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Beatriz de la Flor
EY Spain - Market Risk Team (Financial Services Office)

Javier Ojea-Ferreiro
Joint Research Centre of the European Commission (JRC) and Complutense Institute of Economic Analysis (ICAE)

Eva Ferreira
Department of Quantitative Methods, University of the Basque Country

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We find a leading role played by exchange rate hedging stock losses, which outstrips the position of gold (index) in non-US (US) portfolios. The inclusion of the exchange rate can reduce the ES between 107 and 162 bps. An out-of-sample exercise supports our results.
The implications of this study go beyond risk management decisions. Regulatory and supervisory authorities might find tools to assess the performance of financial assets under market distress scenarios

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JEL Classification C52, C58, C61, F31, G1

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Beatriz de la Flor†
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*All errors are our own. The views expressed are those of the author and do not necessarily reflect those of the JRC.

†Calle Raimundo Fernández de Villaverde, 65, Madrid 28003 (Spain). Email: *beatriz.de.la.flor.de.pablos@es.ey.com*

‡Via E. Fermi, 2749, Ispra VA 21027 (Italy). Email: *javier.ojea-ferreiro@ec.europa.eu*

§Avenida Lehendakari Aguirre, 81, Bilbao 48015 (Spain). Email: *eva.ferreira@ehu.eus*

1 Introduction

Throughout the successive crises of recent decades the high uncertainty in financial markets has encouraged herd behaviour on the part of investors, leading them to hold haven assets such as gold. The safe-haven property of gold has been widely studied by the literature (Gokmenoglu and Fazlollahi, 2015; Reboredo and Ugolini, 2016; Kristoufek and Vosvrda, 2016).

Many studies support the key role of gold as a safe haven against movements in stock markets, in both developed and emerging economies, and especially in the short term. Junttila et al. (2018) analyse the dynamic correlation between these assets based on a VAR model with DCC-GARCH errors while Baur and McDermott (2010), Gürgün and Ünalmis (2014) and Iqbal (2017) employ linear regression to assess the features of gold as a safe haven.

We are aware of the impact of the exchange rate on hedging outcomes when the stock portfolio is not denominated in US dollars, which is the main currency for trading gold. The interrelationship between exchange rate, stock and gold markets means that shocks in one of them cannot be taken in isolation. Indeed, the dependence between these assets is complex, dynamic and presents non-linearities (Christoffersen et al., 2012; Karimalis and Nomikos, 2018).

Reboredo and Rivera-Castro (2014a, 2014b), Reboredo (2013), Qureshi et al. (2018) and Nguyen et al. (2020) analyse the link between gold and exchange rate markets. Reboredo and Rivera-Castro (2014a) design a likelihood ratio test that draws a distinction between hedging and safe-haven characteristics of gold[1]. Reboredo and Rivera-Castro (2014b) and Qureshi et al. (2018) employ the so-called wavelet analysis[2], while Reboredo (2013) and Nguyen et al. (2020) use a copula approach to analyse the link between gold and the depreciation of various currencies. They find that gold acts as hedge and as a weak safe haven against exchange rate movements, with a hedging property decreasing in time (Wang, 2013; Reboredo and Rivera-Castro, 2014b; Qureshi et al., 2018).

Authors such as Lin (2011), Wang et al. (2013), Reboredo et al. (2016) and Ojea-Ferreiro and Reboredo (2021), in turn, use copulas to measure the dependence relation between exchange rates and stock index, finding evidences of dependence on average and tail dependence between them.

However, the related literature on portfolios that consider exchange rate, stock and/or gold markets has focused mainly on the relationship between two of them (Sjaastad, 2008; Baruník et al., 2016). It has either focused on US investors or the gold price has been transformed into the local currency before risk management analysis is conducted in non-US denominated markets. This means that they are missing the complete picture for the risk management of stock portfolios. Ignoring the implication of the introduction of a stochastic component (the exchange rate) into the portfolio dynamics leads to a persistent

[1] Gold acts as a hedge asset (safe haven) when disjointed movements in gold and stock or exchange rate markets occur only on average (in distress scenarios).

[2] The wavelet approach decomposes a serie into different components: lower time scales capture higher frequency time series components which occur over very short periods of time, whilst higher time scales capture lower frequency components occurring over very long periods.

bias in the hedging strategy. By contrast, considering the trend in co-movements between the three assets and properly estimating their dependence is crucial to understanding risk management assignments.

The goal of our study is to analyse the importance of exchange rate risk when non-US investors create a hedging strategy with gold or when US investors include a stock asset not denominated in US dollars in their portfolios.

The questions we endeavor to answer in this study here are: i) what is the hedging role of the exchange rate in a portfolio that includes an asset not denominated in the domestic currency?; ii) what is the cost of ignoring exchange rate risk according to different risk measures?; iii) how does the hedging strategy change depending on the currency in which the portfolio is denominated?; and iv) to what extent are the above questions affected by tail dependence?

To answer these questions, the characteristics of the multivariate distribution, beyond being static and normally distributed, must be known. To model the marginal distribution of each asset we use autoregressive models for the conditional mean and GARCH-type models for the conditional variance. Skewness and kurtosis are captured by the Skewed Student t distribution for innovations. Concerning the dependence between variables, we use copulas to decompose the joint distribution between marginal and dependence components (Cherubini and Luciano, 2001; Sun et al., 2008; Reboredo, 2011; Berger, 2016). More specifically, we employ a vine copula approach to capture asymmetric tail dependence, using data-driven dynamics to take into account the time-varying behaviour of the relationships between assets (see e.g. Aas et al., 2009; Cooke et al., 2011; Aas and Berg, 2011; Yew Low et al., 2013 and Ojea-Ferreiro, 2020).

We build international investment portfolios for non-US and US investors under a minimum-Expected Shortfall (ES) criteria. On the one hand, these non-US (US) portfolios are built using the convolution between the log-returns of the exchange rate and the asset not denominated in the investor's local currency, i.e. gold (equity index). On the other hand, we allow extra exposure to the exchange rate, so that its weight in the portfolio can be divided into one part acquiring gold (stock) and another intended to improve the hedging strategy. Once the portfolio is selected, we explore the influence of the exchange rate on market risk in the tails of the portfolio, by comparing the ES, Conditional ES (CoES), ΔCoVaR and ΔCoES (Adrian and Brunnermeier, 2011) with and without the exchange rate being considered in the investment strategy.

The sample consists of a twenty-year weekly data set including financial returns on gold, exchange rates (domestic vs dollar) and stock index rates in Europe, Japan, Brazil and the UK. The results provide important insight into the leading role of exchange rates in non-US (US) stock portfolios, where they even outstrip the position of gold (index). An extra investment in exchange rate also enhances the hedging strategies in terms of risk. For instance, non-US portfolios with an extra position in the exchange rate show a drop in ES, on average, from 113 to 158 bps, while the ES of US investors' portfolios drops between 107 and 162 bps. Also, an out-of-sample exercise reinforces these results.

We believe that this study has important consequences for practitioners. On the one hand, it has straightforward implications for international investors in terms of portfolio optimisation and risk management decisions. On the other hand, the analysis of the relationship between economic and financial markets is useful for policy makers and the assessment of the performance of domestic stock markets under distress scenarios for the exchange rate might provide a stress test approach for regulatory and supervisory authorities.

The rest of the paper is organised as follows. Section 2 describes the methodology. Section 3 introduces the data used in the empirical application, the results of which are discussed in Section 4. Section 5 sets out the main conclusions.

2 Materials and Methods

2.1 Marginal distributions

For the marginal densities of assets, an ARMA(p,q)-APARCH(1,1) model captures the autocorrelation and heterocedasticity of the series. For innovations, a Hansen (1994)'s Skewed Student t distribution takes into account skewness and kurtosis, which are usual features in financial returns. This model is:

$$\begin{aligned} r_t = \phi_0 + \sum_{i=1}^{p} \phi_i \cdot r_{t-i} + \sum_{j=1}^{q} \theta_i \cdot a_{t-j} + a_t = \mu_t + a_t \\ \sigma_t^\delta = \omega + \alpha \cdot (|a_{t-1}| + \gamma \cdot a_{t-1})^\delta + \beta \cdot \sigma_{t-1}^\delta \end{aligned} \tag{2.1.1}$$

where p and q are non-negative integers and ϕ and θ are the AR and MA parameters, respectively. Note that $\delta = 2$ leads to the GJR-GARCH model and $\delta = 2$ and $\gamma = 0$ to the GARCH model. Moreover, $a_t = \sigma_t \cdot \varepsilon_t$, with ε_t being a Skewed Student variable with mean zero and unit variance (see Annex E).

2.2 Joint distributions

Copula approach enables the trend over the time of complex dependence patterns between two or more variables to be studied. This methodology allows to decompose the joint distribution of d variables into marginal distributions and a copula C that adjust their dependence structure, i.e. $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$, where $F_i(x_i) = u_i \in [0, 1]$ can be any distribution[3].

A vine copula can be understood as a hierarchical dependence structure that decompose the multivariate copula into a cascade of bivariate copulas. This makes the model higly flexible because each pair-copula can be different (Reboredo and Ugolini, 2015a; 2015b). The density function in the trivariate case would be: $f(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{13}(F_1(x_1), F_3(x_3)) \cdot c_{23|1}(F_2(x_2|x_1), F_3(x_3|x_1))$. The joint distribution function is $F_{123}(x_1, x_2, x_3) = \int_{-\infty}^{x_1} C_{23|1}(F_{2|1}(x_2|z), F_{3|1}(x_3|z))\ dF_1(z)$

[3]We use U(0,1) variables obtained from the probability integral transformation of the standardised residuals of the ARMA-APARCH model as marginal distributions.

We select the exchange rate as the conditioning variable of the vine, due to the importance for investors of introducing this asset into their portfolios when investing in an asset denominated in a foreign currency (Ojea-Ferreiro, 2020).

We use eight copulas with different assumptions about the tail dependence between assets[4]: Gaussian (no-tail dependence), Student t (symmetric tail dependence), Gumbel (positive upper tail dependence), 90° rotated Gumbel (negative upper tail dependence), Clayton (positive lower tail dependence), 90° rotated Clayton (negative lower tail dependence), BB1 (positive asymmetric tail dependence) and 90° rotated BB1 model (negative asymmetric tail dependence). The optimal parameters of the copula are estimated by the Inference Functions for Margins (IFM) procedure (Joe, 2014; Ojea-Ferreiro and Reboredo, 2021).

Following Patton (2001, 2006), we assume that the copula model remains fixed over the sample while the parameters vary according to an ARMA(1,10) process to parameterise time-variation in the dependence (see Annex B).

2.2.1 Convolutions

We define the returns of gold denominated in a local currency other than USD (r_{ge}) as the result of the convolution of two dependent stochastic processes, exchange rate (local currency vs US dollar) (r_e) and gold returns in USD (r_g)[5], i.e. $r_{ge} = r_e + r_g$. This means that changes in this variable be due to a variation in the supply and demand of the metal or the appreciation or depreciation of the currency (see Ojea-Ferreiro, 2019). The copula-convolution function (Cherubini et al., 2016) is given by $F_{ge}(r_{ge}) = \int_0^1 C_{g|e}\left(F_g(r_{ge} - F_e^{-1}(u))|u\right) du$

Its density function is $f_{ge}(r_{ge}) = \int_0^1 c_{g,e}\left(u, F_g(r_{ge} - F_e^{-1}(u)) f_g(r_{ge} - F_e^{-1}(u))\right) du$

2.3 Portfolio construction

Assume a non-USD portfolio such that it can be expressed as a proportion $(1-\omega)$ invested in the local equity market (r_s) and a proportion ω invested in gold (expressed in the currency in which the index is traded, r_{ge}). This latter variable is the convolution between exchange rate (r_e) and gold returns (r_g):

$$r_p = \omega \cdot (r_g + r_e) + (1-\omega) \cdot r_s = \omega \cdot r_{ge} + (1-\omega) \cdot r_s \tag{2.3.1}$$

Equivalently, $r_g = \frac{r_p - (1-\omega) r_s - \omega r_e}{\omega}$, so $u_g = F_g\left(\frac{r_p - (1-\omega) F_s^{-1}(u_s) - \omega F_e^{-1}(u_e)}{\omega}\right)$

Proposition 1. *Assume that the correlation matrix between the assets is* $\boldsymbol{R}$ *and there exists an unique copula,* C*, which describes their joint distribution. That copula can be defined as* $C_{g|e,s}(u_g|u_e, u_s)$*. Thus, the cumulative distribution*

[4] Annex A shows more in detail the copulas used in this study.

[5] From the point of view of a US investor, the price of the foreing stock index denominated in US dollars (r_{se}) is a convolution between the return of the stock (r_s) and the USD exchange rate against the currency of the equity market (r_e^*).

function of this portfolio can be expressed in terms of conditional copulas as:

$$F_p(r) = \begin{cases} \int_0^1 \int_0^1 C_{g|e,s}\left(F_g\left(\frac{r_p-(1-\omega)F_s^{-1}(u)-\omega F_e^{-1}(v)}{\omega}\right)\Big|u,v;\boldsymbol{R}\right)c_{e,s}(u,v)dudv & 0<\omega<1 \\ F_{ge}(r) = \int_0^1 C_{g|e}\left(F_g\left(r_p - F_e^{-1}(v)\right)\Big|v;\rho_{g,e}\right)dv & \omega = 1 \\ F_s(r) & \omega = 0 \end{cases} \tag{2.3.2}$$

Assume a non-USD portfolio comprising a stock index, the exchange rate of the investor's own currency against US dollars and gold, where investment in the exchange rate is divided into one part intended to acquire gold, ω_1, and another part intended to improve the hedging strategy, ω_2. Its return is:

$$r_p = \omega_1 \cdot r_g + (\omega_1+\omega_2)\cdot r_e + (1-\omega_1-\omega_2)\cdot r_s \tag{2.3.3}$$

from which it can be deduced that $u_g = F_g\left(\frac{r_p-(1-\omega_1-\omega_2)F_s^{-1}(u_s)-(\omega_1+\omega_2)F_e^{-1}(u_e)}{\omega_1}\right)$

Proposition 2. *Assume a copula C which represents the dependence structure between the exchange rate, stock index and gold. The correlation matrix of the components of the portfolio is $\boldsymbol{R}$. Hence, the cumulative distribution function of the portfolio must be written in terms of conditional copulas as:*

$$F_p(r) = \begin{cases} \int_0^1 \int_0^1 C_{g|e,s}\left(F_g\left(\frac{r_p-(1-\omega_1-\omega_2)F_s^{-1}(u)-(\omega_1+\omega_2)F_e^{-1}(v)}{\omega_1}\right)\Big|u,v;\boldsymbol{R}\right)c_{e,s}(u,v)dudv & \\ & if\ \ 0<\omega_i<1;\quad i=1,2 \\ \int_0^1 C_{s|e}\left(F_s\left(\frac{r_p-\omega_2 F_e^{-1}(v)}{1-\omega_2}\right)\Big|v;\rho_{e,s}\right)dv & if\ \ \omega_1=0;\quad 0<\omega_2<1 \\ \int_0^1 \int_0^1 C_{g|e,s}\left(F_g\left(\frac{r_p-(1-\omega_1)\cdot F_s^{-1}(u)-\omega_1\cdot F_e^{-1}(v)}{\omega_1}\right)\Big|u,v;\boldsymbol{R}\right)c_{e,s}(u,v)dudv & \\ & if\ \ 0<\omega_1<1;\quad \omega_2=0 \\ \int_0^1 C_{g|e}\left(F_g\left(r_p-F_e^{-1}(v)\right)\Big|v;\rho_{g,e}\right)\ dv & if\ \ \omega_1=1;\quad \omega_2=0 \\ F_e(r) & if\ \ \omega_1=0;\quad \omega_2=1 \\ F_s(r) & if\ \ \omega_i=0;\quad i=1,2 \end{cases} \tag{2.3.4}$$

Using the above distributions, we build two portfolios according to a minimum ES criterion, computed by assessment. A discrete number of weights (in total 64.000) are uniformly spaced between 0 and 1 and the optimal weights of the assets are those that provide the mimimum ES value from among them.

2.4 Risk measures

On the one hand, one of the most common tools for measuring the potential losses in a portfolio in an extreme event is the Expected Shortfall (Artzner et al., 1997, 1999), defined as: $ES_p(\alpha) = E\left(r_p|\ r_p \leq VaR_p(\alpha)\right) = \frac{1}{\alpha}\int_0^\alpha VaR_p(q)\ dq$.

On the other hand, we compute measures for the conditional risk of the portfolios on bearish scenarios for the stock market. The Conditional ES (CoES) is based on calculating the expected loss of the portfolio in the β-quantile of the lower tail of its distribution, conditional on another variable, in this case the stock, in its α-quantile (Aas et al., 2009; Aas and Berg, 2011), i.e. $CoES_{p|s}(\beta) = E\left(r_p \middle| r_p \leq CoVaR_{p|s}(\beta,\alpha)\right) = \frac{1}{\beta}\int_0^\beta F_{p|s}^{-1}(u_p)\,du_p = \frac{1}{\beta}\int_0^\beta CoVaR_{p|s}(q,\alpha)\,dq$, where $P\left(r_p < CoVaR_{p|s}(\beta,\alpha)|r_s < VaR_s\left(\alpha\right)\right) = \beta$.

Finally, ΔCoES (Adrian and Brunnermeier, 2011) can be understood as a proxy of the systemic risk of the portfolio because it measures how much the ES of a portfolio changes when the return of the stock component moves from the median to quantile α, i.e. $\Delta CoES_{p|s} = CoES_{p|s}(\beta,\alpha) - CoES_{p|s}(\beta,0.5)$

3 Data

We empirically investigate the role of the exchange rate in a stock portfolio of non-US and US investors using data on the prices of gold (in USD per ounce) and the stock index and exchange rate (unit of the currency against USD) in each region studied. Thus, our data set comprises gold, EUROSTOXX 50, EUR/USD, Nikkei 225, JPY/USD, BOVESPA, BRL/USD, FTSE 100 and GBP/USD. The data are sourced from Datastream. Table 1 (Annex D) provides summary statistics of the distributions of the returns of all the assets.

The sample period runs from 1 January 2000 to 1 June 2021. The (log) changes in the daily prices of each asset are calculated and then aggregated to a weekly frequency. In total the sample yields 1117 weekly returns from each series. The period from 1 January 2000 to 31 December 2019 (1043 observations) is used for the in-sample analysis, while the remaining 74 observations, covering the period from 1 January 2020 to 1 June 2021, are used for an out-of-sample exercise. The use of weekly data is more appropriate for characterising dependence between assets because daily data may be affected by drifts and noise that could mask dependence relations and complicate modelling the marginal distributions (Ang and Chen, 2002; Reboredo, 2013).

4 Results and Discussion

4.1 Joint distribution

Table 1 (Annex F) contains the best marginal model for each asset. To find the model that best captures the behaviour of the variances, we compute GARCH(1,1), GJR(1,1), EWMA, APARCH(1,1) and GARCH(1,1)-M under Normal, Student t and Skewed Student t distribution assumptions. Lag parameters in the mean equation and the model for the variance were chosen so as to maximise the loglikelihood function and minimise AIC and BIC values.

The evidence indicates that the best marginal model is ARMA(p,q)-APARCH(1,1), where the values p and q differ for each asset (see Table 1 in Annex E), except for Nikkei 225 and BOVESPA, where the optimal models for variance are GARCH(1,1) and GJR(1,1), respectively. In any case the residuals of the optimal model follow a Skewed Student t distribution (see Annex E).

To find the vine model that best describes the dependence structure between the assets, we employ eight copula models with different assumptions on tail dependence (see Section 2.1.2) for each pair-copula of the vine structure. We compute these models with constant and time-varying parameters.

The vine structures summarised in Table 2 (Annex F) are those which provide the highest value for the AICc. Figure 1 shows symmetric asset-pair tail dependence in Europe. In Japan a Gaussian copula evidences an absence of conditional tail dependence except in the distribution of exchange rate and gold returns, which is represented by a Student t copula. In the case of Brazil there is only an asymmetric asset-pair tail dependence between exchange rate and gold returns, defined by a 90° rotated Clayton. For the rest of the pair-copulas of this vine structure we find a symmetric dependence relation. Finally, the UK findings show symmetric asset-pair tail dependence between exchange rate and stock and exchange rate and gold, while the conditional distribution function between stock and gold returns is defined by a 90° rotated Gumbel copula.

4.2 Portfolio risk

We study how the composition of minimum-ES portfolios changes when the exchange rate is considered, not only in convolution with the gold returns in the non-US investor's own currency, but also as an additional asset in the portfolio. The left-hand panels in Figure 2 present the trend in the weights of gold denominated in the investor's currency and the index of each region in this portfolio (from here two-asset portfolio). The right-hand panels of Figure 2 analyse the situation in which investment in the exchange rate may be divided into a part intended for the acquisition of gold and a part intended to hedge the portfolio (blue area). In these three-asset portfolios the exchange rate takes a leading position, with gold having zero weight in the cases of Japan and the UK.

We also consider the role of the exchange rate in the portfolio of a US investor composed of a foreign stock index, namely EUROSTOXX 50, Nikkei 225, BOVESPA or FTSE 100, gold and the US Dollar exchange rate against the currency in which the equity index is denominated[6]. The left-hand panels in Figure 3 present the trend the weights of the foreign stock index denominated in US Dollars and gold in the two-asset portfolio. In the right-hand panels in Figure 3 we see that the exchange rate becomes the leading position in all three-asset portfolios. Furthermore, this asset does not replace the other haven asset, namely gold, but the stock position in Europe, Japan and Brazil.

