\{1,2,4\}) & \textit{if } S=\{1,2,4\} \textit{ and } \{a,b\}\subseteq L, \\
v(\{1,2\})+v(\{4\}) & \textit{if } S=\{1,2,4\} \textit{ and } \{a\}\subseteq L,\{b\}\not\subseteq L \\
v(\{2,4\})+v(\{1\}) & \textit{if } S=\{1,2,4\} \textit{ and } \{b\}\subseteq L,\{a\}\not\subseteq L \\
v(\{1,3,4\}) & \textit{if } S=\{1,3,4\} \textit{ and } \{c,d\}\subseteq L, \\
v(\{1,3\})+v(\{4\}) & \textit{if } S=\{1,3,4\} \textit{ and } \{c\}\subseteq L,\{d\}\not\subseteq L \\
v(\{3,4\})+v(\{1\}) & \textit{if } S=\{1,3,4\} \textit{ and } \{d\}\subseteq L,\{c\}\not\subseteq L \\
v(\{2,4\})+v(\{3\}) & \textit{if } S=\{2,3,4\} \textit{ and } \{b\}\subseteq L,\{d\}\not\subseteq L \\
v(\{3,4\})+v(\{2\}) & \textit{if } S=\{2,3,4\} \textit{ and } \{d\}\subseteq L,\{b\}\not\subseteq L \\
v(\{2,4\})+v(\{3,4\})-v(\{4\}) & \textit{if } S=\{2,3,4\} \textit{ and } \{b,d\}\subseteq L, \\
v(\{1,2\})+v(\{3\})+v(\{4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a\}, \\
v(\{2,4\})+v(\{1\})+v(\{3\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{b\}, \\
v(\{1,3\})+v(\{2\})+v(\{4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{c\}, \\
v(\{3,4\})+v(\{1\})+v(\{2\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{d\}, \\
v(\{1,2,4\})+v(\{3\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,b\}, \\
v(\{1,2\})+v(\{1,3\})-v(\{1\})+v(\{4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,c\}, \\
v(\{1,2\})+v(\{3,4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,d\}, \\
v(\{2,4\})+v(\{1,3\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{b,c\}, \\
v(\{2,4\})+v(\{3,4\})-v(\{4\})+v(\{1\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{b,d\}, \\
v(\{1,3,4\})+v(\{2\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{c,d\}, \\
v(\{1,2,4\})+v(\{1,3\})-v(\{1\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,b,c\}, \\
v(\{1,2,4\})+v(\{3,4\})-v(\{4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,b,d\}, \\
v(\{1,3,4\})+v(\{1,2\})-v(\{1\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,c,d\}, \\
v(\{1,3,4\})+v(\{2,4\})-v(\{4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{b,c,d\}, \\
v(\{1,2,4\})+v(\{1,3,4\})-v(\{1,4\}) & \textit{if } S=\{1,2,3,4\} \textit{ and } L=\{a,b,c,d\},
\end{array}
.
$$

Example 5.1.2. *Consider the direcetd communciation situations (N, v, D) as in Example 5.1.1 and v being one of the three games we are using throughout this thesis.*

- *In the messages game, $w_{v_1,D}$ in terms of the unanimity basis is given by*

$$\begin{aligned} w_{v_1,D} &= \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,a\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,3,c\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{2,4,b\}})\right] \\ &+ \left[\mathbf{1} - (\mathbf{1} - u_{\{3,4,d\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,b\}})(\mathbf{1} - u_{\{1,3,4,c,d\}})\right] \\ &= u_{\{1,2,a\}} + u_{\{1,3,c\}} + u_{\{2,4,b\}} + u_{\{3,4,d\}} + u_{\{1,2,4,a,b\}} + u_{\{1,3,4,c,d\}} - u_{\{1,2,3,4,a,b,c,d\}}. \end{aligned} \tag{20}$$

- *In the overhead game, $w_{v_2,D}$ in terms of the unanimity basis is given by*

$$\begin{aligned} w_{v_2,D} &= \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,a\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,3,c\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{2,4,b\}})\right] \\ &+ \left[\mathbf{1} - (\mathbf{1} - u_{\{3,4,d\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,b\}})(\mathbf{1} - u_{\{1,3,4,c,d\}})\right] \\ &- \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,b\}})\right] - \left[\mathbf{1} - (\mathbf{1} - u_{\{1,3,4,c,d\}})\right] \\ &= u_{\{1,2,a\}} + u_{\{1,3,c\}} + u_{\{2,4,b\}} + u_{\{3,4,d\}} - u_{\{1,2,3,4,a,b,c,d\}} \end{aligned} \tag{21}$$

