VECTOR BUNDLES ON $G(1,4)$ WITHOUT INTERMEDIATE COHOMOLOGY

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§0. Introduction

A known result by Horrocks (see [H]) characterizes the line bundles on a projective space as the only indecomposable vector bundles without intermediate cohomology. This result has been generalized by Ottaviani (see [O1], [O2]) to quadrics and Grassmannians. More precisely, he characterizes direct sums of line bundles as those vector bundles without intermediate cohomology and satisfying other cohomological conditions.

More generally, Knörrer (see [K]) has proved for any quadric that the line bundles and spinor bundles (and their twists by line bundles) are characterized by the property of being indecomposable and not having intermediate cohomology (there is an unpublished independent proof of this fact by I. Sols, which has been the starting point of the present work). Buchweitz, Greuel and Schreyer (see [BGS]) proved a "converse" of such results: only in the case of linear spaces and quadrics there are, up to a twist, a finite number of indecomposable vector bundles without intermediate cohomology.

The goal of this paper is to generalize Horrock's result to the Grassmann variety $G(1,4)$ of lines in $\mathbb{P}^4$, in the sense of characterizing those vector bundles on it without intermediate cohomology.

The paper is distributed in three sections. In the first one, we give the preliminaries on vector bundles on $G(1,4)$ that will be needed for the sequel. In the second section, we characterize the universal bundles on $G(1,4)$ as those without intermediate cohomology an verifying other cohomology vanishings. Finally, in the last section we prove our main result, in which –according to the mentioned result in [BGS]– we obtain big families of vector bundles without intermediate cohomology.

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§1. Preliminaries

Let $G = G(1,4)$ denote the Grassmann variety of lines in $\mathbb{P}^4 = \mathbb{P}(V)$, the projective space of hyperplanes of V. We will assume the ground field to have characteristic zero, although all our results are likely to hold in any characteristic different from two. Consider the universal exact sequence on G defining the universal vector bundles $\mathcal{Q}$ and $\mathcal{S}$ of respective ranks two and three:

$$0 \to \check{\mathcal{S}} \to V \otimes \mathcal{O}_G \to \mathcal{Q} \to 0 \tag{1.1}$$

(a check means a dual vector bundle).

The second symmetric power of the above epimorphism induces a long exact sequence

$$\begin{array}{ccccc} 0 \to \mathcal{S}(-1) \to \check{\mathcal{S}} \otimes V & & \to & & S^2V \otimes \mathcal{O}_G \to S^2Q \to 0 \\ & \searrow & & \nearrow & \\ & & M & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array} \tag{1.2}$$

where the rank-twelve vector bundle M is defined to be the corresponding kernel, and we made the identification $\wedge^2 \check{\mathcal{S}} \cong \mathcal{S}(-1)$ (as usual we write $\mathcal{O}_G(1) \cong \wedge^3 \mathcal{S} \cong \wedge^2 \mathcal{Q}$).

On the other hand, taking the second exterior power in the dual universal sequence we have the following natural long exact sequence (defining the rank-seven vector bundle K as a kernel):

$$\begin{array}{ccccc} 0 \to S^2\check{\mathcal{Q}} \to \check{\mathcal{Q}} \otimes V^* & & \to & & \bigwedge^2 V^* \otimes \mathcal{O}_G \to \bigwedge^2 \mathcal{S} \to 0 \\ & \searrow & & \nearrow & \\ & & K & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array} \tag{1.3}$$

It is easy to see that $\mathrm{Ext}^1(\mathcal{S}, K) = V^*$, so that, for any natural numbers i, j there are non-trivial extensions

$$0 \to K^{\oplus i} \to \mathcal{G} \to \mathcal{S}^{\oplus j} \to 0. \tag{1.4}$$

Definition. An indecomposable direct summand of a vector bundle $\mathcal{G}$ as in (1.4) will be called a *vector bundle of type (I).*