Table 1 presents the average ES in the sample of the aforementioned portfolios. A substantial reduction is seen in their tail losses when the exchange rate is included in the hedging strategy, for both non-US and US investors. Summarising, in non-US three-asset portfolios the ES drops, on average, from 113 to 158 bps, while the ES of US portfolios is reduced between 107 and 162 bps.

[6] The USD/x exchange rate is the inverse of x/USD. Thus, $r_{e^*} = log\frac{(x/USD)_t^{-1}}{(x/USD)_{t-1}^{-1}} = -r_e$, where r_e is the logarithmic return of the exchange rate of a local currency against USD.

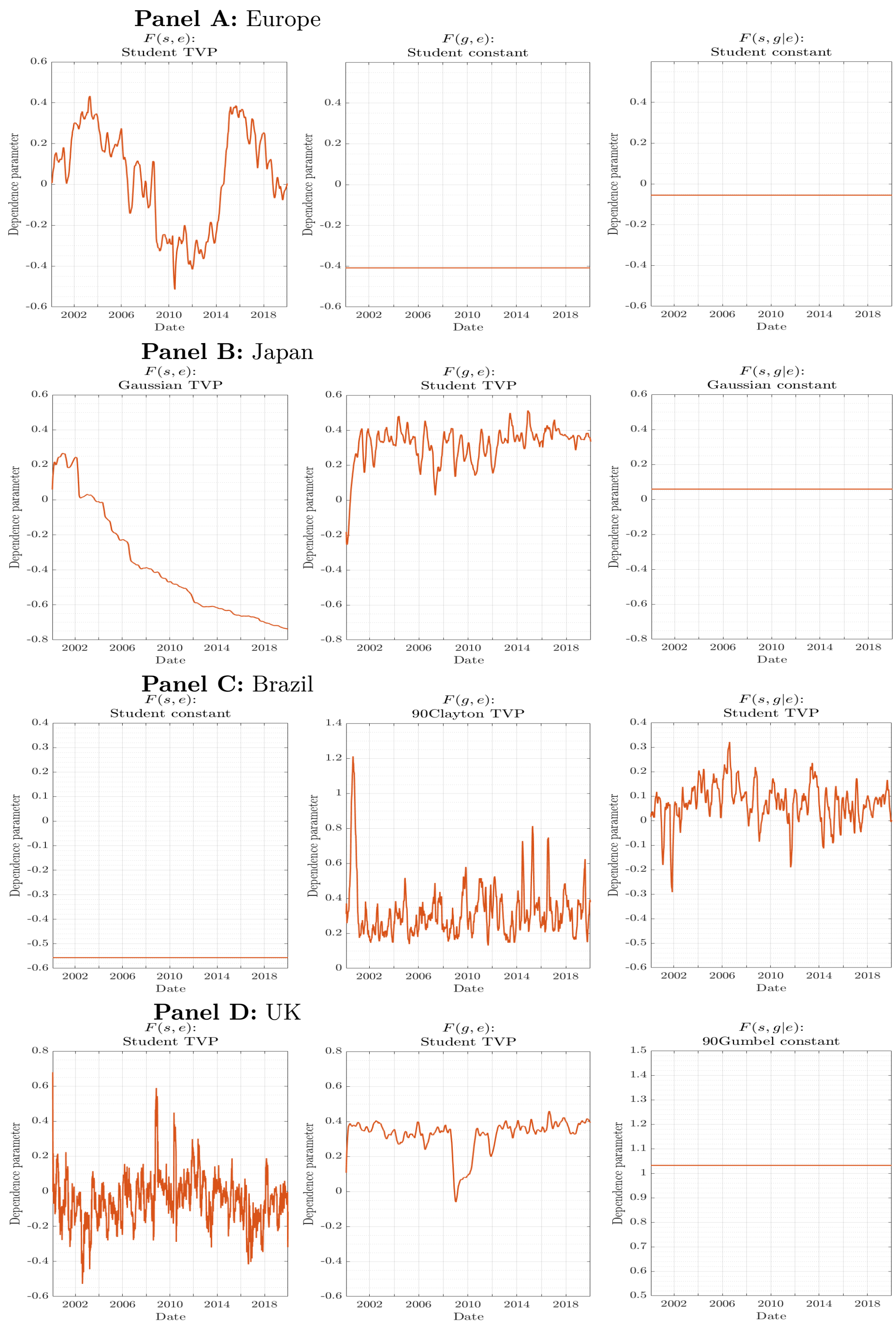
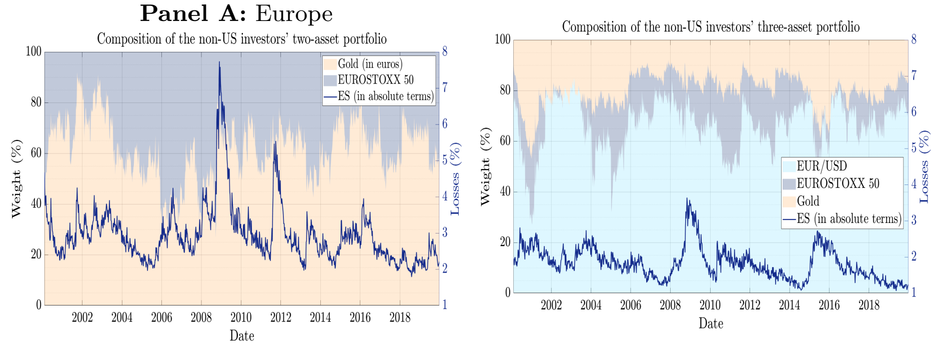
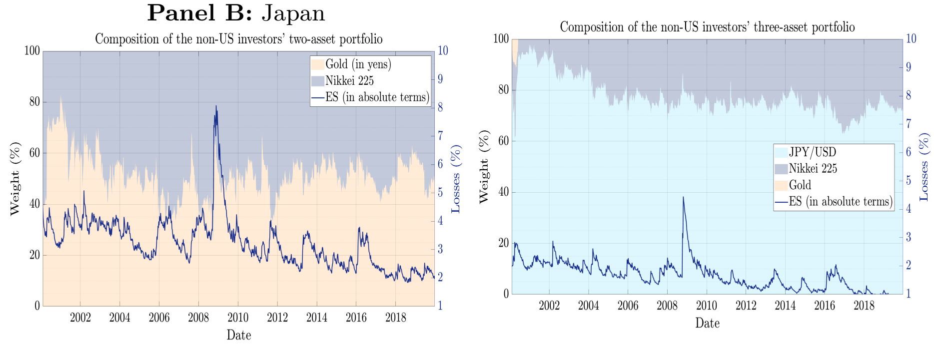


Figure 1: Trend in the optimal parameters of vine copula models. This figure presents the evolution of the constant or time-varing (TVP) parameters of the copulas of the best vine structure selected in each region.

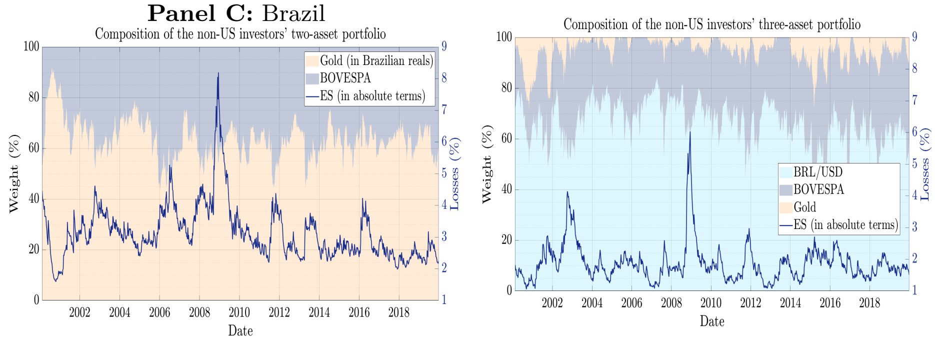
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

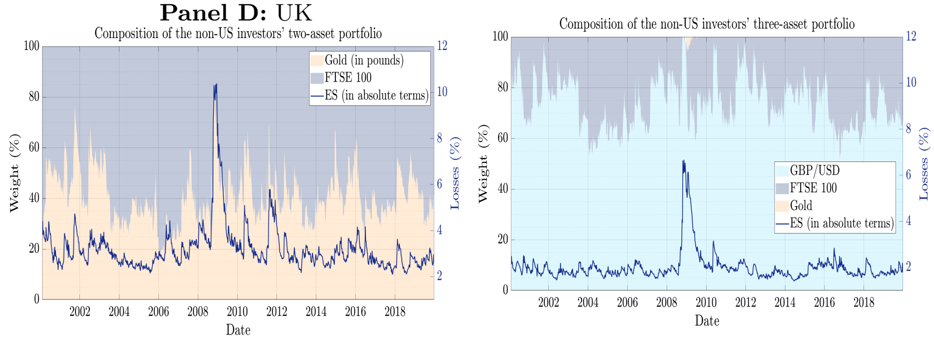
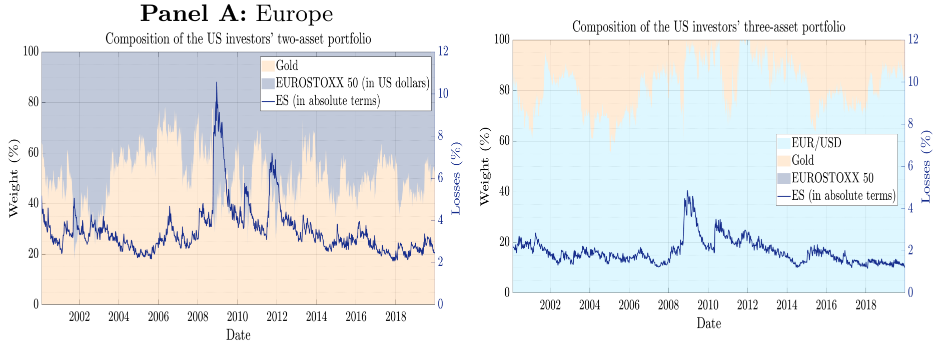
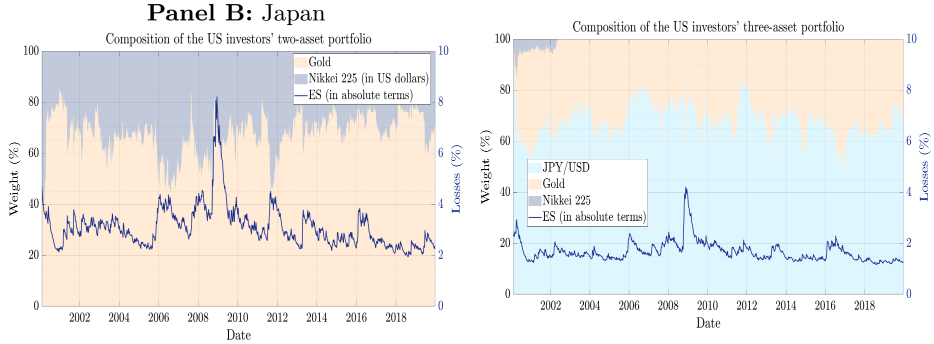


Figure 2: Dynamic composition of minimum-ES portfolios of non-US investors. The left (right) figure in each panel shows the weekly trend in the weights of each asset in the non-US portfolio without (with) extra investment in the exchange rate, i.e. a two-asset (three-asset) portfolio. The optimal weights of each asset are those that minimise the ES(0.1) of the portfolio.

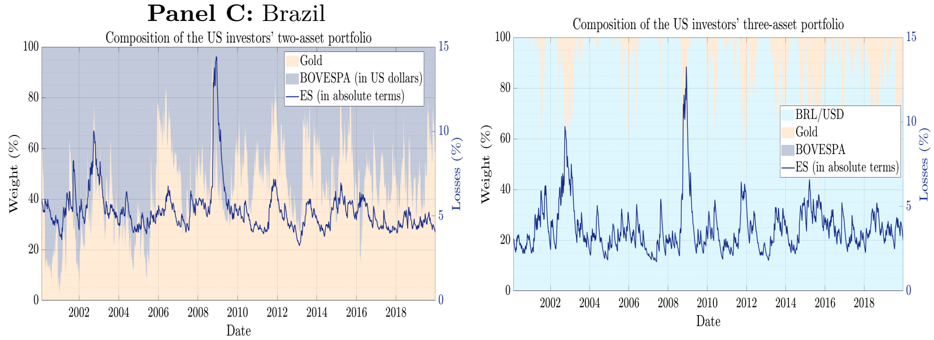
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

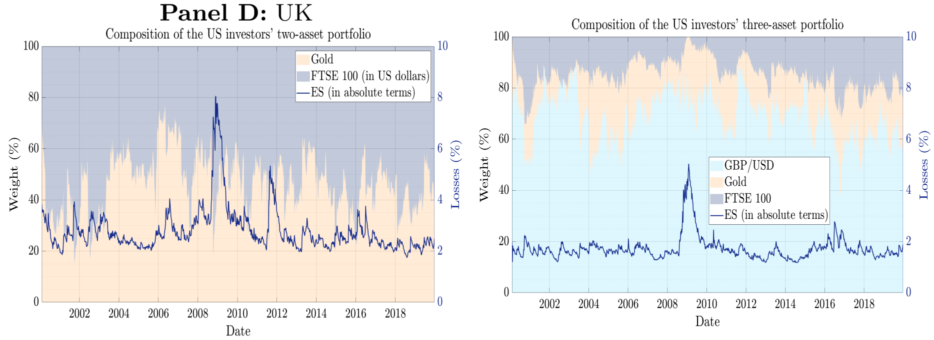


Figure 3: Dynamic composition of minimum-ES portfolios of US investors. The left (right) figure in each panel shows the weekly trend in the weights of each asset in the US portfolio without (with) extra investment in the exchange rate, i.e. a two-asset (three-asset) portfolio. The optimal weights of each asset are those that minimise the ES(0.1) of the portfolio.

Table 1: Expected Shortfall of non-US and US portfolios.

	Panel A: Non-US portfolios		**Panel B:** US portfolios	
	Two-asset portfolio	**Three-asset portfolio**	**Two-asset portfolio**	**Three-asset portfolio**
Europe	-0.0296	-0.0182	-0.0352	-0.0196
Japan	-0.0323	-0.0165	-0.0311	-0.0167
Brazil	-0.0313	-0.0187	-0.0532	-0.0370
UK	-0.0321	-0.0200	-0.0280	-0.0172

NOTE: This table presents the average ES of the non-US and US two- and three-asset portfolios which include the stock index of Europe, Japan, Brazil or the UK.

Several risk measures are useful for the purpose of this study (see Figures in Annex H). We focus on the tail risk of the portfolio conditional on a distress scenario in stock markets to analyse the hedging performance of the exchange rate. Figure 4 shows the difference between the ES and the CoES in a spillover in the stock market of non-US and US portfolios. If CoES values are below the ES then extreme downward movements in the index have a spillover effect on the portfolio. A comparison of the left and right panels of Figure 4 shows that the difference between the unconditional and conditional tail risk is lower for the three-asset portfolios, so they have greater hedging against market risk. Regarding the results for non-US and US investors, portfolios denominated in USD which invest in EUROSTOXX 50 and BOVESPA are more hedged against market risk than non-US portfolios which contain the same index.

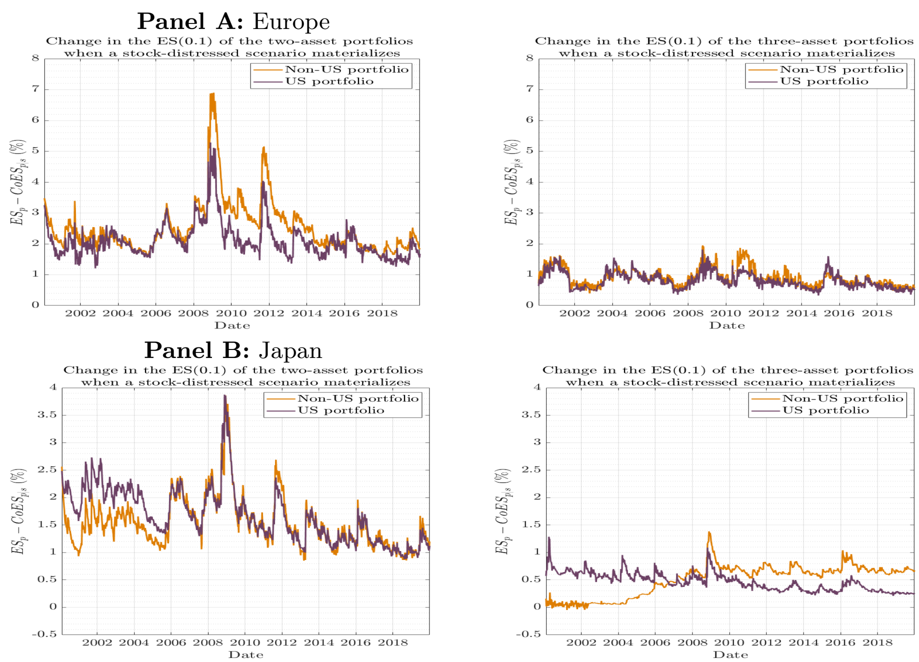
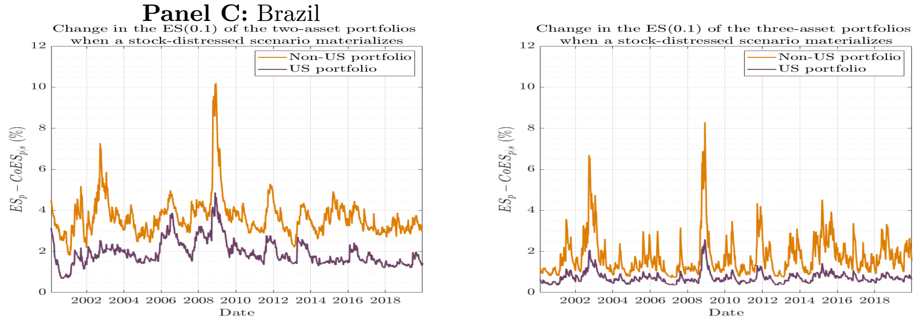


Figure 4: Change in the ES at 10% of non-US and US portfolios when a stock-distressed scenario materialises.

Panel C: Brazil



Panel D: UK

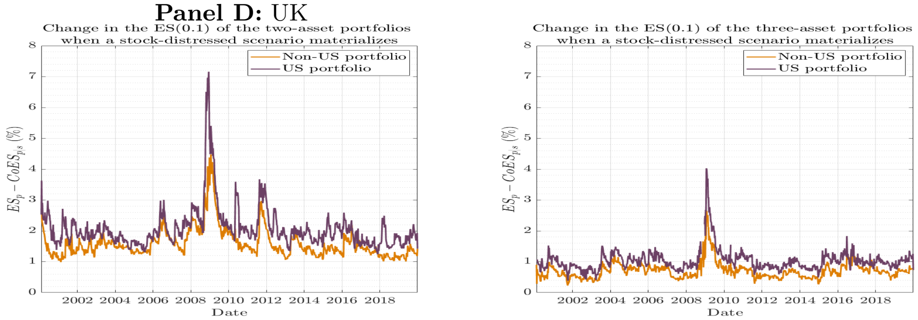


Figure 4 (Cont.): Change in the ES of non-US and US portfolios when a stock-distressed scenario materialises. The left (right) figure in each panel shows the immunisation of the portfolio without (with) extra investment in the exchange rate. The level of immunisation is computed as the difference between the unconditional (ES(0.1)) and conditional (CoES(0.1,0.1)) tail risk of the portfolios in a crash event in the stock market.

It is also possible to quantify the hedging effectiveness of gold and exchange rate in international portfolios in bearish stock markets. Tables 2 and 3 contain the average values of several risk measures of three portfolios, namely a stock portfolio, a two-asset portfolio and a three-asset portfolio, built for non-US and US investors, respectively. The first is a portfolio made up only of the regional stock index. Our findings evidence that including gold and especially the exchange rate in the hedging strategy actually reduces tail (measured by CoES), systemic risk (measured by $\Delta CoVaR_{p|s}$ and $\Delta CoES_{p|s}$), and expected loss on the investment. This reflects that investment in both safe haven assets would cushion the spillover in stock price. In adittion, a comparison between two- and three-asset portfolios show that not only the ES is reduced when the exchange rate is seen as another investment asset (as seen in Table 1), but also the conditional tail risk. In particular, the CoES drops between 225 and 278 bps in non-US portfolios, and between 196 and 355 bps on average in US portfolios.

Table 2: Risk of non-US portfolios in a bearish scenario for stock.

	Panel A: Europe				**Panel B:** Japan		
	Portfolio 1	**Portfolio 2**	**Portfolio 3**		**Portfolio 1**	**Portfolio 2**	**Portfolio 3**
$CoES_p(0.1)$	-0.0912	-0.0505	-0.0263	$CoES_p(0.1)$	-0.0928	-0.0490	-0.0212
$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0269	-0.0234	-0.0144	$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0298	-0.0203	-0.0118
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0468	-0.0336	-0.0197	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0495	-0.0309	-0.0163
$E(r_p\|r_s < VaR_s(0.1))$	-0.0541	-0.0177	-0.0071	$E(r_p\|r_s < VaR_s(0.1))$	-0.0556	-0.0203	-0.0059
	Panel C: Brazil				**Panel D:** UK		
	Portfolio 1	**Portfolio 2**	**Portfolio 3**		**Portfolio 1**	**Portfolio 2**	**Portfolio 3**
$CoES_p(0.1)$	-0.1173	-0.0506	-0.0260	$CoES_p(0.1)$	-0.0758	-0.0529	-0.0303
$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0361	-0.0259	-0.0161	$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0227	-0.0201	-0.0155
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0598	-0.0362	-0.0214	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0391	-0.0309	-0.0215
$E(r_p\|r_s < VaR_s(0.1))$	-0.0668	-0.0157	-0.0049	$E(r_p\|r_s < VaR_s(0.1))$	-0.0455	-0.0236	-0.0105

NOTE: This table contains the average values of CoES, ΔCoVaR, ΔCoES and the expected return of the three non-US portfolios over the sample period. *Portfolio 1* refers to the stock portfolio, *Portfolio 2* to the two-asset portfolio and *Portfolio 3* to the three-asset portfolio.

