

On a nonlocal quasilinear parabolic model related to a current-carrying Stellarator

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1. Introduction

We study the existence of weak solutions of the quasilinear parabolic problem

$$(\mathcal{P}) \begin{cases} \beta(u)_t - \Delta u &= G(u)(t, x) + J(u)(t, x) & \text{in } Q :=]0, T[\times \Omega, \\ u(t, x) &= \gamma & \text{on } \Sigma :=]0, T[\times \partial\Omega, \\ \beta(u(0, x)) &= \beta(u_0(x)) & \text{in } \Omega, \end{cases}$$

where Ω is an open, bounded, and regular set in \mathbb{R}^2 and $T > 0$. Moreover, γ is a negative constant, and the nonlinear functions β , G , and J satisfy the following structural assumptions:

$$\beta(s) := \min(s, 0) = -s_-, \quad (1.1)$$

$$G(u)(t, x) := \left[A - \lambda u_+(t, x)^2 + \int_{|u(t) > 0|}^{u_+(t, x)} g(u_+(t)_*(\sigma), |u_+(t)|_{L^\infty(\Omega)}, [u_+(t)]'_*(\sigma)) d\sigma \right]_+^{1/2}, \quad (1.2)$$

$$J(u)(t, x) := j(u_+(t, x), |u_+(t)|_{L^\infty(\Omega)}, [u_+(t)]'_*(|u(t) > u(t, x)|)), \quad (1.3)$$

with $A > 0$, $\lambda > 0$, $u_+ = \max(u, 0)$, $|E|$ the Lebesgue measure of the set E . So, for example, $|u(t) > u_+(t, x)|$ denotes the measure of the set $\{y \in \Omega : u(t, y) > u_+(t, x)\}$, for a given $x \in \Omega$ and $t \in]0, T[$. The function $u_+(t)_* = [u_+(t)]_*$, defined on the interval $\Omega_* :=]0, |\Omega|$, is the decreasing rearrangement of the function $u_+(t) : \Omega \rightarrow [0, +\infty)$; the latter is defined by $u_+(t)(x) = [u(t, x)]_+$, for $x \in \Omega$ and a fixed $t \in]0, T[$ (see, e. g., [10]). By $[u_+(t)]'_*$ we denote the (weak) derivative of the decreasing rearrangement. We assume that

$$g, j : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \rightarrow \mathbb{R} \text{ are bounded continuous functions,} \quad (1.4)$$

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and we fix a constant C_0 such that

$$\max(|G(u)(t, x)|, |J(u)(t, x)|) \leq \frac{C_0}{2} \quad (1.5)$$

for all admissible functions u , for all $t \in]0, T[$ and $x \in \Omega$.

The above formulation is related to a problem arising in the study of the magnetic confinement of a plasma in a Stellarator device, when the plasma is assumed to be a perfect conductor but with a nonzero net current inside each flux magnetic surface (in contrast with ideal Stellarators).

Taking into account Ohm's and Faraday's laws, the associated Grad-Shafranov equation, obtained after an averaging process from the three-dimensional physical problem, can be formulated as a two-dimensional inverse problem of the form

$$\beta(u)_t - \Delta u = F(u) + F(u)F'(u) + \lambda u_+ \text{ in } Q, \quad (1.6)$$

where $F : \mathbb{R} \rightarrow \mathbb{R}_+$ is an *unknown* function satisfying $F(s) = \sqrt{A}$ (a given positive constant) for any $s \leq 0$ (the set $\{u < 0\}$ corresponds to the vacuum region, separating the plasma from the walls of the device; see, e. g., [2], [6]). The case of an ideal Stellarator, with zero net current within each flux magnetic surface, has been studied recently in [6]. In practice, however, this ideal condition does not hold, and a known current arises in the interior of each magnetic surface (see [4] for a physical modelling and [8] for a mathematical treatment, both for the associated stationary problem). Using the change of variables introduced in [8], the condition of a nonzero current inside each magnetic surface can be expressed in terms of a family of integrals, involving a given function $h : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$:

$$\int_{\{u(t) > s\}} [F(u(t))F'(u(t)) + \lambda u_+(t)] dx = h(s_+, |u_+(t)|_{L^\infty(\Omega)}), \quad s \in [\operatorname{ess\,inf}_\Omega u(t), \operatorname{ess\,sup}_\Omega u(t)], \quad (1.7)$$

for any $t \in [0, T]$; this is known as the *current-carrying condition*. The present paper generalizes the results of [6], which is concerned with the special case $h \equiv 0$.

We point out that the physical model involves some weight functions a and b , which here are assumed to be equal to one, and the diffusion operator is a certain elliptic, second-order operator with variable coefficients. Our problem (\mathcal{P}) thus contains some simplifications. A more general framework will be considered in [7].

