

Convexity analysis in financial instruments

Master Thesis. Máster en Ingeniería Matemática

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1 Introduction

Convexity is a term that appears frequently in the financial literature. It could refer to different ideas, some of them related to risk control and others related to different kinds of bias between financial instruments. Most of times, convexity refers, strictly, to non-linearity in the pricing of financial products. If the pricing input changes, the output price does not change linearly, but depends on the second derivative of the modeling function. The pricing input is usually the yield of the financial instrument. Yield is the income return on an investment. This refers to the interest or dividends received from a security and is usually expressed annually as a percentage based on the investment's cost, its current market value or its face value.

This work have two goals. On the one hand, we are going to study some basic concepts such as duration and convexity. These ideas are essential to understand not only what is financial convexity but how to hedge assets portfolios. Most of financial products are trading in yield, for this reason knowing the price sensitivity to yield changes is necessary to control portfolios risk.

On the other hand, we are going to analyse the price difference between two kinds of contracts: Eurodollar futures and Forward Rate Agreements (FRA). Since this difference is due to a non-linearity in the FRA payoff function, this difference is called convexity bias. FRAs are over the counter contracts whereas Eurodollars are traded on exchanges such as the Chicago Mercantile Exchange (CME). Both of them are loans with similar specifications. For this reason, one could think that the price should be almost identical. However, there is a fact that makes all the difference. Eurodollar futures are mark-to-market, i.e, profits and losses are settled every day whereas FRAs not. Then to know the right price we have to compute the convexity bias between the two instruments. In order to measure this bias we introduce different models and compare its results. This models have been implemented on Matlab program. Some ideas about the implementation are also discussed.

The work division is as follows. It is divided into a introduction, 2 chapters, conclusion, appendix, and bibliography. The first chapter consist in a presentation of the basic concepts and a definition of convexity. Some of the basic concepts that we introduce have been seen in the Master subject "Fundamentos de la matemática financiera" and "Tipos de Interés" in which the basic techniques to price assets were studied. Also the duration and convexity are introduced as a first application of the convexity to hedge bond portfolios. The second chapter focuses on the so called convexity bias, which consist in the systematic advantage to being short Eurodollar futures relative to deposits, swap, or FRAs. As we explained above, this convexity arise due to the non-linear FRA payoff function. This convexity bias implies possible arbitrages. To get a right pricing for a FRA, we have to correct the original price by removing the convexity bias. Different models are introduced to measure this bias. The first model is the most simple mathematically because no stochastic model is needed. The rest of models use diffusion processes and Itô's calculus. these models are Campbell and Temel model, Vasicek, and Cox-Ingersoll-Ross (CIR) model. To get a good understanding of this part the Master subject "Cálculo estocástico y valoración financiera" is really helpful. These models have been implemented on Matlab. The data used in the matlab program has been obtained from Bloomberg. In the CIR model, we have used genetic algorithms. This kind of algorithms are studied in the Master subject "Técnicas avanzadas de optimización". We conclude this chapter providing a comparison between the model results. It is interesting to note that this convexity bias can come to 6 basis point (see "The Convexity Bias in Eurodollar Futures", Burghardt) that is worth more than \$ 200,000 on a 5-year swap. So an investor can lose big amounts of money if this convexity adjustment is not taken into account.

Later we present a conclusion of the results obtained in the models and some possible ways to go further with the convexity bias estimation.

Finally, the appendix explains the main mathematical tools used in this work.

2 A first approach to convexity

2.1 Basic concepts

Spot Rates and Bonds

A **zero coupon bond (ZCB)** with maturity T is a contract which guarantees the payment of one unit of currency at time T . The present value at time t can be expressed as

$$ZCB(t, T) = \frac{1}{1 + (T - t)R(t, T)}, \quad (1)$$

where $R(t, T)$ is simply-compounded **spot rate**, i.e, the price that is quoted for immediate settlement on a commodity, a security or a currency. The **short rate**, see 2.1, is defined as $r(t) = \lim_{T \rightarrow t} R(t, T)$. The ZCB annual-compounded value is

$$ZCB(t, T) = \frac{1}{(1 + R(t, T))^{T-t}}$$

A more general instrument is a **bond**. A bond is a contract which guarantees the payment of a number of cash flows that depend on the nominal, the coupon and the maturity T of the bond. Consider a bond paying fixed riskless cash flows $C_{t_1}, C_{t_2}, \dots, C_{t_n}$ at periods $t = t_1, t_2, \dots, t_n = T$. The bond price at time t is $P(t, T)$. An investor who buys this bond will therefore pay $P(t, T)$ and receive the cash flows $C_{t_1}, C_{t_2}, \dots, C_{t_n}$ in the future. The net present value of the investment is the discounted value of the cash flows, $C_{t_1}, C_{t_2}, \dots, C_{t_n}$ minus the initial bond price P .

The yield to maturity y is the single discount rate that, when applied to all cash flows, gives a net present value of zero. Using annual compounding we have

$$P(t, T) = \frac{C_{t_1}}{1 + y} + \frac{C_{t_2}}{(1 + y)^2} + \dots + \frac{C_{t_n}}{(1 + y)^T} = \sum_{i=1}^n \frac{C_{t_i}}{(1 + y)^i}$$

Normally, the first $n - 1$ cash flows correspond to the coupon times the nominal. The last cash flow corresponds to coupon times the nominal plus the nominal. If c is the bond's coupon and N is the nominal, we have

$$P(t, T) = \frac{cN}{1 + y} + \frac{cN}{(1 + y)^2} + \dots + \frac{cN + N}{(1 + y)^T}$$

The **forward rate** is the interest rate that can be earned on a forward loan starting at T_1 and with maturity at T_2 for $t < T_1 < T_2$. By a non-arbitrage condition it can be shown that the value of the simply-compounded **forward interest rate** prevailing at time t for the period between T_1 and T_2 is

$$f(t; T_1, T_2) = \frac{1}{T_2 - T_1} \left(\frac{ZCB(t, T_1)}{ZCB(t, T_2)} - 1 \right) = \frac{1}{T_2 - T_1} \left(\frac{1 + (T_2 - t)R(t, T_2)}{1 + (T_1 - t)R(t, T_1)} - 1 \right)$$

The Short Rate

Under a short rate model, the stochastic state variable is taken to be the instantaneous forward rate. The short rate, r_t , then, is the (continuously compounded, annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time from time t . Specifying the current short rate does not specify the entire yield curve. However no-arbitrage arguments show that, under some fairly relaxed technical conditions, if we model the evolution of r_t , as a stochastic process under a risk-neutral measure Q , the price at time t of a zero coupon bond maturing at time T is given by

$$ZCB(t, T) = \mathbb{E}^Q \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] \quad (2)$$

where \mathcal{F}_t is the natural filtration for the process. Thus specifying a model for the short rate specifies future bond prices. This means that instantaneous forward rates are also specified by the usual formula, by 2,

$$f(t; t, T) = - \frac{\partial}{\partial T} \ln (ZCB(t, T)) \rightarrow f(t; t, T) = \mathbb{E}^Q [r_T | \mathcal{F}_t] \quad (3)$$

Forward Rate Agreement

Forward rate agreements (FRAs) are over-the-counter contracts whereby two parties agree at time zero on a rate of interest to be paid on a loan starting at time m and maturing at time n . The buyer of the FRA receives the difference between the prevailing rate at time n and the strike rate prespecified in the contract for a given period (typically three months) multiplied by a notional amount. Thus, the buyer of the FRA makes money when rates increase. The payment takes place conventionally at time n and therefore needs to be discounted. The payment follows the money market convention which consists in dividing the number of days in the interest accrual period by 360 days. The reference rate is typically the three-month LIBOR (the London interbank offered rate), which is the rate paid by large international banks on interbank loans. The payoff at time m equals to

$$\frac{1}{1 + L(m, m, n) \frac{n-m}{360}} \times [L(m, m, n) - \bar{R}] \times \frac{n-m}{360} \times N \quad (4)$$

where $L(m, m, n)$ is the LIBOR rate as of time m on a loan starting at time m and expiring at time n , \bar{R} is the prespecified strike rate, and N is the notional principal. If, at time 0, \bar{R} is set at the forward LIBOR prevailing for a loan that starts at time m and ends at time n , in other words, if

$$\bar{R} = L(0, m, n)$$

then, as will be shown below, the value at inception of the FRA is zero. We assume that money can be borrowed and lent at LIBOR. Note that, under this assumption, the payoff at time m of the FRA in (4) is equivalent to a payoff at time n equal to

$$[L(m, m, n) - \bar{R}] \times \frac{n-m}{360} \times N$$

since the cash flow received at time m can be reinvested at the then prevailing LIBOR, $L(m, m, n)$, till time n . The replication strategy is

1. Borrow $\frac{(1 + \bar{R} \times \frac{n-m}{360})N}{1 + L(0, 0, n) \times \frac{n}{360}}$ at time 0 for n days;

2. Deposit $\frac{N}{1+L(0,0,m)\times\frac{m}{360}}$ at time 0 for m days;
3. Deposit at time m the proceeds from the above deposit for $(n-m)$ days.

This way we replicate exactly the cash flow of the FRA. At time 0, the value of the FRA is

$$\frac{(1 + \bar{R} \times \frac{n-m}{360}) N}{1 + L(0, 0, n) \times \frac{n}{360}} - \frac{N}{1 + L(0, 0, m) \times \frac{m}{360}}$$

If the FRA is to have a zero value at inception, we need to set \bar{R} so that the initial cash flows from the portfolio equal zero:

$$\frac{(1 + \bar{R} \times \frac{n-m}{360}) N}{1 + L(0, 0, n) \times \frac{n}{360}} - \frac{N}{1 + L(0, 0, m) \times \frac{m}{360}} = 0$$

Hence the value of \bar{R} is

$$\bar{R} = \left[\frac{1 + L(0, 0, n) \times \frac{n}{360}}{1 + L(0, 0, m) \times \frac{m}{360}} - 1 \right] \times \frac{360}{n - m}$$

that is the forward LIBOR as of time 0 for a loan between time m and time n , that is $L(0, m, n)$.

Swap

A **swap**, generically, is an exchange. In financial parlance, it refers to an exchange of a series of cash flows against another series of cash flows. In a swap one part receives fixed cash flows and pays floating and the other part receives floating cash flows and pays fixed. Consider the case of a T -year swap receiving a fixed coupon c , also called swap rate, and paying six-month LIBOR on a notional amount N . For analytical convenience, we assume that the i period zero-coupon rates used to discount the fixed and floating cash flows are based on the LIBOR curve, that is $R(t, T_\alpha) = L(t, T_\alpha)$. This swap has value

$$V_{\text{swap}} = P_{\text{fixed}} - P_{\text{floating}}$$

where for a pre-specified set of futures dates $T_{\alpha+1}, \dots, T_\beta$ the value of the fixed leg is

$$P_{\text{fixed}} = \frac{cN}{1 + R(t, T_{\alpha+1})} + \dots + \frac{(1 + c)N}{(1 + R(t, T_\beta))^{T_\beta - T_\alpha}}$$

and the value of the floating leg is

$$P_{\text{floating}} = N$$

A swap can be seen as a set of consecutive FRA contracts in which by non-arbitrage condition it is possible to obtain the **forward swap rate** at time t . The forward swap rate can be expressed as

$$S_{\alpha, \beta}(t) = \frac{ZCB(t, T_\alpha) - ZCB(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i ZCB(t, T_i)} = \frac{1 - (1 + R(t, T_\beta))^{T_\alpha - T_\beta}}{\sum_{i=\alpha+1}^{\beta} (1 + R(t, T_{\alpha+i}))^{T_\alpha - i}}, \quad (5)$$

where $t < T_\alpha$ and τ_i is the year fraction between T_{i-1} and T_i .

Eurodollar Futures Contract

Eurodollar futures contracts are traded on major exchanges such as the LIFFE and the Chicago Mercantile Exchange (CME). At expiration date T , the payoff is defined as

$$\text{Futures settlement price} = 100 \times (1 - L_T) \quad (6)$$

where L_T is the 3m-LIBOR at time T . At time $t < T$, we can infer from the futures price before expiration $F(t, T, T + 3m)$ the forward three-month LIBOR $L(t, T, T + 3m)$. This follows directly from (6):

$$L(t, T, T + 3m) = 1 - \frac{F(t, T, T + 3m)}{100}$$

A important feature of Eurodollar futures contracts is that gains and losses are settled every day, in other words, mark-to-market occurs daily in the futures market. This property makes Eurodollar futures very different from FRAs.