Table 3: Risk of US portfolios in a bearish scenario for stock.

	Panel A: Europe				**Panel B:** Japan		
	Portfolio 1	**Portfolio 2**	**Portfolio 3**		**Portfolio 1**	**Portfolio 2**	**Portfolio 3**
$CoES_p(0.1)$	-0.0588	-0.0605	-0.0287	$CoES_p(0.1)$	-0.1101	-0.0464	-0.0218
$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0460	-0.0257	-0.0161	$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0343	-0.0208	-0.0124
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0641	-0.0381	-0.0219	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0575	-0.0312	-0.0178
$E(r_p\|r_s < VaR_s(0.1))$	0.0046	-0.0241	-0.0083	$E(r_p\|r_s < VaR_s(0.1))$	-0.0640	-0.0175	-0.0048
	Panel C: Brazil				**Panel D:** UK		
	Portfolio 1	**Portfolio 2**	**Portfolio 3**		**Portfolio 1**	**Portfolio 2**	**Portfolio 3**
$CoES_p(0.1)$	-0.0920	-0.0899	-0.0544	$CoES_p(0.1)$	-0.0401	-0.0439	-0.0243
$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0489	-0.0335	-0.0284	$\Delta CoVaR_{p\|s}(0.1,\ 0.1)$	-0.0299	-0.0182	-0.0132
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0692	-0.0517	-0.0396	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0441	-0.0274	-0.0179
$E(r_p\|r_s < VaR_s(0.1))$	-0.0266	-0.0433	-0.0236	$E(r_p\|r_s < VaR_s(0.1))$	0.0041	-0.0182	-0.0065

NOTE: This table contains the average values of CoES, ΔCoVaR, ΔCoES and the expected return of the three US portfolios over the sample period. *Portfolio 1* refers to the stock portfolio, *Portfolio 2* to the two-asset portfolio and *Portfolio 3* to the three-asset portfolio.

4.3 Model risk

One of the questions we posed at the beginning of the study is how tail dependence affects this analysis. To answer this question, we propose asumming no tail dependence between the assets, i.e. selecting a time-varying Gaussian copula to model the dependence structure, using the DCC methodology proposed by Engle (2002)[7]. The extent to which fitting using a vine copula methodology is better that a DCC approach can be quantified in terms of the estimation of the tail risk of the portfolios depending on the scenario for the stock component.

Figure 5 shows the difference between the estimation of the tail risk of portfolios using the vine copula (minuend) and the DCC model (subtrahend). On the one hand, a greater overestimation or underestimation of the risk can be seen when the exchange rate is not considered as another potential asset in the portfolio construction. On the other hand, the model risk of assuming an elliptical copula leads to a underestimation of the expected losses, especially in Europe, where the underestimation is 100 bps for the ES and 250 bps for the CoES.

[7]Other authors have used this methodology, like Huang et al. (2009), Joy (2011), Ciner et al. (2013), Baruník et al. (2016) and Ascorbebeitia et al. (2021)

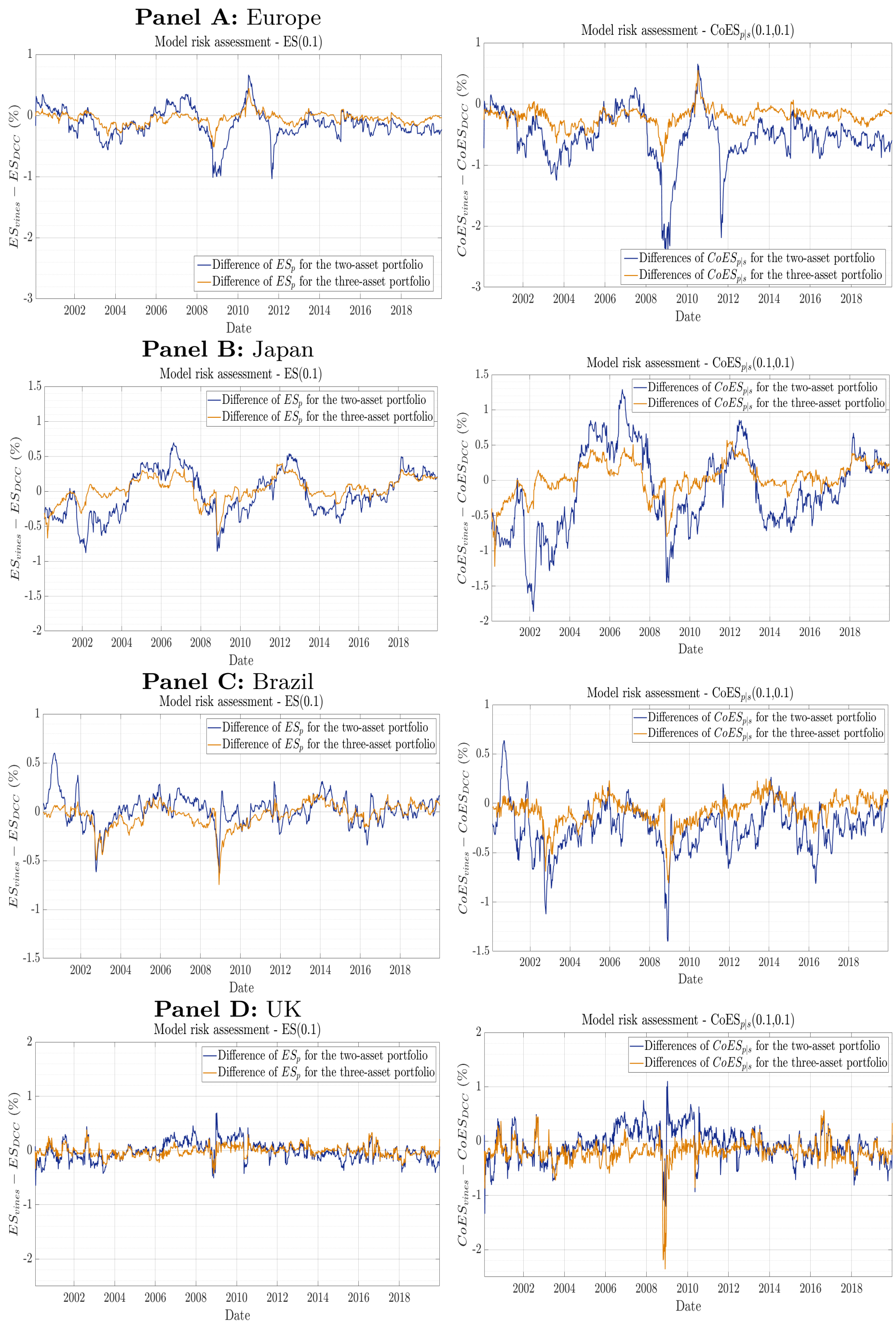


Figure 5: Model risk assessment. The left (right) figure in each panel shows the model risk computed from the difference between the ES(0.1) (CoES(0.1,0.1)) of two- and three-asset portfolios built under a vine copula approach and assuming no-tail dependence across the assets.

4.4 Out-of-sample exercise

Previous results suggest that the exchange rate may be useful in reducing the tail risk of portfolios when there is a spillover in the stock market. We now study the out-of-sample performance of portfolios whithin a sample from 1 January 2020 to 1 June 2021.

We use Monte Carlo simulations to obtain the weekly logarithmic returns for each asset following the optimal marginal models and adjusting their dependence structure by means of the best vine copula models. Once the weekly logarithmic returns are converted to arithmetic, we build two minimum-ES portfolios denominated in a currency other than US dollars. The first is built with gold denominated in the investor's domestic currency and the local stock index, while the second consists of long positions in the index, gold and the exchange rate of local currency against USD (for further information see Annex G).

In line with the in-sample findings, Table 4 provides evidence of the lower unconditional (measured by the ES), conditional (measured by the CoES) or systemic (measured by the ΔCoES) risk of the three-asset portfolio in all regions.

Table 4: Conditional risk measures of portfolios in the out-of-sample period.

Panel A: Europe			**Panel B:** Japan		
	Portfolio 1	**Portfolio 2**		**Portfolio 1**	**Portfolio 2**
$ES_p(0.1)$	-0.0204	-0.0169	$ES_p(0.1)$	-0.0143	-0.0043
$CoES_{p\|s}(0.1,\ 0.1)$	-0.0471	-0.0259	$CoES_{p\|s}(0.1,\ 0.1)$	-0.0299	-0.0077
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0236	-0.0124	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0151	-0.0039
Panel C: Brazil			**Panel D:** UK		
	Portfolio 1	**Portfolio 2**		**Portfolio 1**	**Portfolio 2**
$ES_p(0.1)$	-0.0194	-0.0126	$ES_p(0.1)$	-0.0201	-0.0125
$CoES_{p\|s}(0.1,\ 0.1)$	-0.0404	-0.0241	$CoES_{p\|s}(0.1,\ 0.1)$	-0.0461	-0.0258
$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0197	-0.0115	$\Delta CoES_{p\|s}(0.1,\ 0.1)$	-0.0225	-0.0126

NOTE: This table contains the average value of ES, CoES and ΔCoES of two portfolios over the out-of-sample period in all the regions studied. *Portfolio 1* refers to the two-asset portfolio, while *Portfolio 2* refers to the three-asset portfolio.

5 Conclusions

Gold positions in equity portfolios constitute an effective hedging strategy against downward movements in stock and currency markets. The behaviour of gold in stock portfolios has been widely studied in the literature, but little attention has been paid to how exchange rate movements could affect the hedging strategy of international investors and, therefore, the performance of a portfolio considering the exchange rate as a third asset.

Here we study the role of the exchange rate in the construction of stock portfolios in Europe, Japan, Brazil and the UK. To that end we use a twenty-year sample of weekly data of float exchange rates, stock and gold returns. To estimate the joint distribution, we capture the marginal features of the series using an ARMA-APARCH model, assuming Skewed Student t innovations. For tail dependence vine copula models are employed, using Patton's dynamics to capture the time-varying behaviour of the dependence structure.

We assess the impact of the exchange rate on the performance of portfolios built according to the minimum-ES criterion, comparing the ES with and without including an extra weight in the exchange rate in them. In non-US portfolios the exchange rate acts as a better hedge asset than gold, reducing this tail risk. We find that the exchange rate can also reduce the tail risk of international stock portfolios of US investors. Indeed, introducing the exchange rate into the investment strategy reduces the ES of non-US portfolios, on average, by between 113 bps and 158 bps and the ES of US investor's portfolios by a quantity that ranges from 107 bps to 162 bps. We also compute the cost of not taking into account the exchange rate using conditional risk measures as CoVaR and CoES. Our findings evidence that including the exchange rate in the hedging strategy actually reduces conditional tail risk, systemic risk and the expected loss on the investment when there is a spillover in the stock market.

Summarizing, using the exchange rate as an additional asset in international stock portfolios enhances hedging strategies in terms of tail losses. The exchange rate plays a leading role in all non-US (US) portfolios for hedging stock losses and even outstrips the position of gold (stock).

Finally, we also compute the model risk, measured by the cost of assuming an elliptical copula. In both minimum-ES portfolios this assumption implies a substantial underestimation of the unconditional and conditional expected loss, especially in Europe, where the underestimation is 100 bps for the ES and 250 bps for the CoES. An out-of-sample exercise corroborates that the results are robust to the sample and that including the exchange rate in a risk-averse investor's portfolio reduces potential losses in periods of turmoil.

This study has direct implications for international investors in terms of portfolio optimisation and risk management decisions, although the relevance of our findings goes beyond investment strategies. They also provide important information for policy makers to understand the interdependence between economic and financial variables. Likewise, regulatory and supervisory authorities might find here a powerful stress test approach for assessing the performance of domestic stock markets under distress scenarios for the exchange rate.

Acknowledgments

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References

[1] Aas, K. & Berg, D. (2011). Modeling dependence between financial returns using pair-copula constructions. Joe, H., & Kurowicka, D. (Eds.). Dependence modeling: vine copula handbook (305-328). World Scientific.

[2] Aas, K., Czado, C., Frigessi, A., & Bakken, H. (2009). Pair-copula constructions of multiple dependence. Insurance: Mathematics and economics, 44(2), 182-198.

[3] Acharya, V. V., Pedersen, L. H., Philippon, T., & Richardson, M. (2017). Measuring systemic risk. The review of financial studies, 30(1), 2-47.

[4] Adrian, T., & Brunnermeier, M. K. (2011). CoVaR (No. w17454). National Bureau of Economic Research.

[5] Ang, A., & Chen, J. (2002). Asymmetric correlations of equity portfolios. Journal of financial Economics, 63(3), 443-494.

[6] Artzner, P., Delbaen F., Eber, J.M., & Heath, D. (1997). Thinking Coherently. Risk, 10, November, 68-71.

[7] Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. Mathematical finance, 9(3), 203-228.

[8] Ascorbebeitia Bilbatua, J., Ferreira, E., & Orbe Mandaluniz, S. (2021). The Effect of Dependence on European Market Risk. A Nonparametric Time Varying Approach. Journal of Business & Economic Statistics, 1-11.

[9] Baruník, J., Kočenda, E., & Vácha, L. (2016). Gold, oil, and stocks: Dynamic correlations. International Review of Economics & Finance, 42, 186-201.

[10] Baur, D. G., & McDermott, T. K. (2010). Is gold a safe haven? International evidence. Journal of Banking & Finance, 34(8), 1886-1898.

[11] Berger, T. (2016). On the isolated impact of copulas on risk measurement: Asimulation study. Economic Modelling, 58, 475-481.

[12] Cai, J. J., Einmahl, J. H., de Haan, L., & Zhou, C. (2015). Estimation of the marginal expected shortfall: the mean when a related variable is extreme. Journal of the Royal Statistical Society: Series B: Statistical Methodology, 417-442.

[13] Cherubini, U., Gobbi, F., & Mulinacci, S. (2016). Convolution copula econometrics. Springer.

[14] Cherubini, U., & Luciano, E. (2001). Value-at-risk Trade-off and Capital Allocation with Copulas. Economic notes, 30(2), 235-256.

[15] Christoffersen, P. F. (1998). Evaluating interval forecasts. International economic review, 841-862.

[16] Christoffersen, P., Errunza, V., Jacobs, K., & Langlois, H. (2012). Is the potential for international diversification disappearing? A dynamic copula approach. The Review of Financial Studies, 25(12), 3711-3751.

[17] Ciner, C., Gurdgiev, C., & Lucey, B. M. (2013). Hedges and safe havens: An examination of stocks, bonds, gold, oil and exchange rates. International Review of Financial Analysis, 29, 202-211.

[18] Cooke, R.M., Joe, H. & Aas, K. (2011). Vines arise. Joe, H., & Kurowicka, D. (Eds.). Dependence modeling: vine copula handbook (37-72). World Scientific.

[19] Ding, P. (2016). On the conditional distribution of the multivariate t distribution. The American Statistician, 70(3), 293-295.

[20] Embrechts, P., Frey, R., & McNeil, A. (2011). Quantitative Risk Management.

[21] Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. Journal of Business & Economic Statistics, 20(3), 339-350.

[22] Girardi, G., & Ergün, A. T. (2013). Systemic risk measurement: Multivariate GARCH estimation of CoVaR. Journal of Banking & Finance, 37(8), 3169-3180.

[23] Glosten, L., Jagannathan, R., & Runkle, D., (1993). On the relation between the expected value and the volatility on the nominal excess returns on stocks. Journal of Finance 48(5), 1779-1801.

[24] Gokmenoglu, K. K., & Fazlollahi, N. (2015). The interactions among gold, oil, and stock market: Evidence from S&P500. Procedia Economics and Finance, 25, 478-488.

[25] Gürgün, G., & Ünalmış, İ. (2014). Is gold a safe haven against equity market investment in emerging and developing countries?. Finance Research Letters, 11(4), 341-348.

[26] Hamilton, J. D. (1994). Time series analysis. Princeton university press.

[27] Hansen, B. E. (1994). Autoregressive conditional density estimation. International Economic Review, 705-730.

[28] Hotta, L. K., Lucas, E. C., & Palaro, H. P. (2008). Estimation of VaR using copula and extreme value theory. Multinational Finance Journal, 12(3/4), 205-218.

[29] Huang, J. J., Lee, K. J., Liang, H., & Lin, W. F. (2009). Estimating value at risk of portfolio by conditional copula-GARCH method. Insurance: Mathematics and economics, 45(3), 315-324.

[30] Iqbal, J. (2017). Does gold hedge stock market, inflation and exchange rate risks? An econometric investigation. International Review of Economics & Finance, 48, 1-17.

[31] Joe, H. (2014). Dependence modeling with copulas. CRC press.

[32] Joy, M. (2011). Gold and the US dollar: Hedge or haven?. Finance Research Letters, 8(3), 120-131.

[33] Junttila, J., Pesonen, J., & Raatikainen, J. (2018). Commodity market based hedging against stock market risk in times of financial crisis: The case of crude oil and gold. Journal of International Financial Markets, Institutions and Money, 56, 255-280.

[34] Karimalis, E. N., & Nomikos, N. K. (2018). Measuring systemic risk in the European banking sector: a copula CoVaR approach. The European Journal of Finance, 24(11), 944-975.

[35] Kristoufek, L., & Vosvrda, M. (2016). Gold, currencies and market efficiency. Physica A: Statistical Mechanics and its Applications, 449, 27-34.

[36] Kupiec, P. (1995). Techniques for verifying the accuracy of risk measurement models. The J. of Derivatives, 3(2).

[37] Lin, F. (2011). Tail Dependence between Stock Index Returns and Foreign Exchange Rate Returns- a Copula Approach. Available at SSRN 1931726.

[38] Nguyen, Q. N., Bedoui, R., Majdoub, N., Guesmi, K., & Chevallier, J. (2020). Hedging and safe-haven characteristics of Gold against currencies: An investigation based on multivariate dynamic copula theory. Resources Policy, 68, 101766.

[39] Ojea-Ferreiro, J. (2019). Structural change in the link between oil and the European stock market: implications for risk management. Dependence Modeling, 7(1), 53-125.

[40] Ojea-Ferreiro, J. (2020). Disentangling the role of the exchange rate in oil-related scenarios for the European stock market. Energy Economics, 89, 104776.

[41] Ojea-Ferreiro, J., & Reboredo J.C. (2021). Exchange rates and the global transmission of equity market shocks. JRC Technical Report.

[42] Patton, A. J. (2001). Modelling time-varying exchange rate dependence using the conditional copula. University of California. San Diego, Discussion Paper 01-09.

[43] Patton, A. J. (2006). Modelling asymmetric exchange rate dependence. International economic review, 47(2), 527-556.

[44] Qureshi, S., Rehman, I. U., & Qureshi, F. (2018). Does gold act as a safe haven against exchange rate fluctuations? The case of Pakistan rupee. Journal of Policy Modeling, 40(4), 685-708.

[45] Reboredo, J. C. (2011). How do crude oil prices co-move?: A copula approach. Energy Economics, 33(5), 948-955.

[46] Reboredo, J. C. (2013). Is gold a safe haven or a hedge for the US dollar? Implications for risk management. Journal of Banking & Finance, 37(8), 2665-2676.

[47] Reboredo, J. C., & Rivera-Castro, M. A. (2014a). Can gold hedge and preserve value when the US dollar depreciates?. Economic Modelling, 39, 168-173.

[48] Reboredo, J. C., & Rivera-Castro, M. A. (2014b). Gold and exchange rates: Downside risk and hedging at different investment horizons. International Review of Economics & Finance, 34, 267-279.

[49] Reboredo, J. C., & Ugolini, A. (2015a). A vine-copula conditional value-at-risk approach to systemic sovereign debt risk for the financial sector. The North American Journal of Economics and Finance, 32, 98-123.

[50] Reboredo, J. C., & Ugolini, A. (2015b). Downside/upside price spillovers between precious metals: A vine copula approach. The North American Journal of Economics and Finance, 34, 84-102.

[51] Reboredo, J. C., & Ugolini, A. (2016). Quantile dependence of oil price movements and stock returns. Energy economics, 54, 33-49.

[52] Reboredo, J. C., Rivera-Castro, M. A., & Ugolini, A. (2016). Downside and upside risk spillovers between exchange rates and stock prices. Journal of Banking & Finance, 62, 76-96.

[53] Sjaastad, L. A. (2008). The price of gold and the exchange rates: Once again. Resources Policy, 33(2), 118-124.

[54] Sun, W., Rachev, S., Stoyanov, S. V., & Fabozzi, F. J. (2008). Multivariate skewed Student's t copula in the analysis of nonlinear and asymmetric dependence in the German equity market. Studies in Nonlinear Dynamics & Econometrics, 12(2).

[55] Theodossiou, P. (1998). Financial data and the skewed generalized t distribution. Management Science, 44(12-part-1), 1650-1661.

[56] Wang, K. M. (2013). Can gold effectively hedge risks of exchange rate?. Journal of Business Economics and Management, 14(5), 833-851.

[57] Wang, Y. C., Wu, J. L., & Lai, Y. H. (2013). A revisit to the dependence structure between the stock and foreign exchange markets: A dependence-switching copula approach. Journal of Banking & Finance, 37(5), 1706-1719.