The main goal of this paper is to prove the existence of a weak solution for (\mathcal{P}) . The organization of the rest of the paper is as follows: In Section 2 we explain how (1.6) and (1.7) lead to nonlocal terms of the kind involved in (\mathcal{P}) . In Section 3 we introduce the notion of weak solution and prove some *a priori* estimates for any weak solution. Finally, in Section 4, we state and prove the existence result. For the proof, we shall use a Galerkin method as in [6]. Notice that the equation in (\mathcal{P}) is elliptic-parabolic, depending on the sign of u . So, we start by approximating (\mathcal{P}) by a family of regularized problems $(\mathcal{P})_\alpha$ (obtained by approximating β by suitable strictly increasing functions β_α). Next, we approximate $(\mathcal{P})_\alpha$ by a sequence of finite-dimensional problems $(\mathcal{P})_{\alpha, m}$ and prove their solvability. Using *a priori* estimates, we pass to the limit, first in m , then in α .

2. On the nonlocal terms obtained from the inverse problem

The main goal of this section is to show how the family of conditions (1.7) allows us to write the unknown function F in terms of a nonlocal expression in u . In order to do so, we shall apply some technical results about decreasing and relative rearrangements.

Given a function $b \in L^1(0, T; L^1(\Omega))$, we define the function

$$w(t, \sigma) = \int_{\{u(t) > u(t)_*(\sigma)\}} b(t) dx + \int_0^{\sigma - |u(t) > u(t)_*(\sigma)|} (b(t) |_{\{u(t) = u(t)_*(\sigma)\}})_*(s) ds$$

for $\sigma \in]0, |\Omega|]$, $t \in]0, T[$. The *relative rearrangement* of $b(t)$ with respect to $u(t)$ is defined as the (weak) derivative $\frac{\partial w}{\partial \sigma}(t, \cdot)$, and we denote it by $b(t)_{*u(t)}$ (see, e. g., [10]). This function and the decreasing rearrangement have many useful properties, some of which will be used here (see, for instance, [8], [10], [11], or [12]).

Assuming u to be regular enough, we can apply Theorem 1.1 of [12] in order to obtain the derivative with respect to s of relation (1.7), for a fixed t . In fact,

$$\mu'(s) [F(u(t)) F'(u(t)) + \lambda u_+(t)]_{*u(t)}(\mu(s)) = h'_s(s_+, u_+(t)_*(0)),$$

with $\mu(s) = |u(t) > s|$ the distribution function of $u(t)$ and h'_s the derivative of h with respect to its first variable (we used here that $|u_+(t)|_{L^\infty(\Omega)} = u_+(t)_*(0)$). Now, from Lemma 9 of [8], we get

$$\mu'(s) [F(u(t)_*) F'(u(t)_*) + \lambda u_+(t)_*](\mu(s)) = h'_s(s_+, u_+(t)_*(0)).$$

We also assume that $u(t)$ has no *flat region* (i. e., $|\{\nabla u(t) = 0\}| = 0$ for any fixed t). From this, $u(t)_*(\mu(s)) = u(t)_*(|u(t) > s|) = s$, and so we deduce that $[\mu'(s)]^{-1} = [u_+(t)]'_*(|u(t) > s|)$ for $s \geq 0$ (see, e. g., [8, Lemma 2]). Thus, the last relation can be written as

$$F(s) F'(s) + \lambda s_+ = h'_s(s_+, u_+(t)_*(0)) [u_+(t)]'_*(|u(t) > s|), \quad (2.8)$$

and so, $(\frac{1}{2}F^2)'(s) = -\lambda s_+ + h'_s(s_+, u_+(t)_*(0)) [u_+(t)]'_*(|u(t) > s|)$. (Note that (2.8) holds also for $s < 0$, since F is constant on \mathbb{R}^-). Integrating the last equality on $(0, \sigma_+)$ with $\sigma \in [\text{essinf}_\Omega u(t), \text{esssup}_\Omega u(t)]$, we obtain that

$$\begin{aligned} F(\sigma) &= \left[A - 2\lambda \int_0^{\sigma_+} s ds + 2 \int_0^{\sigma_+} h'_s(s_+, u_+(t)_*(0)) [u_+(t)]'_*(|u(t) > s|) ds \right]_+^{1/2} \\ &= \left[A - \lambda \sigma_+^2 + 2 \int_{|u(t) > 0|}^{|u(t) > \sigma_+|} h'_s(u_+(t)_*(r), u_+(t)_*(0)) ([u_+(t)]'_*(r))^2 dr \right]_+^{1/2}, \end{aligned}$$

where we used the change of variable $s = u_+(t)_*(r)$ (note that $F(\sigma) = F(\sigma_+)$). Taking $\sigma = u(t, x)$, we get

$$F(u(t, x)) = \left[A - \lambda u_+(t, x)^2 + 2 \int_{|u(t) > 0|}^{|u(t) > u_+(t, x)|} h'_s(u_+(t)_*(s), u_+(t)_*(0)) ([u_+(t)]'_*(s))^2 ds \right]_+^{1/2},$$

which is a nonlocal expression in u , to be substituted for the first term on the right-hand side of (1.6).

