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Espacios de aproximación, interpolación límite y espacios de Besov
(Approximation spaces, limiting interpolation and Besov spaces)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Universidad Complutense de Madrid
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Memoria para optar al grado de doctor
con mención de *Doctorado Europeo*
presentada por

Óscar Domínguez Bonilla

Bajo la dirección de los doctores

Fernando Cobos Díaz y Antonio Martínez Martínez

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Sobre esta memoria

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Además, participé entre abril de 2013 y diciembre de 2014 como miembro del equipo investigador del proyecto de título *Interpolación, Espacios de Funciones y Aplicaciones* de referencia MTM2010-15814 financiado por el Ministerio de Ciencia e Innovación, y desde enero de 2015 como miembro del equipo de trabajo del proyecto de título *Interpolación, Aproximación, Entropía y Espacios de Funciones* de referencia MTM2013-42220-P financiado por el Ministerio de Economía y Competitividad.

A lo largo de estos cuatro años, he podido realizar una estancia de investigación en la universidad Friedrich-Schiller-Universität de Jena (Alemania) bajo la supervisión del Profesor H.-J. Schmeisser. Dicha estancia fue subvencionada por la beca FPU AP2012-0779 del Ministerio de Educación y durante su transcurso, pude ampliar mis conocimientos y trabajar junto con un grupo de referencia internacional como es el grupo “Funktionenräume".

Fruto del trabajo de estos cuatro años son los artículos

- F. Cobos, O. Domínguez, *Embeddings of Besov spaces of logarithmic smoothness*, Studia Math. 223 (2014), 193–204.
- F. Cobos, O. Domínguez, *Approximation spaces, limiting interpolation and Besov spaces*, J. Approx. Theory 189 (2015), 43–66.
- F. Cobos, O. Domínguez, *On Besov spaces of logarithmic smoothness and Lipschitz spaces*, J. Math. Anal. Appl. 425 (2015), 71–84.
- F. Cobos, O. Domínguez, *On the relationship between two kinds of Besov spaces with smoothness near zero and some other applications of limiting interpolation*, J. Fourier Anal. Appl., DOI 10.1007/s00041-015-9454-6, published online: 29 December 2015.
- F. Cobos, O. Domínguez, *On Besov spaces modelled on Zygmund spaces*, preprint, Madrid, 2015.
- F. Cobos, O. Domínguez, A. Martínez, *Compact operators and approximation spaces*, Colloq. Math. 136 (2014), 1–11.

- F. Cobos, O. Domínguez, H. Triebel, *Characterizations of logarithmic Besov spaces in terms of differences, Fourier-analytical decompositions, wavelets and semi-groups*, J. Funct. Anal. (2016), http://dx.doi.org/10.1016/j.jfa.2016.03.007.
- O. Domínguez, *Tractable embeddings of Besov spaces into small Lebesgue spaces*, Math. Nachr., DOI 10.1002/mana.201500244, published online: 25 January 2016.

Contents

Resumen

En esta memoria profundizamos en la conexión entre espacios de aproximación y métodos límites de interpolación, aplicando los resultados a problemas en espacios de funciones y, en particular, en espacios de Besov.

La relación simbiótica entre teoría de aproximación y teoría de interpolación es bien conocida como se puede ver en los libros de Bergh y Löfström [10], Triebel [116], Petrushev y Popov [103] y DeVore y Lorentz [50]. El método de interpolación real $(A_0, A_1)_{\theta,q}$ juega un papel importante en este asunto. Se introduce principalmente por medio del K-funcional de Peetre

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1.$$

Normalmente $0 < \theta < 1$ pero para cubrir algunos casos extremos, versiones límites han sido también empleadas, donde $\theta = 0, 1$ y pesos logarítmicos pueden ser incluidos. Ver los artículos de Gomez y Milman [72], Evans y Opic [61], Evans, Opic y Pick [63], Cobos, Fernández-Cabrera, Kühn y Ullrich [33], Cobos y Kühn [40] y Edmunds y Opic [58].

Dado un espacio cuasi Banach X y una familia de aproximación $(G_n)_{n\in\mathbb{N}_0}$ de subconjuntos de X, los espacios de aproximación X_p^α se definen seleccionando aquellos elementos $f \in X$ tales que $(n^{\alpha-1/p}E_n(f))$ pertenece a ℓ_p. Aquí los parámetros cumplen que $0 < \alpha < \infty$ y $0 < p \leq \infty$ y $E_n(f) = \inf\{\|f-g\|_X : g \in G_{n-1}\}$ es el error de mejor aproximación de f por elementos de G_{n-1} (ver Sección 2.1 para mayores detalles). Estos espacios han sido estudiados por Butzer y Scherer [16], Pietsch [104, 105], Petrushev y Popov [103], y DeVore y Lorentz [50], entre otros autores. Los espacios de aproximación límite $X_q^{(0,\gamma)}$ se definen haciendo $\alpha = 0$ e insertando el peso $(1+\log n)^\gamma$ con la sucesión $(E_n(f))$. Han sido investigados por Cobos y Resina [44], Cobos y Milman [41], Cobos y Kühn [38], Fehér y Grässler [65] y las referencias dadas en estos artículos. Como fue probado en [44], incluso cuando $\gamma = 0$, la teoría de los espacios de aproximación límite no se sigue de la teoría de los espacios X_p^α tomando $\alpha = 0$. Los espacios X_p^α y $X_q^{(0,\gamma)}$ nos permiten establecer, de una forma clara y elegante, un número de resultados importantes en espacios de funciones, espacios de sucesiones y espacios de operadores.

Esta memoria comienza fijando la notación, introduciendo los conceptos y construcciones principales, así como también con algunos resultados nuevos en el Capítulo 2. Después, en el Capítulo 3, se estudia la reiteración de construcciones por aproximación. Como fue probado en [104, Teorema 3.2], la construcción $(\cdot)_p^\alpha$ es estable por iteración. Una propiedad similar se tiene para $(\cdot)_q^{(0,\gamma)}$ [65, Teorema 2]. Nosotros estudiamos las propiedades de estabilidad cuando primero aplicamos la construcción $(\cdot)_p^\alpha$ y luego $(\cdot)_q^{(0,\gamma)}$, o viceversa. Probamos que, fuera del caso donde $p = q$, las construcciones no conmutan. Primero establecemos

Teorema 3.2. *Sean X un espacio cuasi Banach y $(G_n)_{n\in\mathbb{N}_0}$ una familia de aproximación en X. Supongamos que $\alpha > 0, 0 < p, q \le \infty$ y $\gamma > -1/q$. Entonces, un elemento $f \in X$ pertenece a $(X_q^{(0,\gamma)})_p^\alpha$ si y solamente si $(n^{\alpha-1/p}(1+\log n)^{\gamma+1/q}E_n(f))$ pertenece a ℓ_p.*

Escribimos $X_p^{(\alpha,\gamma+1/q)}$ para el espacio de aproximación que resulta en el Teorema 3.2. El espacio $(X_p^\alpha)_q^{(0,\gamma)}$ es de una naturaleza diferente. En el Teorema 3.6 obtenemos su caracterización usando espacios de aproximación definidos por espacios de sucesiones de Lorentz pequeños. Para las aplicaciones, las siguientes relaciones entre los espacios $(X_p^\alpha)_q^{(0,\gamma)}$ y $X_q^{(\alpha,\eta)}$ son útiles.

Teorema 3.9. *Sean $\alpha > 0, 0 < p, q \le \infty$ y $\gamma > -1/q$. Entonces, se cumplen las siguientes inyecciones continuas*

$$X_q^{(\alpha,\gamma+1/\min\{p,q\})} \hookrightarrow (X_p^\alpha)_q^{(0,\gamma)} \hookrightarrow X_q^{(\alpha,\gamma+1/\max\{p,q\})}.$$

En la Observación 3.2 probamos que estas inyecciones son las mejores posibles.

Nótese que en el caso diagonal, donde $p = q$, ambas construcciones conmutan con el resultado que

$$(X_p^{(0,\gamma)})_p^\alpha = X_p^{(\alpha,\gamma+1/p)} = (X_p^\alpha)_p^{(0,\gamma)}.$$

En la segunda parte del capítulo, se aplican los resultados previos para estudiar varios problemas en espacios de Besov. Como se pone de manifiesto en los libros de Triebel [117, 118, 120], los espacios de Besov $\mathbf{B}_{p,q}^s$ tienen un papel central en la teoría de espacios de funciones. Sin embargo, para la solución completa de algunas cuestiones naturales como la compacidad de inyecciones límites o espacios definidos en fractales, espacios más generales se han introducido donde la regularidad de las funciones se considera de una forma más delicada que en $\mathbf{B}^s_{p,q}$. Estos espacios de regularidad generalizada han sido estudiados desde hace tiempo y con diferentes puntos de vista. Véase, por ejemplo, los artículos de DeVore, Riemenschneider y Sharpley [51], Gol'dman [71], Merucci [93], Kalyabin y Lizorkin [83], Cobos y Fernandez [32], Edmunds y Haroske [57], Leopold [90], Haroske y Moura [77], Cobos y Kühn [39] y los libros de Triebel [119, 120]. Aquí, trabajamos con espacios de Besov $\mathbf{B}^{s,b}_{p,q}$ de regularidad clásica s y regularidad logarítmica adicional de exponente b. El caso $s = 0$ es de un interés particular para nosotros. Los espacios $\mathbf{B}^{0,b}_{p,q}$ están cercanos a L_p pero poseen más propiedades que L_p debido a su regularidad logarítmica y su estructura de espacio de Besov. Cuando $s = 0$ únicamente el caso $b \geq -1/q$ es de interés porque si $b < -1/q$ entonces $\mathbf{B}^{0,b}_{p,q} = L_p$.

En la segunda parte del Capítulo 3, mejoramos algunos resultados de DeVore, Riemenschneider y Sharpley [51] para espacios $\mathbf{B}^{0,\gamma}_{p,q}$ definidos en el círculo unidad $\mathbb{T}$. Primero probamos en el Teorema 3.10 que para tener que $D^k f$ pertenece a $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ con $\gamma > -1/q$ es suficiente con que $f \in \mathbf{B}^{k,\gamma+1/\min\{2,p,q\}}_{p,q}(\mathbb{T})$. Para $p = 2$ y $2 \leq q < \infty$, este resultado es el mejor posible como se prueba en la Proposición 3.11. Después, investigamos la distribución de los coeficientes de Fourier de funciones de $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$. En el Teorema 3.12 obtenemos que si $1 \leq p \leq 2, 1/p+1/p' = 1, 0 < q \leq \infty, \gamma > -1/q$, y $f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ entonces $(\hat{f}(m))$ pertenece al espacio de sucesiones de Lorentz-Zygmund $\ell_{p',q}(\log \ell)_{\gamma+1/\max\{p',q\}}$. De nuevo, este resultado es el mejor posible para $p = 2$ y $0 < q \leq 2$. El capítulo se completa mostrando que el operador función conjugada H actúa de forma acotada de $\mathbf{B}^{0,\gamma+1}_{p,q}(\mathbb{T})$ en $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ para $\gamma > -1/q$.

Trabajando con espacios de aproximación, es natural estudiar operadores compactos. Este problema fue considerado por Fugarolas [69] y Almira y Luther [2]. Los resultados de [69] caracterizan los subconjuntos compactos de X^α_p para $p < \infty$, mientras que los resultados de [2] se refieren a operadores compactos pero únicamente en el caso Banach. En el Capítulo 4 continuamos esta investigación, prestando especial atención al caso límite. Nuestro método es diferente del dado en [2] y funciona en el caso cuasi Banach.

En el Teorema 4.1 establecemos una condición suficiente para que un operador lineal entre espacios de aproximación sea compacto. La prueba se basa en propiedades de interpolación de operadores compactos. Después consideramos otros tipos de resultados. A saber, probamos una condición necesaria y suficiente para que un operador lineal de un espacio de aproximación en un espacio cuasi Banach sea compacto (Teorema 4.2). También consideramos en los Teoremas 4.7 y 4.8 el caso de un operador actuando de un espacio cuasi Banach en un espacio de aproximación. Como consecuencia, se deduce la compacidad de las inyecciones

$$\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T}) \text{ y } \mathbf{B}^{0,\gamma+\epsilon}_{p,q}(\mathbb{T}) \hookrightarrow \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$$

para $0 < p, q \leq \infty, \gamma > -1/q$ y $\epsilon > 0$.

El Capítulo 5 está dedicado a inyecciones de espacios $\mathbf{B}^{s,b}_{p,q}$. En la Sección 5.1 comenzamos estudiando inyecciones de $\mathbf{B}^{s,b}_{p,q}(\mathbb{T})$ en espacios de funciones de Lorentz-Zygmund $L_{u,v}(\log L)_b(\mathbb{T})$. Si $p,q \geq 1$, esta cuestión fue previamente considerada por DeVore, Riemenschneider y Sharpley [51] usando interpolación de tipo débil. Nuestro método es diferente y cubre el rango completo de parámetros (ver Teoremas 5.2 y 5.3). Después consideramos el caso $s = 0$. Resultados previos sobre este problema se deben a Caetano, Gogatishvili y Opic [17] y Triebel [124]. Los resultados de [17] establecieron inyecciones locales óptimas del espacio $\mathbf{B}^{0,b}_{p,r}$ definido sobre $\mathbb{R}^d$ en $L_{p,q}(\log L)_\beta(\Omega)$ donde $1 \leq p < \infty, 1 \leq r \leq q \leq \infty, b > -1/r, \beta = b + 1/r + 1/\max\{p,q\} - 1/q$ y Ω es cualquier dominio acotado en $\mathbb{R}^d$. Los resultados de [124] se corresponden con inyecciones para otro tipo de espacios de Besov con regularidad próxima a cero. Nosotros consideramos espacios de Besov definidos en el círculo unidad y probamos inyecciones similares a las dadas en [17] las cuales cubren el rango completo de índices si $\beta > 0$, es decir, $0 < p < \infty$ y $0 < r \leq q \leq \infty$. Un rol clave en nuestros argumentos posee la desigualdad de Nikolskiĭ para polinomios trigonométricos y una variante suya. Las propiedades de extrapolación de los espacios $L_{p,q}(\log L)_\beta(\mathbb{T})$ son importantes en el caso $\beta > 0$ (ver Teorema 5.5). Sin embargo, si $\beta < 0$ la descripción de $L_{p,q}(\log L)_\beta(\mathbb{T})$ dada por extrapolación es diferente. Por esta razón, tenemos que emplear otro método basado en interpolación el cual funciona en el caso que $L_{p,q}(\log L)_\beta(\mathbb{T})$ es un espacio de Banach (Teoremas 5.6 y 5.7). Para eliminar esta última restricción, seguimos otra metodología, basada en interpolación límite y espacios del tipo de los espacios de Lebesgue pequeños (Teorema 5.10). También tratamos en el Teorema 5.14 el caso extremo $\mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T})$.

Los espacios $\mathbf{B}^{s,b}_{p,q}$ han sido introducidos usando el módulo de regularidad. Espacios similares $B^{s,b}_{p,q}$ pero introducidos a través del método analítico de Fourier han sido también estudiados en la literatura. Para espacios en $\mathbb{R}^d$, resulta que $\mathbf{B}^{s,b}_{p,q}(\mathbb{R}^d) = B^{s,b}_{p,q}(\mathbb{R}^d)$ si $1 \leq p \leq \infty$ y $s > 0$ (ver [77] y [117]). Sin embargo, si $s = 0$ la relación entre los espacios $\mathbf{B}^{0,b}_{p,q}$ y $B^{0,b}_{p,q}$ no ha sido aun estudiada. Este problema se menciona en el informe de Triebel [124] donde se establecen los primeros resultados sobre esta cuestión para espacios definidos en el cubo unidad (véase también [11]). En la Sección 5.2, se prueban las siguientes inyecciones entre estos espacios.

Teorema 5.16. *Supongamos que* $1 < p < \infty, 0 < q \leq \infty$ *y* $b > -1/q$. *Entonces*

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

En la demostración, se emplea que

$$K(t^k, f; L_p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d)) \sim \min\{1, t^k\}\|f\|_{L_p(\mathbb{R}^d)} + \omega_k(f,t)_p$$

y algunos resultados de interpolación abstracta.

En particular, para $p = q = 2$ y $b > -1/2$, deducimos que $\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d)$. Si $b = -1/2$ hay otro salto en la escala con el resultado que $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d) \neq B^{0}_{2,2}(\mathbb{R}^d)$. De hecho, $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d)$ coincide con el espacio $B^{0,0,1/2}_{2,2}(\mathbb{R}^d)$ definido por la transformada de Fourier y que tiene regularidad del tipo de un logaritmo iterado a la potencia 1/2. El caso general viene dado como sigue.

Teorema 5.18. *Sean $1 < p < \infty$ y $0 < q < \infty$. Entonces*

$$B^{0,1/\min\{2,p,q\}-1/q,1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,1/\max\{2,p,q\}-1/q,1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

También probamos

Teorema 5.22. *Sean $1 < p < \infty$ y $0 < q < \infty$. Entonces*

$$B^{0,0,1/q}_{p,\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,0,1/q}_{p,\max\{2,p,q\}}(\mathbb{R}^d).$$

Además, estudiamos los casos extremos $p = 1$ y $p = \infty$. Los resultados de esta sección también funcionan cuando los espacios de funciones están definidos en el toro d-dimensional $\mathbb{T}^d$.

En la Sección 5.3 se consideran inyecciones entre espacios de Besov con diferentes métricas (Teorema 5.26) y en la Sección 5.4 se estudia la conexión entre los espacios de Besov $B^{1,b}_{p,q}(\mathbb{R}^d)$ y los espacios de Lipschitz logarítmicos $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$. Este problema fue ya considerado por Haroske [75, 76] y Neves [96], entre otros autores. Nuestro método nos permite cubrir algunos casos críticos que surgen de las técnicas empleadas en [75]. Como consecuencia, complementamos y mejoramos resultados anteriores de Haroske [75] en el Teorema 5.31 y el Corolario 5.32.

Sean A_0, A_1 espacios cuasi Banach con $A_1 \hookrightarrow A_0$ y $0 < q \leq \infty, -\infty < \eta < \infty$ y $\theta = 1$ o $\theta = 0$. El espacio de interpolación real límite consiste en todos aquellos $a \in A_0$ que tienen una cuasi norma finita

$$\|a\|_{(A_0,A_1)_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta(1-\log t)^\eta}\right)^q \frac{dt}{t}\right)^{1/q}.$$

Esta construcción produce espacios muy próximos a A_1 si $\theta = 1$ y a A_0 si $\theta = 0$. En los capítulos anteriores hemos establecido varios resultados sobre espacios de interpolación límites. Algunas aplicaciones de estos resultados se dan en el Capítulo 6. Se comienza determinando el espacio dual de $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ para $1 < p < \infty, 1 \leq q < \infty$ y $b > -1/q$. Esto se realiza con la ayuda de los espacios de Lipschitz $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$ y el operador $I_{-1}f = \mathcal{F}^{-1}(1+|x|^2)^{-1/2}\mathcal{F}f$. El resultado obtenido es el siguiente.

Teorema 6.2. *Sean $1 < p < \infty, 1 \leq q < \infty, 1/p + 1/p' = 1 = 1/q + 1/q'$ y $b > -1/q$. El espacio dual $(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'$ consiste en todas las distribuciones f perteneciendo al espacio de Sobolev $H^{-1}_{p'}(\mathbb{R}^d)$ tal que $I_{-1}f \in \mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)$. Además*

$$\|f\|_{(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'} \sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)}.$$

La Sección 6.2 está dedicada a la distribución de los coeficientes de Fourier de funciones en varios espacios. En el Teorema 6.3 se estudia el caso límite cuando f pertenece a $\mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T})$ que no está cubierto por el Teorema 3.12. Después, en los Teoremas 6.4, 6.5 y 6.6, se considera el caso de funciones en espacios cercanos a $L_1(\mathbb{T})$ y a $L_2(\mathbb{T})$. Los resultados extienden estimaciones previas de Hardy y Littlewood y de Bennett para funciones en $L(\log L)_\gamma(\mathbb{T})$ [7] y de Cobos y Segurado para funciones en $L_2(\log L)_{-1/2}(\mathbb{T})$ [46].

El capítulo se concluye probando en el Teorema 6.7 condiciones suficientes para que $D^k f$ pertenezca a $\mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T})$, lo que cierra el caso límite que quedó abierto en el Teorema 3.10.

El Capítulo 7 trata sobre inyecciones de dimensión controlable (tratable) de espacios de Besov sobre el toro d-dimensional en espacios de Lebesgue pequeños. Las inyecciones tratables pueden ser consideradas como una formulación reciente de las así llamadas desigualdades de Sobolev logarítmicas. Estas últimas tienen una larga historia y aplicaciones con un amplio alcance. Véase, por ejemplo, los trabajos de Gross [73], Davies [49], Beckner [5], Cianchi y Pick [23] y Martín y Milman [92]. En particular, la siguiente desigualdad se prueba en [92],

$$\left(\int_0^1 (1-\log t)^{p/2} f^*(t)^p dt\right)^{1/p} \le c\big(\|f\|_{L_p(\mathbb{R}^d)} + \|\nabla f\|_{L_p(\mathbb{R}^d)}\big) \tag{1}$$

para toda $f \in W^{1,p}(\mathbb{R}^d)$ con soporte contenido en el cubo unidad $[0,1]^d$ de $\mathbb{R}^d$, donde la constante $c > 0$ es independiente de f y de la dimensión $d \in \mathbb{N}$. Aquí, $1 \le p < d$ y f^* denota la reordenada no creciente de f. Esto implica la inyección invariante por dimensión del espacio de Sobolev $W_0^{1,p}([0,1]^d)$ en el espacio de Zygmund $L_p(\log L)_{1/2}([0,1]^d)$. Véase también los artículos de Krbec y Schmeisser [88, 89] donde se han propuesto diferentes métodos para conseguir desigualdades de Sobolev logarítmicas con constantes independientes de la dimensión d. Muy recientemente, Fiorenza, Krbec y Schmeisser [68] han mejorado estos resultados empleando los espacios de Lebesgue pequeños.

La versión fraccional de la desigualdad (1) correspondiente a los espacios de Besov $\mathbf{B}_{p,q}^{\alpha}(\mathbb{R}^d)$ con $\alpha > 0$ fue estudiada por Triebel [123, 126]. Él probó que si $1 < p < \infty, 0 < \alpha < M \in \mathbb{N}$, entonces

$$\left(\int_0^1 (1-\log t)^{\alpha p} f^*(t)^p dt\right)^{1/p} \le 2^{\rho d}\left[\|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} \frac{\|\Delta_h^M f\|_{L_p(\mathbb{R}^d)}^p}{|h|^{\alpha p}} \frac{dh}{|h|^d}\right)^{1/p}\right] \tag{2}$$

para toda $f \in \mathbf{B}_{p,p}^{\alpha}(\mathbb{R}^d)$ con soporte dentro de $[0,1]^d$, donde $\rho > 0$ es independiente de la dimensión d. Aquí $|\cdot|$ denota la norma euclídea usual en $\mathbb{R}^d$. Las inyecciones (1) y (2) son también de interés en el campo de la información basada en la complejidad [98, 99]. Nótese que factores del tipo $2^{\rho d}$ con ρ independiente de d se pueden incorporar a la norma de Besov subyacente por un reescalamiento adecuado de la distancia $|x| \to \kappa|x|, \kappa > 0$, independiente de la dimensión.

Es importante resaltar que los resultados de tratabilidad son muy sensibles con respecto a las normas escogidas porque las constantes de equivalencia podrían depender de la dimensión d.

En este capítulo avanzamos en diferentes direcciones. Antes de nada, centrándonos en el caso de espacios de Besov periódicos $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$, en el Teorema 7.9 y el Corolario 7.10 somos capaces de escoger una norma clásica y natural para reducir la constante exponencial con respecto a d en una polinomial. Además, en el Teorema 7.12 extendemos (2) para $p \neq q$ con una inyección en espacios de Lebesgue pequeños adecuadamente elegidos. En el caso especial $p = q$, recuperamos la inyección esperada en un espacio de Zygmund con una dependencia polinomial respecto de d en la constante. Más aun, se tiene en cuenta la influencia del volumen de la bola unidad $|\mathbb{B}^d_r|$ en $\mathbb{T}^d$ relativa a la cuasi norma $|h|_r = \left(\sum_{l=1}^d |h_l|^r\right)^{1/r}$ si $0 < r < \infty$ y $|h|_\infty = \max_{l=1,\dots,d} |h_l|$ para $h = (h_1, \dots, h_d) \in \mathbb{T}^d$. En particular, se deduce el siguiente corolario.

Corolario 7.13. *Sean $\alpha > 0$ y $1 \leq p < \infty$. Sean $0 < r \leq \infty$ y $M \in \mathbb{N}$ con $M > \alpha$. Entonces existen una constante $c > 0$ y un radio $R > 0$ que son independientes de d tales que*

$$\left(\int_0^1 (1 - \log t)^{\alpha p} f^*(t)^p dt\right)^{1/p} \leq c \left[d^{\alpha+1} \|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r \leq 1} \frac{\|\Delta_h^M f\|^p_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha p + d}} \frac{dh}{R^d |\mathbb{B}^d_r|} \right)^{1/p} \right]$$

para todo $f \in \mathbf{B}^{\alpha}_{p,p}(\mathbb{T}^d)$.

En el Teorema 7.16 y el Corolario 7.17 estudiamos inyecciones tratables para los espacios de Besov $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$. Como se indica en la Observación 7.5, la regularidad logarítmica es de hecho suficiente para establecer algunas inyecciones tratables para los espacios de Besov clásicos $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$.

Además de espacios de Besov, espacios de Lizorkin-Triebel y sus extensiones, para tratar algunas situaciones límites, uno también necesita otros tipos de espacios como son los espacios de Sobolev logarítmicos $H^s_p(\log H)_b$, que se introducen tomando como modelo la descripción (por extrapolación) de los espacios de Zygmund $L_p(\log L)_b$ en términos de los espacios de Lebesgue L_p más simples. Véase el libro de Edmunds y Triebel [59, 2.6]. En una forma más abstracta, este método resulta ser también útil combinado con ideas de la teoría de interpolación, como se pone de manifiesto en los artículos [60, 36, 34].

En el Workshop que tuvo lugar en el Departamento de Análisis Matemático de la Universidad Complutense de Madrid en Septiembre de 2014, el Profesor H.-J. Schmeisser propuso estudiar si los espacios de Besov $B^s_p(L_p(\log L)_b)$ sobre $\mathbb{T}^d$ modelados en espacios de Zygmund $L_p(\log L)_b$ (ver Definición 8.1) admiten una descripción en términos de los espacios de Besov clásicos. La respuesta se da en el Capítulo 8 de esta memoria. Más precisamente, en los Teoremas 8.6 y 8.8. Durante el proceso de la prueba, establecemos diversas propiedades de los espacios $B^s_p(L_p(\log L)_b)$ que son de interés independiente, tales como las caracterizaciones por métodos de interpolación límites dadas en el Teorema 8.5 o la descripción de sus duales dada en los Teoremas 8.7 y 8.9. También se prueban las siguientes relaciones con los espacios $B^{s,b}_{p,p}$.

Teorema 8.10. *Sean $1 < p < \infty$ y $s \in \mathbb{R}$.*

(i) *Si $b < 0$ entonces $B^s_p(L_p(\log L)_b(\mathbb{T}^d)) \hookrightarrow B^{s,b}_{p,p}(\mathbb{T}^d)$.*

(ii) *Si $b > 0$ entonces $B^{s,b}_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_p(L_p(\log L)_b(\mathbb{T}^d))$.*

En general, $B^s_p(L_p(\log L)_b(\mathbb{T}^d))$ no coincide con $B^{s,b}_{p,p}(\mathbb{T}^d)$ como probamos en el Contraejemplo 8.1.

En la última sección del capítulo tratamos el caso crítico $s = d/p$ y probamos en el Teorema 8.13 que el espacio $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))$ está inyectado en el espacio de las funciones continuas $C(\mathbb{T}^d)$ si y solamente si $b > 1 - 1/p$.

Como se pone de manifiesto en los libros de Triebel [116, 117, 118, 120, 121], se han obtenido numerosas caracterizaciones para los espacios de Besov $B^s_{p,q}$ en términos de diferencias, núcleos del calor o wavelets, entre otras medias. En el último capítulo de esta memoria estudiamos las caracterizaciones correspondientes para los espacios $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Los resultados son nuevos incluso para los espacios $\mathbf{B}^{0}_{p,q}(\mathbb{R}^d)$. También comparamos los resultados con las caracterizaciones conocidas para los espacios clásicos $B^s_{p,q}(\mathbb{R}^d)$.

En el Teorema 9.1 probamos una caracterización mediante diferencias. Después estudiamos la descomposición analítica de Fourier de $\mathbf{B}^{0,b}_{p,q}$ usando la partición diádica de la unidad usual $(\varphi_j)_{j\in\mathbb{N}_0}$ y la transformada de Fourier. El resultado obtenido es el siguiente.

Teorema 9.7. *Sean $1 < p < \infty, 0 < q \le \infty$ y $b > -1/q$. Entonces, $f \in L_p(\mathbb{R}^d)$ pertenece a $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ si y solamente si*

$$\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty}\left[(1+j)^b\left\|\left(\sum_{\nu=j}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)(\cdot)|^2\right)^{1/2}\right\|_{L_p(\mathbb{R}^d)}\right]^q\right)^{1/q} < \infty.$$

Además, $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ es una cuasi norma equivalente en $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Comparando el Teorema 9.7 con el resultado correspondiente para espacios de Besov clásicos $B^s_{p,q}(\mathbb{R}^d)$, se observa la construcción de Littlewood-Paley truncada adicional que aparece en $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$. Empleando la cuasi norma $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$, se prueba en la Observación 9.3 la razón para el salto en la escala en las igualdades $\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d)$ si $b > -1/2$, mientras $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d) = B^{0,0,1/2}_{2,2}(\mathbb{R}^d)$.

La construcción de Littlewood-Paley truncada también aparece en la caracterización de $\mathbf{B}^{0,b}_{p,q}$ por medio de wavelets que obtenemos en el Teorema 9.12. Usando esta descripción por wavelets, probamos en la Observación 9.6 que, en general, las inyecciones dadas en el Teorema 5.16 son las mejores posibles. Para ser más precisos, si tomamos cualquier $\varepsilon > 0$, tenemos que $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/p+\varepsilon}_{p,q}(\mathbb{R}^d)$ si $p = \max\{2,p,q\}$ y $B^{0,b+1/p-\varepsilon}_{p,q}(\mathbb{R}^d) \not\hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ si $p = \min\{2,p,q\}$.

La última sección de la memoria contiene las caracterizaciones en términos de semigrupos de operadores. Esta vez la construcción de Littlewood-Paley truncada no aparece. En el Teorema 9.13 probamos un resultado abstracto sobre semigrupos de operadores equiacotados fuertemente continuos sobre un espacio de Banach. Después, en el Teorema 9.14, lo particularizamos a los núcleos del calor y en el Teorema 9.15 al caso del semigrupo de Cauchy-Poisson. También se obtienen caracterizaciones de los espacios $B^{0,b}_{p,q}(\mathbb{R}^d)$ en términos de semigrupos. Véase Teoremas 9.16, 9.17 y 9.18.

Parte de los resultados han aparecido en los artículos conjuntos [30], [25], [26], [27], [28], [31] y en mi artículo [55]. Algunos otros resultados forman la prepublicación [29].

Chapter 1

Introduction

In this monograph, we go deeply into the connection between approximation spaces and limiting interpolation methods, applying the results to problems on function spaces and, in particular, on Besov spaces.

The symbiotic relationship between approximation theory and interpolation theory is well known and can be seen in the books by Bergh and Löfström [10], Triebel [116], Petrushev and Popov [103] and DeVore and Lorentz [50]. The real interpolation method $(A_0, A_1)_{\theta,q}$ plays an important role in this matter. It is mainly realized by means of the Peetre's K-functional

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1.$$

Usually $0 < \theta < 1$ but to cover some extreme cases, limiting versions have been also used, where $\theta = 0, 1$ and logarithmic weights may be included. See the papers by Gomez and Milman [72], Evans and Opic [61], Evans, Opic and Pick [63], Cobos, Fernández-Cabrera, Kühn and Ullrich [33], Cobos and Kühn [40] and Edmunds and Opic [58].

Given a quasi-Banach space X and an approximation family $(G_n)_{n\in\mathbb{N}_0}$ of subsets of X, approximation spaces X_p^α are defined by selecting those elements f of X such that $(n^{\alpha-1/p}E_n(f))$ belongs to ℓ_p. Here parameters satisfy $0 < \alpha < \infty$ and $0 < p \le \infty$ and $E_n(f) = \inf\{\|f-g\|_X : g \in G_{n-1}\}$ is the error of best approximation to f by the elements of G_{n-1} (see Section 2.1 for full details). These spaces have been studied by Butzer and Scherer [16], Pietsch [104, 105], Petrushev and Popov [103], and DeVore and Lorentz [50], among other authors. Limiting approximation spaces $X_q^{(0,\gamma)}$ are defined by doing $\alpha = 0$ and inserting the weight $(1+\log n)^\gamma$ with the sequence $(E_n(f))$. They have been investigated by Cobos and Resina [44], Cobos and Milman [41], Cobos and Kühn [38], Fehér and Grässler [65] and the references given in these papers. As it is shown in [44], even when $\gamma = 0$, the theory of limiting approximation spaces does not follow from the theory of spaces X_p^α by taking $\alpha = 0$. Spaces X_p^α and $X_q^{(0,\gamma)}$ allow to establish in an elegant and clear way a number of important results on function spaces, sequence spaces and spaces of operators.

We start this monograph by fixing notation, introducing the main concepts and constructions, as well as some new results in Chapter 2. Then in Chapter 3 we study reiteration of approximation constructions. As it was shown in [104, Theorem 3.2], the construction $(\cdot)_p^\alpha$ is stable by iteration. A similar property holds for $(\cdot)_q^{(0,\gamma)}$ [65, Theorem 2]. We study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa. We show that, outside the case where $p = q$, the constructions do not commute. First we prove

Theorem 3.2. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Suppose that $\alpha > 0, 0 < p, q \le \infty$ and $\gamma > -1/q$. Then an element $f \in X$ belongs to $(X_q^{(0,\gamma)})_p^\alpha$ if and only if $(n^{\alpha-1/p}(1+\log n)^{\gamma+1/q}E_n(f))$ belongs to ℓ_p.*

We write $X_p^{(\alpha,\gamma+1/q)}$ for the resulting approximation space in Theorem 3.2. The space $(X_p^\alpha)_q^{(0,\gamma)}$ has a different shape. It is characterized in Theorem 3.6 by using approximation spaces defined by small Lorentz sequence spaces. For applications, it is useful the following relationships between spaces $(X_p^\alpha)_q^{(0,\gamma)}$ and $X_q^{(\alpha,\eta)}$.

Theorem 3.9. *Let $\alpha > 0, 0 < p, q \le \infty$ and $\gamma > -1/q$. Then the following continuous embeddings hold*

$$X_q^{(\alpha,\gamma+1/\min\{p,q\})} \hookrightarrow (X_p^\alpha)_q^{(0,\gamma)} \hookrightarrow X_q^{(\alpha,\gamma+1/\max\{p,q\})}.$$

These embeddings are the best possible as we show in Remark 3.2.

Note that in the diagonal case, where $p = q$, constructions commute with the result that

$$(X_p^{(0,\gamma)})_p^\alpha = X_p^{(\alpha,\gamma+1/p)} = (X_p^\alpha)_p^{(0,\gamma)}.$$

In the second part of the chapter, we apply the previous results to study several problems on Besov spaces. As it is shown in the books by Triebel [117, 118, 120], Besov spaces $\mathbf{B}_{p,q}^s$ play a central role in the theory of function spaces. However, for the complete solution of some natural questions as compactness in limiting embeddings or spaces on fractals, more general spaces have been introduced where smoothness of functions is considered in a more delicate manner than in $\mathbf{B}_{p,q}^s$. These spaces of generalized smoothness have been studied for long and from different points of view.

See, for example, the papers by DeVore, Riemenschneider and Sharpley [51], Gol'dman [71], Merucci [93], Kalyabin and Lizorkin [83], Cobos and Fernandez [32], Edmunds and Haroske [57], Leopold [90], Haroske and Moura [77], Cobos and Kühn [39] and the books by Triebel [119, 120]. We work here with Besov spaces $\mathbf{B}^{s,b}_{p,q}$ which have classical smoothness s and additional logarithmic smoothness with exponent b. The case $s = 0$ is of particular interest for us. Spaces $\mathbf{B}^{0,b}_{p,q}$ are close to L_p but they have special properties than L_p due to their logarithmic smoothness and their structure of Besov spaces. When $s = 0$ only the case $b \geq -1/q$ is of interest because if $b < -1/q$ then $\mathbf{B}^{0,b}_{p,q} = L_p$.

In the second part of Chapter 3 we improve some results of DeVore, Riemenschneider and Sharpley [51] on spaces $\mathbf{B}^{0,\gamma}_{p,q}$ on the unit circle. First we show in Theorem 3.10 that in order to $D^k f$ belongs to $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ with $\gamma > -1/q$ it suffices that $f \in \mathbf{B}^{k,\gamma+1/\min\{2,p,q\}}_{p,q}(\mathbb{T})$. For $p = 2$ and $2 \leq q < \infty$, this result is the best possible as we show in Proposition 3.11. Then we investigate the distribution of Fourier coefficients of functions of $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$. In Theorem 3.12 we obtain that if $1 \leq p \leq 2, 1/p + 1/p' = 1, 0 < q \leq \infty, \gamma > -1/q$, and $f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ then $(\hat{f}(m))$ belongs to the Lorentz-Zygmund sequence space $\ell_{p',q}(\log \ell)_{\gamma+1/\max\{p',q\}}$. Again, this result is best possible for $p = 2$ and $0 < q \leq 2$. We complete the chapter by showing that the conjugate-function operator H acts boundedly from $\mathbf{B}^{0,\gamma+1}_{p,q}(\mathbb{T})$ into $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ for $\gamma > -1/q$.

Working with approximation spaces, it is natural to study compact operators. This problem was considered by Fugarolas [69] and Almira and Luther [2]. Results of [69] characterize compact subsets of X^α_p for $p < \infty$, while results of [2] refer to compact operators but just in the Banach case. In Chapter 4 we continue this research, paying special attention to the limit case. Our approach is different from the one of [2] and it works in the quasi-Banach case.

In Theorem 4.1 we establish a sufficient condition for a linear operator between approximation spaces to be compact. The proof is based on interpolation properties of compact operators. Then we consider other types of results. Namely, we show a necessary and sufficient condition for a linear operator from an approximation space into a quasi-Banach space to be compact (Theorem 4.2). We also consider in Theorems 4.7 and 4.8 the case of an operator acting from a quasi-Banach space into an approximation space. As a consequence, we derive compactness of the embeddings

$$\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T}) \text{ and } \mathbf{B}^{0,\gamma+\epsilon}_{p,q}(\mathbb{T}) \hookrightarrow \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$$

for $0 < p, q \leq \infty, \gamma > -1/q$ and $\epsilon > 0$.

Chapter 5 is devoted to embeddings of spaces $\mathbf{B}^{s,b}_{p,q}$. In Section 5.1 we start by studying embeddings of $\mathbf{B}^{s,b}_{p,q}(\mathbb{T})$ into Lorentz-Zygmund function spaces $L_{u,v}(\log L)_b(\mathbb{T})$. If $p, q \geq 1$, this question was already considered by DeVore, Riemenschneider and Sharpley [51] by using weak type interpolation. Our approach is different and covers the whole range of parameters (see Theorems 5.2 and 5.3). Then we consider the case $s = 0$. Previous results on this problem are due to Caetano, Gogatishvili and Opic [17] and Triebel [124]. Results of [17] established sharp local embeddings from the space $\mathbf{B}^{0,b}_{p,r}$ defined on $\mathbb{R}^d$ into $L_{p,q}(\log L)_\beta(\Omega)$ where $1 \le p < \infty, 1 \le r \le q \le \infty, b > -1/r, \beta = b + 1/r + 1/\max\{p,q\} - 1/q$ and Ω is any bounded domain in $\mathbb{R}^d$. Results of [124] correspond to embeddings from other kinds of Besov spaces with smoothness close to zero. We work with Besov spaces defined on the unit circle and prove similar embeddings to those of [17] which cover the full range of indices if $\beta > 0$, that is, $0 < p < \infty$ and $0 < r \le q \le \infty$. A key role in our arguments is played by the Nikolskiĭ inequality for trigonometric polynomials and a variant of it. Extrapolation properties of spaces $L_{p,q}(\log L)_\beta(\mathbb{T})$ are important in the case $\beta > 0$ (see Theorem 5.5). However, if $\beta < 0$ the description of $L_{p,q}(\log L)_\beta(\mathbb{T})$ via extrapolation is different. For this reason we have to use another approach based on interpolation which works in the case when $L_{p,q}(\log L)_\beta(\mathbb{T})$ is a Banach space (Theorems 5.6 and 5.7). To overcome this last restriction, we follow another way, based on limiting interpolation and spaces of kind of the small Lebesgue spaces (Theorem 5.10). We also cover in Theorem 5.14 the extreme case $\mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T})$.

Spaces $\mathbf{B}^{s,b}_{p,q}$ have been introduced by using the modulus of smoothness. Similar spaces $B^{s,b}_{p,q}$ but introduced by following the Fourier-analytical approach have been also studied in the literature. For spaces on $\mathbb{R}^d$, it turns out that $\mathbf{B}^{s,b}_{p,q}(\mathbb{R}^d) = B^{s,b}_{p,q}(\mathbb{R}^d)$ if $1 \le p \le \infty$ and $s > 0$ (see [77] and [117]). However if $s = 0$ the relationship between spaces $\mathbf{B}^{0,b}_{p,q}$ and $B^{0,b}_{p,q}$ has not been studied yet. This problem is mentioned in the report by Triebel [124] where the first results on this question are given for spaces defined on the unit cube (see also [11]). In Section 5.2 we establish the following embeddings between these spaces.

Theorem 5.16. *Assume that $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Then*

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

In the proof we use that

$$K(t^k, f; L_p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d)) \sim \min\{1, t^k\}\|f\|_{L_p(\mathbb{R}^d)} + \omega_k(f,t)_p$$

and some abstract interpolation results.

In particular, for $p = q = 2$ and $b > -1/2$, we derive that $\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d)$. If $b = -1/2$ there is another jump in the scale with the result that $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d) \ne B^{0}_{2,2}(\mathbb{R}^d)$. In fact, $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d)$ coincides with the space $B^{0,0,1/2}_{2,2}(\mathbb{R}^d)$ defined by the Fourier transform and having smoothness of the type of an iterated logarithm to the power $1/2$. The general case reads as follows.

Theorem 5.18. *Let $1 < p < \infty$ and $0 < q < \infty$. Then*

$$B^{0,1/\min\{2,p,q\}-1/q,1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,1/\max\{2,p,q\}-1/q,1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

We also show

Theorem 5.22. *Let $1 < p < \infty$ and $0 < q < \infty$. Then*

$$B^{0,0,1/q}_{p,\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,0,1/q}_{p,\max\{2,p,q\}}(\mathbb{R}^d).$$

Moreover, we study the extreme cases $p = 1$ and $p = \infty$. Results of this section also work when the function spaces are defined on the d-dimensional torus $\mathbb{T}^d$.

In Section 5.3 we consider embeddings between Besov spaces with different metrics (Theorem 5.26) and in Section 5.4 we study the connection between Besov spaces $B^{1,b}_{p,q}(\mathbb{R}^d)$ and logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$. This problem was considered by Haroske [75, 76] and Neves [96] among other authors. Our approach allows us to cover some critical cases which come up for the techniques used in [75]. As a consequence, we complement and improve previous results of Haroske [75] in Theorem 5.31 and Corollary 5.32.

Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$ and let $0 < q \le \infty, -\infty < \eta < \infty$ and $\theta = 1$ or $\theta = 0$. The limiting real interpolation space consists of all those $a \in A_0$ having a finite quasi-norm

$$\|a\|_{(A_0,A_1)_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta(1-\log t)^\eta}\right)^q \frac{dt}{t}\right)^{1/q}.$$

This construction produces spaces very close to A_1 if $\theta = 1$ and to A_0 if $\theta = 0$. In the previous chapters we have established several results on limiting interpolation spaces. Some applications of those results are given in Chapter 6. We start by determining the dual space of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ for $1 < p < \infty, 1 \le q < \infty$ and $b > -1/q$. This is done with the help of Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$ and the lift operator $I_{-1}f = \mathcal{F}^{-1}(1+|x|^2)^{-1/2}\mathcal{F}f$. The result reads as follows.

Theorem 6.2. *Let $1 < p < \infty, 1 \le q < \infty, 1/p + 1/p' = 1 = 1/q + 1/q'$ and $b > -1/q$. The dual space $(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'$ consists of all distributions f belonging to the Sobolev space $H^{-1}_{p'}(\mathbb{R}^d)$ such that $I_{-1}f \in \mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)$. Moreover*

$$\|f\|_{(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'} \sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)}.$$

Section 6.2 is devoted to the distribution of Fourier coefficients of functions in several spaces. In Theorem 6.3 we study the limit case when f belongs to $\mathbf{B}^{0,-1/q}_{p,q}(\mathbb{T})$ which is not covered by Theorem 3.12. Then, in Theorems 6.4, 6.5 and 6.6, we consider the case of functions in spaces close to $L_1(\mathbb{T})$ and to $L_2(\mathbb{T})$. The results extend previous estimates of Hardy and Littlewood and of Bennett for functions in $L(\log L)_\gamma(\mathbb{T})$ [7] and of Cobos and Segurado for functions in $L_2(\log L)_{-1/2}(\mathbb{T})$ [46].

We end the chapter by showing in Theorem 6.7 sufficient conditions for $D^k f$ to belongs to $\mathbf{B}^{0,-1/q}_{p,q}(\mathbb{T})$ which close the limit case left open in Theorem 3.10.

Chapter 7 deals with dimension-controllable (tractable) embeddings of Besov spaces on d-dimensional torus into small Lebesgue spaces. Tractable embeddings can be considered as a more recent formulation of the so-called logarithmic Sobolev inequalities. These latter have a long history and far-reaching applications. See, for example, the works by Gross [73], Davies [49], Beckner [5], Cianchi and Pick [23] and Martín and Milman [92]. In particular, the following inequality was proved in [92],

$$\left(\int_0^1 (1-\log t)^{p/2} f^*(t)^p dt\right)^{1/p} \le c\left(\|f\|_{L_p(\mathbb{R}^d)} + \|\nabla f\|_{L_p(\mathbb{R}^d)}\right) \tag{1.1}$$

for all $f \in W^{1,p}(\mathbb{R}^d)$ with support contained in the unit cube $[0,1]^d$ of $\mathbb{R}^d$, where the constant $c > 0$ is independent of f and the dimension $d \in \mathbb{N}$. Here $1 \le p < d$ and f^* denotes the non-increasing rearrangement of f. This yields the dimension-invariant embedding from Sobolev space $W^{1,p}_0([0,1]^d)$ into Zygmund space $L_p(\log L)_{1/2}([0,1]^d)$. See also the papers by Krbec and Schmeisser [88, 89] where different approaches to get logarithmic Sobolev inequalities with constant independent of dimension d were proposed. Very recently, Fiorenza, Krbec and Schmeisser [68] have improved these results by using small Lebesgue spaces.

The fractional version of inequality (1.1) corresponding to Besov spaces $\mathbf{B}^\alpha_{p,p}(\mathbb{R}^d)$ with $\alpha > 0$ was studied by Triebel [123, 126]. He showed that if $1 < p < \infty, 0 < \alpha < M \in \mathbb{N}$, then

$$\left(\int_0^1 (1-\log t)^{\alpha p} f^*(t)^p dt\right)^{1/p} \le 2^{\rho d}\left[\|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} \frac{\|\Delta_h^M f\|^p_{L_p(\mathbb{R}^d)}}{|h|^{\alpha p}}\frac{dh}{|h|^d}\right)^{1/p}\right] \tag{1.2}$$

for all $f \in \mathbf{B}^\alpha_{p,p}(\mathbb{R}^d)$ with support inside $[0,1]^d$, where $\rho > 0$ is independent of the dimension d. Here $|\cdot|$ stands for the usual Euclidean norm in $\mathbb{R}^d$. Embeddings (1.1) and (1.2) are also of interest in the field of information based complexity [98, 99]. Note that factors of type $2^{\rho d}$ with ρ independent of d can be incorporated in the underlying Besov-norm by a suitable dimension-independent rescaling of the distance $|x| \to \kappa|x|, \kappa > 0$.

It is important to remark that tractability results are very sensitive with respect to the chosen norms because equivalence constants may depend on the dimension d.

In this chapter, we are able to proceed in different directions. First of all, by focussing on the case of periodic Besov spaces $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$, in Theorem 7.9 and Corollary 7.10 we are able to choose a natural and classical norm to reduce the exponential constant with respect to d to a polynomial one. Furthermore, in Theorem 7.12 we extend (1.2) to $p \neq q$ with an embedding into suitably chosen small Lebesgue spaces. In the special case $p = q$, we recover the expected embedding into a Zygmund space with a polynomial dependence on d in the constant. In addition, we take into account the influence of the volume of the unit ball $|\mathbb{B}^d_r|$ in $\mathbb{T}^d$ associated to the quasi-norm $|h|_r = \left(\sum_{l=1}^d |h_l|^r\right)^{1/r}$ if $0 < r < \infty$ and $|h|_\infty = \max_{l=1,\dots,d} |h_l|$ for $h = (h_1, \dots, h_d) \in \mathbb{T}^d$. In particular, we derive the next corollary.

Corollary 7.13. *Let $\alpha > 0$ and $1 \le p < \infty$. Let $0 < r \le \infty$ and $M \in \mathbb{N}$ with $M > \alpha$. Then, there exist a constant $c > 0$ and a radius $R > 0$ which are independent of d such that*

$$\left(\int_0^1 (1 - \log t)^{\alpha p} f^*(t)^p dt\right)^{1/p} \le c \left[d^{\alpha+1} \|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r \le 1} \frac{\|\Delta_h^M f\|^p_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha p + d}} \frac{dh}{R^d |\mathbb{B}^d_r|}\right)^{1/p}\right]$$

for all $f \in \mathbf{B}^{\alpha}_{p,p}(\mathbb{T}^d)$.

In Theorem 7.16 and Corollary 7.17 we study tractable embeddings for Besov spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$. As we point out in Remark 7.5, logarithmic smoothness is actually enough to establish some tractable embeddings for classical Besov spaces $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$.

Besides Besov spaces, Triebel-Lizorkin spaces and their extensions, to deal with some limiting situations, one also needs other types of spaces as logarithmic Sobolev spaces $H^s_p(\log H)_b$, which are introduced by taking as a model the description (by extrapolation) of Zygmund spaces $L_p(\log L)_b$ in terms of the more simple Lebesgue spaces L_p. See the book by Edmunds and Triebel [59, 2.6]. In a more abstract way, this approach is also useful combined with ideas of interpolation theory as can be seen in the papers [60, 36, 34].

In the Workshop held at the Department of Mathematical Analysis of Universidad Complutense de Madrid in September 2014, Professor H.-J. Schmeisser suggested to study whether Besov spaces $B^s_p(L_p(\log L)_b)$ over $\mathbb{T}^d$ modelled on Zygmund spaces $L_p(\log L)_b$ (see Definition 8.1) admit a description in terms of classical Besov spaces. The answer is given in Chapter 8 of this monograph. Namely, in Theorems 8.6 and 8.8. In the process of the proof we establish a number of properties of the spaces $B^s_p(L_p(\log L)_b)$ which have independent interest, as the characterizations by limiting interpolation methods given in Theorem 8.5 or the description of their duals given in Theorem 8.7 and 8.9. We also show the following relationship with spaces $B^{s,b}_{p,p}$.

Theorem 8.10. *Let $1 < p < \infty$ and $s \in \mathbb{R}$.*

(i) *If* $b < 0$ *then* $B^s_p(L_p(\log L)_b(\mathbb{T}^d)) \hookrightarrow B^{s,b}_{p,p}(\mathbb{T}^d)$.

(ii) *If* $b > 0$ *then* $B^{s,b}_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_p(L_p(\log L)_b(\mathbb{T}^d))$.

In general, $B^s_p(L_p(\log L)_b(\mathbb{T}^d))$ does not coincide with $B^{s,b}_{p,p}(\mathbb{T}^d)$ as we show in Counterexample 8.1.

In the last section of the chapter we work with the critical case $s = d/p$ and we show in Theorem 8.13 that the space $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))$ is embedded in the space of continuous functions $C(\mathbb{T}^d)$ if and only if $b > 1 - 1/p$.

As one can see in the books by Triebel [116, 117, 118, 120, 121], numerous characterizations have been obtained for Besov spaces $B^s_{p,q}$ in terms of differences, heat kernels or wavelets, among other means. In the last chapter of this monograph we study the corresponding characterizations for spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Results are new even for spaces $\mathbf{B}^0_{p,q}(\mathbb{R}^d)$. We also compare the results with the known characterizations for classical spaces $B^s_{p,q}(\mathbb{R}^d)$.

In Theorem 9.1 we show a characterization by differences. Then we study the Fourier-analytical decomposition of $\mathbf{B}^{0,b}_{p,q}$ using the usual dyadic resolution of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ and the Fourier transform. The result reads as follows.

Theorem 9.7. *Let* $1 < p < \infty, 0 < q \le \infty$ *and* $b > -1/q$. *Then* $f \in L_p(\mathbb{R}^d)$ *belongs to* $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ *if and only if*

$$\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty}\left[(1+j)^b \left\|\left(\sum_{\nu=j}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(\cdot)|^2\right)^{1/2}\right\|_{L_p(\mathbb{R}^d)}\right]^q\right)^{1/q} < \infty.$$

Furthermore, $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ *is an equivalent quasi-norm on* $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Comparing Theorem 9.7 with the corresponding result for classical Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, one observes the additional truncated Littlewood-Paley construction that appears in $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$. Using the quasi-norm $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ we show in Remark 9.3 the reason for the jump in the scale in equalities $\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d)$ if $b > -1/2$, while $\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d) = B^{0,0,1/2}_{2,2}(\mathbb{R}^d)$.

The truncated Littlewood-Paley construction also appears in the characterization of $\mathbf{B}^{0,b}_{p,q}$ by means of wavelets that we obtain in Theorem 9.12. Using this wavelet description, we show in Remark 9.6 that, in general, embeddings given in Theorem 5.16

are the best possible. To be more precise, if we take any $\varepsilon > 0$ we prove that $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/p+\varepsilon}_{p,q}(\mathbb{R}^d)$ if $p = \max\{2, p, q\}$ and $B^{0,b+1/p-\varepsilon}_{p,q}(\mathbb{R}^d) \not\hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ if $p = \min\{2, p, q\}$.

The last section of the monograph contains the characterizations in terms of semi-groups of operators. This time the truncated Littlewood-Paley construction does not appear. In Theorem 9.13 we show an abstract result on a strongly continuous equi-bounded semi-group of operators on a Banach space. Then, in Theorem 9.14, we particularize it to heat kernels and in Theorem 9.15 to the case of the Cauchy-Poisson semi-group. Characterizations of spaces $B^{0,b}_{p,q}(\mathbb{R}^d)$ in terms of semi-groups are also obtained. See Theorems 9.16, 9.17 and 9.18.

Part of these results have appeared in the joint papers [30], [25], [26], [27], [28], [31] and in my paper [55]. Some other results form the preprints [29].

Chapter 2

Preliminaries

In this chapter we fix notation and we introduce the main concepts and constructions used in the monograph. We also establish some new results. In particular, we show some useful interpolation formulae and we characterize Besov spaces $\mathbb{B}^{0,b}_{p,q}$ over $\mathbb{T}$ as approximation spaces. The new results are taken from the papers [25] and [26].

Let X be a (real or complex) linear space. A *quasi-norm* on X is a real-valued non-negative function $\|\cdot\|_X$ defined on X satisfying the following conditions

(1) $\|f\|_X = 0$ if and only if $f = 0$.

(2) $\|\lambda f\|_X = |\lambda| \|f\|_X, \lambda$ scalar and $f \in X$.

(3) There exists a constant $c_X \geq 1$ such that $\|f+g\|_X \leq c_X(\|f\|_X + \|g\|_X)$ for $f, g \in X$.

The pair $(X, \|\cdot\|_X)$ determines a *quasi-normed space* (if $c_X = 1$, then X is a *normed space*). A quasi-normed space is said to be a *quasi-Banach space* if every Cauchy sequence is convergent.

If U, V are non-negative quantities depending on certain parameters, we write $U \lesssim V$ if there is a constant $c > 0$ independent of the parameters in U and V such that $U \leq cV$. We put $U \sim V$ if $U \lesssim V$ and $V \lesssim U$.

The quasi-norms $\|\cdot\|_X^{(1)}$ and $\|\cdot\|_X^{(2)}$ are *equivalent* if $\|f\|_X^{(1)} \sim \|f\|_X^{(2)}$ for all $f \in X$.

For $0 < \rho \leq 1$, a quasi-norm $\|\cdot\|_X$ is called a *ρ-norm* if

$$\|f+g\|_X^\rho \leq \|f\|_X^\rho + \|g\|_X^\rho \text{ for } f, g \in X.$$

Note that condition (3) is satisfied with $c_X = 2^{1/\rho-1}$. Conversely, given a quasi-norm $\|\cdot\|_X$, there exists an equivalent ρ-norm, with $1/\rho = 1+\log_2 c_X$ (see [10, Lemma 3.10.1, p. 59]). It is clear that every ρ-norm is also a η-norm for $0<\eta<\rho\le 1$.