Example 1.1. Consider the following commutative diagram of exact sequences coming from (1.3) and the dual of (1.2), which defines P as a pull-back:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & K & \to & \bigwedge^2 V^* \otimes \mathcal{O}_G & \to & \check{\mathcal{S}}(1) & \to & 0 \\ & & || & & \uparrow & & \uparrow & & \\ 0 & \to & K & \to & P & \to & \mathcal{S} \otimes V^* & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \check{M} & = & \check{M} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\mathrm{Ext}^1(\mathcal{O}_G, \check{M}) = 0$, it follows that the middle vertical exact sequence splits. This shows that $\check{M}$ is a vector bundle of type (I). In fact, the middle horizontal exact sequence is an element in $\mathrm{Ext}^1(\mathcal{S} \otimes V^*, K) \cong \mathrm{Hom}(V, V)$, which is represented by the identity map on V.

Similarly, one can observe that $\mathrm{Ext}^1(\check{K}, K) = V$. This means that, in general, for a vector bundle $\mathcal{G}$ appearing in an exact sequence (split or not) as in (1.4) there are non-trivial extensions

$$0 \to \mathcal{G} \to \mathcal{G}' \to \check{K}^{\oplus l} \to 0.$$

Since $\mathrm{Ext}^1(\check{K}, \mathcal{S}) = 0$, it follows that such a $\mathcal{G}'$ appears in an exact sequence

$$0 \to K^{\oplus i} \to \mathcal{G}' \to \mathcal{S}^{\oplus j} \oplus \check{K}^{\oplus l} \to 0. \tag{1.5}$$

Definition. An indecomposable direct summand of a vector bundle $\mathcal{G}'$ as in (1.5) will be called a *vector bundle of type (II)*.

Finally, the left side of the exact sequence (1.3) shows that $K(1)$ is generated by its global sections. This yields the following exact sequence defining $E(1)$ as a kernel:

$$0 \to E(1) \to V \otimes V^* \otimes \mathcal{O}_G \to K(1) \to 0 \tag{1.6}$$

where E a vector bundle of rank 18. The vector bundle $E(1)$ has the following non-zero extensions groups: $\mathrm{Ext}^1(K(1), E(1))$ (generated by the extension (1.6)), $\mathrm{Ext}^1(\mathcal{Q}, E(1))$, $\mathrm{Ext}^1(\check{K}, E(1))$ and $\mathrm{Ext}^1(\mathcal{S}, E(1))$. We give a general definition containing in particular vector bundles of type (II) and their duals:

Definition. A vector bundle of type (III) will be an indecomposable direct summand of a (maybe trivial) extension

$$0 \to E(1)^{\oplus i} \oplus K^{\oplus j} \oplus \check{\mathcal{S}}^{\oplus k} \to \mathcal{G} \to \mathcal{S}^{\oplus l} \oplus \check{K}^{\oplus m} \oplus \mathcal{Q}^{\oplus n} \to 0$$

Example 1.2. From (1.6) and the first short exact sequence in (1.3) we obtain the following commutative diagram of exact sequences defining P as a pull-back:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & E & \to & V \otimes V^* \otimes \mathcal{O}_G(-1) & \to & K & \to & 0 \\
 & & || & & \uparrow & & \uparrow & & \\
0 & \to & E & \to & P & \to & \check{\mathcal{Q}} \otimes V^* & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & S^2\check{\mathcal{Q}} & = & S^2\check{\mathcal{Q}} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathrm{Ext}^1(\mathcal{O}_G(-1), S^2\check{\mathcal{Q}}) = 0$, it follows that the middle vertical exact sequence splits, and hence $S^2\check{\mathcal{Q}}$ is of type (III).

Definition. A vector bundle F on G is said *not to have intermediate cohomology* if $H^i(G, F(l)) = 0$ for all $l \in \mathbb{Z}$ and $i = 1, \ldots, 5$ (since all cohomology groups are taken on G, we will for short write $H^i(F)$).

Remark. It is easy to see that the vector bundles $\mathcal{Q}, \mathcal{S}, \check{\mathcal{S}}, K, \check{K}, E$ are simple (i.e. their only endomorphisms are multiplications by constants), indecomposable and have no intermediate cohomology. This implies in particular that all vector bundles of type (III) have no intermediate cohomology (in fact, all the vector bundles appearing in this section are, maybe up to a twist, of type (III), as we have remarked). The goal of this paper is to prove that any vector bundle on G without intermediate cohomology is obtained in this way.