Also, setting $s = u(t, x)$ in (2.8), we obtain

$$F(u(t, x)) F'(u(t, x)) + \lambda u_+(t, x) = h'_s(u_+(t, x), u_+(t)_*(0)) [u_+(t)]'_*(|u(t) > u(t, x)|), \quad (2.9)$$

another nonlocal expression in u , which coincides with the sum of the second and third terms on the right-hand side of (1.6). Note that for any $s \in [\operatorname{ess\,inf}_\Omega u(t), \operatorname{ess\,sup}_\Omega u(t)]$,

$$\int_{\{u(t) > s\}} h'_s(u_+(t, x), u_+(t)_*(0)) [u_+(t)]'_*(|u(t) > u(t, x)|) dx = h(s_+, u_+(t)_*(0)),$$

by means of (1.7). Thus, (1.6) is transformed into a nonlocal equation like the one in (\mathcal{P}) . After a truncation argument (as used in [8]), we are led to the assumption of globally bounded functions g and j as in (1.4). The justification of the truncation argument (passing to the limit) is the main goal of [7].

3. On the notion of weak solution and some *a priori* estimates

Definition 3.1 Assume that $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$. We say that a function u is a weak solution of (\mathcal{P}) if the function $w = u - \gamma$ satisfies the following conditions: $w \in L^2(0, T; H_0^1(\Omega))$, $\beta(w + \gamma)_t \in L^2(0, T; H^{-1}(\Omega))$, $\beta(w + \gamma)_t - \Delta w = G(w + \gamma) + J(w + \gamma)$ in $\mathcal{D}'(\Omega)$ for a. e. $t \in]0, T[$, and $\beta(w + \gamma)|_{t=0} = \beta(u_0)$.

We note that if u is a weak solution of (\mathcal{P}) , then $\beta(u) \in C([0, T]; L^2(\Omega))$. We have

Lemma 3.2 If u is any weak solution of (\mathcal{P}) , then

$$|\beta(u(t)) - \beta(\gamma)|_{L^\infty(\Omega)} \leq C_0 t + |\beta(u_0) - \beta(\gamma)|_{L^\infty(\Omega)} \quad \forall t \in [0, T].$$

Proof: For any integer $m \geq 2$, we define $g_m(\sigma) = |\sigma|^{m-2}\sigma$ and denote by T_k the truncation operator given by $T_k(\sigma) = \sigma$ if $|\sigma| \leq k$ and $k \operatorname{sign}(\sigma)$ otherwise. Then we have $w_{m,k} := g_m \circ T_k(\beta(u) - \beta(\gamma)) \in L^\infty(Q) \cap L^2(0, T; H_0^1(\Omega))$ and

$$\langle \beta(w(t) + \gamma)_t, w_{m,k}(t) \rangle + \int_\Omega \nabla w(t) \cdot \nabla w_{m,k}(t) dx = \int_\Omega [G(w(t) + \gamma) + J(w(t) + \gamma)] w_{m,k}(t) dx, \quad (3.10)$$

where $w = u - \gamma$ and $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. By the integration by parts formula (see, e. g., Alt and Luckhaus [1]), we have

$$\frac{d}{dt} y_{m,k}(t) = \langle \beta(w(t) + \gamma)_t, w_{m,k}(t) \rangle, \quad (3.11)$$

where $y_{m,k}(t) = \int_\Omega dx \int_0^{\beta(w(t)+\gamma)-\beta(\gamma)} g_m \circ T_k(\sigma) d\sigma$. Since $\int_\Omega \nabla w(t) \cdot \nabla w_{m,k}(t) dx \geq 0$, and using (1.5), we get

$$\frac{d}{dt} y_{m,k}(t) \leq C_0 \int_\Omega |w_{m,k}(t)| dx \leq C_0 |\Omega|^{\frac{1}{m}} \left(\int_\Omega |w_{m,k}(t)|^{\frac{m}{m-1}} dx \right)^{1-\frac{1}{m}}, \quad (3.12)$$

and so,

$$\int_{\Omega} |w_{m,k}(t)|^{\frac{m}{m-1}} dx = m y_{m,k}(t) - m k^{m-1} \int_{\Omega} (|\beta(w(t) + \gamma) - \beta(\gamma)| - k)_+ dx. \quad (3.13)$$

Then, relation (3.12) leads to $y'_{m,k}(t) \leq C_0 |\Omega|^{\frac{1}{m}} m^{1-\frac{1}{m}} y_{m,k}(t)^{1-\frac{1}{m}}$, from which we conclude that $y_{m,k}^{\frac{1}{m}}(t) \leq C_0 |\Omega|^{\frac{1}{m}} m^{-\frac{1}{m}} t + y_{m,k}^{\frac{1}{m}}(0)$. Using (3.13) and letting $k \rightarrow +\infty$ and $m \rightarrow +\infty$, we get the result. \square