[58] Yew-Low, R. K., Alcock, J., Faff, R., & Brailsford, T. (2013). Canonical vine copulas in the context of modern portfolio management: Are they worth it?. Journal of Banking & Finance, 37(8), 3085-3099.

A Set of copula and conditional copula models

This Appendix presents the principal features of the copula models used in this study. The expression of their distribution and density functions, their tail dependence coefficients and the relation between their dependence parameter and the rank correlation coefficient, i.e. Spearman's rho and Kendall's tau are summarised in Table 2[8].

Gaussian copula

The distribution function of the Gaussian copula is:

$$C(u_1, u_2; \rho) = \Phi_2\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right) = \Phi_2\left(x_1, x_2\right)$$

As elliptical copulas do not have a closed form, they can be expressed as an integral:

$$C(u_1, u_2; \rho) = \int_0^{\Phi^{-1}(u_1)=z_1} \int_0^{\Phi^{-1}(u_2)=z_2} \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2\cdot(1-\rho^2)}\right) dx_1\ dx_2 \tag{A.0.1}$$

where Φ_2 is the cumulative distribution function (cdf) of a bivariate Gaussian distribution, and Φ^{-1} is the inverse cdf of a (univarite) Gaussian distribution.

Deriving a copula distribution function with respect to all marginal distributions gives the copula density:

$$c(u_1, u_2) = \frac{\partial^2 F(x_1, x_2)}{\partial X_1 \partial X_2} = \frac{\partial^2 C\left(F_1(x_1), F_2(x_2)\right)}{\partial F_1(x_1)\partial F_2(x_2)} \cdot \frac{\partial F_1(x_1)}{\partial X_1} \cdot \frac{\partial F_2(x_2)}{\partial X_2} \tag{A.0.2}$$

The density of the Gaussian copula is defined as:

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left\{-\frac{\rho^2\Phi^{-1}(u_1)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2) + \rho^2\Phi^{-1}(u_2)^2}{2(1-\rho^2)}\right\} \tag{A.0.3}$$

This copula model does not allow tail dependence.

[8]From here on the equations shown are for a bivariate copula model, but they can be easily extrapolated to the multivariate case.

Student t copula

The Student t copula has the following distribution and density functions:

$$C(u_1, u_2; \rho, \nu) = T_\nu \left(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2)\right) = T_{2,\nu}\left(x_1, x_2\right) =$$
$$= \int_0^{T_\nu^{-1}(u_1)=z_1} \int_0^{T_\nu^{-1}(u_2)=z_2} \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu \cdot (1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dx_1\ dx_2 \tag{A.0.4}$$

$$c(u_1, u_2; \rho, \nu) = K \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \left[1 + \frac{T_\nu^{-1}(u_1)^2 - 2\rho T_\nu^{-1}(u_1) T_\nu^{-1}(u_2) + T_\nu^{-1}(u_2)^2}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}}$$
$$\left[\left(1 + \frac{T_\nu^{-1}(u_1)^2}{\nu}\right)\left(1 + \frac{T_\nu^{-1}(u_2)^2}{\nu}\right)\right]^{\frac{\nu+1}{2}} \tag{A.0.5}$$

where $K = \Gamma\left(\frac{\nu}{2}\right) \cdot \Gamma\left(\frac{\nu+1}{2}\right)^{-2} \cdot \Gamma\left(\frac{\nu+2}{2}\right)$, $T_{2,\nu}$ is the cumulative distribution function (cdf) of a bivariate Student t distribution, and T_ν^{-1} is the inverse cdf of a (univarite) Student t distribution.

The Student t copula could exhibit symmetric tail dependence, which could be positive or negative, for a small number of degrees of freedom and non-zero correlation. Gaussian and Student t are also known as implicit copulas since they have no explicit closed form.

Clayton copula

The Clayton copula allows positive and asymmetric lower tail dependence. The distribution function of this copula is:

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} + 1\right)^{\frac{-1}{\theta}} \tag{A.0.6}$$

And its density function is:

$$c(u_1, u_2; \theta) = (\theta + 1) \cdot (u_1^{-\theta} + u_2^{-\theta} - 1)^{-2-\frac{1}{\theta}} \cdot u_1^{-\theta-1} \cdot u_2^{-\theta-1} \tag{A.0.7}$$

The dependence parameter is $\theta \in (0, \infty)$. $\theta \to 0$ means independence and $\theta \to \infty$ means perfect dependence.

Gumbel copula

The Gumbel copula only assumes positive dependence in the upper tail of the distribution. Its distribution function is:

$$C(u_1, u_2; \theta) = exp\left\{-\left[(-ln\ u_1)^\theta + (-ln\ u_2)^\theta\right]^{\frac{1}{\theta}}\right\} \tag{A.0.8}$$

And its density function is:

$$c(u_1, u_2; \theta) = (A+\theta-1) \cdot A^{1-2\theta} \cdot e^{-A} \cdot u_1^{-1} \cdot u_2^{-1} \cdot (-ln\ u_1)^{\theta-1} \cdot (-ln\ u_2)^{\theta-1} \tag{A.0.9}$$

where $A = \left[(-ln\ u_1)^{\theta} + (-ln\ u_2)^{\theta}\right]^{\frac{1}{\theta}}$

The dependence parameter is $\theta \in [1, \infty)$. $\theta \to 1$ means independence and $\theta \to \infty$ means perfect dependence.

BB1 copula

The BB1 copula allows asymmetric tail dependence. It is also known as the Clayton-Gumbel copula and has two dependence parameters, $\delta \in (0, \infty)$ (like the Clayton copula) and $\theta \in [1, \infty)$ (for the Gumbel behaviour). The distribution function of this copula is:

$$C(u_1, u_2; \theta, \delta) = \left\{1 + \left[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta}\right]^{\frac{1}{\delta}}\right\}^{\frac{-1}{\theta}} \tag{A.0.10}$$

And its density function is:

$$\begin{aligned} c(u_1, u_2; \theta, \delta) = &\left[1 + (x+y)^{\frac{1}{\delta}}\right]^{\frac{-1}{\theta-2}} (x+y)^{\frac{1}{\delta-2}} \\ &\left[\theta(\delta-1) + (\theta\delta+1)(x+y)^{\frac{1}{\delta}}\right](xy)^{1-\frac{1}{\delta}}(uv)^{-\theta-1} \end{aligned} \tag{A.0.11}$$

with $x = (u_1^{-\theta} - 1)^{\delta}$ and $y = (u_2^{-\theta} - 1)^{\delta}$.

When $\delta = 0$ and $\theta > 1$ there is only upper tail dependence, so in this particular case BB1 copula is the Clayton copula. When $\delta > 0$ and $\theta = 1$ the result is the Gumbel copula, assuming only lower tail dependence. Lastly, if $\delta = 0$ and $\theta = 1$ upper and lower tail independence are obtained.

Archimedean copulas are derived from a generating function. In particular, the generating functions of the three non-elliptical copulas commented above appear in Table 1:

Table 1: Generating functions of the Archimedean copulas employed.

	Generating function
Clayton	$\psi(u;\theta) = \theta^{-1} \cdot (u^{-\theta} - 1)$
	$\psi^{-1}(x;\theta) = (\theta x + 1)^{\frac{-1}{\theta}}$
Gumbel	$\psi(u;\theta) = (-ln\ u)^{\theta};\ \ \theta \geq 1$
	$\psi^{-1}(x;\theta) = exp\left\{-x^{\frac{1}{\theta}}\right\}$
BB1	$\psi(u;\theta,\delta) = (1 + u^{\frac{1}{\delta}})^{\frac{-1}{\theta}}$
	$\psi^{-1}(x;\theta,\delta) = (x^{-\theta} - 1)^{\delta}$

Conditional copulas

In probability theory and statistics, given two jointly distributed random variables X and Y, the conditional probability distribution of Y given X is the probability distribution of Y when X is known to have a particular value, x.

This concept can be applied to copula theory since any d-variate copula can be obtained from the joint distribution of the d variables and their marginal distributions (inverse Sklar's Theorem).

By Sklar's theorem for continuous conditional distributions, there exists a unique conditional copula such as

$$H(x, y|w) = C(F(x|w), G(y|w)|w) \tag{A.0.12}$$

where $x = F^{-1}(u_1)$, $y = F^{-1}(u_2)$, and F and G are the conditional distribution of $X|W$ and $Y|W$, respectively (with W scalar or vector). Note that the conditioning variable(s), W, must be the same for both marginal distributions and the copula (Patton, 2006).

The conditional copula comes from the derivation of the copula function from one input variable, e.g.

$$C_{2|1}(u_2|u_1) = \frac{\partial C(u_1, u_2)}{\partial u_1} \tag{A.0.13}$$

where $C_{2|1}(u_2|u_1)$ indicates the distribution of u_2 given the realisation of u_1.

Rotated copulas

A multivariate copula $C(u_1, ..., u_d)$ represents the joint probability of X_1 being below its u_1-quantile, of X_2 being below its u_2-quantile... That is, copulas express dependence on a quantile scale, which is useful for describing the dependence of extreme outcomes, i.e. the dependence in the tails of the distributions (Embrechts et al., 2011). However, it also could be useful for modelling the dependence between variables when one or more of them are not below the u_i-quantile but above the $1 - u_i$-quantile, i.e. in the upper tail. To that end rotated copulas can be used. This transformation enables other dependences to be captured: positive rather than negative dependence, or lower (upper) tail dependence instead of higher (lower) tail dependence. For instance, if the Gumbel copula is turned 180°, namely turned into the Survival Gumbel copula, the (positive) lower tail dependence of the data can be captured. Rotating the copula 90° or 270° enables negative upper or lower tail dependence, respectively, to be modelled.

Here, we focus on 90° rotated copulas. Given a bivariate copula, $C(u_1, u_2)$, rotation by 90 degrees show the probability of U_1 being in its upper tail while U_2 is in its lower u_2-quantile:

$$C^{90R}(u_1, u_2) = P(U_1 > 1 - u_1, U_2 < u_2) = P(U_2 < u_2) - P(U_1 < 1 - u_1, U_2 < u_2)$$

This rotated copula is defined as:

$$C^{90R}(u_1, u_2) = u_2 - C(1 - u_1, u_2)$$

where $u_2 = P(U_2 < u_2)$ and $C(1 - u_1, u_2) = P(U_1 < 1 - u_1, U_2 < u_2)$

Table 2: Principal features of the main elliptical and Archimedean copulas used in this study.

	Distribution function	Density function
Gaussian	$C(u_1, u_2; \rho) = \Phi_2\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$	$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left\{-\frac{\rho^2\Phi^{-1}(u_1)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2) + \rho^2\Phi^{-1}(u_2)^2}{2(1-\rho^2)}\right\}$
Student t	$C(u_1, u_2; \rho, \nu) = T_\nu\left(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2)\right)$	$c(u_1, u_2; \rho, \nu) = K \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \left[1 + \frac{T_\nu^{-1}(u_1)^2 - 2\rho T_\nu^{-1}(u_1)T_\nu^{-1}(u_2) + T_\nu^{-1}(u_2)^2}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}} \cdot$ $\cdot \left[\left(1 + \frac{T_\nu^{-1}(u_1)^2}{\nu}\right)\left(1 + \frac{T_\nu^{-1}(u_2)^2}{\nu}\right)\right]^{\frac{\nu+1}{2}}$ with $K = \Gamma\left(\frac{\nu}{2}\right) \cdot \Gamma\left(\frac{\nu+1}{2}\right)^{-2} \cdot \Gamma\left(\frac{\nu+2}{2}\right)$
Clayton	$C(u_1, u_2; \delta) = \left(u_1^{-\delta} + u_2^{-\delta} + 1\right)^{\frac{-1}{\delta}}$	$c(u_1, u_2; \delta) = (\delta+1) \cdot (u_1^{-\delta} + u_2^{-\delta} - 1)^{-2-\frac{1}{\delta}} \cdot u_1^{-\delta-1} \cdot u_2^{-\delta-1}$
Gumbel	$C(u_1, u_2; \theta) = exp\left\{-\left(-ln\ u_1^\theta - ln\ u_2^\theta\right)^{\frac{1}{\theta}}\right\}$	$c(u_1, u_2; \theta) = (A + \theta - 1) \cdot A^{1-2\theta} \cdot e^{-A} \cdot u_1^{-1} \cdot u_2^{-1} \cdot$ $\cdot(-log\ u_1)^{\theta-1} \cdot (-log\ u_2)^{\theta-1}$ with $A = \left[(-log\ u_1)^\theta + (-log\ u_2)^\theta\right]^{\frac{1}{\theta}}$
BB1	$C(u_1, u_2; \theta, \delta) = \left\{1 + \left[(u_1^{-\theta} - 1)^\delta + (u_2^{-\theta} - 1)^\delta\right]^{\frac{1}{\delta}}\right\}^{\frac{-1}{\theta}}$	$c(u_1, u_2; \theta, \delta) = \left\{1 + (x+y)^{\frac{1}{\delta}}\right\}^{\frac{-1}{\theta-2}} (x+y)^{\frac{1}{\delta-2}} \left[\theta(\delta-1) + (\theta\delta+1)(x+y)^{\frac{1}{\delta}}\right]$ $\cdot(xy)^{1-\frac{1}{\delta}}(uv)^{-\theta-1}$ with $x = (u_1^{-\theta} - 1)^\delta$ and $y = (u_2^{-\theta} - 1)^\delta$

Table 2 (Cont.): Principal features of the main elliptical and Archimedean copulas used in this study.

	Conditional bivariate copula	Tail dependence	Rank correlation
Gaussian	$C_{2\|1}(u_2\|u_1;\rho) = \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \cdot \Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$	$\lambda_U = \lambda_L = 0$	$\rho = sin\left(\frac{\pi}{2} \cdot \tau\right)$ $\rho = 2 \cdot sin\left(\frac{\pi}{6} \cdot \rho_S\right)$
Student t	$C_{2\|1}(u_2\|u_1;\rho,\nu) = T_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu+\left[T_\nu^{-1}(u_1)\right]^2}} \cdot \frac{T_\nu^{-1}(u_2) - \rho \cdot T_\nu^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$	$\lambda_U = 2 \cdot t_{\nu+1}\left(-\frac{\sqrt{\nu+1} \cdot \sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$ $\lambda_L = \lambda_U$	$\rho = sin\left(\frac{\pi}{2} \cdot \tau\right)$ $\rho = 2 \cdot sin\left(\frac{\pi}{6} \cdot \rho_S\right)$
Clayton	$C_{2\|1}(u_2\|u_1;\delta) = \left(u_1^{-\delta} + u_2^{-\delta} - 1\right)^{-1-\frac{1}{\delta}} \cdot u_1^{-\delta-1}$	$\lambda_U = 0$ $\lambda_L = 2^{\frac{-1}{\delta}}$	$\tau = \frac{\delta}{\delta+2} \Rightarrow \delta = 2 \cdot \frac{\tau}{1-\tau}$
Gumbel	$C_{2\|1}(u_2\|u_1;\theta) = exp\left\{-\left[(-log\ u_1)^\theta + (-log\ u_2)^\theta\right]^{\frac{1}{\theta}}\right\} \cdot \left[(-log\ u_1)^\theta + (-log\ u_2)^\theta\right]^{\frac{1}{\theta}-1} \cdot (-log\ u_1)^{\theta-1} \cdot \frac{1}{u_1}$	$\lambda_U = 2 - 2^{\frac{1}{\theta}}$ $\lambda_L = 0$	$\tau = 1 - \theta^{-1} \Rightarrow \theta = \frac{1}{1-\tau}$
BB1	$C_{2\|1}(u_2\|u_1;\theta,\delta) = \left\{1 + (x+y)^{\frac{1}{\delta}}\right\}^{\frac{-1}{\theta-1}} (x+y)^{\frac{1}{\delta-1}} x^{1-\frac{1}{\delta}} u_1^{-\theta-1}$	$\lambda_L = 2^{\frac{-1}{\delta\theta}}$ $\lambda_U = 2 - 2^{\frac{1}{\delta}}$	$\tau = 1 - \frac{2}{\delta(\theta+2)}$

NOTE: All distribution, density and conditional copula functions are expressed for the particular case of bivariate copula models, but they can be easily extrapolated to the multivariate case. *Tail dependence* refers to the two indicators of the tail dependence of the copulas, namely upper-tail (λ_U) and lower-tail (λ_L) dependence indicators. Both are defined as the value of the copula at the limit of the value of the marginals, i.e., the limit when the uniform variables tend to 1 or to 0, respectively. *Rank correlation* presents the relationship between the Spearman's (ρ) or Kendall's (τ) correlation coefficient and the dependence parameter of each copula.

B Patton's dynamics

Following Patton (2001, 2006), we assume that the functional form of the copula remains fixed over the sample while the parameters vary according to an ARMA(1,10)-type process. Thus the autorregressive term captures any persistence in the dependence parameter and the mean of the product of the last ten observations of the uniform variables captures any variation in dependence.

For the two elliptical copulas, the evolution equation of the correlation parameter is:

$$\rho_{ij,t} = \Lambda\left(\omega + \alpha\frac{1}{10}\sum_{k=1}^{10} F_i^{-1}(u_{i,t-k})F_j^{-1}(u_{j,t-k}) + \beta\rho_{ij,t-1}\right); \quad i \neq j \quad \text{(B.0.1)}$$

where $F_i^{-1}(u_i)$ refers to the inverse cdf. Thus, $F_i^{-1}(u_i) = \Phi^{-1}(u_i)$ for the Gaussian copula and $F_i^{-1}(u_i) = T_\nu^{-1}(u_i)$ for the Student t case. $\Lambda = \frac{1-e^{-x}}{1+e^{-x}}$ is the modified logistic transformation, designed to keep $\rho_{ij,t}$ within the interval (-1,1) at all times. For the Student t copula we assume that the number of degrees of freedom is constant.

The dependence parameter in both the Clayton and Gumbel copulas has the following evolution:

$$\theta_t = \Lambda\left(\omega + \alpha\frac{1}{10}\sum_{k=1}^{10} |u_{i,t-k} - u_{j,t-k}| + \beta\theta_{t-1}\right); \quad i \neq j \quad \text{(B.0.2)}$$

where $\Lambda(x) = \frac{100}{1+e^{-x}}$ for the Clayton copula and $\Lambda(x) = \frac{99}{1+e^{-x}} + 1$ for the Gumbel are logistic transformations to ensure that the parameter is within its feasible region, i.e., $\theta_t > 0$ and $\theta_t > 1$, respectively, at all times[9].

The evolution equation for the BB1 copula is based on the link between these parameters and the tail dependence parameters, both upper and lower:

$$\lambda_t^k = \Lambda\left(\omega_k + \alpha_k\frac{1}{10}\sum_{i=1}^{10} |u_{i,t-k} - u_{j,t-k}| + \beta_k\lambda_{t-1}^k\right); \quad i \neq j; \; k = U, L \quad \text{(B.0.3)}$$

where $\Lambda(x) = \frac{1}{1+e^{-x}}$ is the logistic transformation to keep the tail dependence coefficient within the range $(0, 1)$.

[9]Restricting the feasible range prevents the value of the parameters from going to infinity, thus, improving and accelerating the estimation process.

C Risk measures for conditional trivariate copulas

In this Appendix we calculate the risk measures of international portfolios described in Eq. (2.3.1) and Eq. (2.3.3) using the expression for their cdf, given by Eq. (2.3.2) or Eq. (2.3.4), respectively. On that basis we present the expressions of several risk measures for the former case, i.e. those portfolios defined by Eq. (2.3.1), but they can easily be extrapolated to the second case, using Eq. (2.3.4) instead of Eq. (2.3.2) in the following formulae.

Value at Risk (VaR) is widely used to quantify the market risk of an asset or a portfolio (Hotta et al., 2008). It measures the maximum loss expected over a certain time frame with a confidence level of $(1-\alpha)\%$. Mathematically, this is equivalent to:

$$P\left(r_p < VaR_p\left(\alpha\right)\right) = \alpha \Leftrightarrow VaR_p(\alpha) = F_p^{-1}(\alpha) \tag{C.0.1}$$

where r_p denotes the return of the portfolio.

Conditional VaR (CoVaR) represents the β-quantile of the returns of a portfolio conditional on the return of its component i being below its VaR, i.e., $r_i \leq VaR_i(\alpha)$ (for $i = e, s, g$):

$$P\left(r_p < CoVaR_p(\beta) | r_i < VaR_i\left(\alpha\right)\right) = \beta \tag{C.0.2}$$

This is exactly the definition of tail dependence given by Aas et al. (2009) and Aas and Berg (2011).