- *In the conferences game, $w_{v_3,D}$ in terms of the unanimity basis is given by*

$$\begin{aligned} w_{v_3,D} &= \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,a\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,3,c\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{2,4,b\}})\right] \\ &+ \left[\mathbf{1} - (\mathbf{1} - u_{\{3,4,d\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,b\}})(\mathbf{1} - u_{\{1,3,4,c,d\}})\right] \\ &+ \left[\mathbf{1} - (\mathbf{1} - u_{\{1,2,4,a,b\}})\right] + \left[\mathbf{1} - (\mathbf{1} - u_{\{1,3,4,c,d\}})\right] \\ &= u_{\{1,2,a\}} + u_{\{1,3,c\}} + u_{\{2,4,b\}} + u_{\{3,4,d\}} + 2u_{\{1,2,4,a,b\}} + 2u_{\{1,3,4,c,d\}} - u_{\{1,2,3,4,a,b,c,d\}} \end{aligned} \tag{22}$$

5.2 The mixed directed communication value

We start by defining mixed allocation rules for targeted communication situations.

Definition 5.2.1. *A mixed allocation rule φ on $\mathcal{DCS}^N$ is a map that assigns to every directed communication situation $(N, v, D) \in \mathcal{DCS}^N$ a vector $\varphi(N, v, D) \in \mathbb{R}^{n+|D|}$ containing the payoffs of the players (nodes), $\varphi_i(N, v, D)$, $i \in N$, and the payoff of the arcs, $\varphi_l(N, v, D)$, $l \in D$, in the directed communication situation.*

Remark 5.2.1. *The payoff of arcs can be interpreted as the share of the grand coalition's profit that players must pay to intermediaries who facilitate communication, in situations where the nodes involved in an arc are not the ones who control that arc. Think of telephone communications, for example, where the communicators do not own the telephone lines, or of companies whose freight must pay to cross a waterway, such as the Panama Canal or the Suez Canal.*

Given the prevalence of the Shapley value, we will once again use the Shapley value of the restricted mixed set we have just introduced to define the proposed mixed value.

Definition 5.2.2. *Given a directed communication situation (N, v, D), the mixed directed communication value of (N, v, D), $\rho(N, v, D)$ is defined as,*

$$\rho(N, v, D) = Sh(N \cup D, w_{v,D}).$$

Example 5.2.1. *Consider the directed communication situations in (N, v, D) as in Example 5.1.1. Using the expressions (20), (21) and (22) it is easy to obtain:*

- *In the messages game*

$$\rho(N, v_1, D) = Sh(N \cup D, w_{v_1,D}) = \Big(\frac{113}{120}, \frac{89}{120}, \frac{89}{120}, \frac{113}{120}, \frac{49}{120}, \frac{49}{120}, \frac{49}{120}, \frac{49}{120}\Big)$$

- *In the overhead game*

$$\rho(N, v_2, D) = Sh(N \cup D, w_{v_2,D}) = \Big(\frac{13}{24}\frac{13}{24}, \frac{13}{24}, \frac{13}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24}\Big)$$

- *In the conferences game*

$$\rho(N, v_3, D) = Sh(N \cup D, w_{v_3,D}) = \Big(\frac{161}{120}, \frac{113}{120}, \frac{113}{120}, \frac{161}{120}, \frac{73}{120}, \frac{73}{120}, \frac{73}{120}, \frac{73}{120}\Big).$$

5.3 Characterization of the mixed directed communication value

In this section we characterize the defined value using extensions of classical properties in the literature of games with communication restricted by graphs. First, we introduce the mixed connection efficiency. If in the DC-value and in the position values, for a component C, $v^D(C)$ was distributed among the players, now this payoff must be distributed among players and arcs.

Definition 5.3.1. *A mixed allocation rule, φ, defined on $\mathcal{CS}^N$ satisfies mixed connection efficiency if, for each $(N, v, D) \in \mathcal{DCS}^N$ and each $C \in N/\gamma_D$*

$$\sum_{i \in C} \varphi_i(N, v, D) + \sum_{a \in D_{|C}} \varphi_a(N, v, D) = v^D(C).$$

The mixed fairness property is, of course, inspired in the Myerson's fairness. It states that when two players break their arc, both of them are equally affected in the sense that the variation of the payoffs coincides. Moreover, for each one of these two players, the variation of his payoff and the payoff of the arc also coincides.

Definition 5.3.2. *A mixed allocation rule on $\mathcal{DCS}^N$, φ, satisfies mixed fairness, if for every $(N, v, D) \in \mathcal{DCS}^N$ and every $i, j \in N$ with $a = (i, j)$,*

$$\varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{a\}) = \varphi_j(N, v, D) - \varphi_j(N, v, D \setminus \{a\}) = \varphi_a(N, v, D).$$

We extend now the classical balanced contribution property of Myerson (1977) to mixed allocation rules for directed communication situations. Balanced contributions requires that the effect of isolation of a player on the payoff of another player, is equal to the effect the other way around, that is the effect on the payoff of this player of the isolation of the other player.