We point out that the above statement remains true if we replace β by any nondecreasing Lipschitz function, as in particular $\beta_{\alpha}(s) = \alpha s_+ - s_-$, for a given $\alpha > 0$. In addition, we have

Lemma 3.3 *Let $\beta_{\alpha}(s) = \alpha s_+ - s_-$, with $0 \leq \alpha \leq 1$, and let u be a weak solution of (\mathcal{P}) , but with β replaced by β_{α} . Then, for any $t \in [0, T]$,*

$$|u_+(t)|_{L^{\infty}(\Omega)} \leq \frac{1}{4\pi} C_0 |\Omega|. \quad (3.14)$$

Proof: First, let $\alpha = 0$. We have $\frac{\partial u_-}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$ and $\langle \frac{\partial u_-}{\partial t}(t), (u_+(t) - \theta)_+ \rangle = 0$ for all $\theta > 0$ and a. e. $t \in]0, T[$. Thus, from the equation satisfied by u ,

$$\int_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 dx = \int_{\Omega} [G(u(t)) + J(u(t))] (u_+(t) - \theta)_+ dx.$$

Differentiating with respect to θ and using (1.5), we get

$$-\frac{d}{d\theta} \int_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 dx \leq C_0 |u_+(t) > \theta|.$$

Then, the conclusion holds by standard arguments (see, e. g., [8]). Now assume that $0 < \alpha \leq 1$. We argue as in [10]. Taking $(u_+(t) - \theta)_+$ as a test function and differentiating with respect to θ , we get

$$\alpha \int_{\{u_+(t) > \theta\}} \frac{\partial}{\partial t} u_+(t) dx - \frac{d}{d\theta} \int_{\{u_+(t) > \theta\}} |\nabla u_+(t)|^2 dx = \int_{\{u_+(t) > \theta\}} G(u(t)) dx + \int_{\{u_+(t) > \theta\}} J(u(t)) dx,$$

for a. e. $\theta > 0$. Then arguing as in [5], we obtain that

$$-4\pi s \frac{\partial}{\partial s} u_+(t)_*(s) \leq C_0 s - \alpha \int_0^s \frac{\partial}{\partial t} u_+(t)_*(\sigma) d\sigma, \quad (3.15)$$

for all $s \in]0, |\Omega|[$. If we introduce $K(t, s) = \int_0^s u_+(t)_*(\sigma) d\sigma$, then relation (3.15) leads to

$$\begin{cases} \alpha \frac{\partial K}{\partial t}(t, s) - 4\pi s \frac{\partial^2 K}{\partial s^2}(t, s) \leq C_0 s, \\ K(t, 0) = 0, \quad \frac{\partial K}{\partial s}(t, |\Omega|) = 0. \end{cases}$$

We now define a function $\widehat{K}(s)$, satisfying

$$C_0 s = -4\pi s \frac{d^2 \widehat{K}}{ds^2}, \quad \widehat{K}(0) = 0, \quad \frac{d\widehat{K}}{ds}(|\Omega|) = 0,$$

that is, $\widehat{K}(s) = -\frac{C_0}{4\pi}s^2 + \frac{C_0}{4\pi}|\Omega|$. Then, from the comparison principle (see [5]), we deduce that $\widehat{K}(t, s) \leq \widehat{K}(s)$ for any $s \in [0, |\Omega|]$ and any $t \in [0, T]$. In particular, we get (3.14). \square

Lemma 3.4 *Assume the conditions of Lemma 3.3. Then, for any $t \in [0, T]$,*

$$\begin{aligned} \int_0^t \int_{\Omega} |\nabla u(\sigma, x)|^2 dx d\sigma + \int_{\Omega} dx \int_0^{u_0(x)-\gamma} \beta_{\alpha}(\sigma + \gamma) d\sigma &\leq \\ &\leq \int_{\Omega} (u_0(x) - \gamma) \beta_{\alpha}(u_0(x)) dx + C_0 \int_0^t \int_{\Omega} |u(\sigma, x) - \gamma| dx d\sigma. \end{aligned}$$

Proof: By the *integration by parts formula* we have

$$\frac{d}{dt} y(t) + \int_{\Omega} |\nabla w(t)|^2 dx = \int_{\Omega} [G(w(t) + \gamma) + J(w(t) + \gamma)] w(t) dx, \quad (3.16)$$

where $w = u - \gamma$ and $y(t) = \int_{\Omega} w(t) \beta_{\alpha}(w(t) + \gamma) dx - \int_{\Omega} dx \int_0^{w(t,x)} \beta_{\alpha}(\sigma + \gamma) d\sigma$. Integrating relation (3.16) with respect to t , dropping some nonnegative terms (we have $y(t) \geq 0$ since β_{α} is nondecreasing), and using (1.5), we get the result. \square

