Realization of all Dold's congruences with stability

Francisco R. Ruiz del Portal and José M. Salazar *

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Dedicated to professor José M. Montesinos in the occasion of his 65th birthday and to the memory of professor Julián Martínez.

Abstract

The main goal of this paper is to prove that for each $n > 2$, every sequence of integers satisfying Dold's congruences is realized as the sequence of fixed point indices of the iterates of an orientation preserving $\mathbb{R}^n$-homeomorphism at an isolated stable fixed point. We use Conley index techniques even though stable fixed points are not isolated invariant sets.

1. Introduction.

Let $U \subset \mathbb{R}^2$ be an open set. Given a homeomorphism $f : U \to f(U) \subset \mathbb{R}^2$ and an isolated fixed point $p \in U$, the fixed point index of f at p, $i(f, p)$, is a well defined integer which provides information about the dynamical behavior of f near p. Dancer, Ortega and the first author showed that if p is a stable fixed point then $i(f, p) = 1$, the Euler characteristic of a disc, (see [11] and [19] for proofs in the orientation preserving and reversing cases respectively). For $n > 2$ and $\mathbb{R}^n$-homeomorphisms this result is not longer true. Indeed, if $n > 3$ Bobylek and Krasnosels'kii in [3] (see also the book of Krasnosels'kii and Zabreiko [17]) and Erle ([13]) proved independently that the fixed point index at stable fixed points can be any integer because of the existence of positively invariant neighborhoods (ANR's) with arbitrary Euler characteristic. Bonatti and Villadelprat, in [2], improved substantially the above results showing that for $\mathbb{R}^n$-vector fields, with $n \geq 3$, the index of stable, even in the past and in the future, isolated rest points can be any integer.

The sequence of indices $\{i(f^n, p)\}_{n\in\mathbb{N}}$, when p is not limit of periodic orbits, contains much more dynamical information than just the index $i(f, p)$.

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The problem of the computation of the above sequence has been completely understood for planar homeomorphisms. It is known, see the papers of Le Calvez and Yoccoz, [5], [6], and Le Calvez, [7], for the orientation preserving case and the authors's [20] and [21] for the orientation reversing case, that the sequence is periodic with a very particular pattern. Some results are known for the sequence of indices of $\mathbb{R}^n$-homeomorphisms if $n > 2$. Since Dold, [12], it is known that the sequence $\{i(f^n, p)\}_{n\in\mathbb{N}}$ must satisfy some rules, called Dold's congruences. Shub and Sullivan proved that for C^1-maps the sequence is bounded. Chow, Mallet-Paret and Yorke ([9]) gave bounds about the form of the sequence of indices in terms of the spectrum of the derivative $Df(p)$. Babenko and Bogatyi ([1]) proved that these bounds are sharp in dimension 2 and in a more recent paper Graff and Nowak-Przygodzki have proved ([15]) that for $n = 3$ and C^1-maps, the sequence of fixed point indices follows one among exactly seven different periodic patterns. Le Calvez, Ruiz del Portal and Salazar, in [8], proved that the sequence is periodic for $\mathbb{R}^3$-homeomorphisms at fixed points which are isolated invariant sets and, conversely, any periodic sequence satisfying Dold's congruences is realized as the sequence of fixed point indices of a $\mathbb{R}^3$-homeomorphisms at such a fixed point.

In dimension ≤ 2 all possible sequences realized by homeomorphisms are also realized by diffeomorphisms. One of the goals of this article is to show that in dimension > 2 this fact is very far from remaining true.

A sequence of integers $I = \{I_m\}_{m\in\mathbb{N}}$ satisfies the *Dold's congruences* if

$$\sum_{k|m} \mu(m/k) I_k \cong 0 \,(mod\, m) \quad m = 1, 2, \ldots$$

with $\mu : \mathbb{N} \to \mathbb{Z}$ the Möbius map.

Theorem 1. *[12]. A sequence of integers $I = \{I_m\}_{m\in\mathbb{N}}$ satisfies Dold's congruences if and only if there exists a map $f : X \to X$, with X an ENR, and an open subset $U \subset X$ such that for $U_1 = U, \ldots, U_m = f^{-1}(U_{m-1}) \cap U$, the set $Fix(f^m) \cap U_m$ is compact and $I_m = i_X(f^m, U_m)$.*

It is well known that there are sequences of integers that satisfy Dold's congruences that are not realized as the sequence Lefschetz numbers of the iterates of any map in any compact polyhedron (see Examples 1 and 2 of [1] and pages 67-68 of [16]). On the other hand, recently, the authors (see [22]) have proved that for $\mathbb{R}^3$-homeomorphisms the sequence of indices at stable fixed points can grow arbitrarily fast with negative indices.

In this paper we improve very much this last result providing a new characterization of the sequences of integers which satisfy Dold's congruences. The main theorem of the article is the following:

Main Theorem 1. *For every $n > 2$ and every sequence of integers $\{I_m\}_{m\in\mathbb{N}}$ satisfying Dold's congruences there exists an orientation preserving homeo-*

morphism $h : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $Fix(h) = Per(h) = \{\bar{0}\}$, $\{\bar{0}\}$ *is stable and* $i(h^m, \bar{0}) = I_m$ *for every* $m \in \mathbb{N}$.

We write $\bar{0}$ to denote the point $(0, \dots, 0) \in \mathbb{R}^n$. The paper is organized as follows: the proof of our main result is postponed to Section 3. Section 2 is devoted essentially to prove Proposition 1 i.e. that in many cases, the seven periodic patterns given in [15] can be realized by diffeomorphisms $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Fix(f) = Per(f) = \{\bar{0}\}$ and $\{\bar{0}\}$ is an isolated invariant set. We decided to present this result as a section of this article because it contains useful ingredients we will need in the proof of our Main Theorem. In Section 2 we will provide a sketch of the proof of Proposition 1 and the technical details are left to a final appendix.

2. Some definitions and Proposition 1.

Given $A \subset B \subset N$, $cl(A)$, $cl_B(A)$, $int(A)$, $int_B(A)$, $\partial(A)$ and $\partial_B(A)$ will denote the closure of A, the closure of A in B, the interior of A, the interior of A in B, the boundary of A and the boundary of A in B respectively.

Let $U \subset X$ be an open set. By a *(local) dynamical system* we mean a local homeomorphism $f : U \to X$. The *invariant part* of N, $Inv(N, f)$, is defined as the set of all $x \in N$ such that there is a full orbit γ with $x \in \gamma \subset N$.

$Inv^+(N, f)$ (resp. $Inv^-(N, f)$) will denote the set of all $x \in N$ such that $f^j(x) \in N$ for every $j \in \mathbb{N}$ (resp. $f^{-j}(x)$ is well defined and belongs to N for every $j \in \mathbb{N}$).

A compact set $S \subset X$ is *invariant* if $f(S) = S$. A compact invariant set S is *isolated with respect to* f if there exists a compact neighborhood N of S such that $Inv(N, f) = S$. The neighborhood N is called an *isolating neighborhood of* S.

An *isolating block* N is a compactum such that $cl(int(N)) = N$ and $f^{-1}(N) \cap N \cap f(N) \subset int(N)$. Isolating blocks are a special class of isolating neighborhoods.

We consider the *exit set of* N to be defined as

$$N^- = \{x \in N : f(x) \notin int(N)\}.$$

Let S be an isolated invariant set and suppose $L \subset N$ is a compact pair contained in the interior of the domain of f. The pair (N, L) is called a *filtration pair* for S (see Franks and Richeson paper [14]) provided N and L are each the closure of their interiors and

1) $cl(N \setminus L)$ is an isolating neighborhood of S,
2) L is a neighborhood of N^- in N and
3) $f(L) \cap cl(N \setminus L) = \emptyset$.

Filtration pairs are easy to construct once we have an isolating block N. In fact, for every small enough closed neighborhood L of N^-, (N, L) is a filtration pair.

If X is a locally compact ANR (absolute neighborhood retract for metric spaces), $i_X(f, S)$ will denote the fixed point index of f in a small enough neighborhood of S. The reader is referred to the text of [4], [12] and [18] for information about the fixed point index theory.

In [15] is proved that, if $f : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 map such that $\bar{0}$ is an isolated fixed point for each iteration, there are exactly 7 types of sequences of indices $\{i(f^n, \bar{0})\}_{n\in\mathbb{N}}$. The authors give a list of them and they prove that there are no further restrictions on the sequences except for those given by theorem of Chow, Mallet-Paret and Yorke. In this section we give some examples of each of these 7 sequences in which the C^1 map is a diffeomorphism of $\mathbb{R}^3$ with $\bar{0}$ an isolated invariant set and not only an isolated fixed point. The techniques employed permit us in the forthcoming sections to prove that for every sequence of integers $\{I_n\}_{n\in\mathbb{N}}$, which satisfies the Dold's necessary congruences, we can obtain a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\bar{0}$ is the only periodic orbit and $i(f^n, \bar{0}) = I_n$ for all $n \in \mathbb{N}$. Moreover, we can construct f with $\bar{0}$ to be Lyapunov stable.

Definition 1. Given $k \in \mathbb{N}$, we define the sequence $\{reg_k(n)\}_{n\in\mathbb{N}}$ as follows:

$$reg_k(n) = \begin{cases} k & \text{if } k|n \\ 0 & \text{if } k \nmid n \end{cases}$$

Theorem 2. *[15] Let f be a C^1 self map of $\mathbb{R}^3$. The sequence of indices $\{i(f^n, \bar{0})\}_{n\in\mathbb{N}}$ has one of the following forms:*

(A) $c_A(n) = a_1 reg_1(n) + a_2 reg_2(n)$.

(B) $c_B(n) = reg_1(n) + a_d reg_d(n)$.

(C) $c_C(n) = -reg_1(n) + a_d reg_d(n)$.

(D) $c_D(n) = a_d reg_d(n)$.

(E) $c_E(n) = reg_1(n) - reg_2(n) + a_d reg_d(n)$.

(F) $c_F(n) = reg_1(n) + a_d reg_d(n) + a_{2d} reg_{2d}(n)$ *where d is odd.*

(G) $c_G(n) = reg_1(n) - reg_2(n) + a_d reg_d(n) + a_{2d} reg_{2d}(n)$, *where d is odd.*

In all cases $d \geq 3$ and $a_i \in \mathbb{Z}$.

After some preliminary constructions of special dynamics in adequate sectors of $\mathbb{R}^3$ in the next proposition we will give a method for the construction of diffeomorphisms of $\mathbb{R}^3$, with $\{\bar{0}\}$ an isolated invariant set, for every sequence of types (B), (C) and (D) and for some situations of the remaining types.