Definition 5.3.3. *A mixed allocation rule φ on $\mathcal{DCS}^N$ satisfies balanced contributions if, for all $(N, v, D) \in \mathcal{DCS}^N$ and all $i, j \in N$,*

$$\varphi_i(N, v, D) - \varphi_i(N, v, D_{-j}) = \varphi_j(N, v, D) - \varphi_j(N, v, D_{-i}).$$

We define now the mixed balanced contributions property for directed communication situations. This property states that the harm that the removal of a arc causes a player matches the variation in the payoff of the arc if the player is isolated.

Definition 5.3.4. *A mixed allocation rule, φ, on $\mathcal{DCS}^N$ satisfies the mixed balanced contributions property, if for $i \in N$ and $a \in D$ it holds that*

$$\varphi_a(N,v,D) - \varphi_a(N,v,D_{-i})$$
$$= \varphi_i(N,v,D) - \varphi_i(N,v,D \setminus \{a\}).$$

First, we prove that previous properties are satisfied by ρ.

Proposition 5.3.1. *The mixed directed communication value ρ satisfies mixed connection effiency, mixed fairness, balanced contributions and mixed balanced contributions.*

Proof:

(a) Let us prove that ρ satisfies mixed connection efficiency. Given $(N,v,D) \in \mathcal{DCS}^N$ and $C \in N/\gamma_D$, by definition

$$\rho(N,v,D) = Sh(N \cup D, w_{v,D}).$$

As the weighted Shapley value is efficient:

$$\sum_{i\in C} \rho_i(N,v,D) + \sum_{a\in D_{|C}} \rho_a(N,v,D)$$
$$= \sum_{i\in C} Sh_i(N \cup D, w_{v,D}) + \sum_{a\in D_{|C}} Sh_a(N \cup D, w_{v,D})$$
$$= \sum_{i\in C} Sh_i(C \cup D_{|C}, w_{v_{|C},D_{|C}}) + \sum_{a\in D_{|C}} Sh_a(C \cup D_{|C}, w_{v_{|C},D_{|C}})$$
$$= w_{v_{|C},D_{|C}}(\{C, D_{|C}\}) = v_{|C}^{D_{|C}}(C) = v^D(C),$$

where $D_{|C} = \{(k,l) \in D \mid k,l \in C\}$.

(b) As the mixed directed communication value is clearly linear (in the game), to prove that ρ satisfies mixed fairness, it is sufficient to see that the property holds for directed communication situations of the form (N,u_S,D) with (N,u_S) the unanimity game of the coalition $S \neq \emptyset$. Then, [3]

[3] $(N \cup D, \mathbf{1})$ is the pseudogame in which

$$\mathbf{1}(\{S,L\}) = \begin{cases} 1 & \text{for all } \{S,L\} \neq \{\emptyset,\emptyset\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho(N, u_S, D) = Sh(N \cup D, w_{u_S,D}) = Sh\left[N \cup D, \mathbf{1} - \prod_{i=1}^{t(S,D)} (\mathbf{1} - u_{\{\{Q_i^S\}, Q_i^S\}})\right]$$

if $\mathcal{MCP}(S, N, D) = \{Q_1^S \cdots Q_{t(S,D))}^S\} \neq \emptyset$ and $Sh(N \cup D, w_{u_S,D} \equiv \mathbf{0}^4) = \mathbf{0}^5$, otherwise.

Thus, for $i, j \in N$ and $a = (i, j)$,

$$\rho_i(N, v, D) - \rho_i(N, v, D \setminus \{a\})$$

is zero or is the Shapley value of a linear combination of unanimity games. $(N \cup D, u_{\{T,L\}})$ such that $i, j \in T$ for all T and a belongs to L for all L. If the difference is zero, the result is trivial. In the other case, taking into account the symmetry of the Shapley value we have

$$\rho_i(N, v, D) - \rho_i(N, v, D \setminus \{a\}) = \rho_j(N, v, D) - \rho_j(N, v, D \setminus \{a\})$$

$$= \rho_a(N, v, D) - \rho_a(N, v, D \setminus \{a\}) = \rho_a(N, v, D)$$

and thus the result is proved.

(c) Let us prove that ρ satisfies balanced contributions. For (N, v, D), $i, j \in N$

$$\rho_i(N, v, D) - \rho_i(N, v, D_{-j})$$

and

$$\rho_j(N, v, D) - \rho_j(N, v, D_{-i})$$

are both zero or the Shapley value of a linear combination of unanimity games $(N, u_{\{T,L\}}) \in G^{N \cup D}$ in which i and $j \in T$. The property trivially holds if both quantities vanish. In other case, by the symmetry of the Shapley value, we have that

$$\rho_i(N, v, D) - \rho_i(N, v, D_{-j}) = \rho_j(N, v, D) - \rho_j(N, v, D_{-i})$$

and thus the result.