Let X and Y be quasi-Banach spaces. We denote by $\mathfrak{L}(X,Y)$ the space formed by all bounded linear operators T acting from X into Y. It becomes a quasi-Banach space under the quasi-norm

$$\|T\| = \|T\|_{X,Y} = \sup_{\|f\|_X\le 1} \|Tf\|_Y.$$

2.1 Basic approximation constructions

Let X be a quasi-Banach space. We say that a sequence $(G_n)_{n\in\mathbb{N}_0}$ of subsets of X is an *approximation family* in X if the following conditions hold

$$G_0 = \{0\} \text{ and } \lambda G_n \subseteq G_n \quad \text{for any scalar } \lambda \text{ and } n\in\mathbb{N},$$
$$G_n \subseteq G_{n+1} \quad \text{for any } n\in\mathbb{N}_0 = \mathbb{N}\cup\{0\},$$
$$G_n + G_m \subseteq G_{n+m} \quad \text{for any } n,m\in\mathbb{N}.$$

Given any $f\in X$ and $n\in\mathbb{N}$, we put

$$E_n(f) = E_n(f;X) = \inf\{\|f-g\|_X : g\in G_{n-1}\}.$$

Let $\alpha\ge 0$, $0<p\le\infty$ and $-\infty<\gamma<\infty$. The *approximation space* $X_p^{(\alpha,\gamma)} = (X,G_n)_p^{(\alpha,\gamma)}$ is formed by all those $f\in X$ which have a finite quasi-norm

$$\|f\|_{X_p^{(\alpha,\gamma)}} = \left(\sum_{n=1}^{\infty}(n^\alpha(1+\log n)^\gamma E_n(f))^p n^{-1}\right)^{1/p} \quad \text{if} \quad 0<p<\infty, \tag{2.1}$$

$$\|f\|_{X_p^{(\alpha,\gamma)}} = \sup_{n\ge 1}\left\{n^\alpha(1+\log n)^\gamma E_n(f)\right\} \qquad \text{if} \qquad p=\infty. \tag{2.2}$$

The space $X_p^{(\alpha,\gamma)}$ is a quasi-Banach space with $X_p^{(\alpha,\gamma)} \hookrightarrow X$, where the symbol $\hookrightarrow$ means continuous embedding. The case $\alpha>0$ and $\gamma=0$ corresponds to the classical theory, which have been studied in [16], [104], [103] and [50]. We write simply X_p^α and $\|\cdot\|_{X_p^\alpha}$ if $\gamma=0$. We refer to [2] and [106] for properties of spaces $X_p^{(\alpha,\gamma)}$ with $\alpha>0$. These results are quite similar to the case $\gamma=0$. The case $\alpha=0$ is different. It has been studied in [44], [41], [65] and [2]. Note that $X_q^{(0,\gamma)}$ coincides with X if $\gamma<-1/q$. Moreover, it is not hard to check that

$$X_p^\alpha \hookrightarrow X_q^{(0,\gamma)}, \alpha>0, 0<p,q\le\infty, -\infty<\gamma<\infty. \tag{2.3}$$

Let us give a concrete example. Let $X = \ell_\infty$, the space of bounded sequences and let $G_n = F_n$, the subset of sequences having at most n coordinates different from 0. Then, for any $\xi \in \ell_\infty$, the sequence $(E_n(\xi;\ell_\infty))$ is the non-increasing rearrangement (ξ_n^*) of the sequence ξ. The space X_p^α coincides with the Lorentz sequence space $\ell_{1/\alpha,p}$, the space $X_q^{(0,\gamma)}$ is the Lorentz-Zygmund sequence space $\ell_{\infty,q}(\log \ell)_\gamma$ and $X_p^{(\alpha,\gamma)}$ is $\ell_{1/\alpha,p}(\log \ell)_\gamma$. Recall that for $0 < r, q \le \infty$ and $-\infty < \gamma < \infty$,

$$\ell_{r,q}(\log \ell)_\gamma = \left\{ \xi \in \ell_\infty : \|\xi\|_{\ell_{r,q}(\log \ell)_\gamma} = \left(\sum_{n=1}^{\infty} (n^{1/r}(1+\log n)^\gamma \xi_n^*)^q n^{-1} \right)^{1/q} < \infty \right\} \tag{2.4}$$

(the sum should be replaced by the supremum if $q = \infty$) and $\ell_{r,q} = \ell_{r,q}(\log \ell)_0$ (see [8], [104], [51]).

It is shown in [44] that, even when $\gamma = 0$, the theory of spaces X_p^0 does not follow by taking $\alpha = 0$ in the theory of classical approximation spaces.

One of the basic properties of approximation spaces is a special representation of their elements via elements from the sets G_n. In the case that $\alpha > 0$, the result reads as follows (see [104, Theorem 3.1] and [106, Theorem 3.3]): An element $f \in X$ belongs to $X_p^{(\alpha,\gamma)}$ if and only if

$$f = \sum_{n=0}^{\infty} g_n, g_n \in G_{2^n}, \tag{2.5}$$

with

$$\sum_{n=0}^{\infty} (2^{n\alpha}(1+n)^\gamma \|g_n\|_X)^p < \infty.$$

Moreover,

$$\|f\|^{\text{rep}}_{X_p^{(\alpha,\gamma)}} = \inf \left(\sum_{n=0}^{\infty} (2^{n\alpha}(1+n)^\gamma \|g_n\|_X)^p \right)^{1/p} \tag{2.6}$$

where the infimum is taken over all possible representations (2.5), defines an equivalent quasi-norm to $\|\cdot\|_{X_p^{(\alpha,\gamma)}}$ with equivalence constants depending only on α, γ, p and c_X.

To state the representation theorem for limiting approximation spaces, we introduce the notation $\mu_n = 2^{2^n}, n \in \mathbb{N}_0$. It was proved in [44] and [65] that an element $f \in X$ belongs to $X_p^{(0,\gamma)}$ if and only if there is a representation

$$f = \sum_{n=0}^{\infty} g_n \text{ with } g_n \in G_{\mu_n} \tag{2.7}$$

and

$$\sum_{n=0}^{\infty} (2^{n(\gamma+1/p)} \|g_n\|_X)^p < \infty.$$

Moreover,

$$\|f\|^{\mathrm{rep}}_{X_p^{(0,\gamma)}} = \inf\left(\sum_{n=0}^{\infty}(2^{n(\gamma+1/p)}\|g_n\|_X)^p\right)^{1/p} \tag{2.8}$$

where the infimum is taken over all possible representations (2.7), is an equivalent quasi-norm to $\|\cdot\|_{X_p^{(0,\gamma)}}$ with equivalence constants depending only on γ, p and c_X. This property is useful to get the following result.

Lemma 2.1. *Let X, Y be quasi-Banach spaces which are continuously embedded in a Hausdorff topological vector space. Let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family such that $G_n \subseteq X \cap Y$ for any $n \in \mathbb{N}_0$. Assume that there are constants $c, \beta > 0$ such that*

$$\|g\|_Y \le c(\log(1+n))^{\beta}\|g\|_X, g \in G_n, n \in \mathbb{N}.$$

Then for $0 < q \le \infty$ and $\gamma > -1/q$ we have that

$$X_q^{(0,\beta+\gamma)} \hookrightarrow Y_q^{(0,\gamma)}.$$

Proof. By (2.8), given any $f \in X_q^{(0,\beta+\gamma)}$, we can find a representation $f = \sum_{n=0}^{\infty} g_n$ (in X) with $g_n \in G_{\mu_n}$ such that

$$\left(\sum_{n=0}^{\infty}(2^{n(\beta+\gamma+1/q)}\|g_n\|_X)^q\right)^{1/q} \lesssim \|f\|_{X_q^{(0,\beta+\gamma)}}.$$

Since

$$\|g_n\|_Y \lesssim (\log\mu_n)^{\beta}\|g_n\|_X \sim 2^{n\beta}\|g_n\|_X,$$

we obtain that

$$\left(\sum_{n=0}^{\infty}(2^{n(\gamma+1/q)}\|g_n\|_Y)^q\right)^{1/q} \lesssim \left(\sum_{n=0}^{\infty}(2^{n(\beta+\gamma+1/q)}\|g_n\|_X)^q\right)^{1/q} < \infty.$$

Now it is not hard to show that $\sum_{n=0}^{\infty} g_n$ is convergent in Y. By compatibility, $f = \sum_{n=0}^{\infty} g_n$ also in Y. Therefore, taking the infimum over all possible representations and using again (2.8) we conclude that

$$\|f\|_{Y_q^{(0,\gamma)}} \lesssim \|f\|_{X_q^{(0,\beta+\gamma)}}.$$

□

2.2 Interpolation methods. Some interpolation formulae. Extrapolation spaces

By a *quasi-Banach couple* $\bar{A} = (A_0, A_1)$ we mean two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space $\mathcal{A}$. If both A_0 and A_1 are Banach spaces, we say that $\bar{A}$ is a *Banach couple.* For any quasi-Banach couple, it makes sense to consider $A_0 \cap A_1$ and

$$A_0 + A_1 = \{a \in \mathcal{A} : a = a_0 + a_1, a_j \in A_j\}$$

endowed with the natural quasi-norms

$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

and

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\},$$

respectively.

By an *intermediate space* with respect to $\bar{A} = (A_0, A_1)$ we mean a quasi-Banach space A for which

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1.$$

Given $t > 0$, *Peetre's K- and J-functionals* are defined by

$$\begin{aligned} K(t,a) &= K(t,a;\bar{A}) = K(t,a;A_0,A_1) \\ &= \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\right\}, a \in A_0 + A_1 \end{aligned}$$

and

$$J(t,a) = J(t,a;\bar{A}) = J(t,a;A_0,A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, a \in A_0 \cap A_1,$$

respectively. Note that $K(1,\cdot) = \|\cdot\|_{A_0+A_1}$ and $J(1,\cdot) = \|\cdot\|_{A_0 \cap A_1}$.

Next we introduce real interpolation spaces with the help of the K-functional. Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple. Given $0 < \theta < 1$ and $0 < q \le \infty$, the *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{1/q}$$

(when $q = \infty$ the integral should be replaced by the supremum). See [10], [116], [9] or [14].

Since
$$K(t,a;A_0,A_1) = tK(t^{-1},a;A_1,A_0)$$
we have that
$$(A_0,A_1)_{\theta,q} = (A_1,A_0)_{1-\theta,q}$$
with equality of quasi-norms.

Assume that $\bar{A} = (A_0,A_1)$ is a Banach couple and $1 \le q \le \infty$. By the equivalence theorem [10, Theorem 3.3.1, p. 44], $(A_0,A_1)_{\theta,q}$ can be characterized in terms of the J-functional. More precisely, the space $(A_0,A_1)_{\theta,q}$ coincides with the collection of all those $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that
$$a = \int_0^\infty u(t)\frac{dt}{t} \text{ (convergence in } A_0 + A_1) \tag{2.9}$$
and
$$\int_0^\infty (t^{-\theta}J(t,u(t)))^q \frac{dt}{t} < \infty. \tag{2.10}$$
Furthermore, the norm
$$\|a\|_{\bar{A}_{\theta,q;J}} = \inf\left(\int_0^\infty (t^{-\theta}J(t,u(t)))^q \frac{dt}{t}\right)^{1/q}$$
where the infimum is taken over all $u(t)$ such that (2.9), (2.10) are satisfied, is equivalent to $\|\cdot\|_{(A_0,A_1)_{\theta,q}}$. In particular, if (A_0,A_1) is a Gagliardo couple (see [9, p. 320]) and $q = 1$, it follows from the strong fundamental lemma of interpolation (see [80, p. 20]) that
$$(A_0,A_1)_{\theta,1;J} = \theta(1-\theta)(A_0,A_1)_{\theta,1}. \tag{2.11}$$
Here by $X = \lambda Y$, where X and Y are quasi-normed spaces and $\lambda > 0$, we mean $\|f\|_X \sim \lambda\|f\|_Y$ for all $f \in X$.

We shall also work with the complex interpolation method [10, 116]. Given a Banach couple $\bar{A} = (A_0,A_1)$, let $\mathcal{F}(\overline{A})$ be the space of all functions f from the closed strip $S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ into $A_0 + A_1$ such that f is bounded and continuous on S and analytic on the interior of S and moreover, for $j = 0,1$, the functions $t \to f(j+it)$ are continuous from $\mathbb{R}$ into A_j and tend to zero as $|t| \to \infty$. The norm in $\mathcal{F}(\overline{A})$ is given by
$$\|f\|_{\mathcal{F}(\overline{A})} = \max\{\sup \|f(j+it)\|_{A_j} : j = 0,1\}.$$
For $0 < \theta < 1$, the *complex interpolation space* $[A_0,A_1]_\theta$ is formed by all $a \in A_0$ such that $a = f(\theta)$ for some $f \in \mathcal{F}(\overline{A})$. We provide $[A_0,A_1]_\theta$ with the norm
$$\|a\|_{[A_0,A_1]_\theta} = \inf\{\|f\|_{\mathcal{F}(\overline{A})} : f(\theta) = a, f \in \mathcal{F}(\overline{A})\}.$$

Using that $[\mathbb{C},\mathbb{C}]_\theta = \mathbb{C}$ with equality of norms, one can check that the following embeddings hold with norm less than or equal to 1

$$(A_0,A_1)_{\theta,1;J} \hookrightarrow [A_0,A_1]_\theta \hookrightarrow (A_0,A_1)_{\theta,\infty} \tag{2.12}$$

(see [10, Theorem 3.9.1] and [42, Lemma 1.1]).

The extension of the real interpolation method which is obtained by replacing in the definition t^θ by more general functions $\rho(t)$ is also important. Given a quasi-Banach couple $\bar{A} = (A_0,A_1)$, we define the space $\bar{A}_{\rho,q} = (A_0,A_1)_{\rho,q}$ by the quasi-norm

$$\|a\|_{\bar{A}_{\rho,q}} = \left(\int_0^\infty \left(\frac{K(t,a)}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q}$$

(see [74], [102] and the references given there). Let $\mathbb{A} = (\alpha_0,\alpha_\infty) \in \mathbb{R}^2$, put $\ell(t) = 1+|\log t|$ and write

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{for} \quad t \in (0,1], \\ \\ \ell^{\alpha_\infty}(t) & \text{for} \quad t \in (1,\infty). \end{cases}$$

If $\rho(t) = t^\theta \ell^{\mathbb{A}}(t)$, we put

$$(A_0,A_1)_{\rho,q} = (A_0,A_1)_{\theta,q,\mathbb{A}}.$$

We refer to [61, 63] for details on these spaces which, under suitable conditions on $\mathbb{A}$, are well-defined even if $\theta = 0$ or $\theta = 1$. Note that our notation is slightly different from [61, 63]. In the special case $\alpha_0 = \alpha_\infty = \alpha$, we simply write

$$(A_0,A_1)_{\rho,q} = (A_0,A_1)_{\theta,q,\alpha}.$$

Set $\ell\ell(t) = 1+\log(1+|\log t|)$. In the case that $\rho(t) = t^\theta \ell^\alpha(t)\ell\ell^\delta(t), \theta \in [0,1], \alpha,\delta \in \mathbb{R}$, we write $(A_0,A_1)_{\theta,q,\alpha,\delta}$ to mean $(A_0,A_1)_{\rho,q}$.

Analogously, we can introduce the J-space $\bar{A}_{\rho,q;J} = (A_0,A_1)_{\rho,q;J}$ with respect to the Banach couple $\bar{A} = (A_0,A_1)$ and $1 \le q \le \infty$. Its norm is given by

$$\|a\|_{\bar{A}_{\rho,q;J}} = \inf \left(\int_0^\infty \left(\frac{J(t,u(t))}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q}$$

where the infimum is taken over all strongly measurable functions $u(t)$ with image in $A_0 \cap A_1$ such that the representation (2.9) holds. See [14].

The following *limiting real methods* will be also very useful in our considerations. Let $A_1 \hookrightarrow A_0, 0 < q \le \infty$ and $-\infty < \eta < \infty$. For $\theta = 1$ or $\theta = 0$, the space $\bar{A}_{(\theta,\eta),q} = (A_0,A_1)_{(\theta,\eta),q}$ is formed by all those $a \in A_0$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta(1-\log t)^\eta}\right)^q \frac{dt}{t}\right)^{1/q}$$

(see [72], [33], [40]). To avoid that $\bar{A}_{(1,\eta),q} = \{0\}$, when $\theta = 1$ we suppose that $\eta > 1/q$ if $q < \infty$, and $\eta \geq 0$ if $q = \infty$. In the case that $\theta = 0$, the only case of interest is when $\eta \leq 1/q$ if $q < \infty$, and $\eta < 0$ if $q = \infty$. Otherwise, $\bar{A}_{(0,\eta),q} = A_0$ with equivalence of quasi-norms.

It is clear that any of the above methods, say F, produces intermediate spaces with the *interpolation property* for bounded linear operators. In order to describe this property, let $\bar{B} = (B_0, B_1)$ be another Banach couple. We put $T \in \mathfrak{L}(\bar{A}, \bar{B})$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$, whose restrictions $T : A_j \longrightarrow B_j$ are bounded for $j = 0, 1$. Then, the restriction of T to $F(A_0, A_1)$ is a bounded operator $T : F(A_0, A_1) \longrightarrow F(B_0, B_1)$. In the case that we deal with (ρ, q)-method and $((\theta, \eta), q)$-method, this property can be extended to quasi-Banach couples.

For later use, we establish now some stability properties of limiting methods and we also determine some concrete interpolation spaces.

Subsequently, we put $K(t, a)$ for the K-functional of (A_0, A_1). If we work with a different couple, then we write it explicitly in the notation of the K-functional.

Lemma 2.2. *Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Suppose that $0 < \theta < 1, 0 < q, r \leq \infty, \gamma < -1/q$ and $-\infty < \eta < \infty$. Then we have with equivalence of quasi-norms*

(a) $((A_0, A_1)_{\theta,r}, A_1)_{(1,-\gamma),q} = (A_0, A_1)_{(1,-\gamma),q}$.

(b) $(A_0, (A_0, A_1)_{\theta,r})_{(0,-\eta),q} = (A_0, A_1)_{(0,-\eta),q}$.

Proof. Since $(A_0, A_1)_{\theta,r} \hookrightarrow A_0$, it follows from the interpolation property that

$$((A_0, A_1)_{\theta,r}, A_1)_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{(1,-\gamma),q}.$$

To check the converse embedding, assume first that $0 < r < q$. By Holmstedt's formula [78, Remark 2.1],

$$K(t, a; (A_0, A_1)_{\theta,r}, A_1) \sim \left(\int_0^{t^{1/(1-\theta)}} (s^{-\theta} K(s, a))^r \frac{ds}{s} \right)^{1/r}. \tag{2.13}$$

Hence, we obtain

$$\begin{aligned}
&\|a\|_{((A_0,A_1)_{\theta,r},A_1)_{(1,-\gamma),q}} \\
&\quad\sim \left(\int_0^1 \left[\frac{(1 - \log t)^\gamma}{t} \left(\int_0^{t^{1/(1-\theta)}} (s^{-\theta} K(s, a))^r \frac{ds}{s} \right)^{1/r} \right]^q \frac{dt}{t} \right)^{1/q} \\
&\quad\sim \left(\int_0^1 \left[\frac{(1 - \log t)^{\gamma r}}{t^{(1-\theta)r}} \int_0^t (s^{-\theta} K(s, a))^r \frac{ds}{s} \right]^{q/r} \frac{dt}{t} \right)^{1/q} \\
&\quad\lesssim \left(\int_0^1 \left[\frac{t(1 - \log t)^{\gamma r}}{t^{(1-\theta)r}} (t^{-\theta} K(t, a))^r t^{-1} \right]^{q/r} \frac{dt}{t} \right)^{1/q}
\end{aligned}$$

where we have used Hardy's inequality [8, Theorem 6.4] in the last estimate. Therefore $\|a\|_{((A_0,A_1)_{\theta,r},A_1)_{(1,-\gamma),q}} \lesssim \|a\|_{(A_0,A_1)_{(1,-\gamma),q}}$ and so

$$(A_0, A_1)_{(1,-\gamma),q} \hookrightarrow ((A_0, A_1)_{\theta,r}, A_1)_{(1,-\gamma),q}.$$

If $q \le r$, take $0 < r_0 < q$. Using that $(A_0, A_1)_{\theta,r_0} \hookrightarrow (A_0, A_1)_{\theta,r}$ and the previous case, we derive

$$(A_0, A_1)_{(1,-\gamma),q} \hookrightarrow ((A_0, A_1)_{\theta,r_0}, A_1)_{(1,-\gamma),q} \hookrightarrow ((A_0, A_1)_{\theta,r}, A_1)_{(1,-\gamma),q}.$$

This completes the proof of equality (a). The statement (b) follows from similar arguments. □

Let (Ω, μ) be a σ-finite measure space. Given any measurable function f which is finite almost everywhere, the *non-increasing rearrangement* of f is defined

$$f^*(t) = \inf\{s \ge 0 : \mu\{x \in \Omega : |f(x)| > s\} \le t\}, t \ge 0. \tag{2.14}$$

Suppose that $\mathfrak{g}$ is a slowly varying function on $(0, \infty)$ (see [56, Definition 3.4.32, p. 108]) and let $0 < p, q \le \infty$. The *Lorentz-Karamata space* $L_{p,q;\mathfrak{g}}(\Omega) = L_{p,q;\mathfrak{g}}(\Omega, \mu)$ is formed by all (classes of) measurable functions f on Ω having a finite quasi-norm

$$\|f\|_{L_{p,q;\mathfrak{g}}(\Omega)} = \left(\int_0^{\mu(\Omega)} [t^{1/p}\mathfrak{g}(t) f^*(t)]^q \frac{dt}{t}\right)^{1/q}.$$

See [56, 3.4.3].

If $\mathfrak{g}(t) = (1+|\log t|)^\gamma, \gamma \in \mathbb{R}$ (respectively, $\mathfrak{g}(t) = (1+|\log t|)^\gamma(1+\log(1+|\log t|))^\eta, \gamma, \eta \in \mathbb{R}$) we get the Lorentz-Zygmund space $L_{p,q}(\log L)_\gamma(\Omega)$ (respectively, the generalized Lorentz-Zygmund space $L_{p,q}(\log L)_\gamma(\log\log L)_\eta(\Omega)$). In particular, the special case $p = q$ produces the Zygmund space $L_p(\log L)_\gamma(\Omega)$. If $\gamma = 0$, then $L_{p,q}(\log L)_\gamma(\Omega)$ coincides with the Lorentz space $L_{p,q}(\Omega)$, and if in addition $p = q$, then the space becomes the Lebesgue space $L_p(\Omega)$. See [56, 9, 10, 116]. When $\Omega = \mathbb{Z}$ and μ is the counting measure, we denote these spaces by $\ell_{p,q}(\log \ell)_\gamma, \ell_{p,q}(\log \ell)_\gamma(\log\log \ell)_\eta, \ell_p(\log \ell)_\gamma, \ell_{p,q}$ and ℓ_p, respectively. Note that when $\Omega = \mathbb{T}^d$ equipped with the usual Lebesgue measure, the space $L_{p,q;\mathfrak{g}}(\mathbb{T}^d)$ is formed by 2π-periodic functions in each variable.

The couple of Lebesgue spaces $(L_p(\Omega), L_\infty(\Omega))$ is a quasi-Banach couple. It turns out that generalized Lorentz-Zygmund spaces $L_{r,q}(\log L)_\gamma(\log\log L)_\eta(\Omega)$ can be characterized as interpolation spaces relative to $(L_p(\Omega), L_\infty(\Omega))$. See [102, Lemma 6.1]. Namely, given $0 < \theta < 1, 0 < p < \infty, 0 < q \le \infty$ and $-\infty < \gamma, \eta < \infty$, put $\rho(t) = t^\theta(1 + |\log t|)^{-\gamma}(1 + \log(1 + |\log t|))^{-\eta}$, then it holds that

$$(L_p(\Omega), L_\infty(\Omega))_{\rho,q} = L_{r,q}(\log L)_\gamma(\log\log L)_\eta(\Omega) \tag{2.15}$$

with $1/r = (1 - \theta)/p$.

Lemma 2.3. *Assume that* (Ω, μ) *is a finite measure space, so* $L_\infty(\Omega) \hookrightarrow L_p(\Omega)$. *Let* $0 < p < \infty, 0 < q \le \infty$ *and* $\gamma < -1/q$. *Then we have with equivalence of quasi-norms*

$$(L_p(\Omega), L_\infty(\Omega))_{(1,-\gamma),q} = L_{\infty,q}(\log L)_\gamma(\Omega).$$

Proof. The K-functional of $(L_p(\Omega), L_\infty(\Omega))$ is given by

$$K(t,f) \sim \left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p} \tag{2.16}$$

(see [10, Theorem 5.2.1]). Assume that $p \le q$. Using Hardy's inequality [8, Theorem 6.4], we obtain

$$\begin{aligned}
\|f\|_{(L_p(\Omega),L_\infty(\Omega))_{(1,-\gamma),q}} &\sim \left(\int_0^1 \left[\frac{(1-\log t)^\gamma}{t}\left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[\frac{(1-\log t)^{\gamma p}}{t}\int_0^t f^*(s)^p ds\right]^{q/p} \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 [(1-\log t)^\gamma f^*(t)]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{L_{\infty,q}(\log L)_\gamma(\Omega)}.
\end{aligned}$$

Suppose now $q < p$. Take $0 < r < q$. Since $L_p(\Omega) = (L_r(\Omega), L_\infty(\Omega))_{1-r/p,p}$ (see (2.15)), by Lemma 2.2(a) and the previous case, we derive

$$\begin{aligned}
(L_p(\Omega), L_\infty(\Omega))_{(1,-\gamma),q} &= ((L_r(\Omega), L_\infty(\Omega))_{1-r/p,p}, L_\infty(\Omega))_{(1,-\gamma),q} \\
&= (L_r(\Omega), L_\infty(\Omega))_{(1,-\gamma),q} \\
&= L_{\infty,q}(\log L)_\gamma(\Omega).
\end{aligned}$$

□

Assume now that $\mu(\Omega) = 1$. For $0 < p < \infty, 0 < q \le \infty$ and $\gamma > -1/q$, the *small Lebesgue space* $L^{(p,\gamma,q}(\Omega)$ is the collection of all (classes of) measurable functions f on Ω such that

$$\|f\|_{L^{(p,\gamma,q}(\Omega)} = \left(\int_0^1 \left[(1-\log t)^\gamma \left(\int_0^t f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} < \infty$$

(with the usual modification if $q = \infty$). Note that if $1 < p < \infty, q = 1, \eta > 0$ and $\gamma = \eta(1-1/p) - 1$, we recover spaces $L^{(p,\eta}(\Omega)$ (see [52]). In particular, for $\eta = 1$ we obtain the space $L^{(p}(\Omega)$ (see [67, Corollary 3.3]). For more information about these spaces we refer to [66, 22].

Remark 2.1. Small Lebesgue spaces are subspaces of $L_p(\Omega)$ since

$$\|f\|_{L^{(p,\gamma,q}(\Omega)} = \left(\int_0^1 \left[t^{1/p}(1-\log t)^{\gamma}\left(\frac{1}{t}\int_0^t f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\geq \left(\int_0^1 \left[t^{1/p}(1-\log t)^{\gamma}\right]^q \frac{dt}{t}\right)^{1/q} \left(\int_0^1 f^*(s)^p ds\right)^{1/p} \sim \|f\|_{L_p(\Omega)}.$$

We can provide simple examples showing that the previous embedding is strict. For simplicity, we consider the spaces on the interval $(0,1)$. The function $f(t) = t^{-1/p}(1-\log t)^{\beta}$, $-\gamma - 1/q - 1/p \leq \beta < -1/p$, is such that $f \in L_p$ and $f \notin L^{(p,\gamma,q}$.

In the following result we shall prove that small Lebesgue spaces can be obtained as limiting interpolation spaces between Lebesgue spaces.

Lemma 2.4. *Assume that (Ω,μ) is a finite measure space. Let $0 < p < \infty, 0 < q \leq \infty$ and $\gamma > -1/q$. Then we have with equivalence of quasi-norms*

$$(L_p(\Omega), L_\infty(\Omega))_{(0,-\gamma),q} = L^{(p,\gamma,q}(\Omega).$$

Proof. Inserting (2.16) in the quasi-norm of $(L_p(\Omega), L_\infty(\Omega))_{(0,-\gamma),q}$ and making a change of variables we obtain

$$\|f\|_{(L_p(\Omega),L_\infty(\Omega))_{(0,-\gamma),q}} \sim \left(\int_0^1 \left[(1-\log t)^{\gamma}\left(\int_0^{t^{1/p}} f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\sim \left(\int_0^1 \left[(1-\log t)^{\gamma}\left(\int_0^t f^*(s)^p ds\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} = \|f\|_{L^{(p,\gamma,q}(\Omega)}.$$

□

Next we consider interpolation properties of approximation spaces. It was shown by Peetre and Sparr [101] that the scale of classical approximation spaces is closed under real interpolation. To be more precise, if $0 < \theta < 1, 0 < \alpha_0 \neq \alpha_1 < \infty, 0 < p_0, p_1, q \leq \infty$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, then

$$(X_{p_0}^{\alpha_0}, X_{p_1}^{\alpha_1})_{\theta,q} = X_q^{\alpha} \tag{2.17}$$

with equivalence of quasi-norms. Besides,

$$(X_{p_0}^{\alpha_0}, X)_{\theta,q} = X_q^{\beta}, \beta = (1-\theta)\alpha_0. \tag{2.18}$$

Note that parameters p_0 and p_1 do not play any role in formulae (2.17) and (2.18).

The corresponding results for limiting approximation spaces were proved by Fehér and Grässler [65, Theorems 4 and 5]. Namely, given $0 < \theta < 1, 0 < p_0, p_1, q \leq \infty, \gamma_0 + 1/p_0 \neq \gamma_1 + 1/p_1, \gamma_i > -1/p_i, i = 0, 1$, we have that

$$(X_{p_0}^{(0,\gamma_0)}, X_{p_1}^{(0,\gamma_1)})_{\theta,q} = X_q^{(0,\gamma)} \tag{2.19}$$

where $\gamma + 1/q = (1-\theta)(\gamma_0 + 1/p_0) + \theta(\gamma_1 + 1/p_1)$, and

$$(X_{p_0}^{(0,\gamma_0)}, X)_{\theta,q} = X_q^{(0,\delta)} \tag{2.20}$$

with $\delta + 1/q = (1-\theta)(\gamma_0 + 1/p_0)$.

We shall need to extend the formula (2.18) to interpolation with logarithmic weights. First we establish an auxiliary result.

Lemma 2.5. *Let X be a quasi-Banach space, let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family and $0 < p < \infty$. Then we have*

$$K(n^{1/p}, f; X_p^{1/p}, X) \sim \left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p}, n \in \mathbb{N}, f \in X.$$

Proof. Take any $f \in X$ and let $g \in X$ such that $f - g \in X_p^{1/p}$. Then

$$\begin{aligned}\left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p} &\lesssim \left(\sum_{k=1}^{n} (E_k(f-g) + \|g\|_X)^p\right)^{1/p}\\ &\lesssim \left(\sum_{k=1}^{n} E_k(f-g)^p\right)^{1/p} + n^{1/p}\|g\|_X\\ &\leq \|f-g\|_{X_p^{1/p}} + n^{1/p}\|g\|_X.\end{aligned}$$

Taking the infimum over all $g \in X$ with $f - g \in X_p^{1/p}$, we obtain that

$$\left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p} \lesssim K(n^{1/p}, f; X_p^{1/p}, X).$$

Conversely, choose $g \in G_{n-1}$ satisfying that $\|f-g\|_X \leq 2E_n(f)$. It follows that

$$\begin{aligned}K(n^{1/p}, f; X_p^{1/p}, X) &\leq \|g\|_{X_p^{1/p}} + n^{1/p}\|f-g\|_X\\ &\lesssim \left(\sum_{k=1}^{n} E_k(g)^p\right)^{1/p} + n^{1/p}E_n(f)\\ &\leq \left(\sum_{k=1}^{n} E_k(g)^p\right)^{1/p} + \left(\sum_{k=1}^{n} E_k(f)^p\right)^{1/p}\end{aligned}$$

because $(E_n(f))$ is decreasing. Besides, for $1 \le k \le n$, we get

$$E_k(g) \lesssim E_k(f) + \|f - g\|_X \le 3E_k(f).$$

Consequently,

$$K(n^{1/p}, f; X_p^{1/p}, X) \lesssim \left(\sum_{k=1}^{n} E_k(f)^p \right)^{1/p}.$$

□

Proposition 2.6. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose that $\alpha > 0, 0 < r, q \le \infty, 0 < \theta < 1, -\infty < \gamma < \infty$ and put $\rho(t) = t^\theta(1 + |\log t|)^{-\gamma}$. Then we have with equivalence of quasi-norms*

$$(X_r^\alpha, X)_{\rho,q} = X_q^{((1-\theta)\alpha,\gamma)}.$$

Proof. Let $p > 0$. We claim that

$$(X_p^{1/p}, X)_{\rho,q} = X_q^{((1-\theta)/p,\gamma)}. \tag{2.21}$$

Indeed, since $X_p^{1/p} \hookrightarrow X$, we have that $K(t, f; X_p^{1/p}, X) \sim t\|f\|_X$ for $0 < t < 1$. This yields that

$$\|f\|_{(X_p^{1/p},X)_{\rho,q}} \sim \left(\int_1^\infty \left[\frac{K(t, f; X_p^{1/p}, X)}{t^\theta(1 + \log t)^{-\gamma}} \right]^q \frac{dt}{t} \right)^{1/q}.$$

Whence, using Lemma 2.5 and Hardy's inequality [45, Theorem 1.2], we get

$$\begin{aligned}
\|f\|_{(X_p^{1/p},X)_{\rho,q}} &\sim \left(\sum_{n=1}^{\infty} \left[\frac{K(n^{1/p}, f; X_p^{1/p}, X)}{n^{\theta/p}(1 + \log n)^{-\gamma}} \right]^q \frac{1}{n} \right)^{1/q} \\
&\sim \left(\sum_{n=1}^{\infty} \left[n^{(1-\theta)/p}(1 + \log n)^\gamma \left(\sum_{k=1}^{n} \frac{E_k(f)^p}{n} \right)^{1/p} \right]^q \frac{1}{n} \right)^{1/q} \\
&\sim \left(\sum_{n=1}^{\infty} \left[n^{(1-\theta)/p}(1 + \log n)^\gamma E_n(f) \right]^q \frac{1}{n} \right)^{1/q} \\
&= \|f\|_{X_q^{((1-\theta)/p,\gamma)}}.
\end{aligned}$$

Take $0 < p < 1/\alpha$. Let $(1 - \lambda)/p = \alpha$ and $\mu(t) = t^{(1-\lambda)\theta+\lambda}(1 + |\log t|)^{-\gamma}$. By (2.21) and [102, Corollary 4.4], we get

$$(X_r^\alpha, X)_{\rho,q} = ((X_p^{1/p}, X)_{\lambda,r}, X)_{\rho,q} = (X_p^{1/p}, X)_{\mu,q} = X_q^{((1-\theta)\alpha,\gamma)}$$

where we have used again (2.21) in the last equality. □

We end the section recalling the construction of the Σ-extrapolation method. See [80, 94, 84]. Let $(A_j)_{j\geq 0}$ be a compatible scale of quasi-Banach spaces, that is, there exists a fixed Banach space B such that $A_j \hookrightarrow B$ with embedding constant which is uniform with respect to j. Let $1 \leq q \leq \infty$. Let $(M_j)_{j\geq 0}$ be a sequence formed by positive numbers such that $(M_j^{-1})_{j\geq 0} \in \ell_{q'}, 1/q + 1/q' = 1$. The $\Sigma^{(q)}$-*extrapolation space* $\Sigma^{(q)} M_j A_j$ is formed by all $a \in B$ such that

$$a = \sum_{j=0}^{\infty} a_j, a_j \in A_j, \tag{2.22}$$

and

$$\sum_{j=0}^{\infty} (M_j \|a_j\|_{A_j})^q < \infty. \tag{2.23}$$

Furthermore, $\Sigma^{(q)} M_j A_j$ becomes a quasi-Banach space under the quasi-norm given by

$$\|a\|_{\Sigma^{(q)} M_j A_j} = \inf \left(\sum_{j=0}^{\infty} (M_j \|a_j\|_{A_j})^q \right)^{1/q}$$

where the infimum is taken over all possible representations (2.22) such that (2.23) holds.

Lorentz-Zygmund spaces $L_{p,q}(\log L)_\gamma$ with $\gamma > 0$ can be obtained via Σ-extrapolation of Lorentz spaces. See [84, Theorem 4.4] and [34, Corollary 3.3]. Let Ω be a bounded domain in $\mathbb{R}^d$. Assume that $0 < p < \infty, 0 < q \leq \infty, \gamma > 0$ and $J \in \mathbb{N}_0$ such that $1/p^{\lambda_j} = 1/p - 2^{-j}/d > 0$ for all $j \geq J$. Then $L_{p,q}(\log L)_\gamma(\Omega)$ consists of all measurable functions f on Ω which can be represented as

$$f = \sum_{j=J}^{\infty} f_j, f_j \in L_{p^{\lambda_j},q}(\Omega), \tag{2.24}$$

such that

$$\left(\sum_{j=J}^{\infty} 2^{j\gamma q} \|f_j\|^q_{L_{p^{\lambda_j},q}(\Omega)} \right)^{1/q} < \infty. \tag{2.25}$$

Moreover, the infimum of the expression in (2.25) taken over all admissible representations (2.24) is a quasi-norm equivalent to $\|\cdot\|_{L_{p,q}(\log L)_\gamma(\Omega)}$.

2.3 Besov spaces. Characterization by approximation

In this section we work with the measure space $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$.

For $x, h \in \mathbb{R}^d$ and $k \in \mathbb{N}$, we put

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \text{ and } (\Delta_h^{k+1} f)(x) = \Delta_h^1(\Delta_h^k f)(x). \tag{2.26}$$

The *k-th order modulus of smoothness* of $f \in L_p(\Omega)$ is given by

$$\omega_k(f,t)_p = \sup_{|h|\le t} \|\Delta_h^k f\|_{L_p(\Omega)}, t > 0, \tag{2.27}$$

where $|h|$ stands for the Euclidean norm of h. If $k = 1$, we simply write $\omega(f,t)_p$.

Let $0 < q \le \infty, \alpha \ge 0, k \in \mathbb{N}$ with $k > \alpha$ and $b, d \in \mathbb{R}$. The Besov space $\mathbf{B}_{p,q}^{\alpha,b,d} = \mathbf{B}_{p,q}^{\alpha,b,d}(\Omega)$ is formed by all those $f \in L_p(\Omega)$ such that

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,b,d}(\Omega)} = \|f\|_{L_p(\Omega)} + \left(\int_0^1 [t^{-\alpha}(1-\log t)^b(1+\log(1-\log t))^d \omega_k(f,t)_p]^q \frac{dt}{t}\right)^{1/q} < \infty$$

when $q < \infty$. The usual change is made when $q = \infty$. The notation $\mathbf{B}_{p,q}^{\alpha,b,d}$ is justified by the fact that the definition is independent of $k > \alpha$ (see [77] for $\alpha > 0$. In the case that $\alpha = 0$, it will be proved in Theorem 5.15(a)). Spaces $\mathbf{B}_{p,q}^{\alpha,b,d}$ are a special case of Besov spaces of generalized smoothness, which were considered in [83], [77], [64] and the references given there. In the case that $\alpha > 0$ and $b = d = 0$, we get classical Besov spaces $\mathbf{B}_{p,q}^{\alpha} = \mathbf{B}_{p,q}^{\alpha}(\Omega)$ (see, for example, the books [15], [97] and [116]). If $d = 0$, we simply write $\mathbf{B}_{p,q}^{\alpha,b}$. The space $\mathbf{B}_{p,q}^{0,b}$ has zero classical smoothness ($\alpha = 0$) and logarithmic smoothness with exponent b. See [51]. Since $\omega(f,t)_p \lesssim \|f\|_{L_p(\Omega)}$ for $f \in L_p(\Omega)$, we have that $\mathbf{B}_{p,q}^{0,b}(\Omega) = L_p(\Omega)$ if $b < -1/q$ if $q < \infty$ and $b \le 0$ if $q = \infty$.

Spaces $\mathbf{B}_{p,q}^{\alpha,b}(\mathbb{T})$ as approximation spaces. Indeed, take $X = L_p(\mathbb{T})$ and $G_n = T_n$, the subset of all trigonometric polynomials of order n

$$T_n = \left\{ \sum_{|k|\le n} c_k e^{ikx} : c_k \in \mathbb{C} \right\}.$$

Assume that $\alpha > 0$. An important result in approximation theory states that

$$\mathbf{B}_{p,q}^{\alpha,b}(\mathbb{T}) = (L_p(\mathbb{T}), T_n)_q^{(\alpha,b)} \tag{2.28}$$

with equivalence of quasi-norms. See [97, 51, 107]. In particular, we have that

$$\mathbf{B}_{p,q}^{\alpha}(\mathbb{T}) = (L_p(\mathbb{T}), T_n)_q^{\alpha}. \tag{2.29}$$

Next we show that formula (2.28) also holds in the limit case when $\alpha = 0$.

Lemma 2.7. *Let $0 < p, q \le \infty$ and $b \ge -1/q$. Then we have with equivalence of quasi-norms*

$$\mathbf{B}_{p,q}^{0,b}(\mathbb{T}) = (L_p(\mathbb{T}), T_n)_q^{(0,b)}. \tag{2.30}$$

Proof. If $1 \le p, q \le \infty$ this formula was established in [51, Corollary 7.1] by using weak type interpolation ideas. Next we check the case of the other values of parameters with the help of Jackson and Bernstein-type inequalities.

Assume that $0 < p < 1$. According to Ivanov [79] and Storozhenko, Krotov and Osval'd [113], for any $f \in L_p(\mathbb{T})$ we have

$$E_n(f) \le c\omega\left(f, \frac{\pi}{n}\right)_p \text{ where } c = c(p).$$

Hence

$$\begin{aligned}
\|f\|_{(L_p(\mathbb{T}))_q^{(0,b)}} &= \left(\sum_{n=1}^{\infty}[(1+\log n)^b E_n(f)]^q n^{-1}\right)^{1/q} \\
&\lesssim \left(\sum_{n=1}^{\infty}\left[(1+\log n)^b \omega\left(f, \frac{\pi}{n}\right)_p\right]^q n^{-1}\right)^{1/q} \\
&\lesssim \left(\omega\left(f,\pi\right)_p^q + \int_1^{\infty}\left[(1+\log t)^b \omega\left(f, \frac{\pi}{t}\right)_p\right]^q \frac{dt}{t}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T})} + \left(\int_0^1\left[\left(1+\log\frac{\pi}{t}\right)^b \omega(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{\mathbf{B}_{p,q}^{0,b}(\mathbb{T})}.
\end{aligned}$$

To establish the converse inequality, we first recall that it was also shown by Ivanov [79] and Storozhenko, Krotov and Osval'd [113] that

$$\omega\left(f, \frac{\pi}{n}\right)_p \le \frac{c}{n}\left(\sum_{k=1}^{n} k^{p-1}E_k(f)^p\right)^{1/p}.$$

Therefore

$$\begin{aligned}
\|f\|_{\mathbf{B}_{p,q}^{0,b}(\mathbb{T})} &\lesssim \|f\|_{L_p(\mathbb{T})} + \left(\sum_{n=0}^{\infty}[(1+n)^b\omega(f,2^{-n})_p]^q\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T})} + \left(\sum_{n=0}^{\infty}\left[(1+n)^b 2^{-n}\left(\sum_{k=1}^{2^n}k^{p-1}E_k(f)^p\right)^{1/p}\right]^q\right)^{1/q} \\
&\sim \|f\|_{L_p(\mathbb{T})} + \left(\sum_{n=0}^{\infty}\left[(1+n)^b 2^{-n}\left(\sum_{\nu=0}^{n}2^{\nu p}E_{2^\nu}(f)^p\right)^{1/p}\right]^q\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T})} + \left(\sum_{n=0}^{\infty}\left[(1+n)^{bp} 2^{-np}\sum_{\nu=0}^{n}2^{\nu p}E_{2^\nu}(f)^p\right]^{q/p}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T})} + \left(\sum_{n=0}^{\infty}\left[(1+n)^b E_{2^n}(f)\right]^q\right)^{1/q}
\end{aligned}$$

where we have used in the last inequality a variant of Hardy's inequality (see [103, Lemma 3.10]). Consequently,

$$\begin{aligned}\|f\|_{\mathbf{B}^{0,b}_{p,q}(\mathbb{T})} &\lesssim \left(\sum_{n=0}^{\infty}[(1+n)^b E_{2^n}(f)]^q\right)^{1/q}\\ &\sim \left(\sum_{n=1}^{\infty}[(1+\log n)^b E_n(f)]^q n^{-1}\right)^{1/q}\\ &= \|f\|_{(L_p(\mathbb{T}))^{(0,b)}_q}.\end{aligned}$$

□

Chapter 3

Reiteration of approximation constructions

Iteration of approximation procedures is an important question in the theory of approximation spaces with interesting applications in analysis. Let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in the quasi-Banach space X. Since $G_n \subseteq X_p^\alpha$ and $G_n \subseteq X_q^{(0,\gamma)}$ for any $n \in \mathbb{N}_0$, the sequence $(G_n)_{n\in\mathbb{N}_0}$ is also an approximation family in X_p^α and $X_q^{(0,\gamma)}$. Hence, we can apply again these constructions. Suppose $0 < \alpha, \beta < \infty$ and $0 < p, r \leq \infty$. It was shown by Pietsch [104, Theorem 3.2] that

$$(X_p^\alpha)_r^\beta = X_r^{\alpha+\beta}. \tag{3.1}$$

It is worthwhile mentioning that the parameter p is not important in the previous formula. As a consequence of (3.1) we obtain that the scale formed by classical approximation spaces is stable by iteration.

On the other hand, when we deal with limiting spaces, Fehér and Grässler [65, Theorem 2] proved that

$$(X_q^{(0,\gamma)})_r^{(0,\delta)} = X_r^{(0,\gamma+1/q+\delta)} \tag{3.2}$$

provided that $0 < q, r \leq \infty, \gamma > -1/q$ and $\delta > -1/r$. Note that now the parameter q is reflected in the resulting space. The limiting approximation scale is also closed by reiteration.

Our aim in this chapter is to study the stability properties when we apply first the construction $(\cdot)_p^\alpha$ and then $(\cdot)_q^{(0,\gamma)}$ or vice versa. This is done in Section 3.1. For some classical spaces, like Lebesgue and Besov spaces, the corresponding approximation spaces are identified. Then, in Section 3.2, the new results are applied to investigate several problems on Besov spaces. In many cases, the optimality of the obtained results is shown. The contents of this chapter are part of the papers [26] and [28].

3.1 Reiteration formulae

We start by determining the space that arises applying the construction $(\cdot)_p^\alpha$ to $X_q^{(0,\gamma)}$. For this aim, we first establish an inequality of Jackson's type.

Lemma 3.1. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Suppose that $0<q\le\infty$ and $\gamma>-1/q$. Then there is a constant $c>0$ such that*

$$E_{2n-1}(f;X)\le c(1+\log n)^{-(\gamma+1/q)}E_n(f;X_q^{(0,\gamma)})$$

for all $f\in X_q^{(0,\gamma)}$ and $n\in\mathbb{N}$.

Proof. We can find $g_1,g_2\in G_{n-1}$ such that

$$\|f-g_1\|_{X_q^{(0,\gamma)}}\le 2E_n(f;X_q^{(0,\gamma)})$$

and

$$\|f-g_1-g_2\|_X\le 2E_n(f-g_1;X).$$

So, $g_1+g_2\in G_{2n-2}$ and

$$E_{2n-1}(f;X)\le\|f-g_1-g_2\|_X\le 2E_n(f-g_1;X). \tag{3.3}$$

Since the sequence $(E_n(f-g_1))$ is monotone, we obtain

$$\begin{aligned}\|f-g_1\|_{X_q^{(0,\gamma)}}&\ge\left(\sum_{k=1}^{n}[(1+\log k)^\gamma E_k(f-g_1;X)]^q\,k^{-1}\right)^{1/q}\\&\ge E_n(f-g_1;X)\left(\sum_{k=1}^{n}(1+\log k)^{\gamma q}k^{-1}\right)^{1/q}\\&\gtrsim(1+\log n)^{\gamma+1/q}E_n(f-g_1;X).\end{aligned}$$

Consequently, by (3.3) and the choice of g_1, we conclude that

$$\begin{aligned}E_{2n-1}(f;X)&\le 2E_n(f-g_1;X)\\&\lesssim(1+\log n)^{-(\gamma+1/q)}\|f-g_1\|_{X_q^{(0,\gamma)}}\\&\lesssim(1+\log n)^{-(\gamma+1/q)}E_n(f;X_q^{(0,\gamma)}).\end{aligned}$$

□

Theorem 3.2. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose that $\alpha>0, 0<p,q\le\infty$ and $\gamma>-1/q$. Then*

$$(X_q^{(0,\gamma)})_p^\alpha=X_p^{(\alpha,\gamma+1/q)}$$

with equivalence of quasi-norms.

Proof. Take any $f \in (X_q^{(0,\gamma)})_p^\alpha$. Using Lemma 3.1 we obtain

$$\begin{aligned}
\|f\|_{X_p^{(\alpha,\gamma+1/q)}} &= \left(\sum_{n=1}^{\infty}\left[n^\alpha(1+\log n)^{\gamma+1/q}E_n(f;X)\right]^p n^{-1}\right)^{1/p}\\
&\lesssim \left(\sum_{n=1}^{\infty}\left[n^\alpha(1+\log n)^{\gamma+1/q}E_{2n-1}(f;X)\right]^p n^{-1}\right)^{1/p}\\
&\lesssim \left(\sum_{n=1}^{\infty}\left[n^\alpha E_n(f;X_q^{(0,\gamma)})\right]^p n^{-1}\right)^{1/p}\\
&= \|f\|_{(X_q^{(0,\gamma)})_p^\alpha}.
\end{aligned}$$

Conversely, according to (2.6), given $f \in X_p^{(\alpha,\gamma+1/q)}$, there exists a representation $f = \sum_{n=0}^\infty g_n$ with $g_n \in G_{2^n}$ and

$$\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}(1+n)^{\gamma+1/q}\|g_n\|_X\right]^p\right)^{1/p} \lesssim \|f\|_{X_p^{(\alpha,\gamma+1/q)}}.$$

Since

$$\begin{aligned}
\|g_n\|_{X_q^{(0,\gamma)}} &= \left(\sum_{k=1}^{2^n}\left[(1+\log k)^\gamma E_k(g_n;X)\right]^q k^{-1}\right)^{1/q}\\
&\le \left(\sum_{k=1}^{2^n}(1+\log k)^{\gamma q}k^{-1}\right)^{1/q}\|g_n\|_X\\
&\lesssim (1+n)^{\gamma+1/q}\|g_n\|_X,
\end{aligned}$$

we derive that

$$\begin{aligned}
\left(\sum_{n=0}^{\infty}\left[2^{n\alpha}\|g_n\|_{X_q^{(0,\gamma)}}\right]^p\right)^{1/p} &\lesssim \left(\sum_{n=0}^{\infty}\left[2^{n\alpha}(1+n)^{\gamma+1/q}\|g_n\|_X\right]^p\right)^{1/p}\\
&\lesssim \|f\|_{X_p^{(\alpha,\gamma+1/q)}}.
\end{aligned}$$

This yields that the series $\sum_{n=0}^\infty g_n$ converges to f in $X_q^{(0,\gamma)}$. Finally, by [104, Theorem 3.1], we get

$$\begin{aligned}
\|f\|_{(X_q^{(0,\gamma)})_p^\alpha} &\lesssim \left(\sum_{n=0}^{\infty}\left[2^{n\alpha}\|g_n\|_{X_q^{(0,\gamma)}}\right]^p\right)^{1/p}\\
&\lesssim \|f\|_{X_p^{(\alpha,\gamma+1/q)}}.
\end{aligned}$$

□

Writing down Theorem 3.2 in the special case $1 \le p = q \le \infty$, with X being a Banach space and $(G_n)_{n\in\mathbb{N}_0}$ being a sequence of subspaces of X, we recover a result of Almira and Luther [2, Theorem 6.1].

Example 3.1. If $X = \ell_\infty$ and $G_n = F_n$, the subset of sequences having at most n coordinates different from 0, then for $\alpha > 0, 0 < p, q \le \infty$ and $\gamma > -1/q$, we obtain

$$\begin{aligned}(\ell_{\infty,q}(\log \ell)_\gamma)^\alpha_p &= ((\ell_\infty)^{(0,\gamma)}_q)^\alpha_p = (\ell_\infty)^{(\alpha,\gamma+1/q)}_p\\ &= \ell_{1/\alpha,p}(\log \ell)_{\gamma+1/q}.\end{aligned}$$

Example 3.2. Let $X = L_p(\mathbb{T})$ and $G_n = T_n$, the subset formed by all trigonometric polynomials with degree less than or equal to n. Given $\alpha > 0, 0 < p, q, r \le \infty, \gamma > -1/q$, we derive

$$\begin{aligned}(\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}))^\alpha_r &= ((L_p(\mathbb{T}))^{(0,\gamma)}_q)^\alpha_r\\ &= (L_p(\mathbb{T}))^{(\alpha,\gamma+1/q)}_r\\ &= \mathbf{B}^{\alpha,\gamma+1/q}_{p,r}(\mathbb{T})\end{aligned}$$

where we have used (2.28) in last step.

It is more difficult to determine the resulting space when we apply the approximation constructions in reverse order. We shall do it with the help of interpolation techniques.

Theorem 3.3. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \le \infty$ and $\gamma \ge -1/q$. Then*

$$(X, X^\alpha_p)_{(0,-\gamma),q} = X^{(0,\gamma)}_q \tag{3.4}$$

with equivalence of quasi-norms.

Proof. First, we shall prove that

$$(X, X^{1/p}_p)_{(0,-\gamma),q} = X^{(0,\gamma)}_q. \tag{3.5}$$

Indeed, doing a change of variable, using Lemma 2.5 and Hardy's inequality (see [45, Theorem 1.2]), we derive

$$\begin{aligned}\|f\|_{(X,X^{1/p}_p)_{(0,-\gamma),q}} &= \left(\int_0^1 \Big[(1-\log t)^\gamma t K(t^{-1}, f; X^{1/p}_p, X)\Big]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \left(\int_1^\infty \Big[(1+\log s)^\gamma s^{-1/p} K(s^{1/p}, f; X^{1/p}_p, X)\Big]^q \frac{ds}{s}\right)^{1/q}\\ &\sim \left(\sum_{n=1}^\infty \Big[(1+\log n)^\gamma n^{-1/p} K(n^{1/p}, f; X^{1/p}_p, X)\Big]^q n^{-1}\right)^{1/q}\\ &\sim \left(\sum_{n=1}^\infty \left[(1+\log n)^\gamma \left(\frac{1}{n}\sum_{k=1}^n E_k(f)^p\right)^{1/p}\right]^q n^{-1}\right)^{1/q}\\ &\sim \left(\sum_{n=1}^\infty [(1+\log n)^\gamma E_n(f)]^q n^{-1}\right)^{1/q} = \|f\|_{X^{(0,\gamma)}_q}.\end{aligned}$$

Next let us show (3.4). Take $0 < \rho < 1/\alpha$ and put $\theta = \alpha\rho$. By (2.18), we have that $X_p^\alpha = (X, X_\rho^{1/\rho})_{\theta,p}$. Whence, according to Lemma 2.2(b) and (3.5), we derive

$$\begin{aligned}(X, X_p^\alpha)_{(0,-\gamma),q} &= (X, (X, X_\rho^{1/\rho})_{\theta,p})_{(0,-\gamma),q} \\ &= (X, X_\rho^{1/\rho})_{(0,-\gamma),q} = X_q^{(0,\gamma)}.\end{aligned}$$

□

Now we can determine $(X_p^\alpha)_q^{(0,\gamma)}$ by means of an auxiliary sequence space.