First of all, let us make a partition of $\mathbb{R}^3$ in 5 sectors $\{S_i\}_{i=0}^4$ with a particular dynamics in each of them. We define S_i with spherical coordinates:

$$S_i = \{(\rho, \theta, \phi) : \phi \in [i\pi/5, (i+1)\pi/5]\} \qquad \text{with } i = 0, \dots, 4.$$

It is obvious that $\cup_{i=0}^4 S_i = \mathbb{R}^3$.

Let us denote

$$\phi_{[a,b]} = \{(\rho, \theta, \phi) : \phi \in [a, b]\}.$$

We will construct diffeomorphisms $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f|_{S_k}$ is invariant on each S_k and $f|_{S_k}$ has a special dynamical behavior, which we call *canonical.* Let us describe them:

Given a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $Per(f) = \{\bar{0}\}$ an isolated invariant set, and f invariant on each S_k, let us consider the unit ball $N = B(\bar{0}, 1)$. We define:

Dynamics of type 1. We say that $f|_{S_k}$ has *dynamical behavior of type 1* if $\bar{0}$ is an attracting fixed point for $f|_{S_k}$. We suppose that the exit set L_k of $f|_{N\cap S_k}$ is empty. See figure 1.

Dynamics of type 2. We say that $f|_{S_k}$ ($k \neq 0, 4$) has *dynamical behavior of type 2* if:

- $\bar{0}$ is an attracting fixed point for $f|_{\partial(S_k)}$.
- If we define
$$r_j = \left\{ (\rho, \theta, \phi) : \theta = \theta_0 + \frac{2\pi j}{m}, \phi = (2k+1)\frac{\pi}{10} \right\}$$
with $j = 1, \ldots, m$ and θ_0 constant, then f is invariant on each r_j and $\bar{0}$ is a repelling fixed point for $f|_{r_j}$. We will use the sets r_j with $j = 1, \ldots, m$ to produce sequences reg_m.
- The dynamics of f restricted to an adequate solid cone $U_{j,k} \subset S_k$ with axis r_j and vertex $\bar{0}$ ($U_{j,k}$ is isometric to $\phi_{[0,\delta]}$ for $\delta > 0$ small enough) will be of hyperbolic type with stable manifold the boundary of $U_{j,k}$ and unstable manifold r_j. We have a topological conjugation with the map $\pi : R^+ \to R^+$,
$$\pi(\bar{x}) = A\bar{x} \quad \text{with } A = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
and $R^+ = \{(\rho, \theta, \phi) : \phi \in [0, \pi/2]\}$.

  The exit set L_k of $f|_{N\cap S_k}$ is a family of m balls, $\{L_{1,k}, \ldots, L_{m,k}\}$, invariant under a rotation of angle $\frac{2\pi}{m}$ around the Z axis and such that $L_{j,k} \cap \partial N$ is a closed disc for all j. See figure 1.

Dynamics of type 3. We say that $f|_{S_k}$ ($k \neq 0, 4$) has *dynamical behavior of type 3* if:

- $\bar{0}$ is an attracting fixed point for $f|_{\partial(S_k)}$.
- If we define the family $\{r_j\}$ as above, then f is invariant on each r_j and $\bar{0}$ is an attracting fixed point for $f|_{r_j}$, $j = 1, \ldots, m$.
- The dynamics of f restricted to an adequate solid conical region $U_{j,k} \subset S_k$ defined as above, with axis r_j and vertex $\bar{0}$, will be of hyperbolic type with unstable manifold the boundary of $U_{j,k}$ and stable manifold r_j. We have a topological conjugation with the map $\pi : R^+ \to R^+$,

$$\pi(\bar{x}) = A\bar{x} \quad \text{with } A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

The exit set L_k of $f|_{N\cap S_k}$ is a solid $(m+1)$-torus, that is, the connected sum of $m+1$ 2-dimensional tori, which is invariant under a rotation of angle $\frac{2\pi}{m}$ around the Z axis and such that $L_k \cap \partial N$ is a closed disc with $m+1$ holes. See figure 1.

Dynamics of type 4. We say that $f|_{S_k}$ $(k = 0, 4)$ has *dynamical behavior of type 4* if it is of hyperbolic type with stable manifold $\partial(S_k)$ and unstable manifold the Z axis. The exit set L_k of $f_{S_k \cap N}$ is a closed ball such that $L \cap \partial N$ is a closed disc. See figure 1.

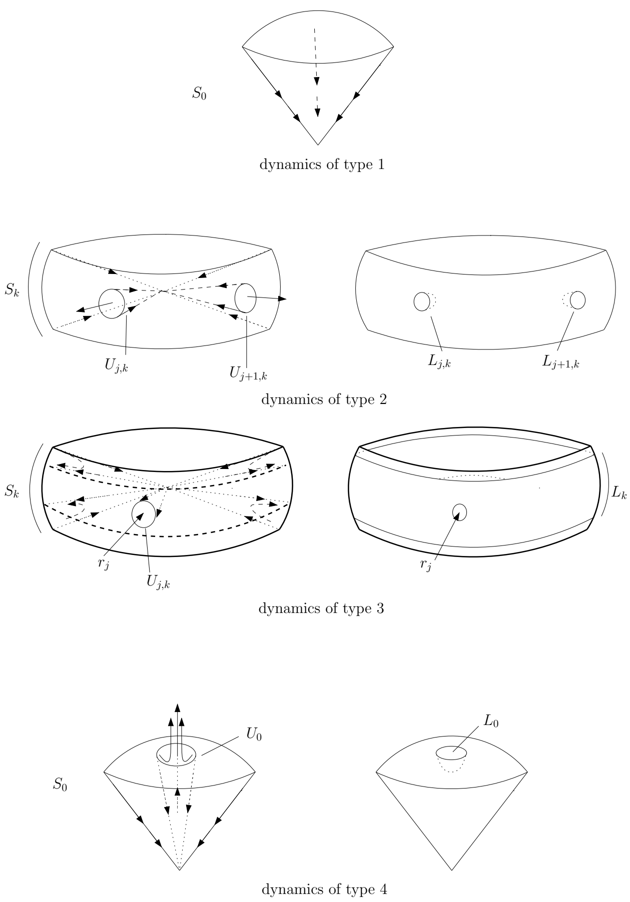


Figure 1

Proposition 1. *There exist diffeomorphisms $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $Fix(f) = Per(f) = \{\bar{0}\}$ an isolated invariant set, for the following cases of the sequences of Theorem 2:*

The case (A) if $a_1 \leq 1$ or $a_2 = 0$.
The cases (B), (C) and (D).
The case (E) if $a_d \leq 0$ or d even.
The cases (F) and (G) if $a_d \leq 0$.

Sketch of the proof of Proposition 1.

Let us select a representative case and prove it in detail. The rest of the cases are variations of this one and we leave the proofs to the reader.

Let us consider the case (A) with $a_1 < 1$ and $a_2 > 0$. The sequence is periodic of period 2 and has the form

$$c_A(n) = \{a_1, a_1 + 2a_2, \dots\}.$$

We will construct a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $Per(f) = \{\bar{0}\}$ an isolated invariant set, such that $i(f^n, \bar{0}) = c_A(n)$ for all $n \in \mathbb{N}$.

The diffeomorphism f will be the composition of two diffeomorphisms $f = s \circ f_0$, where s is a symmetry with respect to the plane $\{z = 0\}$ and f_0 has the following behavior (see figure 2):

- Dynamics of type 1 on S_0 and S_4.
- Dynamics of type 3 on S_1 and S_3. The exit sets L_k of $f_0|_{S_k}$, $k = 1, 3$, are two solid (a_2+1)-tori, invariant under a rotation of angle $\frac{2\pi}{a_2}$ around the Z axis and such that the symmetry with respect to the plane $\{z = 0\}$ sends L_1 to L_3.
- Dynamics of type 2 on S_2. The exit set L_2 is a family of $-a_1+1$ closed balls $\{L_{j,2}\}$, $j = 1, \dots, -a_1 + 1$. L_2 is invariant under a rotation of angle $\frac{2\pi}{-a_1+1}$ around the Z axis and under the symmetry with respect to the plane $\{z = 0\}$.

Let us compute the sequence of indices $\{i(f^n, \bar{0})\}$ for the above f. The first step is to consider a filtration pair (N, L) with N a ball and to identify each connected component of the exit set L of $f|_N$ to a point. We obtain the quotient space N_L and an induced map $\bar{f} : N_L \to N_L$. The space N_L has the homotopy type of a pointed union of $2a_2 + 2$ spheres and $\bar{f}$ has $-a_1 + 1$ attracting fixed points and an attracting periodic orbit of period 2. The fixed points correspond to the closed balls of the exit set and the periodic orbit of period two corresponds to the two solid tori of the exit set. Then

$$\Lambda\left((\bar{f})^{2m+1}\right) = 1 = i_{N_L}(\bar{f}^{2m+1}, \bar{0}) + (-a_1 + 1)$$

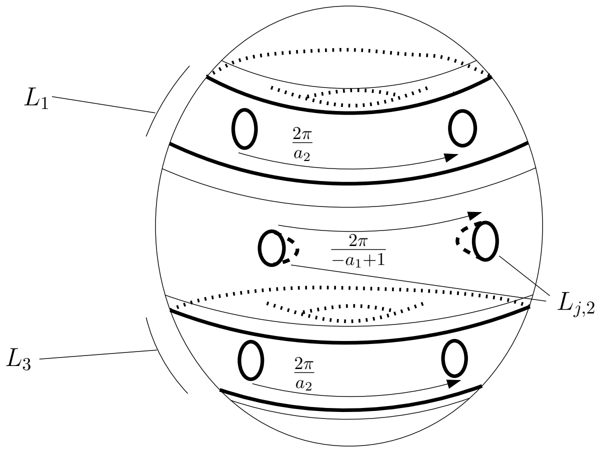
and $i(f^n, \bar{0}) = i_{N_L}(\bar{f}^n, \bar{0}) = a_1$ for n odd.

On the other hand,

$$\Lambda\left((\bar{f})^{2m}\right) = 1 + 2a_2 + 2 = i_{N_L}(\bar{f}^{2m}, \bar{0}) + (-a_1 + 3)$$

and $i(f^n,\bar{0}) = i_{N_L}(\bar{f}^n,\bar{0}) = a_1 + 2a_2$ for n even.

The explicit construction of this map f_0 is not basic in our work and is left to an appendix at the end of the paper.