[4]The null game in $G^{N \cup D}$.
[5]The null vector in $\mathbb{R}^{|N|+|D|}$.

(d) The proof of ρ satisfying mixed balanced contributions follows immediately from the previous one, considering two players and an arc and reproducing the argument. After all, nodes and arcs are all players in the mixed game.

□

Now, we can characterize the mixed directed communication value.

Theorem 5.3.1. *The mixed directed communcation value, ρ, is the unique mixed allocation rule on $\mathcal{DCS}^N$ satisfying mixed connection efficiency and mixed fairness.*

Proof: It has already been proved that the mixed directed communication value satisfies these two properties. Reciprocally, consider a mixed allocation rule, φ, on $\mathcal{DCS}^N$ satisfying both properties. Suppose $(N, v, D) \in \mathcal{DCS}^N$, $C \in N/\gamma_D$, $i, j \in C$. The proof uses induction on $|D|$, the cardinally of D. If $|D| = 0$, then clearly every singleton $\{i\}$, $i \in N$, is a weak component, and thus by mixed connection efficiency, $\varphi_i(N, v, D) = v^D(\{i\}) = v(\{i\}) = \rho_i^d(N, v, D)$.

Proceeding by induction, suppose that the result holds for $(N, v, D') \in \mathcal{DCS}^N$ such that $|D'| \leq k$ and consider (N, v, D) with $|D| = k + 1$. Let $C \in N/\gamma_D$ be a weak component of (N, D), and let (C, T_C) be a spanning tree in $(C, (\gamma_D)_{|C})$.

Then, for every $\{i, j\} \in C$, mixed fairness implies that

$$\begin{aligned} &\varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\})\}) \\ &= \varphi_j(N, v, D) - \varphi_j(N, v, D \setminus \{(t(\{i, j\}), h(\{i, j\})\}), \end{aligned} \tag{23}$$

where $t(e)$ denotes the tail of arc $e \in D$, while $h(e)$ denotes the head of arc $e \in D$.

Notice that $|T_C| = |C| - 1$. Moreover, for every $a \in D_{|C}$ with $i = h(a)$ (or $i = t(a)$), mixed fairness implies that

$$\varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{a\}) = \varphi_a(N, v, D) \tag{24}$$

which are $|D_{|C}|$ equations. For every weak component $C \in N/\gamma_D$, mixed connection efficiency implies that

$$\sum_{i \in C} \varphi_i(N, v, D) + \sum_{a \in D_{|C}} \varphi_a(N, v, D) = v^D(C)$$

Since $\varphi_i(N, v, D\backslash\{(t(\{i,j\}), h(\{i,j\})\})$ and $\varphi_j(N, v, D\backslash\{(t(\{i,j\}), h(\{i,j\})\})$ are uniquely determined by the induction hypothesis, and the $(|C|-1)+|D_{|C}|+1 = |C|+|D_{|C}|$ linear equations (23) and (24) are independent, the numbers (payoffs) $\varphi_i(N, v, D)$, $i \in C$ and $\varphi_a(N, v, D)$, $a \in D_{|C}$ are uniquely determined. The proof of the linear equations are lineal independent is as follow.

The matrix, M, of coefficients formed by the $(|C|-1)+|D_{|C}|+1 = |C|+|D_{|C}|$ equations (23) and (24), defined by blocks, is given by

$$M = \left(\begin{array}{c|c} A & B \\ \hline H & D \end{array}\right)$$

Matrix A is generated by labelling the associated network so that each arc has as its end node the number consecutive to the end node of the previous arc. The matrix A, $c \times c$, is as follows

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Matrix B, $c \times |D_{|C}|$, is given by

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

In matrix H, which order is $|D_{|C}| \times c$, each row has 1 in the position $h(l_j)$, i.e., in the position corresponding to the head of the arc l_j, and 0 in the other positions.

And finally, the matrix D, $|D_{|C}| \times |D_{|C}|$, is the diagonal matrix given by

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

We can add the sum of all rows corresponding to blocks H and D to the last row of blocks A and B, i.e., to the corresponding row of the efficiency equation in which the coefficients are all 1. The result is that in the last row of matrix A is now the out degree of each node plus 1 and in the last row of matrix B, all coefficients are 0. The determinant of a triangular superior block matrix can be calculated as the product of the determinants of the diagonal blocks. And then, in this case,

$$det(M) = det(A) \times det(D).$$

The determinant of matrix D is equal to $(-1)^{|D_{|C}|}$. As $det(A)$ equals $det(A')$ (after adding the sum of the rows of H and D to the last row of A and B) with

$$A' = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ d^O(D)+c & d_2^O(D)+1 & d_3^O(D)+1 & d_4^O(D)+1 & \cdots & d_c^O(D)+1 \end{pmatrix}$$

we have that

$$det(M) = det(A') \times det(D) = \left(d^O(D)+c\right) \cdot \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -1 \end{vmatrix} \cdot (-1)^{|D_{|C}|} \neq 0,$$

and thus, the result is proved. □

Theorem 5.3.2. *The mixed directed communication value is the unique mixed allocation rule on $\mathcal{DCS}^N$ satisfying mixed connection efficiency, balanced contributions and mixed balanced contributions.*