Theorem 3.4. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \le \infty$ and $\gamma \ge -1/q$. Take any $0 < r < 1/\alpha$. Then we have*

$$f \in (X_p^\alpha)_q^{(0,\gamma)} \textit{ if and only if } (E_n(f)) \in (\ell_{1/\alpha,p}, \ell_r)_{(0,-\gamma),q}.$$

Proof. Using Theorem 3.3 and (3.1), we get

$$\begin{aligned}(X_p^\alpha)_q^{(0,\gamma)} &= (X_p^\alpha, (X_p^\alpha)_r^{1/r-\alpha})_{(0,-\gamma),q} \\ &= (X_p^\alpha, X_r^{1/r})_{(0,-\gamma),q}.\end{aligned}$$

In order to estimate the K-functional for the couple $(X_p^\alpha, X_r^{1/r})$, take $0 < \theta < 1$ such that $\alpha = (1-\theta)/r$. By (2.18) we have $X_p^\alpha = (X_r^{1/r}, X)_{\theta,p}$. Whence, according to Holmstedt's formula [78, Remark 2.1] and Lemma 2.5, we derive

$$\begin{aligned}K(2^{n\theta/r}, f; X_r^{1/r}, X_p^\alpha) &= K(2^{n\theta/r}, f; X_r^{1/r}, (X_r^{1/r}, X)_{\theta,p}) \\ &\sim 2^{n\theta/r}\left(\int_{2^{n/r}}^\infty \left[u^{-\theta}K(u, f; X_r^{1/r}, X)\right]^p \frac{du}{u}\right)^{1/p} \\ &\sim 2^{n\theta/r}\left(\sum_{k=n}^\infty \left[2^{-\theta k/r}K(2^{k/r}, f; X_r^{1/r}, X)\right]^p\right)^{1/p} \\ &\sim 2^{n\theta/r}\left(\sum_{k=n}^\infty \left[2^{-\theta k/r}\left(\sum_{j=1}^{2^k} E_j(f)^r\right)^{1/r}\right]^p\right)^{1/p} \\ &\sim 2^{n\theta/r}\left(\sum_{k=n}^\infty \left[2^{-\theta k/r}K(2^{k/r}, (E_j(f)); \ell_r, \ell_\infty)\right]^p\right)^{1/p}\end{aligned}$$

where we have used again Lemma 2.5 in the last equivalence but now with the couple (ℓ_r, ℓ_∞), viewing ℓ_r as $(\ell_\infty)_r^{1/r}$. Hence, reversing the steps and using (2.18), we conclude that

$$\begin{aligned} &K(2^{n\theta/r}, f; X_r^{1/r}, X_p^{\alpha}) \\ &\quad\sim 2^{n\theta/r}\left(\int_{2^{n/r}}^{\infty}\left[u^{-\theta}K\left(u,(E_j(f));\ell_r,\ell_\infty\right)\right]^p\frac{du}{u}\right)^{1/p} \\ &\quad\sim K(2^{n\theta/r},(E_j(f));\ell_r,(\ell_r,\ell_\infty)_{\theta,p}) \\ &\quad\sim K(2^{n\theta/r},(E_j(f));\ell_r,\ell_{1/\alpha,p}). \end{aligned}$$

This yields that

$$K(t,f;X_r^{1/r},X_p^{\alpha}) \sim K(t,(E_j(f));\ell_r,\ell_{1/\alpha,p}), 1 \le t < \infty.$$

Reversing the couple, we obtain that

$$K(t,f;X_p^{\alpha},X_r^{1/r}) \sim K(t,(E_j(f));\ell_{1/\alpha,p},\ell_r), 0 < t \le 1.$$

Therefore

$$\|f\|_{(X_p^\alpha)_q^{(0,\gamma)}} \sim \|f\|_{(X_p^\alpha,X_r^{1/r})_{(0,-\gamma),q}} \sim \|(E_j(f))\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}}$$

which completes the proof. □

We proceed now to study the sequence space that arose in Theorem 3.4.

Definition 3.1. Let $\alpha > 0, 0 < p, q \le \infty$ and $-\infty < \gamma < \infty$. We put

$$\begin{aligned} Z &= Z_{\alpha,p,\gamma,q} \\ &= \left\{\xi \in \ell_\infty : \|\xi\|_Z = \left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}\left(\sum_{j=n}^{\infty}(j^{\alpha}\xi_j^*)^p j^{-1}\right)^{1/p}\right]^q n^{-1}\right)^{1/q} < \infty\right\}. \end{aligned}$$

Note that when $q = 1$, $Z_{\alpha,p,\gamma,1}$ is a *small Lorentz sequence space* in the terminology of Fiorenza and Karadzhov [67].

Lemma 3.5. *Let $\alpha > 0, 0 < p, q \le \infty, \gamma \ge -1/q$ and $0 < r < \min\{1/\alpha, q\}$. Then we have with equivalent quasi-norms*

$$(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q} = Z \cap \ell_{1/\alpha,q}(\log \ell)_\gamma$$

where $\|\xi\|_{Z\cap\ell_{1/\alpha,q}(\log\ell)_\gamma} = \|\xi\|_Z + \|\xi\|_{\ell_{1/\alpha,q}(\log\ell)_\gamma}$.

Proof. Let $1/\delta = 1/r - \alpha$. According to [78, Theorem 4.2], we have that

$$K(t,\xi;\ell_r,\ell_{1/\alpha,p}) \sim \left(\int_0^{t^\delta} \xi^*(v)^r dv\right)^{1/r} + t\left(\int_{t^\delta}^\infty (v^\alpha \xi^*(v))^p \frac{dv}{v}\right)^{1/p}, 0 < t < \infty.$$

Here

$$\xi^*(t) = \begin{cases} \xi_1^* & \text{for} \quad t \in (0,1), \\ \xi_n^* & \text{for} \quad t \in [n-1,n), n = 2,3,... \end{cases}$$

Therefore,

$$\begin{aligned}
\|\xi\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}} &= \left(\int_0^1 \left[(1-\log t)^\gamma t K(t^{-1},\xi;\ell_r,\ell_{1/\alpha,p})\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_1^\infty \left[(1+\log t)^\gamma t^{-1} K(t,\xi;\ell_r,\ell_{1/\alpha,p})\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_1^\infty \left[(1+\log t)^\gamma t^{-1} \left(\int_0^{t^\delta} \xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[(1+\log t)^\gamma \left(\int_{t^\delta}^\infty (v^\alpha \xi^*(v))^p \frac{dv}{v}\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= I_1 + I_2.
\end{aligned}$$

For the term I_1, a change of variables yields

$$I_1 \sim \left(\int_1^\infty \left[t^{-1/\delta}(1+\log t)^\gamma \left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Since

$$\begin{aligned}
&\left(\int_0^1 \left[t^{-1/\delta}(1-\log t)^\gamma \left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\qquad \le \xi_1^* \left(\int_0^1 \left[t^{1/r-1/\delta}(1-\log t)^\gamma\right]^q \frac{dt}{t}\right)^{1/q} \\
&\qquad \lesssim \xi_1^* \\
&\qquad \lesssim \left(\int_0^1 \xi^*(v)^r dv\right)^{1/r} \left(\int_1^\infty \left[t^{-1/\delta}(1+\log t)^\gamma\right]^q \frac{dt}{t}\right)^{1/q} \\
&\qquad \le \left(\int_1^\infty \left[t^{-1/\delta}(1+\log t)^\gamma \left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q},
\end{aligned}$$

we obtain that

$$
\begin{aligned}
I_1 &\sim \left(\int_0^\infty \left[t^{-1/\delta}(1+|\log t|)^\gamma \left(\int_0^t \xi^*(v)^r dv\right)^{1/r}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^\infty \left[t^{-r/\delta}(1+|\log t|)^{\gamma r} \int_0^t \xi^*(v)^r dv\right]^{q/r} \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

Now using the Hardy-type inequality given in [63, Lemma 4.1], we derive

$$
\begin{aligned}
I_1 &\sim \left(\int_0^\infty \left[t^{1-r/\delta}(1+|\log t|)^{\gamma r}\xi^*(t)^r\right]^{q/r} \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^\infty \left[t^\alpha(1+|\log t|)^\gamma\xi^*(t)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\sum_{n=1}^\infty \left[n^\alpha(1+\log n)^\gamma\xi_n^*\right]^q \frac{1}{n}\right)^{1/q}.
\end{aligned}
$$

As for I_2, we get

$$
\begin{aligned}
I_2 &\sim \left(\int_1^\infty \left[(1+\log t)^\gamma \left(\int_t^\infty (v^\alpha\xi^*(v))^p \frac{dv}{v}\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\sum_{n=1}^\infty \int_n^{n+1} \left[(1+\log t)^\gamma \left(\int_t^\infty (v^\alpha\xi^*(v))^p \frac{dv}{v}\right)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_2 &\lesssim \left(\sum_{n=1}^\infty \left[(1+\log n)^\gamma \left(\sum_{j=n}^\infty \int_j^{j+1} (v^\alpha\xi^*(v))^p \frac{dv}{v}\right)^{1/p}\right]^q \frac{1}{n}\right)^{1/q} \\
&\sim \left(\sum_{n=1}^\infty \left[(1+\log n)^\gamma \left(\sum_{j=n}^\infty ((j+1)^\alpha\xi_{j+1}^*)^p \frac{1}{j+1}\right)^{1/p}\right]^q \frac{1}{n}\right)^{1/q}.
\end{aligned}
$$

Similarly,

$$
I_2 \gtrsim \left(\sum_{n=1}^\infty \left[(1+\log n)^\gamma \left(\sum_{j=n+1}^\infty ((j+1)^\alpha\xi_{j+1}^*)^p \frac{1}{j+1}\right)^{1/p}\right]^q \frac{1}{n}\right)^{1/q}.
$$

Consequently,

$$\begin{aligned}
\|\xi\|_{(\ell_{1/\alpha,p},\ell_r)_{(0,-\gamma),q}} &\sim I_1 + I_2\\
&\sim \left(\sum_{n=1}^{\infty}[n^{\alpha}(1+\log n)^{\gamma}\xi_n^*]^q\,\frac{1}{n}\right)^{1/q}\\
&+\left(\sum_{n=1}^{\infty}\left[(1+\log n)^{\gamma}\left(\sum_{j=n+1}^{\infty}((j+1)^{\alpha}\xi_{j+1}^*)^p\frac{1}{j+1}\right)^{1/p}\right]^q\frac{1}{n}\right)^{1/q}\\
&\sim \|\xi\|_{\ell_{1/\alpha,q}(\log\ell)_{\gamma}} + \|\xi\|_Z.
\end{aligned}$$

□

As a direct consequence of Theorem 3.4 and Lemma 3.5, we can now show the following explicit description of $(X_p^{\alpha})_q^{(0,\gamma)}$.

Theorem 3.6. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < p, q \le \infty$ and $\gamma \ge -1/q$. Put*

$$W = W_{\alpha,p,\gamma,q} = \{f \in X : (E_n(f)) \in Z_{\alpha,p,\gamma,q}\}$$

with $\|f\|_W = \|(E_n(f))\|_Z$. Then we have with equivalence of quasi-norms

$$(X_p^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap W.$$

Corollary 3.7. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family. Suppose $\alpha > 0, 0 < q \le \infty$ and $\gamma > -1/q$. Then we have with equivalence of quasi-norms*

$$(X_q^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma+1/q)}$$

Proof. Using equality of the lower parameters, we obtain

$$\begin{aligned}
\|f\|_W &= \left(\sum_{j=1}^{\infty}(j^{\alpha}E_j(f))^q j^{-1}\sum_{n=1}^{j}(1+\log n)^{\gamma q}n^{-1}\right)^{1/q}\\
&\sim \left(\sum_{j=1}^{\infty}(j^{\alpha}(1+\log j)^{\gamma+1/q}E_j(f))^q j^{-1}\right)^{1/q}\\
&= \|f\|_{X_q^{(\alpha,\gamma+1/q)}}.
\end{aligned}$$

Therefore, applying Theorem 3.6, we derive

$$(X_q^{\alpha})_q^{(0,\gamma)} = X_q^{(\alpha,\gamma)} \cap X_q^{(\alpha,\gamma+1/q)} = X_q^{(\alpha,\gamma+1/q)}.$$

□

Corollary 3.7 and Theorem 3.2 show that in the "diagonal case" where $p = q$ the order of application of the approximation constructions is not important.

As we have seen in Theorem 3.6, in general $(X_p^{\alpha})_q^{(0,\gamma)}$ cannot be realized as a space $X_q^{(\alpha,\omega)}$. In applications it is important to know the biggest (respectively, smallest) space $X_q^{(\alpha,\omega)}$ which is contained in (respectively, which contains to) $(X_p^{\alpha})_q^{(0,\gamma)}$. This is done by using the following relationships between different interpolation methods introduced in Chapter 2.

Lemma 3.8. *Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Assume that $0 < \theta < 1, 0 < p, q \le \infty$ and $\gamma < -1/q < \eta$. The following continuous embeddings hold*

(a)
$$(A_0, A_1)_{\theta,q,-\gamma-1/\min\{p,q\}} \hookrightarrow (A_0, (A_0, A_1)_{\theta,p})_{(1,-\gamma),q} \hookrightarrow (A_0, A_1)_{\theta,q,-\gamma-1/\max\{p,q\}},$$
(b)
$$(A_0, A_1)_{\theta,q,-\eta-1/\min\{p,q\}} \hookrightarrow ((A_0, A_1)_{\theta,p}, A_1)_{(0,-\eta),q} \hookrightarrow (A_0, A_1)_{\theta,q,-\eta-1/\max\{p,q\}}.$$

Proof. Take $\alpha > -1/q$. According to [58, Proposition 1] (which also works in the quasi-normed case), we have that

$$(A_0, (A_0, A_1)_{\theta,p})_{(1,-\gamma),q} = (A_0, (A_0, A_1)_{\theta,p})_{1,q,(-\gamma,-\alpha)}.$$

By [61, Theorems 5.9* and 4.7*], we derive that

$$\begin{aligned}(A_0, A_1)_{\theta,q,(-\gamma-\frac{1}{\min\{p,q\}},-\alpha-\frac{1}{\min\{p,q\}})} &\hookrightarrow (A_0, (A_0, A_1)_{\theta,p})_{(1,-\gamma),q} \\ &\hookrightarrow (A_0, A_1)_{\theta,q,(-\gamma-\frac{1}{\max\{p,q\}},-\alpha-\frac{1}{\max\{p,q\}})}.\end{aligned}$$

Now applying again [58, Proposition 1], we see that the space to the left is $(A_0, A_1)_{\theta,q,-\gamma-1/\min\{p,q\}}$ and the space to the right is $(A_0, A_1)_{\theta,q,-\gamma-1/\max\{p,q\}}$. This completes the proof of the embeddings (a). The proof of (b) can be carried out in the same way but using [61, Theorems 5.7 and 4.7]. $\square$

Remark 3.1. Note that if $\gamma = -1/q$ then $(A_0, (A_0, A_1)_{\theta,p})_{(1,-\gamma),q} = \{0\}$, so the left-hand side embedding in Lemma 3.8(a) does not hold. On the other hand, the right-hand side embedding in the same statement is true in this limiting situation. More interesting is the extreme case $\eta = -1/q$ in Lemma 3.8(b). We will return to this point in Chapter 5.

Theorem 3.9. *Let $\alpha > 0, 0 < p, q \le \infty$ and $\gamma > -1/q$. Then*

$$X_q^{(\alpha,\gamma+\frac{1}{\min\{p,q\}})} \hookrightarrow (X_p^{\alpha})_q^{(0,\gamma)} \hookrightarrow X_q^{(\alpha,\gamma+\frac{1}{\max\{p,q\}})}.$$

Proof. Let $\beta > \alpha$. By Theorem 3.3 and (3.1), we obtain that

$$(X_p^\alpha)_q^{(0,\gamma)} = (X_p^\alpha, X_p^\beta)_{(0,-\gamma),q}.$$

Moreover, it follows from (2.18) that $X_p^\alpha = (X, X_p^\beta)_{\theta,p}$ for $\alpha = \theta\beta$. Therefore, Lemma 3.8(b) and Proposition 2.6 yield the wanted embeddings. □

Remark 3.2. Embeddings in Theorem 3.9 are the best possible in the sense that in general for any $\tau > 0$ embeddings

$$X_q^{(\alpha,\gamma+\frac{1}{\min\{p,q\}}-\tau)} \hookrightarrow (X_p^\alpha)_q^{(0,\gamma)}, \tag{3.6}$$

$$(X_p^\alpha)_q^{(0,\gamma)} \hookrightarrow X_q^{(\alpha,\gamma+\frac{1}{\max\{p,q\}}+\tau)} \tag{3.7}$$

do not hold. We show it now by means of counterexamples.

Take $X = \ell_\infty$ and $G_n = F_n$, so $E_n(\xi;\ell_\infty) = \xi_n^*$. As for (3.6), suppose that $\min\{p,q\} = p$. Given any $\tau > 0$, choose $\epsilon > 0$ such that $\tau - \epsilon > 0$. The sequence $\xi = (n^{-\alpha}(1+\log n)^{-(\gamma+1/p-\tau+1/q+\epsilon)})$ belongs to $\ell_{1/\alpha,q}(\log \ell)_{\gamma+1/p-\tau} = X_q^{(\alpha,\gamma+1/p-\tau)}$. However, since

$$\left(\sum_{j=n}^{\infty}(1+\log j)^{-(\gamma+1/p-\tau+1/q+\epsilon)p} j^{-1}\right)^{1/p} \sim (1+\log n)^{-(\gamma-\tau+1/q+\epsilon)}$$

if $\tau < \gamma + 1/q + \epsilon$ and the series diverges otherwise, it follows that

$$\begin{aligned}\|\xi\|_{(X_p^\alpha)_q^{(0,\gamma)}} &\gtrsim \|\xi\|_W \\ &\sim \left(\sum_{n=1}^{\infty}\left[(1+\log n)^\gamma (1+\log n)^{-(\gamma-\tau+1/q+\epsilon)}\right]^q n^{-1}\right)^{1/q} \\ &= \left(\sum_{n=1}^{\infty}(1+\log n)^{q(\tau-\epsilon)-1} n^{-1}\right)^{1/q} = \infty.\end{aligned}$$

Hence, (3.6) does not hold.

As for (3.7), assume that $\max\{p,q\} = p$. Given any $\tau > 0$, let $0 < \epsilon < \tau$ and put

$$\xi = \left(n^{-\alpha}(1+\log n)^{-(\gamma+1/p+1/q+\epsilon)}\right).$$

We claim that $\xi \in (X_p^\alpha)_q^{(0,\gamma)}$. Indeed,

$$\|\xi\|_{X_q^{(\alpha,\gamma)}} = \|\xi\|_{\ell_{1/\alpha,q}(\log\ell)_\gamma} = \left(\sum_{n=1}^{\infty}(1+\log n)^{-q/p-\epsilon q-1} n^{-1}\right)^{1/q} < \infty,$$

and

$$\|\xi\|_W = \left(\sum_{n=1}^{\infty} \left[(1+\log n)^{\gamma} \left(\sum_{j=n}^{\infty} (1+\log j)^{-(\gamma+1/q+\epsilon)p-1} j^{-1} \right)^{1/p} \right]^q n^{-1} \right)^{1/q}$$
$$\sim \left(\sum_{n=1}^{\infty} (1+\log n)^{-\epsilon q-1} n^{-1} \right)^{1/q} < \infty.$$

Hence, according to Theorem 3.6, $\xi \in (X_p^{\alpha})_q^{(0,\gamma)}$. However, $\xi \notin \ell_{1/\alpha,q}(\log \ell)_{\gamma+1/p+\tau} = X_q^{(\alpha,\gamma+\frac{1}{\max\{p,q\}}+\tau)}$.

3.2 Applications to Besov spaces

In this section we apply the previous results to study several problems on Besov spaces.

In what follows we take $X = L_p(\mathbb{T})$ and we choose G_n as the set T_n of all trigonometric polynomials of degree less than or equal to n (see Chapter 2 for precise definitions). Recall that X_r^{α} is the (classical) Besov space $\mathbf{B}_{p,r}^{\alpha}(\mathbb{T})$ (see (2.29)), $X_q^{(0,\gamma)}$ coincides with the Besov space of logarithmic smoothness $\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$ (see (2.30)) and $X_r^{(\alpha,\eta)}$ with $\mathbf{B}_{p,r}^{\alpha,\eta}(\mathbb{T})$ (see (2.28)).

We are interested in the relationship between smoothness of derivatives of f and the smoothness of f. Let $k \in \mathbb{N}$. It is well known that if $f \in \mathbf{B}_{p,q}^{\alpha}(\mathbb{T})$ with $\alpha > k$ then $D^k f \in \mathbf{B}_{p,q}^{\alpha-k}(\mathbb{T})$. This property can be extended to spaces $\mathbf{B}_{p,q}^{\alpha,\gamma}(\mathbb{T})$ for $\alpha > k$ and $-\infty < \gamma < \infty$ [51, page 70]. Now we deal with the limiting situation when $\alpha = k$. According to [51, page 70], if $f \in \mathbf{B}_{p,q}^{k,\gamma+1}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$. The following result shows that sometimes the loss of smoothness is less than a logarithm.

Theorem 3.10. *Let $k \in \mathbb{N}, 1 < p < \infty, 0 < q \leq \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}_{p,q}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$.*

Proof. Clearly $D^k : W^{k,p}(\mathbb{T}) \longrightarrow L_p(\mathbb{T})$ is bounded and $D^k(T_n) \subseteq T_n$. Then $E_m(D^k f; L_p(\mathbb{T})) \leq E_m(f; W^{k,p}(\mathbb{T}))$, and so

$$D^k : (W^{k,p}(\mathbb{T}))_q^{(0,\gamma)} \longrightarrow (L_p(\mathbb{T}))_q^{(0,\gamma)} = \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$$

is bounded. Moreover, it follows from the embedding $B^k_{p,\min\{p,2\}}(\mathbb{T}) \hookrightarrow W^{k,p}(\mathbb{T})$ (see [107, Remark 4, page 164 and Theorem 3.5.4, page 169]) that $(\mathbf{B}^k_{p,\min\{p,2\}}(\mathbb{T}))^{(0,\gamma)}_q \hookrightarrow (W^{k,p}(\mathbb{T}))^{(0,\gamma)}_q$. Finally, by Theorem 3.9,

$$\begin{aligned}\mathbf{B}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}_{p,q}(\mathbb{T}) &= (L_p(\mathbb{T}))^{(k,\gamma+\frac{1}{\min\{q,\min\{p,2\}\}})}_q \\ &\hookrightarrow ((L_p(\mathbb{T}))^k_{\min\{p,2\}})^{(0,\gamma)}_q = (\mathbf{B}^k_{p,\min\{p,2\}}(\mathbb{T}))^{(0,\gamma)}_q.\end{aligned}$$

Therefore, $D^k : \mathbf{B}^{k,\gamma+\frac{1}{\min\{2,p,q\}}}_{p,q}(\mathbb{T}) \longrightarrow \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ is bounded. □

Next we prove that the result given in Theorem 3.10 is in general the best possible.

Proposition 3.11. *Let $2 \le q < \infty$ and $\gamma > -1/q$. For any $\epsilon > 0$ there is a function $f \in \mathbf{B}^{1,\gamma+1/2-\epsilon}_{2,q}(\mathbb{T})$ such that $f' \notin \mathbf{B}^{0,\gamma}_{2,q}(\mathbb{T})$.*

Proof. Since $\mathbf{B}^{1,\eta}_{2,q}(\mathbb{T}) \hookrightarrow \mathbf{B}^{1,\delta}_{2,q}(\mathbb{T})$ if $\delta \le \eta$, we may assume without loss of generality that $0 < \epsilon < b + 1/q$. Take $\beta \in \mathbb{R}$ such that $-\gamma - 1/q - 1/2 \le \beta < -\gamma - 1/q - 1/2 + \epsilon$ and let

$$f(x) = \sum_{k=1}^{\infty} k^{-3/2}(1+\log k)^{\beta}\cos(kx), x \in \mathbb{T}.$$

This function belongs to $L_2(\mathbb{T})$ because

$$\|f\|^2_{L_2(\mathbb{T})} = \pi \sum_{k=1}^{\infty} k^{-3}(1+\log k)^{2\beta} < \infty.$$

We proceed to estimate the modulus of smoothness of order 2 of f with the help of [110, Lemma 3]. Let $S_n f$ be the n-th partial sum of the Fourier series of f and let $S'_n f, S''_n f$ be its first and second derivatives, respectively. We have for $n \in \mathbb{N}$ that

$$\begin{aligned}\omega_2(f,1/n)_2 &\sim \|f - S_n f\|_{L_2(\mathbb{T})} + n^{-2}\|S''_n f\|_{L_2(\mathbb{T})} \\ &\sim \left(\sum_{k=n+1}^{\infty} k^{-3}(1+\log k)^{2\beta}\right)^{1/2} + n^{-2}\left(\sum_{k=1}^{n} k(1+\log k)^{2\beta}\right)^{1/2} \\ &\sim n^{-1}(1+\log n)^{\beta}.\end{aligned}$$

So $\omega_2(f,t)_2 \sim t(1-\log t)^{\beta}$ for $0 < t < 1$ and therefore

$$\int_0^1 [t^{-1}(1-\log t)^{\gamma+1/2-\epsilon}\omega_2(f,t)_2]^q \frac{dt}{t} \sim \int_0^1 (1-\log t)^{(\gamma+1/2-\epsilon+\beta)q}\frac{dt}{t}$$

which is finite by our choice of β. Hence $f \in \mathbf{B}^{1,\gamma+1/2-\epsilon}_{2,q}(\mathbb{T})$.

Nevertheless, $f'(x) = -\sum_{k=1}^{\infty} k^{-1/2}(1+\log k)^{\beta}\sin(kx)$ and

$$\begin{aligned}\omega_1(f',1/n)_2 &\sim \|f'-S_nf'\|_{L_2(\mathbb{T})} + n^{-1}\|S_n'f'\|_{L_2(\mathbb{T})}\\ &\sim \left(\sum_{k=n+1}^{\infty} k^{-1}(1+\log k)^{2\beta}\right)^{1/2} + n^{-1}\left(\sum_{k=1}^{n} k(1+\log k)^{2\beta}\right)^{1/2}\\ &\sim (1+\log n)^{\beta+1/2}\end{aligned}$$

because $\beta+1/2<0$. This yields that

$$\omega_1(f',t)_2 \sim (1-\log t)^{\beta+1/2} \text{ for } 0<t<1.$$

Using now that $\gamma+\beta+1/2+1/q \geq 0$, we derive that

$$\int_0^1 [(1-\log t)^{\gamma}\omega_1(f',t)_2]^q \frac{dt}{t} \sim \int_0^1 (1-\log t)^{(\gamma+\beta+1/2)q}\frac{dt}{t} = \infty.$$

Consequently, $f' \notin \mathbf{B}_{2,q}^{0,\gamma}(\mathbb{T})$. □

Given any integrable function f on $\mathbb{T}$, its *Fourier coefficients* are defined by

$$\hat{f}(m) = c_m = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-imx}dx\,,\ m\in\mathbb{Z}. \tag{3.8}$$

We write $\mathcal{F}$ for the operator assigning to any function f the sequence $\mathcal{F}(f) = (\hat{f}(m))$ of its Fourier coefficients.

Next we use the reiteration results obtained in the previous section to study the behaviour of Fourier coefficients of functions in Besov spaces with logarithmic smoothness.

Theorem 3.12. *Let $1\leq p\leq 2, 1/p' = 1-1/p, 0<q\leq\infty$ and $\gamma>-1/q$. If $f\in\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$ then $(\hat{f}(m))$ belongs to $\ell_{p',q}(\log\ell)_{\gamma+\frac{1}{\max\{p',q\}}}$.*

Proof. By Hausdorff-Young inequality, $\mathcal{F}: L_p(\mathbb{T}) \longrightarrow \ell_{p'}$ is bounded. Besides $\mathcal{F}(T_n)\subseteq F_{2n+1}$, the subset of sequences having at most $2n+1$ coordinates different from 0. Therefore $E_{2(n+1)}(\mathcal{F}(f);\ell_{p'}) \lesssim E_{n+1}(f;L_p(\mathbb{T}))$ for $n\in\mathbb{N}$. It follows that

$$\mathcal{F}:\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T}) = (L_p(\mathbb{T}),T_n)_q^{(0,\gamma)} \longrightarrow (\ell_{p'},F_n)_q^{(0,\gamma)}$$

is bounded. Now, according to Theorem 3.9, we have that

$$\begin{aligned}(\ell_{p'},F_n)_q^{(0,\gamma)} &= ((\ell_\infty)_{p'}^{1/p'})_q^{(0,\gamma)} \hookrightarrow (\ell_\infty)_q^{(1/p',\gamma+\frac{1}{\max\{p',q\}})}\\ &= \ell_{p',q}(\log\ell)_{\gamma+\frac{1}{\max\{p',q\}}}.\end{aligned}$$

This completes the proof. □

The distribution of the Fourier coefficients of functions of $\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ was considered by DeVore, Riemenschneider and Sharpley in [51, Corollary 7.3(i)]. They proved that if $1 \le q \le \infty$ and $f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$ then $(\hat{f}(m)) \in \ell_{p',q}(\log \ell)_\gamma$. Theorem 3.12 improves this result in two ways: On the one hand, if $p \ne 1$ and $q \ne \infty$,

$$\ell_{p',q}(\log \ell)_{\gamma + \frac{1}{\max\{p',q\}}} \subsetneq \ell_{p',q}(\log \ell)_\gamma,$$

on the other hand q can now take values less than 1. Besides it is best possible in general as we show next.

Proposition 3.13. *Let $0 < q \le 2$ and $\gamma > -1/q$. Given any $\epsilon > 0$, there exists $f \in \mathbf{B}^{0,\gamma}_{2,q}(\mathbb{T})$ such that $(\hat{f}(m)) \notin \ell_{2,q}(\log \ell)_{\gamma+1/2+\epsilon}$.*

Proof. Take $\beta \in \mathbb{R}$ such that $\gamma + 1/2 + 1/q < \beta < \gamma + 1/2 + 1/q + \epsilon$ and consider the function

$$f(x) = \sum_{k=1}^{\infty} k^{-1/2}(1 + \log k)^{-\beta} \cos(kx), x \in \mathbb{T}.$$

Since $\beta > 1/2$, we have that

$$\|f\|^2_{L_2(\mathbb{T})} = \pi \sum_{k=1}^{\infty} k^{-1}(1 + \log k)^{-2\beta} < \infty.$$

Using [110, Lemma 3], we obtain that

$$\begin{aligned}\omega(f, 1/n)_2 &\sim \|f - S_n f\|_{L_2(\mathbb{T})} + n^{-1}\|S_n' f\|_{L_2(\mathbb{T})} \\ &\sim \left(\sum_{k=n+1}^{\infty} k^{-1}(1 + \log k)^{-2\beta}\right)^{1/2} \\ &\qquad\qquad + n^{-1}\left(\sum_{k=1}^{n} k(1 + \log k)^{-2\beta}\right)^{1/2} \\ &\sim (1 + \log n)^{-\beta+1/2}\end{aligned}$$

where we have used again that $\beta > 1/2$ in the last equivalence. So

$$\omega(f, t)_2 \sim (1 - \log t)^{-\beta+1/2}, 0 < t < 1.$$

Therefore

$$\int_0^1 [(1 - \log t)^\gamma \omega(f, t)_2]^q \frac{dt}{t} \sim \int_0^1 (1 - \log t)^{(\gamma-\beta+1/2)q} \frac{dt}{t}$$

and the last integral is finite because $\gamma - \beta + 1/2 + 1/q < 0$. Consequently, $f \in \mathbf{B}^{0,\gamma}_{2,q}(\mathbb{T})$.

However, $(\hat{f}(k)) \notin \ell_{2,q}(\log \ell)_{\gamma+1/2+\epsilon}$. Indeed, since $\gamma + 1/2 + \epsilon - \beta + 1/q > 0$, we get

$$\begin{aligned}&\sum_{k=1}^{\infty}[k^{1/2}(1 + \log k)^{\gamma+1/2+\epsilon} k^{-1/2}(1 + \log k)^{-\beta}]^q \frac{1}{k} \\ &\qquad = \sum_{k=1}^{\infty}(1 + \log k)^{(\gamma+1/2+\epsilon-\beta)q} \frac{1}{k} = \infty.\end{aligned}$$

$\square$

We close this chapter with a result on the *conjugate-function operator* H, which is defined on $L_1(\mathbb{T})$ by the principal-value integral

$$Hf(e^{ix}) = \lim_{\epsilon\to 0+} \int_{\epsilon}^{2\pi-\epsilon} f(e^{i(x-y)}) \cot(y/2) dy$$

(see [127, Chapter IV]).

Theorem 3.14. *Let $0 < q \le \infty$ and $\gamma > -1/q$. If $f \in \mathbf{B}^{0,\gamma+1}_{1,q}(\mathbb{T})$ then $Hf \in \mathbf{B}^{0,\gamma}_{1,q}(\mathbb{T})$.*

Proof. According to [127, Theorem IV.3.16], we have that $H : L_1(\mathbb{T}) \longrightarrow L_{1,\infty}(\mathbb{T})$ is bounded. Besides, by [9, Lemma 3.6.9], H maps any trigonometric polynomial in another trigonometric polynomial with the same degree. Hence,

$$H : \mathbf{B}^{0,\gamma+1}_{1,q}(\mathbb{T}) = (L_1(\mathbb{T}))^{(0,\gamma+1)}_q \longrightarrow (L_{1,\infty}(\mathbb{T}))^{(0,\gamma+1)}_q$$

is bounded.

Using Nikolskiĭ inequality in Lorentz spaces given in [109, Theorem 3] (see also [54, Theorem 3.3]), we obtain that

$$\|g\|_{L_1(\mathbb{T})} \lesssim \log(1+n)\|g\|_{L_{1,\infty}(\mathbb{T})}, g \in T_n.$$

Therefore, Lemma 2.1 yields that

$$(L_{1,\infty}(\mathbb{T}))^{(0,\gamma+1)}_q \hookrightarrow (L_1(\mathbb{T}))^{(0,\gamma)}_q = \mathbf{B}^{0,\gamma}_{1,q}(\mathbb{T}).$$

Consequently,

$$H : \mathbf{B}^{0,\gamma+1}_{1,q}(\mathbb{T}) \longrightarrow \mathbf{B}^{0,\gamma}_{1,q}(\mathbb{T})$$

is bounded. □

The result above in the case $1 \le q \le \infty$ can be found in [51, Corollary 6.3 and Remark 8.4]. Other estimates for H can be seen in [8, Section IV.16] and [9, Chapter 3].

Chapter 4

Compact operators and approximation spaces

Let X, Y be quasi-Banach spaces and let $(G_n)_{n\in\mathbb{N}_0}$ and $(F_n)_{n\in\mathbb{N}_0}$ be approximation families in X and Y, respectively. According to [104, Theorem 3.3] if $T \in \mathfrak{L}(X,Y)$ and there is $c > 0$ such that $T(G_n) \subseteq F_m$ whenever $m \geq c\,n$, then T maps X_p^α boundedly into Y_p^α. It is natural to investigate under which conditions T is not only bounded but compact. This question has been already considered by Fugarolas [69] and Almira and Luther [2]. Results of [69] characterize compact subsets of X_p^α for $p < \infty$, while results of [2] refer to compact operators but just in the setting of Banach spaces and $1 \leq p \leq \infty$. In this chapter we continue those investigations, focusing our attention mainly on limiting spaces. Our approach is different from the one of [2] and works for the full range of parameters.

First, we prove in the assumptions of [104, Theorem 3.3], if $T : X \longrightarrow Y$ is compact, then the restriction of T to approximation spaces is also compact. This is a consequence of the interpolation properties of compact operators. We also study operators acting from an approximation space into a quasi-Banach space, and operators with image in an approximation space. Moreover, we give applications of our results to embeddings between Besov spaces.

4.1 Compact operators

The following compactness theorem combines the interpolation properties of approximation spaces with interpolation properties of compact operators.

Theorem 4.1. *Let X, Y be quasi-Banach spaces and let $(G_n)_{n\in\mathbb{N}_0}, (F_n)_{n\in\mathbb{N}_0}$ be approximation families in X and Y, respectively. Suppose that $0 < \alpha < \infty$, $0 < q \le \infty$ and $\gamma > -1/q$. Let $T \in \mathfrak{L}(X, Y)$ such that for some $c > 0$ we have that*

$$T(G_n) \subseteq F_m \quad \textit{whenever } m \ge cn. \tag{4.1}$$

If $T : X \to Y$ is compact, then restrictions

$$T : X_q^{(0,\gamma)} \to Y_q^{(0,\gamma)} \quad \textit{and} \quad T : X_q^{\alpha} \to Y_q^{\alpha}$$

are also compact.

Proof. Let $0 < p \le \infty$ and $\delta > -1/p$ such that $\delta + 1/p > \gamma + 1/q$. Put $\theta = (\gamma + 1/q)/(\delta + 1/p)$. It follows from (2.18) and (2.20) that

$$(X, X_p^{(0,\delta)})_{\theta,q} = X_q^{(0,\gamma)} \tag{4.2}$$

and

$$(X, X_q^{2\alpha})_{1/2,q} = X_q^{\alpha}. \tag{4.3}$$

Assumption (4.1) yields that for any $f \in X$,

$$E_m(Tf; Y) \le \|T\|_{X,Y}\, E_n(f; X) \quad \text{whenever } c(n-1) + 1 \le m < cn + 1,\ n = 1, 2, \ldots$$

(see [104, Theorem 3.3]). Hence restrictions

$$T : X_p^{(0,\delta)} \to Y_p^{(0,\delta)} \quad \text{and} \quad T : X_q^{2\alpha} \to Y_q^{2\alpha}$$

are bounded. Now, using the compactness theorem for the real method in the quasi-Banach case (see [43, Theorem 3.1]), the formulae (4.2) and (4.3), we derive that

$$T : X_q^{(0,\gamma)} \to Y_q^{(0,\gamma)} \quad \text{and} \quad T : X_q^{\alpha} \to Y_q^{\alpha}$$

are compact. □

Remark 4.1. In the particular case where X, Y are Banach spaces, the approximation families $(G_n)_{n\in\mathbb{N}_0}, (F_n)_{n\in\mathbb{N}_0}$ are formed by finite-dimensional subspaces of X and Y, respectively, and $q \ge 1$, Theorem 4.1 was established by Almira and Luther [2, Corollary 7.5].

The following result requires another type of assumptions. As usual, U_X stands for the closed unit ball of X.

Theorem 4.2. *Let X, Y be quasi-Banach spaces an let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X. Let $0 < q \le \infty$, $\gamma > -1/q$ and let $T \in \mathfrak{L}(X,Y)$. Then a necessary and sufficient condition for $T : X_q^{(0,\gamma)} \to Y$ to be compact is that $T(G_n \cap U_X)$ is precompact in Y for any $n \in \mathbb{N}$.*

Proof. Put

$$r_n = \left(\sum_{k=1}^{n} (1+\log k)^{\gamma q} k^{-1}\right)^{1/q}, \ n \in \mathbb{N}.$$

If $f \in G_n \cap U_X$, we have

$$\|f\|_{X_q^{(0,\gamma)}} = \left(\sum_{k=1}^{n} ((1+\log k)^{\gamma} E_k(f))^q \, k^{-1}\right)^{1/q} \le r_n.$$

So $G_n \cap U_X$ is bounded in $X_q^{(0,\gamma)}$. If $T : X_q^{(0,\gamma)} \to Y$ is compact, it follows that $T(G_n \cap U_X)$ is precompact in Y for any $n \in \mathbb{N}$.

In order to show that the condition is sufficient, we recall that without loss of generality we may assume that X and Y are ρ–normed for some $0 < \rho < q$. Let $1/r = 1/\rho - 1/q$.

Take any $\epsilon > 0$. Since $\gamma + 1/q > 0$, there is $N \in \mathbb{N}$ such that

$$\left(\sum_{n>N} 2^{-n(\gamma+1/q)r}\right)^{1/r} \le \frac{\epsilon}{2^{2+1/\rho}\|T\|_{X,Y}}. \tag{4.4}$$

Let $\epsilon_0, \dots, \epsilon_N$ be positive numbers such that $\left(\sum_{n=0}^{N} \epsilon_n^{\rho}\right)^{1/\rho} = \epsilon/2^{1+1/\rho}$. By the assumption on T, for any $n = 0, \dots, N$, there is a finite set $V_n \subseteq Y$ such that

$$T(G_{\mu_n} \cap 2^{1-n(\gamma+1/q)} U_X) \subseteq \bigcup_{v\in V_n} \{v + \epsilon_n U_Y\}. \tag{4.5}$$

Put $W = \left\{\sum_{n=0}^{N} v_n : v_n \in V_n, 0 \le n \le N\right\}$. It is clear that W is finite. Let us check that W is an ϵ-net of $T(U_{X_q^{(0,\gamma)}})$ in Y.

Given any $f \in U_{X_q^{(0,\gamma)}}$, by (2.8), we can find a representation $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in G_{\mu_n}$ and

$$\left(\sum_{n=0}^{\infty} (2^{n(\gamma+1/q)} \|g_n\|_X)^q\right)^{1/q} \le 2.$$

So $\|g_n\|_X \le 2^{1-n(\gamma+1/q)}$ and therefore $g_n \in G_{\mu_n} \cap 2^{1-n(\gamma+1/q)} U_X$. Using (4.5), for $n = 0, \cdots, N$, we can find $v_n \in V_n$ such that $\|Tg_n - v_n\|_Y \le \epsilon_n$. Let $w = \sum_{n=0}^N v_n \in W$. Applying Hölder inequality and using (4.4), we get

$$
\begin{aligned}
\|Tf - w\|_Y &= \left\| \sum_{n=0}^{\infty} Tg_n - \sum_{n=0}^{N} v_n \right\|_Y \\
&\le 2^{1/\rho} \left[\left(\sum_{n=0}^{N} \|Tg_n - v_n\|_Y^\rho \right)^{1/\rho} + \left(\sum_{n>N} \|Tg_n\|_Y^\rho \right)^{1/\rho} \right] \\
&\le 2^{1/\rho} \left[\left(\sum_{n=0}^{N} \epsilon_n^\rho \right)^{1/\rho} + \|T\|_{X,Y} \left(\sum_{n>N} \|g_n\|_X^\rho \right)^{1/\rho} \right] \\
&\le \frac{\epsilon}{2} + 2^{1/\rho} \|T\|_{X,Y} \left(\sum_{n>N} 2^{-n(\gamma+1/q)r} \right)^{1/r} \left(\sum_{n>N} (2^{n(\gamma+1/q)} \|g_n\|_X)^q \right)^{1/q} \\
&\le \frac{\epsilon}{2} + 2^{1/\rho} \|T\|_{X,Y} \frac{\epsilon}{2^{2+1/\rho} \|T\|_{X,Y}} 2 = \epsilon.
\end{aligned}
$$

This shows that $T(U_{X_q^{(0,\gamma)}})$ is precompact in Y and completes the proof. □

Since the closed unit ball of any finite-dimensional topological vector space is compact (see [87, §15.5.(1)]) as a direct consequence of Theorem 4.2 we obtain the following.

Corollary 4.3. *Let X be a quasi-Banach space and let $(G_n)_{n \in \mathbb{N}_0}$ be an approximation family in X. Let $0 < q \le \infty$ and $\gamma > -1/q$. If for each $n \in \mathbb{N}$, the set G_n is a finite-dimensional linear subspace of X, then the embedding $X_q^{(0,\gamma)} \hookrightarrow X$ is compact.*

Remark 4.2. Let $0 < \alpha < \infty$ and $0 < p \le \infty$. In the assumptions of Corollary 4.3, it follows from (2.3) that the embedding $X_p^\alpha \hookrightarrow X$ is also compact. This result was proved by Almira and Luther [2, Theorem 2.1(ii)] in the special case where X is a Banach space and $p \ge 1$.

Corollary 4.4. *Let X, Y be quasi-Banach spaces and let $(G_n)_{n \in \mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. Let $0 < q \le \infty$, $\gamma > -1/q$ and $\epsilon > 0$. If $T \in \mathfrak{L}(X_q^{(0,\gamma)}, Y)$, then $T : X_q^{(0,\gamma+\epsilon)} \to Y$ is compact.*

Proof. Let $\delta = \epsilon - 1/q$. By the reiteration formula (3.2), we have that $X_q^{(0,\gamma+\epsilon)} = \left(X_q^{(0,\gamma)}\right)_q^{(0,\delta)}$. Since Corollary 4.3 yields that the embedding

$$\left(X_q^{(0,\gamma)}\right)_q^{(0,\delta)} \hookrightarrow X_q^{(0,\gamma)}$$

is compact, we conclude that $T : X_q^{(0,\gamma+\epsilon)} \to Y$ is also compact. □

A similar result to Corollary 4.4 is valid for spaces X_p^α. It is a consequence of Remark 4.2 and (3.1).

Corollary 4.5. *Let X be a quasi-Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. If $0 < q_1, q_2 \le \infty$ and $\gamma_1 + 1/q_1 > \gamma_2 + 1/q_2$, then the embedding $X_{q_1}^{(0,\gamma_1)} \hookrightarrow X_{q_2}^{(0,\gamma_2)}$ is compact.*

Proof. Let $\epsilon > 0$ such that $\gamma_1 - \epsilon + 1/q_1 > \gamma_2 + 1/q_2$. By [65, Lemma 2], we have that $X_{q_1}^{(0,\gamma_1-\epsilon)} \hookrightarrow X_{q_2}^{(0,\gamma_2)}$. Then the result follows from Corollary 4.4. □

Next we give applications of these results to Besov spaces. If we write down Corollary 4.3 for these spaces, we derive compactness of embeddings $\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$ for $0 < p, q \le \infty$ and $\gamma > -1/q$.

Corollary 4.5 yields compactness of embeddings

$$\mathbf{B}_{p,q}^{0,\gamma+\epsilon}(\mathbb{T}) \hookrightarrow \mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$$

provided that $0 < p, q \le \infty, \gamma > -1/q$ and $\epsilon > 0$.

Other kind of compact embeddings involving Besov spaces of logarithmic smoothness will be given in the next chapter (see Corollary 5.9).

The following result refers to reflexivity of approximation spaces.

Corollary 4.6. *Let X be a Banach space and let $(G_n)_{n\in\mathbb{N}_0}$ be an approximation family in X, formed by finite-dimensional subspaces. Let $1 < q < \infty$, $0 < \alpha < \infty$ and $\gamma > -1/q$. Then spaces $X_q^{(0,\gamma)}$ and X_q^α are reflexive.*

Proof. Take $1 < p < \infty$ and $\delta + 1/p > \gamma + 1/q$. By Corollary 4.3, embedding $X_p^{(0,\delta)} \hookrightarrow X$ is compact and therefore weakly compact. Consequently, reflexivity of $X_q^{(0,\gamma)}$ follows from formula (2.20) and the interpolation properties of weakly compact operators (see [4, Proposition II.3.1]).

The proof for X_q^α is similar but using now (2.18) and Remark 4.2. □

Writing down Corollary 4.6 for Besov spaces we obtain that $\mathbf{B}_{p,q}^{0,\gamma}(\mathbb{T})$ is reflexive for $1 \le p \le \infty, 1 < q < \infty$ and $\gamma > -1/q$.

The remaining part of the chapter is devoted to operators with image in an approximation space.

Theorem 4.7. *Let X, Y be quasi-Banach spaces and let $(F_n)_{n\in\mathbb{N}_0}$ be an approximation family in Y. Let $0 < \alpha < \infty$, $0 < p < \infty$ and $T \in \mathfrak{L}(X,Y)$. Then $T : X \to Y_p^\alpha$ is compact if and only if the following conditions hold*

(a) *$T : X \to Y$ is compact,*

(b) $\sup\left\{\left(\sum_{m=n}^{\infty}(2^{m\alpha}E_{2^m}(Tf))^p\right)^{1/p} : \|f\|_X \le 1\right\} \to 0$ *as $n \to \infty$.*

Proof. First we show that (b) implies that $T \in \mathfrak{L}(X, Y_p^\alpha)$. There is $N \in \mathbb{N}$ such that $\left(\sum_{m=N}^{\infty}(2^{m\alpha}E_{2^m}(Tf))^p\right)^{1/p} \le 1$ for any $f \in U_X$. Whence, for any $f \in X$, we get

$$\begin{aligned}
\|Tf\|_{Y_p^\alpha} &\sim \left(\sum_{m=0}^{\infty}(2^{m\alpha}E_{2^m}(Tf))^p\right)^{1/p} \\
&\lesssim \left[\left(\sum_{m=0}^{N-1}(2^{m\alpha}E_{2^m}(Tf))^p\right)^{1/p} + \|f\|_X\right] \\
&\lesssim \left[\left(\sum_{m=0}^{N-1}2^{m\alpha p}\right)^{1/p}\|T\|_{X,Y} + 1\right]\|f\|_X \\
&\sim \|f\|_X .
\end{aligned}$$

Consequently, $T \in \mathfrak{L}(X, Y_p^\alpha)$.

Now the result follows by using [69, Theorem 1]. □

Recall that the approximation family $(F_n)_{n\in\mathbb{N}_0}$ of Y is called *linear* if there exists a uniformly bounded sequence of linear projections P_n mapping Y onto F_n. If this is the case, we have that

$$\|h - P_{n-1}h\|_Y \le cE_n(h),\, h \in Y,\, n \in \mathbb{N}, \tag{4.6}$$

with c being a positive constant which is independent of h and n. Next we show that condition (a) in the previous result is not needed if the family $(F_n)_{n\in\mathbb{N}_0}$ is linear and finite-dimensional.

Theorem 4.8. *Let X, Y be quasi-Banach spaces and let $(F_n)_{n\in\mathbb{N}_0}$ be a linear approximation family in Y, formed by finite-dimensional subspaces. Let $0 < \alpha < \infty$, $0 < p < \infty$ and $T \in \mathfrak{L}(X,Y)$. Then the necessary and sufficient condition for $T : X \to Y_p^\alpha$ to be compact is that*

$$\sup\left\{\left(\sum_{m=n}^{\infty}(2^{m\alpha}E_{2^m}(Tf))^p\right)^{1/p} : \|f\|_X \le 1\right\} \to 0 \quad \text{as } n \to \infty.$$

Proof. The argument is similar to [69, Theorem 2]. If the condition holds, then $T \in \mathfrak{L}(X, Y_p^\alpha)$ as we showed in the proof of Theorem 4.7. Take any $\epsilon > 0$ and let us construct an ϵ-net for $T(U_X)$ in Y.

Let P_n be the projection associated to F_n and let c_Y be the constant in the triangle inequality of Y. By (4.6) and the assumption, we can find $n \in \mathbb{N}$ such that

$$\left(\sum_{m=N}^{\infty} (2^{m\alpha} \|Tf - P_{2^m - 1}(Tf)\|_Y)^p \right)^{1/p} \le \frac{\epsilon}{2c_Y}, f \in U_X.$$

In particular, we have that

$$\|Tf - P_{2^N - 1}(Tf)\|_Y \le \frac{\epsilon}{2c_Y}, f \in U_X.$$

Moreover, using compactness of $P_{2^N-1}T : X \to Y$ we can find a finite subset $V = \{h_1, \ldots, h_k\} \subseteq Y$ such that

$$P_{2^N-1}T(U_X) \subseteq \bigcup_{j=1}^{k} \left\{ h_j + \frac{\epsilon}{2c_Y} U_Y \right\}.$$

Therefore, for any $f \in U_X$, if we choose $h_j \in V$ such that $\|P_{2^N-1}Tf - h_j\|_Y \le \epsilon/2c_Y$, we obtain

$$\|Tf - h_j\|_Y \le c_Y(\|Tf - P_{2^N-1}Tf\|_Y + \|P_{2^N-1}Tf - h_j\|_Y) \le \epsilon.$$

This shows that the condition is sufficient. Necessity follows from Theorem 4.7. □

Remark 4.3. It is not hard to check that techniques used in Theorems 4.7 and 4.8 also work to characterize compactness of operators with image in $Y_q^{(0,\gamma)}$ for $0 < q < \infty$ and $\gamma > -1/q$. The corresponding condition to (b) reads now

$$\text{(b}') \quad \sup \left\{ \left(\sum_{m=n}^{\infty} (2^{m(\gamma+1/q)} E_{\mu_m}(Tf))^q \right)^{1/q} : \|f\|_X \le 1 \right\} \to 0 \quad \text{as } n \to \infty.$$

Chapter 5

Embeddings of Besov spaces of logarithmic smoothness

Embedding theorems play a central role in the theory of function spaces, as can be seen in the books by Triebel [117, 118, 120]. The aim of this chapter is to derive new embeddings of Besov spaces of generalized smoothness by using limiting interpolation methods.

In Section 5.1, we investigate embeddings from Besov spaces $\mathbf{B}^{\alpha,b}_{p,r}$ formed by periodic functions into Lorentz-Zygmund spaces. In the case that $\alpha > 0$, this problem was already considered by DeVore, Riemenschneider and Sharpley [51, Corollary 5.5] by applying weak type interpolation techniques. However, their results only cover the Banach case where the parameters are greater or equal than 1. Here we extend those embeddings to the whole rank of parameters. Next we work with Besov spaces involving only logarithimc smoothness, that is, $\alpha = 0$. If the exponent of logarithmic smoothness $b > -1/r$, previous results were obtained by Caetano, Gogatishvili and Opic [17] concerning local embeddings for Besov spaces on $\mathbb{R}^d$ (see also the report by Triebel [124] for other kind of Besov spaces of zero classical smoothness). However the approach given in [17] does not work in the periodic setting and it only covers the Banach case $1 \leq p < \infty$. Then we propose two different methods to obtain similar embeddings to those given in [17] which work for a bigger range of parameters. The first approach is based on the approximation structure of Besov spaces, Nikolskiĭ type inequalities, extrapolation properties of $L_{p,q}(\log L)_\beta$ and interpolation and it covers the full range of indices if $\beta > 0$. This restriction on β can be removed by using limiting interpolation and small Lebesgue spaces. Furthermore, the limit case when $b = -1/r$ will be studied.

Section 5.2 is devoted to compare Besov spaces $B^{s,b}_{p,q}$ defined by using the Fourier transform (see formal definition below) with spaces $\mathbf{B}^{s,b}_{p,q}$ given by means of the modulus of smoothness. If $1 \le p \le \infty$ and $s > 0$, then it is known that $B^{s,b}_{p,q} = \mathbf{B}^{s,b}_{p,q}$ with equivalence of norms (see [77, Theorem 2.5] and [117, 2.5.12]). However, the relationships between these two kinds of spaces when $s = 0$ has not been described yet. With the help of limiting interpolation, we prove among other things that if $b > -1/p$ then the following continuous embeddings hold

$$B^{0,b+1/p}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/2}_{p,p} \text{ if } 1 < p \le 2,$$

$$B^{0,b+1/2}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/p}_{p,p} \text{ if } 2 \le p < \infty.$$

In particular, we have that $B^{0,b+1/2}_{2,2} = \mathbf{B}^{0,b}_{2,2}$ for $b > -1/2$.

We also consider embeddings into $\mathbf{B}^{0,b}_{p,q}$. This will be done in Section 5.3. According to [116, Theorem 2.8.1] or [9, Corollary 5.4.21], if $1 \le p \le r < \infty, 1 \le q \le \infty$ and $s > 0$, then $\mathbf{B}^{n(1/p-1/r)+s}_{p,q} \hookrightarrow \mathbf{B}^{s}_{r,q}$. Note that in the embedding the two spaces have the same differential dimension. The limit case where $s = 0$ has been studied by DeVore, Riemenschneider and Sharpley [51, Corollary 5.3(ii)], where they showed that the embedding holds with a loss of a unit in the exponent of the logarithmic smoothness. To be more precise, if $1 \le p \le r \le \infty, 1 \le q \le \infty$ and $b > -1/q$ then $\mathbf{B}^{n(1/p-1/r),b+1}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}$. This result has been improved recently by Gogatishvili, Opic, Tikhonov and Trebels [70, Corollary 2.8] by showing that the embedding holds with the loss of only $1/\min\{q,r\}$ in the exponent of the logarithmic smoothness of the source space. In this section, we derive the embedding $\mathbf{B}^{n(1/p-1/r),b+1/\min\{q,r\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}$ following a more simple approach than in [70].

Finally, Section 5.4 collects, extends and discusses embeddings between the scales of Besov spaces of logarithmic smoothness and corresponding Lipschitz spaces. In particular, this answers some questions left open in [75] and [76] where the approach was completely different.

Subsequently, if $(\lambda_j)_{j\in\mathbb{N}_0}$ is a sequence of positive numbers, $(A_j)_{j\in\mathbb{N}_0}$ is a sequence of quasi-Banach spaces and $0 < p \le \infty$, we write $\ell_p(\lambda_j A_j)$ for the collection of all vector-valued sequences $a = (a_j)$ such that $a_j \in A_j$ for any $j \in \mathbb{N}_0$ and

$$\|a\|_{\ell_p(\lambda_j A_j)} = \left(\sum_{j=0}^{\infty} (\lambda_j \|a_j\|_{A_j})^p\right)^{1/p} < \infty$$

(as usual, the sum should be replaced by the supremum if $p = \infty$). In the particular case that $\lambda_j = 1, j \in \mathbb{N}_0$, we simply write $\ell_p(A_j)$.

The main part of the results of this chapter are taken from the papers [25, 26, 27, 28]. Results of Section 5.2, 5.3 and 5.4 on spaces on $\mathbb{T}^d$ are new.

5.1 Embeddings of $\mathrm{B}^{\alpha,b}_{p,q}(\mathbb{T})$ into Lorentz-Zygmund spaces

The following interpolation formulae follow from Proposition 2.6, Theorem 3.3 and characterizations of Besov spaces via approximation given by (2.28), (2.29) and (2.30):

$$(L_p(\mathbb{T}), \mathbf{B}^{\alpha}_{p,r}(\mathbb{T}))_{\rho,q} = \mathbf{B}^{\theta\alpha,b}_{p,q}(\mathbb{T}) \quad \text{where} \quad \rho(t) = t^{\theta}(1+|\log t|)^{-b}, \tag{5.1}$$

$$(L_p(\mathbb{T}), \mathbf{B}^{\alpha}_{p,r}(\mathbb{T}))_{(0,-b),q} = \mathbf{B}^{0,b}_{p,q}(\mathbb{T}) \quad \text{if} \quad b \geq -1/q. \tag{5.2}$$

To get some results of this section, we shall use estimates between L_p-quasi-norms of trigonometric polynomials. Nikolskiĭ inequality for trigonometric polynomials (see [97, 3.4.3] and [3]) says that there is a universal constant $c > 1$ such that

$$\|g\|_{L_q(\mathbb{T})} \leq c\, n^{1/p-1/q} \|g\|_{L_p(\mathbb{T})} \text{ for all } 0 < p \leq q \leq \infty, g \in T_n \text{ and } n \in \mathbb{N}. \tag{5.3}$$

We start with known embeddings into $L_\infty(\mathbb{T})$ (see [107, Theorem 3.5.5]). For the sake of completeness, we include the proof.