Assume that the exchange rate return (r_e) is in its α-lower quantile. By Bayes Theorem, a conditional probability can be expresed as the ratio of the joint probability of seeing both scenarios to the probability of observing the conditioning scenario:

$$P\left(r_p < CoVaR_p(\beta) | r_e < VaR_e\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right)}{P\left(r_e < VaR_e\left(\alpha\right)\right)} = \beta$$

where $P\left(r_e < VaR_e\left(\alpha\right)\right) = \alpha$ and $P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right)$ can be understood in terms of the distribution function of the portfolio, i.e.

$$P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\ \left(\alpha\right)\right) = F_p(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right))$$

The cdf of a portfolio can also be written in terms of copulas, so

$$P\left(r_p < CoVaR_p(\beta), r_e < VaR_e\left(\alpha\right)\right) = C(F_p(CoVaR_p(\beta)), \alpha) = C(u_p, u_e)$$

Recall the portfolio return given by Eq. (2.3.1). Thus $r_p < CoVaR_p(\beta)$ is equivalent to $r_g < \frac{CoVaR_p(\beta)-(1-\omega)r_s-\omega r_e}{\omega}$ and so $u_g < F_g\left(\frac{CoVaR_p(\beta)-(1-\omega)F_s^{-1}(u_s)-\omega F_e^{-1}(u_e)}{\omega}\right)$

In addition, the conditional copula definition can be used, by which $C(u_p, u_e) = C_{g|e,s}(u_g|u_e, u_s)$. Thus,

$$P\left(r_p < CoVaR_p(\beta)|r_e < VaR_e\left(\alpha\right)\right) = \beta = \frac{1}{\alpha}\cdot$$
$$\int_0^\alpha \int_0^1 C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)\,F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg| u_e, u_s\right) c_{e,s}(u_e, u_s) du_s du_e \tag{C.0.3}$$

Along the same lines, the CoVaR of the portfolio can be computed conditional on the other components of the portfolio, namely stock and gold, being below their corresponding VaR thresholds:

$$P\left(r_p < CoVaR_p(\beta)|r_s < VaR_s\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_s < VaR_s\left(\alpha\right)\right)}{P\left(r_s < VaR_s\left(\alpha\right)\right)} = \beta = \frac{1}{\alpha}\cdot$$
$$\int_0^1 \int_0^\alpha C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)\,F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg| u_e, u_s\right) c_{e,s}(u_e, u_s) du_s du_e \tag{C.0.4}$$

$$P\left(r_p < CoVaR_p(\beta)|r_g < VaR_g\left(\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_g < VaR_g\left(\alpha\right)\right)}{P\left(r_g < VaR_g\left(\alpha\right)\right)} = \beta = \frac{1}{\alpha}\cdot$$
$$\int_0^1 \int_0^\alpha C_{s|e,g}\left(F_s\left(\frac{CoVaR_p(\beta) - \omega\left[F_g^{-1}\left(u_g\right) + F_e^{-1}\left(u_e\right)\right]}{(1-\omega)}\right)\Bigg| u_e, u_g\right) c_{e,g}(u_e, u_g) du_g du_e \tag{C.0.5}$$

It could be useful to consider the probability of portfolio returns being below their CoVaR at β-level while the component i is in its α-upper quantile. This is equivalent to $P\left(r_p < CoVaR_p(\beta)|r_i > VaR_i\left(\alpha\right)\right)$

Assuming that the exchange rate suffers an extreme upward movement:

$$P\left(r_p < CoVaR_p(\beta)|r_e > VaR_e\left(1-\alpha\right)\right) = \frac{P\left(r_p < CoVaR_p(\beta), r_e > VaR_e\left(1-\alpha\right)\right)}{P\left(r_e > VaR_e\left(1-\alpha\right)\right)} = \beta \tag{C.0.6}$$

where $P\left(r_e > VaR_e\left(1-\alpha\right)\right) = \alpha$. Thus, the bullish CoVaR of the portfolio is obtained from:

$$P\left(r_p < CoVaR_p(\beta)|r_e > VaR_e\left(1-\alpha\right)\right) = \beta = \frac{1}{\alpha}\cdot$$
$$\int_0^1 \int_{1-\alpha}^1 C_{g|e,s}\left(F_g\left(\frac{CoVaR_p(\beta) - (1-\omega)\,F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\Bigg| u_e, u_s\right) c_{e,s}(u_e, u_s) du_e du_s \tag{C.0.7}$$

When building a portfolio, portfolio managers must take into account the aggregate behaviour of part or even all of its components. In other words, it is important to consider the probability of the return of the portfolio being below its CoVaR conditional on more than one of its components being below its VaR level. This probability is:

$$P\left(r_p < CoVaR_p(\gamma)|r_i < VaR_i\left(\alpha\right),\ r_j < VaR_j(\beta)\right) = \gamma \tag{C.0.8}$$

Assume stock and gold prices suffer a sharp drop. This measure can be obtained as follows:

$$P\left(r_p < CoVaR_p(\gamma) | r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) =$$

$$= \frac{P\left(r_p < CoVaR_p(\gamma), r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right)}{P\left(r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right)}$$

In this case, the denominator is not a parameter as in previous cases, but a joint probability, which can also be captured by a copula, i.e. $P\left(r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) = C_{g,s}\left(\alpha, \beta\right)$. Thus,

$$\begin{aligned}&P\left(r_p < CoVaR_p(\gamma) | r_g < VaR_g\left(\alpha\right), r_s < VaR_s\left(\beta\right)\right) = \\ &= \frac{1}{C_{g,s}\left(\alpha,\beta\right)} \int_0^{\beta} \int_0^{\alpha} C_{e|s,g}\left(F_e\left(\frac{CoVaR_p(\gamma) - (1-\omega)\, F_s^{-1}\left(u_s\right) - \omega F_g^{-1}\left(u_g\right)}{\omega}\right) \Bigg| u_s, u_g\right) \\ &c_{s,g}(u_s, u_g)\ du_g du_s = \gamma\end{aligned} \tag{C.0.9}$$

As above, the scenario in which the returns of the portfolio components are in their upper tails can be considered. The probability of the portfolio return being below its CoVaR(γ) conditional on those circumstances is:

$$P\left(r_p < CoVaR_p(\gamma) | r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = \gamma \tag{C.0.10}$$

Again using Bayes Theorem, the probability in Eq. (C.0.10) can be rewritten as:

$$\frac{P\left(r_p < CoVaR_p(\gamma), r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right)}{P\left(r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right)}$$

where the denominator can be capture by a rotated copula as follows:

$$P\left(r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = C_{g,s}^{180R}(\alpha,\beta) = \alpha+\beta-1+C_{g,s}(1-\alpha, 1-\beta)$$

So the CoVaR in this case is:

$$\begin{aligned}&P\left(r_p < CoVaR_p(\gamma) | r_g > VaR_g\left(1-\alpha\right),\ r_s > VaR_s(1-\beta)\right) = \\ &= \frac{1}{\alpha+\beta-1+C_{g,s}(1-\alpha,1-\beta)} \\ &\int_{1-\beta}^{1} \int_{1-\alpha}^{1} C_{e|s,g}\left(F_e\left(\frac{CoVaR_p(\gamma) - (1-\omega)\, F_s^{-1}\left(u_s\right) - \omega F_g^{-1}\left(u_g\right)}{\omega}\right) \Bigg| u_s, u_g\right) \\ &c_{s,g}(u_s, u_g)\ du_g du_s = \gamma\end{aligned} \tag{C.0.11}$$

Adrian and Brunnermeier (2011) propose $\Delta CoVaR$ as a systemic risk measure. **ΔCoVaR** quantifies how much the VaR of a portfolio changes when returns of the asset i moves from the median to quantile α. The definition of this systemic risk of the portfolio given by the authors is:

$$\Delta CoVaR_{\beta}^{p|i} = CoVaR_{\beta}^{r_p|r_i=VaR_i(\alpha)} - CoVaR_{\beta}^{r_p|r_i=VaR_i(0.5)}$$

We instead use the alternative definition of ΔCoVaR proposed by Girardi and Ergün (2013):

$$\Delta CoVaR_{\beta}^{p|i} = CoVaR_{\beta}^{r_p|r_i \leq VaR_i(\alpha)} - CoVaR_{\beta}^{r_p|r_i \leq VaR_i(0.5)} \tag{C.0.12}$$

VaR and CoVaR are not sub-additive measures. Rather, the Expected Shortfall (ES) and the Conditional Expected Shortfall (CoES), respectively, overcome this limitation, i.e. they are coherent risk measures which complement the information provided by VaR and CoVaR (Artzner et al., 1997, 1999). The **Expected Shortfall (ES)** at α-level is the expected return on the portfolio in the worst α% of cases, i.e. when the portfolio return is below its $VaR(\alpha)$. The ES can thus be computed as:

$$ES_p(\alpha) = E\left(r_p |\ r_p \leq VaR_p\left(\alpha\right)\right) = \frac{1}{\alpha}\int_0^{\alpha} F_p^{-1}\left(q\right)\ dq = \frac{1}{\alpha}\int_0^{\alpha} VaR_p\left(q\right)\ dq \tag{C.0.13}$$

In the same sense, the **Conditional Expected Shortfall (CoES)** measures the expected loss when a portfolio obtains a return below its $CoVaR(\beta)$:

$$\begin{aligned} CoES_{p|e}(\alpha,\beta) &= E\left(r_p \middle| r_p \leq CoVaR_{p|e}\left(\alpha,\beta\right)\right) = \\ &= \frac{1}{\beta}\int_0^{\beta} F_{p|e}^{-1}\left(q\right) dq = \frac{1}{\beta}\int_0^{\beta} CoVaR_{p|e}\left(\alpha,q\right) dq \end{aligned} \tag{C.0.14}$$

The conditional inverse cumulative distribution function $F_{p|e}^{-1}(q)$ is such that $F_{p|e}(r_p) = q$, where the cdf of the portfolio conditional on the scenario for the exchange rate, i.e. $F_{p|e}(r_p)$, is:

$$F_{p|e}(r_p) = \int_0^1 \int_0^{\alpha} C_{g|e,s}\left(F_g\left(\frac{CoVaR_p\left(\beta\right) - \left(1-\omega\right)F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega}\right)\middle| u_e, u_s\right) \cdot c_{e,s}(u_e,u_s)\ du_e du_s$$

Finally, the **Marginal Expected Shortfall (MES)** represents the contribution of each asset to portfolio risk, i.e. it is defined as the marginal impact of a certain asset on the ES of a portfolio, so it is useful as a measure of systemic risk (Cai et al., 2015; Acharya et al., 2017). Assume that the portfolio return is given by the weighted sum of the returns of its N components, i.e. $r_p = \sum_N \omega_i r_i$. Its ES can then be rewritten as $ES_p\left(\alpha\right) = E\left(\sum_N \omega_i r_i |\ r_p < VaR_p(\alpha)\right)$. Assuming that ω_i is determinist (or at least known at time $t-1$), it follows that $ES_p\left(\alpha\right) = \sum_N \omega_i E\left(r_i |\ r_p < VaR_p(\alpha)\right)$. The MES of asset i is obtained as:

$$MES_i\left(\alpha\right) = E\left(r_i |\ r_p < VaR_p(\alpha)\right) \tag{C.0.15}$$

If the MES of the component i of the portfolio is multiplied by its weight, ω_i, the contribution of the asset to the ES of the portfolio is obtained.

For instance, the MES of the stock component of the portfolio is:

$$MES_s(\alpha) = E\left(\left. r_s \right| r_p < VaR_p(\alpha)\right) = \int_{-\infty}^{\infty} r_s \cdot f\left(\left. r_s \right| r_p < VaR_p(\alpha)\right) dr_s =$$
$$= \int_{-\infty}^{\infty} r_s \cdot \frac{f\left(r_s, r_p < VaR_p\left(\alpha\right)\right)}{P\left(r_p < VaR_p\left(\alpha\right)\right)} dr_s$$

where $P\left(r_p < VaR_p\left(\alpha\right)\right) = \alpha$ and $f\left(r_s, r_p < VaR_p(\alpha)\right) = f\left(r_p < VaR_p(\alpha) | r_s\right) \cdot f\left(r_s\right)$. In consequence,

$$E\left(\left. r_s \right| r_p < VaR_p(\alpha)\right) = \frac{1}{\alpha} \int_{-\infty}^{\infty} r_s \cdot P\left(r_p < VaR_p(\alpha) | r_s\right) \cdot f\left(r_s\right) dr_s \quad \text{(C.0.16)}$$

Also, $f\left(r_s\right) dr_s = dF\left(r_s\right) = du_s;\ r_s = F^{-1}\left(u_s\right);\ F\left(-\infty\right) = 0;\ F\left(\infty\right) = 1$ and $P(r_p < VaR_p(\alpha)) = F_p(VaR_p(\alpha))$, where the formula for F_p is provided in Eq. (2.3.2). Hence, Eq. (C.0.16) can be rewritten as follows:

$$\begin{aligned} & E\left(\left. r_s \right| r_p < VaR_p(\alpha)\right) = \\ = & \frac{1}{\alpha} \int_0^1 \int_0^1 F_s^{-1}\left(u_s\right)\ C_{g|e,s}\left(F_g\left(\frac{VaR_p\left(\alpha\right) - \left(1-\omega\right) F_s^{-1}\left(u_s\right) - \omega F_e^{-1}\left(u_e\right)}{\omega} \right) \middle| u_e, u_s \right) \\ & c_{e,s}(u_e, u_s)\ du_e du_s \end{aligned} \quad \text{(C.0.17)}$$

D Data analysis

Table 1 provides summary statistics of the distributions of the returns of all the variables analysed. A lower annualised average return can be seen, which is negative in EUR/USD, EUROSTOXX 50 and GBP/USD. BOVESPA returns have the widest range of values, as shown by the interquartile rank and the difference between the historic maximum and minimum. This implies that this asset reaches more extreme returns, both positive and negative, than any other, so it is also the most volatile. For higher moments, all assets present asymmetric and leptokurtic features. On the one hand, the skewness parameter is positive for the exchange rate of the Euros, Yens and Brazilian Reals against USD, and negative for the rest of the assets. On the other hand, all assets have a leptokurtic distribution, with EUR/USD and JPY/USD having the lowest values for the kurtosis statistic. The significance of higher moments in the marginal distributions leads to the rejection of the null hypothesis in the Jarque Bera test for any significance level.

Table 1: Descriptive statistics.

	EUR/USD	EUROSTOXX 50	JPY/USD	Nikkei 225	BRL/USD	BOVESPA	GBP/USD	FTSE 100	Gold
Mean	-0.0001	-0.0002	0.0000	0.0003	0.0008	0.0019	-0.0002	0.0002	0.0016
Standard Deviation	0.0132	0.0297	0.0138	0.0298	0.0214	0.0371	0.0131	0.0236	0.0240
Max	0.0563	0.1359	0.0847	0.1145	0.1258	0.1684	0.0533	0.1258	0.1412
Min	-0.0533	-0.2513	-0.0542	-0.2788	-0.0914	-0.2233	-0.0866	-0.2363	-0.1450
Quantile 1%	-0.0289	-0.0820	-0.0326	-0.0762	-0.0515	-0.0952	-0.0304	-0.0680	-0.0651
Quantile 25%	-0.0087	-0.0163	-0.0090	-0.0165	-0.0110	-0.0201	-0.0079	-0.0123	-0.0110
Median	-0.0007	0.0024	-0.0004	0.0024	-0.0006	0.0045	0.0002	0.0020	0.0027
Quantile 75%	0.0081	0.0165	0.0079	0.0190	0.0109	0.0252	0.0081	0.0132	0.0149
Quantile 99%	0.0355	0.0721	0.0370	0.0647	0.0698	0.0862	0.0312	0.0593	0.0605
Skewness	0.2322	-0.8158	0.3776	-1.0933	0.7359	-0.4628	-0.6566	-1.0565	-0.1768
Kurtosis	3.9955	9.1957	4.9676	10.9444	7.5668	6.5608	6.9278	15.1113	6.7547
Jarque Bera p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Histograms are a very useful graphical tool for showing these features of asset distributions, especially those related to higher moments. Figure 1 shows the histograms of the distributions of the returns of the nine assets analysed, along with scatter plots of the returns on assets per pair. High kurtosis and skewness (frequently negative) can be seen in all distributions, and the scatter plots show some degree of dependence between the returns on the assets.

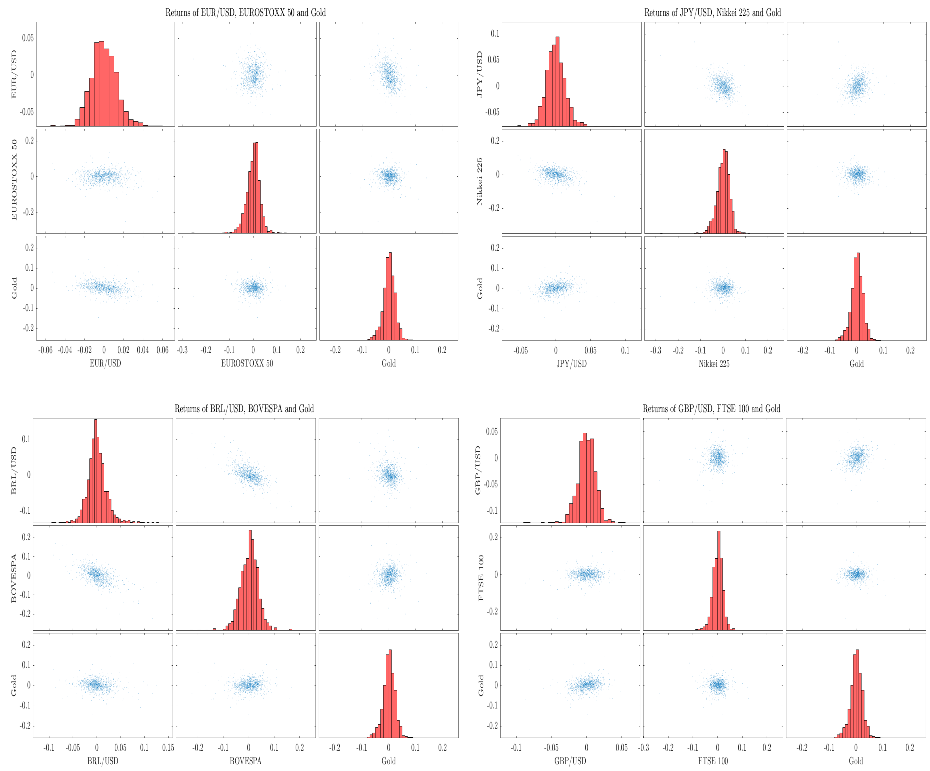


Figure 1: Histograms and scatter plots for the bivariate relationship between assets. The top left panel shows the relation between the returns of European assets, i.e., EUR/USD exchange rate, EUROSTOXX 50 and gold. The top right panel shows the relation between JPY/USD, Nikkei 225 and gold. The graphics in the bottom left panel explain the relationship between BRL/USD, BOVESPA and gold. Finally, the bottom right panel contains the histogram and scatter plot for GBP/USD, FTSE 100 and gold.

In addition, we compute the Kendall's tau correlation coefficient between the three assets of each region on average and when the stock price suffers an extreme downward movement. Table 2 contains the correlation matrix between assets used in each region. Notice that the only positive correlation between gold and the stock index occurs in Brazil, while the correlation between the exchange rate and the index is negative except in Europe.

Table 2: Correlation matrix for the assets used in each region.

	EUR/USD	EUROSTOXX 50	Gold		JPY/USD	Nikkei 225	Gold
EUR/USD	1	0.0382	-0.2693	**JPY/USD**	1	-0.2388	0.1967
EUROSTOXX 50	0.0382	1	-0.0549	**Nikkei 225**	-0.2388	1	-0.0482
Gold	-0.2693	-0.0549	1	**Gold**	0.1967	-0.0482	1
	BRL/USD	**BOVESPA**	**Gold**		**GBP/USD**	**FTSE 100**	**Gold**
BRL/USD	1	-0.3822	-0.1368	**GBP/USD**	1	-0.0153	0.1949
BOVESPA	-0.3822	1	0.1105	**FTSE 100**	-0.0153	1	-0.0109
Gold	-0.1368	0.1105	1	**Gold**	0.1949	-0.0109	1

NOTE: The correlation is measured by Kendall's tau coefficient.

Lastly but not least, Table 3 presents the total correlation of each asset with the other variables in each group. Except in Europe, the exchange rate has the highest total correlation. This seems logical because any investor must acquire exchange rates to form a portfolio which includes any asset traded in a currency other than their own. Moreover, a shock in a market would have consequences such as a fall in the stock index or an increase in the price of gold, and these circumstances always impact the exchange rate.

Table 3: Total correlation of each asset in the data set.

	Pearson	Spearman	Kendall		Pearson	Spearman	Kendall
Total correlation EUR/USD	0.3890	0.4416	0.3075	Total correlation JPY/USD	0.6290	0.6245	**0.4355**
Total correlation EUROSTOXX	0.0959	0.1328	0.0931	Total correlation Nikkei	0.4542	0.4109	0.2870
Total correlation Gold	0.4317	0.4698	**0.3241**	Total correlation Gold	0.3006	0.3561	0.2449
Total correlation BRL/USD	0.7763	0.7390	**0.5190**	Total correlation GBP/USD	0.3231	0.3030	**0.2102**
Total correlation BOVESPA	0.7475	0.7020	0.4926	Total correlation FTSE	0.1110	0.0373	0.0262
Total correlation Gold	0.3189	0.3606	0.2473	Total correlation Gold	0.2840	0.2977	0.2058

NOTE: This table presents the sum of the Pearson, Spearman and Kendall correlation coefficients, respectively, between each asset and the other assets in its group, divided by the currency in which each stock index is denominated. For example, the total correlation of EUROSTOXX 50 is obtained from the sum of the correlation between this index and EUR/USD and between the index and gold.

Figure 2 presents the prices of all assets included in the data set in the sample between 1 January 2000 and 30 December 2019.