Proof: It is already proved that the mixed directed communication value satisfies these three properties. Reciprocally, let us suppose φ is a mixed allocation rule satisfying them, $(N, v, D) \in \mathcal{DCS}^N$ and $C \in N/\gamma_D$. We prove that $\varphi(N, v, D) = \rho(N, v, D)$ by induction on $|D|$, the cardinality of D.

If $|D| = 0$, the result is trivial using the mixed connection efficiency in the (singleton) connected component C. Suppose the result is true for $|D| \leq k-1$ and consider (N, v, D) with $|D| = k$. If C is a singleton, again the mixed connection efficiency determines the value of the node-player. Otherwise, for $i, j \in C$, as φ satisfies the balanced contributions property:

$$\varphi_i(N, v, D) - \varphi_i(N, v, D_{-j}) = \varphi_j(N, v, D) - \varphi_j(N, v, D_{-i})$$

or equivalently

$$\varphi_i(N, v, D) - \varphi_j(N, v, D) = \varphi_i(N, v, D_{-j}) - \varphi_j(N, v, D_{-i}).$$

Using the induction hypothesis (recall that $|D_i| \geq 1$ and $|D_j| \geq 1$ as $i, j \in C$, a connected component),

$$\varphi_i(N, v, D_{-i}) - \varphi_j(N, v, D_{-j}) = \rho_i(N, v, D_{-i}) - \rho_j(N, v, D_{-j}),$$

and using that ρ also satisfies the balanced contributions property,

$$\rho_i(N, v, D_{-i}) - \rho_j(N, v, D_{-j}) = \rho_i(N, v, D) - \rho_j(N, v, D),$$

and thus

$$\varphi_i(N, v, D) - \varphi_j(N, v, D) = \rho_i(N, v, D) - \rho_j(N, v, D), \tag{25}$$

for all $i, j \in C$. Moreover, as φ satisfies the mixed balanced contributions property, given $a \in D$ and $i \in C$

$$\varphi_a(N, v, D) - \varphi_a(N, v, D_{-i}) = \varphi_i(N, v, D) - \varphi_i(N, v, D \setminus \{a\}).$$

Then,

$$\begin{aligned} \varphi_a(N, v, D) - \varphi_i(N, v, D) &= \varphi_a(N, v, D_{-i}) - \varphi_i(N, v, D \setminus \{a\}) \\ &= \rho_a(N, v, D_{-i}) - \rho_i(N, v, D \setminus \{a\}), \end{aligned} \tag{26}$$

the last equality holding by the induction hypothesis. If $|C| = c$ and, without loss of generality, $C = \{1, 2, ..., c\}$, from (25), (26) and the mixed connection efficiency we have the following linear system with $c + |D_{|C}|$ in the variables $\varphi_i(N, v, D), i \in C$, and $\varphi_a(N, v, D), a \in D_{|C}$:

$$\varphi_i(N, v, D) - \varphi_1(N, v, D) = \rho_i(N, v, D) - \rho_1(N, v, D) \text{ for } i = 2, ..., c$$
$$\varphi_a(N, v, D) - \varphi_i(N, v, D) = \rho_a(N, v, D_{-i}) - \rho_i(N, v, D \setminus \{a\}) \text{ for } a \in D_{|C}$$
$$\sum_{i \in N} \varphi_i(N, v, D) + \sum_{a \in D} \varphi_a(N, v, D) = v^D(N).$$

All these equations are linearly independent (the proof mimics the corresponding to Theorem 5.3.1 and then it is omitted) and thus the system has a unique solution that necessarily coincides with ρ. □

5.4 Conclusions

In this chapter, we have introduced a mixed directed communication value for directed communication situations. To do this, in a manner consistent with the definition of the digraph restricted game introduced in Chapter 3, we have defined a pseudo-game which pseudo-players are either players in the original game or arcs in the digraph. The defined allocation rule is the Shapley value of this pseudo-game. Moreover we have obtained two characterizations for this value using as axioms mixed connection efficiency, mixed fairness, balanced contributions and mixed balanced contributions.