Theorem 5.1. *Let $0 < p < \infty$. Then*

$$\mathbf{B}^{1/p}_{p,1}(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T}).$$

Proof. By (2.6), there is a constant $c > 0$ such that for any $f \in \mathbf{B}^{1/p}_{p,1}(\mathbb{T})$ there is a representation $f = \sum_{n=0}^{\infty} g_n$ with $g_n \in T_{2^n}$ and

$$\sum_{n=0}^{\infty} 2^{n/p} \|g_n\|_{L_p(\mathbb{T})} \leq c \|f\|_{\mathbf{B}^{1/p}_{p,1}(\mathbb{T})}.$$

Hence, using (5.3), we derive that

$$\|f\|_{L_\infty(\mathbb{T})} \leq \sum_{n=0}^{\infty} \|g_n\|_{L_\infty(\mathbb{T})} \lesssim \sum_{n=0}^{\infty} 2^{n/p} \|g_n\|_{L_p(\mathbb{T})} \leq c \|f\|_{\mathbf{B}^{1/p}_{p,1}(\mathbb{T})}.$$

□

Theorem 5.2. *Let $0 < p < q < \infty, \alpha = 1/p - 1/q, 0 < r \leq \infty$ and $-\infty < b < \infty$. Then*

$$\mathbf{B}^{\alpha,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{q,r}(\log L)_b(\mathbb{T}).$$

Proof. Combining Theorem 5.1 with the natural embedding $L_p(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$ and interpolating with the function parameter $\rho(t) = t^\theta(1+|\log t|)^{-b}$ where $\theta = 1-p/q$, we derive the continuous embedding

$$(L_p(\mathbb{T}), \mathbf{B}^{1/p}_{p,1}(\mathbb{T}))_{\rho,r} \hookrightarrow (L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{\rho,r}.$$

By (5.1), the space to the left is $\mathbf{B}^{\alpha,b}_{p,r}(\mathbb{T})$ and, according to (2.15), the space to the right is $L_{q,r}(\log L)_b(\mathbb{T})$. This completes the proof. □

Theorem 5.3. *Let $0<p<\infty, 0<r\le\infty$ and $b+1/r<0$. Then*

$$\mathbf{B}^{1/p,b+1/\min\{1,r\}}_{p,r}(\mathbb{T}) \hookrightarrow L_{\infty,r}(\log L)_b(\mathbb{T}).$$

Proof. We interpolate by the limiting method with $\theta = 1$ the embeddings

$$\mathbf{B}^{1/p}_{p,1}(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T}) \quad , \quad L_p(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$$

to obtain that

$$(L_p(\mathbb{T}), \mathbf{B}^{1/p}_{p,1}(\mathbb{T}))_{(1,-b),r} \hookrightarrow (L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{(1,-b),r}.$$

Lemma 2.3 yields that $(L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{(1,-b),r} = L_{\infty,r}(\log L)_b(\mathbb{T})$. On the other hand, take $0<\alpha<p$ and put $\rho(t) = t^{\alpha/p}(1+|\log t|)^{-b-1/\min\{1,r\}}$. Using (5.1) and Lemma 3.8(a), we get

$$\begin{aligned}\mathbf{B}^{1/p,b+1/\min\{1,r\}}_{p,r}(\mathbb{T}) &= (L_p(\mathbb{T}), \mathbf{B}^{1/\alpha}_{p,\alpha}(\mathbb{T}))_{\rho,r}\\ &\hookrightarrow (L_p(\mathbb{T}), (L_p(\mathbb{T}), \mathbf{B}^{1/\alpha}_{p,\alpha}(\mathbb{T}))_{\alpha/p,1})_{(1,-b),r}\\ &= (L_p(\mathbb{T}), \mathbf{B}^{1/p}_{p,1}(\mathbb{T}))_{(1,-b),r}.\end{aligned}$$

Consequently,

$$\mathbf{B}^{1/p,b+1/\min\{1,r\}}_{p,r}(\mathbb{T}) \hookrightarrow L_{\infty,r}(\log L)_b(\mathbb{T}).$$

□

Next we focus our attention on embeddings of Besov spaces with smoothness close to zero into Lorentz-Zygmund spaces.

First we establish a Nikolskiĭ inequality but involving Lorentz-Zygmund spaces (we refer to [109] and [54] for extensions of (5.3) to Lorentz spaces).

Lemma 5.4. *Let $0<q<p<\infty, \gamma>1/p-1/q$ and let $\mu_n = 2^{2^n}$. There is a constant $c>0$ such that*

$$\|g\|_{L_{p,q}(\log L)_\gamma(\mathbb{T})} \le c2^{n(1/q-1/p+\gamma)}\|g\|_{L_p(\mathbb{T})} \text{ for all } g\in T_{\mu_n} \text{ and } n\in\mathbb{N}_0.$$

Proof. It is not hard to check that

$$\|f\|_{L_{p,\infty}(\mathbb{T})} \leq (q/p)^{1/q} \|f\|_{L_{p,q}(\mathbb{T})}, f \in L_{p,q}(\mathbb{T}). \tag{5.4}$$

Moreover, by [12, Theorem 5.1.2], we have that

$$\|h\|_{L_\infty(\mathbb{T})} \leq e^{4ns} h^*(s), \text{ for all } s \in (0, \pi/2] \text{ and } h \in T_n.$$

This inequality yields that

$$\|h\|_{L_{p,\infty}(\mathbb{T})} \geq s^{1/p} e^{-4ns} \|h\|_{L_\infty(\mathbb{T})}, \text{ for all } s \in (0, \pi/2] \text{ and } h \in T_n. \tag{5.5}$$

Indeed

$$\|h\|_{L_{p,\infty}(\mathbb{T})} = \sup_{0<t<2\pi} t^{1/p} h^*(t) \geq s^{1/p} h^*(s) \geq s^{1/p} e^{-4ns} \|h\|_{L_\infty(\mathbb{T})}.$$

We proceed now with the inequality of the statement. Take any $n \in \mathbb{N}_0$ and $g \in T_{\mu_n}$. Let $s = 2^{-2^n}$ and $\rho(t) = (1 + |\log t|)^\gamma$. We have

$$\begin{aligned}\|g\|^q_{L_{p,q}(\log L)_\gamma(\mathbb{T})} &= \int_0^s t^{q/p-1} \rho(t)^q g^*(t)^q dt + \int_s^{2\pi} t^{q/p-1} \rho(t)^q g^*(t)^q dt \\ &= I_1 + I_2.\end{aligned}$$

Using [56, Proposition 3.4.33(v)] and (5.5), (5.4), we derive

$$\begin{aligned}I_1 &\leq \|g\|^q_{L_\infty(\mathbb{T})} \int_0^s t^{q/p-1} \rho(t)^q dt \\ &\sim \|g\|^q_{L_\infty(\mathbb{T})} s^{q/p} \rho(s)^q \leq e^{4\mu_n s q} \rho(s)^q \|g\|^q_{L_p(\mathbb{T})} \\ &= e^{4q} (1 + |\log 2^{-2^n}|)^{\gamma q} \|g\|^q_{L_p(\mathbb{T})} \\ &\sim 2^{n\gamma q} \|g\|^q_{L_p(\mathbb{T})}.\end{aligned}$$

As for I_2, we write

$$I_2 = \int_s^{2\pi} (g^*(t)^p)^{q/p} (\rho(t)^\lambda t^{-1})^{1-q/p} dt$$

where $\lambda = q(1 - q/p)^{-1} = (1/q - 1/p)^{-1}$. By Hölder inequality, we get

$$I_2 \leq \|g\|^q_{L_p(\mathbb{T})} \left(\int_s^{2\pi} \rho(t)^\lambda t^{-1} dt \right)^{1-q/p}.$$

We estimate the integral by splitting it into two sets. Clearly, $\int_1^{2\pi} \rho(t)^\lambda t^{-1} dt = c_1 < \infty$. Furthermore, since $\gamma\lambda > -1$, we have that

$$\int_s^1 (1 - \log t)^{\gamma\lambda} t^{-1} dt \lesssim (1 - \log s)^{1+\gamma\lambda}.$$

Taking into account that $s = 2^{-2^n}$, we obtain

$$\begin{aligned} I_2 &\lesssim \|g\|^q_{L_p(\mathbb{T})}(1-\log s)^{(1+\gamma\lambda)(1-q/p)} \\ &\sim 2^{n(1/q-1/p+\gamma)q}\|g\|^q_{L_p(\mathbb{T})}. \end{aligned}$$

Consequently,

$$\|g\|_{L_{p,q}(\log L)_\gamma(\mathbb{T})} \lesssim 2^{n(1/q-1/p+\gamma)}\|g\|_{L_p(\mathbb{T})}.$$

□

Next we establish the embedding results.

Theorem 5.5. *Let $0 < p < \infty, 0 < r \le q \le \infty, b+1/r > 0$ and let $\beta = b+1/r+1/\max\{p,q\}-1/q > 0$. Then*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_\beta(\mathbb{T}).$$

Proof. By Lemma 2.7 we know that $\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) = (L_p(\mathbb{T}), T_n)^{(0,b)}_r$, so we can work with the representation quasi-norm in terms of trigonometric polynomials . We distinguish two cases.

If $q \le p$, then $\beta = b + 1/r + 1/p - 1/q$ and $\beta > 0$ by assumption. Let $j_0 \in \mathbb{N}$ such that $1/p^{\lambda_j} = 1/p - 2^{-j} > 0$ for all $j \ge j_0$. If $j \ge j_0 + 1$, it follows from [56, Proposition 3.4.4] that

$$\|h\|_{L_{p^{\lambda_j},q}(\mathbb{T})} \le c_j\|h\|_{L_{p^{\lambda_{j-1}}}(\mathbb{T})} \text{ for all } h \in L_{p^{\lambda_{j-1}}}(\mathbb{T}) \tag{5.6}$$

where

$$\begin{aligned} c_j &= (2\pi)^{1/p^{\lambda_j}-1/p^{\lambda_{j-1}}}\left[\frac{p^{\lambda_j}(p^{\lambda_{j-1}}-q)}{q(p^{\lambda_{j-1}}-p^{\lambda_j})}\right]^{1/q-1/p^{\lambda_{j-1}}} \\ &= (2\pi)^{1/2^j}\left[\frac{p-q}{pq}2^j+2\right]^{1/q-1/p+1/2^{j-1}} \\ &\sim 2^{j(1/q-1/p)}. \end{aligned}$$

By (2.8) there is a constant $c > 0$ such that given any $f \in \mathbf{B}^{0,b}_{p,r}(\mathbb{T})$ we can find a representation $f = \sum_{j=0}^\infty g_j$ with $g_j \in T_{\mu_j}$ and

$$\left(\sum_{j=0}^\infty\left(2^{j(b+1/r)}\|g_j\|_{L_p(\mathbb{T})}\right)^r\right)^{1/r} \le c\|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}.$$

Using (5.6) and Nikolskiĭ inequality (5.3), we derive

$$\begin{aligned} \|g_{j-j_0-1}\|_{L_{p^{\lambda_j},q}(\mathbb{T})} &\lesssim 2^{j(1/q-1/p)}2^{2^{j-j_0-1}2^{-j+1}}\|g_{j-j_0-1}\|_{L_p(\mathbb{T})} \\ &\sim 2^{j(1/q-1/p)}\|g_{j-j_0-1}\|_{L_p(\mathbb{T})}. \end{aligned}$$

Therefore, it follows from the extrapolation characterization of Lorentz-Zygmund spaces (see Section 2.2) and $r \le q$ that

$$\begin{aligned}\|f\|_{L_{p,q}(\log L)_\beta(\mathbb{T})} &\lesssim \left(\sum_{j=j_0+1}^{\infty} 2^{j\beta q} \|g_{j-j_0-1}\|^q_{L_{p^{\lambda_j},q}(\mathbb{T})}\right)^{1/q}\\ &\lesssim \left(\sum_{j=j_0+1}^{\infty} 2^{j\beta q} 2^{j(1/q-1/p)q} \|g_{j-j_0-1}\|^q_{L_p(\mathbb{T})}\right)^{1/q}\\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{j(b+1/r)r} \|g_j\|^r_{L_p(\mathbb{T})}\right)^{1/r}\\ &\lesssim \|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}.\end{aligned}$$

Suppose now $p < q$, so $\beta = b + 1/r$. Let $j_0 \in \mathbb{N}$ such that $p < p^{\lambda_j} < q$ for all $j \ge j_0$. According to [56, Proposition 3.4.4]

$$\|h\|_{L_{p^{\lambda_j},q}(\mathbb{T})} \le C\|h\|_{L_{p^{\lambda_{j-1}}}(\mathbb{T})} \text{ for all } h \in L_{p^{\lambda_{j-1}}}(\mathbb{T}) \tag{5.7}$$

where now C is independent of $j \in \mathbb{N}$ with $j \ge j_0 + 1$. Take any $f \in \mathbf{B}^{0,b}_{p,r}(\mathbb{T})$ and choose a representation $f = \sum_{j=0}^{\infty} g_j$ with $g_j \in T_{\mu_j}$ and

$$\left(\sum_{j=0}^{\infty} \left(2^{j(b+1/r)} \|g_j\|_{L_p(\mathbb{T})}\right)^r\right)^{1/r} \le c\|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}.$$

By (5.7) and Nikolskiĭ inequality (5.3), we get

$$\|g_{j-j_0-1}\|_{L_{p^{\lambda_j},q}(\mathbb{T})} \lesssim \|g_{j-j_0-1}\|_{L_{p^{\lambda_{j-1}}}(\mathbb{T})} \lesssim \|g_{j-j_0-1}\|_{L_p(\mathbb{T})}.$$

Consequently,

$$\begin{aligned}\|f\|_{L_{p,q}(\log L)_\beta(\mathbb{T})} &\lesssim \left(\sum_{j=j_0+1}^{\infty} 2^{j(b+1/r)q} \|g_{j-j_0-1}\|^q_{L_{p^{\lambda_j},q}(\mathbb{T})}\right)^{1/q}\\ &\lesssim \left(\sum_{j=j_0+1}^{\infty} 2^{j(b+1/r)q} \|g_{j-j_0-1}\|^q_{L_p(\mathbb{T})}\right)^{1/q}\\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{j(b+1/r)r} \|g_j\|^r_{L_p(\mathbb{T})}\right)^{1/r}\\ &\lesssim \|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}.\end{aligned}$$

This completes the proof. □

If $\beta \le 0$ then the description of $L_{p,q}(\log L)_\beta(\mathbb{T})$ in terms of Lorentz spaces is of different type (see [34, Corollary 3.3]). For this reason we have to use another approach. We start with the case $0 < r \le 1$.

Theorem 5.6. *Let* $1 \le q < p < \infty, 0 < r \le 1, b + 1/r > 0$ *and* $\beta = b + 1/r + 1/p - 1/q$. *Then*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_\beta(\mathbb{T}).$$

Proof. First note that under the assumptions, the quasi-norm of $L_{p,q}(\log L)_\beta(\mathbb{T})$ is equivalent to a norm (see [56, Lemma 3.4.39]). Take any $f \in \mathbf{B}^{0,b}_{p,r}(\mathbb{T})$ and choose a representation $f = \sum_{j=0}^\infty g_j$ with $g_j \in T_{\mu_j}$ and

$$\left(\sum_{j=0}^\infty \left(2^{j(b+1/r)} \|g_j\|_{L_p(\mathbb{T})} \right)^r \right)^{1/r} \le c \|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}.$$

Using Lemma 5.4, we derive

$$\begin{aligned} \|f\|_{L_{p,q}(\log L)_\beta(\mathbb{T})} &\lesssim \sum_{j=0}^\infty \|g_j\|_{L_{p,q}(\log L)_\beta(\mathbb{T})} \\ &\lesssim \sum_{j=0}^\infty 2^{j(b+1/r)} \|g_j\|_{L_p(\mathbb{T})} \\ &\lesssim \left(\sum_{j=0}^\infty \left(2^{j(b+1/r)} \|g_j\|_{L_p(\mathbb{T})} \right)^r \right)^{1/r} \\ &\lesssim \|f\|_{\mathbf{B}^{0,b}_{p,r}(\mathbb{T})}. \end{aligned}$$

□

Next we deal with the case $1 < r \le q$. For this aim, we will use the real interpolation method.

Theorem 5.7. *Let* $1 < r \le q < p < \infty, b + 1/r > 0$ *and* $\beta = b + 1/r + 1/p - 1/q$. *Then*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_\beta(\mathbb{T}).$$

Proof. Take b_0, b_1 such that

$$0 < b_0 + 1 < b + 1/r < b_1 + 1$$

and let $0 < \theta < 1$ with

$$b + 1/r = (1-\theta)(b_0 + 1) + \theta(b_1 + 1) = (1-\theta) b_0 + \theta b_1 + 1.$$

According to Theorem 5.6, we have the continuous embeddings

$$\mathbf{B}^{0,b_i}_{p,1}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_{b_i+1+1/p-1/q}(\mathbb{T}), i = 0, 1.$$

Moreover, by (2.19) and Lemma 2.7, we have that

$$(\mathbf{B}^{0,b_0}_{p,1}(\mathbb{T}), \mathbf{B}^{0,b_1}_{p,1}(\mathbb{T}))_{\theta,r} = \mathbf{B}^{0,\gamma}_{p,r}(\mathbb{T})$$

where $\gamma = (1-\theta)b_0 + \theta b_1 + 1 - 1/r = b$, and according to [102, Example 6.1], if $\tau_i = b_i + 1 + 1/p - 1/q, i = 0, 1$, then

$$\begin{aligned}(L_{p,q}(\log L)_{\tau_0}(\mathbb{T}), L_{p,q}(\log L)_{\tau_1}(\mathbb{T}))_{\theta,r} \\ \hookrightarrow (L_{p,q}(\log L)_{\tau_0}(\mathbb{T}), L_{p,q}(\log L)_{\tau_1}(\mathbb{T}))_{\theta,q} = L_{p,q}(\log L)_{\tau}(\mathbb{T})\end{aligned}$$

where $\tau = (1-\theta)\tau_0 + \theta\tau_1 = b + 1/r + 1/p - 1/q = \beta$. Consequently

$$\mathbf{B}^{0,\gamma}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_\beta(\mathbb{T}).$$

□

Note that if $\beta = 0$ then Theorems 5.6 and 5.7 give the following embedding into Lorentz spaces.

Corollary 5.8. *Let* $1 < p < \infty, 0 < r < \infty$ *and* $-1/r < b \le \min\{-1/p, 1 - 1/r - 1/p\}$. *Put* $1/q = b + 1/r + 1/p$. *Then*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\mathbb{T}).$$

The following compactness result is a direct consequence of the previous embeddings and Corollary 4.4.

Corollary 5.9. *Assume that either*

$$0 < p < \infty, 0 < r \le q \le \infty, b + 1/r > 0 \text{ and } \beta = b + \frac{1}{r} + \frac{1}{\max\{p,q\}} - \frac{1}{q} > 0,$$

or

$$1 \le q < p < \infty, 0 < r < \infty, r \le q, b + 1/r > 0 \text{ and } \beta = b + \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

If $\delta < \beta$ *then embedding*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,q}(\log L)_\delta(\mathbb{T})$$

is compact.

Remark 5.1. Compactness of the restriction operator from Besov spaces $\mathbf{B}^{0,b}_{p,r}(\mathbb{R}^d)$ into Lorentz-Zygmund spaces $L_{p,q}(\log L)_\delta(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^d$, were investigated in [18] and [37].

By Theorem 5.5 we have that $\mathbf{B}^{0,b}_{p,r}(\mathbb{T})$ is continuously embedded into $L_{p,q}(\log L)_\beta(\mathbb{T})$ in the full range of parameters if $\beta = b + 1/r + 1/\max\{p,q\} - 1/q > 0$. In the case that $\beta \le 0$, the approach given above only works when $L_{p,q}(\log L)_\beta(\mathbb{T})$ is a Banach space (see Theorems 5.6 and 5.7). With the help of small Lebesgue spaces and limiting interpolation, we can overcome this restriction.

Theorem 5.10. *Let $0 < p < \infty, 0 < r \le \infty, b > -1/r, r \le q$ and $\gamma = b + 1/r - 1/q$. Then*

$$\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L^{(p,\gamma,q}(\mathbb{T}).$$

Proof. This time we interpolate the embeddings

$$\mathbf{B}^{1/p}_{p,1}(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T}) \quad , \quad L_p(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$$

by the limiting method with $\theta = 0$. Since $\gamma > -1/q$, using (5.2) and Lemma 2.4, it turns out that

$$\mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}) = (L_p(\mathbb{T}), \mathbf{B}^{1/p}_{p,1}(\mathbb{T}))_{(0,-\gamma),q} \hookrightarrow (L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{(0,-\gamma),q} = L^{(p,\gamma,q}(\mathbb{T}).$$

Besides, by [65, Lemma 2], we have $\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T})$. Therefore $\mathbf{B}^{0,b}_{p,r}(\mathbb{T}) \hookrightarrow L^{(p,\gamma,q}(\mathbb{T})$. □

Remark 5.2. With the aim of comparing Theorem 5.10 with embeddings given by Theorems 5.5, 5.6 and 5.7, take $0 < s < p$ and put $\theta = 1 - s/p$ and $\beta = b + 1/r + 1/\max\{p,q\} - 1/q$. According to Lemma 3.8(b) and (2.15), we derive

$$\begin{aligned} L^{(p,\gamma,q}(\mathbb{T}) &= ((L_s(\mathbb{T}), L_\infty(\mathbb{T}))_{\theta,p}, L_\infty(\mathbb{T}))_{(0,-\gamma),q} \\ &\hookrightarrow (L_s(\mathbb{T}), L_\infty(\mathbb{T}))_{\theta,q,-\gamma-1/\max\{p,q\}} \\ &= L_{p,q}(\log L)_\beta(\mathbb{T}). \end{aligned}$$

If $p = q$, Lemma 3.8(b) implies that $L^{(p,\gamma,p}(\mathbb{T}) = L_p(\log L)_\beta(\mathbb{T})$. However if $p \ne q$, say $p < q$, then $L^{(p,\gamma,q}(\mathbb{T}) \subsetneq L_{p,q}(\log L)_\beta(\mathbb{T})$. Indeed, take $1/q < \epsilon < 1/p$ and put $f(t) = t^{-1/p}(1 + |\log t|)^{-\beta-\epsilon}$. Then $f \in L_{p,q}(\log L)_\beta(\mathbb{T})$ but $f \notin L^{(p,\gamma,q}(\mathbb{T})$.

Corollary 5.11. *Let $1 < p < \infty$. Then $\mathbf{B}^{0,-1/p}_{p,1}(\mathbb{T})$ is continuously embedded in the small Lebesgue space $L^{(p}(\mathbb{T})$.*

We finish this section with embeddings from Besov spaces $\mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T})$ into generalized Lorentz-Zygmund spaces. We first need to establish some auxiliary results.

Lemma 5.12. *Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Assume that $0 < \theta < 1, 0 < p \le \infty$ and $0 < q < \infty$. The following continuous embeddings hold*

(a) $(A_0, A_1)_{\theta,q,-\max\{0,1/p-1/q\},-\max\{1/p,1/q\}} \hookrightarrow ((A_0, A_1)_{\theta,p}, A_1)_{(0,1/q),q}$.

(b) $((A_0, A_1)_{\theta,p}, A_1)_{(0,1/q),q} \hookrightarrow (A_0, A_1)_{\theta,q,-\min\{0,1/p-1/q\},-\min\{1/p,1/q\}}$.

Proof. Let $B = ((A_0, A_1)_{\theta,p}, A_1)_{(0,1/q),q}$. Suppose $p < \infty$. The case $p = \infty$ can be carried out similarly.

By Holmstedt's formula (2.13) we have

$$\|a\|_B \sim \left(\int_0^1 \left(\int_0^t \left(\frac{K(s,a)}{s^\theta} \right)^p \frac{ds}{s} \right)^{q/p} \frac{dt}{t(1-\log t)} \right)^{1/q}. \tag{5.8}$$

For $t > 0$, let us denote by $\chi_{(0,t)}$ the characteristic function of the interval $(0,t)$. If $q \le p$, using the embedding $L_{1/(1-\theta),q}([0,1]) \hookrightarrow L_{1/(1-\theta),p}([0,1])$ and that $K(s,a)/s$ is a decreasing function, we derive

$$\begin{aligned}
\|a\|_B &\sim \left(\int_0^1 \left(\int_0^1 \left(s^{1-\theta} \frac{K(s,a)}{s} \chi_{(0,t)}(s) \right)^p \frac{ds}{s} \right)^{q/p} \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&= \left(\int_0^1 \left\| \frac{K(s,a)}{s} \chi_{(0,t)}(s) \right\|_{L_{1/(1-\theta),p}([0,1])}^q \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&\lesssim \left(\int_0^1 \left\| \frac{K(s,a)}{s} \chi_{(0,t)}(s) \right\|_{L_{1/(1-\theta),q}([0,1])}^q \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&= \left(\int_0^1 \int_0^t \left(\frac{K(s,a)}{s^\theta} \right)^q \frac{ds}{s} \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&= \left(\int_0^1 \left(\frac{K(s,a)}{s^\theta} \right)^q \int_s^1 \frac{dt}{t(1-\log t)} \frac{ds}{s} \right)^{1/q} \\
&= \left(\int_0^1 (s^{-\theta} (\log(1-\log s))^{1/q} K(s,a))^q \frac{ds}{s} \right)^{1/q} \\
&\lesssim \|a\|_{(A_0,A_1)_{\theta,q,0,-1/q}}.
\end{aligned}$$

Suppose now $q > p$. Applying the weighted Hardy's inequality of [62, Lemma 3.3] to (5.8) we obtain

$$\begin{aligned}
\|a\|_B &\lesssim \left(\int_0^1 (t^{-\theta}(1-\log t)^{1/p} (\log(1-\log t))^{1/p} K(t,a))^q \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&\lesssim \|a\|_{(A_0,A_1)_{\theta,q,-1/p+1/q,-1/p}}.
\end{aligned}$$

This establishes (a).

In order to prove (b) assume first that $q \geq p$. Using (5.8) and the embedding between Lorentz spaces we obtain

$$\begin{aligned}
\|a\|_B &\sim \left(\int_0^1 \left\| \frac{K(s,a)}{s} \chi_{(0,t)}(s) \right\|^q_{L_{1/(1-\theta),p}([0,1])} \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&\gtrsim \left(\int_0^1 \left\| \frac{K(s,a)}{s} \chi_{(0,t)}(s) \right\|^q_{L_{1/(1-\theta),q}([0,1])} \frac{dt}{t(1-\log t)} \right)^{1/q} \\
&= \left(\int_0^1 (s^{-\theta} (\log(1-\log s))^{1/q} K(s,a))^q \frac{ds}{s} \right)^{1/q} \\
&\sim \|a\|_{(A_0,A_1)_{\theta,q,0,-1/q}}
\end{aligned}$$

where in the last equivalence we have used that $K(t,a) \sim \|a\|_{A_0}$ for $t \geq 1$.

Now suppose that $q < p$. Put $\rho = q/p < 1, \delta = 1/\rho > 1$ and let $\beta > p/q - 1 = 1/\rho\delta' > 0$. Hölder's inequality yields

$$\begin{aligned}
\int_0^t & [(\log(1-\log s))^{-\beta} (s^{-\theta} K(s,a)(1-\log s)^{1/p})^p s^{-1/\rho} (1-\log s)^{-1/\rho}]^\rho ds \\
&\leq \left(\int_0^t [(s^{-\theta} K(s,a)(1-\log s)^{1/p})^p s^{-1} (1-\log s)^{-1}]^{\rho\delta} ds \right)^{1/\delta} \\
&\quad \times \left(\int_0^t [s^{1/\rho-1} (1-\log s)^{1/\rho-1} (\log(1-\log s))^\beta]^{-\rho\delta'} ds \right)^{1/\delta'} \\
&= \left(\int_0^t (s^{-\theta} K(s,a))^p \frac{ds}{s} \right)^{1/\delta} \left(\int_0^t (\log(1-\log s))^{-\beta\rho\delta'} \frac{ds}{s(1-\log s)} \right)^{1/\delta'} \\
&\sim (\log(1-\log t))^{-\beta\rho+1/\delta'} \left(\int_0^t (s^{-\theta} K(s,a))^p \frac{ds}{s} \right)^{q/p}.
\end{aligned}$$

Inserting this estimate in (5.8), we derive

$$\begin{aligned}
\|a\|^q_B &\gtrsim \int_0^1 (\log(1-\log t))^{\beta\rho-1/\delta'} \\
&\quad \times \int_0^t [(\log(1-\log s))^{-\beta} (s^{-\theta} K(s,a)(1-\log s)^{1/p})^p]^\rho \frac{ds}{s(1-\log s)} \frac{dt}{t(1-\log t)} \\
&= \int_0^1 [(\log(1-\log s))^{-\beta} (s^{-\theta} K(s,a)(1-\log s)^{1/p})^p]^\rho \\
&\quad \times \int_s^1 (\log(1-\log t))^{\beta\rho-1/\delta'} \frac{dt}{t(1-\log t)} \frac{ds}{s(1-\log s)} \\
&\sim \int_0^1 [s^{-\theta} K(s,a)(1-\log s)^{1/p}]^q (\log(1-\log s))^{q/p} \frac{ds}{s(1-\log s)} \\
&\sim \|a\|^q_{(A_0,A_1)_{\theta,q,-1/p+1/q,-1/p}}.
\end{aligned}$$

This completes the proof. $\square$

Next we compare the limiting interpolation space $(L_p, L_\infty)_{(0,1/r),r}$ with generalized Lorentz-Zygmund spaces.

Lemma 5.13. *Let (Ω,μ) be a finite measure space, let $0<p<\infty$ and $0<r<\infty$. Then*

$$\begin{aligned} &L_{p,r}(\log L)_{\max\{0,1/p-1/r\}}(\log\log L)_{\max\{1/p,1/r\}}(\Omega)\\ &\qquad\qquad \hookrightarrow (L_p(\Omega), L_\infty(\Omega))_{(0,1/r),r}\\ &\qquad\qquad \hookrightarrow L_{p,r}(\log L)_{\min\{0,1/p-1/r\}}(\log\log L)_{\min\{1/p,1/r\}}(\Omega). \end{aligned}$$

Proof. Take $0<u<p$ and $0<\theta<1$ with $1/p=(1-\theta)/u$. Then $L_p(\Omega)=(L_u(\Omega),L_\infty(\Omega))_{\theta,p}$. Now the result follows from Lemma 5.12 and (2.15). □

Theorem 5.14. *Let $0<p<\infty$ and $0<r<\infty$. Then*

$$\mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T}) \hookrightarrow L_{p,r}(\log L)_{\min\{0,1/p-1/r\}}(\log\log L)_{\min\{1/p,1/r\}}(\mathbb{T}).$$

Proof. By (5.2), $\mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T}) = (L_p(\mathbb{T}), \mathbf{B}^{1/p}_{p,1}(\mathbb{T}))_{(0,1/r),r}$ and, according to Theorem 5.1, $\mathbf{B}^{1/p}_{p,1}(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$. Therefore, using Lemma 5.13 we obtain

$$\begin{aligned} \mathbf{B}^{0,-1/r}_{p,r}(\mathbb{T}) &\hookrightarrow (L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{(0,1/r),r}\\ &\hookrightarrow L_{p,r}(\log L)_{\min\{0,1/p-1/r\}}(\log\log L)_{\min\{1/p,1/r\}}(\mathbb{T}). \end{aligned}$$

□

5.2 Embeddings between Besov spaces with smoothness close to zero

So far we have been working with Besov spaces of generalized smoothness defined by differences. Similar spaces but introduced by following the Fourier-analytical approach have been also studied in the literature. Next we recall this other approach.

Let $\mathcal{D}(\mathbb{T}^d)$ be the set of all complex-valued infinitely differentiable functions on $\mathbb{T}^d$. So, if $f\in\mathcal{D}(\mathbb{T}^d)$ and $x,y\in\mathbb{T}^d$ with $x-y=2k\pi, k\in\mathbb{Z}^d$, then $f(x)=f(y)$. The topology in $\mathcal{D}(\mathbb{T}^d)$ is defined by the family of semi-norms $\|f\|_\alpha = \sup\{|D^\alpha f(x)| : x\in\mathbb{T}^d\}$ where $\alpha=(\alpha_1,\dots,\alpha_d)$ is any multi-index with non-negative coordinates (see [107]). We write $\mathcal{D}'(\mathbb{T}^d)$ for the topological dual of $\mathcal{D}(\mathbb{T}^d)$. Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. By $\mathcal{F}$ we denote the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ and by $\mathcal{F}^{-1}$ the inverse Fourier transform.

Take $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^d : |x| \leq 2\} \text{ and } \varphi_0(x) = 1 \text{ if } |x| \leq 1.$$

For $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$ let

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x). \tag{5.9}$$

Then the sequence $(\varphi_j)_{j=0}^\infty$ forms a dyadic resolution of unity, $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x \in \mathbb{R}^d$.

Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. For $0 < p, q \leq \infty$ and $s, b, d \in \mathbb{R}$, the Besov space $B^{s,b,d}_{p,q}(\mathbb{R}^d)$ (respectively, the periodic Besov spaces $B^{s,b,d}_{p,q}(\mathbb{T}^d)$) consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ (respectively, $f \in \mathcal{D}'(\mathbb{T}^d)$) having a finite quasi-norm

$$\|f\|_{B^{s,b,d}_{p,q}(\Omega)} = \left(\sum_{j=0}^\infty \left(2^{js}(1+j)^b(1+\log(1+j))^d \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L_p(\Omega)} \right)^q \right)^{1/q} \tag{5.10}$$

(with the usual modification if $q = \infty$). See [93], [32], [90], [77]. If $d = 0$ we simply write $B^{s,b}_{p,q}(\Omega)$ instead of $B^{s,b,0}_{p,q}(\Omega)$. In addition, if $b = 0$ then $B^{s,0}_{p,q}(\Omega)$ coincides with the usual Besov space $B^s_{p,q}(\Omega)$.

Remark 5.3. In the periodic setting, note that $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ is equal to the trigonometric polynomial given by $\sum_{m \in \mathbb{Z}^d} \varphi_j(m) \hat{f}(m) e^{imx}$ where $\hat{f}(m)$ are the Fourier coefficients of f given by $\hat{f}(m) = (2\pi)^{-d} f(e^{-imx}), m \in \mathbb{Z}^d$. In particular, if $f \in L_p(\mathbb{T}^d)$ then

$$\hat{f}(m) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-imx} dx$$

(see [107, Remark 3.5.1/1]).

If $1 \leq p \leq \infty$ and $s > 0$, it turns out that $B^{s,b,d}_{p,q}(\Omega) = \mathbf{B}^{s,b,d}_{p,q}(\Omega)$ with equivalence of norms (see [77, Theorem 2.5], [117, 2.5.12] and [107, Theorem 3.5.4]); but if $0 < p < 1$ and $0 < q \leq 1$ then $B^{d(1/p-1)}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{d(1/p-1)}_{p,q}(\mathbb{R}^d)$ as it was shown in [108, Corollary 3.10]. The aim of this section is to study the relationships between these two kinds of spaces when $s = 0$.

First we characterize Besov spaces $\mathbf{B}^{0,b}_{p,q}$ as limiting interpolation spaces between Lebesgue spaces L_p and Sobolev spaces $W^{k,p}$. To get this we will use the following connection between the K-functional for the couple $(L_p, W^{k,p})$ and the k-th order modulus of smoothness. Let $1 \leq p \leq \infty$, it holds that

$$K(t^k, f; L_p(\Omega), W^{k,p}(\Omega)) \sim t^k \|f\|_{L_p(\Omega)} + \omega_k(f,t)_p \tag{5.11}$$

for all $0 < t < 1$ and $f \in L_p(\Omega)$ (see [9, Theorem 5.4.12] and [81, Theorem 1]).

Theorem 5.15. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 \le p \le \infty, 0 < q \le \infty, k \in \mathbb{N}$ and $-\infty < b < \infty$.*

(a) *The space $\mathbf{B}^{0,b}_{p,q}(\Omega)$ does not depend on the choice of $k \in \mathbb{N}$.*

(b) *We have $\mathbf{B}^{0,b}_{p,q}(\Omega) = (L_p(\Omega), W^{k,p}(\Omega))_{(0,-b),q}$ with equivalence of quasi-norms.*

Proof. Let $k \in \mathbb{N}, k > 1$. Put

$$\|f\|_{\mathbf{B}^{0,b}_{p,q}(\Omega)} = \|f\|_{L_p(\Omega)} + \left(\int_0^1 ((1-\log t)^b \omega(f,t)_p)^q \frac{dt}{t}\right)^{1/q} \tag{5.12}$$

and

$$\|f\|^{(k+)}_{\mathbf{B}^{0,b}_{p,q}(\Omega)} = \|f\|_{L_p(\Omega)} + \left(\int_0^1 ((1-\log t)^b \omega_k(f,t)_p)^q \frac{dt}{t}\right)^{1/q}.$$

Our aim is to show the equivalence between the quasi-norms $\|\cdot\|_{\mathbf{B}^{0,b}_{p,q}(\Omega)}$ and $\|\cdot\|^{(k+)}_{\mathbf{B}^{0,b}_{p,q}(\Omega)}$. Since $\omega_k(f,t)_p \le 2^{k-1}\omega(f,t)_p$, it is clear that $\|f\|^{(k+)}_{\mathbf{B}^{0,b}_{p,q}(\Omega)} \lesssim \|f\|_{\mathbf{B}^{0,b}_{p,q}(\Omega)}$. Let us check the converse inequality. Using Marchaud's inequality [81, (2.5)], for $0 < t \le 1$ we obtain

$$\begin{aligned}
\frac{\omega(f,t)_p}{t} &\lesssim \|f\|_{L_p(\Omega)} + \int_t^\infty \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} \\
&\lesssim \|f\|_{L_p(\Omega)} + \int_t^1 \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} + \|f\|_{L_p(\Omega)} \int_1^\infty s^{-1}\frac{ds}{s} \\
&\sim \int_t^1 \frac{\omega_k(f,s)_p}{s}\frac{ds}{s} + \|f\|_{L_p(\Omega)}.
\end{aligned}$$

Therefore, since $\omega_k(f,s)_p/s^k$ is equivalent to a decreasing function, applying Hardy's inequality [8, Theorem 6.4], we get

$$\begin{aligned}
&\left(\int_0^1 ((1-\log t)^b \omega(f,t)_p)^q \frac{dt}{t}\right)^{1/q} \\
&\quad\lesssim \left(\int_0^1 (t(1-\log t)^b)^q \frac{dt}{t}\right)^{1/q} \|f\|_{L_p(\Omega)} \\
&\qquad + \left(\int_0^1 \left[t(1-\log t)^b \int_t^1 \frac{\omega_k(f,s)_p}{s^k} s^{(k-1)-1} ds\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad\lesssim \|f\|_{L_p(\Omega)} + \left(\int_0^1 \left[t^2(1-\log t)^b \frac{\omega_k(f,t)_p}{t^2}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad= \|f\|^{(k+)}_{\mathbf{B}^{0,b}_{p,q}(\Omega)}.
\end{aligned}$$

This proves statement (a).

As for (b), using (5.11), we obtain

$$\begin{aligned}\|f\|_{(L_p(\Omega),W^{k,p}(\Omega))_{(0,-b),q}} &= \left(\int_0^1 [(1-\log t)^b K(t,f;L_p(\Omega),W^{k,p}(\Omega))]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \left(\int_0^1 [(1-\log t)^b K(t^k,f;L_p(\Omega),W^{k,p}(\Omega))]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \left(\int_0^1 [(1-\log t)^b (t^k\|f\|_{L_p(\Omega)} + \omega_k(f,t)_p)]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \|f\|_{L_p(\Omega)} + \left(\int_0^1 [(1-\log t)^b \omega_k(f,t)_p]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \|f\|_{\mathbf{B}^{0,b}_{p,q}(\Omega)}\end{aligned}$$

where we have used (a) in the last equivalence. This completes the proof. □

Remark 5.4. For Besov spaces defined over $\mathbb{T}^d$, the fact that the definition of $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ is independent of $k \in \mathbb{N}$ has also been proved in [70, pp. 1041–1043] by using Jackson inequality and Bernstein inequality.

Remark 5.5. In the case that $\Omega = \mathbb{R}^d$, the interpolation formula given in Theorem 5.15(b) can be extended, replacing Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ by fractional Sobolev spaces $H^s_p(\mathbb{R}^d)$ or even Besov spaces $\mathbf{B}^s_{p,u}(\mathbb{R}^d)$ with $s > 0$. See Theorem 9.3.

As before, we continue assuming that $\mathbf{B}^{0,b}_{p,q}$ is quasi-normed by (5.12).

Now we compare spaces $B^{0,b}_{p,q}$ with spaces $\mathbf{B}^{0,b}_{p,q}$.

Theorem 5.16. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Then*

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\Omega) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\Omega).$$

Proof. Recall that

$$B^s_{p,\min\{p,q\}}(\Omega) \hookrightarrow F^s_{p,q}(\Omega) \hookrightarrow B^s_{p,\max\{p,q\}}(\Omega) \tag{5.13}$$

where $F^s_{p,q}(\Omega)$ stands for the Triebel-Lizorkin space (see [117, Proposition 2.3.2/2(iii)] and [107, (20), p. 164]). Moreover, $F^s_{p,2}(\Omega) = H^s_p(\Omega)$ (see [117, Theorem 2.5.6(i)] and [107, Theorem 3.5.4(v)]) and so $F^0_{p,2}(\Omega) = L_p(\Omega)$. Hence, writing down embeddings (5.13) when $s = 0$ and $q = 2$, we have

$$B^0_{p,\min\{p,2\}}(\Omega) \hookrightarrow L_p(\Omega) \tag{5.14}$$

and

$$L_p(\Omega) \hookrightarrow B^0_{p,\max\{p,2\}}(\Omega). \tag{5.15}$$

According to Theorem 5.15(b), Lemma 3.8 and [32, Theorem 5.3 and Remark 5.4], we derive

$$\begin{aligned}\mathbf{B}^{0,b}_{p,q}(\Omega) &= (L_p(\Omega), W^{1,p}(\Omega))_{(0,-b),q} \hookrightarrow (B^0_{p,\max\{2,p\}}(\Omega), H^1_p(\Omega))_{(0,-b),q}\\ &= ((H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,\max\{2,p\}}, H^1_p(\Omega))_{(0,-b),q}\\ &\hookrightarrow (H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,q,-b-1/\max\{2,p,q\}}\\ &= B^{0,b+1/\max\{2,p,q\}}_{p,q}(\Omega).\end{aligned}$$

Similarly, we have

$$\begin{aligned}B^{0,b+1/\min\{2,p,q\}}_{p,q}(\Omega) &= (H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,q,-b-1/\min\{2,p,q\}}\\ &\hookrightarrow ((H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,\min\{2,p\}}, H^1_p(\Omega))_{(0,-b),q}\\ &= (B^0_{p,\min\{2,p\}}(\Omega), H^1_p(\Omega))_{(0,-b),q}\\ &\hookrightarrow (L_p(\Omega), W^{1,p}(\Omega))_{(0,-b),q}\\ &= \mathbf{B}^{0,b}_{p,q}(\Omega).\end{aligned}$$

□

Remark 5.6. In general $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \neq B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d)$ because in any of the cases

$$\begin{cases} 1<p<\infty \quad , \quad 0<q\le \min\{2,p\} \;,\; b+\frac{1}{\max\{2,p\}}<0<b+\frac{1}{q} \;,\\ 1<p\le 2 \quad , \quad p<q\le\infty \;,\; 0<b+\frac{1}{q}\le \frac{1}{p}-\frac{1}{\max\{2,q\}} \;,\\ 2<p<\infty \quad , \quad 2<q\le\infty \;,\; 0<b+\frac{1}{q}\le\frac{1}{2}-\frac{1}{\max\{p,q\}} \;,\end{cases}$$

the space $B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d)$ does not contain only regular distributions (see [20, Theorem 4.3]).

We write down the case when $p = q$ in Theorem 5.16.

Corollary 5.17. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 < p < \infty$ and $b > -1/p$.*

(a) *If $1 < p \le 2$ then $B^{0,b+1/p}_{p,p}(\Omega) \hookrightarrow \mathbf{B}^{0,b}_{p,p}(\Omega) \hookrightarrow B^{0,b+1/2}_{p,p}(\Omega)$.*

(b) *If $2 \le p < \infty$ then $B^{0,b+1/2}_{p,p}(\Omega) \hookrightarrow \mathbf{B}^{0,b}_{p,p}(\Omega) \hookrightarrow B^{0,b+1/p}_{p,p}(\Omega)$.*

In particular, for $b > -1/2$ we obtain with equivalence of norms

$$B^{0,b+1/2}_{2,2}(\Omega) = \mathbf{B}^{0,b}_{2,2}(\Omega).$$

Remark 5.7. As in Remark 5.6, note that

$$\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d) \neq B^{0,b+1/2}_{p,p}(\mathbb{R}^d) \quad \text{if} \quad 1 < p < 2 \quad \text{and} \quad b + \frac{1}{2} < 0 < b + \frac{1}{p}$$

and

$$\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d) \neq B^{0,b+1/p}_{p,p}(\mathbb{R}^d) \quad \text{if} \quad 2 < p < \infty \quad \text{and} \quad 0 < b + \frac{1}{p} \le \frac{1}{2} - \frac{1}{p}.$$

The limit case when $b = -1/q$ was not considered in Theorem 5.16. Next we focus on the relationships between $\mathbf{B}^{0,-1/q}_{p,q}$ and other kind of Besov spaces with logarithmic smoothness.

Theorem 5.18. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 < p < \infty$ and $0 < q < \infty$. Then*

$$B^{0,1/\min\{2,p,q\}-1/q,1/\min\{2,p,q\}}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,1/\max\{2,p,q\}-1/q,1/\max\{2,p,q\}}_{p,q}(\Omega).$$

Proof. By Theorem 5.15(b), we have that $\mathbf{B}^{0,-1/q}_{p,q}(\Omega) = (L_p(\Omega), H^1_p(\Omega))_{(0,1/q),q}$. Moreover, using (5.15), $L_p(\Omega) \hookrightarrow B^0_{p,\max\{2,p\}}(\Omega) = (H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,\max\{2,p\}}$. Therefore, using Lemma 5.12(b) we derive

$$\begin{aligned}\mathbf{B}^{0,-1/q}_{p,q}(\Omega) &\hookrightarrow ((H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,\max\{2,p\}}, H^1_p(\Omega))_{(0,1/q),q} \\ &\hookrightarrow (H^{-1}_p(\Omega), H^1_p(\Omega))_{1/2,q,-1/\max\{2,p,q\}+1/q,-1/\max\{2,p,q\}} \\ &= B^{0,1/\max\{2,p,q\}-1/q,1/\max\{2,p,q\}}_{p,q}(\Omega).\end{aligned}$$

The proof of the left-hand side embedding of the statement is similar but using now (5.14) and Lemma 5.12(a). $\square$

Remark 5.8. In the previous result we have excluded the case $q = \infty$. Note that $\mathbf{B}^{0,0}_{p,\infty}(\Omega) = L_p(\Omega)$. On the other hand, by [117, Proposition 2.5.7] and [107, (20) and (21), p. 169], we have that $L_p(\Omega) \hookrightarrow B^0_{p,\infty}(\Omega)$ for $1 \le p \le \infty$, and this embedding is known to be strict.

Corollary 5.19. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 < p < \infty$ and $0 < q < \infty$.*

(a) *If $q \le \min\{2,p\}$, then $B^{0,0,1/q}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega)$.*

(b) *If $\max\{2,p\} \le q$, then $\mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,0,1/q}_{p,q}(\Omega)$.*

In particular, we have with equivalence of norms

$$\mathbf{B}^{0,-1/2}_{2,2}(\Omega) = B^{0,0,1/2}_{2,2}(\Omega).$$

The structure of the space $B^{s,b,d}_{p,q}(\Omega)$ has an important peculiarity. To describe it, let φ_j as in (5.9), put $\varphi_{-1} \equiv 0$ and let $\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ for $j \in \mathbb{N}_0$. We put $\lambda_j = 2^{js}(1+j)^b(1+\log(1+j))^d$, and we consider the mapping $\mathfrak{J} : B^{s,b,d}_{p,q}(\Omega) \longrightarrow \ell_q(\lambda_j L_p(\Omega))$ assigning to any $f \in B^{s,b,d}_{p,q}(\Omega)$ the sequence $\mathfrak{J}f = (\mathcal{F}^{-1}(\varphi_j \mathcal{F} f))$ and the mapping $\mathfrak{R} : \ell_q(\lambda_j L_p(\Omega)) \longrightarrow B^{s,b,d}_{p,q}(\Omega)$ defined by $\mathfrak{R}(f_j) = \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\widetilde{\varphi}_j \mathcal{F} f_j)$ (convergence in $\mathcal{S}'$). Using that $\widetilde{\varphi}_j \varphi_j = \varphi_j$ one can check that $\mathfrak{R}\mathfrak{J}f = f$ for any $f \in B^{s,b,d}_{p,q}(\Omega)$. So, in the sense of [116, Definition 1.2.4], the space $B^{s,b,d}_{p,q}(\Omega)$ is a retract of $\ell_q(\lambda_j L_p(\Omega))$, being $\mathfrak{R}$ a retraction from $\ell_q(\lambda_j L_p(\Omega))$ to $B^{s,b,d}_{p,q}(\Omega)$ and $\mathfrak{J}$ the corresponding co-retraction from $B^{s,b,d}_{p,q}(\Omega)$ to $\ell_q(\lambda_j L_p(\Omega))$ (see [32, 1]).

Next we use this property to extend Corollary 5.19(a) to the case $q \nleq \min\{2, p\}$ (respectively, $\max\{2, p\} \nleq q$ in the case (b)). We need some auxiliary results.

Lemma 5.20. *Let A be a Banach space, $\lambda > 1$ and $0 < q < \infty$. Then*

$$(A, \lambda A)_{(0,1/q),q} = (\log(1 + \log \lambda))^{1/q} A$$

with equivalence of quasi-norms where the constants are independent of λ.

Proof. Clearly $K(t, a; A, \lambda A) = \min\{1, t\lambda\}\|a\|_A$. Whence

$$\begin{aligned}\|a\|_{(A,\lambda A)_{(0,1/q),q}} &= \left(\int_0^{1/\lambda} (\lambda t(1-\log t)^{-1/q})^q \frac{dt}{t} + \int_{1/\lambda}^{1} (1-\log t)^{-1} \frac{dt}{t}\right)^{1/q} \|a\|_A \\ &= \left(\int_0^{1/\lambda} (\lambda t(1-\log t)^{-1/q})^q \frac{dt}{t} + \log(1+\log \lambda)\right)^{1/q} \|a\|_A.\end{aligned}$$

Since

$$\int_0^{1/\lambda} (\lambda t(1-\log t)^{-1/q})^q \frac{dt}{t} \lesssim (1+\log\lambda)^{-1}$$

the result follows. □

Subsequently, we put

$$K_p(t, a; A_0, A_1) = \inf\{(\|a_0\|^p_{A_0} + t^p \|a_1\|^p_{A_1})^{1/p} : a = a_0 + a_1, a_j \in A_j\}. \tag{5.16}$$

The K_p-functional is more suitable than the K-functional for the interpolation of vector-valued couples. Note that $K_p(t, a; A_0, A_1) \sim K(t, a; A_0, A_1)$.

Lemma 5.21. *Let $0 < p \leq \infty, 0 < q < \infty$ and let $(A_j), (B_j)$ be sequences of Banach spaces with $B_j \hookrightarrow A_j$ and $\sup_{j \in \mathbb{N}_0} \|I\|_{B_j, A_j} < \infty$, so $\ell_p(B_j) \hookrightarrow \ell_p(A_j)$. Then*

(a) $\ell_{\min\{p,q\}}\left((A_j, B_j)_{(0,1/q),q}\right) \hookrightarrow (\ell_p(A_j), \ell_p(B_j))_{(0,1/q),q}$.

(b) $(\ell_p(A_j), \ell_p(B_j))_{(0,1/q),q} \hookrightarrow \ell_{\max\{p,q\}}\left((A_j, B_j)_{(0,1/q),q}\right)$.

Proof. Let $Z = (\ell_p(A_j), \ell_p(B_j))_{(0,1/q),q}$. Since

$$K_p(t, a; \ell_p(A_j), \ell_p(B_j)) = \left(\sum_{j=0}^{\infty} K_p(t, a_j; A_j, B_j)^p\right)^{1/p},$$

we have that

$$\|a\|_Z \sim \left(\int_0^1 \left[\sum_{j=0}^{\infty}((1-\log t)^{-1/q} K_p(t, a_j; A_j, B_j))^p\right]^{q/p} \frac{dt}{t}\right)^{1/q}.$$

If $q \geq p$, triangle inequality yields that

$$\begin{aligned}\|a\|_Z &\lesssim \left(\sum_{j=0}^{\infty}\left(\int_0^1 [(1-\log t)^{-1/q} K_p(t, a_j; A_j, B_j)]^q \frac{dt}{t}\right)^{p/q}\right)^{1/p} \\ &\sim \left(\sum_{j=0}^{\infty} \|a_j\|^p_{(A_j,B_j)_{(0,1/q),q}}\right)^{1/p}.\end{aligned}$$

If $q < p$, using that $\ell_q \hookrightarrow \ell_p$ we obtain

$$\begin{aligned}\|a\|_Z &\lesssim \left(\int_0^1 \sum_{j=0}^{\infty} [(1-\log t)^{-1/q} K_p(t, a_j; A_j, B_j)]^q \frac{dt}{t}\right)^{1/q} \\ &\sim \left(\sum_{j=0}^{\infty} \|a_j\|^q_{(A_j,B_j)_{(0,1/q),q}}\right)^{1/q}.\end{aligned}$$

This establishes (a). The proof of (b) is similar but using now Minkowski's inequality for integrals when $p \geq q$. □

Theorem 5.22. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $1 < p < \infty$ and $0 < q < \infty$. Then*

$$B^{0,0,1/q}_{p,\min\{2,p,q\}}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,0,1/q}_{p,\max\{2,p,q\}}(\Omega).$$

Proof. Using the interpolation formula given in Theorem 5.15(b), it follows from embeddings (5.15) and $H^1_p(\Omega) \hookrightarrow B^1_{p,\max\{2,p\}}(\Omega)$ (see (5.13) with $s = 1$) that

$$\mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow (B^0_{p,\max\{2,p\}}(\Omega), B^1_{p,\max\{2,p\}}(\Omega))_{(0,1/q),q}.$$

The space $B^k_{p,\max\{2,p\}}(\Omega)$ is a retract of $\ell_{\max\{2,p\}}(2^{jk} L_p(\Omega))$ and for the couple of vector-valued sequence spaces, according to Lemmata 5.21 and 5.20, we get

$$\begin{aligned}&(\ell_{\max\{2,p\}}(L_p(\Omega)), \ell_{\max\{2,p\}}(2^j L_p(\Omega)))_{(0,1/q),q} \\ &\qquad \hookrightarrow \ell_{\max\{2,p,q\}}\left((L_p(\Omega), 2^j L_p(\Omega))_{(0,1/q),q}\right) \\ &\qquad = \ell_{\max\{2,p,q\}}((\log(1+j))^{1/q} L_p(\Omega)).\end{aligned}$$

Consequently, we conclude that $\mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,0,1/q}_{p,\max\{2,p,q\}}(\Omega)$.

To establish the left-hand side embedding of the statement, we use now (5.14) and $B^1_{p,\min\{2,p\}}(\Omega) \hookrightarrow H^1_p(\Omega)$ (see (5.13) with $s = 1$). Hence

$$(B^0_{p,\min\{2,p\}}(\Omega), B^1_{p,\min\{2,p\}}(\Omega))_{(0,1/q),q} \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega).$$

Since

$$\ell_{\min\{2,p,q\}}((\log(1+j))^{1/q} L_p(\Omega)) \hookrightarrow (\ell_{\min\{2,p\}}(L_p(\Omega)), \ell_{\min\{2,p\}}(2^j L_p(\Omega)))_{(0,1/q),q},$$

we conclude that $B^{0,0,1/q}_{p,\min\{2,p,q\}}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega)$. □

Remark 5.9. Corollary 5.19 can be also derived from Theorem 5.22. However, if $0 < q < \max\{2,p\}$, then Theorem 5.18 yields that

$$\mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,1/\max\{2,p\}-1/q,1/\max\{2,p\}}_{p,q}(\mathbb{R}^d) \tag{5.17}$$

while Theorem 5.22 shows that

$$\mathbf{B}^{0,-1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,0,1/q}_{p,\max\{2,p\}}(\mathbb{R}^d) \tag{5.18}$$

and spaces to the right-hand side in (5.17) and (5.18) are not comparable. Indeed, since

$$\left(\frac{(1+j)^{1/\max\{2,p\}-1/q}(1+\log(1+j))^{1/\max\{2,p\}}}{(1+\log(1+j))^{1/q}}\right) \notin \ell_{(1/q-1/\max\{2,p\})^{-1}},$$

it follows from [19, Proposition 5.3(i)] that

$$B^{0,0,1/q}_{p,\max\{2,p\}}(\mathbb{R}^d) \not\hookrightarrow B^{0,1/\max\{2,p\}-1/q,1/\max\{2,p\}}_{p,q}(\mathbb{R}^d).$$

On the other hand, we also have

$$\left(\frac{(1+\log(1+j))^{1/q}}{(1+j)^{1/\max\{2,p\}-1/q}(1+\log(1+j))^{1/\max\{2,p\}}}\right) \notin \ell_\infty.$$

Therefore

$$B^{0,1/\max\{2,p\}-1/q,1/\max\{2,p\}}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,0,1/q}_{p,\max\{2,p\}}(\mathbb{R}^d).$$

Theorems 5.16, 5.18 and 5.22 require the assumption $1 < p < \infty$. We close this section with the corresponding results in the extreme cases $p = 1$ and $p = \infty$.