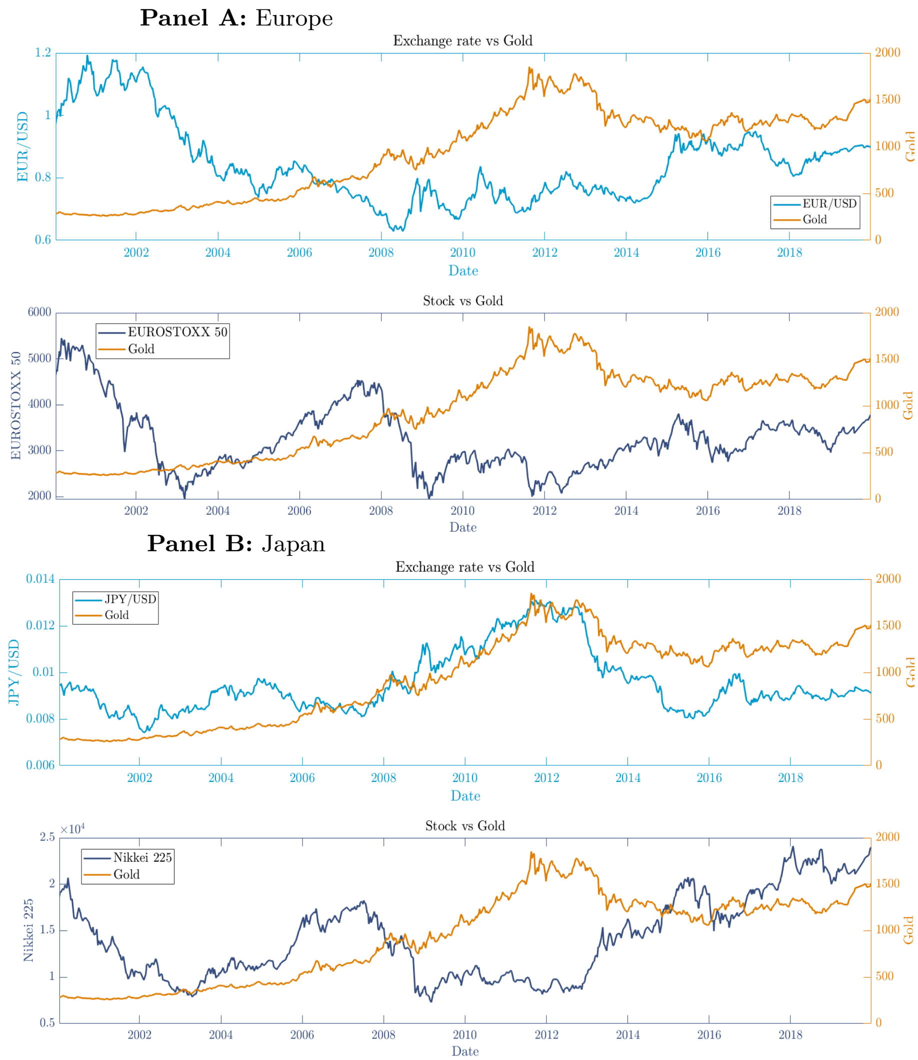
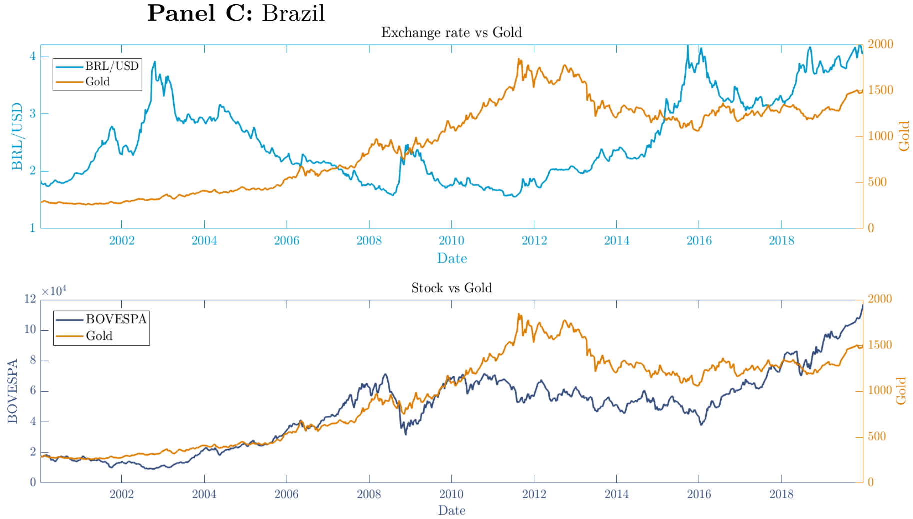


Figure 2: Prices for financial series between 2000 and 2019.

Panel C: Brazil



Panel D: UK

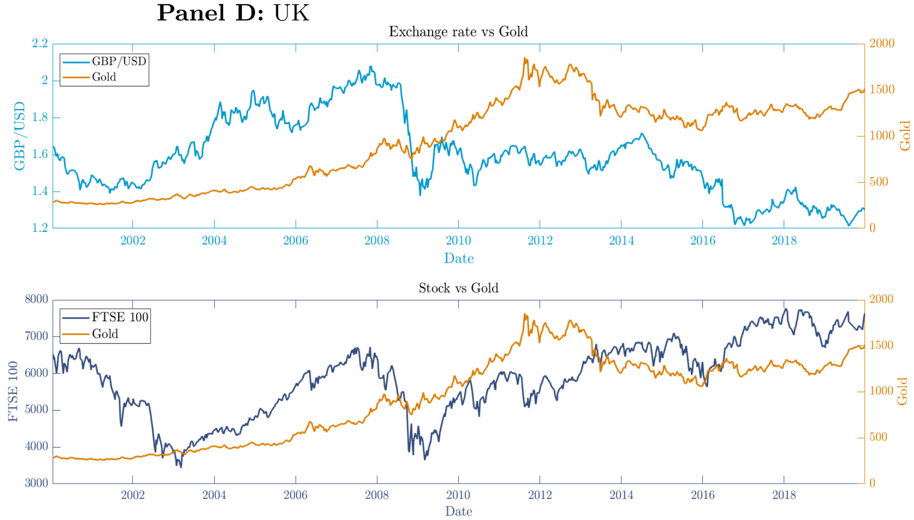


Figure D.2 (Cont.): Prices for financial series between 2000 and 2019.

E Marginal model selection

For marginal distributions we consider an ARMA(p,q) model for the mean and a GARCH(m,s) model for the variance of the distribution.

ARMA(p,q) models are very useful for describing the dynamic of a variable. They can be broken down into autoregressive (AR) and moving average (MA) parts. The AR part relates the return at time t, r_t, with the last p values, while the moving average term tries to explain this variable as a weighted sum of the last q values of the innovations. Given these definitions, the dynamic of the return on an asset can be written as:

$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i \cdot r_{t-i} + \sum_{j=1}^{q} \theta_i \cdot a_{t-j} + a_t \tag{E.0.1}$$

where p and q are non-negative integers and ϕ and θ are AR and MA parameters, respectively. Also, $a_t = \sigma_t \cdot \varepsilon_t$, with ε_t being a random variable with zero mean and unit variance (white noise).

Changes in variance (σ_t^2) are very important in understanding financial markets, while investors demand higher expected returns as compensation for taking on more risks in their investments (Hamilton, 1994). We compare several models to take into account heterocedasticity, some of which are nested in others, to fit the dynamics of the volatility as accurately as possible. They are introduced below.

1. GARCH (*Generalized AutoRegressive Conditional Heteroscedasticity*) models are used to adjust the variance of a series, and they also contain autoregressive and a moving average terms:

$$\begin{aligned} r_t &= \mu_t + a_t = \mu_t + \sigma_t \cdot \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha \cdot a_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 \end{aligned} \tag{E.0.2}$$

where ε_t is a white noise which follows a certain distribution with null mean and unit variance. To fulfill the stacionarity conditions, it must happen that $\omega \geq 0$, α, $\beta > 0$ and $\alpha + \beta < 1$.

This model has certain limitations, such as not taking into account the skewness of the residuals or the leverage effect. To overcome them, some extensions of the standard GARCH model have been drawn.

2. The GJR-GARCH model (Glosten et al., 1993) is one of the most widely used extensions because it includes the leverage effect. This model is expressed as follows:

$$\sigma_t^2 = \omega + \left(\alpha + \gamma \cdot \mathbf{1}_{\{\varepsilon_{t-1}<0\}}\right) \cdot \varepsilon_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 \tag{E.0.3}$$

Asymmetry is captured by γ. When γ is positive, negative shocks introduce more volatility than positive shocks of the same size (Huang et al., 2009). Because of that, it is expected that $\gamma > 0$ in the GJR model. Other conditions for the stationarity of the model are $\omega \geq 0$, $\alpha, \beta > 0$ and $\alpha + \beta + \gamma \cdot \mathbf{1}_{\{\varepsilon_{t-1}<0\}} < 1$.

3. The EWMA (*Equally-Weighted Moving Average*) model is a particular case of GARCH where $\omega = 0$ and $\alpha = 1 - \beta$, i.e. the accompanying parameters act as weights:

$$\sigma_t^2 = (1-\lambda) \cdot a_{t-1}^2 + \lambda \cdot \sigma_{t-1}^2 \tag{E.0.4}$$

This model does not present mean reversion, i.e. it has no term for the average variance in the long term.

4. The APARCH(1,1) (*Asymmetric Power ARCH*) model is defined as follows:

$$\sigma_t^\delta = \omega + \alpha \cdot (|\varepsilon_{t-1}| + \gamma \cdot \varepsilon_{t-1})^\delta + \beta \cdot \sigma_{t-1}^\delta \tag{E.0.5}$$

This model is stationary only if δ is positive. When $\delta = 2$ and $\gamma = 0$, APARCH is reduced to the GARCH model. Furthermore, for any value of γ, when $\delta = 2$ APARCH becomes the GJR-GARCH model.

5. The GARCH-M model seeks to capture the fact that in finance the return on an asset depends on its volatility, so greater volatility should imply a greater expected return. The GARCH(1,1)-M model does not include a variation on the standard GARCH in the variance equation, but in the mean equation:

$$r_t = \mu_t + c \cdot \sigma_t^2 + a_t = \mu_t + c \cdot \sigma_t^2 + \sigma_t \cdot \varepsilon_t \tag{E.0.6}$$

where $\mu_t = \phi_0 + \sum_{i=1}^p \phi_i \cdot r_{t-i} + \sum_{j=1}^q \theta_i \cdot a_{t-j}$ and c represents the risk premium. A positive value of this parameter means that the greater the volatility is, the greater the return on the asset will be. There are no additional conditions for the parameter c compared to the GARCH for it to be a stationary model.

To select the model best which captures the behaviour of each series, we compute GARCH(1,1), GJR(1,1), EWMA, APARCH(1,1) and GARCH(1,1)-M models for each series under different distribution assumptions, i.e. Normal, Student t and Hansen's Skewed Student t distribution (Hansen, 1994). In total we test 15 different models for each series.

The density function under **Gaussian distributed assumption** is:

$$f(\varepsilon_t|\Omega_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \cdot e^{-\frac{\varepsilon_t^2}{2}} \tag{E.0.7}$$

where, σ_t^2 is adjusted by the GARCH(m,s)-type model specificated and the standardised residual is defined as $\varepsilon_t = \frac{a_t}{\sigma_t} = \frac{r_t - \mu_t}{\sigma_t}$, where μ_t summarises the ARMA(p,q) model.

Student t distribution is widely applied in studies based on data with outliers or heavy tails (Ding, 2016). If innovations are assumed to follow this distribution, then the following density function is obtained:

$$f(\varepsilon_t|\Omega_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\sigma_t^2\pi(\nu-2)}} \cdot \left(1 + \frac{\varepsilon_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \tag{E.0.8}$$

Finally, **Hansen's (1994) Skewed Student t** is a flexible distribution accommodating the skewness and excess kurtosis often present in financial data (Theodossiou, 1998). The density function of this distribution is given by:

$$f(\varepsilon_t|\nu,\lambda) = \begin{cases} bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{a+b\varepsilon_t}{1-\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, & \varepsilon_t < -\dfrac{a}{b} \\ bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{a+b\varepsilon_t}{1+\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, & \varepsilon_t \geq -\dfrac{a}{b} \end{cases} \tag{E.0.9}$$

where $2 < \nu < \infty$ and $-1 < \lambda < 1$. The parameter ν controls the tails of the density, and the skewness parameter, λ, represents the rate of descent of the density around $\varepsilon = 0$. The constants a, b and c are given by:

$$a = 4\lambda c\frac{\nu-2}{\nu-1}, \qquad b^2 = 1+3\lambda^2-a^2, \qquad c = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\dfrac{\nu}{2}\right)}$$

Note that this distribution becomes a Student t distribution when $\lambda = 0$, i.e. when there is no asymmetry, and a Normal distribution if also $\nu \to \infty$.

We use various criteria to select the best GARCH model. Firstly, we calculate the maximum loglikelihood. Secondly, Akaike and Swarchtz (or Bayesian) information criteria are used as useful estimators of prediction error, i.e., given a set of models, these criteria estimate the quality of each model, relative to each of the other models in the set. Thirdly, we obtain the parametric VaR of the assets assuming its variance is adjusted by the GARCH model selected. The parametric VaR is defined as $VaR_{i,t}(\alpha) = F_i^{-1}(\alpha)\cdot\sigma_{i,t}+\mu_{i,t}$, where $F^{-1}(\alpha)$ is the inverse of the cdf of the distribution used. Once the VaR has been calculated, we apply two backtesting tests, namely tose of Kupiec (1995) and Christoffersen (1998). In both cases the goal is to confirm whether a model is suitable for calculating the VaR of an asset or portfolio, so the null hypothesis is that the number of exceedances of the VaR, i.e. the observations in which the returns of the asset are below the VaR, is statistically equal to the level of significance of the test.

The evidence indicates that the best marginal model is ARMA(p,q)-APARCH(1,1) (see Table 1). In any case the residuals of the models follow a Skewed Student t distribution, capturing higher moments of the financial series.

It must be noted that for the loglikelihood function of gold, Akaike and Swchartz's criteria are not conclusive as to whether the best model is GARCH-M or APARCH with Skewed Student t innovations. Following the results of Kupiec and Christoffersen's tests, we have finally selected the latter model for the variance. In the case of the European equity index, even though the maximum loglikelihood and the information criteria show that the best model is GJR(1,1), following VaR backtesting criteria the model which best fits the tails

Table 1: Optimal marginal model selected for each returns series.

	AR(p)	MA(q)	GARCH(m,s)
EUR/USD	4	11	APARCH(1,1)
EUROSTOXX50	1	13	APARCH(1,1)
JPY/USD	1	1	APARCH(1,1)
Nikkei225	1	1	GARCH(1,1)
BRL/USD	1	1	APARCH(1,1)
BOVESPA	1	1	GJR(1,1)
GBP/USD	1	1	APARCH(1,1)
FTSE100	1	13	APARCH(1,1)
Gold	1	1	APARCH(1,1)

NOTE: This table contains the ARMA(p,q)-GARCH(m,s) model for each asset in the data set employed in this study. In all cases the innovations of the model follow a Hansen's (1994) Skewed Student t distribution.

of the distribution is the APARCH(1,1) with Skewed Student t innovations. In addition, for Nikkei 225 and BOVESPA the optimal model proves to be an APARCH(1,1) with Skewed Student t innovations following maximum loglikelihood and Akaike and Swchartz's criteria, but the VaR backtesting tests select GARCH(1,1) and GJR(1,1), respectively, as the best models to fit the tails of the distributions. It must be borne in mind that GJR is a particular case of the APARCH model in which $\delta = 2$, and if it also holds that $\gamma = 0$, APARCH becomes a standard GARCH model.

Once these optimal models are adjusted it is confirmed that there is no information contained in their innovations, but all the information of a series is contained in the models for the mean or for the variance. We use two tests to check that the model is actually suitable. Firstly, we use the Ljung-Box test to confirm whether the lags of the acf and pacf of the residuals are not significant and the model is thus well specified, i.e. the residuals are white noise. The alternative hypothesis is that there is a non-zero autocorrelation coefficient. Secondly, Engle's ARCH test assesses the null hypothesis that a series of residuals exhibits no conditional heteroscedasticity against the alternative of an ARCH(1) model. The null hypothesis cannot be rejected in the tests, indicating that the model is suitable and there is no ARCH component in the residuals.

We also test the distribution of the model. On the one hand, a Kolmogorov-Smirnov test checks whether independent random samples, X_1 and X_2, are drawn from the same underlying continuous population. X_1 represents the residuals of the model and X_2 represents a simulation of the same length obtained from the theoretical distribution. On the other hand, the Anderson-Darling is a statistical test of whether a sample is drawn from a given probability distribution, in particular from a Normal, Student t or Skewed Student t.

F Estimates of marginal and vine copula models

Table 1: Estimates of optimal marginal models and backtesting results.

	EUR/USD	**EUROSTOXX 50**	**JPY/USD**	**Nikkei 225**	**BRL/USD**	**BOVESPA**	**GBP/USD**	**FTSE 100**	**Gold**
Panel A									
ϕ_0	-0.0004	0.0011*	0.0000	0.0009	0.0002	0.0021**	-0.0001	0.0007*	0.0013***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
ϕ_1	0.0522	-0.1110***	-0.0360	-0.0392	0.9027	-0.0830	0.0156	0.0257**	0.2161
	(0.03)	(0.03)	(0.06)	(0.36)	(0.43)	(0.03)	(0.13)	(0.03)	(0.32)
θ_1	0.0735***	0.0130	0.0000	0.0030	-0.8518***	0.0523	0.0008	-0.0924*	-0.2463*
	(0.03)	(0.03)	(0.05)	(0.36)	(0.42)	(0.03)	(0.12)	(0.03)	(0.32)
ω	0.0000	0.0000	0.0000	0.0000	0.0016	0.0000	0.0000	0.0000	0.0001
	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
α	0.0487***	0.0958***	0.0410***	0.0788***	0.1171***	0.0651***	0.0695***	0.1198***	0.0771***
	(0.02)	(0.03)	(0.03)	(0.02)	(0.04)	(0.02)	(0.04)	(0.04)	(0.03)
β	0.9221***	0.8808***	0.8893***	0.8783***	0.8706***	0.9069***	0.8778***	0.8441***	0.9011***
	(0.09)	(0.03)	(0.20)	(0.05)	(0.04)	(0.04)	(0.19)	(0.04)	(0.03)
γ	0.0055	0.0075***	0.0020	-	-0.7587***	0.0000	0.0012	0.0137	-0.4064***
	(0.10)	(0.02)	(0.04)		(0.32)	(0.01)	(0.13)	(0.10)	(0.26)
δ	2.8720***	2.0112***	3.1884***	-	0.8713***	-	2.5787***	1.8440***	1.6293***
	(15.14)	(0.00)	(30.30)		(0.50)		(11.18)	(0.69)	(1.40)
Skewness (λ)	0.1357***	-0.2766***	0.1074**	-0.2404***	0.1779**	-0.2186***	-0.1706	-0.2123***	-0.0790***
	(0.04)	(0.04)	(0.04)	(0.04)	(0.05)	(0.04)	(0.04)	(0.04)	(0.04)
Tail (ν)	27.1315***	9.8980***	12.7823***	8.4621***	6.9605***	9.4092***	27.9858***	7.2266***	6.7724***
	(19.76)	(18.19)	(35.09)	(8.13)	(12.09)	(3.28)	(18.19)	(5.38)	(1.42)
Panel B									
Loglikelihood	3082.45	2361.02	3020.92	2267.14	2694.92	2041.18	3118.77	2596.70	2510.01
AIC	6184.90	4740.04	6061.83	4550.28	5409.84	4100.36	6257.53	5213.41	5040.02
BIC	6234.39	4784.58	6111.32	4589.87	5459.33	4144.90	6307.02	5262.90	5089.51
LBQ test	0.8663	0.8882	0.5463	0.8773	$0.0586^{(1)}$	0.5667	0.7037	0.5855	0.8676
ARCH test	0.8964	0.5030	0.3753	$0.0462^{(2)}$	0.6739	0.5855	0.8639	0.6818	0.9203
KS test	0.8078	0.9420	0.8402	0.7381	0.7015	0.9947	0.6273	0.6273	0.3573
AD test	0.6118	0.6992	0.7270	0.7423	0.6749	0.7740	0.6113	0.4156	0.4397
Kupiec									
VaR(5%)	$0.0542^{(1)}$	$0.0007^{(3)}$	0.3118	$0.0252^{(2)}$	0.4788	0.1061	$0.0252^{(2)}$	$0.0002^{(3)}$	0.3118
VaR(10%)	0.4085	0.8286	0.6850	$0.0442^{(2)}$	$0.0677^{(1)}$	0.3511	0.4085	0.5383	0.3030
Christoffersen									
VaR(5%)	0.6634	0.2803	0.4648	0.7633	0.1422	0.5865	0.1885	0.3975	0.9684
VaR(10%)	0.4717	0.4746	0.8169	0.1195	$0.0169^{(2)}$	0.1386	0.6348	0.0522	0.1288

NOTE: Panel A presents the optimal parameters of each marginal model, obtained from maximum likelihood. The standard deviation is calculated by Monte Carlo simulation. ***/**/* indicates statistical significance at 1/5/10%. Panel B contains the maximum loglikelihood function value, Akaike and Swchartz information criteria and the *p*-values of Ljung-Box (LBQ), ARCH, Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) tests for the goodness of fit of thoose models, as well as Kupiec's and Christoffersen's tests for the VaR backtesting to confirm that the models are also appropriate for calculating risk measures. The values with $^{(3)}/^{(2)}/^{(1)}$ represent the rejection of the null hypothesis for a level of significance of 1/5/10%.

Table 2: Parameters of optimal vine copula structures.