Pair (N,L) associated to f

Figure 2

As a final remark, let us comment the case (G) $\quad c_G(n) = reg_1(n) - reg_2(n) + a_d reg_d(n) + a_{2d} reg_{2d}(n)$ with d odd and $a_d \leq 0, a_{2d} \geq 0$. The sequence is periodic of period $2d$ and has the form

$$\{\overbrace{1,-1,\ldots,1+da_d}^{d},\overbrace{-1,1,\ldots,-1+da_d+2da_{2d}}^{d},\ldots\}.$$

Let f_0 be a diffeomorphism of $\mathbb{R}^3$ with the following dynamical behavior:

- Dynamics of type 4 on S_0 and S_4.
- Dynamics of type 3 on S_1 and S_3 with da_{2d} solid conical regions $\{U_{j,k}\}$ in each S_k for $k = 1,3$. The exit sets are two solid $(da_{2d}+1)$-tori L_1 and L_3 defined as in the case (A).
- Dynamics of type 2 on S_2 with $-da_d$ solid conical regions $\{U_{j,2}\}$, defined as in the case (A).

It is easy to see that the map $f = s \circ g \circ f_0$ with s the symmetry with respect to the plane $\{z = 0\}$ and g a rotation around the Z axis of angle $\frac{2\pi}{d}$ is a diffeomorphism such that $i(f^n,\bar{0}) = c_G(n)$ for all $n \in \mathbb{N}$. The proof is analogous to the given for the case (A) with $a_1 < 1$ and $a_2 > 0$.

The next table is a summary of the form of f in all the cases of Proposition 1. The map s represents the symmetry with respect to the plane $\{z = 0\}$ and the map g is the rotation around the Z axis of angle $2\pi/d$.

		Cases	Dynamics of f_0
Case A	$f = s \circ f_0$	$a_1 \leq 1$	S_0, S_4 type 1 S_1, S_3 type 3 (exit set a solid $(\lvert a_2 \rvert + 1)$-torus) S_2 type 2 (exit set $-a_1 + 1$ closed balls)
		$a_2 = 0, a_1 > 1$	S_0, S_1, S_3, S_4 type 1 S_2 type 3 (exit set a solid a_1-torus)
Case B	$f = g \circ f_0$	$a_d \geq 0$	S_0, S_1, S_3, S_4 type 1 S_2 type 3 (exit set a solid $(da_d + 1)$-torus)
		$a_d < 0$	S_0, S_1, S_3, S_4 type 1 S_2 type 2 (exit set $-da_d$ closed balls)
Case C	$f = g \circ f_0$	$a_d \geq 0$	S_0, S_4 type 4 S_1, S_3 type 1 S_2 type 3 (exit set a solid $(da_d + 1)$-torus)
		$a_d < 0$	S_0, S_4 type 4 S_1, S_3 type 1 S_2 type 2 (exit set $-da_d$ closed balls)
Case D	$f = g \circ f_0$	$a_d \geq 0$	S_0 type 4 S_1, S_3, S_4 type 1 S_2 type 3 (exit set a solid $(da_d + 1)$- torus)
		$a_d < 0$	S_0 type 4 S_1, S_3, S_4 type 1 S_2 type 2 (exit set $-da_d$ closed balls)
Case E	$f = s \circ g \circ f_0$	d even and $a_d < 0$ or d odd and $a_d \leq 0$	S_0, S_4 type 4 S_1, S_3 type 1 S_2 type 2 (exit set $-da_d$ closed balls)
		d even and $a_d \geq 0$	S_0, S_4 type 4 S_1, S_3 type 1 S_2 type 3 (exit set a solid $(da_d + 1)$- torus)
Case F	$f = s \circ g \circ f_0$	$a_d \leq 0, a_{2d} < 0$	S_0, S_4 type 1 S_1, S_3 type 2 (exit set $-da_{2d}$ closed balls) S_2 type 2 (exit set $-da_d$ closed balls)
		$a_d \leq 0, a_{2d} \geq 0$	S_0, S_4 type 1 S_1, S_3 type 3 (exit set a solid $(da_{2d} + 1)$-torus) S_2 type 2 (exit set $-da_d$ closed balls)
Case G	$f = s \circ g \circ f_0$	$a_d \leq 0, a_{2d} < 0$	S_0, S_4 type 4 S_1, S_3 type 2 (exit set $-da_{2d}$ closed balls) S_2 type 2 (exit set $-da_d$ closed balls)
		$a_d \leq 0, a_{2d} \geq 0$	S_0, S_4 type 4 S_1, S_3 type 3 (exit set a solid $(da_{2d} + 1)$-torus) S_2 type 2 (exit set $-da_d$ closed balls)

Question 1. Is Proposition 1 optimal? In other words, if the sequence of fixed point indices of the iterates of a $\mathbb{R}^3$-diffeomorphism, f, at an isolated fixed point p such that $Fix(f) = Per(f) = \{p\}$, does not follow one of the patterns of Proposition 1, then $\{p\}$ is not an isolated invariant set?.

3. Proof of the Main Theorem

Let $I = \{I_m\}_{m\in\mathbb{N}}$ be a sequence of integers which satisfies Dold's congruences. The sequence of algebraic multiplicities of I, $A = \{a_m\}_{m\in\mathbb{N}}$, is a sequence of integers such that

$$a_m = \frac{1}{m}\sum_{k|m}\mu(m/k)I_k \qquad I_m = \sum_{k|m} k a_k.$$

First of all note that it suffices to prove the theorem for $\mathbb{R}^3$-homeomorphisms. Indeed, once we have a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Fix(f) = Per(f) = \{\bar{0}\}$, $\{\bar{0}\}$ is stable and $i(f^m, p) = I_m$ for every $m \in \mathbb{N}$, we consider a diffeomorphism $\beta : \mathbb{R}^{n-3} \to \mathbb{R}^{n-3}$ such that $Fix(\beta) = \{\bar{0}\}$ is a global attractor. Now, the mapping $(f, \beta) : \mathbb{R}^n \to \mathbb{R}^n$ is the desired homeomorphism.

3.1. Construction of the $\mathbb{R}^3$-homeomorphism.

Let us prove that for any sequence of integers $I = \{I_m\}_{m\in\mathbb{N}}$, which satisfies Dold's necessary congruences, there exists a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $i(f^m, \bar{0}) = I_m$ for every $m \in \mathbb{N}$ and such that $Per(f) = \{\bar{0}\}$.

Given the sequence of algebraic multiplicities A, let us construct a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with the above properties. Let $N = B(\bar{0}, 1)$ be the unit closed ball, and let A^+ and A^- be the subsequences of positive and negative elements of A. We will suppose that the subsequence of non-zero elements of A, $A^\pm$, is infinite (if this is not the case, the sequence I is periodic and the proof is analogous).

Our first step is to make a partition of $\mathbb{R}^3$ in different solid regions. Let

$$R_n = \{(\rho, \theta, \phi) : \phi \in [0, \phi_n]\} = \phi_{[0,\phi_n]} \subset \mathbb{R}^3, \qquad \phi_n < \pi/2$$

be solid regions such that $\phi_n < \phi_{n+1}$ and $\lim_{n\to\infty} \phi_n = \pi/2$, that is,

$$R_n \subsetneq R_{n+1} \subsetneq \cdots \subset R^+ \quad \text{and} \quad cl(\bigcup R_n) = R^+$$

with $R^+ = \{(\rho, \theta, \phi) : \phi \in [0, \pi/2]\}$.

We define the different solid regions S_n, in which f will have a characteristic dynamical behavior, in the next way:

Let
$S_0 = R_0 = \phi_{[0,\pi/4]}$,
$S_n = cl(R_n \setminus R_{n-1}) = \phi_{\left[\phi_{n-1}=\frac{\pi}{2}-\frac{\pi}{2^{n+1}},\phi_n=\frac{\pi}{2}-\frac{\pi}{2^{n+2}}\right]}$ for $n \geq 1$.
$S_\infty = \phi_{[\pi/2,\pi]} = R^-$.
We have a partition of $\mathbb{R}^3$,

$$\mathbb{R}^3 = \bigcup_{m=0}^{\infty} S_m$$

with $S_n \cap S_{n+1} = \{(\rho, \theta, \phi) : \phi = \phi_n\}$ and $S_i \cap S_j = \{\bar{0}\}$ if $j \notin \{i-1, i, i+1\}$.

We will define the homeomorphism f as the composition of two homeomorphisms $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ with $f = f_0 \circ g_0$.

The behavior of f_0 in the regions $\bigcup_{\substack{n \in 2\mathbb{N} \\ i \neq 0}} S_n$ is the following:

$$f_0(\rho, \theta, \phi) = (k_n(\phi)\rho, \theta, \phi)$$

with

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{n+1}}, \frac{\pi}{2} - \frac{\pi}{2^{n+2}}\right] \to \left[1 - \frac{1}{2^{n-2}}, 1 - \frac{1}{2^n}\right]$$

an increasing, bijective and linear map.

On the other hand, the dynamics of f_0 in S_∞ is

$$f_0(\rho, \theta, \phi) = \left(\left(\frac{3}{2} - \frac{\phi}{\pi}\right)\rho, \theta, \phi\right).$$

In other words, **the dynamics of $f_0|_{S_n}$, with $n \neq 0$ even, and $f_0|_{S_\infty}$ are of type 1**. See figure 3.

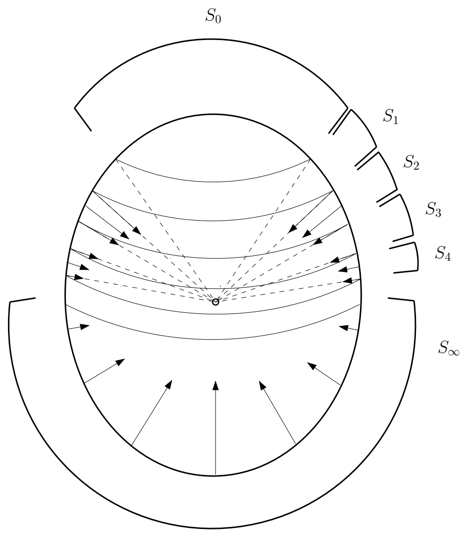


Figure 3

If $A^\pm = \{a_{j_m}\}_{m \in \mathbb{N}}$, let $\{r_m\}_{m \in \mathbb{N}} = \{p_m/j_m\}_{m \in \mathbb{N}}$ be a sequence of rational numbers converging to an irrational number r. We can construct the sequence $\{r_m\}_{m \in \mathbb{N}}$ with $0 < r < 1$ in the following way:

For each j_m we consider a partition of the unit interval $[0, 1]$ in j_m intervals of length $1/j_m$ and select $p_m < j_m$ as the natural number such that $d(p_m/j_m, r) = \min\{d(n/j_m, r)\}$ with $n \in \mathbb{N}$. Then, the sequence $\{p_m/j_m\}_{m\in\mathbb{N}} \to r$ when $m \to \infty$.

We consider, for each a_{j_m} of A^-, a **dynamics of type 2** in the solid region S_n, with $n = 2m-1$ odd, with a family of $j = 1, \ldots, -j_m a_{j_m}$ isometric solid regions $\{U_{j,m}\}_j$, as in section 2, invariant under a rotation around the vertical axis of angle $\frac{2\pi}{-j_m a_{j_m}}$.