Chapter 6

Conclusions and Future Research

6.1 Conclusions (in English)

The main objective of this paper is to create new values or assignment rules for situations in which the players of a TU game are restricted in their ability to communicate by means of a directed network. Starting from the classical values for communication situations, we tried to extend them to this new scenario and we have walked on the steps that generated the Myerson value, the positional value and the mixed value.

We have introduced a new concept of connectivity for the communication of players in a directed network. Under this concept only players who are on a directed path can communicate. This assumption allows modelling what happens in supply chains, in attribution models or in vaccination policies.

Based on this definition of connectivity, three TU games restricted to digraphs have been proposed. One for players, one for arcs and one for players and arcs. The Shapley value in the first of these games, the DC-value, has been characterised in terms of connection efficiency, fairness and balanced contributions. The DC-value has been successfully used, in this report, to define game-parameterised centrality measures in directed graphs and also to introduce measures of efficiency and vulnerability in networks given by a digraph. Both measures deserve a more in-depth analysis that has not been included in this report but will undoubtedly be discussed in the near future.

For the arc game, and assuming that the direction of the arcs introduces a possible asymmetry between the players maintaining them, we have introduced a family of positional values parameterised by $\alpha \in [0, 1]$. These values have been characterised using the properties of connection efficiency and α-balanced arc contributions.

Finally, a mixed value for directed communication situations has also been introduced from the same idea of connectivity. After creating a restricted set of nodes and arcs, the Shapley value has been proposed as an assignment rule. The value of arcs can be assumed as the cost that players have to pay to those who control communications when the players do not own them. This can happen in telephone conversations or in payments for crossing channels such as the Panama or Suez Canal. This value has also been characterised using properties such as mixed connection efficiency, mixed fairness, balanced contributions and mixed balanced contributions.

The following table (Table 6.2) shows the properties satisfied by each of the values proposed in this report and which of them have been used to characterise them (marked with a C_i those used in the ith characterisation). When a property appears as an axiom in the characterization of different values it shows a certain robustness since it is satisfied by different allocations rules and with the help of other axioms implies these different allocations. As mentioned throughout the report, the properties proposed for each of the values are extensions of the classical properties of efficiency, fairness, balanced contributions and balanced link contributions.

	DC-value	**Family position values**	**Mixed directed communication value**
Connection efficiency	✓ C_1, C_2	✓ C_1	×
Mixed connection efficiency	×	×	✓ C_1, C_2
Fairness	✓ C_1	✓	✓
Mixed fairness	×	×	✓C_1
Balanced contributions	✓C_2	×	✓ C_2
Balanced out-arc contributions	×	✓	×
Balanced in-arc contributions	×	✓	×
α-balanced arc contributions	×	✓ C_1	×
Mixed balanced contributions	×	×	✓ C_2
Balanced link contributions	×	✓	×

Table 6.1: Summary of properties

Thus, firstly, for the DC-value, connection efficiency and fairness provide a first characterisation. And connection efficiency and balanced contributions provide a second characterisation. In this way, the results obtained allow us to establish a deep parallelism between the Myerson value and the DC-value. The two differ only in the efficiency, which is logically strongly influenced by the type of connection.

The family of positional values introduced meets the properties of connection efficiency, balanced out-arc contributions, balanced in-arc contributions, α-balanced link contributions and balanced link contributions. From these, the properties of connection efficiency and α-balanced link contributions have been used to characterise the family of values. Again, we find a deep parallelism between the results obtained and those corresponding to the positional value for undirected communication situations. The nuances in this case come, again, from the side of connection efficiency and from the fact that directed arcs introduce a natural asymmetry in communication that means that initiators and receivers of communication may be rewarded differently.

Finally, for the mixed directed communication value, it has to fulfils the properties of mixed connection efficiency, fairness, mixed fairness, balanced contributions and mixed balanced contributions. This value has been characterised firstly through mixed connection efficiency and mixed fairness and secondly through mixed connection efficiency, balanced contributions and mixed balanced contributions.

6.2 Future research lines (in English)

The results obtained allow us to be optimistic in the analysis of TU games with restricted communication through directed networks. As has been said, the use of the DC-value to measure actor centrality in directed social networks with a game-theoretic perspective deserves further analysis by studying properties of these measures and relating them to those corresponding to the undirected case. Likewise, the analysis of the efficiency and vulnerability of networks with a game-theoretic perspective promises interesting results, being a field currently in ferment. We cannot forget a new element that has appeared in networks, namely polarisation, which is undoubtedly a buzzword. The analysis of polarisation in networks from a game-theoretic perspective is a very interesting field to which we will undoubtedly devote ourselves in the future, being even one of the objectives of the research project requested for the coming years.