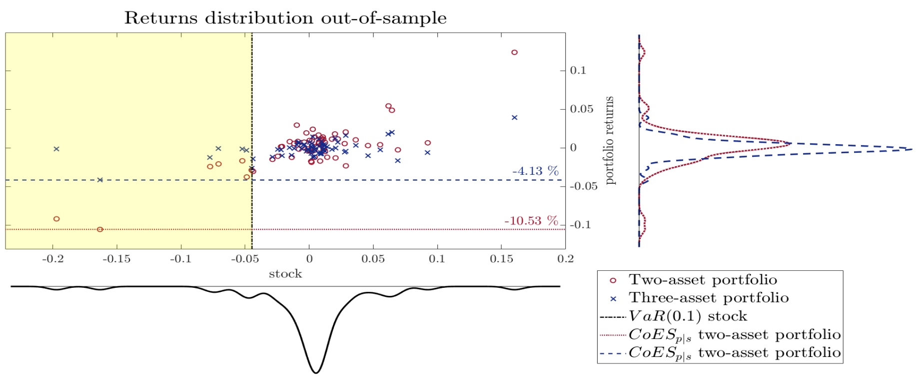
	Panel A: Europe		**Panel B:** Japan		**Panel C:** Brazil		**Panel D:** UK	
$F(e,s)$	**Student time-varying**		**Gaussian time-varying**		**Student constant**		**Student time-varying**	
	ω	0.0009 (0.25)	ω	-0.1049*** (0.79)	ρ	-0.5563* (0.67)	ω	-0.1249 (0.11)
	α	0.0497** (0.10)	α	-1.0757 (0.96)	ν	18.0147 (40.19)	α	0.8501*** (0.21)
	β	1.9855 (1.34)	β	3.7174 (1.22)			β	-1.0765 (0.80)
	ν	10.2934*** (0.73)					ν	12.2711*** (1.51)
$F(e,g)$	**Student constant**		**Student time-varying**		**90R Clayton time-varying**		**Student time-varying**	
	ρ	-0.4084*** (0.09)	ω	0.0487*** (0.29)	ω	-5.6029* (3.45)	ω	-0.0010 (0.46)
	ν	12.0966*** (0.30)	α	0.0704* (0.11)	α	-2.4810 (1.85)	α	-0.0271 (0.14)
			β	1.8611 (0.93)	β	1.5340 (0.87)	β	2.1265 (1.16)
			ν	9.9106*** (1.94)			ν	8.3432*** (1.04)
$F(s,g\|e)$	**Student constant**		**Gaussian constant**		**Student time-varying**		**90R Gumbel constant**	
	ρ	-0.0552* (0.08)	ρ	0.0586* (0.07)	ω	0.0356 (4.03)	θ	1.0321*** (0.02)
	ν	10.9945*** (0.78)			α	0.1276 (3.22)		
					β	1.3942 (4.83)		
					ν	15.7284 (41.24)		

NOTE: The table reports the estimates and the standard deviation (in parentheses) for the parameters of the optimal copula according to AICc values. ***/**/* indicates statistical significance at 1/5/10%. The estimates of the parameters are obtained using the IFM approach. First, we estimate the optimal parameters of each marginal distribution individually and then those parameters are plugged into the marginal distribution to obtain the optimal parameters of the copula. Its main drawback is that in its second step copula parameters are estimated from the pseudo-observations obtained from the estimated marginals, i.e. the optimal parameters are conditional on the estimates of the parameters of the marginal distributions. To overcome this limitation, we use a Monte Carlo procedure to simulate and re-estimate the model a large number of times in order to obtain the standard deviation of the parameters estimated.

G Out-of-sample exercise

Figure 1 shows the joint distribution of gold, the exchange rate and the local stock index of each region in the out-of-sample period. The x-axis shows the unconditional distribution of the stock returns, where negative skewness can be seen, especially in Japan and Brazil. The yellowish area indicates the scenario where the price of the index falls sharply and its returns are in its 10%-percentile. The y-axis indicates the conditional distribution histogram (smoothed by a kernel function) of both portfolios provided there is a distress scenario for the stock market. The three-asset portfolio is more leptokurtic than the two-asset one in all regions, showing fatter tails. Furthermore, in line with in-sample findings, the three-asset portfolio performs better in terms of CoES than the two-asset portfolio.

Panel A: Europe



Panel B: Japan

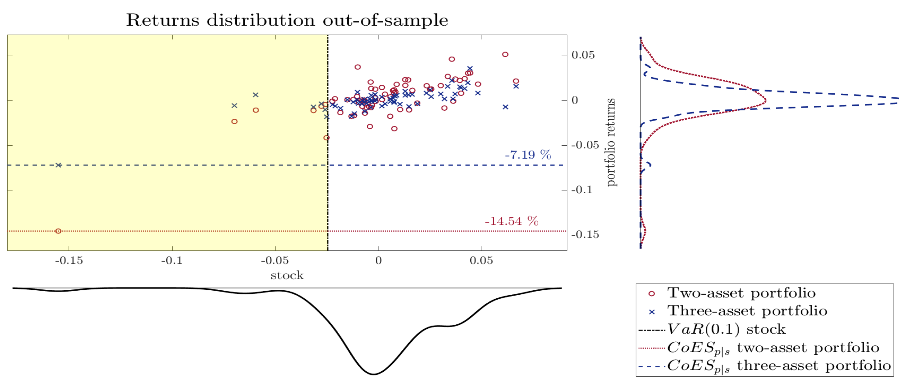
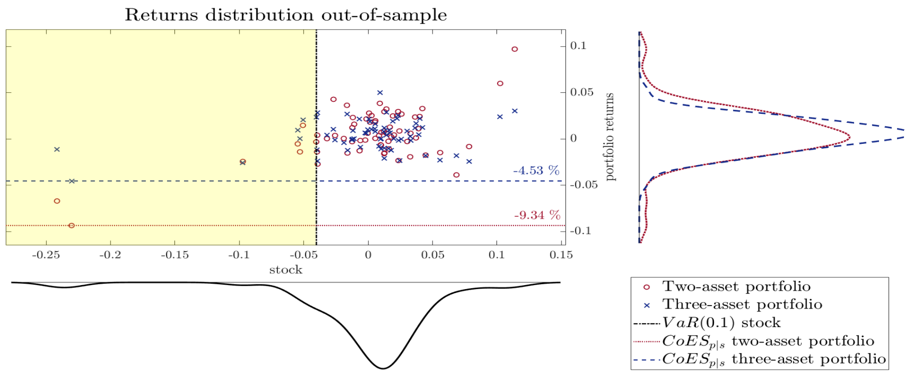


Figure 1: Empirical joint distribution of assets during the out-of-sample period.

Panel C: Brazil



Panel D: UK

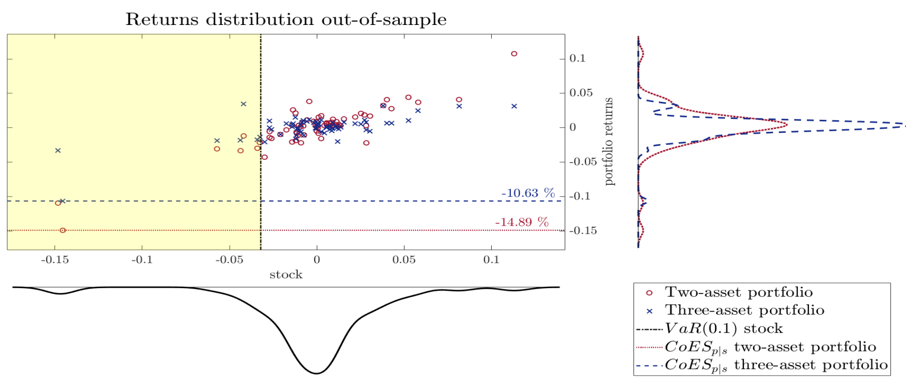
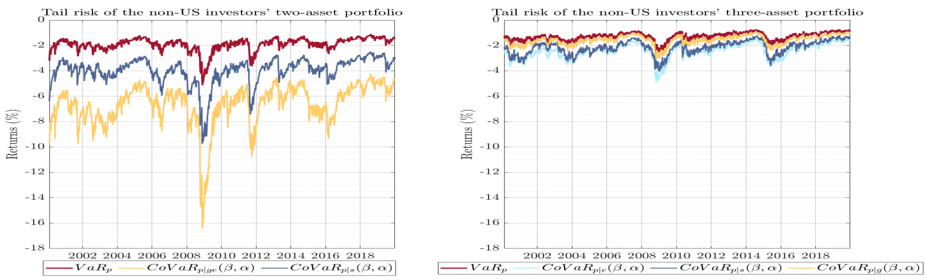


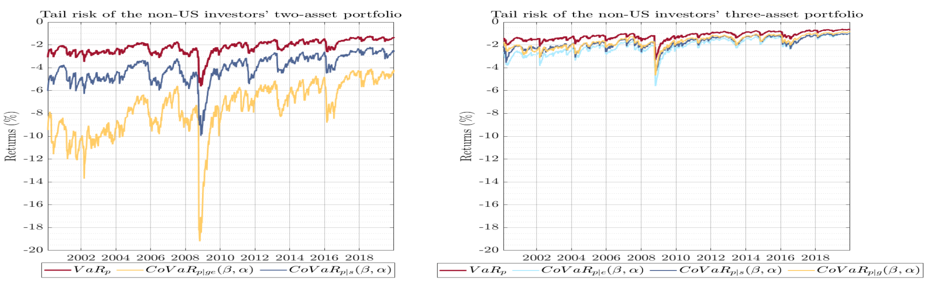
Figure 1 (Cont.): Empirical joint distribution of assets during the out-of-sample period. This figure presents the histogram (smoothed by a kernel function) of the local stock index returns of each region and the histograms of the returns of the two-asset and three-asset portfolios conditional on a bearish scenario of that stock index. The yellowish area indicates the scenario where the stock returns are in the 10%-percentile. Also are shown the CoES of the two portfolios conditional on the stock component being below the VaR(0.10) threshold with a significance level of 10%.

H Extra Figures

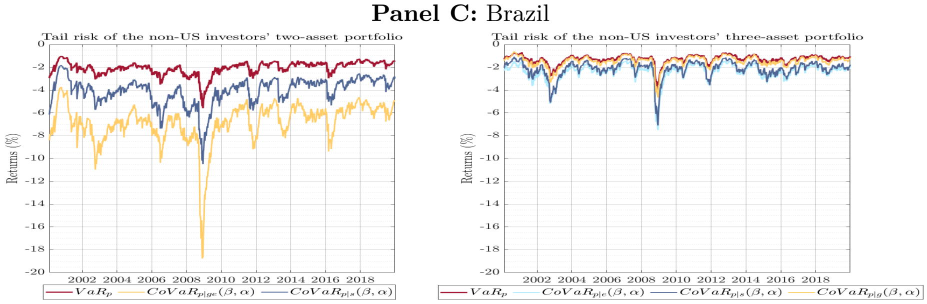
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

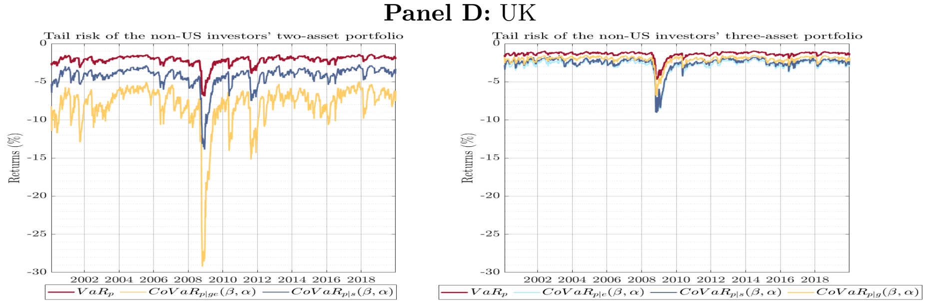
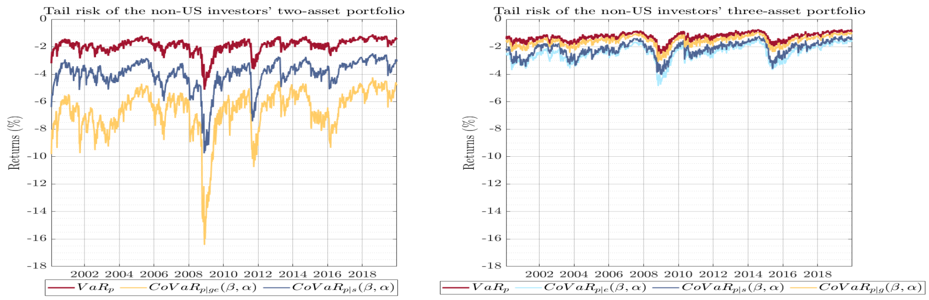
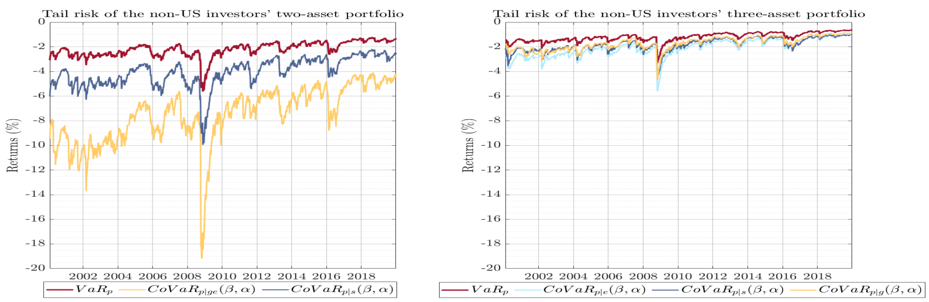


Figure 1: VaR(0.1) and CoVaR(0.1,0.1) of non-US portfolios conditional on one asset being in the lower tail. We see that downside risk is higher in the two-asset portfolios. CoVaR values are systematically below the VaR, then downward movements in the price of any asset have a spillover effect in the portfolio.

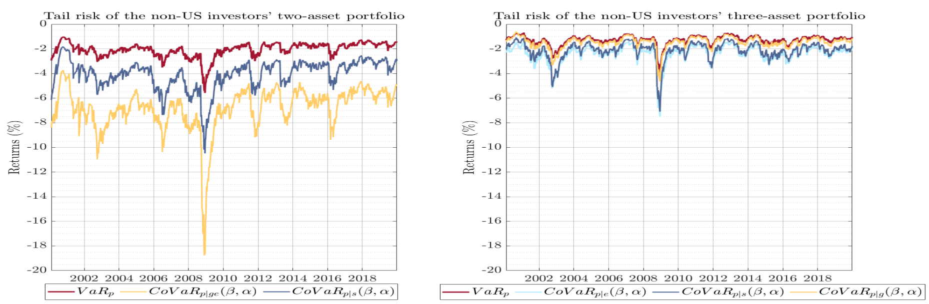
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

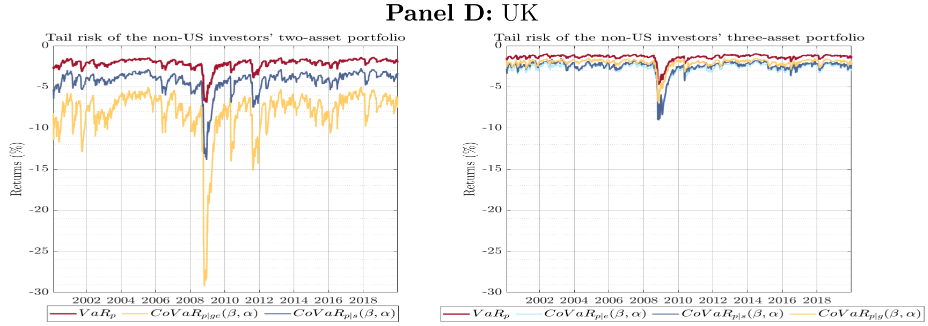


Figure 2: VaR(0.1) and CoVaR(0.1,0.1) of US portfolios conditional on one asset being in the lower tail. We see that downside risk is higher in the two-asset portfolios. CoVaR values are systematically below the VaR, then downward movements in the price of any asset have a spillover effect in the portfolio.

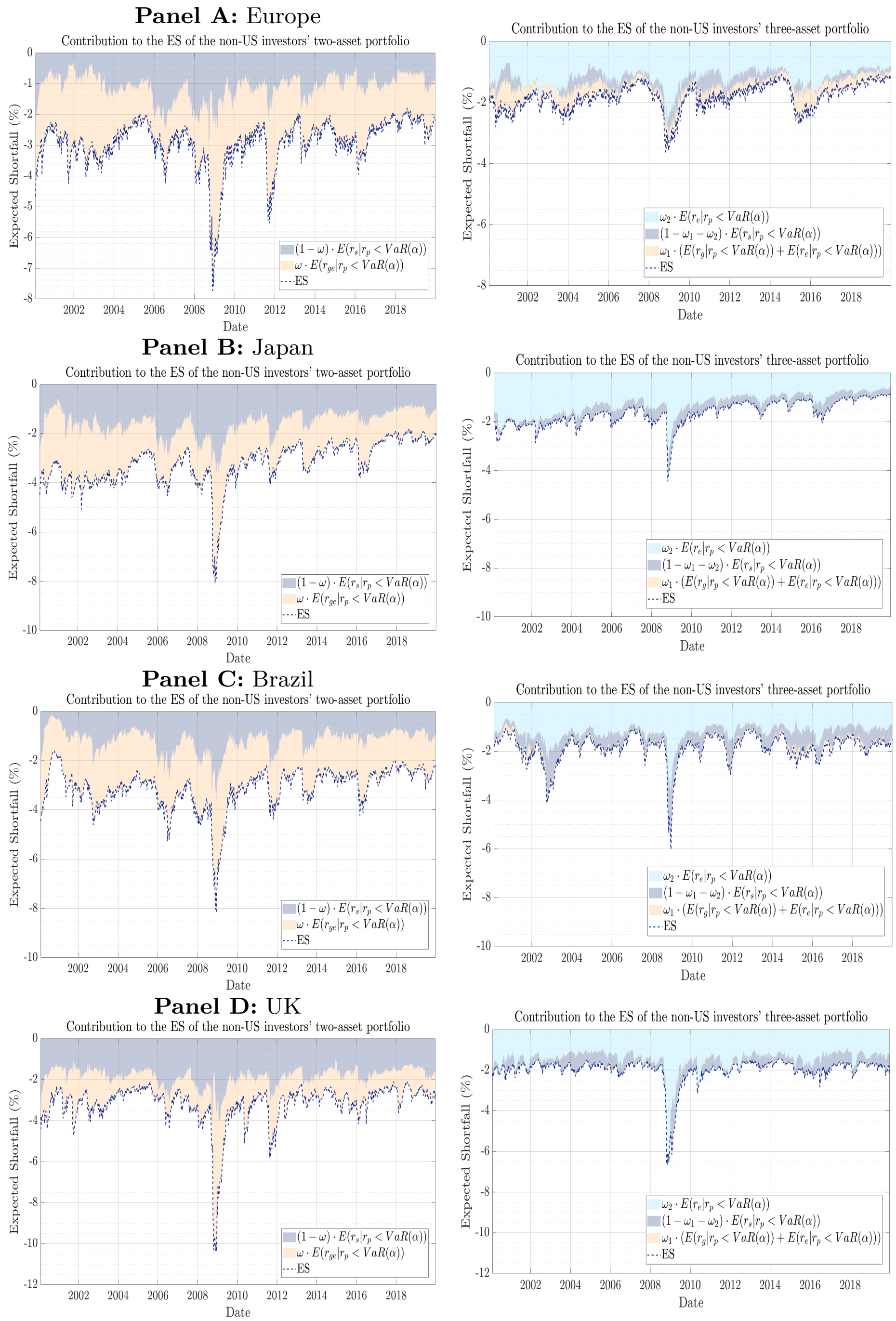


Figure 3: Contribution of each asset to the ES of minimum-ES portfolios of non-US investors. The left (right) figure in each panel shows the contribution of each asset to the ES of the two- (three-) asset portfolios in Figure 2. These contributions are obtained by weighting the Marginal Expected Shortfall (MES) by exposure to each asset in the portfolio (see Annex C).