Risk Free Rates. Zero Coupon Curve

The **risk free rate** is assumed to be the inter-bank market interest rate of the corresponding currency. Represents the interest an investor would expect from an absolutely risk free investment over a specified period of time. The LIBOR curve and the Treasury yield curve are the most widely-used proxies for the risk-free interest rates. Although not theoretically risk-free, LIBOR is considered a good proxy against which to measure the risk/return tradeoff for other short-term floating rate instruments. We use the Libor **zero-coupon rate curve (ZCC)** to obtain discount factors. Let L_k be the Libor rate over an interval $[T_k, T_{k+1}]$, then the corresponding zero coupon bond (ZCB) value is, by (1):

$$ZCB(T_k, T_{k+1}) = \frac{1}{1 + \tau_k L_k}, \quad (7)$$

where $\tau_k = T_{k+1} - T_k$. ZCBs are not traded assets so their value must be inferred from the prices of traded contracts such as Libor rates, interest rate futures and swap rates. We construct the Libor ZCC using yearly swap rate data. Let $S_{0,1}, \dots, S_{0,n}$ be the yearly swap rates for 1-year to n -years. From equation (5) it can be seen that if we take the 1-year swap rate we can easily obtain the 1-year ZCB, since

$$S_{0,1} = \frac{1 - ZCB(T_0, T_1)}{\tau_1 ZCB(T_0, T_1)} = \frac{1}{\tau_1} \left(\frac{1}{ZCB(T_0, T_1)} - 1 \right) = L_1.$$

Thus, from (7) we get

$$ZCB(T_0, T_1) = \frac{1}{1 + \tau_1 S_{0,1}}.$$

Now assume that $ZCB(T_0, T_i)$ have been determined for $i = 1, \dots, k$, and let

$$A_k = \sum_{i=1}^k \tau_i ZCB(T_0, T_i).$$

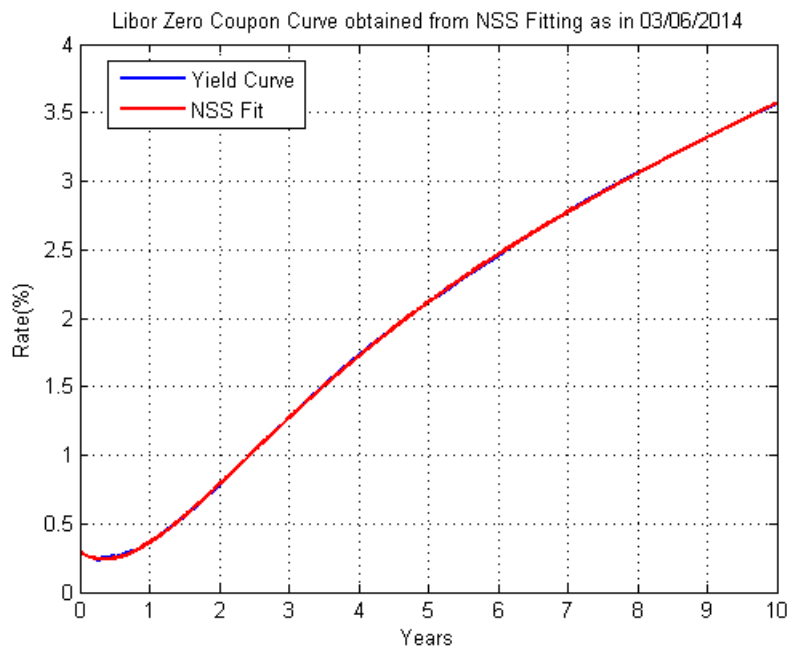
Then, from equation (5) we can express the $(k + 1)$ -year swap rate as

$$S_{0,k+1} = \frac{1 - ZCB(T_0, T_{k+1})}{A_k + \tau_{k+1} ZCB(T_0, T_{k+1})},$$

implying that we can determine recursively all $ZCB(T_0, T_k)$ values in which we have swap rate data as

$$ZCB(T_0, T_{k+1}) = \frac{1 - S_{k+1}A_k}{1 + \tau_{k+1}S_{k+1}},$$

To determine $ZCB(T_0, T_h)$ values for times T_h in which there isn't a T_h -years swap rate data, we need to use an interpolation technique. For this purpose we use the Nelson-Siegel-Svensson approach (see Appendix 5.1). You can see the Libor ZCC in Figure 1:



Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Rate(%)	0,36	0,78	1,27	1,72	2,11	2,46	2,77	3,05	3,32	3,57

Figure 1: Libor Zero Coupon Curve as in 03/06/2014.

2.2 Bond Duration and Convexity

Now that we know how to express the price of a bond with respect to its yield to maturity, we can define the duration of a bond. Duration is a measure of the sensitivity of bond price to a change in interest rates. We postulate a flat term structure of interest rates, that is, the yield-to-maturity curve, the zero-coupon curve, and the forward curve are all flat and identical. Differentiating P with respect to y gives

$$\frac{dP}{dy} = \frac{-1}{1+y} \sum_{i=1}^T \frac{iC_i}{(1+y)^i}$$

$\frac{dP}{dy}$ is known as \$duration (or \$Dur). If we multiply both sides of the last equation $\frac{dy}{P}$ we get

$$\frac{dP}{P} = -\frac{1}{1+y} \frac{\sum_{i=1}^T \frac{iC_i}{(1+y)^i}}{\sum_{i=1}^T \frac{C_i}{(1+y)^i}} dy$$

Other measure is the modified duration $ModD$. Modified duration is defined as the relative sensitivity $\frac{1}{P} \frac{dP}{dy}$. We define Macaulay duration D as

$$D \equiv \frac{\sum_{i=1}^T \frac{iC_i}{(1+y)^i}}{\sum_{i=1}^T \frac{C_i}{(1+y)^i}}$$

This measure was suggested by F.R. Macaulay as measure of interest rate risk in 1938.

The relationship between bond price relative changes and interest rate changes can therefore be formulated as

$$\frac{dP}{P} = \frac{-D}{1+y} dy \quad (8)$$

It is important to note that in this equation the differential dy belies three assumptions

1. Changes in interest rates are infinitesimal.
2. The term structure of interest rates y is flat; and
3. Changes in the term structure are parallel.

As shown in the definition, the duration is the weighted average time (expressed in years) until maturity of a bond, with the weights being the present value of the cash flows divided by the bond price. Indeed, (8) can be rewritten as

$$D = -\frac{1+y}{P} \frac{dP}{dy} = -(1+y) \frac{d \ln P}{dy} = \frac{d \ln P}{d \ln P_1},$$

where $P_1 = \frac{1}{1+y}$. Duration is then interpreted as the elasticity of the bond price vis-a-vis a one-year zero-coupon bond price. We can extend the definition of duration to portfolios. The duration and the modified duration enable us to compute the absolute P&L and the relative P&L of a Portfolio \mathbf{P} for a small change Δy of the yield to maturity

$$\text{Absolute P\&L} \simeq \$Dur \times \Delta y$$

$$\text{Relative P\&L} \simeq -MD \times \Delta y$$

Note that we are using a one-order Taylor expansion,

$$\begin{aligned} dP(y) &= P(y+dy) - P(y) = P'(y)dy + o(y) \simeq \$Dur \times dy \\ \frac{dP(y)}{P(y)} &= \frac{P'(y)}{P(y)} dy + o(y) \simeq -MD \times dy \end{aligned}$$

Another standard measure is the basis point value, BPV, (also called DV01), which is the change in the bond given a basis point change in the bond's yield. BPV is given by

$$BPV = \frac{MD \times P}{10,000} = \frac{-\$Dur}{10,000}$$

The first definition of convexity arises as the next step of the duration. The first-order Taylor expansion gives a good approximation of the bond price only when the variation of its yield to maturity is small.

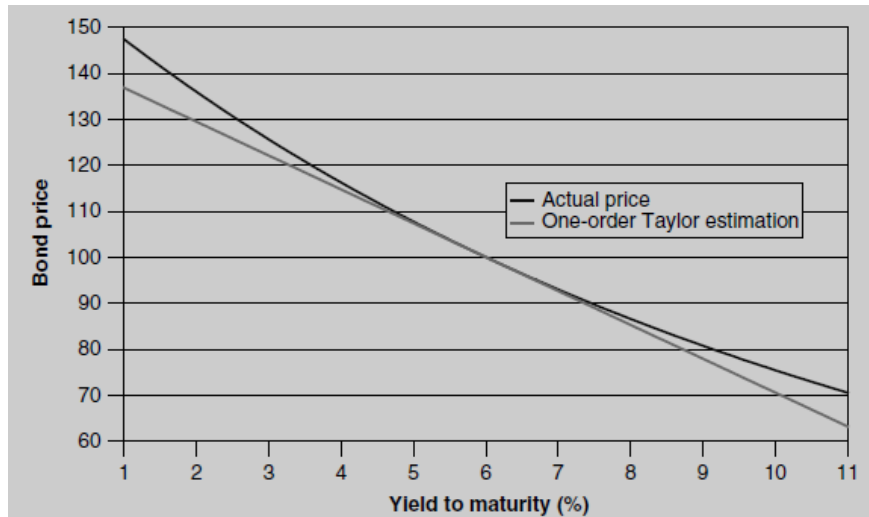


Figure 2: The convexity of the bond price

If one is concerned about the impact of a larger move dy on a bond portfolio value, one needs to write (at least) a second-order version of the Taylor expansion

$$\begin{aligned} dP(y) &= P'(y)dy + \frac{1}{2}P''(y)(dy)^2 + o((dy)^2) \\ &\simeq \$Dur(P)dy + \frac{1}{2}\$Conv(P)(dy)^2 \end{aligned} \quad (9)$$

where

$$P''(y) = \frac{d^2P}{dy^2} = \sum_{i=1}^T \frac{i(i+1)C_i}{(1+y)^{i+2}}$$

the second derivative of the bond value function with respect to yield to maturity, also denoted by $\$Conv(P)$ is known as the $\$convexity$ of the bond P . Dividing equation (9) by $P(y)$, we obtain an approximation of the relative change in the value of the portfolio as

$$\frac{dP(y)}{P(y)} \simeq -MD(P(y))dy + \frac{1}{2}RC(P)(dy)^2$$

where

$$RC(P) = \frac{P''(y)}{P(y)}$$

is called the relative convexity.

We can get a better hedge by using both duration and convexity. One needs to introduce two hedging assets with prices in $\$$ denoted by H_1 and H_2 , and YTM by y_1 and y_2 , respectively, in order to hedge at the first and second order, the interest-rate risk of a portfolio with price in $\$$ denoted by \mathbf{P} , and YTM by y . The goal is to obtain a portfolio that is both $\$duration$ -neutral and $\$convexity$ -neutral. The optimal quantity (ϕ_1, ϕ_2) of these two assets to hold is then given by the solution to the following system of equations, at each date, assuming that $dy = dy_1 = dy_2$:

$$\left. \begin{aligned} \phi_1 H_1'(y_1) + \phi_2 H_2'(y_2) &= -P'(y) \\ \phi_1 H_1''(y_1) + \phi_2 H_2''(y_2) &= -P''(y) \end{aligned} \right\}$$

which translates into

$$\left. \begin{aligned} \phi_1 \$Dur(H_1(y_1)) + \phi_2 \$Dur(H_2(y_2)) &= -\$Dur(P(y)) \\ \phi_1 \$Conv(H_1(y_1)) + \phi_2 \$Conv(H_2(y_2)) &= -\$Conv(P(y)) \end{aligned} \right\}$$

2.3 The Free Lunch in the Duration Model

The duration model, as pointed out, assumed parallel shifts of a flat term structure of interest rates. We now question whether these types of movements of the term structure are acceptable. In our parlance, "acceptable" means that the term structure changes that we postulate disallow riskless arbitrage. If such arbitrage is made possible by our assumptions, then these assumptions need to be revised to reflect the simple reality of bond markets: there are very few riskless arbitrage opportunities around. This free lunch arises from the fact that in a world where movements in a flat term structure are parallel, a strategy where one purchases a high-convexity portfolio and sell a low-convexity portfolio should always make money.

Consider a portfolio A of q_i intermediate duration zero-coupon bonds with maturity t and price P_i . Now consider a portfolio B with q_s short duration bonds with maturity $t - \Delta t$ and price P_s and q_l long duration bonds with maturity $t + \Delta t$ and price P_l . Then we have:

$$\begin{aligned} P_l q_l + P_s q_s &= P_i \text{ (equal value constraint)} \\ P_l q_l M D_l + P_s q_s M D_s &= P_i M D_i \text{ (duration-matching constraint)} \end{aligned}$$

This system of two equations with three unknowns (q_s , q_i and q_l) determines the composition of the portfolios. It follows that

$$P_s q_s = P_l q_l = \frac{P_i q_i}{2}$$

The relative convexity of a zero-coupon bond is

$$RC = \frac{T(T+1)}{(1+y)^2} \quad (10)$$

where T is the maturity of the bond. Now if we use (10) we obtain that the convexity of the portfolio B is

$$RC(B) = \frac{(t+\Delta t)(t+\Delta t+1) + (t-\Delta t)(t-\Delta t+1)}{2(1+y)^2} = \frac{t^2 + t + (\Delta t)^2}{(1+y)^2} \quad (11)$$

whereas the convexity of the intermediate bond portfolio A is

$$RC(A) = \frac{t^2 + t}{(1+y)^2} \quad (12)$$

It follows that $RC(A) < RC(B)$ and that the convexity difference is proportional to the square of the difference between the maturity of the long zero coupon and that of the short zero coupon. For this reason portfolio B dominates portfolio A , i.e., one could create a money machine out of the following arbitrage: sell portfolio A and buy against portfolio B . Figure 3 illustrates the argument.