4. On the existence of a weak solution

The main result of this paper is the following:

Theorem 4.1 *Assume that $u_0 - \gamma \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$. Then there exists a weak solution u of (\mathcal{P}) . Moreover, $u \in L^{\infty}(Q)$, $u - \gamma \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, and $\beta(u)_t \in L^2(Q)$.*

We will start by proving a similar result for $(\mathcal{P})_{\alpha}$, that is, problem (\mathcal{P}) with β replaced by β_{α} (as defined in Lemma 3.3), for $0 < \alpha \leq 1$:

Theorem 4.2 *Assume that $u_0 - \gamma \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ and $0 < \alpha \leq 1$. Then there exists a weak solution u^{α} of $(\mathcal{P})_{\alpha}$. Moreover, $u^{\alpha} \in L^2(0, T; H^2(\Omega)) \cap L^{\infty}(Q)$ and $u_t^{\alpha} \in L^2(Q)$.*

The proof of Theorem 4.2 will be obtained by means of a Galerkin method as in [6]. First, we shall find solutions w_m of some auxiliary, finite-dimensional problems $(\mathcal{P})_{\alpha, m}$.

4.1. On the finite-dimensional problems $(\mathcal{P})_{\alpha,m}$

Let $(\lambda_k, \varphi_k)_{k \geq 1}$ be the eigenvalues and eigenfunctions associated with $-\Delta$ on Ω with zero boundary conditions, i. e., $-\Delta \varphi_k = \lambda_k \varphi_k$ and $\varphi_k \in H_0^1(\Omega)$. For $m \geq 1$, we denote by V_m the vector space spanned by $\{\varphi_1, \dots, \varphi_m\}$. For all $v \in V_m$ we shall use the decomposition $v = \sum_{i=1}^m v^i \varphi_i$.

For a fixed α with $0 < \alpha \leq 1$, we consider the following finite-dimensional approximations to problem $(\mathcal{P})_\alpha$: To find $w_m \in L^1(0, T; V_m)$, $w_m(t) = \sum_{i=1}^m w_m^i(t) \varphi_i$, satisfying

$$(\mathcal{P})_{\alpha,m} \begin{cases} \int_{\Omega} \left(\frac{\partial}{\partial t} \beta_\alpha(w_m(t) + \gamma) \right) \varphi_k dx + \int_{\Omega} \nabla w_m(t) \cdot \nabla \varphi_k dx = \\ = \int_{\Omega} G(w_m(t) + \gamma) \varphi_k dx + \int_{\Omega} J(w_m(t) + \gamma) \varphi_k dx, & k = 1, \dots, m, \\ w_m(0) = P_m(u_0 - \gamma), \end{cases}$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m .

Theorem 4.3 *For each $m \geq 1$ there exists a solution w_m of problem $(\mathcal{P})_{\alpha,m}$. Furthermore, there exists a number k_0 such that $w_m \not\equiv 0$ for all $m \geq k_0$.*

Problem $(\mathcal{P})_{\alpha,m}$ can be written as a nonlinear differential system for the functions $w_m^1(t), \dots, w_m^m(t)$. Indeed, these functions satisfy

$$\begin{aligned} \sum_{i=1}^m a_{ik}(w_m(t)) \frac{d}{dt} w_m^i(t) + \sum_{i=1}^m b_{ik} w_m^i(t) &= \widehat{\mathcal{I}}_k(w_m(t)), \\ w_m^k(0) &= \text{the } k^{\text{th}} \text{ component of } P_m(u_0 - \gamma), \quad k = 1, \dots, m, \end{aligned} \quad (4.17)$$

where $a_{ik}(v) := \int_{\Omega} \beta'_\alpha(v + \gamma) \varphi_i \varphi_k dx$, $b_{ik} := \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_k dx$, and $\widehat{\mathcal{I}}_k(v) := \int_{\Omega} G(v + \gamma) \varphi_k dx + \int_{\Omega} J(v + \gamma) \varphi_k dx$, for $v \in V_m$ and $i, k = 1, \dots, m$.

To prove the existence of a solution of this initial-value problem, we need the following

Lemma 4.4 *The functions $\widehat{\mathcal{I}}_k : V_m \rightarrow \mathbb{R}$, $k = 1, \dots, m$, are continuous.*

Proof: This is a consequence of assumption (1.4) and of the following Lemma 4.5 (see [8] for details). \square