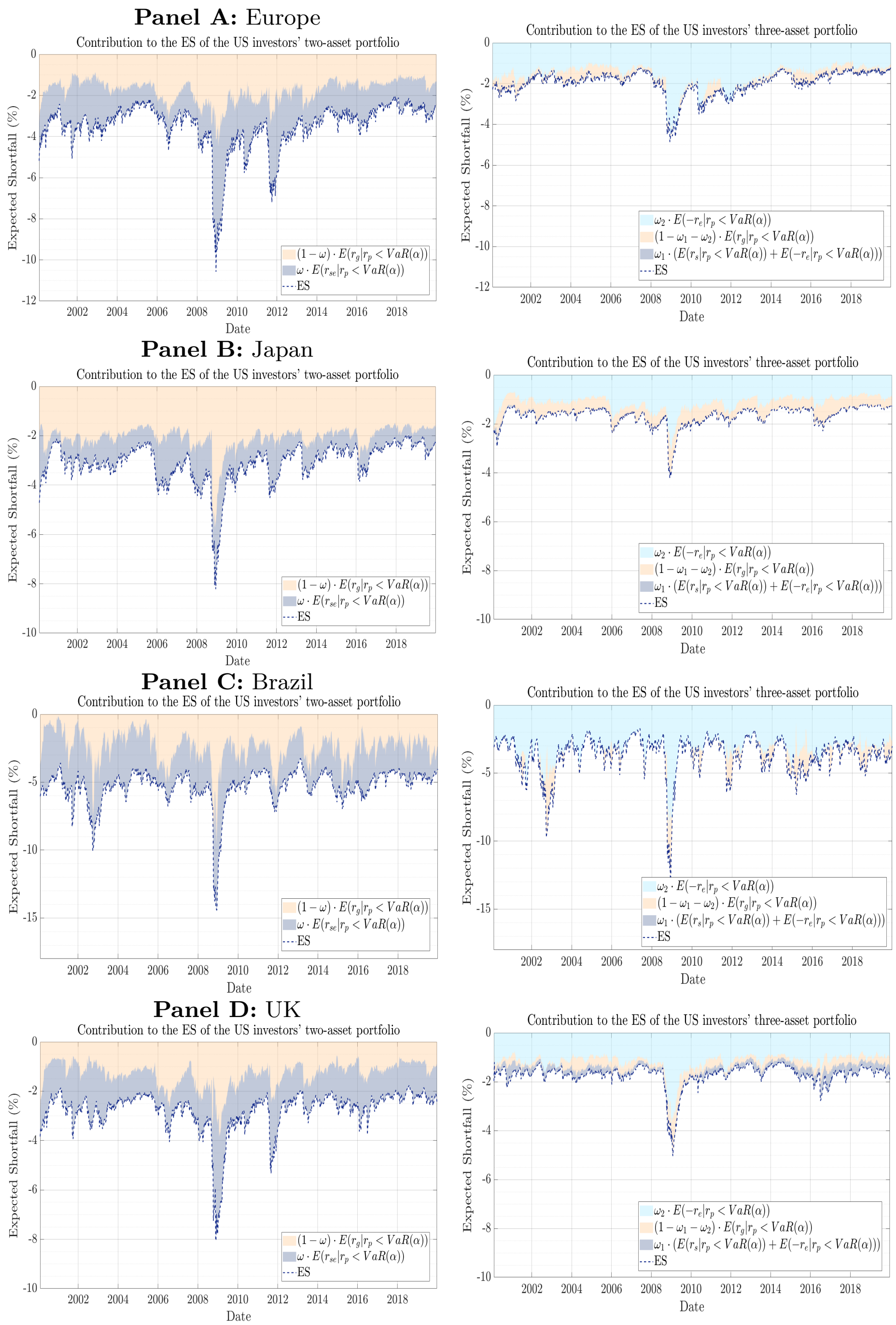
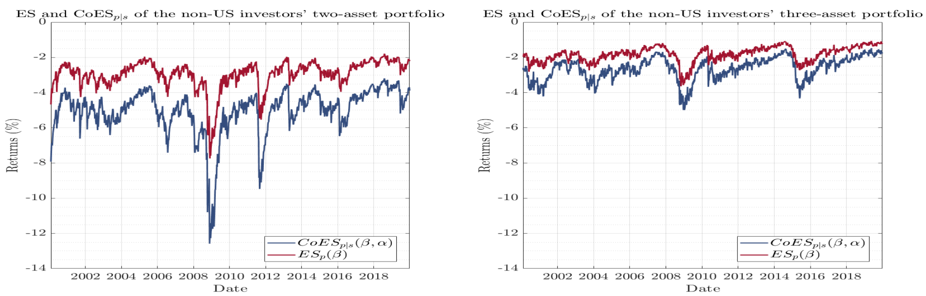
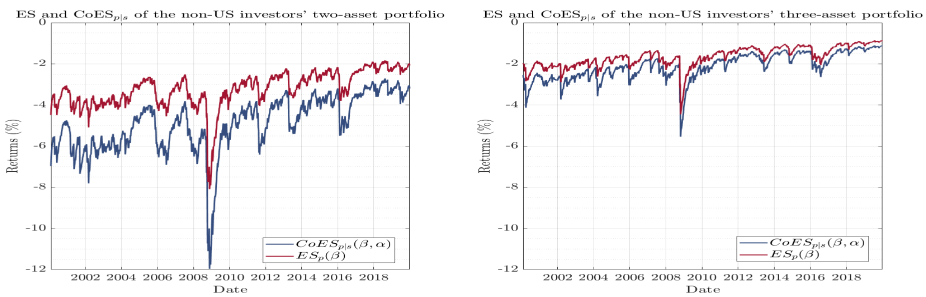


Figure 4: Contribution of each asset to the ES of minimum-ES portfolios of US investors. The left (right) figure in each panel shows the contribution of each asset to the ES of the two- (three-) asset portfolios in Figure 3. These contributions are obtained by weighting the Marginal Expected Shortfall (MES) by exposure to each asset in the portfolio (see Annex C).

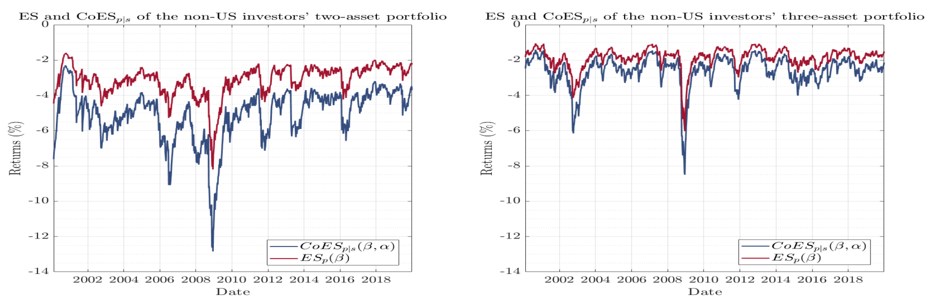
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

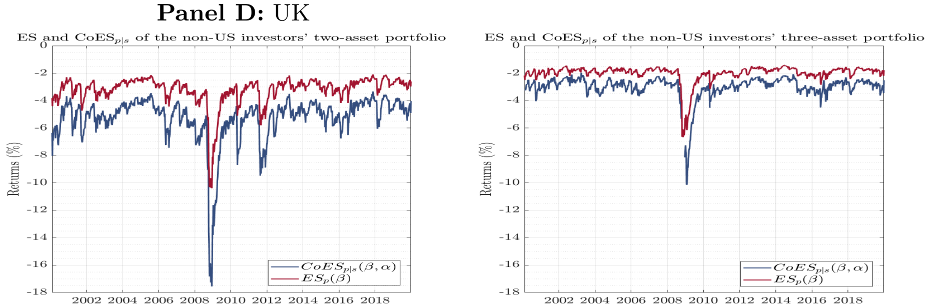
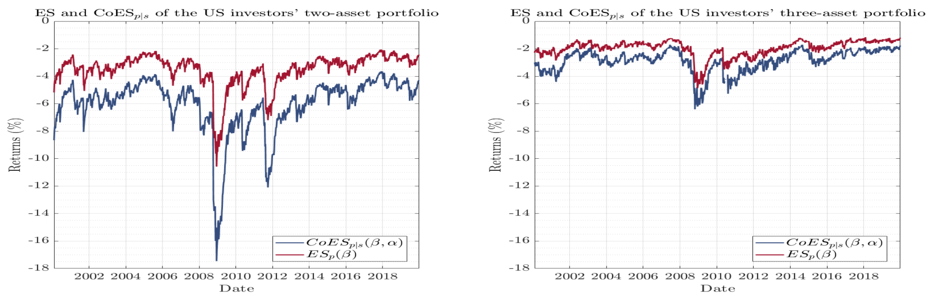
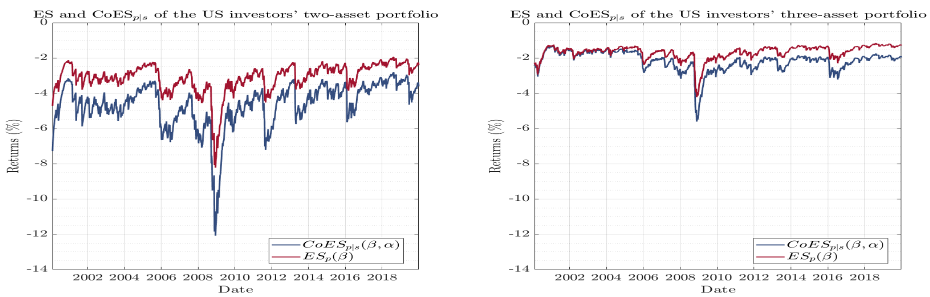


Figure 5: ES(0.1) and CoES(0.1,0.1) of non-US portfolios conditional on a bearish scenario in stock markets. The left (right) figures show the unconditional and conditional tail risks of the two- (three-) asset portfolios. We see that including this asset in the minimum-ES portfolio reduces the impact of a sharp drop in the equity index, improving the hedging strategy. Notice that CoES values are systematically below the ES threshold in all portfolios.

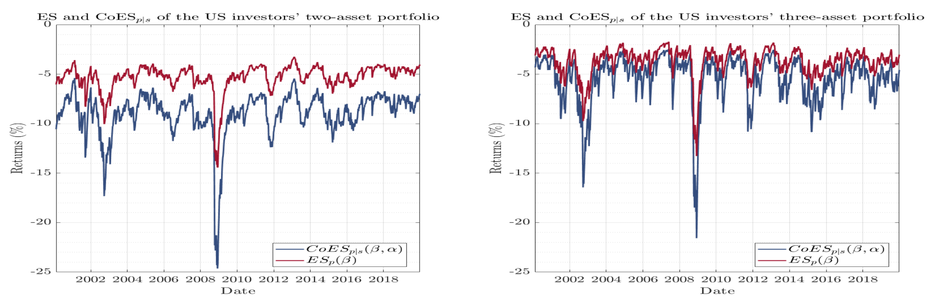
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

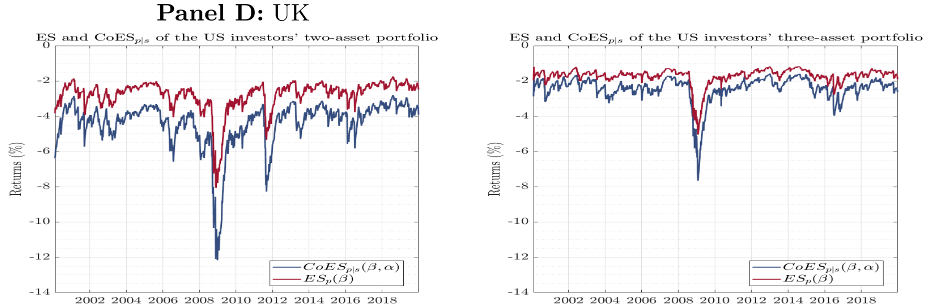
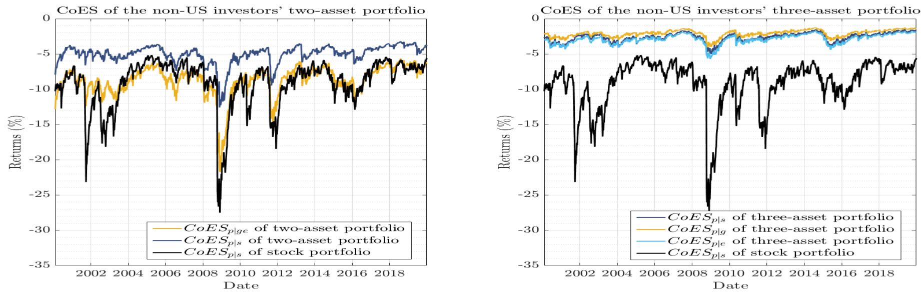
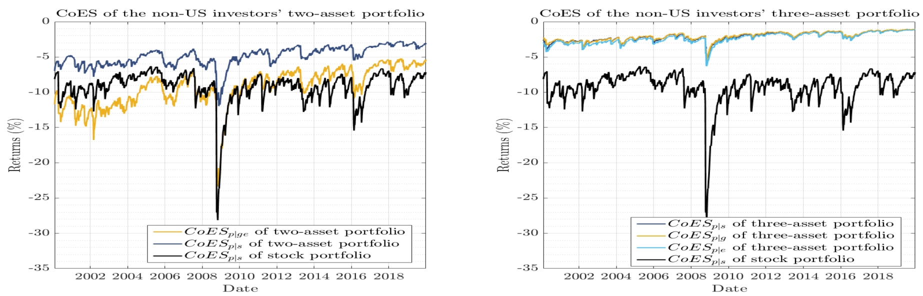


Figure 6: ES(0.1) and CoES(0.1,0.1) of non-US portfolios conditional on a bearish scenario in stock markets. The left (right) figures show the unconditional and conditional tail risks of the two-(three-) asset portfolios. We see that including this asset in the minimum-ES portfolio reduces the impact of a sharp drop in the equity index, improving the hedging strategy. Notice that CoES values are systematically below the ES threshold in all portfolios.

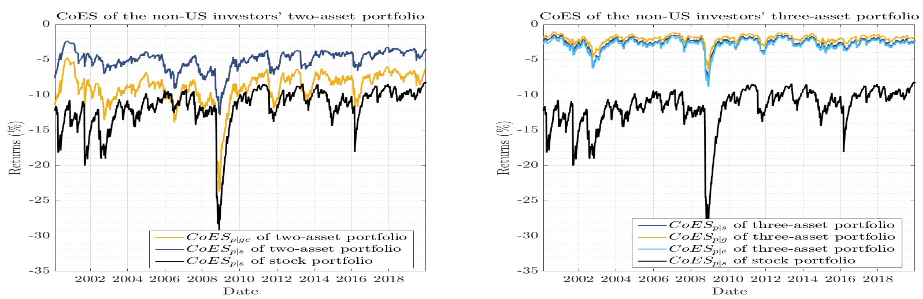
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

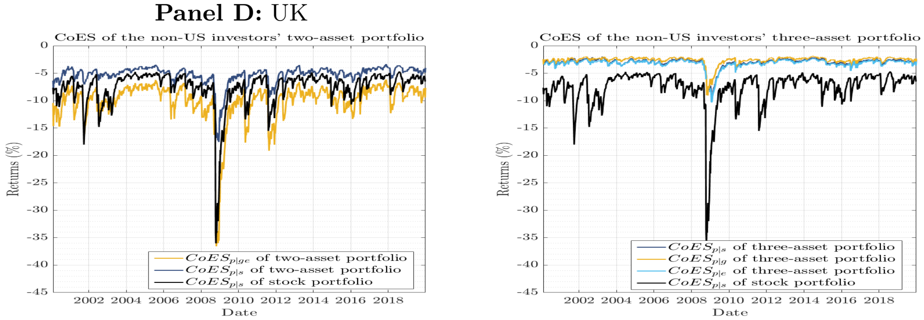
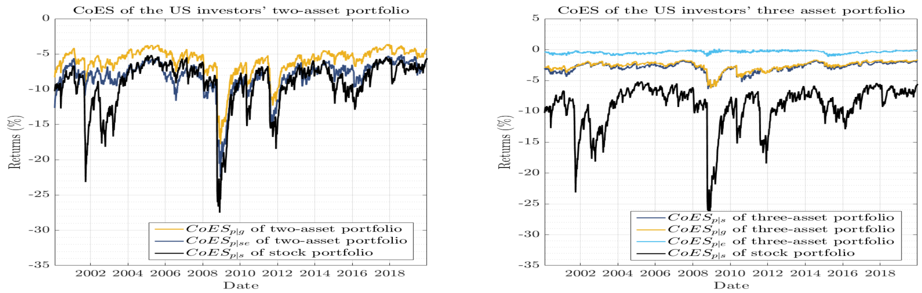
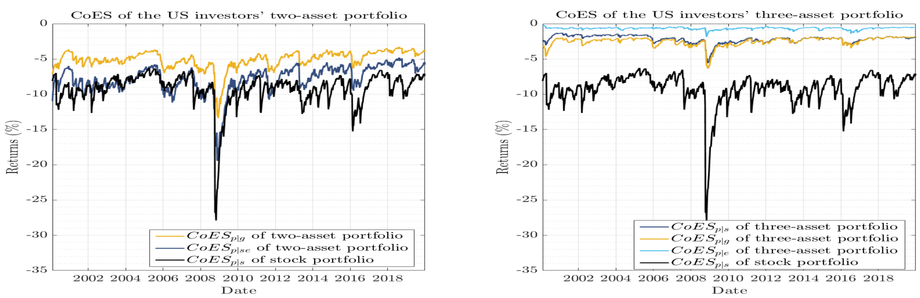


Figure 7: CoES(0.1,0.1) of non-US portfolios in a bearish scenario for one of their components. Yellow, dark blue and light blue lines show the CoES of two-asset (left panel) and three-asset portfolio (right panel) conditional on a bearish scenario of gold, stock and the exchange rate, respectively. The black lines represent the ES of the stock portfolio. A comparison of the CoES in the two minimum-ES portfolios with the ES of the stock portfolio reveals that in the second case (right panels) it pays to assume exchange rate and commodity risks in order to hedge the market risk.

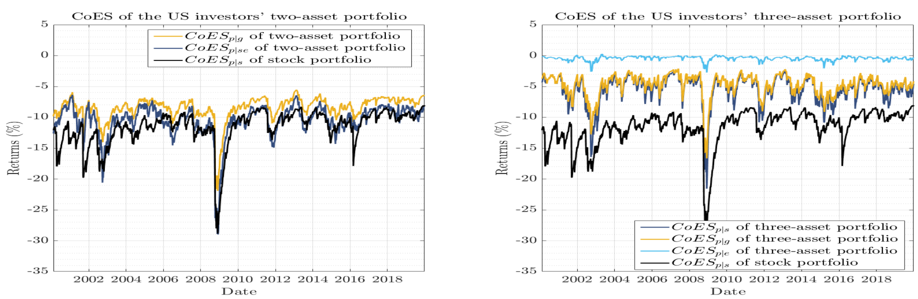
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

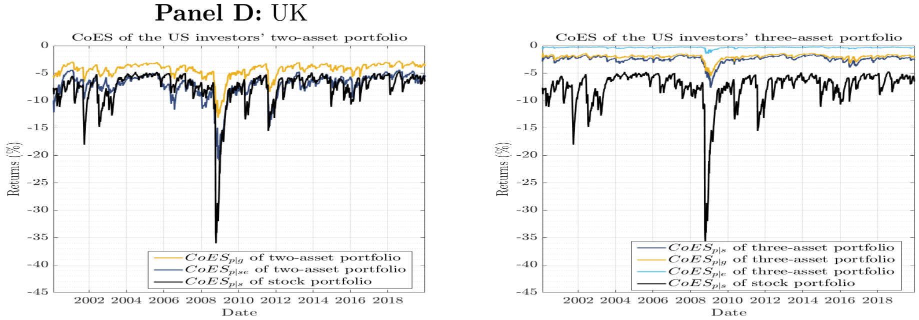
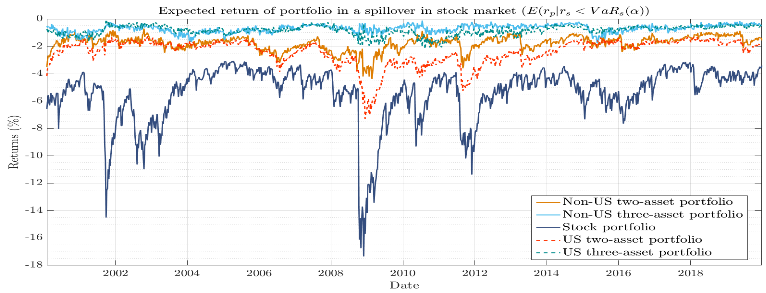
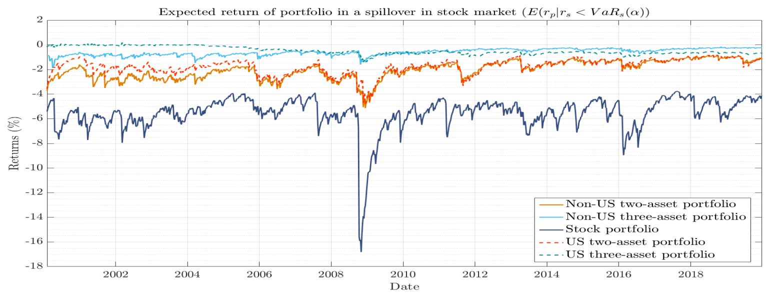


Figure 8: CoES(0.1,0.1) of US portfolios in a bearish scenario for one of their components. Yellow, dark blue and light blue lines show the CoES of two-asset (left panel) and three-asset portfolio (right panel) conditional on a bearish scenario of gold, stock and the exchange rate, respectively. The black lines represent the ES of the stock portfolio. A comparison of the CoES in the two minimum-ES portfolios with the ES of the stock portfolio reveals that in the second case (right panels) it pays to assume exchange rate and commodity risks in order to hedge the market risk.

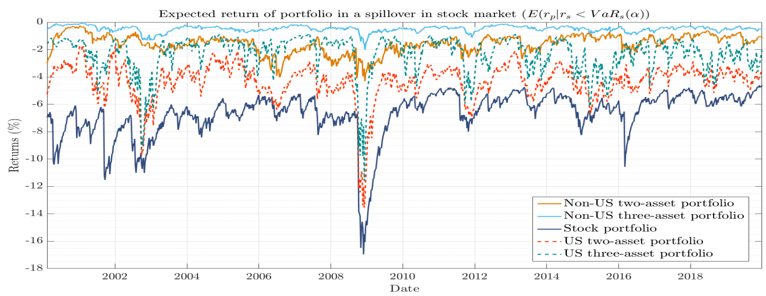
Panel A: Europe



Panel B: Japan



Panel C: Brazil



Panel D: UK

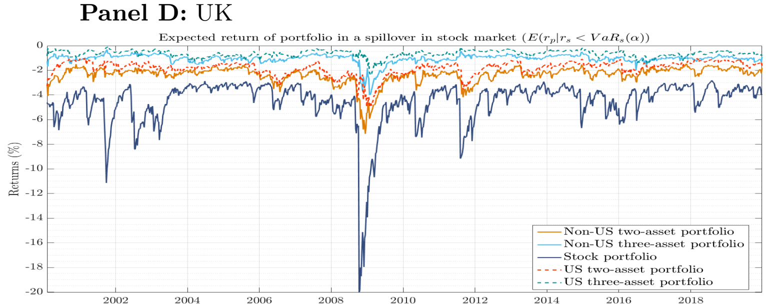


Figure 9: Expected return of portfolios in a bearish scenario for stock markets. This figure shows the expected return on the stock, two-asset and three-asset portfolios, highlighting the great reduction in the losses caused by the spillover in the equity market when the investor decides to acquire gold and especially exchange rate.