Table 1.3. For the reader's convenience, we list here the only non-zero intermediate cohomology of the above five vector bundles when tensored with $\mathcal{Q}$ and $\check{\mathcal{S}}$:
$h^1(\mathcal{Q} \otimes \check{\mathcal{S}}(-1)) = h^5(\mathcal{S} \otimes \mathcal{Q}(-5)) = h^1(\check{\mathcal{S}} \otimes \mathcal{Q}(-1)) = h^2(\check{\mathcal{S}} \otimes \check{\mathcal{S}}(-1)) = h^2(K \otimes \mathcal{Q}(-2)) =$
$h^3(K \otimes \check{\mathcal{S}}(-2) = h^4(\check{K} \otimes \mathcal{Q}(-4)) = h^5(\check{K} \otimes \check{\mathcal{S}}(-4)) = h^3(E \otimes \mathcal{Q}(-2)) = h^4(E \otimes \check{\mathcal{S}}(-2) = 1.$
$h^1(K \otimes \check{\mathcal{S}}) = h^1(E \otimes \mathcal{Q}) = h^2(E \otimes \check{\mathcal{S}}) = h^1(E \otimes \check{\mathcal{S}}(1)) = 5.$

Most of the above equalities can be derive from the others by using the universal exact sequence (1.1) or the Serre duality, taking into account that the canonical line bundle on G is $\omega_G = \mathcal{O}_G(-5)$.

§2. Characterization of the universal bundles

We start by recalling Ottaviani's characterization of direct sums of line bundles, when particularized to $G(1,4)$.

Theorem 2.1. *(Ottaviani, [O1], [O2]) Let F be a vector bundle on $G(1,4)$. Then F is a direct sum of line bundles if and only if the following conditions hold:*
- *a) F has no intermediate cohomology*
- *b) $H^i(F \otimes \mathcal{Q}(l)) = 0$ for any $i = 1, \ldots, 5$ and $l \in \mathbb{Z}$*
- *c) $H^1(F \otimes \check{\mathcal{S}}(l)) = 0$ for any $l \in \mathbb{Z}$.*

Remark. Ottaviani's original statement is not as we gave it. Instead of conditions b) and c), his conditions are (see [O1], Theor. 1 (c) for $k = 1$, $n = 4$):
- b') $H^i(F \otimes \mathcal{S}(l)) = 0$ for any $i = 3, 4, 5$ and $l \in \mathbb{Z}$
- c') $H^i(F \otimes \check{\mathcal{S}}(l)) = 0$ for any $i = 1, 2, 3$ and $l \in \mathbb{Z}$.

These are clearly equivalent to b) and c) in our statement by taking cohomology in the universal exact sequence (1.1) and its dual tensored with $F(l)$, and using the assumption that F has no intermediate cohomology.

The idea for proving the main theorem is to successively remove the six extra cohomological conditions appearing in b) and c) to eventually characterize those vector bundles without intermediate cohomology. Each time we remove a condition, a new family of vector bundles will appear. Table 1.3 indicates which vector bundles must appear each time. We will characterize in this section the universal vector bundles, by removing –one by one– the conditions in Theorem 2.1 that they do not satisfy.

The first condition we remove will be c), which is the only condition that $\mathcal{Q}$ does not verify. Hence, $\mathcal{Q}$ should be characterized by conditions a) and b) in Ottaviani's theorem. Notice that then we obtain a result completely analogous to Horrock's theorem, the role of line bundles being played now by line bundles and their tensor products with $\mathcal{Q}$. We will prove this result in detail, the others being sketched as long as they are similar (in fact, this proof will have a difficulty at the beginning not appearing in the remaining proofs). The precise statement is:

Theorem 2.2. *Let F be an indecomposable vector bundle on $G(1,4)$. Then F is, up to a twist with a line bundle, either the trivial line bundle or $\mathcal{Q}$ if and only if the following conditions hold:*

a) F has no intermediate cohomology

b) $H^i(F\otimes\mathcal{Q}(l))=0$ for any $i=1,\dots,5$ and $l\in\mathbb{Z}$.