The dynamics in the set $cl(S_{2m-1} \setminus \bigcup_j U_{j,m})$ are

$$f_0(\rho, \theta, \phi) = \left(\left(1 - \frac{1}{2^{2m-2}}\right)\rho, \theta, \phi\right)$$

and the dynamical behavior in the regions $U_{j,m}$ is hyperbolic, topologically conjugated with the map $\pi : R^+ \to R^+$,

$$\pi(\bar{x}) = A\bar{x} \quad \text{ with } A = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and commutes with a rotation of angle $\frac{2\pi}{-j_m a_{j_m}}$ around the Z axis.

The behavior of f_0 on the set S_n must be such that $d(f_0(\bar{x}), \bar{x}) \leq k_n \|\bar{x}\|$ for all $\bar{x} \in S_n$ where $k_n \to 0$ when $n \to \infty$.

If we restrict our interest to the dynamics in the unit ball N, we obtain a family of $k = 1, \ldots, -j_m a_{j_m}$ closed balls $\{L_{j,m}\}_k$ which are the exit set for $f_0|_{S_n}$. The balls have in S_n a constant angle $\frac{2\pi}{-j_m a_{j_m}}$ around the Z axis.

If $a_{j_m} \in A^+$ we consider in S_n, with $n = 2m-1$ odd, a family of $j_m a_{j_m}$ isometric solid regions $\{U_{j,m}\}$, such that $f_0|_{S_n}$ has **dynamics of type 3**, in each of the solid regions $U_{j,m}$ it is hyperbolic, topologically conjugated with the map $\pi : R^+ \to R^+$ defined as

$$\pi(\bar{x}) = A\bar{x} \quad \text{ with } A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

We construct the map in the set S_n in such a way that $d(f_0(\bar{x}), \bar{x}) \leq k_n \|\bar{x}\|$ for all $\bar{x} \in S_n$ where $k_n \to 0$ when $n \to \infty$.

If we pay attention to the unit ball N (which is an isolating block), the exit region of $f_0|_{S_n}$ is a solid $(j_m a_{j_m}+1)$-torus L_m such that $L_m \cap \partial N$ is a disc with $j_m a_{j_m} + 1$ holes which contains in its interior the set $(\bigcup \partial U_{j,m}) \cap \partial N$. The holes of $L_m \cap \partial N$ are distributed in the following way: one of them contains the north pole, and the remaining $j_m a_{j_m}$ are separated by a fixed angle $\frac{2\pi}{j_m a_{j_m}}$ around the Z axis.

Let us observe that as $n \in \mathbb{N}$ increases, the points move slower by f_0, that is, $f_0(x) \approx x$ if $x \in S_n \cap N$ with n big enough and $f_0(x) = x$ if $x \in N \cap \{z = 0\}$.

The map $f_0|_{S_0}$ has *dynamics of type 4*.

The exit set of $f_0|_{S_0\cap N}$ is a closed ball $L_0 \subset S_0 \cap N$ with $L_0 \cap \partial N$ a closed disc.

It is not difficult to see that the map $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism, limit of homeomorphisms $\{f_{0,n}\}_n$ with $f_{0,n} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f_{0,n}(\bar{x}) = \begin{cases} f_0(\bar{x}) & \text{if } \bar{x} \in R_{2n} \cup R^-_{2n-2} \\ \left(1 - \frac{1}{2^{2n}}\right)\bar{x} & \text{if } \bar{x} \in \mathbb{R}^3 \setminus (R_{2n} \cup R^-_{2n-2}) \end{cases}$$

with $R^-_n = \{\bar{x} \in \mathbb{R}^3 : -\bar{x} \in R_n\}$.

Let us observe that $Fix(f_{0,n}) = Per(f_{0,n}) = Inv(N, f_{0,n}) = \{\bar{0}\}$ and $Fix(f_0) = Per(f_0) = Inv(N, f_0) = N \cap \{x_3 = 0\}$ with $N^-(f_0) = \{\bar{x} \in N : f_0(\bar{x}) \notin int(N)\} = \bigcup L_{j,m} \cup \bigcup L_m \cup (\{x_3 = 0\} \cap \partial(N)) \cup L_0$.

The homeomorphism $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is defined in the following way:

$g_0|_{S_n}$ with $n = 2m - 1$ odd is a rotation around the Z axis with angle $2\pi\frac{p_m}{j_m}$, that is,

$$g_0|_{S_n}(\rho, \theta, \phi) = (\rho, \theta + 2\pi\frac{p_m}{j_m}, \phi)$$

$g_0|_{S_\infty}$ and $g_0|_{S_0}$ are rotations around the Z axis with angles $2\pi r$ and $2\pi\frac{p_1}{j_1}$ respectively, that is,

$$g_0|_{S_0}(\rho, \theta, \phi) = (\rho, \theta + 2\pi\frac{p_1}{j_1}, \phi) \quad \text{and} \quad g_0|_{S_\infty}(\rho, \theta, \phi) = (\rho, \theta + 2\pi r, \phi).$$

The dynamical behavior of $g_0|_{S_n}$ with $n = 2m$ even is as follows. Since $g_0|_{\partial R_{n-1}}$ and $g_0|_{\partial R_n}$ are rotations with angles $2\pi\frac{p_m}{j_m}$ and $2\pi\frac{p_{m+1}}{j_{m+1}}$, we define $g_0|_{S_n}$ as

$$g_0|_{S_n}(\rho, \theta, \phi) = (\rho, \theta + k_n(\phi), \phi)$$

where

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{2m+1}}, \frac{\pi}{2} - \frac{\pi}{2^{2m+2}}\right] \to \left[2\pi\frac{p_m}{j_m}, 2\pi\frac{p_{m+1}}{j_{m+1}}\right]$$

is a bijective and linear map. See figure 4.

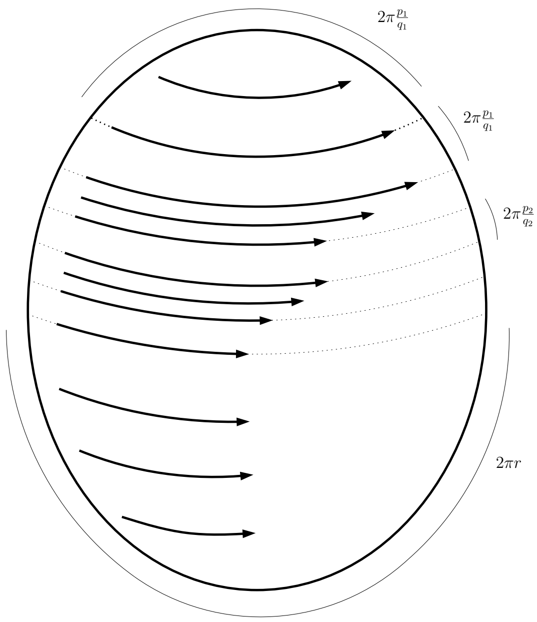


Figure 4

The map $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ constructed is a homeomorphism, limit of homeomorphisms $\{g_{0,n}\}_n$ with $g_{0,n} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows:

$$g_{0,n}(\rho, \theta, \phi) = \begin{cases} g_0(\rho, \theta, \phi) & \text{if } (\rho, \theta, \phi) \in R_{2n} \cup R^-_{2n-2} \\ (\rho, \theta + k_n(\phi), \phi) & \text{if } (\rho, \theta, \phi) \notin R_{2n} \cup R^-_{2n-2} \end{cases}$$

with

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{2n+2}}, \frac{\pi}{2} + \frac{\pi}{2^{2n}}\right] \to \left[2\pi\frac{p_{n+1}}{j_{n+1}}, 2\pi r\right].$$

The map $f = f_0 \circ g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism with $Fix(f) = Per(f) = \{\bar{0}\}$ and $Inv(N, f) = N \cap \{x_3 = 0\}$.

Our aim is to prove that $i(f^n, \bar{0}) = I_n$ for every $n \in \mathbb{N}$.

Let us consider the sequence of homeomorphisms $\{f_n\}_{n\in\mathbb{N}}$ where $f_n = f_{0,n} \circ g_{0,n} : \mathbb{R}^3 \to \mathbb{R}^3$. It is obvious that $Fix(f_n) = Per(f_n) = Inv(N, f_n) = \{\bar{0}\}$ and $f_n \to f$ when $n \to \infty$.

The computation of the index $i(f^d, 0)$ needs the following proposition.

Proposition 2. *Let X be a metric ANR, W an open subset of X and $F : cl(W) \times [0, 1] \to X$ a continuous and compact map such that $F(x, t) \neq x$ for $(x, t) \in \partial(W) \times [0, 1]$. Then $i_X(F_t, W)$ is constant for $0 \leq t \leq 1$.*

If we fix $d \in \mathbb{N}$, the map $f^d|_N : N \to \mathbb{R}^3$ can be approximated by the maps $f^d_n|_N : N \to \mathbb{R}^3$. It is easy to see that there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we can obtain a homotopy $H : N \times I \to \mathbb{R}^3$ with $H_0 = f^d$, $H_1 = f_n^d$ and $H(\bar{x}, t) \neq \bar{x}$ for all $\bar{x} \in \partial N$ and $t \in [0, 1]$. From the last proposition we obtain that $i(f^d, \bar{0}) = i(f_{n_0}^d, \bar{0}) = i(f_n^d, \bar{0})$. Let us select n_0 big enough, in such a way that if $j_m | d$ then $2m - 1 \leq 2n_0$.

Our aim is to compute $i(f_{n_0}^d, \bar{0})$. Let us observe that the exit regions of N for $f_{n_0}|_N$ is a finite family of closed balls L_0 and $\{L_{j,m}\}$, and a finite family of solid $(j_m a_{j_m} + 1)$-tori $\{L_m\}$. Let us identify each of these components to points l_0, $\{l_{j,m}\}$ and $\{l_m\}$. In this way for an adequate filtration pair (N, L) we obtain a quotient space N_L and an induced map $\bar{f}_{n_0} : N_L \to N_L$ with $i_{N_L}(\bar{f}_{n_0}^d, \bar{0}) = i(f_{n_0}^d, \bar{0})$. The points $\{l_{j,m}\}$ are attracting fixed points for an adequate iteration of the induced map $\bar{f}_{n_0}$ and the points l_0 and $\{l_m\}$ are attracting fixed points for $\bar{f}_{n_0}$. The identification of each $\{L_m\}$ to a point produces a space with the homotopy type of a pointed union of $j_m a_{j_m} + 1$ spheres. In this way, the quotient space N_L has the homotopy type of a pointed union of $\displaystyle\sum_{\substack{a_{j_m} \in A^+ \\ 2m-1 \leq 2n_0}} (j_m a_{j_m} + 1)$ spheres.