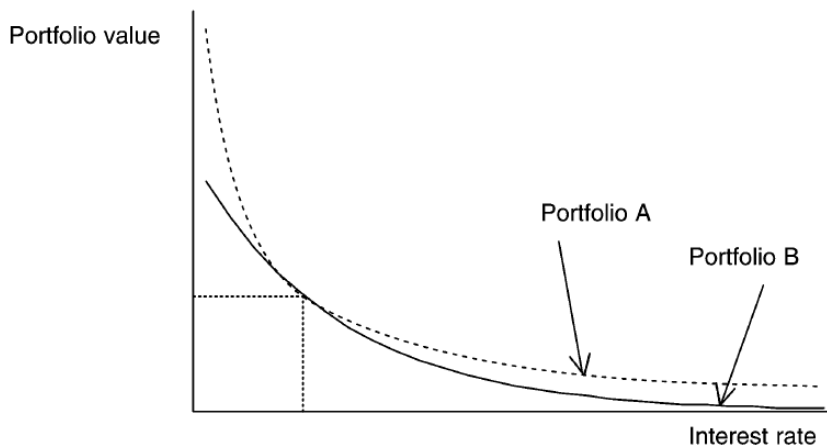


Figure 3: Arbitrage in the duration model

Hence the need for a more sophisticated interest rate models.

3 The Convexity Bias in Eurodollar Futures

A major difference between FRAs and futures stems from the fact that the payoff of a futures contract is linear, whereas it is nonlinear for an FRA (see Figure 4).

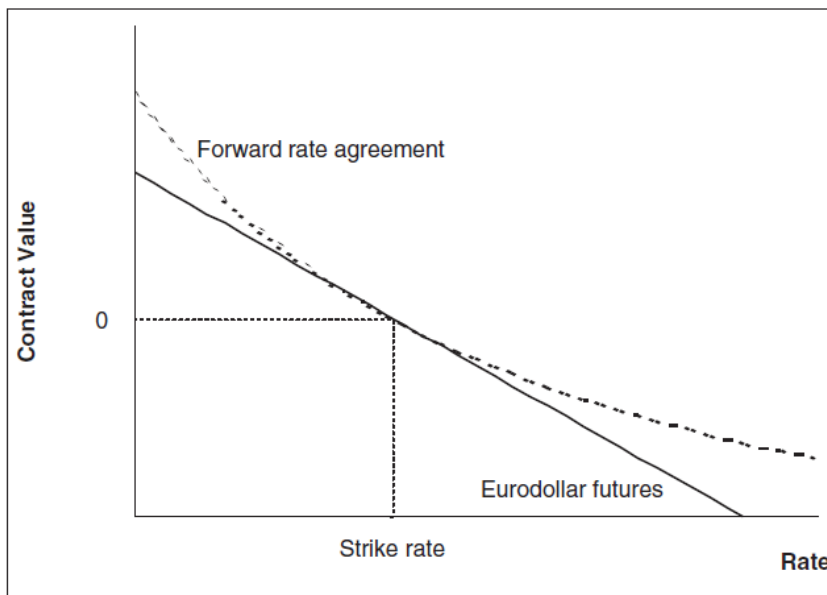


Figure 4: Convexity adjustment

Indeed, a FRA can be viewed as an m -day forward zero-coupon bond with maturity $(n - m)$ days. The price-yield relationship in the FRA case is similar to that of a bond. For example, dollar-based investors expecting a rate decline may try to take advantage of this view by paying LIBOR and receiving the fixed strike rate of a USD FRA or alternatively, may purchase a Eurodollar

futures contract. As shown in Figure 4, no matter where the rate goes in the future, the value of the forward rate agreement will be higher than the value of the futures contract if the strike rate is the same for both contracts. Because the investor is paying LIBOR and receiving fixed on both contracts, the strike rate must be lower in the FRA than in the futures contract to eliminate the advantage from the FRA nonlinear payoff. In other words, the forward rate is lower than the futures rate. The non-linearity advantage is called the convexity advantage. We provide a numerical example to illustrate the so-called convexity adjustment.

Example 3.1 *Suppose that the three-month and six-month LIBORs are 4% and 4.2%, respectively, and $m = 91$; $n = 183$. The strike rate of the FRA is $\bar{R} = 4.35\%$, and $N = 100$ million USD. We want to calculate the convexity advantage of this FRA over an "equivalent" futures contract. By equivalent, we mean a contract with the same interest rate sensitivity and the same strike rate. Let us posit an investor seeking to profit from a decline in rates. This investor will sell an FRA and will have the following payoff at maturity of the FRA:*

$$V(m) = \frac{1}{1 + L(m, m, n) \frac{n-m}{360}} \times [\bar{R} - L(m, m, n)] \times \frac{n-m}{360} \times N$$

The value of the FRA position at time zero is $V(0) = 0$, by construction since $\bar{R} = L(m, m, n)$. If all rates decrease by 1 basis point (0.01%) instantaneously at time 0, then the new value is

$$V_1(0) = \frac{1}{1 + 4.19\% \times \frac{183}{360}} \times [4.35\% - 4.34\%] \times \frac{92}{360} \times 100mn = 2,502 \text{ USD}$$

How many Eurodollar futures will have the same interest rate sensitivity as the above FRA? We know that one contract earns 25 USD per basis point: we therefore need $2,502/25 = 100.08$ contracts. Let us now calculate the change in value of both contracts for large interest rate changes: if rates fall instantaneously by 100 basis points (1%), then the value of the FRA is

$$V_2(0) = \frac{1}{1 + 3.2\% \times \frac{183}{360}} \times [4.35\% - 3.35\%] \times \frac{92}{360} \times 100mn = 251,465 \text{ USD}$$

whereas the value of the Eurodollar futures is

$$25 \times 100 \times 100.08 = 250,200 \text{ USD}$$

The convexity advantage is therefore $251,465 - 250,200 = 1,265 \text{ USD}$. If all rates rose instantaneously by 100 basis points relative to their initial values, then the value of the FRA is

$$V_3(0) = \frac{1}{1 + 5.2\% \times \frac{183}{360}} \times [4.35\% - 5.35\%] \times \frac{92}{360} \times 100mn = -248,974 \text{ USD}$$

Here again, the FRA has a convexity advantage equal to $-248,974 - (-250,200) = 1,226 \text{ USD}$.

When rates either rise or fall, therefore one gains more from paying LIBOR and receiving fixed in an FRA than from buying a Eurodollar futures contract. There is a systematic advantage to being short Eurodollar futures relative to deposits, swaps, or FRAs. Because of this advantage, which we characterize as a convexity bias, Eurodollar futures prices should be lower than their so-called fair value. The next question is how to size a convexity adjustment that will compensate for the discrepancy between FRAs and futures.

3.1 Burghardt model

This model allows us to have a measure of the convexity advantage without the need of a stochastic rate model. That is much more simple mathematically. The model is based in considering that the difference in the performance of a swap (or a FRA) and the performance of a Eurodollar futures contract depends on three things:

1. The size of the change in the forward rate.
2. The size of the change in the term rate (or zero coupon bond price).
3. The correlation between the two.

To understand this fact better we consider the following example:

Example 3.2 *Imagine the following structure of Eurodollar futures prices and rates shown in Figure 5.*

Quarter	Eurodollar futures			
	Expiration	Price	Implied futures rate (percent)	Days in period
1	6/13/94	95.44	4.56	98
2	9/19/94	94.84	5.16	91
3	12/19/94	94.14	5.86	84
4	3/13/95	93.91	6.09	98
5	6/19/95	93.61	6.39	91
6	9/18/95	93.36	6.64	91
7	12/18/95	93.12	6.88	91
8	3/18/96	93.08	6.92	91
9	6/17/96	92.98	7.02	91
10	9/16/96	92.89	7.11	91
11	12/16/96	92.74	7.26	91
12	3/17/97	92.72	7.28	91
13	6/16/97	92.63	7.37	91
14	9/15/97	92.55	7.45	91
15	12/15/97	92.42	7.58	91
16	3/16/98	92.42	7.58	91
17	6/15/98	92.34	7.66	91
18	9/14/98	92.28	7.72	91
19	12/14/98	92.16	7.84	91
20	3/15/99	92.17	7.83	91

Swap payment on: 6/14/99

Figure 5: Structure of Eurodollar futures (June 13, 1994)

These were the final settlement or closing prices on Monday, June 13, 1994. Each of the implied futures rates roughly correspond to a three-month period. The actual number of days covered by each of the futures contracts is shown in the right hand column. The rates implied by a strip of Eurodollar futures prices together with an initial spot rate can be used to calculate the terminal value of \$1 invested today. For example,

$$TW_T = \left[1 + R_0 \left(\frac{D_0}{360} \right) \right] \times \left[1 + F_1 \left(\frac{D_1}{360} \right) \right] \times \dots \times \left[1 + F_n \left(\frac{D_n}{360} \right) \right]$$

where

TW_T is the terminal wealth of \$1 invested today for T years

R_0 is spot LIBOR to first futures expiration

F_i is the futures rate for each Eurodollar contract, $i = 0, \dots, n$

D_i is the actual number of days in each period, $i = 0, \dots, n$

From this value of terminal wealth, we can calculate Eurodollar rates in several forms including money market, semiannual bond equivalent, and continuously compounded. All three are zero-coupon bond rates implied by a set of Eurodollar futures prices. The zero-coupon bond price is:

$$P = \frac{1}{TW_T}$$

In the example data we get $TW_T = 1.41509$ and $P = 0.70667$. To calculate the hedge ratio:

$$\text{Hedge Ratio} = A \times 0.0001 \times (\text{Days}/360) \times \text{Zero-Coupon Bond Price}/\$25$$

In our case, Hedge Ratio = 71.45. Note that the person who is long the swap (that is, the person who pays fixed and receives floating) receives \$2,527.78 ($0.0001 \times (91/360) \times \$100,000,000$) for each basis point that 3-month LIBOR is above 7.83 percent. For each basis point that 3-month LIBOR is below 7.83 percent, the person who is long the swap pays \$2,527.78. That is the DV01.

To compute the semiannual bond equivalent yield we have to compute the value of R_S that satisfies

$$\left[1 + \frac{R_S}{2}\right]^{2T} = TW_T$$

that is

$$R_S = [TW_T^{\frac{1}{2T}} - 1] \times 2$$

When all the forward rates increase 10 basis points the zero-coupon bond falls to \$0.7315 and R_S is 0.07169204. To \$0.70667 the value of R_S is 0.0706577 and the difference is 10.3 basis points. The value of a swap is:

$$\text{Swap Value} = A \times \left[(X - F) \times \frac{\text{Days}}{360} \right] \times \text{Zero-Coupon Bond Price}$$

Finally we have a table to summarize these numbers (see Figure 6):

Interest rate changes		Short swap P/L				
Forward rate	Term rate on zero-coupon bond	Nominal value	Price of zero-coupon bond	Present value	Short Eurodollar P/L	Net
(basis points)	(basis points)	(as of 6/14/99)	(as of 6/13/94)	(as of 6/13/94)	(71.45 contracts)	
		(notional principal amount = \$100 million)				
10	10.3	(\$25,278)	0.70315	(\$17,774)	\$17,863	\$89
0	0	0	0.70667	\$0	\$0	\$0
-10	-10.3	\$25,278	0.71020	\$17,952	(\$17,863)	\$89

Figure 6: Swap and Eurodollar futures P/Ls

We see how the net profit is symmetric respect to the yield change. Remember that in the last example (see example 3.1) the profits were nonsymmetric. That was because we did not consider the correlation between forward and spot rates, that is, we have to consider that when forward change, spot change. If we repeat the last computations we get:

Zero-coupon yield change (bp)	Forward rate change (bp)				
	-10	-5	0	5	10
10	(\$86)	(\$43)	\$0	\$43	\$86
5	(\$43)	(\$22)	\$0	\$22	\$43
0	\$0	\$0	\$0	\$0	\$0
-5	\$43	\$22	\$0	(\$22)	(\$43)
-10	\$86	\$43	\$0	(\$43)	(\$86)

Figure 7: Net P/Ls for a Short Swap Hedged with Short Eurodollar Futures

In this example we have seen how a changes in forward rates and a change in a spot rate (considering its correlation) change the net profit in the difference between swap and Eurodollar futures. Now we are going to construct a measure to compute the convexity adjustment. We have seen that the net present value of a forward swap that receives fixed and pays floating for a 3-month period is:

$$NPV = A \times (X - F) \times (Days/360) \times P_z$$

where A is the swap's notional principal amount. X is the fixed rate at which the swap is struck, F is the forward rate, $Days$ is the actual number of days in the swap period to which the floating and fixed rates apply, and P_z is the fractional price of a zero-coupon bond that matures on the swap payment date. The interest rates in this expression are expressed in percent (that is, 7 percent would be 0.07). If we multiply and divide this expression by \$1,000,000 as well as by 90, we get

$$NPV = (A/\$1mn) \times [(X - F) \times 10,000] \times [(Days/90) \times (90/360) \times \$100] \times P_z$$

which is fairly messy but allows us to arrive at

$$NPV = (A/\$1mn) \times (X^* - F^*) \times (Days/90) \times \$25 \times P_z$$

in which X^* and F^* are expressed in basis points. When a typical swap is transacted, we begin with $X^* = F^*$ so that the net present value of the swap is zero. When interest rates change, both F^* and P_z change, and both contribute to the swap's profit or loss. For a change of ΔF^* in the forward rate and ΔP_z in the price of the zero, the profit on the forward swap is

$$\Delta NPV = -(A/\$1mn) \times (Days/90) \times \$25 \times \Delta F^* \times (P_z + \Delta P_z)$$