Theorem 5.23. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $0 < q \le \infty, b > -1/q$ and $p = 1$ or ∞. Then*

$$B^{0,b+1/\min\{1,q\}}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\Omega) \hookrightarrow B^{0,b}_{p,q}(\Omega).$$

Proof. Suppose $p = 1$. The case $p = \infty$ can be treated analogously. Take any $0 < \theta < 1$. By Theorem 5.15(b) and Lemma 2.2(b), we have that

$$\begin{aligned}\mathbf{B}^{0,b}_{1,q}(\Omega) &= (L_1(\Omega), W^{1,1}(\Omega))_{(0,-b),q} = (L_1(\Omega), (L_1(\Omega), W^{1,1}(\Omega))_{\theta,\infty})_{(0,-b),q} \\ &= (L_1(\Omega), B^{\theta}_{1,\infty}(\Omega))_{(0,-b),q}.\end{aligned}$$

Moreover, $B^0_{1,1}(\Omega) \hookrightarrow L_1(\Omega) \hookrightarrow B^0_{1,\infty}(\Omega)$ (see [117, Proposition 2.5.7] and [107, (21), p. 169]). Consequently, using Lemma 3.8(b) and [32, Theorem 5.3], we derive

$$\begin{aligned}\mathbf{B}^{0,b}_{1,q}(\Omega) &\hookrightarrow (B^0_{1,\infty}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{(0,-b),q} \\ &= ((B^{-\theta}_{1,\infty}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{1/2,\infty}, B^{\theta}_{1,\infty}(\Omega))_{(0,-b),q} \\ &\hookrightarrow (B^{-\theta}_{1,\infty}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{1/2,q,-b} \\ &= B^{0,b}_{1,q}(\Omega).\end{aligned}$$

Proceeding similarly, we get

$$\begin{aligned}\mathbf{B}^{0,b}_{1,q}(\Omega) &\hookleftarrow (B^0_{1,1}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{(0,-b),q} \\ &= ((B^{-\theta}_{1,\infty}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{1/2,1}, B^{\theta}_{1,\infty}(\Omega))_{(0,-b),q} \\ &\hookleftarrow (B^{-\theta}_{1,\infty}(\Omega), B^{\theta}_{1,\infty}(\Omega))_{1/2,q,-b-1/\min\{1,q\}} \\ &= B^{0,b+1/\min\{1,q\}}_{1,q}(\Omega).\end{aligned}$$

□

The following result can be derived proceeding as in Theorem 5.23 but using now Lemma 5.12.

Theorem 5.24. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $0 < q < \infty$ and $p = 1$ or ∞. Then*

$$B^{0,1/\min\{1,q\}-1/q,1/\min\{1,q\}}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,-1/q}_{p,q}(\Omega).$$

The corresponding result to Theorem 5.22 reads as follows.

Theorem 5.25. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Assume that $0 < q < \infty$ and $p = 1$ or ∞. Then*

$$B^{0,0,1/q}_{p,\min\{1,q\}}(\Omega) \hookrightarrow \mathbf{B}^{0,-1/q}_{p,q}(\Omega) \hookrightarrow B^{0,0,1/q}_{p,\infty}(\Omega).$$

5.3 Embeddings between Besov spaces with different metrics

A classical embedding for function spaces, which goes back to the Hardy-Littlewood theorem on Lipschitz classes, says that given $1 \le p < r < \infty, 1 \le q \le \infty$ and $0 < s_1 < s_0 < \infty$, then

$$\mathbf{B}^{s_0}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{s_1}_{r,q}(\Omega) \text{ if } s_0 - \frac{d}{p} = s_1 - \frac{d}{r}, \tag{5.19}$$

where $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ (see [97, 6.3 and 6.10.1]). The generalization to Besov spaces of logarithmic smoothness $\mathbf{B}^{s,b}_{p,q}(\Omega)$ was obtained by DeVore, Riemenschneider and Sharpley [51, Corollary 5.3(i)] by using weak type interpolation techniques. For s_0, s_1, p, r, q satisfying the above conditions and $-\infty < b < \infty$, we have

$$\mathbf{B}^{s_0,b}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{s_1,b}_{r,q}(\Omega). \tag{5.20}$$

The aim of this section is to study the limiting case when $s_1 = 0$ in embedding (5.20). This problem has been considered in [51, Corollary 5.3(ii)] and [70, Corollary 2.8]. We follow a more simple approach than in [70] based on limiting interpolation.

Theorem 5.26. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $1 \le p < r < \infty, 0 < q \le \infty, b > -1/q$ and $\alpha = d(1/p - 1/r)$. Then*

$$\mathbf{B}^{\alpha, b+1/\min\{q,r\}}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{0,b}_{r,q}(\Omega).$$

Proof. According to [9, Corollary 5.4.20] and [51, Corollary 5.5(i)], we have

$$\mathbf{B}^{\alpha}_{p,r}(\Omega) \hookrightarrow L_r(\Omega). \tag{5.21}$$

On the other hand, let $k \in \mathbb{N}$ such that $k > \alpha$ and $0 < \theta < 1$ such that $\theta k > \alpha$. By [9, Corollaries 5.4.13], [107, (3), p. 173] and (5.19), we derive

$$W^{k,p}(\Omega) \hookrightarrow (L_p(\Omega), W^{k,p}(\Omega))_{\theta,p} = \mathbf{B}^{\theta k}_{p,p}(\Omega) \hookrightarrow \mathbf{B}^{\theta k - \alpha}_{r,p}(\Omega). \tag{5.22}$$

Interpolating embeddings (5.21) and (5.22) by the limiting real method we get

$$(\mathbf{B}^{\alpha}_{p,r}(\Omega), W^{k,p}(\Omega))_{(0,-b),q} \hookrightarrow (L_r(\Omega), \mathbf{B}^{\theta k-\alpha}_{r,p}(\Omega))_{(0,-b),q}.$$

The target space in this embedding can be determined by using Lemma 2.2(b) and Theorem 5.15(b). Indeed,

$$\begin{aligned}(L_r(\Omega), \mathbf{B}^{\theta k-\alpha}_{r,p}(\Omega))_{(0,-b),q} &= (L_r(\Omega), (L_r(\Omega), W^{k,r}(\Omega))_{\frac{\theta k-\alpha}{k},p})_{(0,-b),q} \\ &= (L_r(\Omega), W^{k,r}(\Omega))_{(0,-b),q} = \mathbf{B}^{0,b}_{r,q}(\Omega).\end{aligned}$$

As for the domain space, Lemma 3.8(b) yields

$$\begin{aligned}\mathbf{B}^{\alpha,b+1/\min\{q,r\}}_{p,q}(\Omega) &= (L_p(\Omega), W^{k,p}(\Omega))_{\alpha/k,q,-b-1/\min\{q,r\}}\\ &\hookrightarrow ((L_p(\Omega), W^{k,p}(\Omega))_{\alpha/k,r}, W^{k,p}(\Omega))_{(0,-b),q}\\ &= (\mathbf{B}^{\alpha}_{p,r}(\Omega), W^{k,p}(\Omega))_{(0,-b),q}.\end{aligned}$$

This completes the proof. □

5.4 Embeddings between Besov and Lipschitz spaces

Next we recall the definition of logarithmic Lipschitz spaces introduced by Haroske [75, 76] and we show that they can be generated by interpolation from the couple $(L_p, W^{1,p})$.

Definition 5.1. Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $1 \le p \le \infty, 0 < q \le \infty$ and $\alpha > 1/q\,(\alpha \ge 0$ if $q = \infty)$. The space $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)$ is formed by all functions $f \in L_p(\Omega)$ having a finite quasi-norm

$$\|f\|_{\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)} = \|f\|_{L_p(\Omega)} + \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Note that if $p = q = \infty$ and $\alpha = 0$, we recover classical Lipshitz spaces formed by all functions $f \in L_\infty(\Omega)$ for which

$$\sup_{0<|h|<\frac{1}{2}} \sup_{x\in\Omega} \frac{|f(x+h)-f(x)|}{|h|} < \infty.$$

Lemma 5.27. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $1 \le p \le \infty, 0 < q \le \infty$ and $\alpha > 1/q\,(\alpha \ge 0$ if $q = \infty)$. Then*

$$(L_p(\Omega), W^{1,p}(\Omega))_{(1,\alpha),q} = \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)$$

with equivalent quasi-norms.

Proof. Using (5.11) we derive

$$\begin{aligned}\|f\|_{(L_p(\Omega),W^{1,p}(\Omega))_{(1,\alpha),q}} &= \left(\int_0^1 \left[\frac{K(t,f;L_p(\Omega),W^{1,p}(\Omega))}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \left(\int_0^1 (1-\log t)^{-\alpha q}\frac{dt}{t}\right)^{1/q} \|f\|_{L_p(\Omega)} + \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \|f\|_{\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)}.\end{aligned}$$

□

The next result describes the position of Lipschitz spaces between Besov spaces with classical smoothness 1 and additional logarithmic smoothness.

Theorem 5.28. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $1 < p < \infty, 0 < q \le \infty$ and $\alpha > 1/q$. Then*

$$B^{1,-\alpha+1/\min\{2,p,q\}}_{p,q}(\Omega) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega) \hookrightarrow B^{1,-\alpha+1/\max\{2,p,q\}}_{p,q}(\Omega).$$

Proof. By Lemmata 5.27, 3.8(a) and [32, Theorem 5.3 and Remark 5.4], we obtain

$$\begin{aligned}\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega) &= (L_p(\Omega), W^{1,p}(\Omega))_{(1,\alpha),q} \hookrightarrow (L_p(\Omega), B^1_{p,\max\{2,p\}}(\Omega))_{(1,\alpha),q}\\ &= (L_p(\Omega), (L_p(\Omega), W^{2,p}(\Omega))_{1/2,\max\{2,p\}})_{(1,\alpha),q}\\ &\hookrightarrow (L_p(\Omega), W^{2,p}(\Omega))_{1/2,q,\alpha-1/\max\{2,p,q\}}\\ &= B^{1,-\alpha+1/\max\{2,p,q\}}_{p,q}(\Omega).\end{aligned}$$

Similarly, we derive

$$\begin{aligned}B^{1,-\alpha+1/\min\{2,p,q\}}_{p,q}(\Omega) &= (L_p(\Omega), W^{2,p}(\Omega))_{1/2,q,\alpha-1/\min\{2,p,q\}}\\ &\hookrightarrow (L_p(\Omega), (L_p(\Omega), W^{2,p}(\Omega))_{1/2,\min\{2,p\}})_{(1,\alpha),q}\\ &= (L_p(\Omega), B^1_{p,\min\{2,p\}}(\Omega))_{(1,\alpha),q}\\ &\hookrightarrow (L_p(\Omega), W^{1,p}(\Omega))_{(1,\alpha),q}\\ &= \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega).\end{aligned}$$

□

Corollary 5.29. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Let $1 < p < \infty$ and $\alpha > 1/p$.*

(a) *If $1 < p \le 2$ then $B^{1,-\alpha+1/p}_{p,p}(\Omega) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,p}(\Omega) \hookrightarrow B^{1,-\alpha+1/2}_{p,p}(\Omega)$.*

(b) *If $2 \le p < \infty$ then $B^{1,-\alpha+1/2}_{p,p}(\Omega) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,p}(\Omega) \hookrightarrow B^{1,-\alpha+1/p}_{p,p}(\Omega)$.*

In particular, if $\alpha > 1/2$ we have

$$B^{1,-\alpha+1/2}_{2,2}(\Omega) = \mathrm{Lip}^{(1,-\alpha)}_{2,2}(\Omega).$$

Next we recall a result of Haroske [75, Proposition 16].

Proposition 5.30. *Let $1 \le p \le \infty, 0 < q, v \le \infty, \alpha > 1/q$ and $\beta > 1/v$. Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,v}(\mathbb{R}^d) \text{ if, and only if, } \begin{cases}\beta - \frac{1}{v} \ge \alpha - \frac{1}{q} & \text{and} \quad v \ge q,\\ \beta - \frac{1}{v} > \alpha - \frac{1}{q} & \text{and} \quad v < q.\end{cases}$$

In the remaining part of this section we show that combining Proposition 5.30 with the previous results we can derive some complements and improvements of the embedding results given in [75] for logarithmic Lipschitz spaces over $\mathbb{R}^d$.

Theorem 5.31. *Let $1 < p < \infty, 0 < q, v \leq \infty$ and $\alpha > 1/v$. Then*

$$B^1_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}(\mathbb{R}^d) \;\; \textit{if} \begin{cases} 0 < q \leq \min\{2,p\}, \\ \min\{2,p\} < q, v < q \quad \textit{and} \quad \alpha > \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}, \\ \min\{2,p\} < q \leq v \quad \textit{and} \quad \alpha \geq \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}. \end{cases}$$

Proof. If $0 < q \leq \min\{2,p\}$, we obtain

$$\begin{aligned} B^1_{p,q}(\mathbb{R}^d) &\hookrightarrow B^1_{p,\min\{2,p\}}(\mathbb{R}^d) \hookrightarrow (L_p(\mathbb{R}^d), B^1_{p,\min\{2,p\}}(\mathbb{R}^d))_{(1,\alpha),v} \\ &\hookrightarrow (L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{(1,\alpha),v} = \mathrm{Lip}^{(1,-\alpha)}_{p,v}(\mathbb{R}^d). \end{aligned}$$

If $\min\{2,p\} < q$, let $\beta = 1/\min\{2,p\}$. By Theorem 5.28 and Proposition 5.30, we derive

$$B^1_{p,q}(\mathbb{R}^d) = B^{1,-\beta+\frac{1}{\min\{2,p,q\}}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}(\mathbb{R}^d).$$

□

Remark 5.10. Theorem 5.31 confirms a conjecture of Haroske in [75, Remark 12] and closes a problem also mentioned in [76, page 115] by showing that if $1 < p < \infty$ the embedding $B^1_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}(\mathbb{R}^d)$ holds not only for $\alpha = 1/q' + 1/v = 1 + 1/v - 1/q$ but even for smaller values of α.

Remark 5.11. Note that the argument given in Theorem 5.31 when $0 < q \leq \min\{2,p\}$ works for any $\alpha \geq 0$ if $v = \infty$. So when $q = \min\{2,p\}$ we recover a result proved by Neves [96, Proposition 5.6] using different techniques.

Next for $1 \leq q < \infty$ we cover a limit case left open in [75, Corollary 23(i)] (see also [76, Corollary 7.20(i)]).

Corollary 5.32. *Let $1 < p < \infty, 1 \leq q \leq \infty$ and $0 < \beta = \alpha - 1/q$. Then*

$$B^{1,-\beta}_{p,1}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d).$$

Proof. Using again Theorem 5.28 and Proposition 5.30 we obtain

$$B^{1,-\beta}_{p,1}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\beta-1)}_{p,1}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d).$$

□

Proceeding as in Corollary 5.32, we can also derive the embedding

$$B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$$

provided that $1 < p < \infty, 0 < q \leq \infty$ and $\alpha > 1/q$. This improves [75, (29), p. 793] because

$$B^{1,-\alpha+1/q}_{p,\min\{1,q\}}(\mathbb{R}^d) \hookrightarrow B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}}(\mathbb{R}^d).$$

Note also that from Theorem 5.28 we can recover [75, (41), p. 796] for $1 < p < \infty$. Besides, Theorem 5.28 also yields that if $\alpha > 1/q$ then

$$\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{1,-\alpha+1/q}_{p,q}(\mathbb{R}^d) \text{ if } \max\{2,p\} \leq q,$$

and

$$B^{1,-\alpha+1/q}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d) \text{ if } q \leq \min\{2,p\}.$$

Chapter 6

Some other applications of limiting interpolation

In this chapter we give some other interesting applications of limiting real interpolation methods. Section 6.1 contains the duality characterization of Besov spaces $\mathbf{B}^{0,b}_{p,q}$ for $1 < p < \infty, 1 \le q < \infty$ and $b > -1/q$. This is done with the help of Lipschitz spaces with logarithmically perturbed smoothness $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ (see Definition 5.1). In Section 6.2 we investigate the distribution of Fourier coefficients of periodic functions with additional smoothness or integrability properties. First, we complement Theorem 3.12 by analyzing the extreme case $\gamma = -1/q$. We remark that an additional iterated logarithm arises in this limit setting. We also obtain Hausdorff-Young type results for spaces close to L_1 and L_2, which extend previous estimates by Hardy and Littlewood and Bennett for functions in Zygmund spaces $L(\log L)_\gamma$ [7, Theorem 1.6(a)] and Cobos and Segurado for functions in $L_2(\log L)_{-1/2}$ [46, Theorem 8.5]. Finally, in Section 6.3, we continue studying the relationships between smoothness of functions and smoothness of their derivatives. As it was shown in Proposition 3.11, the mapping $f \to f'$ does not map, in general, $\mathbf{B}^{1,\gamma}_{p,q}$ into $\mathbf{B}^{0,\gamma}_{p,q}$. In order to get derivatives belonging to $\mathbf{B}^{0,\gamma}_{p,q}$, a shift in the exponent of logarithmic smoothness of $\mathbf{B}^{1,\gamma}_{p,q}$ is needed. This has been proved in Theorem 3.10 when $\gamma > -1/q$. Then, in this section we deal with the remaining case $\gamma = -1/q$. The results of this chapter have appeared in the papers [26, 27, 28].

6.1 Duality

Let $1 < p < \infty, 1 \le q < \infty$ and $-\infty < b < \infty$. Since $B^{0,b}_{p,q}(\mathbb{R}^d) = (H^{-1}_p(\mathbb{R}^d), H^1_p(\mathbb{R}^d))_{1/2,q,-b}$, using the duality formula for spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ (see [48, Theorem 3.1] or [102, Theorem

2.4]) and that $(H^s_p(\mathbb{R}^d))' = H^{-s}_{p'}(\mathbb{R}^d)$ [116, Theorem 2.6.1], it follows that

$$(B^{0,b}_{p,q}(\mathbb{R}^d))' = B^{0,-b}_{p',q'}(\mathbb{R}^d) \text{ where } 1/p + 1/p' = 1 = 1/q + 1/q'$$

(see also [64, Theorem 3.1.10]). Here $(X)'$ denotes the dual space of the Banach space X.

In this section, we shall determine the dual space of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ with the help of logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{R}^d)$ (see Definition 5.1). We first establish an auxiliary result.

Lemma 6.1. *Let A_0, A_1 be Banach spaces with A_1 continuously and densely embedded in A_0. Assume that $1 \le q < \infty, 1/q + 1/q' = 1$, and $\eta > -1/q$. Then we have with equivalence of norms*

$$(A_0, A_1)'_{(0,-\eta),q} = (A'_1, A'_0)_{(1,\eta+1),q'}.$$

Proof. Since $A_1 \hookrightarrow A_0$, we have that $K(t,a;A_0,A_1) \sim \|a\|_{A_0}$ for $t \ge 1$. Take any $\tau < -1/q$. It follows that

$$\begin{aligned}\left(\int_1^\infty [(1+\log t)^\tau K(t,a;A_0,A_1)]^q \frac{dt}{t}\right)^{1/q} \\ &\sim \left(\int_1^\infty (1+\log t)^{\tau q} \frac{dt}{t}\right)^{1/q} \|a\|_{A_0} \\ &\sim \|a\|_{A_0} \lesssim \|a\|_{(A_0,A_1)_{(0,-\eta),q}}.\end{aligned}$$

This yields that

$$(A_0, A_1)_{(0,-\eta),q} = (A_0, A_1)_{0,q,(-\eta,-\tau)} = (A_1, A_0)_{1,q,(-\tau,-\eta)}.$$

Since $\tau + 1/q < 0 < \eta + 1/q$, we can apply the duality formula established in [47, Theorem 5.6] (with a slightly different notation) to derive

$$(A_0, A_1)'_{(0,-\eta),q} = (A_1, A_0)'_{1,q,(-\tau,-\eta)} = (A'_1, A'_0)_{1,q',(\eta+1,\tau+1)}.$$

Density of the embedding $A_1 \hookrightarrow A_0$ implies that $A'_0 \hookrightarrow A'_1$. So $K(t,g;A'_1,A'_0) \sim \|g\|_{A'_1}$ for $t \ge 1$. Now, using that $K(t,g)/t$ is a decreasing function we get

$$\begin{aligned}\left(\int_1^\infty [t^{-1}(1+\log t)^{-\tau-1} K(t,g;A'_1,A'_0)]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\sim \|g\|_{A'_1} \sim K(1,g;A'_1,A'_0)\left(\int_0^1 (1-\log t)^{(-\eta-1)q'} \frac{dt}{t}\right)^{1/q'} \\ &\le \|g\|_{(A'_1,A'_0)_{(1,\eta+1),q'}}.\end{aligned}$$

Consequently, $(A_0, A_1)'_{(0,-\eta),q} = (A'_1, A'_0)_{(1,\eta+1),q'}$.

□

Now we are ready to describe the dual space of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Recall that the usual lift operator I_s is defined by

$$I_s f = \mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F}f, -\infty < s < \infty. \tag{6.1}$$

Theorem 6.2. *Let $1 < p < \infty, 1 \le q < \infty$ and $b > -1/q$. The space $(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'$ consists of all $f \in H^{-1}_{p'}(\mathbb{R}^d)$ such that $I_{-1}f \in \mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)$ with $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover,*

$$\|f\|_{(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))'} \sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)}.$$

Proof. By Theorem 5.15(b) and Lemma 6.1, we derive

$$(\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d))' = ((L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{(0,-b),q})' = (H^{-1}_{p'}(\mathbb{R}^d), L_{p'}(\mathbb{R}^d))_{(1,b+1),q'}.$$

On the other hand, lift operators

$$I_{-1} : H^{-1}_{p'}(\mathbb{R}^d) \longrightarrow L_{p'}(\mathbb{R}^d), I_{-1} : L_{p'}(\mathbb{R}^d) \longrightarrow W^{1,p'}(\mathbb{R}^d)$$

are bijective and bounded. Hence

$$\begin{aligned} K(t, f; H^{-1}_{p'}(\mathbb{R}^d), L_{p'}(\mathbb{R}^d)) &\sim K(t, I_{-1}f; L_{p'}(\mathbb{R}^d), W^{1,p'}(\mathbb{R}^d)) \\ &\sim t\|I_{-1}f\|_{L_{p'}(\mathbb{R}^d)} + \omega(I_{-1}f, t)_{p'} \end{aligned}$$

for $0 < t < 1$, where we have used (5.11) for the last equivalence. Consequently

$$\begin{aligned} \|f\|_{(\mathbf{B}^{0,b}_{p,q})'(\mathbb{R}^d)} &\sim \left(\int_0^1 (1-\log t)^{(-b-1)q'} \frac{dt}{t}\right)^{1/q'} \|I_{-1}f\|_{L_{p'}(\mathbb{R}^d)} \\ &\quad + \left(\int_0^1 \left[\frac{\omega(I_{-1}f, t)_{p'}}{t(1-\log t)^{b+1}}\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}(\mathbb{R}^d)}. \end{aligned}$$

□

6.2 Fourier coefficients

Recall that given an integrable function f on $\mathbb{T}$, we denote by $(\hat{f}(m)) = (c_m)$ (see (3.8)) the sequence formed by its Fourier coefficients. In this section we study the distribution of Fourier coefficients in some extreme cases by using limiting interpolation.

Applying reiteration of approximation constructions, we have shown in Theorem 3.12 that if $1 \le p \le 2, 1/p + 1/p' = 1, 0 < q \le \infty$ and $\gamma > -1/q$, then

$$f \in \mathbf{B}^{0,\gamma}_{p,q}(\mathbb{T}) \text{ implies that } (\hat{f}(m)) \in \ell_{p',q}(\log \ell)_{\gamma+1/\max\{p',q\}}.$$

Besides, in Proposition 3.13, we have established that this result is the best possible in general.

Next we study the limit case when $\gamma = -1/q$.

Theorem 6.3. *Let $1 \le p \le 2, 1/p+1/p' = 1$ and $0 < q < \infty$. If $f \in \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{T})$ then $(\hat{f}(m))$ belongs to $\ell_{p',q}(\log \ell)_{-1/q+1/\max\{p',q\}}(\log\log \ell)_{1/\max\{p',q\}}$.*

Proof. Let $\mathcal{F}(f) = (\hat{f}(m))$. By the Hausdorff-Young inequality, $\mathcal{F} : L_p(\mathbb{T}) \longrightarrow \ell_{p'}$ is bounded. Take any $\alpha > 0$ and let $1/r = \alpha + 1/p'$. It follows from [104, Theorem 3] that $\mathcal{F} : \mathbf{B}^{\alpha}_{p,r}(\mathbb{T}) \longrightarrow \ell_r$ is also bounded. Therefore

$$\mathcal{F} : (L_p(\mathbb{T}), \mathbf{B}^{\alpha}_{p,r}(\mathbb{T}))_{(0,1/q),q} \longrightarrow (\ell_{p'}, \ell_r)_{(0,1/q),q} \tag{6.2}$$

is bounded as well. By (5.2), we know that $(L_p(\mathbb{T}), \mathbf{B}^{\alpha}_{p,r}(\mathbb{T}))_{(0,1/q),q} = \mathbf{B}^{0,-1/q}_{p,q}(\mathbb{T})$. So it is enough to check that the target space in (6.2) is contained in the generalized Lorentz-Zygmund sequence space of the statement. Take $0 < \theta < 1$ with $1/p' = \theta/r$. Using Lemma 5.12 we derive

$$\begin{aligned}(\ell_{p'}, \ell_r)_{(0,1/q),q} &= ((\ell_\infty, \ell_r)_{\theta,p'}, \ell_r)_{(0,1/q),q} \\ &\hookrightarrow (\ell_\infty, \ell_r)_{\theta,q,-\min\{0,1/p'-1/q\},-\min\{1/p',1/q\}} \\ &= \ell_{p',q}(\log \ell)_{-1/q+1/\max\{p',q\}}(\log\log \ell)_{1/\max\{p',q\}}\end{aligned}$$

where the last equality follows from (2.15). □

In the remaining part of this section we study Fourier coefficients of functions in spaces close to $L_1(\mathbb{T})$ and to $L_2(\mathbb{T})$.

It was shown by Hardy and Littlewood (case $\gamma = 1$) and Bennett (case $\gamma > 0$) that if $f \in L(\log L)_\gamma(\mathbb{T})$ then $\sum_{n=1}^{\infty}(1 + \log n)^{\gamma-1} c_n^* n^{-1} < \infty$ (see [7, Theorem 1.6(a)]). Here (c_n^*) denotes the non-increasing rearrangement (see (2.14)) of the sequence $(\hat{f}(m)) = (c_m)$. Next we extend this result to functions in $L_{1,q}(\log L)_\gamma(\mathbb{T})$.

Theorem 6.4. *Let $0 < q \le \infty$ and $\gamma > -1/q$. If $f \in L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}}(\mathbb{T})$ then*

$$\sum_{n=1}^{\infty}((1 + \log n)^\gamma c_n^*)^q n^{-1} < \infty.$$

Proof. Since

$$\mathcal{F}: L_1(\mathbb{T}) \longrightarrow \ell_\infty \quad \text{and} \quad \mathcal{F}: L_2(\mathbb{T}) \longrightarrow \ell_2$$

are bounded operators, we obtain that

$$\mathcal{F}: (L_1(\mathbb{T}), L_2(\mathbb{T}))_{(0,-\gamma),q} \longrightarrow (\ell_\infty, \ell_2)_{(0,-\gamma),q} \text{ boundedly.}$$

Take $0 < p < 1$. By Lemma 2.2(b) we have

$$\begin{aligned}(L_1(\mathbb{T}), L_2(T))_{(0,-\gamma),q} &= (L_1(\mathbb{T}), (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{1/2,2})_{(0,-\gamma),q} \\ &= (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{(0,-\gamma),q} \\ &= ((L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{1-p,1}, L_\infty(\mathbb{T}))_{(0,-\gamma),q}.\end{aligned}$$

On the other hand, (2.15) and Lemma 3.8(b) yield that

$$\begin{aligned}L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}}(\mathbb{T}) &= (L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{1-p,q,-\gamma-1/\min\{1,q\}} \\ &\hookrightarrow ((L_p(\mathbb{T}), L_\infty(\mathbb{T}))_{1-p,1}, L_\infty(\mathbb{T}))_{(0,-\gamma),q}.\end{aligned}$$

Hence

$$\mathcal{F}: L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}}(\mathbb{T}) \longrightarrow (\ell_\infty, \ell_2)_{(0,-\gamma),q}$$

is bounded. Now we work with the space at the right-hand side. We have

$$\begin{aligned}(\ell_\infty, \ell_2)_{(0,-\gamma),q} &= (\ell_\infty, (\ell_\infty, \ell_1)_{1/2,2})_{(0,-\gamma),q} \\ &= (\ell_\infty, \ell_1)_{(0,-\gamma),q} = \ell_{\infty,q}(\log \ell)_\gamma\end{aligned}$$

where the last equality follows from Theorem 3.3. Consequently,

$$\mathcal{F}: L_{1,q}(\log L)_{\gamma+1/\min\{1,q\}}(\mathbb{T}) \longrightarrow \ell_{\infty,q}(\log \ell)_\gamma$$

is bounded, which completes the proof. □

Note that when $q = 1$ and $\gamma > -1$, Theorem 6.4 recovers [7, Theorem 1.6(a)].

Now we cover the limit case $\gamma = -1/q$.

Theorem 6.5. *Let* $0 < q < \infty$. *If*

$$f \in L_{1,q}(\log L)_{-1/q+1/\min\{1,q\}}(\log\log L)_{1/\min\{1,q\}}(\mathbb{T}) \quad \textit{then} \quad \sum_{n=1}^{\infty} \frac{(c_n^*)^q}{n(1+\log n)} < \infty.$$

Proof. By the interpolation property,

$$\mathcal{F}: (L_1(\mathbb{T}), L_2(\mathbb{T}))_{(0,1/q),q} \longrightarrow (\ell_\infty, \ell_2)_{(0,1/q),q}$$

is also bounded. According to Lemma 2.2(b),

$$\begin{aligned}(L_1(\mathbb{T}), L_2(\mathbb{T}))_{(0,1/q),q} &= (L_1(\mathbb{T}), (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{1/2,2})_{(0,1/q),q} \\ &= (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{(0,1/q),q}.\end{aligned}$$

Hence, Lemma 5.13 yields that

$$L_{1,q}(\log L)_{-1/q+1/\min\{1,q\}}(\log\log L)_{1/\min\{1,q\}}(\mathbb{T}) \hookrightarrow (L_1(\mathbb{T}), L_2(\mathbb{T}))_{(0,1/q),q}.$$

As for the sequence space, it follows from Theorem 3.3 that $(\ell_\infty, \ell_2)_{(0,1/q),q} = \ell_{\infty,q}(\log \ell)_{-1/q}$.

This completes the proof. □

Theorem 6.6. *Let $0 < q \le \infty$ and $\gamma < -1/q$. If $f \in L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}}(\mathbb{T})$ then*

$$\sum_{n=1}^{\infty} (n^{1/2}(1+\log n)^{\gamma+1/\max\{2,q\}} c_n^*)^q n^{-1} < \infty.$$

Proof. This time we interpolate by limiting method with $\theta = 1$ to obtain that

$$\mathcal{F} : (L_1(\mathbb{T}), L_2(\mathbb{T}))_{(1,-\gamma),q} \longrightarrow (\ell_\infty, \ell_2)_{(1,-\gamma),q}$$

is bounded. By (2.15) and Lemma 3.8(a), we get that

$$\begin{aligned}L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}}(\mathbb{T}) &= (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{1/2,q,-\gamma-1/\min\{2,q\}} \\ &\hookrightarrow (L_1(\mathbb{T}), (L_1(\mathbb{T}), L_\infty(\mathbb{T}))_{1/2,2})_{(1,-\gamma),q} \\ &= (L_1(\mathbb{T}), L_2(\mathbb{T}))_{(1,-\gamma),q}.\end{aligned}$$

Besides, Lemma 3.8(a) and Proposition 2.6 yield that

$$\begin{aligned}(\ell_\infty, \ell_2)_{(1,-\gamma),q} &= (\ell_\infty, (\ell_\infty, \ell_1)_{1/2,2})_{(1,-\gamma),q} \\ &\hookrightarrow (\ell_\infty, \ell_1)_{1/2,q,-\gamma-1/\max\{2,q\}} \\ &= \ell_{2,q}(\log \ell)_{\gamma+1/\max\{2,q\}}.\end{aligned}$$

Therefore

$$\mathcal{F} : L_{2,q}(\log L)_{\gamma+1/\min\{2,q\}}(\mathbb{T}) \longrightarrow \ell_{2,q}(\log \ell)_{\gamma+1/\max\{2,q\}}$$

boundedly. □

Writing down Theorem 6.6 in the special case $q = 2$ and $\gamma = -1$, we recover a result of Cobos and Segurado [46, Theorem 8.5].

6.3 Derivatives on Besov spaces $\mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T})$

In this section we give a sufficient condition on smoothness of functions so that their derivatives belong to $\mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T})$. Using approximation techniques, it is proved in Theorem 3.10 that if $f \in \mathbf{B}_{p,q}^{k,\gamma+1/\min\{2,p,q\}}(\mathbb{T})$ then $D^k f \in \mathbf{B}_{p,q}^{0,b}(\mathbb{T})$ provided that $k \in \mathbb{N}, 1 < p < \infty, 0 < q \leq \infty$ and $\gamma > -1/q$. Note that there is a loss of smoothness in order to have $D^k f \in \mathbf{B}_{p,q}^{0,b}(\mathbb{T})$. Besides, this result is best possible in general as can be seen in Proposition 3.11.

Next we investigate the case when $\gamma = -1/q$ by applying limiting interpolation.

Theorem 6.7. *Let $1 < p < \infty, 0 < q < \infty$ and $k \in \mathbb{N}$.*

$$If \quad f \in \mathbf{B}_{p,q}^{k,-1/q+1/\min\{2,p,q\},1/\min\{2,p,q\}}(\mathbb{T}) \quad then \quad D^k f \in \mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T}).$$

Proof. Take any $\alpha \in \mathbb{R}$ with $\alpha > k$. Then $D^k : H_p^k(\mathbb{T}) \longrightarrow L_p(\mathbb{T})$ is bounded and $D^k : \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}) \longrightarrow \mathbf{B}_{p,q}^{\alpha-k}(\mathbb{T})$ is also bounded (see [51, page 70]). Interpolating we derive that

$$D^k : (H_p^k(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{(0,1/q),q} \longrightarrow (L_p(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha-k}(\mathbb{T}))_{(0,1/q),q}$$

is bounded as well. By (5.2), we have that $(L_p(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha-k}(\mathbb{T}))_{(0,1/q),q} = \mathbf{B}_{p,q}^{0,-1/q}(\mathbb{T})$. To work with the source space, take $0 < \alpha_0 < k$ and let $0 < \theta < 1$ such that $k = (1-\theta)\alpha_0 + \theta\alpha$. Using [106, Proposition 5.5], Lemma 5.12(a) and (5.13), we derive

$$\begin{aligned}
&\mathbf{B}_{p,q}^{k,-1/q+1/\min\{2,p,q\},1/\min\{2,p,q\}}(\mathbb{T}) \\
&\quad = (\mathbf{B}_{p,q}^{\alpha_0}(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{\theta,q,1/q-1/\min\{2,p,q\},-1/\min\{2,p,q\}} \\
&\quad \hookrightarrow ((\mathbf{B}_{p,q}^{\alpha_0}(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{\theta,\min\{2,p\}}, \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{(0,1/q),q} \\
&\quad = (\mathbf{B}_{p,\min\{2,p\}}^{k}(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{(0,1/q),q} \\
&\quad \hookrightarrow (H_p^k(\mathbb{T}), \mathbf{B}_{p,q}^{\alpha}(\mathbb{T}))_{(0,1/q),q}.
\end{aligned}$$

This completes the proof. $\square$

Chapter 7

Tractable embeddings of Besov spaces into small Lebesgue spaces

It is known that Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ can be embedded into the Lebesgue space $L_{p^*}(\mathbb{R}^d)$ if $k = d(1/p - 1/p^*)$. This critical index p^* is bigger than p but $p^* \to p$ as $d \to \infty$. Triebel raised the question if there is a dimension-invariant embedding for Sobolev spaces. A positive answer was given by Martín and Milman [92] in terms of Zygmund spaces, who proved that

$$\left(\int_0^1 (1-\log t)^{p/2} f^*(t)^p dt\right)^{1/p} \le c\|f\|_{W^{1,p}(\mathbb{R}^d)}$$

for $f \in W^{1,p}(\mathbb{R}^d), 1 < p < \infty$, with supp $f \subset \{x \in \mathbb{R}^d : 0 \le x_l \le 1\}$. This is a recent formulation of logarithmic Sobolev inequalities [73, 49, 5, 92, 23]. The main point is the extra factor $(1-\log t)^{1/2}$ independently of the dimension d, where c can also be chosen independent of d. Hence, it shows a logarithm gain in the integrability. The step from $W^{1,p}(\mathbb{R}^d)$ to related Besov spaces $\mathbf{B}^\alpha_{p,p}(\mathbb{R}^d)$ goes back to Triebel [123, 126] where the following counterpart was obtained

$$\left(\int_0^1 (1-\log t)^{\alpha p} f^*(t)^p dt\right)^{1/p} \le 2^{\rho d}\left[\|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} \frac{\|\Delta_h^M f\|^p_{L_p(\mathbb{R}^d)}}{|h|^{\alpha p + d}}\,dh\right)^{1/p}\right] \tag{7.1}$$

for all $f \in \mathbf{B}^\alpha_{p,p}(\mathbb{R}^d)$ with supp $f \subset \{x \in \mathbb{R}^d : 0 \le x_l \le 1\}$, where $\rho > 0$ is independent of the dimension d. Note that such a relation is very sensitive with respect to the norm used for the Besov space because we must control the equivalence constants with respect to d, what might be not so obvious.

The main aim of this chapter is to provide tractable embeddings for Besov spaces $\mathbf{B}^\alpha_{p,q}$ defined over $\mathbb{T}^d$. We obtain a similar inequality to (7.1) with the important fact that we are able to reduce the exponential constant to a polynomial one. Furthermore, we can extend (7.1) to $p \neq q$ by replacing the Zygmund space on left-hand side by suitable small Lebesgue spaces. In the particular case that $p = q$, we recover the expected tractable embedding (7.1) with a polynomial dependence on d in the constant. This will be done in Section 7.2 by using the approximation structure of Besov spaces, interpolation and extrapolation properties of small Lebesgue spaces. Previously, in Section 7.1, we obtain the extrapolation description of small Lebesgue spaces in terms of classical Lebesgue spaces, paying special attention to the dependence with respect to d of the equivalence constant.

We also derive tractable embeddings for Besov spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ with classical smoothness zero. These spaces are very close to $L_p(\mathbb{T}^d)$, but as we showed in Theorem 5.10 for $d = 1$, we can still improve the integrability properties of their functions. Then, in Section 7.3, we generalize Theorem 5.10 to higher dimensions with a special emphasis on the behaviour of the embedding constant with respect to the dimension d. Furthermore, we remark that logarithmic smoothness is enough to establish some tractability results for classical Besov spaces $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ obtained in Section 7.2.

The main results of this chapter form the paper [55].

7.1 Extrapolation description of small Lebesgue spaces

This section is devoted to the extrapolation characterization of small Lebesgue spaces $L^{(p,\gamma,q}(\Omega)$ in terms of classical Lebesgue spaces $L_r(\Omega)$, where Ω is any bounded domain in $\mathbb{R}^d$. To get this, we shall use limiting interpolation methods given by the K-functional and their equivalent representations via the J-functional. Moreover, our approach allows us to control the equivalence constants with respect to the dimension d.

The following limiting J-methods will be useful in our considerations.

Definition 7.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple with $A_1 \hookrightarrow A_0$. For $1 \le q \le \infty$ and $-\infty < \eta < \infty$, the space $\bar{A}_{(0,\eta),q;J} = (A_0, A_1)_{(0,\eta),q;J}$ is formed by all $a \in A_0$ having a finite norm

$$\|a\|_{\bar{A}_{(0,\eta),q;J}} = \inf \left(\int_0^1 \left(\frac{J(t, u(t))}{(1 - \log t)^{\eta}} \right)^q \frac{dt}{t} \right)^{1/q}$$

where the infimum is taken over all strongly measurable functions $u(t)$ with values in A_1 such that $a = \int_0^1 u(t) \frac{dt}{t}$ in A_0.

Remark 7.1. Spaces $\bar{A}_{(0,\eta),q;J}$ were already considered by Cobos, Fernández-Cabrera, Kühn and Ullrich [33] in the special case that $\eta = 1$.

Let us consider the weight ρ given by

$$\rho(t) = \begin{cases} (1 - \log t)^\eta & \text{if} \quad 0 < t < 1, \\ \\ t^\theta & \text{if} \quad t \geq 1 \end{cases}$$

where $0 < \theta < 1$ and $-\infty < \eta < \infty$ are fixed. It is not hard to check that

$$(A_0, A_1)_{\rho,q;J} = (A_0, A_1)_{(0,\eta),q;J} \tag{7.2}$$

with equivalence constants which are independent of A_0 and A_1.

We can still describe $(A_0, A_1)_{(0,\eta),q}$ in terms of the J-functional but we should modify the exponent of the logarithm. This can be seen in [33, Theorem 7.6] for $\eta = 0$ (see also [47]). However, we are interested in the dependence with respect to the couple in the equivalence result and then we give a detailed proof below. For this purpose, we shall need two lemmata.

Lemma 7.1. *Let $\alpha > 0, -\infty < \eta < \infty, 1 \leq q \leq \infty$ and $b_k \geq 0$ for $k = 0, 1, \ldots$ Then,*

$$\sum_{j=0}^{\infty} \left(2^{-j\alpha} (1+j)^\eta \sum_{k=0}^{j} b_k \right)^q \lesssim \sum_{j=0}^{\infty} (2^{-j\alpha} (1+j)^\eta b_j)^q.$$

Proof. The proof is similar to that given in [103, Lemma 3.10] for the case $\eta = 0$. Assume that $1 < q < \infty$ and let $q' = \frac{q}{q-1}$. By Hölder's inequality we derive

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(2^{-j\alpha} (1+j)^\eta \sum_{k=0}^{j} b_k \right)^q &= \sum_{j=0}^{\infty} 2^{-j\alpha q} (1+j)^{\eta q} \left(\sum_{k=0}^{j} 2^{k\frac{\alpha}{2}} 2^{-k\frac{\alpha}{2}} b_k \right)^q \\
&\leq \sum_{j=0}^{\infty} 2^{-j\alpha q} (1+j)^{\eta q} \left(\sum_{k=0}^{j} 2^{-k\frac{\alpha}{2} q} b_k^q \right) \left(\sum_{k=0}^{j} 2^{k\frac{\alpha}{2} q'} \right)^{q/q'} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-j\alpha q} (1+j)^{\eta q} 2^{j\frac{\alpha}{2} q} \sum_{k=0}^{j} 2^{-k\frac{\alpha}{2} q} b_k^q \\
&= \sum_{k=0}^{\infty} 2^{-k\frac{\alpha}{2} q} b_k^q \sum_{j=k}^{\infty} 2^{-j\frac{\alpha}{2} q} (1+j)^{\eta q} \\
&\sim \sum_{k=0}^{\infty} 2^{-k\alpha q} (1+k)^{\eta q} b_k^q.
\end{aligned}$$

The cases $q = 1$ and $q = \infty$ are easier so we omit the details. $\square$

The following result can be considered as the discrete version of limiting Hardy's inequality given in [6, Theorem 6.2].

Lemma 7.2. *Let $1 \le q \le \infty, \eta > -1/q$ and $b_k \ge 0$ for $k = 0, 1, \ldots$ Then,*

$$\sum_{j=0}^{\infty}(1+j)^{\eta q}\left(\sum_{k=j+1}^{\infty} b_k\right)^q \lesssim \sum_{j=0}^{\infty}(1+j)^{(\eta+1)q} b_j^q.$$

Proof. We consider the step function $\psi(s) = \frac{b_k}{2^{-k-1}}$ for $s \in [2^{-k-1}, 2^{-k}), k \ge 0$. We obtain the estimates,

$$\begin{aligned}
\int_0^1 \left((1-\log t)^{\eta} \int_0^t \psi(s)ds\right)^q \frac{dt}{t} &= \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} \left((1-\log t)^{\eta} \int_0^t \psi(s)ds\right)^q \frac{dt}{t} \\
&\sim \sum_{j=0}^{\infty}(1+j)^{\eta q} \int_{2^{-j-1}}^{2^{-j}} \left(\int_0^t \psi(s)ds\right)^q \frac{dt}{t} \\
&\gtrsim \sum_{j=0}^{\infty}(1+j)^{\eta q} \left(\int_0^{2^{-j-1}} \psi(s)ds\right)^q \\
&= \sum_{j=0}^{\infty}(1+j)^{\eta q} \left(\sum_{k=j+1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \psi(s)ds\right)^q \\
&= \sum_{j=0}^{\infty}(1+j)^{\eta q} \left(\sum_{k=j+1}^{\infty} \frac{b_k}{2^{-k-1}} \int_{2^{-k-1}}^{2^{-k}} ds\right)^q \\
&= \sum_{j=0}^{\infty}(1+j)^{\eta q} \left(\sum_{k=j+1}^{\infty} b_k\right)^q .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^1 (t(1-\log t)^{\eta+1}\psi(t))^q \frac{dt}{t} &= \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} (t(1-\log t)^{\eta+1}\psi(t))^q \frac{dt}{t} \\
&\sim \sum_{j=0}^{\infty} \left(2^{-j}(1+j)^{\eta+1} \frac{b_j}{2^{-j-1}}\right)^q \\
&\sim \sum_{j=0}^{\infty}(1+j)^{(\eta+1)q} b_j^q.
\end{aligned}$$

Now it is sufficient to apply [6, Theorem 6.2]. □

Next we show the equivalence theorem for $((0,\eta),q)$-method.

Theorem 7.3. *Let A_0, A_1 be Banach spaces with $A_1 \hookrightarrow A_0$ and let $1 \le q \le \infty, \eta > -1/q$. Then*

$$(A_0, A_1)_{(0,-\eta),q} = (A_0, A_1)_{(0,-\eta-1),q;J}$$

with equivalence constants which are independent of A_0 and A_1.

Proof. Let $a \in (A_0, A_1)_{(0,-\eta),q}$. Since $\eta > -1/q$, it follows that $\lim_{t\to 0} K(t,a) = 0$. Consider the sequence $\lambda_\nu = 2^{-2^\nu}$ for $\nu = 0, 1, \ldots$ Choose a representation $a = a_{0,\nu} + a_{1,\nu}$ such that

$$\|a_{0,0}\|_{A_0} + \|a_{1,0}\|_{A_1} \le 2K(1,a) \tag{7.3}$$

and

$$\|a_{0,\nu}\|_{A_0} + \lambda_{\nu-1}\|a_{1,\nu}\|_{A_1} \le 2K(\lambda_{\nu-1}, a) \tag{7.4}$$

for $\nu = 1, 2, \ldots$ Put $u_0 = a_{1,0}, u_\nu = a_{1,\nu} - a_{1,\nu-1}$ for $\nu = 1, 2, \ldots$ By construction, the sequence $(u_\nu)_{\nu\ge 0} \subseteq A_1$ and $\sum_{\nu=0}^\infty u_\nu = a$ in A_0. Consider the partition of the unit interval given by $L_0 = (1/2, 1], L_\nu = (\lambda_\nu, \lambda_{\nu-1}]$ for $\nu = 1, 2, \ldots$ and put

$$u(t) = \begin{cases} \frac{1}{\log 2} u_0 & \text{if} \quad t \in L_0, \\ \\ \frac{1}{2^{\nu-1}\log 2} u_\nu & \text{if} \quad t \in L_\nu, \nu = 1, 2, \ldots \end{cases}$$

Then, we obtain

$$\begin{aligned} \int_0^1 u(t)\frac{dt}{t} &= \sum_{\nu=1}^\infty \int_{L_\nu} u(t)\frac{dt}{t} + \int_{L_0} u(t)\frac{dt}{t} \\ &= \sum_{\nu=1}^\infty \int_{L_\nu} \frac{1}{2^{\nu-1}\log 2} u_\nu \frac{dt}{t} + \int_{L_0} \frac{1}{\log 2} u_0 \frac{dt}{t} \\ &= \sum_{\nu=1}^\infty \frac{1}{2^{\nu-1}\log 2} u_\nu \log\left(\frac{\lambda_{\nu-1}}{\lambda_\nu}\right) + u_0 \\ &= \sum_{\nu=0}^\infty u_\nu = a. \end{aligned}$$

Assume that $t \in L_0$. By (7.3) we derive

$$\begin{aligned} J(t, u(t)) &= J\left(t, \frac{1}{\log 2} u_0\right) \sim J(t, u_0) \le J(1, u_0) \\ &\le \|u_0\|_{A_0} + \|u_0\|_{A_1} \\ &= \|a - a_{0,0}\|_{A_0} + \|a_{1,0}\|_{A_1} \\ &\le \|a\|_{A_0} + \|a_{0,0}\|_{A_0} + \|a_{1,0}\|_{A_1} \\ &\lesssim K(1,a) \end{aligned}$$

and consequently,

$$\begin{aligned}\int_{L_0}((1-\log t)^{\eta+1}J(t,u(t)))^q\frac{dt}{t} &\lesssim \left(\int_{L_0}(1-\log t)^{(\eta+1)q}\frac{dt}{t}\right)K(1,a)^q\\ &\sim \left(\int_{L_0}(1-\log t)^{\eta q}\frac{dt}{t}\right)K(1,a)^q\\ &\lesssim \left(\int_{L_0}(1-\log t)^{\eta q}\frac{dt}{t}\right)K(1/2,a)^q\\ &\le \int_{L_0}((1-\log t)^{\eta}K(t,a))^q\frac{dt}{t}.\end{aligned}$$

Let $t\in L_\nu$ for $\nu=1,2,\ldots$ Then it follows from (7.4) that

$$\begin{aligned}J(t,u(t)) &\le J\left(\lambda_{\nu-1},\frac{1}{2^{\nu-1}\log 2}u_\nu\right)\\ &\le \frac{1}{2^{\nu-1}\log 2}(\|a_{0,\nu-1}-a_{0,\nu}\|_{A_0}+\lambda_{\nu-1}\|a_{1,\nu}-a_{1,\nu-1}\|_{A_1})\\ &\le \frac{1}{2^{\nu-1}\log 2}(\|a_{0,\nu-1}\|_{A_0}+\|a_{0,\nu}\|_{A_0}+\lambda_{\nu-1}\|a_{1,\nu}\|_{A_1}+\lambda_{\nu-1}\|a_{1,\nu-1}\|_{A_1})\\ &\lesssim \frac{1}{2^{\nu-1}\log 2}\begin{cases}K(\lambda_{\nu-2},a) & \text{if}\quad \nu=2,3,\ldots\\ K(1,a) & \text{if}\quad \nu=1.\end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{L_1}((1-\log t)^{\eta+1}J(t,u(t)))^q\frac{dt}{t} &\lesssim \left(\int_{L_1}(1-\log t)^{(\eta+1)q}\frac{dt}{t}\right)K(1,a)^q\\ &\lesssim \left(\int_{L_1}(1-\log t)^{\eta q}\frac{dt}{t}\right)K(1/4,a)^q\\ &\le \int_{L_1}((1-\log t)^{\eta}K(t,a))^q\frac{dt}{t}\end{aligned}$$

and for $\nu=2,3,\ldots$ we have that

$$\begin{aligned}\int_{L_\nu}((1-\log t)^{\eta+1}J(t,u(t)))^q\frac{dt}{t} &\lesssim \int_{L_\nu}\left((1-\log t)^{\eta+1}\frac{1}{2^{\nu-1}\log 2}K(\lambda_{\nu-2},a)\right)^q\frac{dt}{t}\\ &\sim (1+2^\nu)^{(\eta+1)q}2^\nu\log 2\,\frac{K(\lambda_{\nu-2},a)^q}{2^{\nu q}}\\ &\sim \left(\int_{L_{\nu-2}}(1-\log t)^{\eta q}\frac{dt}{t}\right)K(\lambda_{\nu-2},a)^q\\ &\le \int_{L_{\nu-2}}((1-\log t)^{\eta}K(t,a))^q\frac{dt}{t}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\int_0^1 ((1-\log t)^{\eta+1} J(t,u(t)))^q \frac{dt}{t} &= \sum_{\nu=0}^{\infty} \int_{L_\nu} ((1-\log t)^{\eta+1} J(t,u(t)))^q \frac{dt}{t} \\ &\lesssim \int_0^1 ((1-\log t)^{\eta} K(t,a))^q \frac{dt}{t} \\ &= \|a\|^q_{(A_0,A_1)_{(0,-\eta),q}}.\end{aligned}$$

Let us prove the converse estimate. We can work with the corresponding discrete version of each method since the related equivalence constants depend only on η and q. Let $a \in (A_0, A_1)_{(0,-\eta-1),q;J}$. Take a representation $a = \sum_{k=0}^{\infty} u_k$ with $u_k \in A_1$ and

$$\left(\sum_{k=0}^{\infty} [(1+k)^{\eta+1} J(2^{-k}, u_k)]^q\right)^{1/q} \le 2\|a\|_{(A_0,A_1)_{(0,-\eta-1),q;J}}.$$

It follows from

$$\begin{aligned}K(2^{-j},a) &\le \sum_{k=0}^{\infty} K(2^{-j},u_k) \\ &\le \sum_{k=0}^{j} 2^{-j+k} J(2^{-k},u_k) + \sum_{k=j+1}^{\infty} J(2^{-k},u_k)\end{aligned}$$

and Lemmata 7.1 and 7.2 that

$$\begin{aligned}\left(\sum_{j=0}^{\infty}[(1+j)^{\eta} K(2^{-j},a)]^q\right)^{1/q} &\le \left(\sum_{j=0}^{\infty}\left[(1+j)^{\eta} 2^{-j} \sum_{k=0}^{j} 2^k J(2^{-k},u_k)\right]^q\right)^{1/q} \\ &\quad + \left(\sum_{j=0}^{\infty}\left[(1+j)^{\eta} \sum_{k=j+1}^{\infty} J(2^{-k},u_k)\right]^q\right)^{1/q} \\ &\lesssim \left(\sum_{j=0}^{\infty}[(1+j)^{\eta} 2^{-j} 2^j J(2^{-j},u_j)]^q\right)^{1/q} \\ &\quad + \left(\sum_{j=0}^{\infty}[(1+j)^{\eta+1} J(2^{-j},u_j)]^q\right)^{1/q} \\ &\lesssim \left(\sum_{j=0}^{\infty}[(1+j)^{\eta+1} J(2^{-j},u_j)]^q\right)^{1/q} \\ &\lesssim \|a\|_{(A_0,A_1)_{(0,-\eta-1),q;J}}.\end{aligned}$$

The proof is complete. □

Assume that Ω is a bounded domain in $\mathbb{R}^d$. As it was shown by Fiorenza and Karadzhov [67, Theorems 3.2 and 8.2], the small Lebesgue spaces $L^{(p}(\Omega)$ arise in the extrapolation of Lebesgue spaces. Next we extend this result to $L^{(p,\gamma,q}(\Omega)$ using the $\Sigma^{(q)}$-extrapolation method. Moreover, this approach allows us to control how the related equivalence constants depend on the dimension d which will be crucial in our arguments. Recall the notation $1/p^{\lambda_j} = 1/p - 2^{-j}/d > 0$, where $J \le j \in \mathbb{N}_0$, $J \in \mathbb{N}_0$ such that $2^J > p/d$.

Proposition 7.4. *Let $1 \le p < \infty, 1 \le q \le \infty$ and $\gamma > -1/q$. If f is a function which admits a representation*

$$f = \sum_{j=J}^{\infty} f_j, f_j \in L_{p^{\lambda_j}}(\Omega), \tag{7.5}$$

such that

$$\sum_{j=J}^{\infty} 2^{j(\gamma+1/q)q} \|f_j\|^q_{L_{p^{\lambda_j}}(\Omega)} < \infty, \tag{7.6}$$

then f belongs to $L^{(p,\gamma,q}(\Omega)$. Moreover,

$$\|f\|_{L^{(p,\gamma,q}(\Omega)} \lesssim d^{\gamma+1/q} \inf \left(\sum_{j=J}^{\infty} 2^{j(\gamma+1/q)q} \|f_j\|^q_{L_{p^{\lambda_j}}(\Omega)} \right)^{1/q} \tag{7.7}$$

where the infimum is taken over all possible representations (7.5) such that (7.6) holds.

Proof. Since the formula (2.16) holds with equivalence constants independent of Ω (but may depend on p), it is a simple consequence of Fubini's theorem that

$$(L_p(\Omega), L_\infty(\Omega))_{\theta,p} = \theta^{-1/p} L_{r,p}(\Omega) \tag{7.8}$$

whenever $1/r = (1-\theta)/p$, with equivalence constants which are independent of θ and Ω.