If $a_{j_m} \in A^-$, the action of the map $\bar{f}_{n_0}$ on the family of points $\{l_{j,m}\}_j$ with $j = 1, \ldots, -j_m a_{j_m}$ give us a union of $-a_{j_m}$ cycles of length j_m,

$$\{l_{j,m}\}_j = \bigcup_q \{l(q, 1), \ldots, l(q, j_m)\}$$

with $q = 1, \ldots, -a_{j_m}$ such that

$$\bar{f}_{n_0}(l(q, p)) = l(q, p + 1)$$

for $p = 1, \ldots, j_m$.

It is obvious that

$$i_{N_L}(\bar{f}_{n_0}^d, l_{j,m}) = \begin{cases} 1 & \text{if } d \in j_m\mathbb{N} \\ 0 & \text{if } d \notin j_m\mathbb{N} \end{cases}$$

and, if $a_{j_m} \in A^+$,

$$i_{N_L}(\bar{f}_{n_0}^d, l_m) = 1 \qquad \text{for all } d \in \mathbb{N}.$$

Let us consider the elements of A^- and A^+ indexed by the sets $I^-(A)$ and $I^+(A)$ respectively. We have that

$$\Lambda(\bar{f}_{n_0}^d) = i_{N_L}(\bar{f}_{n_0}^d, 0) + \sum_{\substack{2m-1 \leq 2n_0 \\ j_m \in I^+(A)}} i_{N_L}(\bar{f}_{n_0}^d, l_m) +$$

$$+ \sum_{\substack{j=1,\ldots,j_m a_{j_m} \\ 2m-1 \leq 2n_0 \text{ and } j_m \in I^-(A)}} i_{N_L}(\bar{f}_{n_0}^d, l_{j,m}) + i_{N_L}(\bar{f}_{n_0}^d, l_0) =$$

$$= i_{N_L}(\bar{f}^d_{n_0}, 0) + \quad \#\{j_m \in I^+(A) : 2m-1 \le 2n_0\} + \sum_{j_m | d \text{ and } j_m \in I^-(A)} -j_m a_{j_m} + 1.$$

On the other hand,

$$\Lambda(\bar{f}^d_{n_0}) = 1 + tr(((\bar{f}_{n_0})_*)_2) = 1 + \sum_{j_m \in I^+(A) \text{ and } j_m | d} (j_m a_{j_m} + 1)$$

$$+ \quad \#\{j_m \in I^+(A) : j_m \nmid d \text{ and } 2m-1 \le 2n_0\} = 1 + \sum_{j_m \in I^+(A) \text{ and } j_m | d} j_m a_{j_m} +$$

$$+ \quad \#\{j_m \in I^+(A) : 2m - 1 \le 2n_0\}.$$

Then,

$$i(f^d, \bar{0}) = i_{N_L}(\bar{f}^d_{n_0}, \bar{0}) = \sum_{j_m \in I^+(A) \text{ and } j_m | d} j_m a_{j_m} + \sum_{j_m \in I^-(A) \text{ and } j_m | d} j_m a_{j_m} =$$

$$\sum_{j_m \in I^+(A) \cup I^-(A) \text{ and } j_m | d} j_m a_{j_m} = \sum_{k | d} k a_k = I_d.$$

3.2. Stability.

Given a sequence of indices $I = \{I_m\}$ which satisfies the Dold's congruences, let us see that there exist a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $i(h^m, \bar{0}) = I_m$ for every $m \in \mathbb{N}$ and such that $Per(h) = \{\bar{0}\}$ is Lyapunov stable.

The construction of h will be analogous to the construction of $f = g_0 \circ f_0$ given in the above section. In fact, h will have the form $h = g_0 \circ h_0$ with g_0 the same map of section 3.1. We only have to construct h_0 as an adequate modification of f_0.

Consider the solid cylinder $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1, z \in [a, b]\}$ and the flow induced by the constant vector field $Y = (0, 0, 1)$. Denote by $\sigma(B)$, $\tau(B)$ and $\beta(B)$ the lateral, top and bottom boundaries of B respectively.

A *flow box* (U, g) for a vector field X at a point p consists of a neighborhood U of p and a diffeomorphism $g : B \to U$ such that:

i) X is transverse to $g(\beta(B))$.

ii) There is a positive constant c such that $\phi(ct, g(x)) = g(\psi(t, x))$ where $\phi(t, \cdot)$ and $\psi(t, \cdot)$ denote the flows induced by X and Y on B respectively. When it is clear from the context, we shall omit the diffeomorphism g.

Let U and V be two flow boxes with $V \subset U$. Then V is called *a shrinkage* of U if $\sigma(V) \subset int(U)$, $\tau(V) \subset \tau(U)$ and $\beta(V) \subset \beta(U)$.

Let us recall the following version of Wilson's theorem ([24]) that we will need.

Theorem 3. *Let X be a C^∞ $\mathbb{R}^3$-vector field. Let U be a flow box of X and let V be a shrinkage of U. Then, there exist a C^∞ vector field X^1 on U such that:*

a) X^1 coincides with X on a neighborhood of ∂U.

b) The limit sets of X^1 are a finite collection of invariant circles on which the restricted flow is minimal.

c) Every trajectory of X^1 which intersects $\beta(V)$ remains in positive time inside U.

d) Each trajectory of X^1 which leaves U in positive and negative time coincides as a point set with some trajectory of X in a neighborhood of ∂U.

Let us consider the map f_0 of section 3.1 and let X be a vector field associated to f_0 with $X(\bar{x}, 1) = f_0(\bar{x})$. Given the sets S_0 (dynamics of type 4) and $\{S_n\}$ (with $n = 2m - 1$ odd, and $a_{j_m} \in A^-$ which give us dynamics of type 2), we call $U^- = U_0 \cup \bigcup_{a_{j_m} \in A^-} U_{j,m}$. Our aim is to modify $X|_{U^-}$ in such a way that $\bar{0}$ be a stable fixed point in U^- for the new vector field X_1.

Let us construct $X_1|_{U_{j,m}}$ with $a_{j_m} \in A^-$ (the construction of $X_1|_{U_0}$ is analogous).

We can suppose, without loss of generality that $U_{j,m} = \{(x_1, x_2, x_3) : x_2 \geq 0\}$, with unstable manifold the set $\{x_1 = 0, x_3 = 0\}$.

If we work with spherical coordinates, let us consider, for every natural number $n \geq 2$, the sets

$$B_n = \{(\rho, \theta, \phi) : \rho \in [1/n, 1/(n-1)]\}$$

and

$$D_{\delta,l} = \left\{(\rho, \theta, \phi) : \rho = 1/l, \theta \in \left[\frac{\pi}{2} - \delta\frac{\pi}{8}, \frac{\pi}{2} + \delta\frac{\pi}{8}\right], \phi \in \left[\frac{\pi}{2} - \delta\frac{\pi}{8}, \frac{\pi}{2} + \delta\frac{\pi}{8}\right]\right\}$$

for $\delta \in \{1, 2\}$.

Let us define, for each positive even integer k, the sets

$$U_k = \{\varphi(\bar{x}, t) : \bar{x} \in D_{2,k}, t \geq 0\} \cap B_k \text{ and } V_k = \{\varphi(\bar{x}, t) : \bar{x} \in D_{1,k}, t \geq 0\} \cap B_k$$

where φ represents the continuous dynamical system obtained from X.

The set V_k is a shrinkage of U_k and $U_k \cap U_{k'} = \emptyset$ if $k \neq k'$.

For each k even, let $X_{1,k}$ be the vector field obtained from the Wilson's theorem applied to X on the pair (U_k, V_k). See figure 5.

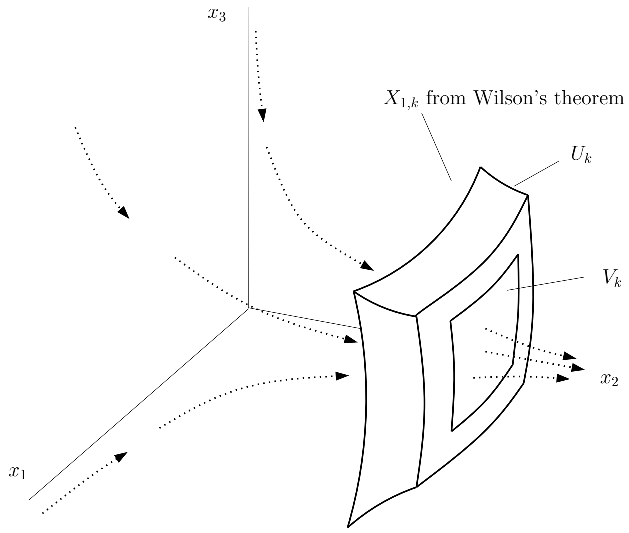


Figure 5

Now let $G : U_{j,m} \to U_{j,m}$ be the vector field defined as $G(\bar{x}) = X(\bar{x})$ if $\bar{x} \notin \bigcup_{k\in 2\mathbb{N}} U_k$ and $G(\bar{x}) = X_{1,k}(\bar{x})$ if $\bar{x} \in U_k$. Finally consider a flat enough (in $\bar{0}$) smooth non-negative real map γ, depending of $\|\bar{x}\|^2$, such that $\gamma^{-1}(0) = \{\bar{0}\}$ to obtain $X_1 = \gamma G$ to be smooth.

Let ψ_m be the flow in $U_{j,m}$ associated to X_1. The set of periodic orbits of ψ_m is countable. Then we can choose a positive and decreasing sequence $t_m \to 0$ such that $Fix(\psi_m(t_m, \cdot)) = Per(\psi_m(t_m, \cdot)) = \{\bar{0}\}$. Since each $D_{2,k}$ is a section that captures every orbit in $int(U_{j,m})$ near $\bar{0}$, it is clear that $\bar{0}$ is Lyapunov stable on U^- for the map

$$h_0|_{U^-\cup\{x_3=0\}} : U^- \cup \{x_3 = 0\} \to U^- \cup \{x_3 = 0\}$$

defined as the homeomorphism obtained by pasting copies of the homeomorphisms $\psi_m(t_m, \cdot) : U_{j,m} \to U_{j,m}$ and such that h_0 is the identity map on $\{x_3 = 0\}$.

Let us consider the sets S_n with $n = 2m - 1$ odd and $a_{j_m} \in A^+$. We call $S^+ = \bigcup_{a_{j_m}\in A^+} S_n$. Now, our aim is to construct $h_0|_{S^+}$. The dynamics of f_0 on each S_n is of type 3. We will modify $X|_{S^+}$ in such a way that $\bar{0}$ will be a stable fixed point in this region.