Proof. Let F be a vector bundle satisfying a) and b). We will prove our result by induction on $\sum_l h^1(F\otimes\check{\mathcal{S}}(l))$. If this sum is zero, then we are in the hypotheses of Ottaviani's Theorem 2.1, so that F is a direct sum of line bundles.

So assume that $h^1(F\otimes\check{\mathcal{S}}(l))\neq 0$ for some $l\in\mathbb{Z}$. By changing if necessary F with a twist, we can assume $l=0$. Then we have a non-zero element in $\mathrm{Ext}^1(\mathcal{S},F)$, which yields a non-trivial extension

$$0\to F\to P\to\mathcal{S}\to 0. \tag{2.1}$$

We first claim that P also verifies conditions a) and b). Indeed the only vanishing to check is that of $H^5(P\otimes\mathcal{Q}(-5))$, since $h^5(\mathcal{S}\otimes\mathcal{Q}(-5))=1$. To prove this vanishing, we first dualize and suitably twist the exact sequences (1.1), (1.2), (1.3), and using the natural isomorphisms $\bigwedge^2\mathcal{S}\cong\check{\mathcal{S}}(1)$ and $S^2\check{\mathcal{Q}}\cong S^2\mathcal{Q}(-2)$ we get a long exact sequence

$$0\to\mathcal{Q}(-5)\to\mathcal{O}_G(-4)^{\oplus 5}\to\mathcal{O}_G(-3)^{\oplus 10}\to\mathcal{Q}(-3)^{\oplus 5}\to\mathcal{O}_G(-1)^{\oplus 15}\to\check{M}(-1)\to 0.$$

The fact that F and $\mathcal{S}$ (and hence also P) satisfy conditions a) and, for $i=2,3$, also b) easily implies that there is a commutative diagram

$$\begin{array}{ccc} H^5(P\otimes\mathcal{Q}(-5)) & \to & H^5(\mathcal{S}\otimes\mathcal{Q}(-5)) \\ \uparrow & & \uparrow \\ H^1(P\otimes\check{M}(-1)) & \to & H^1(\mathcal{S}\otimes\check{M}(-1)) \end{array}$$

where the vertical maps are isomorphisms. Since we just need to prove that the map on the top is zero, it suffices to prove the same for the map in the bottom. By looking at the dual of the first short exact sequence in (1.2), that map appears in a commutative diagram

$$\begin{array}{ccc} H^0(P\otimes\check{\mathcal{S}}) & \to & H^0(\mathcal{S}\otimes\check{\mathcal{S}}) \\ \downarrow & & \downarrow \\ H^1(P\otimes\check{M}(-1)) & \to & H^1(\mathcal{S}\otimes\check{M}(-1)). \end{array}$$

The claim follows by observing that the vertical map on the left is an epimorphism (its cokernel lies in $H^1(P\otimes\mathcal{S}(-1)\otimes V^*)=H^2(P\otimes\check{\mathcal{Q}}(-1)\otimes V^*)=0$), while the map on the top is zero since the extension (2.1) was non-trivial and $\mathcal{S}$ is simple.

It is also immediate to check that $h^1(P\otimes\check{\mathcal{S}}(l))=h^1(F\otimes\check{\mathcal{S}}(l))$ for any l except $l=0$, for which $h^1(P\otimes\check{\mathcal{S}})=h^1(F\otimes\check{\mathcal{S}})-1$. The latter follows from the exact sequence

$$H^0(P\otimes\check{\mathcal{S}})\to H^0(\mathcal{S}\otimes\check{\mathcal{S}})\to H^1(F\otimes\check{\mathcal{S}})\to H^1(P\otimes\check{\mathcal{S}})\to H^1(\mathcal{S}\otimes\check{\mathcal{S}})=0,$$

in which, as we observed, the first map is zero and $h^0(\mathcal{S}\otimes\check{\mathcal{S}})=1$.