Note that $p^{\lambda_j} = p + \frac{2^{-j}p^2}{d-2^{-j}p}, j \ge J$. Consequently, we can rewrite (7.7) as

$$\|f\|_{L^{(p,\gamma,q}(\Omega)} \lesssim d^{\gamma+1/q} \|f\|_{\Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+\frac{2^{-j}p^2}{d-2^{-j}p}}(\Omega)}. \tag{7.9}$$

A simple change of variables yields that

$$\Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+\frac{2^{-j}p^2}{d-2^{-j}p}}(\Omega) = d^{-(\gamma+1/q)} \Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j}}(\Omega). \tag{7.10}$$

The following estimates hold with constants which are independent of j and Ω,

$$\|f\|_{L_{p+2^{-j}}(\Omega)} \lesssim \|f\|_{L_{p+2^{-j},p}(\Omega)}$$

for all $f \in L_{p+2^{-j},p}(\Omega)$, and

$$\|f\|_{L_{p+2^{-j},p}(\Omega)} \lesssim \|f\|_{L_{p+2^{1-j}}(\Omega)}$$

for all $f \in L_{p+2^{1-j}}(\Omega)$ (see [56, Propositions 3.4.3 and 3.4.4]). Consequently,

$$\Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j}}(\Omega) = \Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j},p}(\Omega)$$

and by (7.8) we can rewrite the last expression as

$$\begin{aligned} \Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j}}(\Omega) &= \Sigma^{(q)} 2^{j(\gamma+1/q)} 2^{-j/p} (L_p(\Omega), L_\infty(\Omega))_{2^{-j},p} \\ &= \Sigma^{(q)} 2^{j(\gamma+1/q)} 2^{-j} (L_p(\Omega), L_\infty(\Omega))_{2^{-j},1} \\ &= \Sigma^{(q)} 2^{j(\gamma+1/q)} (L_p(\Omega), L_\infty(\Omega))_{2^{-j},1;J} \end{aligned} \tag{7.11}$$

where the second equality follows from [84, Theorem 2.10] with equivalence constants which are independent of the couple and the last equality is a consequence of (2.11). Since

$$(L_p(\Omega), L_\infty(\Omega))_{2^{-j},1;J} \hookrightarrow 2^{j/q'} (L_p(\Omega), L_\infty(\Omega))_{2^{-j},q;J}$$

with embedding constant equals to 1 (see [80]), it follows from (7.11) and [84, Theorems 2.2 and 2.3] that

$$\Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j}}(\Omega) \hookrightarrow (L_p(\Omega), L_\infty(\Omega))_{\rho,q;J}$$

where $\rho(t) \sim (1-\log t)^{-\gamma-1}$ if $0 < t \le 1$ and $\rho(t) \sim t^{1/2}$ if $t > 1$. By (7.2), Theorem 7.3 and Lemma 2.4, we derive

$$\Sigma^{(q)} 2^{j(\gamma+1/q)} L_{p+2^{-j}}(\Omega) \hookrightarrow (L_p(\Omega), L_\infty(\Omega))_{(0,-\gamma),q} = L^{(p,\gamma,q}(\Omega). \tag{7.12}$$

Finally, it follows from (7.12), (7.10) and (7.9) that (7.7) is satisfied. The proof is complete. □

As we shall prove in the next lemma, the scale formed by small Lebesgue spaces contains Zygmund spaces $L_p(\log L)_a(\Omega)$ with $a > 0$.

Lemma 7.5. *Let $1 \le p < \infty$ and $a > 0$. We have*

$$L^{(p,a-1/p,p}(\Omega) = L_p(\log L)_a(\Omega)$$

with equivalence constants which are independent of Ω.

Proof. Define the weight

$$w(t) = \begin{cases} (1-\log t)^{ap-1} & \text{if} \quad 0 < t < 1, \\ t^{-ap} & \text{if} \quad t \ge 1. \end{cases}$$

We are going to show that

$$\int_0^1 (1-\log t)^{ap-1} \int_0^t f^*(s)^p ds \frac{dt}{t} \sim \int_0^\infty w(t) \int_0^t f^*(s)^p ds \frac{dt}{t}. \tag{7.13}$$

One estimate is clear. Let us prove the converse estimate. Since $f^*(s) = 0$ for $s \geq 1$, we get

$$\begin{aligned}
\int_1^\infty t^{-ap} \int_0^t f^*(s)^p ds \frac{dt}{t} &= \left(\int_1^\infty t^{-ap} \frac{dt}{t}\right)\left(\int_0^1 f^*(s)^p ds\right) \\
&\sim \left(\int_0^1 t(1-\log t)^{ap-1} \frac{dt}{t}\right)\left(\int_0^1 f^*(s)^p ds\right) \\
&\leq \int_0^1 t(1-\log t)^{ap-1} \frac{1}{t} \int_0^t f^*(s)^p ds \frac{dt}{t} \\
&= \int_0^1 (1-\log t)^{ap-1} \int_0^t f^*(s)^p ds \frac{dt}{t}.
\end{aligned}$$

Consequently, using (7.13) and Fubini's theorem we derive

$$\begin{aligned}
\|f|L^{(p,a-1/p,p}(\Omega)\|^p &\sim \int_0^\infty w(t) \int_0^t f^*(s)^p ds \frac{dt}{t} \\
&= \int_0^\infty f^*(s)^p \int_s^\infty w(t) \frac{dt}{t} ds \\
&= \int_0^1 f^*(s)^p \int_s^\infty w(t) \frac{dt}{t} ds \\
&\sim \int_0^1 f^*(s)^p (1-\log s)^{ap} ds \\
&= \|f\|^p_{L_p(\log L)_a(\Omega)}.
\end{aligned}$$

□

As a consequence of Proposition 7.4 and Lemma 7.5, we derive the following result for Zygmund spaces.

Corollary 7.6. *Let $1 \leq p < \infty$ and $a > 0$. If f is a function which admits a representation*

$$f = \sum_{j=J}^\infty f_j, f_j \in L_{p^{\lambda_j}}(\Omega), \tag{7.14}$$

such that

$$\sum_{j=J}^\infty 2^{jap} \|f_j\|^p_{L_{p^{\lambda_j}}(\Omega)} < \infty, \tag{7.15}$$

then f belongs to $L_p(\log L)_a(\Omega)$. Moreover,

$$\|f\|_{L_p(\log L)_a(\Omega)} \lesssim d^a \inf \left(\sum_{j=J}^\infty 2^{jap} \|f_j\|^p_{L_{p^{\lambda_j}}(\Omega)}\right)^{1/p} \tag{7.16}$$

where the infimum is taken over all possible representations (7.14) such that (7.15) holds.

Remark 7.2. Using different techniques, Triebel showed in [123, Proposition 2.5(ii)] that if $1 < p < \infty$ we can change $\lesssim$ by $\sim$ in (7.16).

7.2 Tractable embeddings of classical Besov spaces

In this section we establish tractable embeddings from classical periodic Besov spaces $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ with $\alpha > 0, 1 \leq p < \infty, 1 \leq q \leq \infty$, into small Lebesgue spaces.

Under the previous assumptions on parameters, it is well known that $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ can be characterized as approximation spaces in $L_p(\mathbb{T}^d)$ with respect to the family $(T_n)_{n\in\mathbb{N}_0}$ of trigonometric polynomials. See [107, 3.7.1]. To be more precise, let

$$T_n = \left\{ \sum_{\substack{-n\leq k_l\leq n \\ l=1,\dots,d}} c_{k_1,\dots,k_d} e^{i(k_1x_1+\dots+k_dx_d)} : c_{k_1,\dots,k_d} \in \mathbb{C} \right\}, n \in \mathbb{N}_0, \tag{7.17}$$

be the subset given by all trigonometric polynomials of order n relative to the variables $x_1, \dots, x_d$. Then, the space $\mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ coincides with $(L_p(\mathbb{T}^d), T_n)^{\alpha}_q = (L_p(\mathbb{T}^d))^{\alpha}_q$ with equivalence constants which depend on the dimension d.

Theorem 7.7. *Let $\alpha > 0, 1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then*

$$\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} \lesssim d^{\alpha} \|f\|^{\text{rep}}_{(L_p(\mathbb{T}^d))^{\alpha}_q}$$

for all $f \in (L_p(\mathbb{T}^d))^{\alpha}_q$.

Proof. Let $f \in (L_p(\mathbb{T}^d))^{\alpha}_q$. Choose a representation $f = \sum_{j=0}^{\infty} f_j$ with $f_j \in T_{2^j}$ such that

$$\left(\sum_{j=0}^{\infty} [2^{j\alpha} \|f_j\|_{L_p(\mathbb{T}^d)}]^q \right)^{1/q} \leq 2\|f\|^{\text{rep}}_{(L_p(\mathbb{T}^d))^{\alpha}_q}.$$

Since we are interested in higher dimensions, we may suppose that $J = 0$ in the construction of p^{λ_j}. Let p_0 be the smallest integer larger than or equal to $p/2$. By Nikolskiĭ inequality for trigonometric polynomials [107, Proposition 3.3.2] we derive

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} [2^{j\alpha} \|f_j\|_{L_{p^{\lambda_j}}(\mathbb{T}^d)}]^q \right)^{1/q} \\
&\leq \left(\sum_{j=0}^{\infty} \left[2^{j\alpha} (2p_0 2^j + 1)^{d\left(\frac{1}{p} - \frac{1}{p^{\lambda_j}}\right)} \|f_j\|_{L_p(\mathbb{T}^d)} \right]^q \right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} \left[2^{j\alpha} (2p_0 2^j + 1)^{2^{-j}} \|f_j\|_{L_p(\mathbb{T}^d)} \right]^q \right)^{1/q} \\
&\lesssim \left(\sum_{j=0}^{\infty} \left[2^{j\alpha} \|f_j\|_{L_p(\mathbb{T}^d)} \right]^q \right)^{1/q} \\
&\lesssim \|f\|^{\text{rep}}_{(L_p(\mathbb{T}^d))^{\alpha}_q}.
\end{aligned}$$

The estimate given by Proposition 7.4 yields that

$$\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} \lesssim d^{\alpha}\|f\|^{\mathrm{rep}}_{(L_p(\mathbb{T}^d))^{\alpha}_q}.$$

The proof is complete. □

Taking into account the equivalence between different norms in $(L_p(\mathbb{T}^d))^{\alpha}_q$ (see Section 2.1), we can write the previous result in the following way.

Corollary 7.8. *Let $\alpha > 0, 1 \le p < \infty$ and $1 \le q \le \infty$. Then*

$$\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} \lesssim d^{\alpha}\|f\|_{(L_p(\mathbb{T}^d))^{\alpha}_q}$$

for all $f \in (L_p(\mathbb{T}^d))^{\alpha}_q$.

For $0 < r \le \infty$ and $h = (h_1, \dots, h_d) \in \mathbb{R}^d$, we put

$$|h|_r = \left(\sum_{l=1}^{d} |h_l|^r\right)^{1/r} \quad \text{if } r < \infty$$

and $|h|_\infty = \max_{l=1,\dots,d}|h_l|$.

Let Δ^k_h be the iterated difference of order k with step h introduced in (2.26). The modulus of smoothness of order k of the function $f \in L_p(\mathbb{T}^d)$ with respect to the $|\cdot|_r$-quasi-norm in $\mathbb{T}^d$ is given by

$$\omega_k(f,t)_{p,r} = \sup_{|h|_r \le t} \|\Delta^k_h f\|_{L_p(\mathbb{T}^d)}, t > 0.$$

In the particular case that $r = 2$, we get the modulus of smoothness with respect to the Euclidean norm $\omega_k(f,t)_p$ considered in (2.27). It is clear that different values of r give equivalent modulus of smoothness but equivalence constants depend on the dimension d.

The Jackson-Stechkin inequality estimates the value of the best approximation of a function $f \in L_p(\mathbb{T}^d)$ by trigonometric polynomials of degree $\le j-1$ in terms of its modulus of smoothness of order M. It reads

$$E_j(f)_p \lesssim d\,\omega_M\left(f, \frac{\pi}{j-1}\right)_{p,r}, j \ge 2. \tag{7.18}$$

A detailed proof may be found in [97, 5.3] (see also [114]) but without the dependence with respect to d on the right-hand side of (7.18). However, the proof shows that one can take d as constant which depends on the dimension and it is independent of $0 < r \le \infty$.

The following result is a dimension-controllable embedding from Besov spaces into small Lebesgue spaces.

Theorem 7.9. *Let $\alpha > 0, 1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $0 < r \leq \infty$ and $M \in \mathbb{N}$ with $M > \alpha$. Then*

$$\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} \lesssim d^{\alpha+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1 (t^{-\alpha}\omega_M(f,t)_{p,r})^q \frac{dt}{t}\right)^{1/q}\right]$$

for all $f \in \mathbf{B}^{\alpha}_{p,q}(\mathbb{T}^d)$.

Proof. By (7.18) and basic properties of modulus of smoothness, we obtain that

$$\begin{aligned}
\|f\|_{(L_p(\mathbb{T}^d))^{\alpha}_q} &= \left(\sum_{j=1}^{\infty}[j^{\alpha}E_j(f)_p]^q j^{-1}\right)^{1/q} \\
&\sim \|f\|_{L_p(\mathbb{T}^d)} + \left(\sum_{j=3}^{\infty}[j^{\alpha}E_j(f)_p]^q j^{-1}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=3}^{\infty}\left[j^{\alpha}\omega_M\left(f, \frac{\pi}{j-1}\right)_{p,r}\right]^q j^{-1}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\left[(j+2)^{\alpha}\omega_M\left(f, \frac{\pi}{j+1}\right)_{p,r}\right]^q (j+2)^{-1}\right)^{1/q} \\
&\leq \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\left[(3j)^{\alpha}\omega_M\left(f, \frac{\pi}{j+1}\right)_{p,r}\right]^q (j+1)^{-1}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\int_j^{j+1}\left[t^{\alpha}\omega_M\left(f, \frac{\pi}{t}\right)_{p,r}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_1^{\infty}\left[t^{\alpha}\omega_M\left(f, \frac{\pi}{t}\right)_{p,r}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_0^1 [t^{-\alpha}\omega_M(f, \pi t)_{p,r}]^q \frac{dt}{t}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_0^1 [t^{-\alpha}\omega_M(f, t)_{p,r}]^q \frac{dt}{t}\right)^{1/q}
\end{aligned}$$

Inserting this estimate in Corollary 7.8 we get the desired result. □

In view of Lemma 7.5 we obtain the following dimension-controllable estimate of Besov spaces into Zygmund spaces.

Corollary 7.10. *Let $\alpha > 0$ and $1 \leq p < \infty$. Let $0 < r \leq \infty$ and $M \in \mathbb{N}$ with $M > \alpha$. Then*

$$\|f\|_{L_p(\log L)_{\alpha}(\mathbb{T}^d)} \lesssim d^{\alpha+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1 (t^{-\alpha}\omega_M(f,t)_{p,r})^p \frac{dt}{t}\right)^{1/p}\right]$$

for all $f \in \mathbf{B}^{\alpha}_{p,p}(\mathbb{T}^d)$.

Let $\mathbb{B}_r^d(R)$ be the ball of radius $R>0$ in $\mathbb{T}^d$, centered at the origin, that is,

$$\mathbb{B}_r^d(R)=\{x\in\mathbb{T}^d:|x|_r<R\}$$

and let $|\mathbb{B}_r^d(R)|$ be its volume with respect to the Lebesgue measure. If $R=1$ we simply write $\mathbb{B}_r^d$ instead of $\mathbb{B}_r^d(1)$.

Our next aim is to extend inequality (7.1) to Besov spaces $\mathbf{B}^\alpha_{p,q}(\mathbb{T}^d)$ with $p\neq q$ into suitably chosen small Lebesgue spaces. Furthermore, we will get a polynomial constant with respect to d rather than an exponential constant.

First we establish an auxiliary result.

Lemma 7.11. *Let $1\le p<\infty, 1\le q\le\infty, 0<r\le\infty$ and $M\in\mathbb{N}$. If $f\in L_p(\mathbb{T}^d)$, there exist positive constants c_1,c_2 which are independent of d such that*

$$\begin{aligned}c_1 2^{-c_2 d}|\mathbb{B}_r^d|^{1/q}\omega_M(f,t)_{p,r}&\le\left(t^{-d}\int_{|h|_r\le t}\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}dh\right)^{1/q}\\&\le|\mathbb{B}_r^d|^{1/q}\omega_M(f,t)_{p,r}\end{aligned}$$

for all $t>0$.

Proof. It is clear that

$$\int_{|h|_r\le t}\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}dh\le t^d|\mathbb{B}_r^d|\sup_{|h|_r\le t}\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}.$$

Therefore,

$$\left(t^{-d}\int_{|h|_r\le t}\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}dh\right)^{1/q}\le|\mathbb{B}_r^d|^{1/q}\omega_M(f,t)_{p,r}$$

for all $t>0$.

We establish the converse estimate. Firstly, assume that $q=1$. Let $|h|_r\le t, |\xi|_r\le t$. We apply L_p norm to the identity

$$\Delta_h^M f(x)=\sum_{i=1}^M(-1)^i\binom{M}{i}\left(\Delta^M_{i(\xi-h)/M}f(x+ih)-\Delta^M_{h+i(\xi-h)/M}f(x)\right)$$

(see [103, p. 192]) to obtain

$$\begin{aligned}&\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)}\\&\quad\le\sum_{i=1}^M\binom{M}{i}\left(\|\Delta^M_{i(\xi-h)/M}f(\cdot+ih)\|_{L_p(\mathbb{T}^d)}+\|\Delta^M_{h+i(\xi-h)/M}f\|_{L_p(\mathbb{T}^d)}\right)\\&\quad=\sum_{i=1}^M\binom{M}{i}\left(\|\Delta^M_{i(\xi-h)/M}f\|_{L_p(\mathbb{T}^d)}+\|\Delta^M_{h+i(\xi-h)/M}f\|_{L_p(\mathbb{T}^d)}\right).\qquad(7.19)\end{aligned}$$

On the other hand, applying L_p norm to the identity

$$\Delta_h^M f(x) = \sum_{i_1=0}^{1} \cdots \sum_{i_M=0}^{1} \Delta_{h/2}^M f(x + i_1 h/2 + \cdots + i_M h/2)$$

(see [103, p. 42]) we derive that

$$\begin{aligned}\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)} &\le \sum_{i_1=0}^{1} \cdots \sum_{i_M=0}^{1} \|\Delta_{h/2}^M f(\cdot + i_1 h/2 + \cdots + i_M h/2)\|_{L_p(\mathbb{T}^d)} \\ &\sim \|\Delta_{h/2}^M f\|_{L_p(\mathbb{T}^d)}.\end{aligned}$$

Consequently, $\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)} \lesssim \|\Delta_{h/2^\lambda}^M f\|_{L_p(\mathbb{T}^d)}$ for any $\lambda \in \mathbb{N}$. Let us take $\lambda = 1$ if $1 \le r \le \infty$ and $\lambda = \left[\frac{1}{r}\right] + 1$ if $0 < r < 1$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$. Integrating over $|\xi|_r \le t$ in (7.19) we derive

$$\begin{aligned}\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)} t^d |\mathbb{B}_r^d| &\le \sum_{i=1}^{M} \binom{M}{i} \int_{|\xi|_r \le t} \|\Delta_{i(\xi-h)/M}^M f\|_{L_p(\mathbb{T}^d)} d\xi \\ &\qquad + \binom{M}{i} \int_{|\xi|_r \le t} \|\Delta_{h+i(\xi-h)/M}^M f\|_{L_p(\mathbb{T}^d)} d\xi \\ &\lesssim \sum_{i=1}^{M} \binom{M}{i} \int_{|\xi|_r \le t} \|\Delta_{i(\xi-h)/(M2^\lambda)}^M f\|_{L_p(\mathbb{T}^d)} d\xi \\ &\qquad + \binom{M}{i} \int_{|\xi|_r \le t} \|\Delta_{((M-i)h+i\xi)/(M2^\lambda)}^M f\|_{L_p(\mathbb{T}^d)} d\xi \\ &\lesssim \sum_{i=1}^{M} \binom{M}{i} \left(\frac{2^\lambda M}{i}\right)^d \int_{|\xi|_r \le t} \|\Delta_\xi^M f\|_{L_p(\mathbb{T}^d)} d\xi \\ &= (2^\lambda M)^d \left(\int_{|\xi|_r \le t} \|\Delta_\xi^M f\|_{L_p(\mathbb{T}^d)} d\xi\right) \sum_{i=1}^{M} \binom{M}{i} i^{-d} \\ &\lesssim (2^\lambda M)^d \int_{|\xi|_r \le t} \|\Delta_\xi^M f\|_{L_p(\mathbb{T}^d)} d\xi.\end{aligned}$$

Therefore

$$\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)} |\mathbb{B}_r^d| \lesssim 2^{cd} t^{-d} \int_{|\xi|_r \le t} \|\Delta_\xi^M f\|_{L_p(\mathbb{T}^d)} d\xi$$

for all $|h|_r \le t$. Taking the supremum over all $|h|_r \le t$ we obtain the desired estimate for $q = 1$.

Assume now that $1 < q < \infty$. By Hölder's inequality we get

$$\begin{aligned}\omega_M(f,t)_{p,r} &\lesssim 2^{cd} |\mathbb{B}_r^d|^{-1} t^{-d} \int_{|h|_r \le t} \|\Delta_h^M f\|_{L_p(\mathbb{T}^d)} dh \\ &\le 2^{cd} |\mathbb{B}_r^d|^{-1} t^{-d} \left(\int_{|h|_r \le t} \|\Delta_h^M f\|_{L_p(\mathbb{T}^d)}^q dh\right)^{1/q} (t^d |\mathbb{B}_r^d|)^{1-1/q} \\ &= 2^{cd} |\mathbb{B}_r^d|^{-1/q} \left(t^{-d} \int_{|h|_r \le t} \|\Delta_h^M f\|_{L_p(\mathbb{T}^d)}^q dh\right)^{1/q}.\end{aligned}$$

If $q = \infty$ the estimates are trivial. □

Remark 7.3. In the particular case $1 \le p = q < \infty$ with $r = 2$ and $M = 1$, the estimates given by Lemma 7.11 with explicit equivalence constants were obtained in [86].

Theorem 7.12. *Let $\alpha > 0, 1 \le p < \infty$ and $1 \le q \le \infty$. Let $0 < r \le \infty$ and $M \in \mathbb{N}$ with $M > \alpha$. Then, there exists a radius $R > 0$ which is independent of d such that*

$$\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} \lesssim d^{\alpha+1}\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r\le 1} \frac{\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha q+d}} \frac{dh}{|\mathbb{B}_r^d(R)|}\right)^{1/q}$$

for all $f \in \mathbf{B}^\alpha_{p,q}(\mathbb{T}^d)$.

Proof. Using Lemma 7.11 and Fubini's theorem we get

$$\begin{aligned}
&\int_0^1 (t^{-\alpha}\omega_M(f,t)_{p,r})^q \frac{dt}{t} \\
&\lesssim \int_0^1 t^{-\alpha q} 2^{cd} |\mathbb{B}_r^d|^{-1} t^{-d} \int_{|h|_r\le t} \|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)} dh \frac{dt}{t} \\
&= \int_{|h|_r\le 1} \|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)} 2^{cd} |\mathbb{B}_r^d|^{-1} \int_{|h|_r}^1 t^{-\alpha q-d} \frac{dt}{t} dh \\
&\le \int_{|h|_r\le 1} \|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)} 2^{cd} |\mathbb{B}_r^d|^{-1} \frac{|h|_r^{-\alpha q-d}}{\alpha q+d} dh \\
&\le \int_{|h|_r\le 1} \frac{\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha q+d}} \frac{dh}{2^{-cd}|\mathbb{B}_r^d|}.
\end{aligned}$$

Let $R = 2^{-c-\frac{\alpha+1}{\log 2}q}$. By Theorem 7.9 we derive

$$\begin{aligned}
\|f\|_{L^{(p,\alpha-1/q,q}(\mathbb{T}^d)} &\lesssim d^{\alpha+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1 (t^{-\alpha}\omega_M(f,t)_{p,r})^q \frac{dt}{t}\right)^{1/q}\right] \\
&\lesssim d^{\alpha+1}\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r\le 1} \frac{\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha q+d}} \frac{dh}{R^d|\mathbb{B}_r^d|}\right)^{1/q} \\
&= d^{\alpha+1}\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r\le 1} \frac{\|\Delta_h^M f\|^q_{L_p(\mathbb{T}^d)}}{|h|_r^{\alpha q+d}} \frac{dh}{|\mathbb{B}_r^d(R)|}\right)^{1/q}.
\end{aligned}$$

This completes the proof. □

Applying Lemma 7.5 for the diagonal case $p = q$ we obtain the following dimension-controllable embedding into Zygmund spaces which is a possible extension of [126, Theorem] from $1 < p < \infty, 0 < \alpha < 1/p$ to $1 \le p < \infty, \alpha > 0$.

Corollary 7.13. *Let $\alpha > 0$ and $1 \leq p < \infty$. Let $0 < r \leq \infty$ and $M \in \mathbb{N}$ with $M > \alpha$. Then, there exists a radius $R > 0$ which is independent of d such that*

$$\|f\|_{L_p(\log L)_\alpha(\mathbb{T}^d)} \lesssim d^{\alpha+1}\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_{|h|_r \leq 1} \frac{\|\Delta_h^M f\|_{L_p(\mathbb{T}^d)}^p}{|h|_r^{\alpha p+d}} \frac{dh}{|\mathbb{B}_r^d(R)|}\right)^{1/p} \tag{7.20}$$

for all $f \in \mathbf{B}^\alpha_{p,p}(\mathbb{T}^d)$.

Remark 7.4. If $\alpha > 0, 1 < p < \infty$ and $r = 2$, the previous result was obtained by Triebel [123, Theorem 3.7] using different techniques but the influence of the volume $|\mathbb{B}_2^d| \sim d^{-d/2}$ in (7.20) is overlooked (see also [126, Remark 2.7]).

7.3 Tractable embeddings of Besov spaces with logarithmic smoothness

In this section we deal with tractable embeddings of periodic Besov spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ with zero classical smoothness and logarithmic smoothness of exponent b with $1 \leq p < \infty, 1 \leq q \leq \infty$ and $b > -1/q$, into small Lebesgue spaces. By construction, the space $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ is a subspace of $L_p(\mathbb{T}^d)$. However, there is an improvement of the integrability properties of the functions in $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ as we show in Chapter 5. Indeed, applying limiting interpolation procedures, we prove in Theorem 5.10 that $\mathbf{B}^{0,b}_{p,q}(\mathbb{T})$ is continuously embedded in $L^{(p,b,q}(\mathbb{T})$. Next we propose a new approach which allows us to control the embedding constant with respect to the dimension d.

Under the above assumptions on parameters, we can describe $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$ as the limiting approximation space $(L_p(\mathbb{T}^d), T_n)^{(0,b)}_q = (L_p(\mathbb{T}^d))^{(0,b)}_q$ with equivalence constants which depend on the dimension d (see [51, Corollary 7.1] and Lemma 2.7).

Next we show a counterpart to Theorem 7.7 for the limiting case $\alpha = 0$.

Theorem 7.14. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ and $b > -1/q$. Then*

$$\|f\|_{L^{(p,b,q}(\mathbb{T}^d)} \lesssim d^{b+1/q}\|f\|^{\mathrm{rep}}_{(L_p(\mathbb{T}^d))^{(0,b)}_q}$$

for all $f \in (L_p(\mathbb{T}^d))^{(0,b)}_q$.

Proof. Let $f \in (L_p(\mathbb{T}^d))_q^{(0,b)}$. Choose a representation $f = \sum_{j=0}^\infty f_j$ with $f_j \in T_{\mu_j}$ such that

$$\left(\sum_{j=0}^{\infty}[2^{j(b+1/q)}\|f_j\|_{L_p(\mathbb{T}^d)}]^q\right)^{1/q} \leq 2\|f\|^{\mathrm{rep}}_{(L_p(\mathbb{T}^d))_q^{(0,b)}}. \tag{7.21}$$

Let p_0 be the smallest integer larger than or equal to $p/2$. By Nikolskiĭ inequality [107, Proposition 3.3.2] we derive

$$\begin{aligned}\|f_j\|_{L_{p^{\lambda_j}}(\mathbb{T}^d)} &\leq (2p_0 2^{2^j}+1)^{d\left(\frac{1}{p}-\frac{1}{p^{\lambda_j}}\right)}\|f_j\|_{L_p(\mathbb{T}^d)} \\ &\sim \|f_j\|_{L_p(\mathbb{T}^d)}\end{aligned}$$

for $j = 0, 1, \ldots$ This estimate and Proposition 7.4 yield that

$$\begin{aligned}\|f\|_{L^{(p,b,q}(\mathbb{T}^d)} &\lesssim d^{b+1/q}\left(\sum_{j=0}^{\infty}[2^{j(b+1/q)}\|f_j\|_{L_{p^{\lambda_j}}(\mathbb{T}^d)}]^q\right)^{1/q} \\ &\lesssim d^{b+1/q}\left(\sum_{j=0}^{\infty}[2^{j(b+1/q)}\|f_j\|_{L_p(\mathbb{T}^d)}]^q\right)^{1/q}.\end{aligned}$$

By (7.21) we get the desired result. □

We write down the estimate given by the previous result but in terms of the equivalent norm in $(L_p(\mathbb{T}^d))_q^{(0,b)}$ given by (2.1) and (2.2).

Corollary 7.15. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ and $b > -1/q$. Then*

$$\|f\|_{L^{(p,b,q}(\mathbb{T}^d)} \lesssim d^{b+1/q}\|f\|_{(L_p(\mathbb{T}^d))_q^{(0,b)}}$$

for all $f \in (L_p(\mathbb{T}^d))_q^{(0,b)}$.

As in the classical case, we are interested in stating the dimension-controllable estimate in terms of the modulus of smoothness $\omega_M(f,t)_{p,r}$. To get this aim, we rely on Jackson-Stechkin inequality.

Theorem 7.16. *Let $1 \leq p < \infty, 1 \leq q \leq \infty$ and $b > -1/q$. Let $0 < r \leq \infty$ and $M \in \mathbb{N}$. Then*

$$\|f\|_{L^{(p,b,q}(\mathbb{T}^d)} \lesssim d^{b+1/q+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1((1-\log t)^b\omega_M(f,t)_{p,r})^q\frac{dt}{t}\right)^{1/q}\right]$$

for all $f \in \mathbf{B}^{0,b}_{p,q}(\mathbb{T}^d)$.

Proof. Using (7.18) and basic properties of the modulus of smoothness, we derive

$$
\begin{aligned}
\|f\|_{(L_p(\mathbb{T}^d))_q^{(0,b)}} &= \left(\sum_{j=1}^{\infty}[(1+\log j)^b E_j(f)_p]^q j^{-1}\right)^{1/q} \\
&\sim \|f\|_{L_p(\mathbb{T}^d)} + \left(\sum_{j=3}^{\infty}[(1+\log j)^b E_j(f)_p]^q j^{-1}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\left[(1+\log(j+2))^b \omega_M\left(f, \frac{\pi}{j+1}\right)_{p,r}\right]^q (j+2)^{-1}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\left(\int_j^{j+1}(1+\log t)^{bq}dt\right)\omega_M\left(f, \frac{\pi}{j+1}\right)_{p,r}^q (j+1)^{-1}\right)^{1/q} \\
&\leq \|f\|_{L_p(\mathbb{T}^d)} + d\left(\sum_{j=1}^{\infty}\int_j^{j+1}(1+\log t)^{bq}\omega_M\left(f, \frac{\pi}{t}\right)_{p,r}^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_1^{\infty}(1+\log t)^{bq}\omega_M\left(f, \frac{\pi}{t}\right)_{p,r}^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_0^{1}(1-\log t)^{bq}\omega_M\left(f, \pi t\right)_{p,r}^q \frac{dt}{t}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{T}^d)} + d\left(\int_0^{1}(1-\log t)^{bq}\omega_M\left(f, t\right)_{p,r}^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

Finally, by Corollary 7.15 we obtain

$$
\begin{aligned}
\|f\|_{L^{(p,b,q}(\mathbb{T}^d)} &\lesssim d^{b+1/q}\|f\|_{(L_p(\mathbb{T}^d))_q^{(0,b)}} \\
&\lesssim d^{b+1/q+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1((1-\log t)^b\omega_M(f,t)_{p,r})^q\frac{dt}{t}\right)^{1/q}\right].
\end{aligned}
$$

The proof is complete. □

By Lemma 7.5 we derive dimension-controllable estimates into Zygmund spaces in the diagonal case.

Corollary 7.17. *Let $1 \leq p < \infty$ and $b > -1/p$. Let $0 < r \leq \infty$ and $M \in \mathbb{N}$. Then*

$$
\|f\|_{L_p(\log L)_{b+1/p}(\mathbb{T}^d)} \lesssim d^{b+1/p+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1((1-\log t)^b\omega_M(f,t)_{p,r})^p\frac{dt}{t}\right)^{1/p}\right]
$$

for all $f \in \mathbf{B}_{p,p}^{0,b}(\mathbb{T}^d)$.

Remark 7.5. Since $\mathbf{B}^{\alpha}_{p,p}(\mathbb{T}^d) \hookrightarrow \mathbf{B}^{0,b}_{p,p}(\mathbb{T}^d)$ is a matter of (dimension-controlled) elementary embedding, we can use Corollary 7.17 to improve on the result in Corollary 7.10. In fact, for $\alpha, \beta > 0, 0 < r \leq \infty, 1 \leq p < \infty$ and $M \in \mathbb{N}$ with $M > \alpha$, it holds

$$\|f\|_{L_p(\log L)_\beta(\mathbb{T}^d)} \lesssim d^{\beta+1}\left[\|f\|_{L_p(\mathbb{T}^d)} + \left(\int_0^1 (t^{-\alpha}\omega_M(f,t)_{p,r})^p \frac{dt}{t}\right)^{1/p}\right]$$

for all $f \in \mathbf{B}^{\alpha}_{p,p}(\mathbb{T}^d)$.

Chapter 8

Besov-Zygmund spaces and logarithmic Besov spaces

Logarithmic Sobolev spaces $H_p^s(\log H)_b$ can be introduced as extrapolation spaces associated to the scale of (fractional) Sobolev spaces H_p^s, taking as a model the description of Zygmund spaces $L_p(\log L)_b$ in terms of the more simple Lebesgue spaces L_p. In applications, to deal with some limiting situations, the refined tuning of the scale H_p^s given by spaces $H_p^s(\log H)_b$ is desirable. See the book by Edmunds and Triebel [59, 2.6]. In a more abstract way, this approach is also useful combined with ideas of interpolation theory as can be seen in the papers [60], [36] and [34].

This chapter deals with Besov spaces with refined smoothness and integrability. We introduce in Section 8.1 the logarithmic Besov spaces $B_p^s(\log B)_b$ over $\mathbb{T}^d$ via extrapolation mimicking the characterization of logarithmic Sobolev spaces as extrapolation spaces. Then, we prove that $B_p^s(\log B)_b$ coincides (equivalence of norms) with Besov-Zygmund spaces $B_p^s(L_p(\log L)_b)$, which are defined replacing the Lebesgue spaces L_p by Zygmund spaces $L_p(\log L)_b$ in the Fourier-analytical definition of classical Besov spaces $B_{p,p}^s = B_p^s(L_p)$. The case $b > 0$ can be reduced to the case $b < 0$ by duality. Furthermore, we characterize $B_p^s(L_p(\log L)_b)$ as limiting interpolation spaces of classical Besov spaces which allows us to apply the abstract extrapolation theory.

In Section 8.2, we compare $B_p^s(L_p(\log L)_b)$ with the Besov spaces $B_{p,p}^{s,b}$ which include logarithmic smoothness (see Chapter 5). We prove that $B_p^s(L_p(\log L)_b)$ is continuously embedded into $B_{p,p}^{s,b}$ if $b < 0$ and vice versa if $b > 0$. Moreover, it is shown that equality does not hold.

Finally, in Section 8.3, we work with the critical case $s = d/p$ and we show that the space $B_p^{d/p}(L_p(\log L)_b)$ is embedded into the space C of continuous functions if $b > 1 - 1/p$. This can not be improved with respect to the parameter b.

The main results of this chapter form the paper [29].

8.1 Spaces $B_p^s(L_p(\log L)_b)$

We start by introducing Besov spaces modelled on Zygmund spaces.

Definition 8.1. Let $1 < p < \infty$ and $s, b \in \mathbb{R}$. The space $B_p^s(L_p(\log L)_b(\mathbb{T}^d))$ is formed by all those $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{B_p^s(L_p(\log L)_b(\mathbb{T}^d))} = \left(\sum_{j=0}^{\infty} 2^{jsp} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^p_{L_p(\log L)_b(\mathbb{T}^d)}\right)^{1/p} < \infty.$$

Here $\varphi_0, \varphi_1, \ldots$ are the functions given by (5.9). Note that $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ is identified with the trigonometric polynomial $\sum_{m \in \mathbb{Z}^d} \varphi_j(m) \hat{f}(m) e^{imx}$ (see Remark 5.3).

It is worthwhile mentioning that spaces $B_p^s(L_p(\log L)_b(\mathbb{T}^d))$ make sense for $0 < p < \infty$ but due to the tools we use here, we consider only a smaller range of indices.

Spaces $B_p^s(L_p(\log L)_b(\mathbb{T}^d))$ form a more refined scale than the scale of spaces $B_{p,p}^s(\mathbb{T}^d)$ introduced in Chapter 5. In fact, since $L_p(\log L)_\epsilon(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d) \hookrightarrow L_p(\log L)_{-\epsilon}(\mathbb{T}^d)$ for any $\epsilon > 0$, we have that

$$B_p^s(L_p(\log L)_\epsilon(\mathbb{T}^d)) \hookrightarrow B_{p,p}^s(\mathbb{T}^d) \hookrightarrow B_p^s(L_p(\log L)_{-\epsilon}(\mathbb{T}^d)).$$

Having in mind the extrapolation representation of Zygmund spaces in terms of L_p spaces (see [59, 2.6.2]) and the construction of logarithmic Sobolev spaces [59, 2.6.3], it is natural to introduce the following logarithmic Besov spaces.

Definition 8.2. Let $1 < p < \infty$ and $j_0 \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$,

$$\frac{1}{p^{\sigma_j}} = \frac{1}{p} + \frac{2^{-j}}{d} < 1 \text{ and } \frac{1}{p^{\lambda_j}} = \frac{1}{p} - \frac{2^{-j}}{d} > 0.$$

(i) Let $b < 0$. The space $B_p^s(\log B)_b(\mathbb{T}^d)$ is formed by all those $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{B_p^s(\log B)_b(\mathbb{T}^d)} = \left(\sum_{j=j_0}^{\infty} 2^{jbp} \|f\|^p_{B^s_{p^{\sigma_j},p}(\mathbb{T}^d)}\right)^{1/p} < \infty.$$

(ii) Let $b > 0$. The space $B^s_p(\log B)_b(\mathbb{T}^d)$ is the collection of all $f \in \mathcal{D}'(\mathbb{T}^d)$ which can be represented as

$$f = \sum_{j=j_0}^{\infty} f_j \text{ (convergence in } \mathcal{D}'(\mathbb{T}^d)), \text{ with } f_j \in B^s_{p^{\lambda_j},p}(\mathbb{T}^d), \tag{8.1}$$

and

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|f_j\|^p_{B^s_{p^{\lambda_j},p}(\mathbb{T}^d)}\right)^{1/p} < \infty. \tag{8.2}$$

The norm $\|f\|_{B^s_p(\log B)_b(\mathbb{T}^d)}$ is given by taking the infimum over the expressions (8.2) with respect to all representations satisfying (8.1) and (8.2).

Note that $B^s_p(\log B)_b(\mathbb{T}^d)$ is independent of j_0 with equivalent norms.

Next we show that the spaces given by Definitions 8.1 and 8.2 are the same. We start with some auxiliary results.

Lemma 8.1. *Let $1 < r < p < \infty$ and $b < 0$. Then*

$$(L_r(\mathbb{T}^d), L_p(\mathbb{T}^d))_{(1,-(b-1/p)),p} = L_p(\log L)_b(\mathbb{T}^d)$$

with equivalence of norms.

Proof. Let $1/\alpha = 1/r - 1/p$. By Holmstedt's formula [78, Theorem 4.1], we have

$$\begin{aligned}\|f\|_{(L_r(\mathbb{T}^d),L_p(\mathbb{T}^d))_{(1,-(b-1/p)),p}} &\sim \left(\int_0^1 \left[t^{-1}(1-\log t)^{b-1/p}\left(\int_0^{t^\alpha} f^*(s)^r ds\right)^{1/r}\right]^p \frac{dt}{t}\right)^{1/p} \\ &+ \left(\int_0^1 \left[(1-\log t)^{b-1/p}\left(\int_{t^\alpha}^1 f^*(s)^p ds\right)^{1/p}\right]^p \frac{dt}{t}\right)^{1/p} \\ &= I_1 + I_2.\end{aligned}$$

For the second term we obtain

$$\begin{aligned}I_2 &= \left(\int_0^1 f^*(s)^p \int_0^{s^{1/\alpha}} (1-\log t)^{bp-1} \frac{dt}{t} ds\right)^{1/p} \\ &\sim \left(\int_0^1 (1-\log s)^{bp} f^*(s)^p ds\right)^{1/p} \sim \|f\|_{L_p(\log L)_b(\mathbb{T}^d)}.\end{aligned}$$

As for I_1, making a change of variables and then using Hardy's inequality [8, Theorem 6.4], we get

$$
\begin{aligned}
I_1 &\sim \left(\int_0^1 \left[t^{-r/\alpha}(1-\log t)^{br-r/p}\int_0^t f^*(s)^r ds\right]^{p/r}\frac{dt}{t}\right)^{1/p} \\
&\lesssim \left(\int_0^1 [t^{1-r/\alpha}(1-\log t)^{br-r/p} f^*(t)^r]^{p/r}\frac{dt}{t}\right)^{1/p} \\
&= \left(\int_0^1 [t^{1/p}(1-\log t)^{b-1/p} f^*(t)]^p\frac{dt}{t}\right)^{1/p} \\
&\lesssim \|f\|_{L_p(\log L)_b(\mathbb{T}^d)}.
\end{aligned}
$$

This completes the proof. □

The following result can be checked with similar arguments.

Lemma 8.2. *Let $1 < p < r < \infty$ and $b < 0$. Then*

$$(L_p(\mathbb{T}^d), L_r(\mathbb{T}^d))_{(0,b+1/p),p} = L_p(\log L)_{-b}(\mathbb{T}^d)$$

with equivalence of norms.

We shall need a couple of results on interpolation of vector-valued sequence spaces. The K_p-functional defined in (5.16) will be useful in the proof.

Lemma 8.3. *Let $1 < p < \infty, b > 1/p$ and let $(A_j), (B_j)$ be sequences of Banach spaces with $B_j \hookrightarrow A_j$ and $\sup_{j\in\mathbb{N}_0} \|I\|_{B_j,A_j} < \infty$, so $\ell_p(B_j) \hookrightarrow \ell_p(A_j)$. Then*

$$(\ell_p(A_j), \ell_p(B_j))_{(1,b),p} = \ell_p((A_j, B_j)_{(1,b),p})$$

(equivalent norms).

Proof. Let $a = (a_j) \in \ell_p(A_j)$. We have

$$K_p(t, a; \ell_p(A_j), \ell_p(B_j)) \sim \left(\sum_{j=0}^\infty K_p(t, a_j; A_j, B_j)^p\right)^{1/p}.$$

Therefore,

$$
\begin{aligned}
&\|a\|_{(\ell_p(A_j),\ell_p(B_j))_{(1,b),p}} \\
&\quad\sim \left(\int_0^1 \sum_{j=0}^\infty t^{-p} K_p(t, a_j; A_j, B_j)^p (1-\log t)^{-bp}\frac{dt}{t}\right)^{1/p} \\
&\quad= \left(\sum_{j=0}^\infty \int_0^1 \left(\frac{K_p(t, a_j; A_j, B_j)}{t(1-\log t)^b}\right)^p \frac{dt}{t}\right)^{1/p} \\
&\quad\sim \|a\|_{\ell_p((A_j,B_j)_{(1,b),p})}.
\end{aligned}
$$

□

Note that in Lemma 8.3 the assumption $b > 1/p$ is needed in order not to have that the interpolation spaces are $\{0\}$. The corresponding result for the method $((0,b),p)$ can be proved similarly and, since the method $((0,b),p)$ is always meaningful, it does not require any condition on b.

Lemma 8.4. *Let $1 < p < \infty, b \in \mathbb{R}$ and let $(A_j),(B_j)$ be sequences of Banach spaces with $B_j \hookrightarrow A_j$ and $\sup_{j\in\mathbb{N}_0} \|I\|_{B_j,A_j} < \infty$. Then*

$$(\ell_p(A_j),\ell_p(B_j))_{(0,b),p} = \ell_p((A_j,B_j)_{(0,b),p})$$

(equivalent norms).

Theorem 8.5. *Let $1 < q < r < p < \infty, s \in \mathbb{R}$ and $b < 0$. Then we have, with equivalence of norms,*

(i) $(B^s_{r,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{(1,-(b-1/p)),p} = B^s_p(L_p(\log L)_b(\mathbb{T}^d))$.

(ii) $(B^s_{q,q}(\mathbb{T}^d), B^s_{r,q}(\mathbb{T}^d))_{(0,b+1/q),q} = B^s_q(L_q(\log L)_{-b}(\mathbb{T}^d))$.

Proof. Let φ_j be the functions given by (5.9). Write $\varphi_{-1} \equiv 0$ and $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. Consider the mapping $\mathfrak{J}f = (\mathcal{F}^{-1}(\varphi_j\mathcal{F}f))$. The restrictions,

$$\begin{aligned}
&\mathfrak{J} : B^s_{r,p}(\mathbb{T}^d) \longrightarrow \ell_p(2^{js}L_r(\mathbb{T}^d)),\\
&\mathfrak{J} : B^s_{p,p}(\mathbb{T}^d) \longrightarrow \ell_p(2^{js}L_p(\mathbb{T}^d)),\\
&\mathfrak{J} : B^s_p(L_p(\log L)_b(\mathbb{T}^d)) \longrightarrow \ell_p(2^{js}L_p(\log L)_b(\mathbb{T}^d))
\end{aligned}$$

are bounded. Let $\mathfrak{R}(f_j) = \sum_{j=0}^\infty \mathcal{F}^{-1}(\tilde{\varphi}_j\mathcal{F}f_j)$. This mapping satisfies that the restrictions

$$\begin{aligned}
&\mathfrak{R} : \ell_p(2^{js}L_r(\mathbb{T}^d)) \longrightarrow B^s_{r,p}(\mathbb{T}^d),\\
&\mathfrak{R} : \ell_p(2^{js}L_p(\mathbb{T}^d)) \longrightarrow B^s_{p,p}(\mathbb{T}^d),\\
&\mathfrak{R} : \ell_p(2^{js}L_p(\log L)_b(\mathbb{T}^d)) \longrightarrow B^s_p(L_p(\log L)_b(\mathbb{T}^d))
\end{aligned}$$

are bounded and that $\mathfrak{R}(\mathfrak{J}f) = f$. That is, $B^s_{r,p}(\mathbb{T}^d)$ (respectively, $B^s_{p,p}(\mathbb{T}^d)$ and $B^s_p(L_p(\log L)_b(\mathbb{T}^d)))$ is a retract of $\ell_p(2^{js}L_r(\mathbb{T}^d))$ (respectively, of $\ell_p(2^{js}L_p(\mathbb{T}^d))$ and $\ell_p(2^{js}L_p(\log L)_b(\mathbb{T}^d)))$ (see [116, Definition 1.2.4]). For the vector-valued spaces, according to Lemmata 8.3 and 8.1, we obtain

$$\begin{aligned}
(\ell_p(2^{js}L_r(\mathbb{T}^d)), &\ell_p(2^{js}L_p(\mathbb{T}^d)))_{(1,-(b-1/p)),p}\\
&= \ell_p(2^{js}(L_r(\mathbb{T}^d), L_p(\mathbb{T}^d))_{(1,-(b-1/p)),p})\\
&= \ell_p(2^{js}L_p(\log L)_b(\mathbb{T}^d)).
\end{aligned}$$

Therefore, using [116, Theorem 1.2.4], we conclude that

$$(B^s_{r,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{(1,-(b-1/p)),p} = B^s_p(L_p(\log L)_b(\mathbb{T}^d)).$$

The proof of (ii) is similar but using now Lemmata 8.4 and 8.2. □

Now we are ready to show that if $b<0$ then spaces given by Definitions 8.1 and 8.2 are equal.

Theorem 8.6. *Let $1<p<\infty$, $s\in\mathbb{R}$ and $b<0$. Then $B^s_p(L_p(\log L)_b(\mathbb{T}^d)) = B^s_p(\log B)_b(\mathbb{T}^d)$ with equivalence of norms.*

Proof. Let j_0 and p^{σ_j} as in Definition 8.2. Put $1/p^0 = 1/p + 2^{-j_0}/d$. Then $B^s_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_{p^0,p}(\mathbb{T}^d)$ and $(B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))$ is a Gagliardo couple (see [9, p. 320]). Indeed, take any bounded sequence $(f_n)\subseteq B^s_{p,p}(\mathbb{T}^d)$ with $\lim_{n\to\infty}\|f-f_n\|_{B^s_{p^0,p}(\mathbb{T}^d)} = 0$ for some $f\in B^s_{p^0,p}(\mathbb{T}^d)$, and let us check that $f\in B^s_{p,p}(\mathbb{T}^d)$. Since the space $B^s_{p,p}(\mathbb{T}^d)$ is reflexive [107, Theorem 3.5.6], there is $g\in B^s_{p,p}(\mathbb{T}^d)$ such that (f_n) converges weakly to g in $B^s_{p,p}(\mathbb{T}^d)$. Hence (f_n) converges weakly to f and to g in $B^s_{p^0,p}(\mathbb{T}^d)$. This yields that $f=g\in B^s_{p,p}(\mathbb{T}^d)$.

For $j\ge j_0$ put $\theta_j = 1-2^{-j}$. So $(1-\theta_j)/p^0 + \theta_j/p = 1/p + 2^{-(j+j_0)}/d = 1/p^{\sigma_{j+j_0}}$. Using that $[\ell_p(2^{ks}L_{p^0}(\mathbb{T}^d)), \ell_p(2^{ks}L_p(\mathbb{T}^d))]_{\theta_j} = \ell_p(2^{ks}L_{p^{\sigma_{j+j_0}}}(\mathbb{T}^d))$ with equal norms (see [10, Theorems 5.6.3 and 5.1.1]) and the property of retraction, we derive that

$$\|f\|_{B^s_p(\log B)_b(\mathbb{T}^d)} \sim \left(\sum_{j=j_0}^{\infty}\left(2^{jb}\|f\|_{[B^s_{p^0,p}(\mathbb{T}^d),B^s_{p,p}(\mathbb{T}^d)]_{\theta_j}}\right)^p\right)^{1/p}.$$

That is to say, the space $B^s_p(\log B)_b(\mathbb{T}^d)$ is an extrapolation space of the type $\delta^{(p)+}_{1-2^{-j_0},1}(2^{jb}[B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d)]_{\theta_j})$ in the notation of [84, p. 72]. Moreover, by (2.11) and (2.12), we know that

$$\theta_j(1-\theta_j)(B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{\theta_j,1;K} \hookrightarrow [B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d)]_{\theta_j} \hookrightarrow (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{\theta_j,\infty}$$

with constants independent of j. Therefore, according to [84, Theorem 3.2, (61) and (62) in page 74], we have that

$$B^s_p(\log B)_b(\mathbb{T}^d) = (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{\rho,p} \tag{8.3}$$

where

$$\begin{aligned}\frac{1}{\rho(t)} &= \left(\sum_{j=j_0}^{\infty}[t^{-(1-2^{-j})}2^{-j(-b+1/p)}]^p\right)^{1/p}\\ &= t^{-1}\left(\sum_{j=j_0}^{\infty}2^{jbp}2^{-j}t^{2^{-j}p}\right)^{1/p}\\ &= t^{-1}V(t).\end{aligned}$$

Let us compute $V(t)$. For $t \geq 1$, we have $V(t) \leq t^{2^{-j_0}} \left(\sum_{j=j_0}^{\infty} 2^{jbp}\right) \lesssim t^{2^{-j_0}}$ because $b < 0$. Since we also have $V(t) \gtrsim t^{2^{-j_0}}$, we obtain that $V(t) \sim t^{2^{-j_0}}$. For the case $0 < t < 1$ we follow an idea of [84, p. 82]. We have

$$\begin{aligned} V(t)^p &\sim \int_0^{2^{-j_0}} t^{\sigma p} \sigma^{-bp} d\sigma \sim \int_0^{2^{-j_0}} e^{-\sigma p (1-\log t)} \sigma^{-bp} d\sigma \\ &= \int_0^{2^{-j_0}(1-\log t)} e^{-\sigma p} \left(\frac{\sigma}{1-\log t}\right)^{-bp} \frac{d\sigma}{1-\log t} \\ &= (1-\log t)^{bp-1} \int_0^{2^{-j_0}(1-\log t)} e^{-\sigma p} \sigma^{-bp} d\sigma. \end{aligned}$$

Moreover

$$\int_0^\infty e^{-\sigma p} \sigma^{-bp} d\sigma = \int_1^\infty t^{-p} (\log t)^{-bp} \frac{dt}{t} < \infty.$$

Hence

$$V(t) \sim (1-\log t)^{b-1/p} \text{ for } 0 < t < 1.$$

Consequently,

$$\frac{1}{\rho(t)} \sim \begin{cases} t^{-1+2^{-j_0}} & \text{for} \quad t \geq 1, \\ t^{-1}(1-\log t)^{b-1/p} & \text{for} \quad 0 < t < 1. \end{cases}$$

To determine the interpolation space in (8.3), note that the embedding $B^s_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_{p^0,p}(\mathbb{T}^d)$ yields that

$$K(t, f; B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d)) \sim \|f\|_{B^s_{p^0,p}(\mathbb{T}^d)} \text{ for } 1 \leq t < \infty. \tag{8.4}$$

Whence

$$B^s_p(\log B)_b(\mathbb{T}^d) = (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{\rho,p} = (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{(1,-(b-1/p)),p}. \tag{8.5}$$

Finally, applying Theorem 8.5(i) we derive that $B^s_p(\log B)_b(\mathbb{T}^d) = B^s_p(L_p(\log L)_b(\mathbb{T}^d))$. □

We shall use duality in order to establish the corresponding result when $b > 0$. We start by determining the dual space of $B^s_p(\log B)_b(\mathbb{T}^d)$.

Let $1 < p < \infty, s \in \mathbb{R}$ and $b < 0$. By (8.5) and (8.4) we have

$$\begin{aligned} B^s_p(L_p(\log L)_b(\mathbb{T}^d)) &= B^s_p(\log B)_b(\mathbb{T}^d) = (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{(1,-(b-1/p)),p} \\ &= (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{1,p,(-b+1/p,0)}. \end{aligned} \tag{8.6}$$

Then, as a consequence of [47, Corollary 3.7], we obtain that $B^s_{p,p}(\mathbb{T}^d)$ is dense in $B^s_p(\log B)_b(\mathbb{T}^d)$. Therefore, by [107, Theorem 3.5.1(i)], the set $\mathcal{D}(\mathbb{T}^d)$ is dense in $B^s_p(\log B)_b(\mathbb{T}^d)$ and

$$\mathcal{D}(\mathbb{T}^d) \hookrightarrow B^s_p(\log B)_b(\mathbb{T}^d) \hookrightarrow \mathcal{D}'(\mathbb{T}^d).$$

So

$$\mathcal{D}(\mathbb{T}^d) \hookrightarrow (B^s_p(\log B)_b(\mathbb{T}^d))' \hookrightarrow \mathcal{D}'(\mathbb{T}^d).$$

Theorem 8.7. *Let $1 < p < \infty, 1/p + 1/p' = 1, s \in \mathbb{R}$ and $b < 0$. Then we have with equivalent norms*

$$(B^s_p(\log B)_b(\mathbb{T}^d))' = B^{-s}_{p'}(L_{p'}(\log L)_{-b}(\mathbb{T}^d)).$$

Proof. By (8.6), [47, Theorem 5.6] and [107, Theorem 3.5.6], we get

$$\begin{aligned}(B^s_p(\log B)_b(\mathbb{T}^d))' &= ((B^s_{p^0,p}(\mathbb{T}^d))', (B^s_{p,p}(\mathbb{T}^d))')_{1,p',(1,b+1/p')}\\ &= (B^{-s}_{(p^0)',p'}(\mathbb{T}^d), B^{-s}_{p',p'}(\mathbb{T}^d))_{1,p',(1,b+1/p')}\\ &= (B^{-s}_{p',p'}(\mathbb{T}^d), B^{-s}_{(p^0)',p'}(\mathbb{T}^d))_{0,p',(b+1/p',1)}\\ &= (B^{-s}_{p',p'}(\mathbb{T}^d), B^{-s}_{(p^0)',p'}(\mathbb{T}^d))_{(0,b+1/p'),p'}\end{aligned}$$

where the last equality is a consequence of the fact that

$$K(t, f; B^{-s}_{p',p'}(\mathbb{T}^d), B^{-s}_{(p^0)',p'}(\mathbb{T}^d)) \sim \|f\|_{B^{-s}_{p',p'}(\mathbb{T}^d)} \text{ for } 1 \le t < \infty.$$

Now, applying Theorem 8.5(ii), we conclude that

$$(B^s_p(\log B)_b(\mathbb{T}^d))' = B^{-s}_{p'}(L_{p'}(\log L)_{-b}(\mathbb{T}^d)).$$

□

Theorem 8.8. *Let $1 < q < \infty, s \in \mathbb{R}$ and $b > 0$. Then $B^s_q(L_q(\log L)_b(\mathbb{T}^d)) = B^s_q(\log B)_b(\mathbb{T}^d)$ with equivalence of norms.*

Proof. Let $1/q + 1/p = 1$. Define p^{σ_j} and q^{λ_j} as in Definition 8.2. Then $1/q^{\lambda_j} + 1/p^{\sigma_j} = 1$ for $j \ge j_0$. We know by Theorem 8.7 that $B^s_q(L_q(\log L)_b(\mathbb{T}^d)) = (B^{-s}_p(\log B)_{-b}(\mathbb{T}^d))'$. Next we compute this dual space following a different way. By Definition 8.2, the space $B^{-s}_p(\log B)_{-b}(\mathbb{T}^d)$ is isometric to the diagonal D of $\ell_p(2^{-jb}B^{-s}_{p^{\sigma_j},p}(\mathbb{T}^d))$. Therefore, its dual space is

$$\begin{aligned}(B^{-s}_p(\log B)_{-b}(\mathbb{T}^d))' = D' &= (\ell_p(2^{-jb}B^{-s}_{p^{\sigma_j},p}(\mathbb{T}^d)))'/D^\perp\\ &= \ell_q(2^{jb}B^s_{q^{\lambda_j},q}(\mathbb{T}^d))/D^\perp.\end{aligned}$$

Whence, functionals G belonging to $(B^{-s}_p(\log B)_{-b}(\mathbb{T}^d))'$ are those given by a sequence $(f_j) \in \ell_q(2^{jb}B^s_{q^{\lambda_j},q}(\mathbb{T}^d))$ by the formula $G(\varphi) = \sum_{j=j_0}^\infty < f_j, \varphi >, \varphi \in \mathcal{D}(\mathbb{T}^d)$, and the norm of G is

$$\|G\| = \inf\left\{\left(\sum_{j=j_0}^\infty 2^{jbq}\|f_j\|^q_{B^s_{q^{\lambda_j},q}(\mathbb{T}^d)}\right)^{1/q} : G = \sum_{j=j_0}^\infty f_j\right\}.$$

This shows that $B^s_q(L_q(\log L)_b(\mathbb{T}^d)) = B^s_q(\log B)_b(\mathbb{T}^d)$. □

We close this section by characterizing the dual space of $B_p^s(L_p(\log L)_b(\mathbb{T}^d))$ when $b > 0$.