Lemma 4.5 *Let $(v_n)_{n \geq 1}$ be a sequence in $V_m \setminus \{0\}$ and let v be in $V_m \setminus \{0\}$, such that $v_n \rightarrow v$ in V_m . Then one has that*

$$\begin{aligned} v_n &\xrightarrow[n \rightarrow \infty]{} v && \text{strongly in } \mathcal{C}^k(\overline{\Omega}) \quad \forall k \in \mathbb{N} \cup \{0\}, \\ v_{n*} &\xrightarrow[n \rightarrow \infty]{} v_* && \text{strongly in } \mathcal{C}(\overline{\Omega}_*), \\ v'_{n+*} &\xrightarrow[n \rightarrow \infty]{} v'_{+*} && \text{strongly in } L^q(\Omega_*) \quad \forall 1 \leq q < \infty, \\ v'_{n+*}(|v_n|) &\xrightarrow[n \rightarrow \infty]{} v'_{+*}(|v|) && \text{strongly in } L^q(\Omega) \quad \forall 1 \leq q < \infty. \end{aligned}$$

Proof: See [8, Lemma 22]. Regarding the notation, recall that $\Omega_* :=]0, |\Omega|[$. Moreover, given a (measurable) function v on Ω and a function ϕ on $\overline{\Omega}_*$, we denote by $\phi(|v > v(\cdot)|)$ the function $x \mapsto \phi(|v > v(x)|)$, defined on Ω . \square

Next, we will establish some *a priori* estimates.

Lemma 4.6 *Let $w_m(t) = \sum_{i=1}^m w_m^i(t) \varphi_i$, where $(w_m^1(t), \dots, w_m^m(t))$ is a (local) solution of the initial-value problem (4.17), for some $m \geq 1$. Then we have*

$$\begin{aligned} \int_0^t |\nabla w_m(\sigma)|_{L^2(\Omega)}^2 d\sigma + 2 \int_{\Omega} dx \int_0^{w_m(0)} \beta_{\alpha}(\sigma + \gamma) d\sigma &\leq \\ &\leq 2 \int_{\Omega} w_m(0) \beta_{\alpha}(w_m(0) + \gamma) dx + \frac{C_0^2 |\Omega| t}{\lambda_1} \end{aligned}$$

and

$$\int_0^t |w'_m(s)|_{L^2(\Omega)}^2 ds \leq \frac{1}{\alpha} |\nabla w_m(0)|_{L^2(\Omega)}^2 + \frac{1}{\alpha^2} C_0^2 |\Omega|,$$

for all t in the solution's interval of existence.

Proof: The first statement follows from an estimate similar to the one in Lemma 3.4 and from Poincaré's inequality. To prove the second statement, we multiply the k^{th} equation in (4.17) by $\frac{d}{dt} w_m^k(t)$. Summing over k , we get

$$\begin{aligned} \int_{\Omega} \beta'_{\alpha}(w_m(t) + \gamma) |w'_m(t)|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w_m(t)|^2 dx &= \\ = \int_{\Omega} [G(w_m(t) + \gamma) + J(w_m(t) + \gamma)] w'_m(t) dx. \end{aligned}$$

Since $\beta'_{\alpha} \geq \alpha$, it follows that $\alpha |w'_m(t)|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} |\nabla w_m(t)|_{L^2(\Omega)}^2 \leq C_0 |w'_m(t)|_{L^1(\Omega)}$. Applying Hölder's and Young's inequalities, we get

$$\alpha |w'_m(t)|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} |\nabla w_m(t)|_{L^2(\Omega)}^2 \leq \frac{\alpha}{2} |w'_m(t)|_{L^2(\Omega)}^2 + \frac{C_0^2 |\Omega|}{2\alpha}.$$

Integrating, we have

$$\int_0^t |w'_m(s)|_{L^2(\Omega)}^2 ds + \frac{1}{\alpha} |\nabla w_m(t)|_{L^2(\Omega)}^2 \leq \frac{1}{\alpha} |\nabla w_m(0)|_{L^2(\Omega)}^2 + \frac{1}{\alpha^2} C_0^2 |\Omega|,$$

which proves the assertion. \square

Proof of Theorem 4.3: Since $\{\varphi_1, \dots, \varphi_m\}$ is a basis for V_m and since $\beta_{\alpha} \in W^{1,\infty}(\mathbb{R})$ with $0 < \alpha \leq \beta'_{\alpha} \leq 1$, the matrix of coefficients $a_{ik}(w_m(t))$ in (4.17) is invertible. So, by the Cauchy-Peano theorem, the initial-value problem (4.17) has a maximal solution $(w_m^1(t), \dots, w_m^m(t))$, defined on some interval $[0, T_m]$. From the *a priori* estimates given in Lemma 4.6, we have that, in fact, $T_m = T$; that is, $w_m(t) = \sum_{i=1}^m w_m^i(t) \varphi_i$ is a solution of $(\mathcal{P})_{\alpha, m}$. To finish the proof, we observe that there exists a number k_0 with

$\int_{\Omega} \varphi_{k_0}(x) dx \neq 0$, since $(\varphi_k)_{k \geq 1}$ is complete in $L^2(\Omega)$. Now suppose that $m \geq k_0$ and $w_m \equiv 0$. Then $G(w_m + \gamma)$ and $J(w_m + \gamma)$ are constants, and the k_0^{th} equation in (4.17) implies that $\int_{\Omega} \varphi_{k_0}(x) dx = 0$. Thus, $w_m \not\equiv 0$ if $m \geq k_0$. \square

Corollary 4.7 *For $m \geq 1$, let w_m be a solution of $(\mathcal{P})_{\alpha, m}$. Then we have:*

- (a) $(w_m)_{m \geq 1}$ is bounded in $L^2(0, T; H_0^1(\Omega))$.
- (b) $(\frac{\partial w_m}{\partial t})_{m \geq 1}$ is bounded in $L^2(Q)$.
- (c) $(w_m)_{m \geq 1}$ is bounded in $Y := H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$.