Continuing the research on mixed values in situations of directed communication, a future line of research is to explore how a family of ρ^α values could be defined in which the payoff of the arcs includes the asymmetry between the initiator and receiver of the communication as a function of $\alpha \in [0,1]$. This in turn would allow us to analyse the relationship between the properties balanced out-arc contributions, balanced in-arc contributions and α-balanced link contributions with these values.

Based on the results on mixed value in López et al. (2022), in which they propose a link encapsulation method for communication situations restricted to multigraphs in which the players are the nodes and the links, we are currently working on how the mixed value varies depending on the type of connection in the multigraph: serial, parallel or other. An open line of research from this thesis is to explore the behaviour of values introduced in directed multigraphs, on which we are already working with Professor René van den Brink of the Vrije Universiteit Amsterdam. In particular, we explore the possibility of replacing the relative degrees, used for the defined family of position values, by other power or centrality measures for directed graphs and also other weights determined exogenously by, for example, bargaining, political, military, etc., power can be taken into consideration. Moreover, other types of axioms can be considered, such as monotonicity axioms related to adding/deleting arcs, or related to changes in contributions in the game.

We are also working on the extension of position value to multigraphs with results in the process of publication, which promise to be generalisable to situations with directed multigraphs.

Finally, cost allocation in river pollution (communication that is necessarily targeted) as an application of the results obtained in the thesis is a line of research that we are pursuing with Professor René van den Brink of the Vrije Universiteit Amsterdam and Professor Takayuki Oishi of Meiji Gakuin University in Tokyo.

6.3 Conclusiones (en Español)

Esta memoria tiene como objetivo principal, crear nuevos valores o reglas de asignación para situaciones en las que los jugadores de un juego TU tienen restringida su capacidad de comunicación mediante una red dirigida. Partiendo de los valores clásicos paras situaciones de comunicación, tratamos de extenderlos a este nuevo escenario y hemos caminado sobre los pasos que generaron el valor de Myerson, el valor posicional y el valor mixto.

Hemos introducido un concepto nuevo de conectividad para la comunicación de jugadores en una red dirigida. Bajo este concepto solo pueden comunicarse aquellos jugadores que estén en un trayectoria dirigida (path). Esta suposición permite modelar lo que ocurre en las cadenas de suministro, en los modelos de atribución o en las políticas de vacunación.

A partir de esta definición de conectividad, se han propuesto tres juegos TU restringidos a los dígrafos. Uno para jugadores, otro para arcos y otro para jugadores y arcos. El valor de Shapley en el primero de estos juegos, el DC-value, ha sido caracterizado en términos de connection efficiency, fairness and balanced contributions. El DC-value ha sido utilizado, en esta memoria, con éxito para definir medidas de centralidad parametrizadas por el juego en grafos dirigidos y también para introducir medidas de eficiencia y vulnerabilidad en redes dadas por un dígrafo. Tanto unas medidas como otras merecen un análisis más profundo que no ha sido incluido en esta memoria pero al que sin duda nos dedicaremos en un futuro próximo.

Para el juego de los arcos, y asumiendo que el sentido de los arcos introduce una posible asimetria entre los jugadores que los mantienen, hemos introducido una familia de valores posicionales parametrizada por $\alpha \in [0,1]$. Estos valores han sido caracterizados utilizando las propiedades de connection efficiency and α-balanced arc contributions.

Finalmente también se ha introducido, a partir de la misma idea de conectividad, un valor mixto para situaciones de comunicación dirigidas. Después de crear un juego restringido de nodos y arcos, el valor de Shapley ha sido propuesto como regla de asignación. El valor de los arcos puede ser asumido como el coste que los jugadores tienen que pagar a los que controlan las comunicaciones cuando los jugadores no son los dueños de ellas. Esto puede pasar en las conversaciones telefónicas o en los pagos por cruzar canales como el de Panamá o Suez. Este valor también ha sido caracterizado utilizando propiedades como la mixed connection efficiency, mixed fairness, balanced contributions y mixed balanced contributions.

En la siguiente tabla (Table 6.2) aparecen las propiedades satisfechas por cada uno de los valores propuestos en esta memoria y cuales de ellas han sido utilizadas para caracterizarlos (marcado con una C_i las utilizadas en la i-ésima caracterización). Cuando una propiedad aparece como axioma en la caracterización de diferentes valores muestra una cierta robustez ya que es satisfecha por diferentes reglas de reparto y con la ayuda de otros axiomas implica esos diferentes repartos. Como se ha dicho a lo largo de la memoria, las propiedades propuestas para cada uno de los valores son extensiones de las propiedades clásicas de eficiencia, equidad, contribuciones equilibradas y contribuciones equilibradas de arcos.