Theorem 8.9. *Let $1 < p < \infty, 1/p + 1/p' = 1, s \in \mathbb{R}$ and $b > 0$. Then we have with equivalent norms*

$$(B_p^s(L_p(\log L)_b(\mathbb{T}^d)))' = B_{p'}^{-s}(L_{p'}(\log L)_{-b}(\mathbb{T}^d)).$$

Proof. Take any $r > p$. Using Theorem 8.5(ii) and the fact that

$$K(t, f; B_{p,p}^s(\mathbb{T}^d), B_{r,p}^s(\mathbb{T}^d)) \sim \|f\|_{B_{p,p}^s(\mathbb{T}^d)} \text{ for } 1 \le t < \infty,$$

we obtain

$$\begin{aligned} B_p^s(L_p(\log L)_b(\mathbb{T}^d)) &= (B_{p,p}^s(\mathbb{T}^d), B_{r,p}^s(\mathbb{T}^d))_{(0,-b+1/p),p} \\ &= (B_{p,p}^s(\mathbb{T}^d), B_{r,p}^s(\mathbb{T}^d))_{0,p,(-b+1/p,1)} \\ &= (B_{r,p}^s(\mathbb{T}^d), B_{p,p}^s(\mathbb{T}^d))_{1,p,(1,-b+1/p)}. \end{aligned}$$

Then, according to [47, Theorem 5.6], [107, Theorem 3.5.6] and Theorem 8.5, we derive

$$\begin{aligned} (B_p^s(L_p(\log L)_b(\mathbb{T}^d)))' &= (B_{r',p'}^{-s}(\mathbb{T}^d), B_{p',p'}^{-s}(\mathbb{T}^d))_{1,p',(b+1/p',0)} \\ &= (B_{r',p'}^{-s}(\mathbb{T}^d), B_{p',p'}^{-s}(\mathbb{T}^d))_{(1,b+1/p'),p'} \\ &= B_{p'}^{-s}(L_{p'}(\log L)_{-b}(\mathbb{T}^d)). \end{aligned}$$

□

8.2 Relation with spaces $B_{p,p}^{s,b}$

In this section we describe the relationship between Besov spaces of logarithmic smoothness $B_{p,p}^{s,b}$ (see (5.10)) and logarithmic Besov spaces $B_p^s(L_p(\log L)_b)$.

Theorem 8.10. *Let $1 < p < \infty$ and $s \in \mathbb{R}$.*

(i) *If $b < 0$ then $B_p^s(L_p(\log L)_b(\mathbb{T}^d)) \hookrightarrow B_{p,p}^{s,b}(\mathbb{T}^d)$.*

(ii) *If $b > 0$ then $B_{p,p}^{s,b}(\mathbb{T}^d) \hookrightarrow B_p^s(L_p(\log L)_b(\mathbb{T}^d))$.*

Proof. Suppose first $b<0$. As we have seen in (8.5), we have that $B^s_p(L_p(\log L)_b(\mathbb{T}^d)) = (B^s_{p^0,p}(\mathbb{T}^d), B^s_{p,p}(\mathbb{T}^d))_{(1,-(b-1/p)),p}$ where $p^0 = (1/p+2^{-j_0}/d)^{-1}$. Let $s^0 = s-2^{-j_0}$ and $s_0 = s+2^{-j_0}$. Since $p^0<p$ and $s-d/p^0 = s^0-d/p$, it follows from [107, Theorem 3.5.5(i)] that $B^s_{p^0,p}(\mathbb{T}^d) \hookrightarrow B^{s^0}_{p,p}(\mathbb{T}^d)$. Besides, $(B^{s^0}_{p,p}(\mathbb{T}^d), B^{s_0}_{p,p}(\mathbb{T}^d))_{1/2,p} = B^s_{p,p}(\mathbb{T}^d)$ by [107, Theorem 3.6.1(1)]. Therefore, using Lemma 3.8(a), we derive

$$\begin{aligned} B^s_p(L_p(\log L)_b(\mathbb{T}^d)) &\hookrightarrow (B^{s^0}_{p,p}(\mathbb{T}^d), (B^{s^0}_{p,p}(\mathbb{T}^d), B^{s_0}_{p,p}(\mathbb{T}^d))_{1/2,p})_{(1,-(b-1/p)),p} \\ &= (B^{s^0}_{p,p}(\mathbb{T}^d), B^{s_0}_{p,p}(\mathbb{T}^d))_{1/2,p,-b} \\ &= B^{s,b}_{p,p}(\mathbb{T}^d) \end{aligned}$$

where the last equality follows from the property of retractions of Besov spaces proceeding as in [32, Theorem 5.3]. This establishes (i).

In fact, $B^s_p(L_p(\log L)_b(\mathbb{T}^d))$ is densely embedded in $B^{s,b}_{p,p}(\mathbb{T}^d)$. Indeed, we have

$$B^{s_0}_{p,p}(\mathbb{T}^d) = B^{s^0}_{p,p}(\mathbb{T}^d) \cap B^{s_0}_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_{p,p}(\mathbb{T}^d) \hookrightarrow B^s_p(L_p(\log L)_b(\mathbb{T}^d))$$

and $B^{s_0}_{p,p}(\mathbb{T}^d)$ is dense in $B^{s,b}_{p,p}(\mathbb{T}^d) = (B^{s^0}_{p,p}(\mathbb{T}^d), B^{s_0}_{p,p}(\mathbb{T}^d))_{1/2,p,-b}$ as can be easily checked by using [74, Theorem 2.2].

Taking duals in (i) and using Theorem 8.7, we derive

$$(B^{s,b}_{p,p}(\mathbb{T}^d))' \hookrightarrow (B^s_p(L_p(\log L)_b(\mathbb{T}^d)))' = B^{-s}_{p'}(L_{p'}(\log L)_{-b}(\mathbb{T}^d))$$

where $1/p+1/p'=1$. The space $(B^{s,b}_{p,p}(\mathbb{T}^d))'$ can be determined with the help of [48, Theorem 3.1] or [102, Theorem 2.4]. We have

$$\begin{aligned} (B^{s,b}_{p,p}(\mathbb{T}^d))' &= ((B^{s^0}_{p,p}(\mathbb{T}^d), B^{s_0}_{p,p}(\mathbb{T}^d))_{1/2,p,-b})' \\ &= (B^{-s^0}_{p',p'}(\mathbb{T}^d), B^{-s_0}_{p',p'}(\mathbb{T}^d))_{1/2,p',b} \\ &= B^{-s,-b}_{p',p'}(\mathbb{T}^d). \end{aligned}$$

This completes the proof. □

Next we show that equality does not hold in Theorem 8.10.

Counterexample 8.1. We work in the one dimensional case, $d=1$, with $s>0, 1<p<\infty$ and $b<0$. As we have seen in (8.5), $B^s_p(L_p(\log L)_b(\mathbb{T})) = (B^s_{p^0,p}(\mathbb{T}), B^s_{p,p}(\mathbb{T}))_{(1,-(b-1/p)),p}$. So $B^s_p(L_p(\log L)_b(\mathbb{T})) \hookrightarrow B^s_{p^0,p}(\mathbb{T})$. To check that in Theorem 8.10(i) we have $B^{s,b}_{p,p}(\mathbb{T}) \not\hookrightarrow B^s_p(L_p(\log L)_b(\mathbb{T}))$, it suffices to show that $B^{s,b}_{p,p}(\mathbb{T}) \not\hookrightarrow B^s_{p^0,p}(\mathbb{T})$.

Let $h(x) = \sum_{k=-2^{n_0}}^{2^{n_0}} \lambda_k e^{ikx}$ be a trigonometric polynomial of degree 2^{n_0} with $\|h\|_{L_p(\mathbb{T})} = 1$ and $\|h\|_{L_{p^0}(\mathbb{T})} = c>0$. For $n \geq n_0+1$, put $h_n(x) = h(x)e^{3i2^n x}$. Let (P_n) be the sequence of Fourier projections and put $Q_n = P_{2^{n+1}-1} - P_{2^n-1}$. Note that $h_n \in Q_{n+1}(L_p(\mathbb{T}))$ for $n \geq n_0 + 1$.

Let (β_n) be a decreasing sequence of positive numbers such that

$$\left(\sum_{n=1}^{\infty}(2^{ns}\beta_n)^p\right)^{1/p} = \infty \text{ and } \left(\sum_{n=1}^{\infty}(2^{ns}(1+n)^b\beta_n)^p\right)^{1/p} < \infty \tag{8.7}$$

and consider the function $f(x) = \sum_{n=n_0+1}^{\infty} \beta_{n+1} h_n(x)$. Then f belongs to $B^{s,b}_{p,p}(\mathbb{T})$ because, according to (2.28), (2.6) and (8.7),

$$\begin{aligned}\|f\|_{B^{s,b}_{p,p}(\mathbb{T})} &\lesssim \left(\sum_{n=n_0+1}^{\infty}[2^{(n+2)s}(2+n)^b\beta_{n+1}\|h_n\|_{L_p(\mathbb{T})}]^p\right)^{1/p} \\ &\lesssim \left(\sum_{n=n_0+1}^{\infty}[2^{(n+1)s}(2+n)^b\beta_{n+1}]^p\right)^{1/p} < \infty.\end{aligned}$$

But f does not belong to $B^s_{p^0,p}(\mathbb{T})$. Indeed, using the Linear Representation Theorem [104, p. 122], we obtain

$$\begin{aligned}\|f\|_{B^s_{p^0,p}(\mathbb{T})} &\gtrsim \left(\sum_{n=n_0+1}^{\infty}[2^{ns}\|Q_n f\|_{L_{p^0}(\mathbb{T})}]^p\right)^{1/p} \\ &\sim \left(\sum_{n=n_0+1}^{\infty}[2^{(n+1)s}\beta_{n+1}\|h_n\|_{L_{p^0}(\mathbb{T})}]^p\right)^{1/p} \\ &= c\left(\sum_{n=n_0+1}^{\infty}[2^{(n+1)s}\beta_{n+1}]^p\right)^{1/p} = \infty.\end{aligned}$$

8.3 Embeddings into C

Let $C(\mathbb{T}^d)$ be the space of uniformly continuous functions on $\mathbb{T}^d$ with the usual norm. In this section we consider the critical case $s = d/p$ and we characterize the embedding from $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))$ into $C(\mathbb{T}^d)$. We start with some auxiliary results.

Lemma 8.11. *Let A be a Banach space, let $\lambda > 1, 1 \leq p < \infty$ and $\gamma > -1/p$. Then*

$$(A, \lambda A)_{(0,-\gamma),p} = (1 + \log \lambda)^{\gamma+1/p} A$$

with equivalence of norms where the constants are independent of λ.

Proof. Using that $K(t,a;A,\lambda A)=\min\{1,t\lambda\}\|a\|_A$, we obtain

$$\begin{aligned}\|a\|_{(A,\lambda A)_{(0,-\gamma),p}} &= \left(\int_0^{1/\lambda}(t\lambda(1-\log t)^{\gamma})^p\frac{dt}{t}+\int_{1/\lambda}^1(1-\log t)^{\gamma p}\frac{dt}{t}\right)^{1/p}\|a\|_A\\ &= \left(\int_0^{1/\lambda}(t\lambda(1-\log t)^{\gamma})^p\frac{dt}{t}+\frac{(1+\log\lambda)^{\gamma p+1}-1}{\gamma p+1}\right)^{1/p}\|a\|_A.\end{aligned}$$

Moreover

$$\lambda^p\int_0^{1/\lambda}t^p(1-\log t)^{\gamma p}\frac{dt}{t}\lesssim(1+\log\lambda)^{\gamma p}.$$

Whence

$$\|a\|_{(A,\lambda A)_{(0,-\gamma),p}}\sim(1+\log\lambda)^{\gamma+1/p}\|a\|_A.$$

□

Lemma 8.12. *Let $1\le p\le\infty, 1<q<\infty, 1/q+1/q'=1$ and $b>1/q'$. Then $B^{d/p,b}_{p,q}(\mathbb{T}^d)\hookrightarrow C(\mathbb{T}^d)$.*

Proof. Let (φ_j) be the functions given by (5.9). For any $f\in B^{d/p,b}_{p,q}(\mathbb{T}^d)$, the function $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ is a trigonometric polynomial of order less than or equal to 2^{j+1} (see Remark 5.3). Hence, according to Nikol'skiĭ inequality [107, Remark 3.3.2/2] and Hölder's inequality we obtain

$$\begin{aligned}\|f\|_{C(\mathbb{T}^d)} &\le\sum_{j=0}^{\infty}\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)\|_{C(\mathbb{T}^d)}\lesssim\sum_{j=0}^{\infty}2^{jd/p}\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)\|_{L_p(\mathbb{T}^d)}\\ &\le\left(\sum_{j=0}^{\infty}[2^{jd/p}(1+j)^b\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)\|_{L_p(\mathbb{T}^d)}]^q\right)^{1/q}\left(\sum_{j=0}^{\infty}(1+j)^{-bq'}\right)^{1/q'}\\ &\lesssim\|f\|_{B^{d/p,b}_{p,q}(\mathbb{T}^d)}.\end{aligned}$$

□

Remark 8.1. For function spaces defined over $\mathbb{R}^d$, embedding $B^{d/p,b}_{p,q}\hookrightarrow C$ has been proved by Kalyabin [82, Theorem 2] and Caetano and Moura [21, Proposition 3.13(i)].

Theorem 8.13. *Let $1<p<\infty, 1/p+1/p'=1$ and $b\in\mathbb{R}$. Then a necessary and sufficient condition for the embedding $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))\hookrightarrow C(\mathbb{T}^d)$ to hold is that $b>1/p'$.*

Proof. The condition is sufficient:

By Theorem 8.5, $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))=(B^{d/p}_{p,p}(\mathbb{T}^d),B^{d/p}_{p_0,p}(\mathbb{T}^d))_{(0,-b+1/p),p}$ where $p_0=(1/p-2^{-j_0}/d)^{-1}$. On the other hand, according to [107, Theorem 3.5.5(i)], we know that

$B^{d/p}_{p,p}(\mathbb{T}^d) \hookrightarrow B^0_{\infty,p}(\mathbb{T}^d)$ and $B^{d/p}_{p_0,p}(\mathbb{T}^d) \hookrightarrow B^{2^{-j_0}}_{\infty,p}(\mathbb{T}^d)$. Therefore, we get $B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d)) \hookrightarrow (B^0_{\infty,p}(\mathbb{T}^d), B^{2^{-j_0}}_{\infty,p}(\mathbb{T}^d))_{(0,-b+1/p),p}$.

Lemmata 8.4 and 8.11 yield that

$$\begin{aligned}(\ell_p(L_\infty(\mathbb{T}^d)), \ell_p(2^{k2^{-j_0}} L_\infty(\mathbb{T}^d)))_{(0,-b+1/p),p} \\ &= \ell_p((L_\infty(\mathbb{T}^d), 2^{k2^{-j_0}} L_\infty(\mathbb{T}^d))_{(0,-b+1/p),p}) \\ &= \ell_p((1+k)^b L_\infty(\mathbb{T}^d)).\end{aligned}$$

Whence, by the property of retraction of Besov spaces, we obtain that

$$(B^0_{\infty,p}(\mathbb{T}^d), B^{2^{-j_0}}_{\infty,p}(\mathbb{T}^d))_{(0,-b+1/p),p} = B^{0,b}_{\infty,p}(\mathbb{T}^d).$$

Finally, by Lemma 8.12, we derive that

$$B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d)) \hookrightarrow B^{0,b}_{\infty,p}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d).$$

The condition is necessary:

Since $(1+k)^{-1/p'} \notin \ell_{p'}$, we can find a sequence a positive numbers (z_k) such that

$$\sum_{k=0}^{\infty} z_k = \infty \text{ and } \sum_{k=0}^{\infty} z_k^p (1+k)^{p/p'} < \infty. \tag{8.8}$$

Consider the Féjer's kernels $F_n(u) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{iju}$ and put

$$K_n(u) = \frac{1}{n+1} F_n(u) = \left(\frac{\sin\left(\frac{n+1}{2}u\right)}{(n+1)\sin\left(\frac{1}{2}u\right)} \right)^2, u \in \mathbb{T},$$

(see [97, p. 86]) which are trigonometric polynomials with order n. Moreover, $\|K_n\|_{L_p(\mathbb{T})} \lesssim n^{-1/p}$. Let R_n be the trigonometric polynomial in $\mathbb{T}^d$ given by

$$R_n(x) = \prod_{j=1}^{d} K_{2^n}(x_j), x = (x_1, ..., x_d) \in \mathbb{T}^d,$$

and consider the function $f(x) = \sum_{k=0}^{\infty} z_k R_k(x)$. We have

$$\|f\|_{L_p(\mathbb{T}^d)} \le \sum_{k=0}^{\infty} z_k \|R_k\|_{L_p(\mathbb{T}^d)} \lesssim \sum_{k=0}^{\infty} z_k 2^{-dk/p} < \infty.$$

So $f \in L_p(\mathbb{T}^d)$. In fact, f belongs to $B^{d/p,1/p'}_{p,p}(\mathbb{T}^d)$ because, according to [51, Corollary 7.1(i)], (2.6) and (8.8), we get

$$\begin{aligned}\|f\|_{B^{d/p,1/p'}_{p,p}(\mathbb{T}^d)} &\lesssim \|f\|_{L_p(\mathbb{T}^d)} + \left(\sum_{k=0}^{\infty}(2^{kd/p}(1+k)^{1/p'} z_k \|R_k\|_{L_p(\mathbb{T}^d)})^p\right)^{1/p} \\ &\lesssim \|f\|_{L_p(\mathbb{T}^d)} + \left(\sum_{k=0}^{\infty}(1+k)^{p/p'} z_k^p\right)^{1/p} < \infty.\end{aligned}$$

By Theorem 8.10(ii), we derive that $f \in B^{d/p}_p(L_p(\log L)_{1/p'}(\mathbb{T}^d))$. However f does not belong to $C(\mathbb{T}^d)$ since

$$f(0) = \sum_{k=0}^{\infty} z_k R_k(0) = \sum_{k=0}^{\infty} z_k = \infty.$$

This establishes that the condition is necessary because for any $b \leq 1/p'$ we have that $B^{d/p}_p(L_p(\log L)_{1/p'}(\mathbb{T}^d)) \subseteq B^{d/p}_p(L_p(\log L)_b(\mathbb{T}^d))$. □

Chapter 9

Equivalent characterizations of $\mathbf{B}^{0,b}_{p,q}$

There are many characterizations of classical Besov spaces $\mathbf{B}^{s}_{p,q}(\mathbb{R}^d)$ given by the modulus of smoothness in terms of other means. For instance, spaces $\mathbf{B}^{s}_{p,q}(\mathbb{R}^d)$ can be described in terms of the Fourier transform, heat kernels, wavelets or characterizations by approximation. See the monographs by Triebel [116, 117, 118, 121].

In this chapter we work with Besov spaces $\mathbf{B}^{0,b}_{p,q}$ defined on $\mathbb{R}^d$, with logarithmic smoothness, and show several equivalent characterizations.

As it was shown in Section 5.2, the situation when $s = 0$ is tricky because equivalent approaches in the classical setting differ in this limit case. As a consequence, in some characterizations, a new ingredient appears: an additional truncated Littlewood-Paley construction.

The plan of the chapter is as follows. In Section 9.1 we show another characterization of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ by differences. In particular, we derive that spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ coincide with those introduced by Besov [11] for $b = 0$. In Section 9.2 we obtain more results on the structure of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. In Sections 9.3, 9.4 and 9.5 we deal with characterizations of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ in terms of Fourier-analytical decompositions, wavelets and semi-groups of operators, respectively. From the abstract results on semi-groups we derive characterizations in terms of the heat kernels and the Cauchy-Poisson semi-group. We also continue studying the relationships between $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ and their Fourier-analytically defined counterparts $B^{0,b}_{p,q}(\mathbb{R}^d)$ given in Section 5.2. Moreover we discuss the structural differences of diverse quasi-norms of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ and their counterparts in $B^{s}_{p,q}(\mathbb{R}^d)$ and $B^{0,b}_{p,q}(\mathbb{R}^d)$.

It is worthwhile mentioning that in some results of this chapter, the extreme value $b = -1/q$ and the cases $p = 1, \infty$ are not considered. The reason is that $b = -1/q$ may give rise to jumps in the scale and $p = 1, \infty$ sometimes require different type of arguments (see Chapters 5 and 6).

The main results of this chapter form the paper [31].

9.1 Characterization of $\mathrm{B}^{0,b}_{p,q}$ by differences

Let $0 < \alpha < k \in \mathbb{N}$. It is well known that the quasi-norms

$$\|f\|_{\mathbf{B}^{\alpha}_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (t^{-\alpha}\omega_k(f,t)_p)^q \frac{dt}{t}\right)^{1/q} \tag{9.1}$$

are equivalent on $\mathbf{B}^{\alpha}_{p,q}(\mathbb{R}^d)$. The corresponding result on Besov spaces involving only logarithmic smoothness $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ has been proved in Theorem 5.15(a).

According to [117, Theorem 2.5.12], the quasi-norm (9.1) is equivalent to

$$\|f\|^{(k)}_{\mathbf{B}^{\alpha}_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} |h|^{-\alpha q} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)} \frac{dh}{|h|^d}\right)^{1/q}. \tag{9.2}$$

The next result shows the characterization by differences in $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ corresponding to (9.2).

Theorem 9.1. *Let $1 \le p \le \infty$, $0 < q \le \infty$, $b \ge -1/q$ and $k \in \mathbb{N}$. Then*

$$\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} (1-\log|h|)^{bq} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)} \frac{dh}{|h|^d}\right)^{1/q}$$

is an equivalent quasi-norm on $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. We shall use that

$$\omega_k(f,t)_p \sim \left(t^{-d}\int_{|h|\le t} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)}\, dh\right)^{1/q}$$

(see [85, (2.4) and Appendix A]). We have

$$\begin{aligned}
\|f\|_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 \left[(1-\log t)^b \omega_k(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (1-\log t)^{bq} t^{-d} \int_{|h|\le t} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)}\, dh \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)} \int_{|h|}^1 t^{-d}(1-\log t)^{bq} \frac{dt}{t}\, dh\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\le 1} \|\Delta^k_h f\|^q_{L_p(\mathbb{R}^d)} |h|^{-d}(1-\log|h|)^{bq}\, dh\right)^{1/q} \\
&= \|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\leq \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_{|h|\leq 1} (1-\log|h|)^{bq}\omega_k(f,|h|)^q_p \,\frac{dh}{|h|^d}\right)^{1/q} \\
&= \|f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{j=0}^{\infty}\int_{2^{-j-1}<|h|\leq 2^{-j}} (1-\log|h|)^{bq}\omega_k(f,|h|)^q_p \,\frac{dh}{|h|^d}\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{j=0}^{\infty}(1+j)^{bq}\omega_k(f,2^{-j})^q_p\right)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (1-\log t)^{bq}\omega_k(f,t)^q_p \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

□

Remark 9.1. The special case of the semi-quasi-norm

$$\left(\int_{|h|\leq 1} (1-\log|h|)^{bq}\left\|\Delta^k_h f\right\|^q_{L_p(\mathbb{R}^d)} \frac{dh}{|h|^d}\right)^{1/q}$$

when $b=0$ and $k=1$ has been used by Besov [11].

9.2 Characterization of $\mathrm{B}^{0,b}_{p,q}$ by approximation and limiting interpolation

The structure of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ as approximation space will be useful. We describe it now.

Let $G_0 = \{0\}$ and for $n \in \mathbb{N}$ put

$$G_n = \{g \in L_p(\mathbb{R}^d) : \text{supp } \mathcal{F}g \subseteq \{x : |x| \leq n\}\}. \tag{9.3}$$

So if $g \in G_n$ then g is an entire analytic function of exponential type n. Given $f \in L_p(\mathbb{R}^d)$ and $n \in \mathbb{N}$, let $E_n(f)_p = E_n(f; L_p(\mathbb{R}^d))$ be the n-th approximation error of f by terms of G_n. For $1 \leq p \leq \infty$, $\alpha > 0$, $0 < q \leq \infty$ and $-\infty < b < \infty$, a classical result in approximation theory states that

$$(L_p(\mathbb{R}^d))^{\alpha}_q = (L_p(\mathbb{R}^d), G_n)^{\alpha}_q = \mathbf{B}^{\alpha}_{p,q}(\mathbb{R}^d) \tag{9.4}$$

with equivalence of quasi-norms (see [97, 5.6], [116, Theorem 2.5.4] and [117, Theorem 2.5.3]).

The following result can be proved using ideas of [97, 5.6] and [116, 2.5.4] but also by means of interpolation as we do.

Lemma 9.2. *Let $1 \le p \le \infty, 0 < q \le \infty$ and $b \ge -1/q$. Then we have*

$$(L_p(\mathbb{R}^d))_q^{(0,b)} = (L_p(\mathbb{R}^d), G_n)_q^{(0,b)} = \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d).$$

Proof. Take any $0 < \alpha < 1$. By (9.4), we know that $(L_p(\mathbb{R}^d))_p^\alpha = \mathbf{B}^\alpha_{p,p}(\mathbb{R}^d)$. Whence, using (3.4) and Lemma 2.2(b), we derive

$$\begin{aligned}(L_p(\mathbb{R}^d))_q^{(0,b)} &= (L_p(\mathbb{R}^d), (L_p(\mathbb{R}^d))_p^\alpha)_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d), \mathbf{B}^\alpha_{p,p}(\mathbb{R}^d))_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d), (L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{\alpha,p})_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{(0,-b),q}\\ &= \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)\end{aligned}$$

where the last equality follows from Theorem 5.15(b). □

In the proof of the previous lemma we have used that $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ coincides with $(L_p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d))_{(0,-b),q}$ (see Theorem 5.15(b)). We can show now that this result can be extended from Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ to any fractional Sobolev space $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, any Triebel-Lizorkin space $F^s_{p,u}(\mathbb{R}^d)$, or any Besov space $\mathbf{B}^s_{p,u}(\mathbb{R}^d)$. We refer to the books by Triebel [116, 117] for more information about spaces $F^s_{p,u}(\mathbb{R}^d)$.

Theorem 9.3. *Let $s > 0$, $1 \le p < \infty$, $0 < u \le \infty$, $0 < q \le \infty$ and $b \ge -1/q$. For $A^s_{p,u}(\mathbb{R}^d) = F^s_{p,u}(\mathbb{R}^d)$ or $\mathbf{B}^s_{p,u}(\mathbb{R}^d)$ we have with equivalence of quasi-norms $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), A^s_{p,u}(\mathbb{R}^d))_{(0,-b),q}$.*

Proof. Using (5.13), (9.4), (3.4) and Lemma 9.2, we derive

$$\begin{aligned}(L_p(\mathbb{R}^d), A^s_{p,u}(\mathbb{R}^d))_{(0,-b),q} &\hookrightarrow (L_p(\mathbb{R}^d), \mathbf{B}^s_{p,\max(p,u)}(\mathbb{R}^d))_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d), (L_p(\mathbb{R}^d))^s_{\max(p,u)})_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d))_q^{(0,b)}\\ &= \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d).\end{aligned}$$

For the converse embedding, we obtain

$$\begin{aligned}\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) &= (L_p(\mathbb{R}^d))_q^{(0,b)}\\ &= (L_p(\mathbb{R}^d), (L_p(\mathbb{R}^d))^s_{\min(p,u)})_{(0,-b),q}\\ &= (L_p(\mathbb{R}^d), \mathbf{B}^s_{p,\min(p,u)}(\mathbb{R}^d))_{(0,-b),q}\\ &\hookrightarrow (L_p(\mathbb{R}^d), A^s_{p,u}(\mathbb{R}^d))_{(0,-b),q}.\end{aligned}$$

□

9.3 Fourier-analytical decomposition of $\mathbf{B}^{0,b}_{p,q}$

Let (φ_j) be the smooth dyadic resolution of unity given in (5.9). Recall that (see (5.10)), for $1 \le p \le \infty$, $0 < q \le \infty$ and $b \in \mathbb{R}$, the Besov spaces $B^{0,b}_{p,q}(\mathbb{R}^d)$ are defined by

$$B^{0,b}_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^{0,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} ((1+j)^b \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L_p(\mathbb{R}^d)})^q \right)^{1/q} < \infty \right\}.$$

Spaces $B^{0,b}_{p,q}(\mathbb{R}^d)$ also have logarithmic smoothness but they are different from $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ although they are closely related (see Theorems 5.16 and 5.18). The characterization of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ in terms of Fourier-analytical decompositions has not been studied yet, even for the classical space $\mathbf{B}^{0}_{p,q}(\mathbb{R}^d)$. This will be done in this section with the help of the limiting real method for $\theta = 0$, but first we need some auxiliary results. The first one refers to interpolation of couples of vector-valued spaces $L_p(\mathbb{R}^d, A)$ which is formed by the collection of all strongly measurable functions f such that

$$\|f\|_{L_p(\mathbb{R}^d,A)} = \left(\int_{\mathbb{R}^d} \|f(x)\|_A^p dx \right)^{1/p} < \infty.$$

Lemma 9.4. *Let A_0, A_1 be Banach spaces with $A_1 \hookrightarrow A_0$. Let $1 \le p < \infty$ and $b \in \mathbb{R}$ with $b \ge -1/p$. Then we have with equivalence of norms*

$$(L_p(\mathbb{R}^d, A_0), L_p(\mathbb{R}^d, A_1))_{(0,-b),p} = L_p(\mathbb{R}^d, (A_0, A_1)_{(0,-b),p}).$$

Proof. Consider the collection $\mathcal{S}$ of all functions $v(x) = \sum_{j=1}^N a_j \chi_{\Omega_j}(x)$, where $N \in \mathbb{N}, a_j \in A_1$, the measure of Ω_j is finite and $\Omega_j \cap \Omega_k = \varnothing$ if $j \ne k$. As usual, χ_Ω denotes the characteristic function of $\Omega \subset \mathbb{R}^d$. Then the set $\mathcal{S}$ is dense in $(L_p(\mathbb{R}^d, A_0), L_p(\mathbb{R}^d, A_1))_{(0,-b),p}$ and in $L_p(\mathbb{R}^d, (A_0, A_1)_{(0,-b),p})$. Indeed, take $\tau < -1/p$. It turns out that

$$\begin{aligned}(L_p(\mathbb{R}^d, A_0), L_p(\mathbb{R}^d, A_1))_{(0,-b),p} &= (L_p(\mathbb{R}^d, A_0), L_p(\mathbb{R}^d, A_1))_{0,p,(-b,-\tau)} \\ &= (L_p(\mathbb{R}^d, A_1), L_p(\mathbb{R}^d, A_0))_{1,p,(-\tau,-b)}.\end{aligned}$$

Consequently, since $\tau + 1/p < 0 \le b + 1/p$, it follows from [47, Corollary 3.7] that $L_p(\mathbb{R}^d, A_1)$ is dense in $(L_p(\mathbb{R}^d, A_0), L_p(\mathbb{R}^d, A_1))_{(0,-b),p}$ and $L_p(\mathbb{R}^d, (A_0, A_1)_{(0,-b),p})$, then so is $\mathcal{S}$.

Let $v \in S$. We make use of the K_p-functional (5.16) to obtain that

$$
\begin{aligned}
&\|v\|^p_{(L_p(\mathbb{R}^d,A_0),L_p(\mathbb{R}^d,A_1))_{(0,-b),p}} \\
&\quad\sim \int_0^1 \Big((1-\log t)^b \inf_{\substack{v=v_0+v_1\\ v_j\in L_p(\mathbb{R}^d,A_j)}} \{\|v_0\|^p_{L_p(\mathbb{R}^d,A_0)} + t^p\|v_1\|^p_{L_p(\mathbb{R}^d,A_1)}\}^{1/p}\Big)^p \frac{dt}{t} \\
&\quad= \int_0^1 (1-\log t)^{bp} \int_{\mathbb{R}^d} \inf_{\substack{v(x)=v_0(x)+v_1(x)\\ v_j(x)\in A_j}} (\|v_0(x)\|^p_{A_0} + t^p\|v_1(x)\|^p_{A_1}) dx \frac{dt}{t} \\
&\quad= \int_{\mathbb{R}^d}\int_0^1 (1-\log t)^{bp} K_p(t,v(x);A_0,A_1)^p \frac{dt}{t} dx \\
&\quad\sim \int_{\mathbb{R}^d} \|v(x)\|^p_{(A_0,A_1)_{(0,-b),p}} dx \\
&\quad= \|v\|^p_{L_p(\mathbb{R}^d,(A_0,A_1)_{(0,-b),p})}.
\end{aligned}
$$

This completes the proof. □

Next consider the sequence space ℓ_2 on $\mathbb{N}_0$ and for $\lambda > 0$ write

$$
\ell_2^\lambda = \Big\{\xi = (\xi_j)_{j\in\mathbb{N}_0} : \|\xi\|_{\ell_2^\lambda} = \Big(\sum_{j=0}^\infty (2^{\lambda j}|\xi_j|)^2\Big)^{1/2} < \infty\Big\}.
$$

If A is a Banach space, the vector-valued sequence space $\ell_2^\lambda(A)$ is defined by

$$
\ell_2^\lambda(A) = \Big\{x = (x_j)_{j\in\mathbb{N}_0} \subset A : \|x\|_{\ell_2^\lambda(A)} = \Big(\sum_{j=0}^\infty (2^{\lambda j}\|x_j\|_A)^2\Big)^{1/2} < \infty\Big\}.
$$

Lemma 9.5. *Let $0 < q < \infty$ and $b \in \mathbb{R}$. Then we have with equivalence of quasi-norms*

$$
(\ell_2,\ell_2^\lambda)_{(0,-b),q} = \Big\{\xi = (\xi_j) : \|\xi\| = \Big(\sum_{j=0}^\infty \Big[(1+j)^b\Big(\sum_{\nu=j}^\infty |\xi_\nu|^2\Big)^{1/2}\Big]^q\Big)^{1/q} < \infty\Big\}
$$

and $\|\cdot\|$ is an equivalent quasi-norm on $(\ell_2,\ell_2^\lambda)_{(0,-b),q}$.

Proof. Consider first the case $\lambda = 1$. Since

$$
K_2(t,\xi;\ell_2,\ell_2^1) \sim \Bigg(\sum_{\nu=0}^\infty (\min(1,t2^\nu)|\xi_\nu|)^2\Bigg)^{1/2},
$$

we have for $j \geq 0$ that

$$
K_2(2^{-j},\xi;\ell_2,\ell_2^1) \sim \Big(\sum_{\nu=0}^j (2^{\nu-j}|\xi_\nu|)^2\Big)^{1/2} + \Big(\sum_{\nu=j+1}^\infty |\xi_\nu|^2\Big)^{1/2}.
$$

Hence

$$
\begin{aligned}
\|\xi\|_{(\ell_2,\ell_2^1)_{(0,-b),q}} &\sim \Big(\sum_{j=0}^{\infty}[(1+j)^b K_2(2^{-j},\xi)]^q\Big)^{1/q} \\
&\sim \left(\sum_{j=0}^{\infty}\left[(1+j)^b\Big(\sum_{\nu=0}^{j}(2^{\nu-j}|\xi_\nu|)^2\Big)^{1/2}\right]^q\right)^{1/q} \\
&\qquad + \left(\sum_{j=0}^{\infty}\left[(1+j)^b\Big(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\Big)^{1/2}\right]^q\right)^{1/q}.
\end{aligned}
$$

In the last expression, the first term is dominated by the second term. Indeed, if $q/2 \le 1$, we obtain

$$
\begin{aligned}
&\Big(\sum_{j=0}^{\infty}\Big[(1+j)^{2b}\sum_{\nu=0}^{j}(2^{\nu-j}|\xi_\nu|)^2\Big]^{q/2}\Big)^{1/q} \\
&\qquad \le \Big(\sum_{j=0}^{\infty}(1+j)^{qb}2^{-qj}\sum_{\nu=0}^{j}2^{q\nu}|\xi_\nu|^q\Big)^{1/q} \\
&\qquad = \Big(\sum_{\nu=0}^{\infty}2^{q\nu}|\xi_\nu|^q\sum_{j=\nu}^{\infty}(1+j)^{qb}2^{-qj}\Big)^{1/q} \\
&\qquad \lesssim \Big(\sum_{\nu=0}^{\infty}(1+\nu)^{qb}|\xi_\nu|^q\Big)^{1/q} \\
&\qquad \le \left(\sum_{j=0}^{\infty}\left[(1+j)^b\Big(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\Big)^{1/2}\right]^q\right)^{1/q}.
\end{aligned}
$$

If $1 < q/2$ we use the Hardy's inequality given in Lemma 7.1. We get

$$
\begin{aligned}
&\left(\sum_{j=0}^{\infty}\left[(1+j)^b 2^{-j}\Big(\sum_{\nu=0}^{j}(2^{\nu}|\xi_\nu|)^2\Big)^{1/2}\right]^q\right)^{1/q} \\
&\qquad = \Big(\sum_{j=0}^{\infty}\Big[(1+j)^{2b}2^{-2j}\sum_{\nu=0}^{j}(2^{\nu}|\xi_\nu|)^2\Big]^{q/2}\Big)^{1/q} \\
&\qquad \lesssim \Big(\sum_{j=0}^{\infty}[(1+j)^b 2^{-j}2^{j}|\xi_j|]^q\Big)^{1/q} \\
&\qquad \le \left(\sum_{j=0}^{\infty}\left[(1+j)^b\Big(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\Big)^{1/2}\right]^q\right)^{1/q}.
\end{aligned}
$$

Consequently,

$$
\|\xi\|_{(\ell_2,\ell_2^1)_{(0,-b),q}} \sim \left(\sum_{j=0}^{\infty}\left[(1+j)^b\Big(\sum_{\nu=j}^{\infty}|\xi_\nu|^2\Big)^{1/2}\right]^q\right)^{1/q}.
$$

The general case $0 < \lambda$ follows by appropiate modifications. $\square$

Remark 9.2. Note that if A is a Banach space then a similar argument gives

$$\begin{aligned}(\ell_2(A), \ell_2^\lambda(A))_{(0,-b),q} \\ = \Big\{x = (x_j) : \|x\| = \Big(\sum_{j=0}^\infty \Big[(1+j)^b \Big(\sum_{\nu=j}^\infty \|x_\nu\|_A^2\Big)^{1/2}\Big]^q\Big)^{1/q} < \infty\Big\}.\end{aligned}$$

We are ready to establish the Fourier-analytical description of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. We start with the diagonal case where $p = q$.

Theorem 9.6. *Let $1 < p < \infty$ and $b \geq -1/p$. Then $f \in L_p(\mathbb{R}^d)$ belongs to $\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)$ if and only if*

$$\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)} = \left(\sum_{j=0}^\infty (1+j)^{bp} \left\|\left(\sum_{\nu=j}^\infty |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p} < \infty.$$

Moreover, $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)}$ is an equivalent norm on $\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)$.

In particular, if $b = 0$ we obtain

$$\mathbf{B}^0_{p,p}(\mathbb{R}^d) = \left\{f \in L_p(\mathbb{R}^d) : \|f\|^{\varphi+}_{\mathbf{B}^0_{p,p}(\mathbb{R}^d)} = \left(\sum_{j=0}^\infty \left\|\left(\sum_{\nu=j}^\infty |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p} < \infty\right\}.$$

Proof. We know that $L_p(\mathbb{R}^d) = F^0_{p,2}(\mathbb{R}^d)$ is a retract of $L_p(\mathbb{R}^d, \ell_2)$ and that $W^{1,p}(\mathbb{R}^d) = F^1_{p,2}(\mathbb{R}^d)$ is a retract of $L_p(\mathbb{R}^d, \ell_2^1)$, the corresponding coretraction operator being $\mathfrak{J} f = (\mathcal{F}^{-1}(\varphi_j \mathcal{F} f))$ (see [116, p. 185]). For the vector-valued $L_p(\mathbb{R}^d)$-spaces, using Lemmata 9.4 and 9.5, we obtain

$$\begin{aligned}(L_p(\mathbb{R}^d, \ell_2), L_p(\mathbb{R}^d, \ell_2^1))_{(0,-b),p} = L_p(\mathbb{R}^d, (\ell_2, \ell_2^1)_{(0,-b),p}) \\ = \left\{f = (f_j) : \|f\| = \left\|\left(\sum_{j=0}^\infty (1+j)^{bp} \left(\sum_{\nu=j}^\infty |f_\nu(x)|^2\right)^{p/2}\right)^{1/p}\right\|_{L_p(\mathbb{R}^d)} < \infty\right\}.\end{aligned}$$

Since $\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{(0,-b),p}$ (see Theorem 5.15(b)), and

$$\begin{aligned}\|\mathfrak{J} f\|_{L_p(\mathbb{R}^d, (\ell_2, \ell_2^1)_{(0,-b),p})} \\ = \left(\sum_{j=0}^\infty (1+j)^{bp} \int_{\mathbb{R}^d} \Big(\sum_{\nu=j}^\infty |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(x)|^2\Big)^{p/2} dx\right)^{1/p} \\ = \left(\sum_{j=0}^\infty (1+j)^{bp} \left\|\left(\sum_{\nu=j}^\infty |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p},\end{aligned}$$

the wanted characterization follows from [116, Theorem 1.2.4]. □

Remark 9.3. It is shown in Corollary 5.17 that

$$\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d) = B^{0,b+1/2}_{2,2}(\mathbb{R}^d) \text{ if } b > -1/2.$$

In the limit case $b = -1/2$, we have proved in Corollary 5.19 that

$$\mathbf{B}^{0,-1/2}_{2,2}(\mathbb{R}^d) = B^{0,0,1/2}_{2,2}(\mathbb{R}^d).$$

Using Theorem 9.6 we can see clearly the reason for this jump in the scale: It is owing to the asymptotic behaviour of $d_\nu = \sum_{j=0}^{\nu}(1+j)^{2b}$ which behaves as $(1+\nu)^{2b+1}$ if $b > -1/2$ and as $\log(1+\nu)$ if $b = -1/2$. Indeed

$$\begin{aligned}
\|f\|^2_{\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d)} &\sim \sum_{j=0}^{\infty}(1+j)^{2b} \left\| \left(\sum_{\nu=j}^{\infty} |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(\cdot)|^2 \right)^{1/2} \right\|^2_{L_2(\mathbb{R}^d)} \\
&= \sum_{j=0}^{\infty}(1+j)^{2b} \int_{\mathbb{R}^d} \Big(\sum_{\nu=j}^{\infty} |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(x)|^2 \Big) dx \\
&= \int_{\mathbb{R}^d} \Big(\sum_{\nu=0}^{\infty} |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(x)|^2 \sum_{j=0}^{\nu}(1+j)^{2b} \Big) dx \\
&= \sum_{\nu=0}^{\infty} \left(d_\nu^{1/2} \Big(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(x)|^2 dx \Big)^{1/2} \right)^2 \\
&\sim \begin{cases} \|f\|^2_{B^{0,b+1/2}_{2,2}(\mathbb{R}^d)} & \text{if} \quad b > -1/2, \\ \|f\|^2_{B^{0,0,1/2}_{2,2}(\mathbb{R}^d)} & \text{if} \quad b = -1/2. \end{cases}
\end{aligned}$$

Next we study the non-diagonal case $p \neq q$. We work with the vector-valued sequence spaces

$$\ell^s_p(A_j) = \left\{ (a_j)_{j\in\mathbb{N}_0} : \|(a_j)\|_{\ell^s_p(A_j)} = \Big(\sum_{j=0}^{\infty} (2^{js}\|a_j\|_{A_j})^p \Big)^{1/p} < \infty \right\}.$$

Here $s \in \mathbb{R}$ and $(A_j)_{j\in\mathbb{N}_0}$ is a sequence of Banach spaces. It is well-known that for $0 < \theta < 1, -\infty < s_0 \neq s_1 < \infty$ and $0 < q \le \infty$, we have with equivalence of quasi-norms

$$(\ell^{s_0}_p(A_j), \ell^{s_1}_p(A_j))_{\theta,q} = \ell^s_q(A_j) \quad , \quad s = (1-\theta)s_0 + \theta s_1, \tag{9.5}$$

(see [116, Theorem 1.18.2] where (9.5) is proved for $A_j = A, j \in \mathbb{N}_0$; arguments work also in the general case).

Theorem 9.7. *Let $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Then $f \in L_p(\mathbb{R}^d)$ belongs to $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ if and only if*

$$\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} \left[(1+j)^b \left\| \left(\sum_{\nu=j}^{\infty} |\mathcal{F}^{-1}(\varphi_\nu \mathcal{F}f)(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \right]^q \right)^{1/q} < \infty \tag{9.6}$$

(usual modification if $q = \infty$). Furthermore, $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ is an equivalent quasi-norm on $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. Take b_0, b_1 such that $-1/p < b_0 < b + 1/q - 1/p < b_1$. We can find $0 < \theta < 1$ such that

$$b + 1/q = (1-\theta)(b_0 + 1/p) + \theta(b_1 + 1/p). \tag{9.7}$$

With this choice of parameters, Lemma 9.2 and (2.19) yield that

$$\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) = (\mathbf{B}^{0,b_0}_{p,p}(\mathbb{R}^d), \mathbf{B}^{0,b_1}_{p,p}(\mathbb{R}^d))_{\theta,q}. \tag{9.8}$$

Moreover, it follows from Theorem 9.6 and Lemma 9.5 that $\mathbf{B}^{0,b_i}_{p,p}(\mathbb{R}^d)$ is a retract of $L_p(\mathbb{R}^d, (\ell_2, \ell_2^1)_{(0,-b_i),p})$, $i = 0, 1$. Note also that

$$\begin{aligned}
&\|(\mathcal{F}^{-1}(\varphi_j \mathcal{F} f))\|_{L_p(\mathbb{R}^d,(\ell_2,\ell_2^1)_{(0,-b_i),p})} \\
&\qquad \sim \left(\sum_{j=0}^{\infty}(1+j)^{b_i p}\left\|\left(\sum_{\nu=j}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p} \\
&\qquad = \left(\sum_{k=0}^{\infty}\sum_{j=2^k}^{2^{k+1}-1} j^{b_i p}\left\|\left(\sum_{\nu=j-1}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p}.
\end{aligned}$$

If $2^k \le j \le 2^{k+1}-1$ then $j \sim 2^k$ and so $\sum_{j=2^k}^{2^{k+1}-1} j^{b_i p} \sim 2^{k b_i p} 2^k = 2^{k(b_i+1/p)p}$. Whence,

$$\begin{aligned}
&\|(\mathcal{F}^{-1}(\varphi_j \mathcal{F} f))\|_{L_p(\mathbb{R}^d,(\ell_2,\ell_2^1)_{(0,-b_i),p})} \\
&\qquad \sim \left(\sum_{k=0}^{\infty} 2^{k(b_i+1/p)p}\left\|\left(\sum_{\nu=2^k-1}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|^p_{L_p(\mathbb{R}^d)}\right)^{1/p}.
\end{aligned}$$

Write $\mathfrak{U}f = ((\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f))_{\nu \ge 2^k-1})_{k\in\mathbb{N}_0}$ and $A_k = L_p(\mathbb{R}^d, \ell_2)$ for $k \in \mathbb{N}_0$. The previous considerations and Theorem 9.6 show that

$$\mathfrak{U} : \mathbf{B}^{0,b_i}_{p,p}(\mathbb{R}^d) \longrightarrow \ell_p^{b_i+1/p}(A_k)$$

is bounded for $i = 0, 1$. Interpolating this operator and using (9.5), (9.7) and (9.8), we derive that

$$\mathfrak{U} : \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \longrightarrow \ell_q^{b+1/q}(A_k)$$

is also bounded. Therefore one has by (9.6)

$$\begin{aligned}
\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \left(\sum_{k=0}^{\infty}\left[2^{k(b+1/q)}\left\|\left(\sum_{\nu=2^k-1}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)(\cdot)|^2\right)^{1/2}\right\|_{L_p(\mathbb{R}^d)}\right]^q\right)^{1/q} \\
&= \|\mathfrak{U}f\|_{\ell_q^{b+1/q}(A_k)} \\
&\lesssim \|f\|_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

To check the converse inequality, note that using $L_p(\mathbb{R}^d,\ell_2)$-Fourier multipliers and Littlewood-Paley theorem based on (φ_τ) where $\varphi_\tau\varphi_\nu = 0$ if $|\tau-\nu|>1$ one has

$$\Big\| \sum_{\nu=j+1}^{\infty} \mathcal{F}^{-1}(\varphi_\nu \mathcal{F} f)\Big\|_{L_p(\mathbb{R}^d)} \lesssim \Big\| \Big(\sum_{\tau=j}^{\infty} |\mathcal{F}^{-1}(\varphi_\tau \mathcal{F} f)(\cdot)|^2\Big)^{1/2}\Big\|_{L_p(\mathbb{R}^d)}.$$

Hence

$$\begin{aligned}
(\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)})^q &= \sum_{j=0}^{\infty}\Bigg[(1+j)^b \Big\|\Big(\sum_{\nu=j}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)(\cdot)|^2\Big)^{1/2}\Big\|_{L_p(\mathbb{R}^d)}\Bigg]^q\\
&\sim \|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q}\Big\|\Big(\sum_{\nu=2^j}^{\infty}|\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)(\cdot)|^2\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\\
&\gtrsim \|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q}\Big\|\sum_{\nu=2^j+1}^{\infty}\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)\Big\|^q_{L_p(\mathbb{R}^d)}\\
&= \|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q}\Big\|f-\sum_{\nu=0}^{2^j}\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)\Big\|^q_{L_p(\mathbb{R}^d)}.
\end{aligned}$$

Let $\mu_j = 2^{2^j}$. Since

$$\text{supp } \mathcal{F}\Big(\sum_{\nu=0}^{2^j}\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)\Big) = \text{supp } \sum_{\nu=0}^{2^j}\varphi_\nu\mathcal{F}f \subseteq \{x : |x|\le \mu_{j+1}\},$$

we have $\sum_{\nu=0}^{2^j}\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f) \in G_{\mu_{j+1}}$ (see (9.3)) and

$$\Big\|f-\sum_{\nu=0}^{2^j}\mathcal{F}^{-1}(\varphi_\nu\mathcal{F}f)\Big\|_{L_p(\mathbb{R}^d)} \ge \inf_{g\in G_{\mu_{j+2}-1}}\|f-g\|_{L_p(\mathbb{R}^d)} = E_{\mu_{j+2}}(f)_p.$$

Consequently, using [65, Lemma 1] and Lemma 9.2, we derive that

$$\begin{aligned}
\|f\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\gtrsim \Big(\|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty}2^{j(b+1/q)q}E_{\mu_j}(f)^q_p\Big)^{1/q}\\
&\sim \|f\|_{L_p(\mathbb{R}^d)^{(0,b)}_q}\\
&\sim \|f\|_{\mathbf{B}^{0,b}_{p,q}}.
\end{aligned}$$

This completes the proof. $\square$

Remark 9.4. Comparing Theorem 9.7 with the corresponding result for classical Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ (see (5.10)) one observes the additional truncated Littlewood-Paley construction that appears in $\|\cdot\|^{\varphi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$. To provide a better understanding of Theorem 9.7, consider the Banach case $1 \le q \le \infty$. We remark first that one can replace in (5.10)

$$\left|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right| \quad \text{by} \quad \Big(\sum_{\nu=j}^{\infty}\left|\mathcal{F}^{-1}\big(\varphi_\nu \mathcal{F} f\big)(\cdot)\right|^2\Big)^{1/2},$$

obtaining equivalent norms. This follows from

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} 2^{jsq}\left\|\Big(\sum_{k=0}^{\infty}\left|\mathcal{F}^{-1}\big(\varphi_{j+k}\mathcal{F} f\big)(\cdot)\right|^2\Big)^{1/2}\right\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q}\\
&\qquad\le \left(\sum_{j=0}^{\infty} 2^{jsq}\left\|\sum_{k=0}^{\infty}\left|\mathcal{F}^{-1}\big(\varphi_{j+k}\mathcal{F} f\big)(\cdot)\right|\right\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q}\\
&\qquad\le \sum_{k=0}^{\infty} 2^{-ks}\left(\sum_{j=0}^{\infty} 2^{(j+k)sq}\left\|\mathcal{F}^{-1}(\varphi_{j+k}\mathcal{F} f)\right\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q}\\
&\qquad\lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^d)},
\end{aligned}$$

where we used the triangle inequality for weighted $\ell_q(L_p(\mathbb{R}^d))$ spaces and $s > 0$. On the other hand one cannot replace in (9.6)

$$\Big(\sum_{\nu=j}^{\infty}\left|\mathcal{F}^{-1}\big(\varphi_\nu \mathcal{F} f\big)(\cdot)\right|^2\Big)^{1/2} \quad \text{by} \quad \left|\mathcal{F}^{-1}\big(\varphi_j \mathcal{F} f\big)(\cdot)\right|$$

because in the special case $p = q = 2$ and $b > 0$ such a change yields the norm

$$\Big(\sum_{j=0}^{\infty}((1+j)^b\|\mathcal{F}^{-1}(\varphi_j\mathcal{F} f)\|_{L_p(\mathbb{R}^d)})^q\Big)^{1/q}$$

which characterizes $B^{0,b}_{2,2}(\mathbb{R}^d)$ and, as we have seen in Remark 9.3, the space $B^{0,b}_{2,2}(\mathbb{R}^d)$ does not coincide with $\mathbf{B}^{0,b}_{2,2}(\mathbb{R}^d)$ but with $\mathbf{B}^{0,b-1/2}_{2,2}(\mathbb{R}^d)$.

9.4 Characterization of $\mathbf{B}^{0,b}_{p,q}$ in terms of wavelets

We start by collecting some basic notation of wavelets following closely [121, 1.2.1, pp. 13–14]. There one finds further explanations and related references. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all (complex-valued) continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u (inclusively). Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \qquad u \in \mathbb{N}, \tag{9.9}$$

be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v\, dx = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u.$$

Recall that ψ_F is called the *scaling function* (father wavelet) and ψ_M the associated *wavelet* (mother wavelet). The extension of these wavelets from $\mathbb{R}$ to $\mathbb{R}^d$, $2 \le d$, is based on the usual tensor procedure. Let

$$G = (G_1, \dots, G_d) \in G^0 = \{F, M\}^d,$$

which means that G_r is either F or M. Let

$$G = (G_1, \dots, G_d) \in G^j = \{F, M\}^{d*}, \qquad j \in \mathbb{N},$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^d elements, whereas G^j with $j \in \mathbb{N}$ has $2^d - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jd/2} \prod_{r=1}^{d} \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, \quad m \in \mathbb{Z}^d, \tag{9.10}$$

where (now) $j \in \mathbb{N}_0$. We always assume the ψ_F and ψ_M in (9.9) have $L_2(\mathbb{R})$-norm 1. Then

$$\Psi = \left\{ \Psi^j_{G,m} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^d \right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$ and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^d} \lambda^{j,G}_m\, 2^{-jd/2}\, \Psi^j_{G,m}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jd/2} \int_{\mathbb{R}^d} f(x)\, \Psi^j_{G,m}(x)\, dx = 2^{jd/2} \left(f, \Psi^j_{G,m} \right) \tag{9.11}$$

in the corresponding expansion, adapted to our needs, where $2^{-jd/2}\Psi^j_{G,m}$ are uniformly bounded functions (with respect to j and m).

Let

$$1 < p < \infty, 0 < q \le \infty \text{ and } s > 0.$$

The space $b^s_{p,q}$ consists of all sequences

$$\lambda = \left\{ \lambda^{j,G}_m \in \mathbb{C} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^d \right\}$$

such that

$$\|\lambda\|_{b^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^d} \left| \lambda^{j,G}_m \right|^p \right)^{q/p} \right)^{1/q} \tag{9.12}$$

is finite. Let $u > s$. Then $B^s_{p,q}(\mathbb{R}^d)$ consists of all $f \in L_p(\mathbb{R}^d)$ which can be represented as

$$f = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jd/2} \Psi^j_{G,m}, \qquad \lambda \in b^s_{p,q}, \tag{9.13}$$

unconditional convergence being in $L_p(\mathbb{R}^d)$. This representation is unique, with $\lambda^{j,G}_m$ given by (9.11) and

$$I: \quad f \mapsto \Big\{ 2^{jd/2} \big(f, \Psi^j_{G,m} \big) \Big\} \tag{9.14}$$

is an isomorphic map of $B^s_{p,q}(\mathbb{R}^d)$ onto $b^s_{p,q}$. Details and explanations may be found in [121, Theorem 1.20, pp. 15–16].