We can therefore apply the induction hypothesis to P and conclude that it decomposes as a direct sum of summands of the type $\mathcal{O}_G(l)$ and $\mathcal{Q}(l)$. We next consider the following commutative diagram defining P' as a pull-back (and in which the right vertical map is the dual of (1.1)):

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & F & \to & P & \to & \mathcal{S} & \to & 0 \\ & & || & & \uparrow & & \uparrow & & \\ 0 & \to & F & \to & P' & \to & \mathcal{O}_G^{\oplus 5} & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & \check{\mathcal{Q}} & = & \check{\mathcal{Q}} & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The middle horizontal exact sequence splits since F has not intermediate cohomology. The middle vertical exact sequence also splits, since $\mathrm{Ext}^1(P,\check{\mathcal{Q}})\cong H^5(P\otimes\mathcal{Q}(-5))^*=0$. Hence $P\oplus\check{\mathcal{Q}}\cong F\oplus\mathcal{O}_G^{\oplus 5}$, from which the theorem follows. □

Theorem 2.3. *Let F be an indecomposable vector bundle on $G(1,4)$. Then F is, up to a twist with a line bundle, either the trivial line bundle, or $\mathcal{Q}$, or $\mathcal{S}$ if and only if the following conditions hold:*

a) F has no intermediate cohomology

b) $H^i(F\otimes\mathcal{Q}(l))=0$ for any $i=1,2,3,4$ and $l\in\mathbb{Z}$.

Proof. We prove it by induction on $\sum_l h^5(F\otimes\mathcal{Q}(l))$, the zero case being Theorem 2.2. Hence we assume $h^5(F\otimes\mathcal{Q}(l))\neq 0$ for some l, and we can suppose without loss of generality that $l=-5$. Hence, by Serre duality, there is a non-zero element in $\mathrm{Ext}^1(\mathcal{Q},\check{F})$. This produces a non-trivial extension

$$0\to\check{F}\to P\to\mathcal{Q}\to 0. \tag{2.2}$$

It is now immediate (in contrast with the proof of Theorem 2.2) to prove that $\check{P}$ still satisfies conditions a) and b) (this is because $\mathcal{Q}\otimes\mathcal{Q}$ has no intermediate cohomology). Also, it holds that $h^5(\check{P}\otimes\mathcal{Q}(l))=h^5(F\otimes\mathcal{Q}(l))$ for any l, except for $l=-5$ for which $h^5(\check{P}\otimes\mathcal{Q}(-5))=h^5(F\otimes\mathcal{Q}(-5))-1$ (the proof being, by using Serre duality, as in Theorem 2.2). Therefore, by induction hypothesis, $\check{P}$ decomposes as a direct sum of summands of the type $\mathcal{O}_G(l)$, $\mathcal{Q}(l)$ and $\mathcal{S}(l)$.

We consider now the following commutative diagram defining P' as a pull-back:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & \check{F} & \to & P & \to & \mathcal{Q} & \to & 0\\
 & & || & & \uparrow & & \uparrow & & \\
0 & \to & \check{F} & \to & P' & \to & \mathcal{O}_G^{\oplus 5} & \to & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \check{\mathcal{S}} & = & \check{\mathcal{S}} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

As in the proof of Theorem 2.2, the middle horizontal exact sequence splits. Therefore, our result will follow if the middle vertical exact sequence also splits (this is the only difficulty that did not appear in Theorem 2.2 and that will appear in the rest of the proofs). To see this, we study the direct summands of $\mathrm{Ext}^1(P,\check{\mathcal{S}})\cong H^1(\check{P}\otimes\check{\mathcal{S}})$ corresponding to the decomposition of P into direct summands. Only a summand of the type $\mathcal{Q}\subset P$ (if it exists) produces a non-zero summand $\mathrm{Ext}^1(\mathcal{Q},\check{\mathcal{S}})\subset\mathrm{Ext}^1(P,\check{\mathcal{S}})$. But then the corresponding component of the element of $\xi\in\mathrm{Ext}^1(P,\check{\mathcal{S}})$ defined by the vertical extension must be zero. Indeed, ξ is the image of the universal extension (1.1) under the map $\pi^*:\mathrm{Ext}^1(\mathcal{Q},\check{\mathcal{S}})\to\mathrm{Ext}^1(P,\check{\mathcal{S}})$ induced by the projection $\pi:P\to\mathcal{Q}$ in (2.2). Since (2.2) is non-split and the only endomorphisms of $\mathcal{Q}$ are the multiplications by a constant, it follows that the restriction of π to any direct summand $\mathcal{Q}\subset P$ is zero.