Because the change in the value of one Eurodollar futures contract is equal to $-\$25 \times \Delta F^*$, the number of futures contract needed to hedge against unexpected changes in rates would be

$$\text{Hedge Ratio} = -(A/\$1mn) \times (Days/90) \times P_z$$

This hedge ratio makes sense. The minus sign indicates that the hedger must short the contracts, $A/\$1mn$ captures the nominal number of contracts required, $Days/90$ reflects the importance of the day count in the swap, and P_z provides the present value correction for the difference in timing

of the cash flows on the futures and the swap. Given this hedge ratio, the profit on the short Eurodollar futures position would be

$$(A/\$1mn) \times (Days/90) \times P_z \times (\Delta F^* + Drift) \times \$25$$

where the drift represents the systematic change in the Eurodollar futures rate relative to the forward rate needed to compensate for the convexity difference between the swap and the futures contract. To eliminate any possibility of a free lunch in this hedge, the expected profit of the hedged swap must be zero. Put differently, the expected profit on the swap must exactly offset the expected profit on the Eurodollar position. Because the $[(A/\$1mn) \times (Days/90) \times \$25]$ is common to both the profit on the swap and the profit on the Eurodollar position, this part of both expressions cancel out. The result of setting the two combined profits equal to zero and rearranging shows us that

$$\mathbb{E}[\Delta F^* \times (P_z + \Delta P_z)] = \mathbb{E}[P_z \times (\Delta F^* + Drift)]$$

Because P_z is a known number, we can solve for the drift by dividing through by P_z within the expectations to get

$$\mathbb{E}[Drift] = \mathbb{E}[\Delta F^* \times (\Delta P_z/P_z)]$$

If we combine this expression with the fact that the average move in forward rates and term rates will be zero and use the formula for correlation, we arrive at the rule of thumb:

$$\begin{aligned} \text{Corr}(\Delta F^*, \Delta P_z/P_z) &= \frac{\text{Cov}(\Delta F^*, \Delta P_z/P_z)}{\sqrt{\text{Var}(\Delta F^*)\text{Var}(\Delta P_z/P_z)}} \\ &= \frac{\mathbb{E}[\Delta F^* \Delta P_z/P_z] - \mathbb{E}[\Delta F^*] - \mathbb{E}[\Delta P_z/P_z]}{\sqrt{\text{Var}(\Delta F^*)\text{Var}(\Delta P_z/P_z)}} \end{aligned}$$

Then:

$$\mathbb{E}[Drift] = \sqrt{\text{Var}(\Delta F^*)} \times \sqrt{\text{Var}(\Delta P_z/P_z)} \times \text{Corr}(\Delta F^*, \Delta P_z/P_z) \quad (13)$$

To end this model we are going to go through some practical consideration:

- This model assumes nothing about the distribution of rates changes.
- The drift is expressed in basis points per period if the standard deviation of ΔF^* is in basis points per period.
- To use volatilities from the options market, relative or percentage rate volatilities must be converted to absolute rate volatilities by multiplying by the level of the interest rate. Also the standard deviations of the zero coupon yield changes and eurodollar rate changes should be annualized.
- $\Delta P_z/P_z$ is the unexpected return on a zero-coupon bond over the period. It should be expressed as a fraction (for example, as 0.015). The easiest way to compute the standard deviation of the zero's continuously compounded yield and duration.
- The length of the period over which you calculate changes in rates is not terribly important as long as the duration for the zero-coupon bond is chosen to be its average years to maturity over the period. A period of one day would be theoretically correct, because mark-to-market actually occurs daily in the futures market. But this would be computationally overkill. Using a quarterly period produces almost the same result as daily calculations but involves a lot less work.

Burghardt model Implementation

Now we are going to compute the convexity bias between the dates January, 1st 2013 and June, 1st 2014 for different maturities. Eurodollar futures rates have been obtained from Bloomberg and zero coupon rates have been constructed by using futures rates as it has been explained above (see example 3.2). We have chosen a rolling window of three years to compute the volatilities and correlations, i.e, we use three years of data to estimate the volatilities and the correlations of our model. The implementation of this model consist in applying (13) to the data obtained from Bloomberg. Next we show the results of this models for different maturities of the swap. We analyse maturities from 1 to 10 years and show the proper graphics for this maturities.

Burghardt model results: 1 Year Convexity

If the maturity is 1 year we have the following convexity bias:

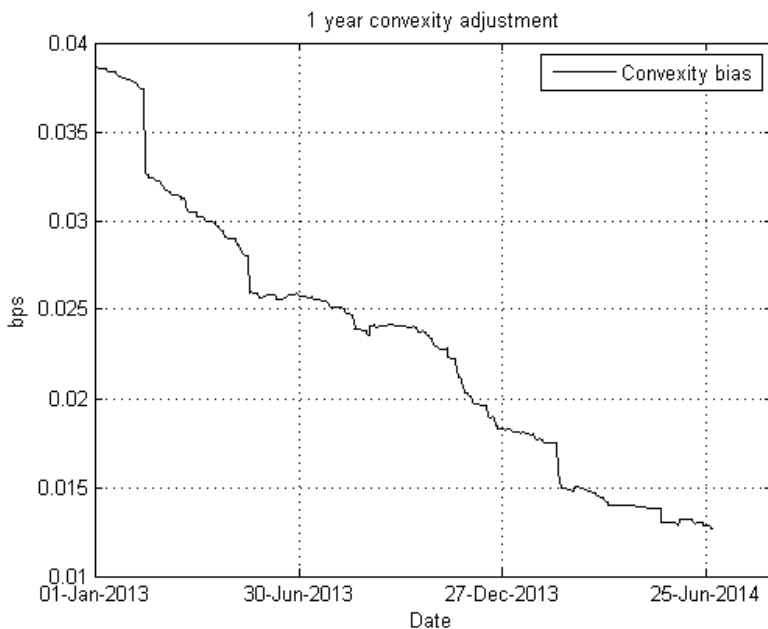


Figure 8: Burghardt model 1 year convexity bias

So we can observe that the convexity adjustment is less than 0.05 basis point for maturities of 1 year. This difference is so small that it could be not considered in order to price a FRA (or swap) with this maturity. Also we can observe that in the first part of the period the convexity is greater. Eurodollar futures rates and its convexity adjusted value is shown below:

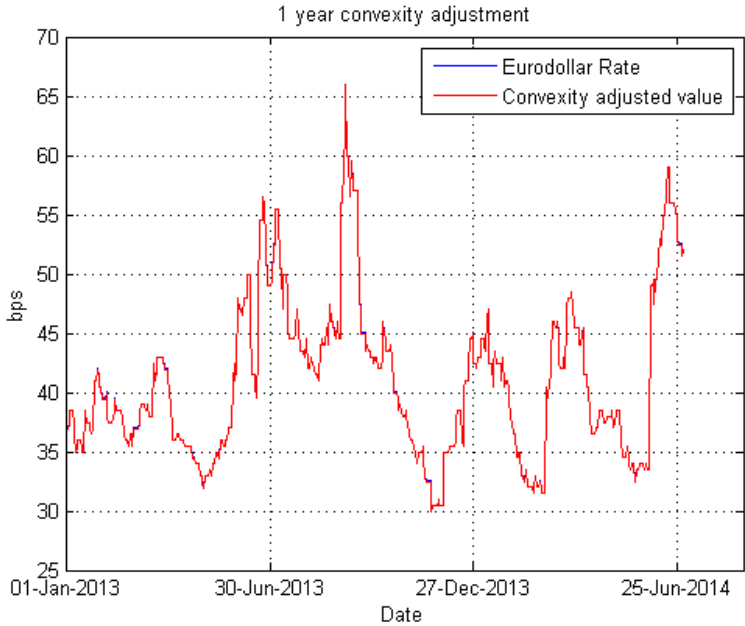


Figure 9: Burghardt model 1 year convexity adjustment

Burghardt model results: 5 Year Convexity

If the maturity is 5 years we obtain the following results:

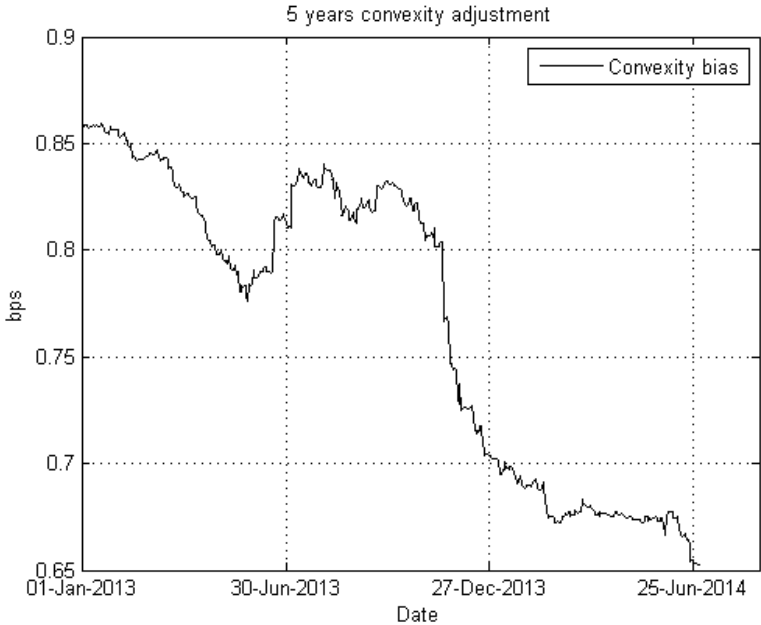


Figure 10: Burghardt model 5 years convexity bias

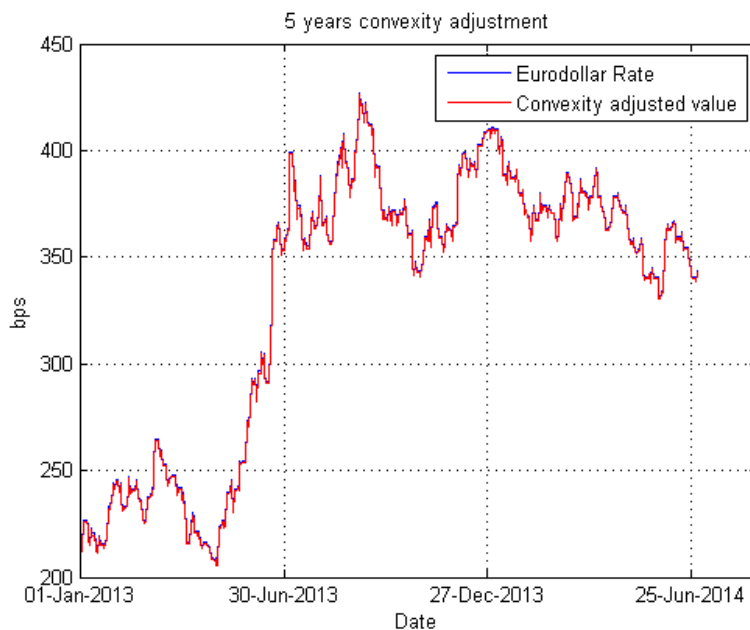


Figure 11: Burghardt model 5 years convexity adjustment

In this case the convexity is not so small. It is greater than 0.5 basis points and almost 1 basis point in the first part of the period. Also we can appreciate a decreasing shape of the time series.

Burghardt model results: 10 Year Convexity

If the maturity is 10 years we have:

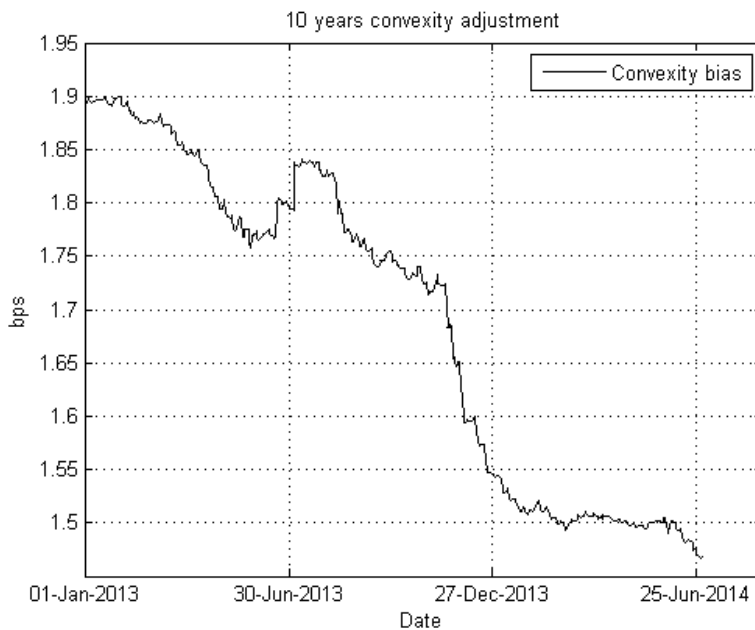


Figure 12: Burghardt model 10 years convexity bias

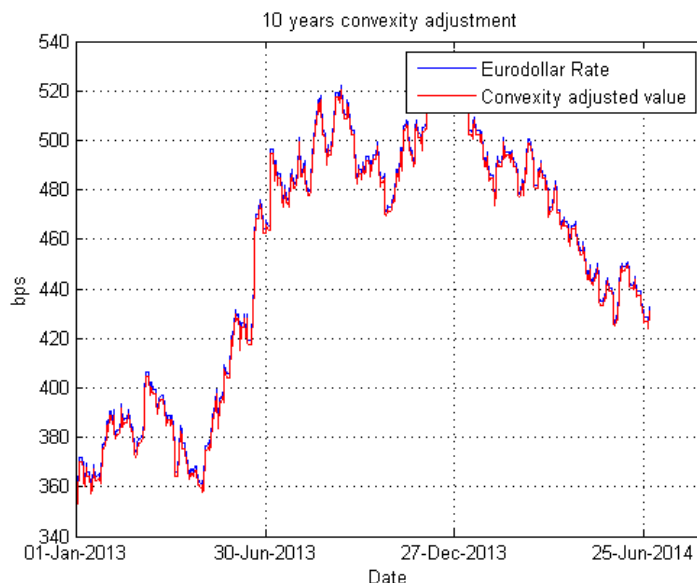


Figure 13: Burghardt model 10 years convexity adjustment

In this case the convexity is an important factor to take into account. It moves around 1.5 basis points and it is almost 2 basis points in the first part of the period. A difference of 1.5 basis points could be very important in some financial instruments. We also observe the decreasing shape. A table with the value of the convexity adjustment for the rest of the years is provided below.