Let $\chi_{j,m}$ be the characteristic function of the dyadic cube $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^d$ in $\mathbb{R}^d$ with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner. For $s = 0,1$ and $1 < p < \infty$, we write $f^s_{p,2}$ for the space of all sequences $\lambda = (\lambda^{j,G}_m)$ with $j \in \mathbb{N}_0, G \in G^j$ and $m \in \mathbb{Z}^d$ such that

$$\|\lambda\|_{f^s_{p,2}} = \left\| \Big(\sum_{j,G,m} 2^{js2} |\lambda^{j,G}_m \chi_{j,m}(\cdot)|^2 \Big)^{1/2} \right\|_{L_p(\mathbb{R}^d)} < \infty.$$

We put

$$\ell^s_2(\ell_2) = \Big\{ ((\mu^{j,G}_m)_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}})_{j\in\mathbb{N}_0} : \|((\mu^{j,G}_m)_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}})\|_{\ell^s_2(\ell_2)} = \Big(\sum_j 2^{js2} \sum_{G,m} |\mu^{j,G}_m|^2 \Big)^{1/2} < \infty \Big\}.$$

Lemma 9.8. *For $s = 0,1$ and $1 < p < \infty$, the space $f^s_{p,2}$ can be identified with a complemented subspace $\Delta^s_{p,2}$ of $L_p(\mathbb{R}^d, \ell^s_2(\ell_2))$. The projection onto $\Delta^s_{p,2}$ associates to each $h(\cdot) = (h_j(\cdot))_{j\in\mathbb{N}_0} = ((h^{j,G}_m(\cdot))_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}})_{j\in\mathbb{N}_0} \in L_p(\mathbb{R}^d, \ell^s_2(\ell_2))$ the function Ph defined by*

$$Ph(x) = \bigg(\Big(2^{jd} \int_{Q_{j,m}} h^{j,G}_m(y) dy \, \chi_{j,m}(x) \Big)_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}} \bigg)_{j\in\mathbb{N}_0}.$$

Proof. Given any $\lambda \in f^s_{p,2}$, let $R(\lambda)$ be the function defined by $R(\lambda)(x) = (g_j(x))$ where $g_j(x) = (\lambda^{j,G}_m \chi_{j,m}(x))_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}}$. Since

$$\begin{aligned}
\|\lambda\|_{f^s_{p,2}} &= \left\| \left(\sum_{j\in\mathbb{N}_0} 2^{js2} \sum_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}} |\lambda^{j,G}_m|^2 \chi_{j,m}(\cdot) \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\
&= \left\| \left(\sum_{j\in\mathbb{N}_0} 2^{js2} \|g_j(\cdot)\|^2_{\ell_2} \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\
&= \|R(\lambda)\|_{L_p(\mathbb{R}^d, \ell^s_2(\ell_2))}
\end{aligned}$$

we have that $f^s_{p,2}$ is isometric to the subspace $\Delta^s_{p,2} = \{R(\lambda) : \lambda \in f^s_{p,2}\}$ of $L_p(\mathbb{R}^d, \ell^s_2(\ell_2))$. It is easy to check that $Ph = h$ for any $h \in \Delta^s_{p,2}$. Let us show that P is bounded in $L_p(\mathbb{R}^d, \ell^s_2(\ell_2))$. We have that

$$2^{jd}\int_{Q_{j,m}} |h^{j,G}_m(y)|dy\, \chi_{j,m}(x) \lesssim (\mathcal{M}h^{j,G}_m)(x)\,,\; x \in \mathbb{R}^d,$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Using the vector-valued estimate for $\mathcal{M}$ (see [112, Theorem 1.1.1, p. 51]), we obtain

$$\begin{aligned}
\|Ph\|_{L_p(\mathbb{R}^d,\ell^s_2(\ell_2))} &\le \left\| \Big(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^d}} \Big(2^{jd}\int_{Q_{j,m}} |2^{js}h^{j,G}_m(y)|dy\Big)^2 \chi_{j,m}(\cdot)\Big)^{1/2}\right\|_{L_p(\mathbb{R}^d)} \\
&\lesssim \left\| \Big(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^d}} (\mathcal{M}(2^{js}|h^{j,G}_m|)(\cdot))^2\Big)^{1/2}\right\|_{L_p(\mathbb{R}^d)} \\
&\lesssim \left\| \Big(\sum_{j\in\mathbb{N}_0} \sum_{\substack{G\in G^j\\ m\in\mathbb{Z}^d}} 2^{2js}|h^{j,G}_m(\cdot)|^2\Big)^{1/2}\right\|_{L_p(\mathbb{R}^d)} \\
&= \|h\|_{L_p(\mathbb{R}^d,\ell^s_2(\ell_2))}.
\end{aligned}$$

In addition, this also shows that if $h \in L_p(\mathbb{R}^d, \ell^s_2(\ell_2))$ then $Ph \in \Delta^s_{p,2}$. The proof is completed. □

Lemma 9.9. *Let $1 < p < \infty$ and $b \ge -1/p$. Then $\lambda = (\lambda^{j,G}_m)$ belongs to $(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}$ if and only if*

$$\|\lambda\| = \left(\int_{\mathbb{R}^d} \sum_{j=0}^{\infty} (1+j)^{bp} \Big(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}} |\lambda^{\nu,G}_m|^2 \chi_{\nu,m}(x)\Big)^{p/2} dx \right)^{1/p}$$

is finite. Moreover, $\|\lambda\|$ defines an equivalent norm in $(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}$.

Proof. By Lemmata 9.4, 9.5 and Remark 9.2, we have that

$$(L_p(\mathbb{R}^d, \ell_2(\ell_2)), L_p(\mathbb{R}^d, \ell^1_2(\ell_2)))_{(0,-b),p} = L_p(\mathbb{R}^d, (\ell_2, \ell^1_2)_{(0,-b),p}(\ell_2)).$$

Hence, according to Lemma 9.8 and the theorem on interpolation of complemented subspaces [116, Theorem 1.17.1], we derive that

$$\|\lambda\|_{(f^0_{p,2}, f^1_{p,2})_{(0,-b),p}} \sim \left(\int_{\mathbb{R}^d} \sum_{j=0}^{\infty} (1+j)^{bp} \Big(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}} |\lambda^{\nu,G}_m|^2 \chi_{\nu,m}(x)\Big)^{p/2} dx \right)^{1/p}.$$

□

In what follows we work with the sequence of wavelets $(\Psi^j_{G,m})$ defined in (9.10) with $u > 1$ and the sequence space $\mathbf{b}^{0,b}_{p,q}$ given by the quasi-norm

$$\|\lambda\|_{\mathbf{b}^{0,b}_{p,q}} = \Bigg(\sum_{j=0}^{\infty}(1+j)^{bq}\Big\|\Big(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m|^2\,\chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\Bigg)^{1/q}. \tag{9.15}$$

Here $1 < p < \infty, 0 < q \le \infty$ and $b \ge -1/q$.

Theorem 9.10. *Let $1 < p < \infty$ and $b \ge -1/p$. Then f belongs to $\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)$ if, and only if, it can be represented as $f = \sum_{j,G,m}\lambda^{j,G}_m 2^{-jd/2}\Psi^j_{G,m}$ (unconditional convergence being in $L_p(\mathbb{R}^d)$) with $\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jd/2}(f,\Psi^j_{G,m})$ and*

$$\begin{aligned}\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)} &= \|(\lambda^{j,G}_m)\|_{\mathbf{b}^{0,b}_{p,p}}\\ &= \Bigg(\sum_{j=0}^{\infty}(1+j)^{bp}\Big\|\Big(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^p_{L_p(\mathbb{R}^d)}\Bigg)^{1/p}\end{aligned}$$

is finite. Moreover, $\|\cdot\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)}$ defines an equivalent norm in $\mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)$.

Proof. The unconditional convergence in $L_p(\mathbb{R}^d)$ for any sequence $(\lambda^{j,G}_m) \in \mathbf{b}^{0,b}_{p,p}$ follows from a corresponding assertion for $L_p(\mathbb{R}^d)$ based on $f^0_{p,2}$ according to [121, Theorem 1.20] and $\mathbf{b}^{0,b}_{p,p} \hookrightarrow f^0_{p,2}$ as a consequence of Lemma 9.9.

Let D be the operator defined by

$$D((\lambda^{j,G}_m)) = \sum_{j,G,m}\lambda^{j,G}_m 2^{-jd/2}\Psi^j_{G,m}.$$

According to [121, Theorem 1.20], the restrictions

$$D : f^0_{p,2} \longrightarrow L_p(\mathbb{R}^d) \text{ and } D : f^1_{p,2} \longrightarrow W^{1,p}(\mathbb{R}^d)$$

are isomorphisms. Interpolating and using Theorem 5.15(b) (see also Theorem 9.3), we obtain that

$$D : (f^0_{p,2}, f^1_{p,2})_{(0,-b),p} \longrightarrow (L_p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d))_{(0,-b),p} = \mathbf{B}^{0,b}_{p,p}(\mathbb{R}^d)$$

is also an isomorphism. As for the source space of this operator, by Lemma 9.9, we know that

$$\begin{aligned}&\|(\lambda^{j,G}_m)\|_{(f^0_{p,2},f^1_{p,2})_{(0,-b),p}}\\ &\sim \Bigg(\int_{\mathbb{R}^d}\sum_{j=0}^{\infty}(1+j)^{bp}\Big(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(x)\Big)^{p/2}dx\Bigg)^{1/p}\\ &= \Bigg(\sum_{j=0}^{\infty}(1+j)^{bp}\Big\|\Big(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^p_{L_p(\mathbb{R}^d)}\Bigg)^{1/p}.\end{aligned}$$

Furthermore, $\lambda^{j,G}_m = \lambda^{j,G}_m(f)$ is again covered by [121, Theorem 1.20]. This completes the proof. □

In order to study the case $p \neq q$, we first introduce some notation and we establish an auxiliary result.

For $1 < p < \infty$ and $j \in \mathbb{N}$, let $P_{2^j} : L_p(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ be the operator defined by

$$P_{2^j} f = \sum_{\nu=0}^{j-1} \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^d}} \lambda^{\nu,G}_m(f) 2^{-\nu d/2} \Psi^\nu_{G,m}.$$

As $\{\Psi^\nu_{G,m}\}$ is an unconditional Schauder basis in $L_p(\mathbb{R}^d)$, we have that

$$\sup\{\|P_{2^j}\|_{L_p(\mathbb{R}^d,),L_p(\mathbb{R}^d)} : j \in \mathbb{N}\} < \infty.$$

Let $V_0 = \{0\}$ and for $n = 2, 3, \ldots$ with $2^m \leq n < 2^{m+1}, m \in \mathbb{N}$, put

$$V_{n-1} = P_{2^m}(L_p(\mathbb{R}^d)) = \Big\{g \in L_p(\mathbb{R}^d) : g = \sum_{\nu=0}^{m-1} \sum_{\substack{G \in G^\nu \\ m \in \mathbb{Z}^d}} c^{\nu,G}_m 2^{-\nu d/2} \Psi^\nu_{G,m} \text{ with } c^{\nu,G}_m \in \mathbb{C}\Big\}.$$

Then $(L_p(\mathbb{R}^d), V_n)$ determines an approximation scheme. Put

$$E^\Psi_n(f)_p = \inf\{\|f - g\|_{L_p(\mathbb{R}^d)} : g \in V_{n-1}\}\,,\ n \in \mathbb{N}\,,$$

and define the space $(L_p(\mathbb{R}^d), V_n)^{(0,b)}_q$ as (2.1)–(2.2) but replacing the sequence $(E_n(f))$ by $(E^\Psi_n(f)_p)$.

Note that

$$E^\Psi_{2^j}(f)_p \sim \|f - P_{2^j} f\|_{L_p(\mathbb{R}^d)} \quad , \quad j \in \mathbb{N}.$$

Indeed, given any $g \in V_{2^j-1}$, we have

$$\begin{aligned}\|f - P_{2^j} f\|_{L_p(\mathbb{R}^d)} &\leq \|f - g\|_{L_p(\mathbb{R}^d)} + \|g - P_{2^j} f\|_{L_p(\mathbb{R}^d)} \\ &= \|f - g\|_{L_p(\mathbb{R}^d)} + \|P_{2^j}(g - f)\|_{L_p(\mathbb{R}^d)} \\ &\lesssim \|f - g\|_{L_p(\mathbb{R}^d)}.\end{aligned}$$

Lemma 9.11. *Let $1 < p < \infty$, $0 < q \leq \infty$ and $b > -1/q$. Then we have with equivalence of norms*

$$(L_p(\mathbb{R}^d), V_n)^{(0,b)}_q = \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d).$$

Proof. We start with the case $1 < p = q < \infty$. Recall that $\mu_j = 2^{2^j}, j \in \mathbb{N}_0$. We have

$$\begin{aligned}
\|f\|^p_{(L_p(\mathbb{R}^d),V_n)^{(0,b)}_p} &\sim \|f\|^p_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty}(2^{j(b+1/p)}E^{\Psi}_{\mu_j}(f)_p)^p \\
&\sim \|f\|^p_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty}(2^{j(b+1/p)}\|f - P_{\mu_j}f\|_{L_p(\mathbb{R}^d)})^p \\
&= \|f\|^p_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty}2^{j(b+1/p)p}\Big\|\sum_{\nu=2^j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}\lambda^{\nu,G}_m(f)2^{-\nu d/2}\Psi^{\nu}_{G,m}\Big\|^p_{L_p(\mathbb{R}^d)} \\
&\sim \|f\|^p_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty}2^{j(b+1/p)p}\Bigg\|\Big(\sum_{\nu=2^j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m(f)|^2\chi_{\nu,m}(\cdot)\Big)^{1/2}\Bigg\|^p_{L_p(\mathbb{R}^d)}
\end{aligned}$$

where we have used [120, Theorem 1.64] (or [121, Theorem 1.20]) in the last equivalence. Now the result follows from Theorem 9.10. Note that the above argument works even if $b = -1/p$.

To establish the remaining case $p \neq q$, choose b_0, b_1 such that $-1/p < b_0 < b+1/q-1/p < b_1$ and take $0 < \theta < 1$ with

$$b + 1/q = (1-\theta)(b_0 + 1/p) + \theta(b_1 + 1/p).$$

According to (9.8), (2.19) and the result just proved for the diagonal case, we obtain that

$$\begin{aligned}
\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) &= (\mathbf{B}^{0,b_0}_{p,p}(\mathbb{R}^d), \mathbf{B}^{0,b_1}_{p,p}(\mathbb{R}^d))_{\theta,q} \\
&= ((L_p(\mathbb{R}^d), V_n)^{(0,b_0)}_p, (L_p(\mathbb{R}^d), V_n)^{(0,b_1)}_p)_{\theta,q} \\
&= (L_p(\mathbb{R}^d), V_n)^{(0,b)}_q.
\end{aligned}$$

□

Theorem 9.12. *Let $1 < p < \infty, 0 < q \leq \infty$ and $b > -1/q$. Then f belongs to $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ if, and only if, it can be represented as $f = \sum_{j,G,m}\lambda^{j,G}_m 2^{-jd/2}\Psi^j_{G,m}$ (unconditional convergence in $L_p(\mathbb{R}^d)$) with $\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jd/2}(f, \Psi^j_{G,m})$ and*

$$\begin{aligned}
\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &= \|(\lambda^{j,G}_m)\|_{\mathbf{b}^{0,b}_{p,q}} \\
&= \Bigg(\sum_{j=0}^{\infty}(1+j)^{bq}\Bigg\|\Big(\sum_{\nu=j}^{\infty}\sum_{\substack{G\in G^\nu\\ m\in\mathbb{Z}^d}}|\lambda^{\nu,G}_m|^2\chi_{\nu,m}(\cdot)\Big)^{1/2}\Bigg\|^q_{L_p(\mathbb{R}^d)}\Bigg)^{1/q} < \infty.
\end{aligned}$$

Moreover, $\|\cdot\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ defines an equivalent quasi-norm in $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. The unconditional convergence in $L_p(\mathbb{R}^d)$ is covered by the related argument at the beginning of the proof of Theorem 9.10 and the above interpolation (9.8).

Using Lemma 9.11, we obtain

$$\begin{aligned}
\|f\|_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \|f\|_{(L_p(\mathbb{R}^d),V_n)^{(0,b)}_q} \\
&\sim \Big(\|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} E^{\Psi}_{\mu_j}(f)^q_p\Big)^{1/q} \\
&\sim \Big(\|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} \|f - P_{\mu_j} f\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q} \\
&\sim \left(\|f\|^q_{L_p(\mathbb{R}^d)} + \sum_{j=0}^{\infty} 2^{j(b+1/q)q} \Big\|\Big(\sum_{\nu=2^j}^{\infty} \sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^d}} |\lambda^{\nu,G}_m|^2 \chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q} \\
&\sim \left(\sum_{j=0}^{\infty} (1+j)^{bq} \Big\|\Big(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^d}} |\lambda^{\nu,G}_m|^2 \chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q}.
\end{aligned}$$

□

Remark 9.5. Comparing Theorem 9.12 with the corresponding result for classical Besov spaces given in (9.11)-(9.14), we observe again an additional truncated Littlewood-Paley-type construction. The corresponding sequence space being $\mathbf{b}^{0,b}_{p,q}$ quasi-normed by (9.15) in contrast to $b^s_{p,q}$ in (9.12).

In order to take a closer look into these sequence spaces, consider the Banach case $1 \le q \le \infty$ and notice that the norm (9.12) of $b^s_{p,q}$ can be rewritten as

$$\|\lambda\|_{b^s_{p,q}} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^j} \Big\|\sum_{m\in\mathbb{Z}^d} |\lambda^{j,G}_m|\, \chi_{j,m}(\cdot)\Big\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q}. \tag{9.16}$$

Then it follows by the same arguments as in Remark 9.4 for some $0 < c_1 < c_2 < \infty$,

$$\begin{aligned}
c_1\, \|\lambda\|_{b^s_{p,q}} &\le \left(\sum_{j=0}^{\infty} 2^{jsq} \Big\|\Big(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G^\nu \\ m\in\mathbb{Z}^d}} |\lambda^{\nu,G}_m|^2\, \chi_{\nu,m}(\cdot)\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\right)^{1/q} \\
&\le c_2\, \|\lambda\|_{b^s_{p,q}}, \qquad \lambda \in b^s_{p,q},
\end{aligned} \tag{9.17}$$

where we used again $s > 0$. Hence one can replace $b^s_{p,q}$ in (9.12) by the middle term in (9.17). Afterwards one can compare $b^s_{p,q}$ with $\mathbf{b}^{0,b}_{p,q}$ according to (9.15).

To continue with the description of relationships between classical and logarithmic spaces, let

$$\big(\lambda^j f\big)(x) = \sum_{\substack{G\in G^j \\ m\in\mathbb{Z}^d}} \big|\lambda^{j,G}_m(f)\big|\, \chi_{j,m}(x)$$

and

$$\left(\lambda^j f\right)_+(x) = \Big(\sum_{\nu=j}^{\infty} \sum_{\substack{G\in G_\nu \\ m\in\mathbb{Z}^d}} \left|\lambda_m^{\nu,G}(f)\right|^2 \chi_{\nu,m}(x)\Big)^{1/2}.$$

Then

$$\|f\|^{\Psi}_{B^s_{p,q}(\mathbb{R}^d)} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\lambda^j f\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q} \tag{9.18}$$

and

$$\|f\|^{\Psi+}_{B^s_{p,q}(\mathbb{R}^d)} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\lambda^j f)_+\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q} \tag{9.19}$$

are equivalent norms in $B^s_{p,q}(\mathbb{R}^d)$. This is covered by (9.12), (9.14) combined with (9.16), (9.17). The norm-generating basic ingredient in the refined norm (9.19) is *monotonically decreasing* in j, in contrast to their original counterpart in (9.18).

If one switches from $B^s_{p,q}(\mathbb{R}^d)$ to their logarithmic counterpart $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$, then as it is shown in Theorem 9.12, the corresponding norm to (9.19) is

$$\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \Big(\sum_{j=0}^{\infty} (1+j)^{bq} \big\|(\lambda^j f)_+\big\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q}.$$

On the other hand, according to [1, Theorem 13] and the comments in [122, 1.3.3, pp. 54–60], the counterparts of (9.18), hence

$$\|f\|^{\Psi}_{B^{0,b}_{p,q}(\mathbb{R}^d)} = \Big(\sum_{j=0}^{\infty} (1+j)^{bq} \|\lambda^j f\|^q_{L_p(\mathbb{R}^d)}\Big)^{1/q}$$

is an equivalent norm in the space $B^{0,b}_{p,q}(\mathbb{R}^d)$ which does not coincide with $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

To finish this section we consider the embeddings

$$B^{0,b+1/\min\{2,p,q\}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}(\mathbb{R}^d).$$

established in Theorem 5.16 for $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. As an application of the characterizations by means of wavelets, we show next two results on the optimality of the embeddings above.

Remark 9.6. Let $1 < p < \infty, 0 < q \le \infty$ and $b > -1/q$. Suppose that $p = \max\{2,p,q\}$. We are going to show that for any $\varepsilon > 0$ we have $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) \not\hookrightarrow B^{0,b+1/p+\varepsilon}_{p,q}(\mathbb{R}^d)$.

Given ε choose β such that $b + 1/q + 1/p < \beta \le b + 1/q + 1/p + \varepsilon$ and put

$$\lambda^{j,G}_m = \begin{cases} 2^{jd/p}(1+j)^{-\beta} & \text{if } j \in \mathbb{N}_0, G = (M,\dots,M) \\ & \text{and } m = (0,\dots,0), \\ 0 & \text{otherwise}. \end{cases}$$

Then $f = \sum_{j,G,m} \lambda^{j,G}_m 2^{-jd/2} \Psi^j_{G,m}$ belongs to $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Indeed, according to Theorem 9.12, it is enough to show that

$$\begin{aligned}&\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}\\ &= \Big(\sum_{j=0}^{\infty} (1+j)^{bq} \Big\| \Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \chi_{\nu,(0,\dots,0)}(\cdot) \Big)^{1/2} \Big\|^q_{L_p(\mathbb{R}^d)} \Big)^{1/q} < \infty.\end{aligned}$$

We claim that if $x \neq 0$ and $j \in \mathbb{N}_0$, then

$$\begin{aligned}&\Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/2}\\ &\qquad\sim \Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^p \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/p} \qquad (9.20)\end{aligned}$$

with constants in the equivalence which are independent of x and j. Indeed, assume $2 \le p$. The case $p < 2$ can be carried out similarly. Since $\ell_2 \hookrightarrow \ell_p$, it is clear that

$$\Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^p \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/p} \le \Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/2}.$$

To check the converse inequality, we distinguish two cases. If $x \notin 2^{-j}(0,1)^d = Q_{j,(0,\dots,0)}$, then $\chi_{\nu,(0,\dots,0)}(x) = 0$ for $\nu \ge j$. So

$$\Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/2} = 0 = \Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^p \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/p}.$$

Suppose now that $x \in 2^{-j}(0,1)^d$ and let $\nu_x \in \mathbb{N}_0$ the bigger value such that $x \in Q_{\nu_x,(0,\dots,0)}$. We have

$$\begin{aligned}\Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/2} &= \Big(\sum_{\nu=j}^{\nu_x} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \Big)^{1/2}\\ &\le \Big(\sum_{\nu=0}^{\nu_x} (2^{\nu d/p}(1+\nu)^{-\beta})^2 \Big)^{1/2}\\ &\lesssim 2^{\nu_x d/p}(1+\nu_x)^{-\beta}\\ &\le \Big(\sum_{\nu=j}^{\nu_x} (2^{\nu d/p}(1+\nu)^{-\beta})^p \Big)^{1/p}\\ &= \Big(\sum_{\nu=j}^{\infty} (2^{\nu d/p}(1+\nu)^{-\beta})^p \chi_{\nu,(0,\dots,0)}(x) \Big)^{1/p}\end{aligned}$$

which establishes (9.20).

Consequently, since $-\beta p+1<0$ and $(b-\beta+1/p)q<-1$, we get

$$\begin{aligned}
\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \bigg(\sum_{j=0}^{\infty}(1+j)^{bq}\Big\|\Big(\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^p\chi_{\nu,(0,\dots,0)}(\cdot)\Big)^{1/p}\Big\|^q_{L_p(\mathbb{R}^d)}\bigg)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{bq}\Big(\int_{\mathbb{R}^d}\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^p\chi_{\nu,(0,\dots,0)}(x)dx\Big)^{q/p}\Big)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{bq}\Big(\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^p 2^{-\nu d}\Big)^{q/p}\Big)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{bq}\Big(\sum_{\nu=j}^{\infty}(1+\nu)^{-\beta p}\Big)^{q/p}\Big)^{1/q}\\
&\sim \Big(\sum_{j=0}^{\infty}(1+j)^{bq}(1+j)^{(-\beta p+1)q/p}\Big)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{(b-\beta+1/p)q}\Big)^{1/q}<\infty.
\end{aligned}$$

Therefore, $f\in\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. But, by our choice of β, we have

$$\begin{aligned}
&\Big(\sum_{j=0}^{\infty}2^{-jdq/p}(1+j)^{(b+1/p+\varepsilon)q}(2^{jd/p}(1+j)^{-\beta})^q\Big)^{1/q}\\
&\qquad\qquad = \Big(\sum_{j=0}^{\infty}(1+j)^{(b+1/p+\varepsilon-\beta)q}\Big)^{1/q}=\infty.
\end{aligned}$$

Hence, according to [1, Theorem 13], we derive that $f\notin B^{0,b+1/p+\varepsilon}_{p,q}(\mathbb{R}^d)$.

Assume this time that $p=\min\{2,p,q\}$. Let us show that for any $\varepsilon>0$ we have $B^{0,b+1/p-\varepsilon}_{p,q}(\mathbb{R}^d)\not\hookrightarrow\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Take β such that $b+1/p+1/q-\varepsilon<\beta\le b+1/p+1/q$ and put as before,

$$\lambda^{j,G}_m=\begin{cases}2^{jd/p}(1+j)^{-\beta} & \text{if } j\in\mathbb{N}_0\,,\ G=(M,\dots,M)\text{ and } m=(0,\dots,0),\\ 0 & \text{otherwise,}\end{cases}$$

and $f=\sum_{j,G,m}\lambda^{j,G}_m 2^{-jd/2}\Psi^j_{G,m}$. Using (9.20) we derive

$$\begin{aligned}
\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} &= \bigg(\sum_{j=0}^{\infty}(1+j)^{bq}\Big\|\Big(\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^2\chi_{\nu,(0,\dots,0)}(\cdot)\Big)^{1/2}\Big\|^q_{L_p(\mathbb{R}^d)}\bigg)^{1/q}\\
&\sim \bigg(\sum_{j=0}^{\infty}(1+j)^{bq}\Big\|\Big(\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^p\chi_{\nu,(0,\dots,0)}(\cdot)\Big)^{1/p}\Big\|^q_{L_p(\mathbb{R}^d)}\bigg)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{bq}\Big(\sum_{\nu=j}^{\infty}(2^{\nu d/p}(1+\nu)^{-\beta})^p 2^{-\nu d}\Big)^{q/p}\Big)^{1/q}\\
&= \Big(\sum_{j=0}^{\infty}(1+j)^{bq}\Big(\sum_{\nu=j}^{\infty}(1+\nu)^{-\beta p}\Big)^{q/p}\Big)^{1/q}.
\end{aligned}$$

This sum is ∞ if $-\beta p + 1 \geq 0$. If $-\beta p + 1 < 0$, we have

$$\|f\|^{\Psi+}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} \sim \Big(\sum_{j=0}^{\infty} (1+j)^{bq} \Big(\sum_{\nu=j}^{\infty} (1+\nu)^{-\beta p} \Big)^{q/p} \Big)^{1/q}$$
$$\sim \Big(\sum_{j=0}^{\infty} (1+j)^{(b-\beta+1/p)q} \Big)^{1/q} = \infty.$$

So, Theorem 9.12 yields that $f \notin \mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. However

$$\Big(\sum_{j=0}^{\infty} 2^{-jdq/p} (1+j)^{(b+1/p-\varepsilon)q} (2^{jd/p}(1+j)^{-\beta})^q \Big)^{1/q}$$
$$= \Big(\sum_{j=0}^{\infty} (1+j)^{(b+1/p-\varepsilon-\beta)q} \Big)^{1/q} < \infty$$

which means, according to [1, Theorem 13], that $f \in B^{0,b+1/p-\varepsilon}_{p,q}(\mathbb{R}^d)$.

9.5 Semi-groups of operators

First we show an abstract result on semi-groups of operators and limiting real interpolation, and then we apply it to heat kernels and spaces $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Let X be a (complex) Banach space and let $\{T(t) : 0 \leq t < \infty\}$ be a family of linear and bounded operators from X into itself. The family $\{T(t)\}_{t\geq 0}$ is called to be a *strongly continuous equi-bounded semi-group* of operators in X if

(i) $T(t+s) = T(t)T(s)\,, \quad t,s \geq 0\,.$

(ii) $T(0) = \mathrm{id} \quad$ (identity in X).

(iii) $\|T(t)f\|_X \leq M\|f\|_X\,, \quad t \geq 0\,,\ f \in X\,.$

(iv) $\lim_{t\to 0+} T(t)f = f\,, \quad f \in X.$

The infinitesimal generator Λ of the semi-group $\{T(t)\}_{t\geq 0}$ is defined by

$$\Lambda f = \lim_{t\downarrow 0} t^{-1}\big(T(t)f - f\big)$$

whenever that the limit exists. The domain $D(\Lambda)$ of Λ consists of all those $f \in X$ for which the limit exists. The domain $D(\Lambda^m)$ of the m-th power of Λ is a Banach space endowed with the norm

$$\|f\|_{D(\Lambda^m)} = \|f\|_X + \|\Lambda^m f\|_X\,, \quad m = 1,2,\ldots\,.$$

The semi-group $\{T(t)\}_{t\geq 0}$ is said to be *analytic* (or *holomorphic*) if in addition to (i)-(iv), it satisfies

(v) $T(t)f \in D(\Lambda)$, for all $f \in X$ and $t > 0$.

(vi) $t\|\Lambda T(t)f\|_X \le N\|f\|_X$, $0 < t < \infty$, $f \in X$.

See [15, 116, 118] for more details on semi-groups of operators.

Consider the following modulus of continuity of order $m \in \mathbb{N}$

$$\overline{\omega}_m(t^m, f) = \sup_{0 \le s \le t} \|[T(s) - \mathrm{id}]^m f\|_X.$$

This modulus is related to the K-functional of the couple $\big(X, D(\Lambda^m)\big)$. Indeed, it is shown in [15, Proposition 3.4.1] that

$$K(t^m, f; X, D(\Lambda^m)) \lesssim \overline{\omega}_m(t^m, f) + \min(1, t^m)\|f\|_X \tag{9.21}$$

and

$$\overline{\omega}_m(t^m, f) \lesssim K(t^m, f; X, D(\Lambda^m)). \tag{9.22}$$

On the other hand, if the semi-group is analytic and we consider the modified K-functional given by

$$\widetilde{K}(t, f) = \widetilde{K}(t, f; X, D(\Lambda^m)) = \inf_{f_1 \in D(\Lambda^m)} \{\|f - f_1\|_X + t\|\Lambda^m f_1\|_X\},$$

then it follows from [53, Theorem 5.1] that

$$\widetilde{K}(t^m, f; X, D(\Lambda^m)) \sim \|[T(t) - \mathrm{id}]^m f\|_X. \tag{9.23}$$

Theorem 9.13. *Let X be a Banach space, let $\{T(t)\}_{t \ge 0}$ be a strongly continuous equi-bounded semi-group of operators in X, let $m \in \mathbb{N}$, $0 < q \le \infty$ and $b \ge -1/q$. The quasi-norm*

$$\|f\|_1 = \|f\|_X + \Big(\int_0^1 \Big((1 - \log t)^b \overline{\omega}_m(t^m, f)\Big)^q \frac{dt}{t}\Big)^{1/q}$$

is equivalent to the interpolation quasi-norm $\|\cdot\|_{(X, D(\Lambda^m))_{(0,-b),q}}$ on $(X, D(\Lambda^m))_{(0,-b),q}$.

In addition, if the semi-group $\{T(t)\}_{t \ge 0}$ is analytic then

$$\|f\|_2 = \|f\|_X + \Big(\int_0^1 \Big((1 - \log t)^b \|[T(t) - \mathrm{id}]^m f\|_X\Big)^q \frac{dt}{t}\Big)^{1/q}$$

is also an equivalent quasi-norm on $(X, D(\Lambda^m))_{(0,-b),q}$.

Proof. Making a change of variable and using (9.21), we obtain

$$\begin{aligned}
\|f\|_{(X,D(\Lambda^m))_{(0,-b),q}} &= \Big(\int_0^1 \Big((1-\log t)^b K(t,f) \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \Big(\int_0^1 \Big((1-\log t)^b K(t^m,f) \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\lesssim \Big(\int_0^1 \Big((1-\log t)^b \overline{\omega}_m(t^m,f) \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\qquad + \Big(\int_0^1 \Big((1-\log t)^b t^m \Big)^q \frac{dt}{t} \Big)^{1/q} \|f\|_X \\
&\sim \|f\|_1 .
\end{aligned}$$

To check the converse inequality, note that $(X, D(\Lambda^m))_{(0,-b),q} \hookrightarrow X$. Moreover, by (9.22), we have

$$\begin{aligned}
\Big(\int_0^1 \Big((1-\log t)^b \overline{\omega}_m(t^m,f) \Big)^q \frac{dt}{t} \Big)^{1/q} &\lesssim \Big(\int_0^1 \Big((1-\log t)^b K(t^m,f) \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\sim \|f\|_{(X,D(\Lambda^m))_{(0,-b),q}} .
\end{aligned}$$

Consequently, $\|f\|_{(X,D(\Lambda^m))_{(0,-b),q}} \sim \|f\|_1$.

Assume now that the semi-group $\{T(t)\}_{t\geq 0}$ is analytic. To complete the proof we first show that

$$K(t,f) \sim t\|f\|_X + \widetilde{K}(t,f)\,, \quad 0 < t \leq 1\,,\ f \in X. \tag{9.24}$$

Indeed, take any $f_1 \in D(\Lambda^m)$. Using the triangle inequality in X and that $t \leq 1$, we obtain

$$\begin{aligned}
K(t,f) &\leq \|f - f_1\|_X + t\|f_1\|_X + t\|\Lambda^m f_1\|_X \\
&\leq 2\|f - f_1\|_X + t\|f\|_X + t\|\Lambda^m f_1\|_X .
\end{aligned}$$

Taking the infimum over all $f_1 \in D(\Lambda^m)$ it follows that $K(t,f) \lesssim t\|f\|_X + \widetilde{K}(t,f)$. Conversely,

$$\begin{aligned}
t\|f\|_X + \widetilde{K}(t,f) &\leq t\|f\|_X + \|f - f_1\|_X + t\|\Lambda^m f_1\|_X \\
&\leq t\|f - f_1\|_X + t\|f_1\|_X + \|f - f_1\|_X + t\|\Lambda^m f_1\|_X \\
&\leq 2\|f - f_1\|_X + t\|f_1\|_{D(\Lambda^m)} .
\end{aligned}$$

Therefore, we derive (9.24).

Now (9.24) and (9.23) yield that $\|\cdot\|_2 \sim \|\cdot\|_{(X,D(\Lambda^m))_{(0,-b),q}}$. This completes the proof.

$\square$

Consider next the Gauss-Weierstrass semi-group

$$W(t)f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy = \mathcal{F}^{-1}\big(e^{-t|\xi|^2}\mathcal{F}f(\xi)\big)(x)\,,\ t > 0,\ x \in \mathbb{R}^d\,,$$
$$W(0) = \mathrm{id}\,.$$

Basic information about the use of $\{W(t)\}_{t\geq 0}$ in connection with function spaces may be found in [116, 2.5.2, pp. 190–192] including (historical) references. See also [15, 4.3.2], [118, 2.6.4] and [125, 3.6.6]. The semi-group $\{W(t)\}_{t\geq 0}$ is analytic in $L_p(\mathbb{R}^d)$ for $1 < p < \infty$ and its infinitesimal generator is the Laplacian operator $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$. Hence, $D(\Lambda^m) = W_p^{2m}(\mathbb{R}^d)$ for $m = 1, 2, \ldots$.

Let $1 < p < \infty, 0 < q \leq \infty, s > 0$ and $s/2 < m \in \mathbb{N}$. An equivalent characterization of Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ in terms of the Gauss-Weierstrass semi-group is

$$\|f\|^{(m)*}_{B^s_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \bigg(\int_0^1 t^{-\frac{s}{2}q} \|[W(t)-\mathrm{id}]^m f\|^q_{L_p(\mathbb{R}^d)} \frac{dt}{t}\bigg)^{1/q}. \tag{9.25}$$

See [15, Theorem 3.4.6 and Section 4.3.2] and [116, 1.13.2]. Another equivalent quasi-norm is

$$\|f\|^{(m)}_{B^s_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \bigg(\int_0^1 t^{-\frac{s}{2}q} \|t^m \partial_t^m W(t) f\|^q_{L_p(\mathbb{R}^d)} \frac{dt}{t}\bigg)^{1/q} \tag{9.26}$$

(see [116, Theorem 2.5.2], [118, Theorem 2.6.4] and the references within). Here $\partial_t = \partial/\partial t$ and $\partial_t^m = \partial^m/\partial t^m$.

Next we specify Theorem 9.13 for the case of the Gauss-Weierstrass semi-group.

Since $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^{2m,p}(\mathbb{R}^d))_{(0,-b),q}$ (see Theorem 5.15(b) or Theorem 9.3) as a direct consequence of Theorem 9.13 we obtain the following characterization of $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ by means of heat kernels.

Theorem 9.14. *Let $1 < p < \infty$, $0 < q \leq \infty$, $b \geq -1/q$ and $m \in \mathbb{N}$. Then $f \in L_p(\mathbb{R}^d)$ belongs to $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ if, and only if,*

$$\|f\|^{(m)*}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \bigg(\int_0^1 \Big((1-\log t)^b \|[W(t)-\mathrm{id}]^m f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\bigg)^{1/q}$$

is finite. Furthermore, $\|\cdot\|^{(m)*}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ is an equivalent quasi-norm in $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Comparing Theorem 9.14 with the corresponding result for classical Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ given in (9.25) we observe that the truncated Littlewood-Paley construction does not appear this time.

Now we consider harmonic extensions, that is, the case of the Cauchy-Poisson semi-group

$$P(t)f(x) = c_d \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2+t^2)^{\frac{d+1}{2}}} f(y)dy = \mathcal{F}^{-1}(e^{-t|\xi|}\mathcal{F}f(\xi))(x)\,,\ t>0,\ x\in\mathbb{R}^d\,,$$

where $c_d\big\|(1+|x|^2)^{-(d+1)/2}\big\|_{L_1(\mathbb{R}^d)} = 1$, with $P(0) = \mathrm{id}$. This is also an analytic semi-group in $L_p(\mathbb{R}^d)$ for $1<p<\infty$ (see [116, 2.5.3] and [118, 2.6.4]). The corresponding characterization reads as follows.

Theorem 9.15. *Let $1<p<\infty$, $0<q\le\infty$, $b\ge -1/q$ and $m\in\mathbb{N}$. Then $f\in L_p(\mathbb{R}^d)$ belongs to $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ if, and only if,*

$$\|f\|^{(m)\diamond}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b \|[P(t)-\mathrm{id}]^m f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}$$

is finite. Furthermore, $\|\cdot\|^{(m)\diamond}_{\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)}$ is an equivalent quasi-norm in $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. For the semi-group $\{P(t)\}_{t\ge0}$ we have

$$\Lambda^{2m}f = (-1)^m\Delta^m f \quad \text{and} \quad D(\Lambda^{2m}) = W^{2m,p}(\mathbb{R}^d)\,.$$

Using again that $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^{2m,p}(\mathbb{R}^d))_{(0,-b),q}$ and Theorem 9.13, we obtain that the wanted result holds for any even natural number m. To complete the proof, write

$$\|f\|_m = \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b \|[P(t)-\mathrm{id}]^m f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}.$$

It suffices to show that for any $m\in\mathbb{N}$ the quasi-norms $\|\cdot\|_m$ and $\|\cdot\|_{m+1}$ are equivalent on $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$.

Take any $f\in\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$. Using (9.23), we obtain

$$\begin{aligned}\|[P(t)-\mathrm{id}]^m f\|_{L_p(\mathbb{R}^d)} &\sim \widetilde{K}(t^m, f; L_p(\mathbb{R}^d), D(\Lambda^m)) \\ &\sim \sup_{0\le s\le t}\|[P(s)-\mathrm{id}]^m f\|_{L_p(\mathbb{R}^d)} = \overline{\omega}_m(t^m, f).\end{aligned}$$

By [51, (4.10)], we have that $\overline{\omega}_{m+1}(t^{m+1}, f) \lesssim \overline{\omega}_m(t^m, f)$. Hence

$$\begin{aligned}\|f\|_{m+1} &\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b\,\overline{\omega}_{m+1}(t^{m+1}, f)\Big)^q \frac{dt}{t}\Big)^{1/q} \\ &\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b\,\overline{\omega}_m(t^m, f)\Big)^q \frac{dt}{t}\Big)^{1/q} \\ &\sim \|f\|_m\,.\end{aligned}$$

In order to establish the converse inequality, note that

$$\overline{\omega}_m(t^m, f) \lesssim t^m \int_t^\infty s^{-m-1}\,\overline{\omega}_{m+1}(s^{m+1}, f)ds$$

(see [115, Theorem 1.4, (1.7)]. Therefore

$$\begin{aligned}
\|f\|_m &\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b\,\overline{\omega}_m(t^m, f)\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)}\| + \Big(\int_0^1 \Big((1-\log t)^b t^m \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big((1-\log t)^b t^m \int_t^1 \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\quad + \Big(\int_0^1 \Big((1-\log t)^b t^m \int_1^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&= \|f\|_{L_p(\mathbb{R}^d)} + J_1 + J_2.
\end{aligned}$$

Since $\overline{\omega}_{m+1}(s^{m+1}, f)/s^{m+1}$ is equivalent to the decreasing function $\widetilde{K}(s^{m+1}, f)/s^{m+1}$, we can still apply the extension of the Hardy inequality established in [8, Theorem 6.4] to derive that

$$\begin{aligned}
J_1 &\lesssim \Big(\int_0^1 \Big(t^{m+1}(1-\log t)^b\, \frac{\overline{\omega}_{m+1}(t^{m+1}, f)}{t^{m+1}}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\sim \Big(\int_0^1 \Big((1-\log t)^b \|[P(t)-\mathrm{id}]^{m+1} f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\le \|f\|_{m+1}\,.
\end{aligned}$$

As for J_2, using that

$$\overline{\omega}_{m+1}(s^{m+1}, f) \sim \widetilde{K}(s^{m+1}, f; L_p(\mathbb{R}^d), D(\Lambda^{m+1})) \le \|f\|_{L_p(\mathbb{R}^d)}$$

we get

$$J_2 \lesssim \Big(\int_0^1 \Big((1-\log t)^b t^m\Big)^q \frac{dt}{t}\Big)^{1/q} \|f\|_{L_p(\mathbb{R}^d)} \lesssim \|f\|_{m+1}\,.$$

This yields that $\|f\|_m \lesssim \|f\|_{m+1}$ and completes the proof.

□

Comparing (9.25) with Theorem 9.14, one might think that the counterpart of the quasi-norm (9.26) for logarithmically perturbed Besov spaces is given by replacing in (9.26) the term $t^{-\frac{s}{2}q}$ by $(1-\log t)^{bq}$. However, the involved spaces are not $\mathbf{B}^{0,b}_{p,q}(\mathbb{R}^d)$ but $B^{0,b}_{p,q}(\mathbb{R}^d)$. We study this question in the rest of the chapter.

The following result refers to abstract semi-groups.

Theorem 9.16. *Let X be a Banach space, let $\{T(t)\}_{t\geq 0}$ be an analytic semi-group of operators in X, let $0 < s/2 < m \in \mathbb{N}$, $0 < q \leq \infty$ and $b \in \mathbb{R}$. The quasi-norm*

$$\|f\|_3 = \|f\|_X + \Big(\int_0^1 \Big(t^{m-\frac{s}{2}} (1-\log t)^b \|\Lambda^m T(t) f\|_X \Big)^q \frac{dt}{t} \Big)^{1/q}$$

is equivalent to the interpolation quasi-norm on $(X, D(\Lambda^m))_{s/2m,q,-b}$.

Proof. Since $\|f\|_X \leq \|f\|_{D(\Lambda^m)}$ for any $f \in D(\Lambda^m)$, we have that $K(t,f) = \|f\|_X$ for any $f \in X$ and $t \geq 1$. This yields that

$$\begin{aligned}\|f\|_{(X,D(\Lambda^m))_{s/2m,q,-b}} &\sim \|f\|_X + \Big(\int_0^1 \Big(t^{-s/2m} (1-\log t)^b K(t,f) \Big)^q \frac{dt}{t} \Big)^{1/q} \\ &\sim \|f\|_X + \Big(\int_0^1 \Big(t^{-s/2} (1-\log t)^b K(t^m,f) \Big)^q \frac{dt}{t} \Big)^{1/q}.\end{aligned}$$

Using that

$$t^m \|\Lambda^m T(t) f\|_X \lesssim K(t^m, f) \tag{9.27}$$

(see [15, Lemma 3.5.4]), we get that $\|f\|_3 \lesssim \|f\|_{(X,D(\Lambda^m))_{s/2m,q,-b}}$. To check the converse inequality, note that

$$\|[T(t) - \mathrm{id}]^m f\|_X \lesssim \int_0^t \tau^{m-1} \|\Lambda^m T(\tau) f\|_X d\tau \tag{9.28}$$

(see [15, Lemma 3.5.5]). Hence, using (9.21), we obtain for $0 < t < 1$

$$\begin{aligned}K(t^m, f) &\lesssim \overline{\omega}_m(t^m, f) + t^m \|f\|_X \\ &\lesssim \int_0^t \tau^{m-1} \|\Lambda^m T(\tau) f\|_X d\tau + t^m \|f\|_X.\end{aligned}$$

This implies that

$$\begin{aligned}\|f\|_{(X,D(\Lambda^m))_{s/2m,q,-b}} &\lesssim \|f\|_X \\ &\quad + \Big(\int_0^1 \Big(t^{-s/2} (1-\log t)^b \int_0^t \tau^{m-1} \|\Lambda^m T(\tau) f\|_X d\tau \Big)^q \frac{dt}{t} \Big)^{1/q}.\end{aligned}$$

By (iii), we have for $0 < \tau < \mu$ that

$$\|\Lambda^m T(\mu) f\|_X = \|T(\mu - \tau) \Lambda^m T(\tau) f\|_X \leq M \|\Lambda^m T(\tau) f\|_X.$$

Hence, using the extension of the Hardy inequality established in [8, Theorem 6.4], we derive that

$$\begin{aligned}\|f\|_{(X,D(\Lambda^m))_{s/2m,q,-b}} &\lesssim \|f\|_X + \Big(\int_0^1 \Big(t^{1-\frac{s}{2}} (1-\log t)^b t^{m-1} \|\Lambda^m T(t) f\|_X \Big)^q \frac{dt}{t} \Big)^{1/q} \\ &= \|f\|_3.\end{aligned}$$

□

Next we apply Theorem 9.16 to the Gauss-Weierstrass semi-group $\{W(t)\}_{t\geq 0}$. Recall that I_s denotes the lift operator (6.1).

Theorem 9.17. *Let* $1 < p < \infty$, $0 < q \leq \infty$, $b \in \mathbb{R}$ *and* $m \in \mathbb{N}$. *Then*

$$\|f\|^{(m)}_{B^{0,b}_{p,q}(\mathbb{R}^d)} = \|I_{-2}f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^m(1-\log t)^b \Big\| \frac{\partial^m W(t)f}{\partial t^m} \Big\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}$$

is an equivalent quasi-norm on $B^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. According to [95, Proposition 1.8], the operator I_{-2} is an isomorphism from $B^{0,b}_{p,q}(\mathbb{R}^d)$ onto $B^{2,b}_{p,q}(\mathbb{R}^d)$. The classical smoothness of $B^{2,b}_{p,q}(\mathbb{R}^d)$ is $2 > 0$ so, by [77, Theorem 2.5], we know that

$$\|f\|_{B^{2,b}_{p,q}(\mathbb{R}^d)} \sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \big(t^{-2}(1-\log t)^b \omega_{2(m+1)}(f,t)_p\big)^q \frac{dt}{t}\Big)^{1/q}. \tag{9.29}$$

Moreover, for $f \in L_p(\mathbb{R}^d)$ and $0 < t < 1$, we have

$$K(t^{2(m+1)}, f; L_p(\mathbb{R}^d), W^{2(m+1),p}(\mathbb{R}^d)) \sim t^{2(m+1)}\|f\|_{L_p(\mathbb{R}^d)} + \omega_{2(m+1)}(f,t)_p \tag{9.30}$$

(see (5.11)). It follows from (9.29) and (9.30) that

$$B^{2,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^{2(m+1),p}(\mathbb{R}^d))_{1/(m+1),q,-b}, \tag{9.31}$$

so

$$B^{2,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), D(\Lambda^{m+1}))_{1/(m+1),q,-b}$$

where $\Lambda = \Delta$ is the infinitesimal generator of the semi-group $\{W(t)\}_{t\geq 0}$. Applying Theorem 9.16 we get

$$\begin{aligned}\|f\|_{B^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \|I_{-2}f\|_{B^{2,b}_{p,q}(\mathbb{R}^d)} \\ &\sim \|I_{-2}f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \big(t^m(1-\log t)^b \|\Delta^{m+1}W(t)I_{-2}f\|_{L_p(\mathbb{R}^d)}\big)^q \frac{dt}{t}\Big)^{1/q}.\end{aligned}$$

Next we use that

$$\Delta^{m+1}W(t)I_{-2}f = \Delta^{m+1}I_{-2}W(t)f = \Delta I_{-2}\Delta^m W(t)f$$

and that, according to [111, p. 133],

$$\|\Delta I_{-2}g\|_{L_p(\mathbb{R}^d)} = \|\Delta(\mathrm{id}-\Delta)^{-1}g\|_{L_p(\mathbb{R}^d)} \sim \|g\|_{L_p(\mathbb{R}^d)}.$$

This yields that

$$\|f\|_{B^{0,b}_{p,q}(\mathbb{R}^d)} \sim \|I_{-2}f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^m(1-\log t)^b \Big\| \frac{\partial^m W(t)f}{\partial t^m} \Big\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}.$$

$\square$

Note that the operator I_{-2} is necessary because in general $B^{0,b}_{p,q}(\mathbb{R}^d)$ may not contain only regular distributions (see [20, Theorem 4.3]).

Finally we consider the case of the Cauchy-Poisson semi-group $\{P(t)\}_{t\geq 0}$.

Theorem 9.18. *Let $1<p<\infty$, $0<q\leq\infty$, $b\in\mathbb{R}$ and $m\in\mathbb{N}$. Then*

$$\|f\|^{(m)\diamond}_{B^{0,b}_{p,q}(\mathbb{R}^d)} = \|I_{-2}f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^m(1-\log t)^b\Big\|\frac{\partial^m P(t)f}{\partial t^m}\Big\|_{L_p(\mathbb{R}^d)}\Big)^q\frac{dt}{t}\Big)^{1/q}$$

is an equivalent quasi-norm on $B^{0,b}_{p,q}(\mathbb{R}^d)$.

Proof. This time $\Lambda^{2(m+1)}f = (-1)^{m+1}\Delta^{m+1}f$ and $D(\Lambda^{2(m+1)}) = W^{2(m+1),p}(\mathbb{R}^d)$. By (9.31), we get

$$B^{2,b}_{p,q}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), W^{2(m+1),p}(\mathbb{R}^d))_{1/(m+1),q,-b} = (L_p(\mathbb{R}^d), D(\Lambda^{2(m+1)}))_{1/(m+1),q,-b}\,.$$

Therefore, applying Theorem 9.16 with $s=4$, we obtain

$$\|f\|_{B^{2,b}_{p,q}(\mathbb{R}^d)} \sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{2(m+1)-2}(1-\log t)^b\|\Lambda^{2(m+1)}P(t)f\|_{L_p(\mathbb{R}^d)})^q\frac{dt}{t}\Big)^{1/q}.$$

This means that for any even natural number m with $m\geq 4$ we have

$$\|f\|_{B^{2,b}_{p,q}(\mathbb{R}^d)} \sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b\|\Lambda^m P(t)f\|_{L_p(\mathbb{R}^d)}\Big)^q\frac{dt}{t}\Big)^{1/q}. \tag{9.32}$$

Write $\|f\|^\diamond_m$ for the quasi-norm on the right-hand side of (9.32). We claim that for any $m\in\mathbb{N}$ with $m>2$ we have

$$\|\cdot\|^\diamond_m \sim \|\cdot\|^\diamond_{m+1} \quad\text{on}\quad B^{2,b}_{p,q}(\mathbb{R}^d)\,. \tag{9.33}$$

Indeed, by (vi), given any $f\in L_p(\mathbb{R}^d)$ we have

$$\|\Lambda^{m+1}P(t)f\|_{L_p(\mathbb{R}^d)} = \|\Lambda P(t/2)\Lambda^m P(t/2)f\|_{L_p(\mathbb{R}^d)} \lesssim t^{-1}\|\Lambda^m P(t/2)f\|_{L_p(\mathbb{R}^d)}.$$

Whence

$$\begin{aligned}\|f\|^\diamond_{m+1} &\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b\|\Lambda^m P(t/2)f\|_{L_p(\mathbb{R}^d)}\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^{1/2} \Big(t^{m-2}(1-\log t)^b\|\Lambda^m P(t)f\|_{L_p(\mathbb{R}^d)}\Big)^q\frac{dt}{t}\Big)^{1/q} \leq \|f\|^\diamond_m.\end{aligned}$$

Conversely, by (9.27) and (9.24)

$$\begin{aligned}\|f\|^\diamond_m &\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{-2}(1-\log t)^b K(t^m,f;L_p(\mathbb{R}^d),D(\Lambda^m))\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{-2}(1-\log t)^b \widetilde{K}(t^m,f;L_p(\mathbb{R}^d),D(\Lambda^m))\Big)^q\frac{dt}{t}\Big)^{1/q}.\end{aligned}$$

As we have seen in the proof of Theorem 9.15,

$$\widetilde{K}(t^m, f; L_p(\mathbb{R}^d), D(\Lambda^m)) \sim \overline{\omega}_m(t^m, f)\,.$$

Moreover, by [115, Theorem 1.4, (1.7)]

$$\overline{\omega}_m(t^m, f) \lesssim t^m \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\,.$$

Now proceeding as in the proof of Theorem 9.15, using that $m > 2$ and the extension of the Hardy inequality [8, Theorem 6.4], we obtain

$$\begin{aligned}
\|f\|_m^\diamond &\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b \int_t^\infty \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b \int_t^1 \frac{\overline{\omega}_{m+1}(s^{m+1}, f)}{s^{m+1}} ds\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\qquad + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b\Big)^q \frac{dt}{t}\Big)^{1/q} \|f\|_{L_p(\mathbb{R}^d)} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-1}(1-\log t)^b \frac{\overline{\omega}_{m+1}(t^{m+1}, f)}{t^{m+1}}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\sim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{-2}(1-\log t)^b \|[P(t)-\mathrm{id}]^{m+1} f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{-2}(1-\log t)^b \int_0^t s^m \|\Lambda^{m+1} P(s) f\|_{L_p(\mathbb{R}^d)} ds\Big)^q \frac{dt}{t}\Big)^{1/q}
\end{aligned}$$

where we have used (9.28) in the last inequality. The extension of the Hardy inequality implies now that

$$\|f\|_m^\diamond \lesssim \|f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{-1}(1-\log t)^b t^m \|\Lambda^{m+1} P(t) f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q} = \|f\|_{m+1}^\diamond.$$

This proves (9.33).

Now to complete the proof of the theorem we can proceed as in Theorem 9.17 with the help of the lift operator I_{-2}. Indeed, given any natural number $m > 2$, since

$$\Lambda^m P(t) I_{-2} f = \Lambda^2 \Lambda^{m-2} I_{-2} P(t) f = -\Delta I_{-2} \Lambda^{m-2} P(t) f\,,$$

by (9.32) and (9.33) we obtain

$$\begin{aligned}
\|f\|_{B^{0,b}_{p,q}(\mathbb{R}^d)} &\sim \|I_{-2} f\|_{B^{2,b}_{p,q}(\mathbb{R}^d)} \sim \|I_{-2} f\|_m^\diamond \\
&= \|I_{-2} f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b \|\Delta I_{-2} \Lambda^{m-2} P(t) f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&\sim \|I_{-2} f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b \|\Lambda^{m-2} P(t) f\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q} \\
&= \|I_{-2} f\|_{L_p(\mathbb{R}^d)} + \Big(\int_0^1 \Big(t^{m-2}(1-\log t)^b \Big\|\frac{\partial^{m-2} P(t) f}{\partial t^{m-2}}\Big\|_{L_p(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\Big)^{1/q}.
\end{aligned}$$

This finishes the proof. $\square$

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