$\square$

Finally, we prove a theorem characterizing the universal vector bundles on $G(1,4)$ by means of their cohomology vanishings.

Theorem 2.4. *Let F be an indecomposable vector bundle on $G(1,4)$. Then F is, up to a twist with a line bundle, either the trivial line bundle, or $\mathcal{Q}$, or $\mathcal{S}$, or $\check{\mathcal{S}}$ if and only if the following conditions hold:*

a) F has no intermediate cohomology
b) $H^i(F \otimes \mathcal{Q}(l)) = 0$ for any $i = 2, 3, 4$ and $l \in \mathbb{Z}$.

Proof. We use now induction on $\sum_l h^1(F \otimes \mathcal{Q}(l))$, the zero case being now Theorem 2.3. Again we can assume $h^1(F \otimes \mathcal{Q}(-1)) \neq 0$. Therefore, there is a non-zero element in $\mathrm{Ext}^1(\mathcal{Q}, F)$, which yields a non-trivial extension

$$0 \to F \to P \to \mathcal{Q} \to 0.$$

As before, P still satisfies conditions a) and b) and $h^1(P \otimes \mathcal{Q}(l)) = h^1(F \otimes \mathcal{Q}(l))$ for any l, except for $l = -1$ for which $h^1(P \otimes \mathcal{Q}(-1)) = h^1(F \otimes \mathcal{Q}(-1)) - 1$. Hence, by induction hypothesis, P decomposes as a direct sum of summands of the type $\mathcal{O}_G(l)$, $\mathcal{Q}(l)$, $\mathcal{S}(l)$ and $\check{\mathcal{S}}(l)$.

Finally, we consider the following commutative diagram defining P' as a pull-back:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P & \to & \mathcal{Q} & \to & 0 \\
 & & || & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P' & \to & \mathcal{O}_G^{\oplus 5} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \check{\mathcal{S}} & = & \check{\mathcal{S}} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle horizontal exact sequence splits as usual, and the splitting of the middle vertical exact sequence is proved as in Theorem 2.3. This proves the theorem. □

§3. Vector bundles without intermediate cohomology

We continue here the strategy of the previous section. The difference now is that we will not obtain a finite number of vector bundles when removing any of the conditions in Theorem 2.4 b).

Theorem 3.1. *Let F be an indecomposable vector bundle on $G(1, 4)$. Then F is, up to a twist with a line bundle, either the trivial line bundle or $\mathcal{Q}$, $\mathcal{S}$, $\check{\mathcal{S}}$ or a vector bundle of type (I) if and only if the following conditions hold:*
a) F has no intermediate cohomology
b) $H^i(F \otimes \mathcal{Q}(l)) = 0$ for any $i = 3, 4$ and $l \in \mathbb{Z}$.

Proof. We use induction on $\sum_l h^2(F \otimes \mathcal{Q}(l))$, the zero case being Theorem 2.4. We can assume $h^2(F \otimes \mathcal{Q}(-2)) \neq 0$. Hence, by using the universal exact sequence (1.1) there is a non-zero element in $\mathrm{Ext}^1(\check{\mathcal{S}}(1), F)$, which yields a non-trivial extension

$$0 \to F \to P \to \check{\mathcal{S}}(1) \to 0.$$

As in the proofs of the preceeding section, P still satisfies conditions a) and b) and $h^2(P \otimes \mathcal{Q}(l)) = h^2(F \otimes \mathcal{Q}(l))$ for any l, except for $l = -2$ for which $h^2(P \otimes \mathcal{Q}(-2)) = h^2(F \otimes \mathcal{Q}(-2)) - 1$. By induction hypothesis, P decomposes as a direct sum of summands of the type $\mathcal{O}_G(l)$, $\mathcal{Q}(l)$, $\mathcal{S}(l)$, $\check{\mathcal{S}}(l)$ and twists of vector bundles of type (I).