Proof: (a) and (b) follow from the estimates in Lemma 4.6. To prove (c), we note that w_m satisfies

$$P_m(\beta_{\alpha}(w_m(t) + \gamma)_t) - \Delta w_m(t) = P_m(G(w_m(t) + \gamma) + J(w_m(t) + \gamma)), \quad (4.18)$$

for a. e. $t \in]0, T[$, where P_m is the orthogonal projection from $L^2(\Omega)$ onto V_m . From (b) and the fact that $\alpha \leq \beta'_{\alpha} \leq 1$, we get that $\beta_{\alpha}(w_m + \gamma)_t$ is bounded in $L^2(Q)$. From (4.18), we then infer that Δw_m remains in a bounded set of $L^2(Q)$. The rest is standard. \square

4.2. Passing to the limit as $m \rightarrow \infty$

Proof of Theorem 4.2: For a fixed α with $0 < \alpha \leq 1$, let $(w_m)_{m \geq 1}$ be a sequence of solutions of $(\mathcal{P})_{\alpha, m}$. By Corollary 4.7, $(w_m)_{m \geq 1}$ has a subsequence (still denoted by $(w_m)_{m \geq 1}$) that converges, weakly in Y , to a function $w^{\alpha} \in Y$. So, by compactness results (see [9], [14]), we get (again for some subsequence)

$$w_m \rightarrow w^{\alpha} \text{ strongly in } L^2(0, T; H_0^1(\Omega)) \cap L^2(0, T; W^{1,p}(\Omega)) \text{ for all } p \geq 1.$$

As G and J are bounded, there exist $G_{\alpha}, J_{\alpha} \in L^{\infty}(Q)$ such that $G(w_m + \gamma) \rightharpoonup G_{\alpha}$ and $J(w_m + \gamma) \rightharpoonup J_{\alpha}$ weakly-star in $L^{\infty}(Q)$. Thus, w^{α} is a solution of the *limit problem*

$$\begin{cases} \beta_{\alpha}(w + \gamma)_t - \Delta w^{\alpha} = G_{\alpha} + J_{\alpha}, \\ w^{\alpha}(0) = u_0 - \gamma \text{ and } w^{\alpha} \in Y. \end{cases}$$

To verify that $G_{\alpha} = G(w^{\alpha} + \gamma)$ and $J_{\alpha} = J(w^{\alpha} + \gamma)$, we argue as in the proof of the continuity of the maps $\widehat{\mathcal{I}}_k$ (see Lemma 4.4) and use the fact that $w_m \rightarrow w^{\alpha}$ strongly in $L^2(0, T; W^{1,p}(\Omega))$ for all $p \geq 1$ and Lemma 7 of [8]. So, $u^{\alpha} = w^{\alpha} + \gamma$ is a weak solution of $(\mathcal{P})_{\alpha}$; moreover, $u^{\alpha} - \gamma \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $u_t^{\alpha} \in L^2(Q)$. Finally, by Lemma 3.2 and Lemma 3.3, $u^{\alpha} \in L^{\infty}(Q)$. \square

4.3. Passing to the limit as $\alpha \rightarrow 0$

Let $(u^{\alpha})_{0 < \alpha \leq 1}$ be a family of solutions of $(\mathcal{P})_{\alpha}$, according to Theorem 4.2, and let $w^{\alpha} = u^{\alpha} - \gamma$.

Lemma 4.8 *We have*

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_{-}^{\alpha}(t)|^2 dx = \int_{\Omega} \frac{\partial}{\partial t} u_{-}^{\alpha}(t) \Delta u^{\alpha}(t) dx.$$

Moreover, the family $(u_{-}^{\alpha})_{0 < \alpha \leq 1}$ is bounded in $L^{\infty}(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$.