Given S_n with $n = 2m - 1$ and $a_{j_m} \in A^+$, we suppose, without loss of generality, that

$$S_n = \{(\rho, \theta, \phi) : \phi \in [2\pi/5, 3\pi/5]\}.$$

The exit set of $N \cap S_n$ is a solid $(j_m a_{j_m} + 1)$-torus. Let us observe the repelling behavior given in section 2 (see figure 1) for the set

$$U = \left\{ (\rho, \theta, \phi) : \phi \in \left[\frac{2\pi}{5} + \epsilon, \frac{3\pi}{5} - \epsilon \right] \right\} \setminus \bigcup int(U_{j,m}).$$

Let $D = \partial(N) \cap U$ and let us suppose that the holes of the equator have the center in the points of coordinates

$$\rho = 1, \quad \theta = \frac{2j\pi}{j_m a_{j_m}} \text{ with } j = 1, \dots, j_m a_{j_m}, \quad \phi = \pi/2.$$

We will suppose, without loss of generality, that

$$U_{j,m} = \left\{ (\rho, \theta, \phi) : \theta \in \left[\frac{2j\pi}{j_m a_{j_m}} - \epsilon_1, \frac{2j\pi}{j_m a_{j_m}} + \epsilon_1 \right], \phi \in \left[\frac{\pi}{2} - \epsilon_1, \frac{\pi}{2} + \epsilon_1 \right] \right\}.$$

Let us construct the sets

$$D^j_{\delta,l} = \left\{ (\rho, \theta, \phi) : \rho = 1/l, \theta \in \left[\frac{2j\pi}{j_m a_{j_m}} + \epsilon_1 - \delta\epsilon_2, \frac{2(j+1)\pi}{j_m a_{j_m}} - \epsilon_1 + \delta\epsilon_2 \right], \right.$$

$$\left. \phi \in \left[\frac{2\pi}{5} + \epsilon - \delta\epsilon_2, \frac{3\pi}{5} - \epsilon + \delta\epsilon_2 \right] \right\}$$

and

$$D^{j,\mathrm{up}}_{\delta,l} = \left\{ (\rho, \theta, \phi) : \rho = 1/l, \theta \in \left[\frac{2j\pi}{j_m a_{j_m}} - \epsilon_1 - \delta\epsilon_2, \frac{2j\pi}{j_m a_{j_m}} + \epsilon_1 + \delta\epsilon_2 \right], \right.$$

$$\left. \phi \in \left[\frac{2\pi}{5} + \epsilon - \delta\epsilon_2, \frac{\pi}{2} - \epsilon_1 + \delta\epsilon_2 \right] \right\}$$

$$D^{j,\mathrm{low}}_{\delta,l} = \left\{ (\rho, \theta, \phi) : \rho = 1/l, \theta \in \left[\frac{2j\pi}{j_m a_{j_m}} - \epsilon_1 - \delta\epsilon_2, \frac{2j\pi}{j_m a_{j_m}} + \epsilon_1 + \delta\epsilon_2 \right], \right.$$

$$\left. \phi \in \left[\frac{\pi}{2} + \epsilon_1 - \delta\epsilon_2, \frac{3\pi}{5} - \epsilon + \delta\epsilon_2 \right] \right\}$$

for $j = 1, \dots, j_m a_{j_m}$, $\delta \in \{1, 2\}$ and $\epsilon_2 \simeq 0$ small enough such that the sets of the two families

$$\{D^{j,\mathrm{up}}_{2,l}, D^{j,\mathrm{low}}_{2,l}\} \text{ and } \{D^j_{2,l}\}$$

are pairwise disjoint. See figure 6.

For every natural number $n \geq 2$, let

$$B_n = \{(\rho, \theta, \phi) : \rho \in [1/n, 1/(n-1)]\}.$$

Let us define, for each positive integer $k = 4m - 2 \geq 2$, the sets

$$U_k^j = \{\varphi(\bar{x}, t) : \bar{x} \in D_{2,k}^j, t \geq 0\} \cap B_k \text{ and } V_k^j = \{\varphi(\bar{x}, t) : \bar{x} \in D_{1,k}^j, t \geq 0\} \cap B_k$$

and, for each positive integer $k = 4m$, the sets

$$U_k^{j,\mathrm{up}} = \{\varphi(\bar{x}, t) : \bar{x} \in D_{2,k}^{j,\mathrm{up}}, t \geq 0\} \cap B_k \text{ and } V_k^{j,\mathrm{up}} = \{\varphi(\bar{x}, t) : \bar{x} \in D_{1,k}^{j,\mathrm{up}}, t \geq 0\} \cap B_k$$

$$U_k^{j,\mathrm{low}} = \{\varphi(\bar{x}, t) : \bar{x} \in D_{2,k}^{j,\mathrm{low}}, t \geq 0\} \cap B_k \text{ and } V_k^{j,\mathrm{low}} = \{\varphi(\bar{x}, t) : \bar{x} \in D_{1,k}^{j,\mathrm{low}}, t \geq 0\} \cap B_k$$

where φ represents the continuous dynamical system obtained from X.

The sets V_k^j, $V_k^{j,\mathrm{up}}$ and $V_k^{j,\mathrm{low}}$ are shrinkages of U_k^j, $U_k^{j,\mathrm{up}}$ and $U_k^{j,\mathrm{low}}$ respectively, which are pairwise disjoint. See figure 6.

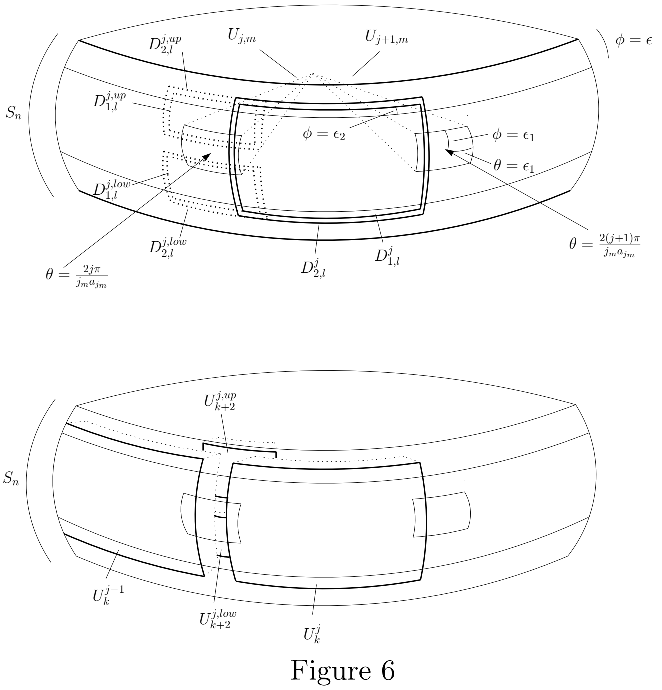


Figure 6

For each $k = 4m$ or $k = 4m - 2$, let X_k^j, $X_k^{j,\mathrm{low}}$ and $X_k^{j,\mathrm{up}}$ be the vector fields obtained from the Wilson's theorem applied on the pairs (U_k^j, V_k^j), $(U_k^{j,\mathrm{low}}, V_k^{j,\mathrm{low}})$ and $(U_k^{j,\mathrm{up}}, V_k^{j,\mathrm{up}})$ respectively.

Let $\widetilde{X} : \bigcup U_k^j \cup \bigcup U_k^{j,\mathrm{low}} \cup \bigcup U_k^{j,\mathrm{up}} \to \bigcup U_k^j \cup \bigcup U_k^{j,\mathrm{low}} \cup \bigcup U_k^{j,\mathrm{up}}$ be the vector field obtained from the above construction and let $G : S_n \to S_n$ be the vector field defined as $G(\bar{x}) = X(\bar{x})$ if $\bar{x} \notin \bigcup U_k^j \cup \bigcup U_k^{j,\mathrm{low}} \cup \bigcup U_k^{j,\mathrm{up}}$ and $G(\bar{x}) = \widetilde{X}(\bar{x})$ in other case. Let us consider, as above, a flat enough (in $\bar{0}$)

smooth non-negative real map γ, depending of $||\bar{x}||^2$, such that $\gamma^{-1}(0) = \{\bar{0}\}$ to obtain the field $X_1 = \gamma G$ to be smooth.

Let ψ_n be the flow in S_n associated to X_1. The set of periodic orbits of ψ_n is countable. Then we can choose, as above, a positive and decreasing sequence $t_n \to 0$ such that $Fix(\psi_n(t_n, \cdot)) = Per(\psi_n(t_n, \cdot)) = \{\bar{0}\}$.

Given $m_0 \in \mathbb{N}$ fixed, since the family $\bigcup_j D^j_{1,4m_0-2} \cup \bigcup_j D^{j,\text{low}}_{1,4m_0} \cup \bigcup_j D^{j,\text{up}}_{1,4m_0}$ captures every exit orbit of N in S_n near $\bar{0}$, we have that $\bar{0}$ is Lyapunov stable on the set $S^+ = \bigcup_{a_{jm} \in A^+} S_n$ for the map

$$h_0|_{S^+ \cup \{x_3=0\}} : S^+ \cup \{x_3 = 0\} \to S^+ \cup \{x_3 = 0\}$$

defined as the homeomorphism obtained by pasting copies of the homeomorphisms $\psi_n(t_n, \cdot) : S_n \to S_n$ and such that h_0 is the identity map on $\{x_3 = 0\}$. Then we have defined the homeomorphism $h_0|_{U^- \cup S^+ \cup \{x_3=0\}}$. Let us extend it to a homeomorphism $h_0 : \mathbb{R}^3 \to \mathbb{R}^3$ such that it is decreasing in each ray $\{\lambda \bar{x} : \lambda \geq 0, \bar{x} \in \mathbb{R}^3 \setminus (S^+ \cup U^- \cup \{x_3 = 0\})\}$. It is clear that $\{\bar{0}\}$ is Lyapunov stable for h_0.

Let $h = g_0 \circ h_0$. We obtain in this way a $\mathbb{R}^3$-homeomorphism such that $Fix(h) = Per(h) = \{\bar{0}\}$ and $\bar{0}$ is Lyapunov stable. It is easy to see that also h is limit of a sequence of homeomorphisms for which every closed ball centered in $\bar{0}$ and large enough radius is still an isolating block with the same exit sets and the same behavior than in section 3.1. Then, the sequence of fixed point indices of the iterates of h and f coincide. $\square$.

Appendix. Some technical details of the proof of Proposition 1. Construction of f_0 of case (A).