Now we consider the following commutative diagram defining P' as a pull-back, and in which the right column is the second short exact sequence in (1.3):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P & \to & \check{\mathcal{S}}(1) & \to & 0 \\
 & & || & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P' & \to & \mathcal{O}_G^{\oplus 10} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & K & = & K & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle horizontal exact sequence splits as usual, because F has not intermediate cohomology. As for the middle vertical exact sequence, the main difference now with the proofs in the preceeding sections is that the element $\mathrm{Ext}^1(P, K)$ defining the extension can have non-zero components diferent from those corresponding to possible summands $\check{\mathcal{S}}(1) \subset P$ (which as usual we know to produce a zero coordinate). Indeed, any direct summand of type (I) (we include here possible direct summands $\mathcal{S} \subset P$) will yield a non-zero summand of $\mathrm{Ext}^1(P, K)$. We decompose $P = P_1 \oplus P_2$, where P_1 is a sum of vector bundles of type (I) and P_2 does not have any summand of type (I). Since our extension lives in $\mathrm{Ext}^1(P, K) = \mathrm{Ext}^1(P_1, K) \oplus \mathrm{Ext}^1(P_2, K)$ and the second coordinate is zero, it follows that $P' \cong P_1' \oplus P_2$, where P_1' appears in an exact sequence

$$0 \to K \to P_1' \to P_1 \to 0.$$

But from this exact sequence it is immediate to see that P_1' is a direct sum of vector bundles of type (I) (since P_1 is too). This completes the proof. □

Theorem 3.2. *Let F be an indecomposable vector bundle on $G(1, 4)$. Then F is, up to a twist with a line bundle, either the trivial line bundle, or $\mathcal{Q}$, or $\mathcal{S}$, or $\check{\mathcal{S}}$, or a vector* *bundle of type (II), or the dual of a vector bundle of type (II) if and only if the following conditions hold:*

a) F has no intermediate cohomology

b) $H^3(F \otimes \mathcal{Q}(l)) = 0$ for any $l \in \mathbb{Z}$.

Proof. We use induction on $\sum_l h^4(F \otimes \mathcal{Q}(l))$, the zero case being Theorem 3.1. We can assume $h^2(\check{F} \otimes \check{\mathcal{Q}}(-1)) = h^4(F \otimes \mathcal{Q}(-4)) \neq 0$. Hence, by using the universal exact sequence (1.1) there is a non-zero element in $\mathrm{Ext}^1(\check{\mathcal{S}}(1), \check{F})$, which yields a non-trivial extension

$$0 \to \check{F} \to P \to \check{\mathcal{S}}(1) \to 0.$$

As usual, $\check{P}$ still satisfies conditions a) and b) and $h^4(\check{P} \otimes \mathcal{Q}(l)) = h^4(F \otimes \mathcal{Q}(l))$ for any l, except for $l = -4$ for which $h^4(P \otimes \mathcal{Q}(-4)) = h^4(F \otimes \mathcal{Q}(-4)) - 1$. By induction hypothesis, P decomposes as a direct sum of summands of the type $\mathcal{O}_G(l)$, $\mathcal{Q}(l)$, $\mathcal{S}(l)$, $\check{\mathcal{S}}(l)$ and twists of vector bundles of type (II) or their duals.

Now we consider the following commutative diagram defining P' as a pull-back:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & \check{F} & \to & P & \to & \check{\mathcal{S}}(1) & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & \check{F} & \to & P' & \to & \mathcal{O}_G^{\oplus 10} & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & K & = & K & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Once more, the middle horizontal exact sequence splits, and the only non-zero components (corresponding to direct summands of P) in the element $\mathrm{Ext}^1(P, K)$ defining the middle vertical extension can be those coming from possible summands of type (II) of P (as in the preceeding proofs, the coordinates corresponding to possible factors $\check{\mathcal{S}}(1)$ are zero). As in the proof of Theorem 3.1, we also consider $\mathcal{S}$ and $\check{K}$ as vector bundles of type (II). We decompose $P = P_1 \oplus P_2$, where P_1 is a sum of vector bundles of type (II) and P_2 does not have any summand of type (II). Hence, it follows that $P' \cong P_1' \oplus P_2$, where P_1' appears in an exact sequence

$$0 \to K \to P_1' \to P_1 \to 0.$$

But, again as in the proof of Theorem 3.1, P_1' is a direct sum of vector bundles of type (II), which completes the proof. □

We can finally state and prove our result characterizing vector bundles on $G(1,4)$ without intermediate cohomology.