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Average Bias (bps)	0,02	0,12	0,3	0,54	0,77	0,95	1,12	1,30	1,51	1,69

It is important to understand the shape of the convexity bias across the period of time. To understand it, we should observe how the volatilities moves across the period:

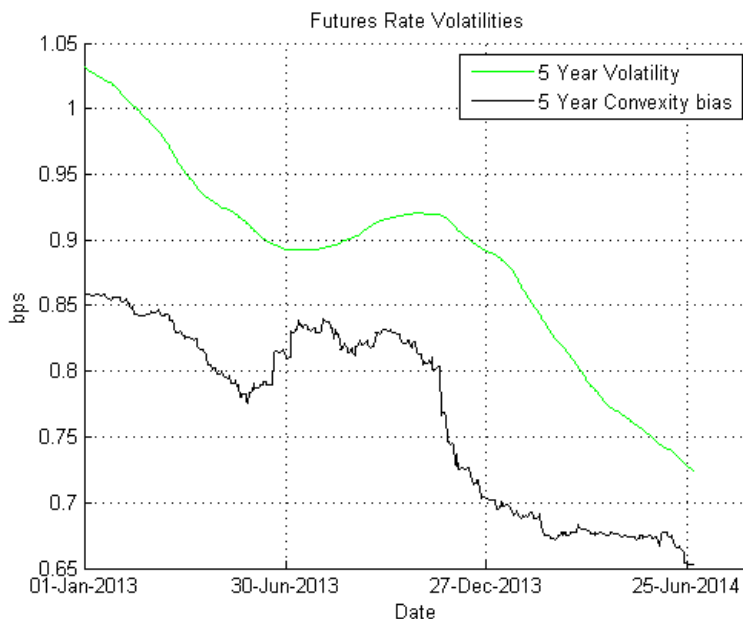


Figure 14: 5 year futures rate volatility

Note how this shape is very similar to the convexity adjustment for the 5 years. In fact, in this case the volatility explains around the 98% of the variability of convexity bias volatility. In conclusion, the Burghardt model give us a simple way to estimate the convexity bias as well as a good understanding of the two most important factors in this bias, maturity and volatility. Finally, we can construct the following linear regression where the dependent variable is the convexity bias and the independent variable is the swap (or FRA) maturity in years. As it could be seen in Figure 15 the regression model is

$$\text{Convexity bias} = -0.219 + 0.189 \times \text{Maturity} + 0.0002 \times \text{Maturity}^2$$

where the R squared is 99%.

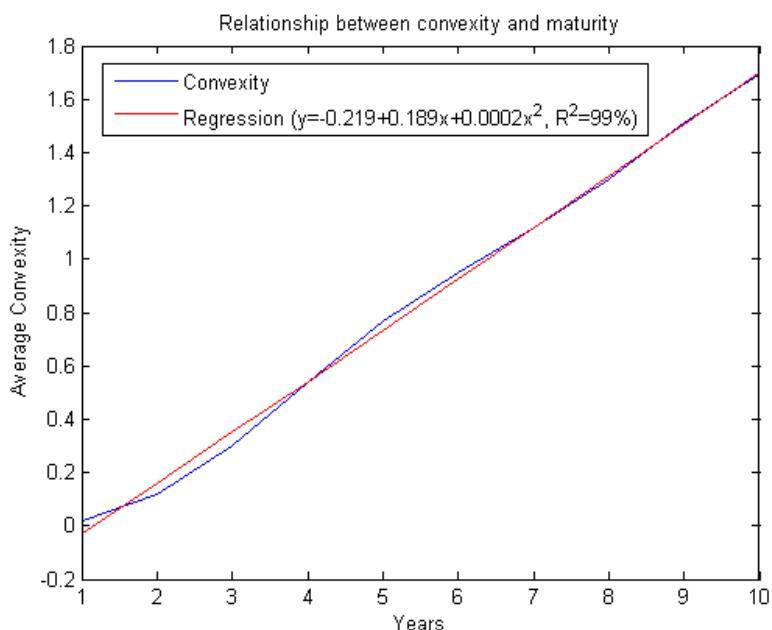


Figure 15: Relationship with maturity

It will be better understood why we are using a second order linear regression when we study the next model. This model gives us an easy way to estimate the convexity bias for any maturity. Without this regression will be impossible to estimate the bias for more than 10 years by using Eurodollar futures rates. It is because there is no Eurodollar future with maturity longer than 10 years. However we could get a interpolation of the Eurodollar curve to get a logical value for these non existing ultra long term Eurodollar futures. With this estimation we could approach the true value of the corresponding long term FRA. Note that a long term FRA could exist in a over-the-counter market, so this model makes sense to price this possible assets. The value of the convexities obtained by this regression model is in the next table:

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Rate(%)	0,00	0,16	0,35	0,54	0,73	0,92	1,11	1,31	1,50	1,69
Tenor	11y	12y	13y	14y	15y	16y	17y	18y	19y	20y
Rate(%)	1,89	2,08	2,28	2,47	2,67	2,87	3,06	3,26	3,46	3,66
Tenor	21y	22y	23y	24y	25y	26y	27y	28y	29y	30y
Rate(%)	3,86	4,06	4,25	4,45	4,66	4,86	5,06	5,26	5,46	5,67

3.2 Campbell and Temel model

Campbell and Temel model is a simple model that gives a simple rule for the convexity adjustment. We assume continuous compounding and a flat yield curve. All forward rates are equal to f and have the following stochastic dynamics (see Appendix 5.3):

$$df = \sigma dW \quad (14)$$

Under these assumptions, the value of a bond $P(t, T)$ is

$$P(t, T) = e^{-f(T-t)} \quad (15)$$

where the time variables are defined on the timeline (in years). The payoff at $(T - 1)$ is the discounted value of a payoff $(f - \bar{f})$ at time T where \bar{f} is the prespecified strike rate and $f(T - 1, T - 1, T)$ is one-year rate prevailing at $T - 1$. $P(T - 1, T)$ is the value at time $T - 1$ of a bond maturing at T with face value 1. It is easy to see, using previous argument, that the value of the FRA at time t is

$$V = P(t, T) \times [f(t, T - 1, T) - \bar{f}]$$

For simplicity we denote $f(t, T - 1, T)$ by f . For V to be equal to zero at t , date of inception of the FRA, we set $\bar{f} = f$. Also for ease of notation, set $P(t, T) = P$. If both f and P follow Itô processes, then we note from (14) that the change in V can be obtained using a two-dimensional Itô's lemma (see Appendix 5.2):

$$dV = (f - \bar{f})dP + Pd f + dP d f = Pd f + dP d f$$

Consider now a futures contract valued at F , with the same characteristics as the FRA. As explained above, we need to subtract from the rate f a convexity adjustment k_f to do away with the forward-futures discrepancy. The futures contract pays M when rates change by one unit. Then the change in F is

$$dF = -M(df - dk_f) \quad (16)$$

We now construct a portfolio composed of one FRA contract and N futures contracts. The change $d\Pi$ in the value of the portfolio is

$$d\Pi = dV + NdF = Pd f + dP d f - NMdf + NMdk_f$$

We pick $N = P/M$ to get

$$d\Pi = dP d f + Pdk_f$$

$d\Pi = 0$ will set the futures-forward discrepancy to zero. Then

$$dk_f = -\frac{dP}{P}df \quad (17)$$

Applying Itô's lemma to (15) we get

$$\begin{aligned} dP &= -(T-t)Pdf + (T-t)^2P(df)^2 + fPdt \\ &= -(T-t)Pdf + [(T-t)^2P\sigma^2 + fP]dt \end{aligned}$$

where the last line follows from (14): $(df)^2 = (\sigma dW)^2 = \sigma^2 dt$. Therefore

$$dP d f = -(T-t)P(df)^2 = -(T-t)P\sigma^2 dt \quad (18)$$

From (17) and (18)

$$dk_f = (T-t)\sigma^2 dt$$

and the convexity adjustment $k_f(T)$ applicable to a contract with accrual until T is

$$\begin{aligned} k_f(T) &= k_f(t) + \int_t^T (T-t)\sigma^2 dt \\ &= \left[\left(Tt - \frac{t^2}{2} \right) \sigma^2 \right]_t^T = -\frac{\sigma^2}{2}(T-t)^2 \end{aligned}$$

This way we have got a simple rule to compute the convexity bias. Note that this bias depends on maturity and volatility as we had seen with the Burghardt model. This quadratic relationship with the volatility was the reason why we used a second order regression to construct the Burghardt regression model. Before using this model we present a short exercise for a better understanding.

Example 3.3 *The Eurodollar futures contract on the three-month LIBOR is expiring in 14 months. The futures rate is 4.2%. What is the forward rate applicable to a FRA, when the basis point volatility is $\sigma = 1\%$? First note that $(T-t) = 17/12$ years. The forward hence should be lower than the futures by*

$$\frac{(1\%)^2}{2} \times \left(\frac{17}{12} \right)^2 = 1 \text{ basis point} = 0.01\%$$

The convexity adjustment is 1 basis point. The forward rate is

$$4.20\% - 0.01\% = 4.19\%$$

Campbell and Temel model Implementation

We take the same time period that we used for the Burghardt model, from January, 1st 2013 to June, 1st 2014. The only data we need are the Eurodollar futures rates that we have to use to compute the volatilities. As in the last model we will use periods of 3 years to do the calculations.

Campbell model results: 1 Year Convexity

The results for 1 year of maturity are:

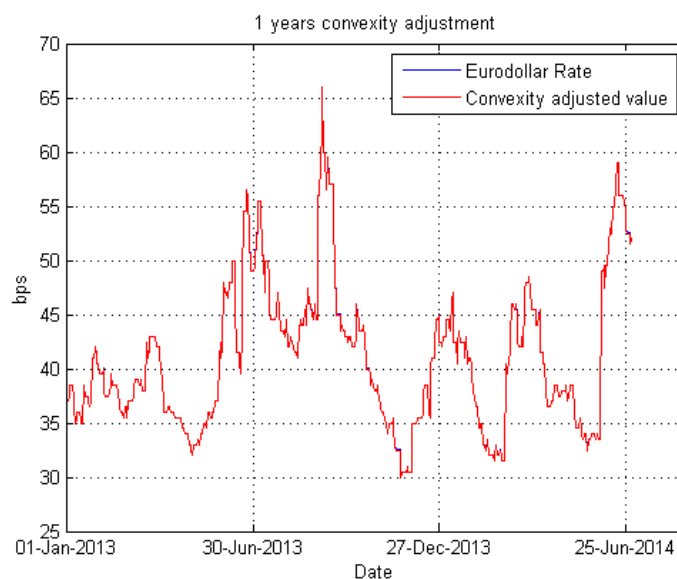


Figure 16: Campbell model 1 Year Convexity adjustment

And the convexity bias is:

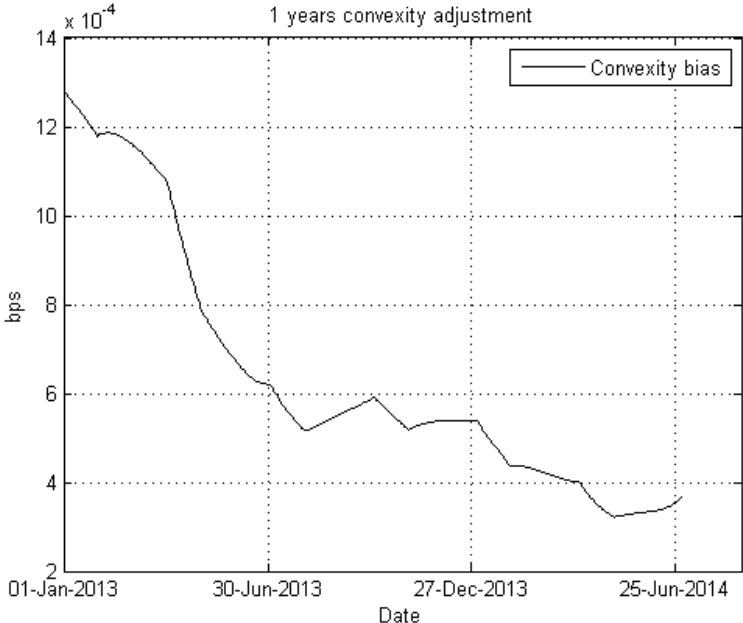


Figure 17: Campbell model 1 Year Convexity bias

As we can observe above in Figure 16 and Figure 17 the convexity bias in this case is almost insignificant. It moves around the 0.0006 basis points. We can observe too how the decreasing volatility is reflected in the behavior of the convexity.