Let us construct f_0 on S_2 (dynamics of type 2). We consider the family of $-a_1+1$ solid conical regions $\{U_{j,2}\}_j$ (isometric to $\phi_{[0,\delta]}$) such that $U_{j,2} \cap \partial S_2 = \bar{0}$. We put the regions $\{U_{j,2}\}$ invariant under a rotation around the Z axis of angle $\frac{2\pi}{-a_1+1}$. Each region is also invariant under the symmetry with respect to the plane $\{z = 0\}$.

Let us consider the following vector field $X_1 : \mathbb{R}^3 \to \mathbb{R}^3$ which vanish in $\bigcup U_{j,2} \cup \bigcup_{k \neq 2} S_k$ and generates an attracting flow at $\bar{0}$ in $\mathbb{R}^3 \setminus \bigcup U_{j,2} \cup \bigcup_{k \neq 2} S_k$:

$$X_1(\bar{x}) = -H_1(\bar{x})\bar{x}$$

with

$$H_1(\bar{x}) = \begin{cases} h_1\left(\frac{\bar{x}}{||\bar{x}||}\right) ||\bar{x}||^2 & \text{if } \bar{x} \neq \bar{0} \\ 0 & \text{if } \bar{x} = \bar{0} \end{cases}$$

where $h_1 : S^2 \to [0, 1]$ is a smooth map defined in the following way:

Let $E(U_{j,2}) \subset S_2$ be a solid conical region, isomorphic to $U_{j,2}$, with $U_{j,2} \subsetneq E(U_{j,2})$, and such that has the same axes and vertex than $U_{j,2}$. We suppose that the sets $E(U_{j,2})$ are isometric, $E(U_{j,2}) \cap \partial(S_2) = E(U_{j,2}) \cap E(U_{j',2}) = \bar{0}$. Let us define $A_{j,2} = S^2 \cap U_{j,2}$ and $B_{j,2} = S^2 \cap E(U_{j,2})$. They are two discs with the same "center", $A_{j,2} \subset int_{S^2}(B_{j,2})$ and $B_{j,2} \cap B_{j',2} = \emptyset$. Given $\epsilon > 0$ small enough, let $A_2 = S_2 \cap S^2$, $B_2(\epsilon) = \phi_{[2\pi/5-\epsilon, 3\pi/5+\epsilon]} \cap S^2$. In the same way we define the sets A_k and $B_k(\epsilon)$ for $k = 0, \ldots, 4$. We select $\epsilon > 0$ in such a way that $B_k(2\epsilon) \cap B_{j,2} = \emptyset$ for all j and for all $k \in \{1, 3\}$.

The construction of h_1 is such that $h_1^{-1}(0) = \bigcup A_{j,2} \cup \bigcup_{k \neq 2} A_k$, $h_1^{-1}(1) = S^2 \setminus \bigcup int(B_{j,2}) \cup \bigcup_{k \neq 2} int(B_k(\epsilon))$ and, if $0 < r < 1$, $h_1^{-1}(r) = \bigcup C_{r,j,2} \cup C_{r,up,2} \cup C_{r,low,2}$ with $C_{r,j,2} \simeq C_{r,up,2} \simeq C_{r,low,2} \simeq S^1$, $C_{r,j,2} \subset B_{j,2} \setminus A_{j,2}$, $C_{r,up,2} \subset \phi_{(2\pi/5, 2\pi/5+\epsilon)} \cap A_2$, $C_{r,low,2} \subset \phi_{(3\pi/5-\epsilon, 3\pi/5)} \cap A_2$.

The vector field X_1 is a smooth vector field because H_1 is smooth. See figure 7.

Let us construct, for each $U_{j,2}$, another smooth vector field $X_{j,2} : \mathbb{R}^3 \to \mathbb{R}^3$ which vanish in $\mathbb{R}^3 \setminus E(U_{j,2})$. The dynamical behavior will be of hyperbolic type (with an unstable manifold of dimension 1 and a stable manifold of dimension 2) in $U_{j,2}$, with fixed points in $\mathbb{R}^3 \setminus int(E(U_{j,2}))$ and $\bar{0}$ will be an attractor fixed point for the orbits in the region $int(E(U_{j,2})) \setminus int(U_{j,2})$. See figure 7.

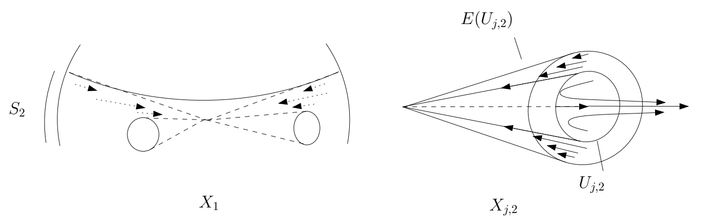


Figure 7

We will reduce the construction of $X_{j,2}$ to the following equivalent case:

We suppose without loss of generality that

$$E(U_{j,2}) = \{\bar{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 \geq x_3^2\} \cup \{\bar{x} \in \mathbb{R}^3 : x_3 \geq 0\} \quad \text{and} \quad U_{j,2} = \{\bar{x} \in \mathbb{R}^3 : x_3 \geq 0\}.$$

Then

$$X_{j,2}(x_1, x_2, x_3) = (-x_1 f_1(\bar{x}), -x_2 f_1(\bar{x}), -x_3 f_2(\bar{x})).$$

The maps f_1 and f_2 are smooth maps defined in the following way:

$$f_1(x_1, x_2, x_3) = \begin{cases} x_3^2 (x_3 + \sqrt{x_1^2 + x_2^2})^2 & \text{if } -\frac{1}{2}\sqrt{x_1^2 + x_2^2} \geq x_3 \geq -\sqrt{x_1^2 + x_2^2} \\ \frac{1}{16}(x_1^2 + x_2^2)^2 & \text{if } x_3 \geq -\frac{1}{2}\sqrt{x_1^2 + x_2^2} \\ 0 & \text{if } x_3 < -\sqrt{x_1^2 + x_2^2} \end{cases}$$

$$f_2(x_1, x_2, x_3) = \begin{cases} x_3^2 (x_3 + \sqrt{x_1^2 + x_2^2})^2 & \text{if } 0 \geq x_3 \geq -\sqrt{x_1^2 + x_2^2} \\ -x_3^2 & \text{if } x_3 \geq 0 \\ 0 & \text{if } x_3 \leq -\sqrt{x_1^2 + x_2^2} \end{cases}$$

The vector field $X_{j,2}$ determines fixed points for the associated flow in $cl(\mathbb{R}^3 \setminus E(U_{j,2}))$. In the region $int(E(U_{j,2})) \setminus int(U_{j,2})$ each orbit converges to $\bar{0}$, and in $U_{j,2}$ the dynamical behavior is of hyperbolic type. See figure 7.

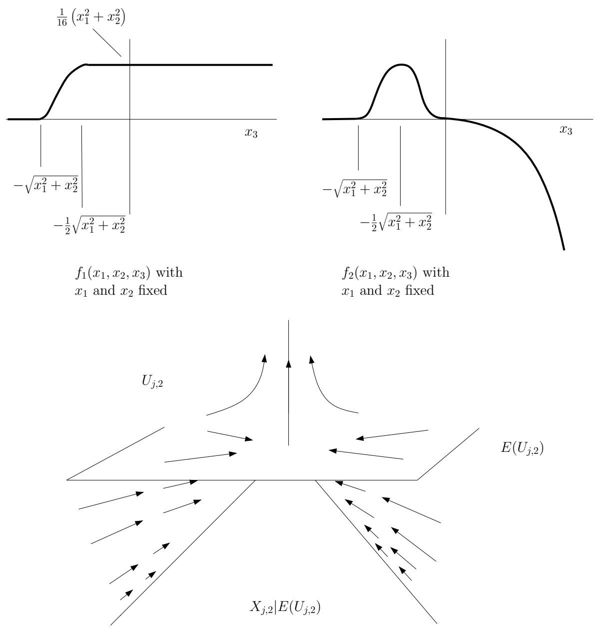


Figure 8

The smooth vector field

$$X_{S_2} = X_{-a_1+1,2} \circ \cdots \circ X_{1,2} \circ X_1$$

generates in $int(S_2)$ the dynamics of type 2 which we want for f and vanish on $\mathbb{R}^3 \setminus int(S_2)$.

Let us construct the dynamics of type 3 on S_1 (the construction on S_3 is analogous). We consider a family of a_2 isometric solid conical regions $\{U_{j,1}\}_j$ contained in S_1 and a_2 isometric solid cones $\{U_{j,3}\}_j$ contained in S_3, in such a way that they are isometric to $\phi_{[0,\delta]}$ and such that $U_{j,k} \cap \partial S_k = \bar{0}$ $(k = 1, 3)$. We put the regions $\{U_{j,k}\}$, with k fixed, invariant under a rotation around the Z axis of angle $\frac{2\pi}{a_2}$ and such that $U_{j,1}$ goes to $U_{j,3}$, for every j, under the symmetry with respect to the plane $\{z = 0\}$.

We also consider a_2 solid cones $\{E(U_{j,1})\}$ with $U_{j,1} \subsetneq E(U_{j,1}) \subset S_1$, and a_2 solid conical regions $\{E(U_{j,3})\}$ with $U_{j,3} \subsetneq E(U_{j,3}) \subset S_3$, with the corresponding discs $A_{j,k} \subsetneq B_{j,k}$ for $k = 1, 3$, constructed in the same way than the regions $\{E(U_{j,2})\}$, $A_{j,2}$ and $B_{j,2}$. We select $\epsilon > 0$ such that $B_{j,k} \cap B_{k'}(2\epsilon) = \emptyset$ for all $k \neq k'$.

We define smooth vector fields $X_{j,1} : \mathbb{R}^3 \to \mathbb{R}^3$ in such a way that the associated flow is the inverse of the flow obtained with $X_{j,2}$. These vector fields vanish in $\mathbb{R}^3 \setminus E(U_{j,1})$, the dynamical behavior in $U_{j,1}$ is hyperbolic with unstable manifold the boundary of $U_{j,1}$ and stable manifold the axis of $U_{j,1}$, and $\bar{0}$ is a repelling fixed point in $int(E(U_{j,1}) \setminus U_{j,1})$. See figure 9.