Theorem 3.3. *An indecomposable vector bundle on $G(1,4)$ without intermediate cohomology is, up to a twist with a line bundle, a vector bundle of type (III).*

Proof. Let F be a vector bundle on $G(1,4)$ without intermediate cohomology. We use induction on $\sum_l h^3(F\otimes\mathcal{Q}(l))$, the zero case being Theorem 3.2. Without loss of generality, we assume $h^3(F\otimes\mathcal{Q}(-3))\neq 0$. Hence, by using the dual of the exact sequences (1.1) and (1.3) and the identification $\wedge^2\mathcal{S}\cong\check{S}(1)$, there is a non-zero element in $\mathrm{Ext}^1(K(1),F)$, which yields a non-trivial extension

$$0\to F\to P\to K(1)\to 0. \tag{3.1}$$

As usual, $\check{P}$ has no intermediate cohomology and $h^3(\check{P}\otimes\mathcal{Q}(l))=h^3(F\otimes\mathcal{Q}(l))$ for any l, except for $l=-3$ for which $h^3(P\otimes\mathcal{Q}(-3))=h^3(F\otimes\mathcal{Q}(-3))-1$. Hence, by induction hypothesis, P decomposes as a direct sum of summands of twists of vector bundles of type (III).

Now we consider the following commutative diagram defining P' as a pull-back, and in which the right column is the exact sequence (1.6):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P & \to & K(1) & \to & 0\\
 & & || & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & P' & \to & \mathcal{O}_G^{\oplus 25} & \to & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & E(1) & = & E(1) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle horizontal exact sequences splits once more because F has not intermediate cohomology. As for the middle vertical exact sequence, we decompose $P=P_1\oplus P_2$, where P_1 is a sum of vector bundles of type (III) and the summands of P_2 are not of type (III), but twists of vector bundles of type (III) with a non-trivial line bundle. As in the other proofs in this section, it suffices to prove that the element $\xi\in\mathrm{Ext}^1(P,E(1))=\mathrm{Ext}^1(P_1,E(1))\oplus\mathrm{Ext}^1(P_2,E(1))$ corresponding to the middle vertical extension has zero as its second component in this decomposition.

The extra difficulty now is that, if a summand $H\subset P_2$ produces a non-zero summand $\mathrm{Ext}^1(H,E(1))\subset\mathrm{Ext}^1(P_2,E(1))$, it is not neccesarily $H=K(1)$, but it could happen, more generally, that $H(-1)$ is a vector bundle of type (III). In this case, H is a direct summand of a vector bundle $\mathcal{G}(1)$ fitting in an exact sequence

$$0\to E(2)^{\oplus i}\oplus K(1)^{\oplus j}\oplus\check{\mathcal{S}}(1)^{\oplus k}\to\mathcal{G}(1)\to\mathcal{S}(1)^{\oplus l}\oplus\check{K}(1)^{\oplus m}\oplus\mathcal{Q}(1)^{\oplus n}\to 0$$

By contradiction, assume that the component of ξ in $Ext^1(H, E(1))$ is not zero. Then, at least one summand $K(1)$ of that exact sequence must produce a non-zero map

$$K(1) \subset E(2)^{\oplus i} \oplus K(1)^{\oplus j} \oplus \check{\mathcal{S}}(1)^{\oplus k} \to \mathcal{G}(1) \to H \subset P \to K(1)$$

(the projection $P \to K(1)$ being that of (3.1)). Since the only endomorphisms of K are multiplications by constants, this implies that the projection $P \to K(1)$ has a section, so that the extension (3.1) must be trivial, which is a contradiction. □

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