Campbell model results: 5 Year Convexity

The results of the model for 5 year maturity are:

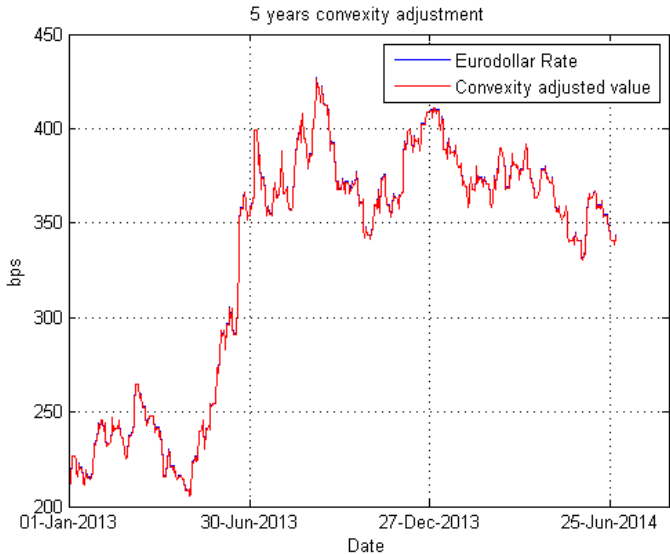


Figure 18: Campbell model 5 Year Convexity adjustment

and the convexity bias:

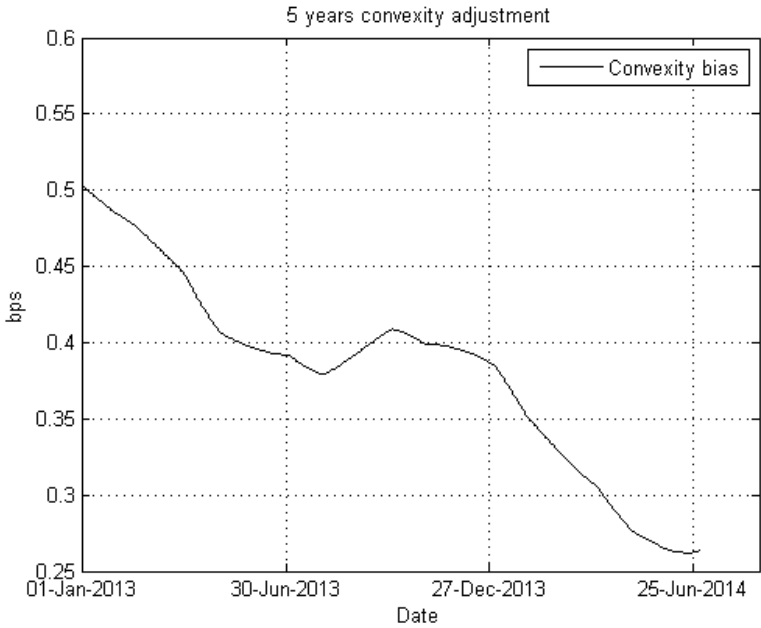


Figure 19: Campbell model 5 Year Convexity bias

In this case the volatility is not so insignificant and it moves around 0.5 basis points. As in the rest of figures we observe the shape of the volatility.

Campbell model results: 10 Year Convexity

The result of the model for 10 year maturity is:

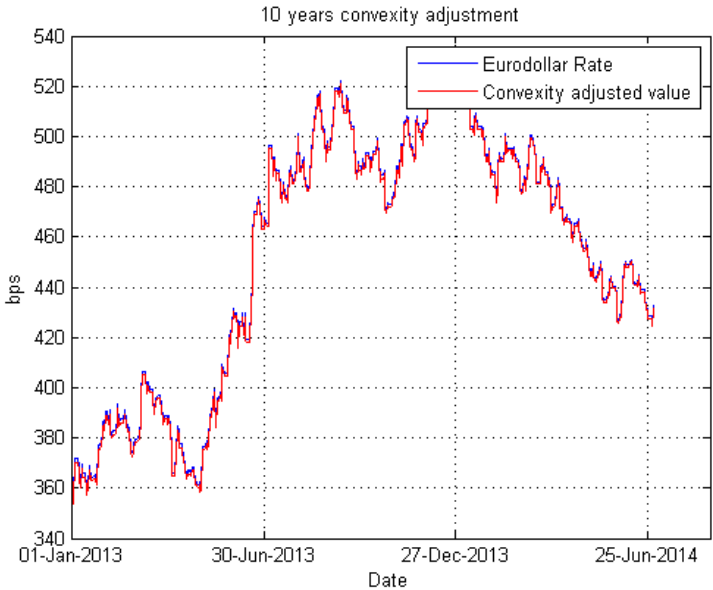


Figure 20: Campbell model 10 Year Convexity adjustment

and the convexity bias:

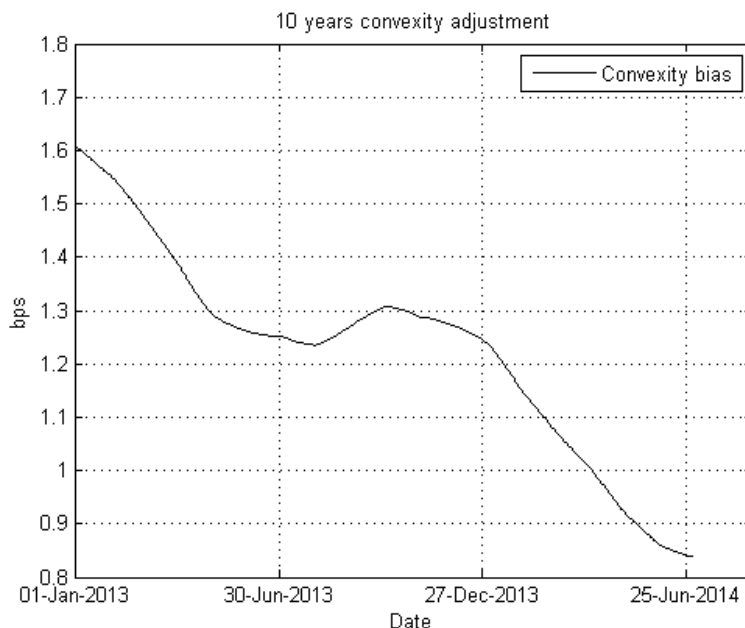


Figure 21: Campbell model 10 Year Convexity bias

We obtain a similar bias to the Burghardt model 10 year adjustment. This bias is around 1.5 basis points, a little lower than in the Burghardt model in which the bias came to 1.8. The value for the rest of the maturities is provided in the following table.

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Average Bias (bps)	0,00	0,01	0,10	0,22	0,38	0,53	0,67	0,83	1,02	1,22

Following the steps of the last model we can construct a linear regression to explain how the bias moves with longer maturities. In this case we obtain that the linear regression model is

$$\text{Convexity bias} = -0.092 + 0.05 \times \text{Maturity} + 0.008 \times \text{Maturity}^2$$

where the R squared is 99% (see Figure 22). This model give us an easy way to estimate the convexity bias for any maturity, even greater than 10 years. The values are in the next table:

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Rate(%)	0,00	0,04	0,13	0,24	0,36	0,50	0,66	0,83	1,02	1,23
Tenor	11y	12y	13y	14y	15y	16y	17y	18y	19y	20y
Rate(%)	1,45	1,69	1,95	2,22	2,51	2,81	3,13	3,47	3,83	4,20
Tenor	21y	22y	23y	24y	25y	26y	27y	28y	29y	30y
Rate(%)	4,60	5,00	5,41	5,85	6,30	6,77	7,26	7,76	8,28	8,81

In this case the values are a bit greater than in the Burghardt regression model.

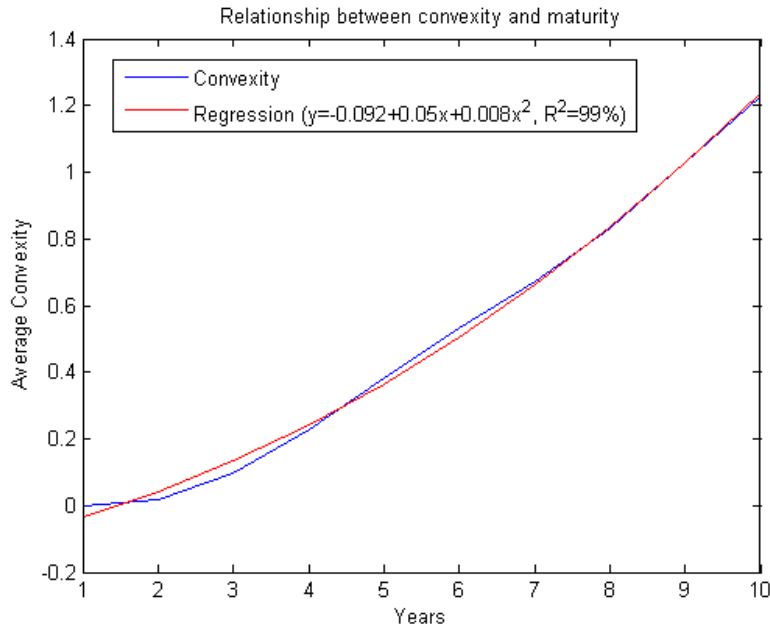


Figure 22: Relationship with maturity

3.3 Vasicek model

In this section we use Vasicek Stochastic model to compute the convexity bias. As the rest of models, this model use the mark-to-market feature of the Eurodollar futures contracts to price them. Intuitively, the mark-to-market feature resets the value of the futures contract to zero at each instant. The Vasicek model assumes that $r(t)$ follows the Ornstein-Uhlenbeck process,

$$dr = \kappa(\mu - r)dt + \sigma dz$$

Vasicek shows that under this process the prices of discount bonds are given by:

$$P(t, T) = A(t - T)\exp(-B(t - T)r(t)), \quad (19)$$

where $A(x)$ and $B(x)$ are obtained by solving the proper differential equations (see Appendix 5.4)

$$\begin{aligned} B(x) &= \frac{1 - e^{-\kappa x}}{\kappa} \\ A(x) &= \exp\left[(B(x) - x)\left(\mu^* - \frac{\sigma^2}{2\kappa^2}\right) - \frac{\sigma^2 B(x)^2}{4\kappa}\right] \\ \mu^* &= \mu - \frac{\lambda\sigma}{\kappa} \end{aligned}$$

and λ is the market price of interest rate risk per unit of σ , assumed constant. Under this stochastic dynamics we have to compute the forward rate and the futures eurodollar rate to know the difference between them. Equation (19) implies that the Vasicek forward rate is

$$f(t, T_1, T_2) = \frac{360}{T_2 - T_1} \left[\frac{P(t, T_1)}{P(t, T_2)} - 1 \right] = \frac{360}{T_2 - T_1} \left[\left(\frac{A(T_1)}{A(T_2)} \right) \exp[(B(T_2) - B(T_1))r(t)] \right] \quad (20)$$

Under the assumptions of the Vasicek model, the "no-arbitrage" Eurodollar futures rate at date t is

$$F(t, T_1, T_2) = \frac{360}{T_2 - T_1} \left[\frac{1}{A(T_2 - T_1)} \mathbb{E}[\exp((B(T_2 - T_1)r(T_1)))] - 1 \right], \quad (21)$$

where

$$\mathbb{E}[\exp(B(T_2 - T_1)r(T_1))] = \exp\left[B(T_2 - T_1)\mathbb{E}[r(T_1)] + \frac{1}{2}B(T_2 - T_1)^2\text{Var}[r(T_1)]\right],$$

$$\mathbb{E}[r(T_1)] = \exp(-\kappa T_1)r(t) + (1 - \exp(-\kappa T_1))\mu^*,$$

and

$$\text{Var}[r(T_1)] = \frac{\sigma^2(1 - \exp(-2\kappa T_1))}{2\kappa}$$

This way we can compute the convexity bias by subtracting (20) from (21).

$$\text{Bias} = F(t, T_1, T_2) - f(t, T_1, T_2)$$

Vasicek model Implementation

To use Vasicek model we need to estimate the parameters μ , κ and σ . There are a lot of ways to calibrate a stochastic model, here we are going to use the so called momentum method. In the momentum method we get the value of the unknown parameters from the equations of different $r(T_1)$ momentums. Indeed, we are not going to use only $r(T_1)$ momentums but it covariance also. One can compute the covariance for two different times t and s :

$$\text{Cov}[r(t), r(s)] = \frac{\sigma^2 e^{-\kappa(t+s)}}{2\kappa} (e^{2\kappa \max(t,s)} - 1) \quad (22)$$