We define the vector field $X_2 : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$X_2(\bar{x}) = H_2(\bar{x})\bar{x}$$

with $H_2 : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$H_2(\bar{x}) = \begin{cases} h_2\left(\frac{\bar{x}}{||\bar{x}||}\right) ||\bar{x}||^2 & \text{if } \bar{x} \neq \bar{0} \\ 0 & \text{if } \bar{x} = \bar{0} \end{cases}$$

where $h_2 : S^2 \to [0,1]$ is a smooth map defined in the following way:

$$h_2^{-1}(0) = \bigcup A_{j,1} \cup \bigcup_{k\neq 1} B_k(\epsilon)$$

$$h_2^{-1}(1) = S^2 \setminus \left(\bigcup int_{S^2}(B_{j,1}) \cup \bigcup_{k\neq 1} int_{S^2}(B_k(2\epsilon)) \right).$$

and $h_2^{-1}(r) = \bigcup C_{r,j,1} \cup C_{r,up,1} \cup C_{r,low,1}$ with $C_{r,j,1} \simeq C_{r,up,1} \simeq C_{r,low,1} \simeq S^1$, $C_{r,j,1} \subset B_{j,1} \setminus A_{j,1}$, $C_{r,up,1} \subset A_1 \cap \phi_{(\pi/5+\epsilon,\pi/5+2\epsilon)}$, $C_{r,low,1} \subset A_1 \cap \phi_{(2\pi/5-2\epsilon,2\pi/5-\epsilon)}$.

The smooth vector field X_2 vanish on $\bigcup U_{j,1} \cup \phi_{[0,\pi/5+\epsilon]} \cup \phi_{[2\pi/5-\epsilon,\pi]}$ and $\bar{0}$ is a repelling fixed point for $\mathbb{R}^3 \setminus \left(\bigcup U_{j,1} \cup \phi_{[0,\pi/5+\epsilon]} \cup \phi_{[2\pi/5-\epsilon,\pi]}\right)$. See figure 9.

Let $X_3 : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field which vanish on $\mathbb{R}^3 \setminus \phi_{[\pi/5-\epsilon,\pi/5+2\epsilon]}$ and has the behavior of figure 9 on $\phi_{[\pi/5-\epsilon,\pi/5+2\epsilon]}$.

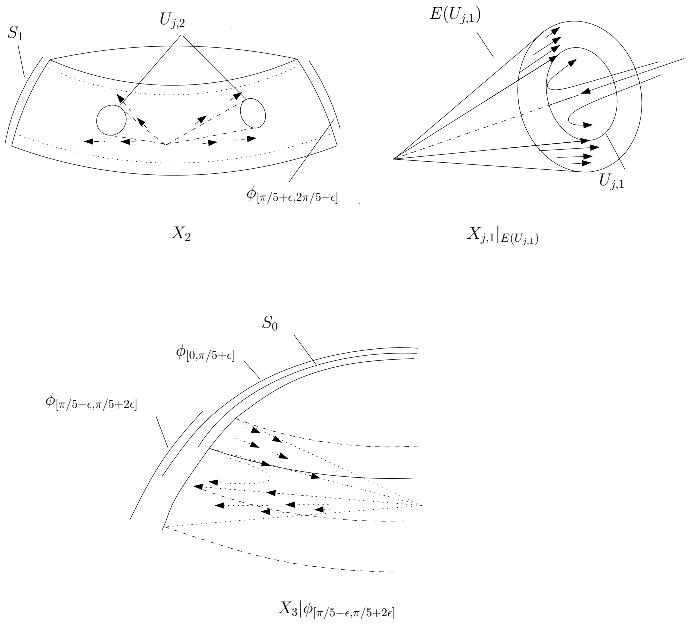


Figure 9

In this construction we will suppose, without loss of generality, that

$$\phi_{[\pi/5-\epsilon,\pi/5+2\epsilon]} \equiv \{(x_1,x_2,x_3) : x_1^1 + x_2^2 \geq x_3^2\}$$

$$\phi_{[\pi/5-\epsilon,\pi/5]} \equiv \{(x_1,x_2,x_3) : x_3 \geq 0, x_3^2 \leq x_1^1 + x_2^2 \leq 4x_3^2\}$$

$$\phi_{[\pi/5,\pi/5+\epsilon]} \equiv \{(x_1,x_2,x_3) : x_3 \geq 0, x_1^1 + x_2^2 \geq 4x_3^2\}$$

$$\phi_{[\pi/5,\pi/5+\epsilon]} \equiv \{(x_1,x_2,x_3) : x_3 \leq 0\}.$$

We define

$$X_3(x_1,x_2,x_3) = (x_1 g_1(\bar{x}), x_2 g_1(\bar{x}), x_3 g_2(\bar{x}))$$

The maps g_1 and g_2 are smooth maps defined in the following way:

$$g_1(x_1,x_2,x_3) =$$

$$= \begin{cases} -x_3(x_3+\sqrt{x_1^2+x_2^2})^2(x_3-\sqrt{x_1^2+x_2^2})^2 & \text{if } |x_3| \in \left[\frac{1}{2}\sqrt{x_1^2+x_2^2}, \sqrt{x_1^2+x_2^2}\right] \\ \mu(x_1,x_2,x_3) & \text{if } |x_3| < \frac{1}{2}\sqrt{x_1^2+x_2^2} \\ 0 & \text{if } |x_3| > \sqrt{x_1^2+x_2^2} \end{cases}$$

with

$$\mu(x_1,x_2,x_3) = -\frac{9}{16}(x_1^2+x_2^2)^{3/2}\frac{\mu_0(x_3)}{\mu_0(x_3)+\mu_0(\sqrt{x_1^2+x_2^2}/2-x_3)}+\frac{9}{32}(x_1^2+x_2^2)^{3/2}$$

and

$$\mu_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

$$g_2(x_1,x_2,x_3) =$$

$$= \begin{cases} -x_3(x_3+\sqrt{x_1^2+x_2^2})^2(x_3-\sqrt{x_1^2+x_2^2})^2 & \text{if } |x_3| \leq \sqrt{x_1^2+x_2^2} \\ 0 & \text{if } |x_3| > \sqrt{x_1^2+x_2^2} \end{cases}$$

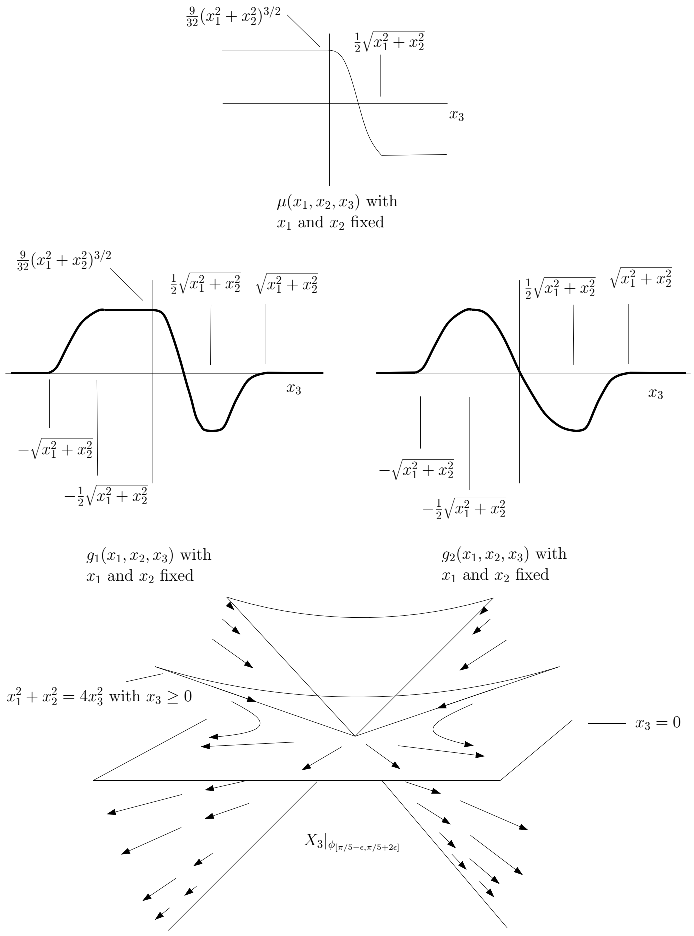


Figure 10

The vector field X_3 vanish on $\mathbb{R}^3 \setminus \phi_{(\pi/5-\epsilon,\pi/5+2\epsilon)}$. The point $\bar{0}$ is an attracting fixed point on $\phi_{(\pi/5-\epsilon,\pi/5]}$ and a repelling fixed point on $\phi_{[\pi/5+\epsilon,\pi/5+2\epsilon)}$. The behavior of X_3 on $\phi_{[\pi/5,\pi/5+\epsilon]}$ is of hyperbolic type. See figure 9.

In the same way we define a vector field X_4 which vanish on $\mathbb{R}^3 \setminus \phi_{[2\pi/5-\epsilon,2\pi/5+2\epsilon]}$, such that $\bar{0}$ is an attracting fixed point on $\phi_{[2\pi/5,2\pi/5+\epsilon)}$, and a repelling fixed point on $\phi_{(2\pi/5-2\epsilon,2\pi/5-\epsilon]}$. The behavior of X_4 on $\phi_{[2\pi/5-\epsilon,2\pi/5]}$ is of hyperbolic type.

The smooth vector field

$$X_{S_1} = X_4 \circ X_3 \circ X_2 \circ X_{a_2,1} \circ \cdots \circ X_{1,1}$$

has the dynamics of type 3 which we want for f on S_1 and vanish on $\mathbb{R}^3 \setminus \phi_{(\pi/5-\epsilon,2\pi/5+\epsilon)}$.

In an analogous way we construct a smooth vector field X_{S_3} which has dynamics of type 3 on S_3 and vanish on $\mathbb{R}^3 \setminus \phi_{(3\pi/5-\epsilon,4\pi/5+\epsilon)}$.

We only have to obtain the dynamics on S_0 and S_4, which are of type 1. Let us construct an adequate smooth vector field X_{S_0} (the construction of X_{S_4} is analogous). We define

$$X_{S_0}(\bar{x}) = -H_3(\bar{x})\bar{x}$$

with

$$H_3(\bar{x}) = \begin{cases} h_3\left(\frac{\bar{x}}{||\bar{x}||}\right)||\bar{x}||^2 & \text{if } \bar{x} \neq \bar{0} \\ 0 & \text{if } \bar{x} = \bar{0} \end{cases}$$

The map $h_3 : S^2 \to [0,1]$ is a smooth map such that $h_3^{-1}(0) = \mathbb{R}^3 \setminus int(S_0)$, $h_3^{-1}(1) = (0,0,1)$ and, if $0 < r < 1$, $h_3^{-1}(r) \simeq S^1$.

The smooth vector field

$$X = X_{S_0} \circ X_{S_4} \circ X_{S_3} \circ X_{S_1} \circ X_{S_2}$$

is the one we are looking for. The map $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is the diffeomorphism obtained by considering the time one flow induced by X, that is,

$$f_0(\bar{x}) = \varphi(1, \bar{x}).$$

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Francisco R. Ruiz del Portal
Departamento de Geometría y Topología, Facultad de CC.Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain.
E-mail: R_Portal@mat.ucm.es

José Manuel Salazar.
Departamento de Matemáticas. Universidad de Alcalá. Alcalá de Henares. Madrid 28871, Spain.
E-mail: josem.salazar@uah.es