Proof: The desired equality follows from an integration by parts. To obtain an estimate for $\frac{\partial}{\partial t} u_{-}^{\alpha}$, we multiply the differential equation in $(\mathcal{P})_{\alpha}$ by $\frac{\partial}{\partial t} u_{-}^{\alpha}$ and integrate over Ω . Using the equality stated in the lemma and (1.5), we find that

$$\left| \frac{\partial}{\partial t} u_{-}^{\alpha}(t) \right|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_{-}^{\alpha}(t)|^2 dx \leq C_0 \left| \frac{\partial}{\partial t} u_{-}^{\alpha}(t) \right|_{L^1(\Omega)}.$$

From this we deduce, after integration with respect to t and a simple estimate, that

$$\int_0^t \left| \frac{\partial}{\partial t} u_{-}^{\alpha}(\sigma) \right|_{L^2(\Omega)}^2 d\sigma + |\nabla u_{-}^{\alpha}(t)|_{L^2(\Omega)}^2 \leq |\nabla(u_0)_{-}|_{L^2(\Omega)}^2 + T|\Omega|C_0^2.$$

The last assertion of the lemma now follows. \square

With similar reasoning as in the proof of Corollary 4.7(c), we infer

Lemma 4.9 *The family $(u^{\alpha})_{0 < \alpha \leq 1}$ is bounded in $L^2(0, T; H^2(\Omega))$.*

Proof of Theorem 4.1: Due to the boundedness properties of $u^{\alpha} = w^{\alpha} + \gamma$ (see Lemma 3.2, Lemma 3.3, and Lemma 3.4), there exists $w \in L^2(0, T; H_0^1(\Omega)) \cap L^{\infty}(Q)$ such that $w^{\alpha} \rightharpoonup w$ weakly in $L^2(0, T; H_0^1(\Omega))$ and weakly-star in $L^{\infty}(Q)$. Also, there exists $z \in L^2(Q)$ such that $(w^{\alpha} + \gamma)_{-} \rightarrow z$ strongly in $L^2(Q)$. To verify that $z = (w + \gamma)_{-}$, consider the maximal-monotone operator $A : L^2(Q) \rightarrow L^2(Q)$, defined by

$$Av := -(v + \gamma)_{-} = \min(0, v + \gamma), \quad \text{for } v \in L^2(Q).$$

By the previous arguments, $w^{\alpha} + \gamma \rightharpoonup w + \gamma$ weakly in $L^2(Q)$ and $-Aw^{\alpha} \rightarrow z$ strongly in $L^2(Q)$. Thus, by the theory of maximal-monotone operators (see [3]), we conclude that $z = -Aw^{\alpha}$, that is, $z = (w + \gamma)_{-}$. As a consequence of Lemma 4.8 and Lemma 4.9, we have $(w + \gamma)_{-} \in L^{\infty}(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and $w \in L^2(0, T; H^2(\Omega))$. To identify the nonlinear terms in the equations, after passing to the limit as $\alpha \rightarrow 0$, we apply a compactness result due to Rakotoson and Temam (see [13]). We deduce that $w^{\alpha} \rightarrow w$ strongly in $L^2(Q)$. Then, from Lemma 4.8 and the boundedness of w^{α} in $L^{\infty}(Q)$, we have

$$\lim_{\alpha \searrow 0} \int_Q w^{\alpha} \frac{\partial}{\partial t} (w^{\alpha} + \gamma)_{-} dx dt = \int_Q w \frac{\partial}{\partial t} (w + \gamma)_{-} dx dt$$

and

$$\lim_{\alpha \searrow 0} \alpha \int_Q w^{\alpha} \frac{\partial}{\partial t} (w^{\alpha} + \gamma)_{+} dx dt = 0.$$

Furthermore, $(G + J)(w^\alpha + \gamma)$ converges, weakly in $L^2(Q)$, to some function h ; thus,

$$\lim_{\alpha \searrow 0} \int_Q w^\alpha (G + J)(w^\alpha + \gamma) = \int_Q wh.$$

Multiplying the differential equation in $(\mathcal{P})_\alpha$ by w^α , integrating over Q , and letting $\alpha \rightarrow 0$, we deduce

$$\lim_{\alpha \searrow 0} \int_Q |\nabla w^\alpha|^2 dx dt = \int_Q w \frac{\partial}{\partial t} (w + \gamma)_- dx dt + \int_Q h w dx dt = \int_Q |\nabla w|^2 dx dt.$$

From the weak convergence of w^α to w in $L^2(0, T; H_0^1(\Omega))$ and from the last equality, we get that $w^\alpha \rightarrow w$ strongly in $L^2(0, T; H_0^1(\Omega))$. Thus, we may assume that $w^\alpha(t) \rightarrow w(t)$ strongly in $H_0^1(\Omega)$ for a. e. $t \in]0, T[$. In fact, as w^α remains in a bounded set of $L^2(0, T; H^2(\Omega))$, Gagliardo-Nirenberg interpolation shows that

$$w^\alpha(t) \rightarrow w(t) \text{ strongly in } W^{1,p}(\Omega) \text{ for a. e. } t \in]0, T[\text{ and } 1 \leq p < \infty.$$

Finally, we argue as in the proof of Theorem 4.2 to show that $h = G(w + \gamma) + J(w + \gamma)$. We conclude that $u = w + \gamma$ is a weak solution of (\mathcal{P}) with the properties claimed in the theorem. \square

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