	DC-value	**Family position values**	**Mixed directed communication value**
Connection efficiency	✓ C_1, C_2	✓ C_1	×
Mixed connection efficiency	×	×	✓ C_1, C_2
Fairness	✓ C_1	×	✓
Mixed fairness	×	×	✓C_1
Balanced contributions	✓C_2	×	✓ C_2
Balanced out-arc contributions	×	✓	×
Balanced in-arc contributions	×	✓	×
α-balanced arc contributions	×	✓ C_1	×
Mixed balanced contributions	×	×	✓ C_2
Balanced link contributions	×	✓	×

Table 6.2: Resumen de propiedades

Así, en primer lugar, para el DC-value, connection efficiency y fairness permiten una primera caracterización. Y connection efficiency y balanced contributions proporcionan una segunda caracterización. De esta manera los resultados obtenidos permiten establecer un hondo paralelismo entre el valor de Myerson y el DC-value. Ambos se separan solo en la eficiencia, que lógicamente viene fuertemente influida por el tipo de conexión.

La familia de valores posicionales introducida, cumple las propiedades de connection efficiency, balanced out-arc contributions, balanced in-arc contributions, α-balanced link contributions and balanced link contributions. De ellas se han utilizado las propiedades de connection efficiency y α-balanced link contributions para caracterizar la familia de valores. De nuevo, encontramos un profundo paralelismo entre los resultados obtenidos y los correspondientes al valor posicional para situaciones de comunicación no dirigidas. Los matices en este caso vienen, de nuevo, del lado de la eficiencia en la conexión y del hecho de que los arcos dirigidos introducen una natural asimetría en la comunicación que hace que iniciadores y receptores de esta puedan ser retribuidos de manera diferente.

Por último, para el valor de comunicación dirigida mixto, se tiene que cumple las propiedades de mixed connection efficiency, fairness, mixed fairness, balanced contributions y mixed balanced contributions. Este valor ha sido caracterizado en primer lugar a través de mixed connection efficiency y mixed fairness y en segundo lugar a través de mixed connection efficiency, balanced contributions y mixed balanced contributions.

6.4 Futuras líneas de investigación (en Español)

Los resultados obtenidos permiten ser optimistas en el análisis de los juegos TU con comunicación restringida por redes dirigidas. Como se ha dicho el uso del DC-value para medir centralidad de actores en redes sociales dirigidas con perspectiva juego-teórica merece un análisis más profundo estudiando propiedades de estas medidas y relacionándolas con las correspondientes al caso no dirigido. Así mismo el análisis de la eficiencia y vulnerabilidad de las redes con una perspectiva juego-teórica promete resultados interesantes siendo un campo actualmente en ebullición. No podemos olvidar un elemento nuevo que ha aparecido en las redes y que es la polarización, palabra sin duda alguna de moda. El análisis de la polarización en redes con una perspectiva juego-teórica es un campo muy interesante al que sin duda nos vamos a dedicar en el futuro, siendo incluso uno de los objetivos del proyecto de investigación solicitado para los próximos años.

Continuando con la investigación sobre los valores mixtos en situaciones de comunicación dirigida, una futura línea de investigación es explorar cómo se podría definir un familia de valores ρ^α en los que el valor de los arcos incluyera la asimetría entre el iniciador y el receptor de la comunicación en función de $\alpha \in [0, 1]$. Ello a su vez permitiría analizar la relación entre las propiedades balanced out-arc contributions, balanced in-arc contributions y α-balanced link contributions con dichos valores.

A partir de los resultados sobre el valor mixto en López et al. (2022), en el que proponen un método de encapsulado de links para situaciones de comunicación restringida a multigrafos en las que los jugadores son los nodos y los links, actualmente estamos trabajando en como varía el valor mixto en función del tipo de conexión en el multigrafo: serie, paralelo u otras. Una línea de investigación abierta a partir de este tesis, es explorar el comportamiento de los valores introducidos en multigrafos dirigidos, en la que ya estamos trabajando con el profesor René van den Brink de la Universidad Libre de Ámsterdam. En particular, exploramos la posibilidad de sustituir los grados relativos, utilizados para la familia de valores posicionales definida, por otras medidas de poder o centralidad para grafos dirigidos y también tener en cuenta otros pesos determinados exógenamente por ejemplo, por el poder de negociación, político, militar, etc. Además, se pueden considerar otros tipos de axiomas, como los axiomas de monotonicidad relacionados con la adición/eliminación de arcos, o relacionados con los cambios en las contribuciones en el juego.

También estamos trabajando en la extensión del valor posicional a multigrafos con resultados en vías de publicación, que prometen ser generalizos a situaciones con multigrafos dirigidos.

Finalmente, la asignación de costes en la contaminación de ríos (comunicación que es necesariamente dirigida) como aplicación de los resultados obtenidos en la tesis es una línea de investigación a la que nos estamos dedicando con el profesor René van den Brink de la Universidad Libre de Ámsterdam y con el profesor Takayuki Oishi de la Universidad Meiji Gakuin de Tokio.

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