Now we can apply (22) to times T_1 and $T_1 - 1$ and get the following equation system:

$$\begin{aligned} \mathbb{E}[r(T_1)] &= \exp(-\kappa T_1)r(t) + (1 - \exp(-\kappa T_1))\mu^* \\ \text{Var}[r(T_1)] &= \frac{\sigma^2(1 - \exp(-2\kappa T_1))}{2\kappa} \\ \text{Cov}(r(T_1), r(T_1 - 1)) &= \frac{\sigma^2 e^{-\alpha(2T_1 - 1)}}{2\alpha} (e^{2\alpha(2T_1 - 1)} - 1) \end{aligned}$$

by solving this non-linear equation system we obtain:

$$\begin{aligned} \kappa &= \left(\frac{1}{T_1 - 1}\right) \ln \left(\frac{\frac{\rho}{S} + \sqrt{\left(\frac{\rho}{S}\right)^2 + 4}}{2}\right) \\ \sigma &= \sqrt{\frac{2S\kappa}{1 - e^{-2\kappa T_1}}} \\ \mu^* &= \frac{\bar{x} - r(t)e^{-\kappa T_1}}{1 - e^{-\kappa T_1}} \end{aligned} \quad (23)$$

where \bar{x} is the mean of the data set, S is the standard deviation and ρ is the covariance between the time series and the same series in the last instant. Note that this non-linear system could not have a analytical solution. For this reason in some cases is really useful to use numerical methods of calibration. In our model we choose $\lambda = 0$, i.e., we suppose that there is no market price of interest rate risk. Now we can use (23) to compute the parameters that get the better fitting of our LIBOR 3-months time series. The obtained parameters are $(\kappa, \mu, \sigma) = (1.6, 26.7, 0.110)$. If we simulate different paths (around 1000 paths) and do the mean for all the path we obtain the following fitting:

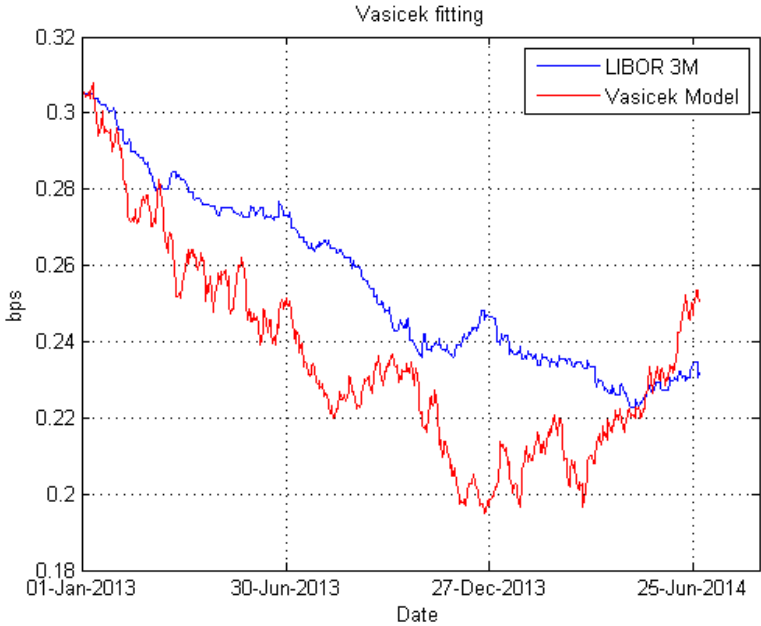


Figure 23: LIBOR 3-months Vasicek fitting

so we can conclude that the parameters explain quite good the Libor 3-month behavior in that period.

Vasicek model results: 1 Year Convexity

If we take one year of maturity, we have:

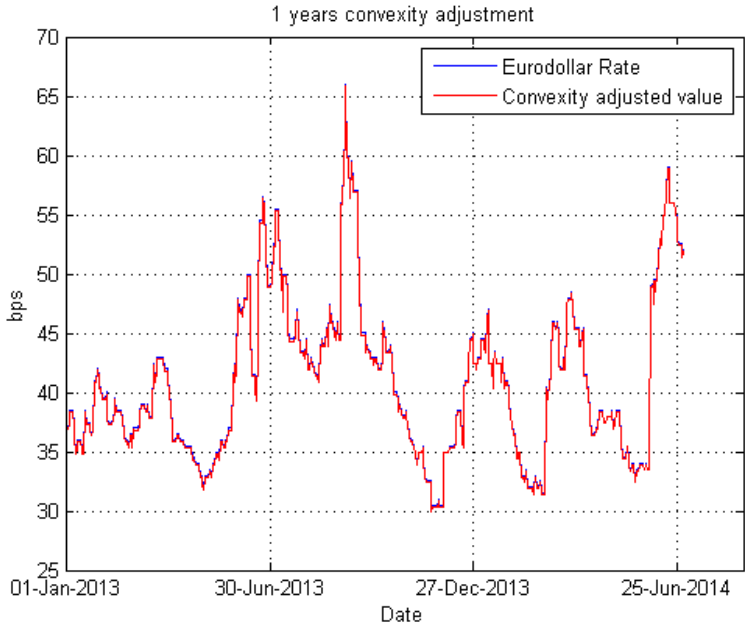


Figure 24: Vasicek model 1 Year Convexity adjustment

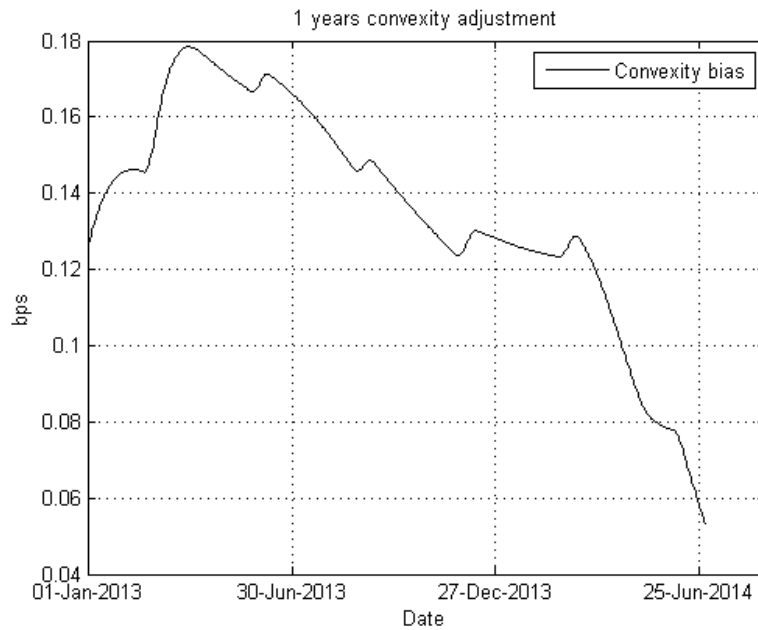


Figure 25: Vasicek model 1 Year Convexity bias

In this case we obtain a bigger value of convexity than the obtained with Burghardt or Campbell model. Note that the decreasing shape of the Figure 25 is no more because of the volatility movements. Remember that we have calculated a fixed value of σ that is what we use to price the forward and the Eurodollar futures so we are not using different volatilities as we did in the last models.

Vasicek model results: 5 Year Convexity

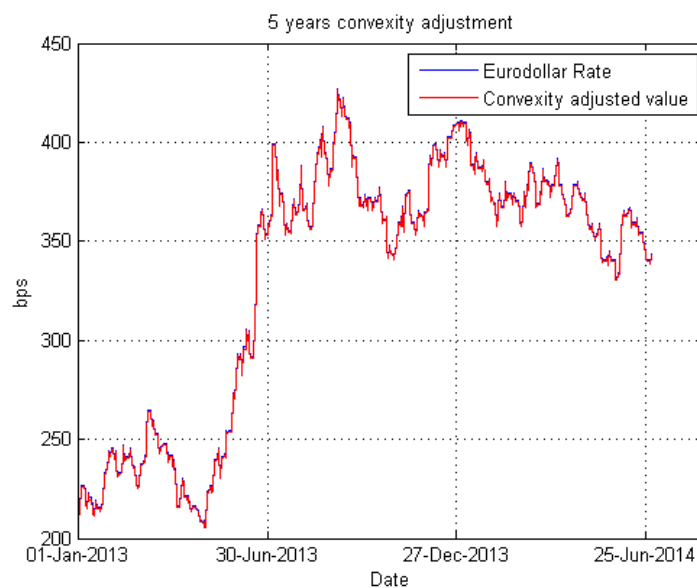


Figure 26: Vasicek model 5 Year Convexity adjustment

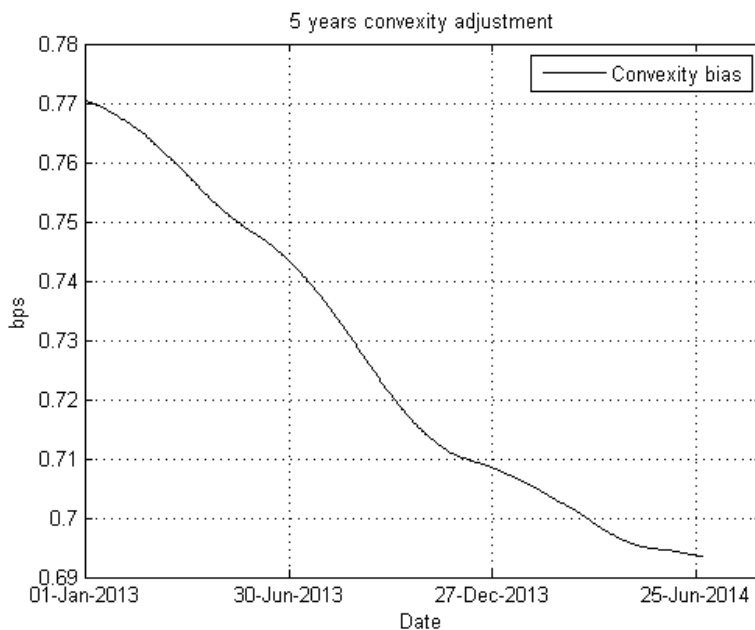


Figure 27: Vasicek model 5 Year Convexity bias

Here we can observe a average bias of 0.75 basis points. This value is greater than the value obtained with Campbell but lower than the value obtained with Burghardt model.

Vasicek model results: 10 Year Convexity

Finally, the results for the 1 years case are:

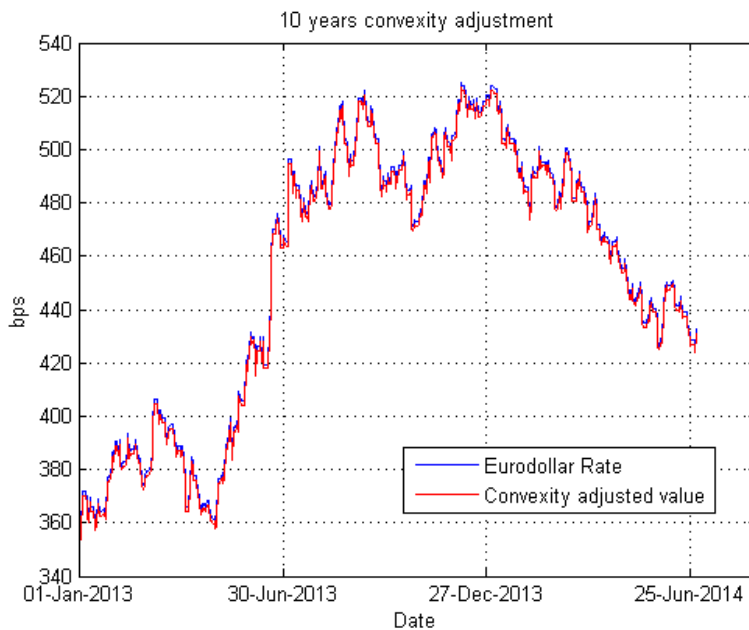


Figure 28: Vasicek model 10 Year Convexity adjustment

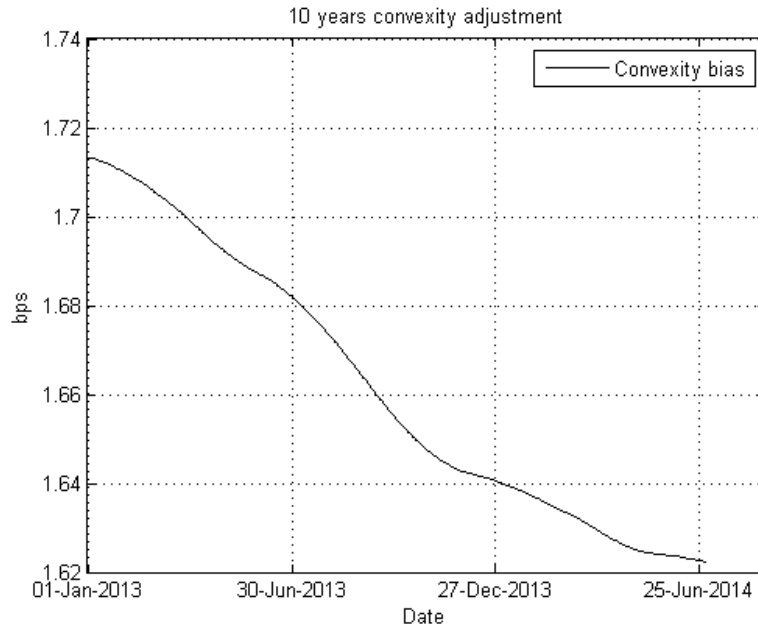


Figure 29: Vasicek model 10 Year Convexity bias

This value is again an intermediate value between Campbell and Burghardt models. As we commented above the decreasing shape can not be explained in this model by the volatility shape as we are using the same constant σ for all the volatilities in the time period. The table with the rest of values is in the net table:

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Average Bias (bps)	0,11	0,24	0,32	0,47	0,69	0,82	0,96	1,18	1,33	1,69

3.4 Cox-Ingersoll-Ross model

Cox-Ingersoll-Ross model is an improvement of Vasicek model. We first compute closed-form solutions for the Eurodollar futures and forward rate difference using Cox-Ingersoll-Ross term structure model. The Cox-Ingersoll-Ross model assumes that $r(t)$ follows the square root process,

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz$$

Under this process, the time t prices of zero coupon bonds paying one dollar at time T are given by:

$$P(t, T) = A(T - t)\exp(-B(T - t)r(t)) \quad (24)$$

where $A(x)$ and $B(x)$ are obtained by solving the proper differential equations (see Appendix 5.5)

$$\begin{aligned} A(x) &= \left[\frac{2\gamma \exp[(\kappa^* + \gamma)x/2]}{(\kappa^* + \gamma)(\exp(\gamma x) - 1) + 2\gamma} \right]^{2\kappa^*\mu^*/\sigma^2} \\ B(x) &= \left[\frac{2(\exp(\gamma x) - 1)}{(\kappa^* + \gamma)(\exp(\gamma x) - 1) + 2\gamma} \right] \\ \gamma &= [(\kappa)^2 + 2\sigma^2]^{1/2} \\ \mu &= \kappa\mu/(\kappa + \lambda) \end{aligned}$$

and

λ = the market price of interest rate risk

Equation (24) implies that the forward rates in the CIR model are given by:

$$f(t, T_1, T_2) = \frac{360}{T_2 - T_1} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) = \frac{360}{T_2 - T_1} \left[\frac{A(T_1)}{A(T_2)} \exp[(B(T_2) - B(T_1))r(t)] - 1 \right] \quad (25)$$

To avoid ambiguity in exposition, we will compare the forward rate with what we term the "futures rate", which is the price of a futures contract on the LIBOR rate. Since there is no cash outlay at the time a futures contract is initiated, the expected change in the futures rate equals the premium for bearing interest rate risk. Therefore,

$$\mathbb{E}[dF] = \lambda \frac{\partial F}{\partial r} r dt,$$

where subscripts denote partial derivatives. By Itô's Lemma (see Appendix 5.2),

$$\mathbb{E}[dF] = \left(\frac{\partial F}{\partial r} \kappa(\mu - r) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial r^2} \sigma^2 r \right) dt$$

Hence, under the CIR interest rate process, the equilibrium futures rate should satisfy the following partial differential equation:

$$\frac{\partial F}{\partial r} \kappa^*(\mu^* - r) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial r^2} \sigma^2 r = 0 \quad (26)$$

This partial differential equation is subjected to the boundary condition

$$F(T_1, T_1, T_2) = L(T_1, T_1, T_2)$$

Under the assumptions of the CIR model, the solution of this partial differential equation at date t is

$$F(t, T_1, T_2) = \frac{360}{T_2 - T_1} \left[\frac{1}{A(T_2 - T_1)} \mathbb{E}[\exp((B(T_2 - T_1)r(T_1)))] - 1 \right] \quad (27)$$

where

$$\mathbb{E}[\exp((B(T_2 - T_1)r(T_1)))] = \frac{\exp \left[B(T_2 - T_1) \frac{r(t) \exp(-\kappa^* T_1)}{1 - B(T_2 - T_1) \sigma^2 \frac{b_c(T_1)}{2}} \right]}{\left[1 - B(T_2 - T_1) \sigma^2 \frac{b_c(T_1)}{2} \right]^{\frac{2\kappa^* \mu^*}{\sigma^2}}}$$

and where

$$b_c(x) = \frac{1 - \exp(-\kappa^* x)}{\kappa^*}$$

To solve equation (26) the following ideas could be useful. The solution to the partial differential equation (26) is given by the risk-neutral expectation of the terminal futures price. In this case, with the boundary condition, the solution is:

$$F(t, T_1, T_2) = \mathbb{E}[L(T_1, T_1, T_2)]$$

where \mathbb{E} denotes expectation taken with respect to the distribution of $r(T_1)$ conditional on $r(t)$, with the dynamics of r given by the following "risk-neutral" process:

$$dr = \kappa^*(\mu^* - r)dt + \sigma \sqrt{r} dz.$$

CIR shows that the distribution of $r(T_1)$ conditional on $R(t)$ given this stochastic process is a noncentral chi-square. The moment generating function for the non-central chi-square distribution $\chi^2(\lambda, k)$ is

$$\mathbb{E}[e^{tX}] = \frac{\exp\left(\frac{\lambda t}{1-2t}\right)}{(1-2t)^{k/2}}$$

The LIBOR $L(T_1, T_1, T_2)$ interest rate is given by

$$L(T_1, T_1, T_2) = \frac{360}{T_2 - T_1} \left[\frac{1}{P(T_1, T_2)} - 1 \right] = \frac{360}{T_2 - T_1} \left[\frac{1}{A(T_2 - T_1)} \exp((B(T_2 - T_1)r(T_1)) - 1) \right]$$

Thus, $\mathbb{E}[L(T_1, T_1, T_2)]$ as defined above, is a linear function of the moment generating function of $r(T_1)$ conditional on $r(t)$. Then, the value of the convexity is given by

$$\text{Bias} = F(t, T_1, T_2) - f(t, T_1, T_2)$$

Cox-Ingersoll-Ross model Implementation

Now we put in practice the model between the same dates and taking $\lambda = 0$. In this case we will not use the momentum method to calibrate the model but we estimate the μ , κ and σ parameters by minimizing the following function:

$$F(\kappa, \mu, \sigma) = \sum_{i=0}^T (\mathbb{E}[r(t_i)] - L(t_i))^2$$

where $dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz$, $r(t_0) = L_0$ and L_0, L_1, \dots, L_T are the Libor 3-months rates for the instants t_0, t_1, \dots, t_T . This way we are looking for the μ , κ and σ parameters that minimize the distance between the model rate and the real market rate. To minimize this function we could compute an analytical expression for $\mathbb{E}[r(t)]$ but we prefer to work with a numerical approach. We will simulate a big number of paths for r and do the average to estimate the value of the function.

The minimizing process has been performed by using the following **genetic algorithm**.

- 1.) (**Create population**) We construct a set of N elements of the form (κ, μ, σ) . To simplify we choose an even number of elements N . Also we compute the value of the function F for each element.
- 2.) (**Selection operator**) We get the 25% of the population with the lowest function value. The rest of the set is eliminated.
- 3.) (**Reproduction operator**) Each element generates a new element that is the father plus a mutation component multiplied by a random number with distribution $\mathcal{N}(0, 1)$. Half population has been created by adding a random noise to the other half. We have to compute only $\frac{N}{4}$ new values of F . This is a great advantage since this function requires a lot of time to be evaluated. Now the set has $\frac{N}{2}$ elements.
- 4.) (**Mutation operator**) We use no mutation operator since we are adding noise to our sample data in the reproduction phase. Finally in the last iteration our set has only 2 elements, we select the element with lower function value.

The value of the found parameters is $(\kappa, \mu, \sigma) = (0.6, 25.1, 0.0703)$. Note that these parameters are very similar to Vasicek parameters. The path generated with these parameters is (see Figure 30):

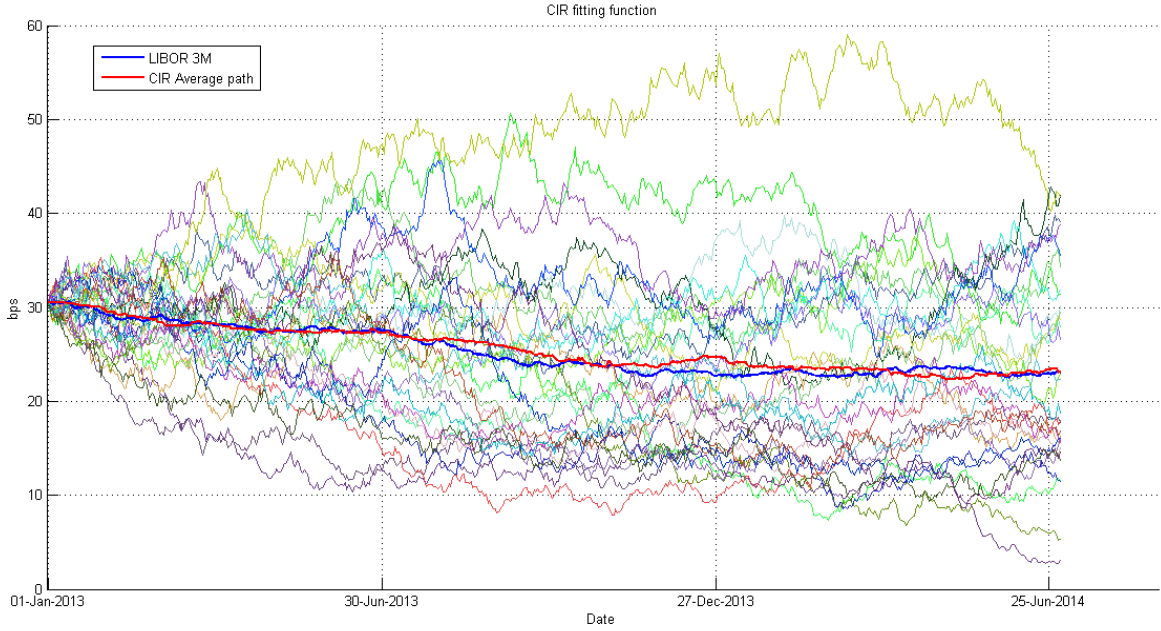


Figure 30: CIR paths simulation and fitting

Here we can observe how the average of the simulated paths approach the LIBOR 3-months in the chosen period. In the following graphic (Figure 31) we can observe better this fitting.

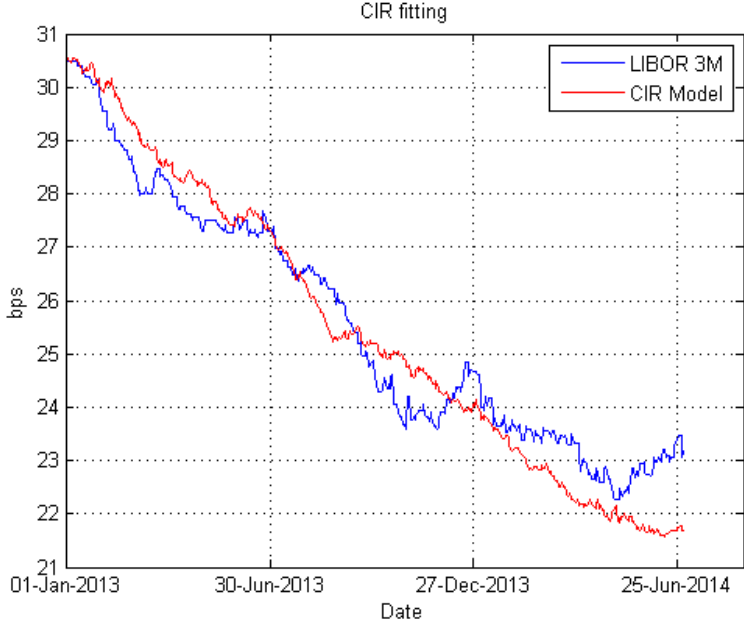


Figure 31: CIR Fitting

With the calculated parameters we can estimate the convexity for the different maturities.

CIR model results: 1 Year Convexity

The model results for 1 year of maturity are:

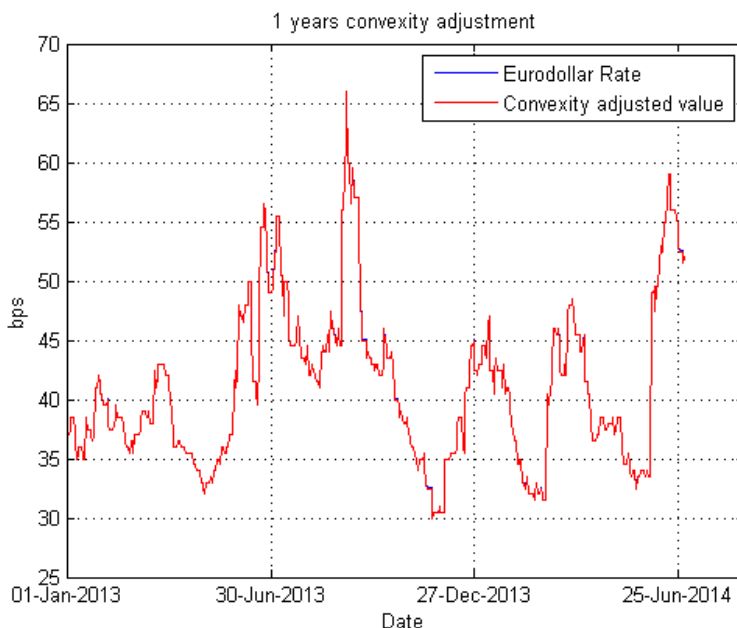


Figure 32: CIR model 1 year convexity adjustment

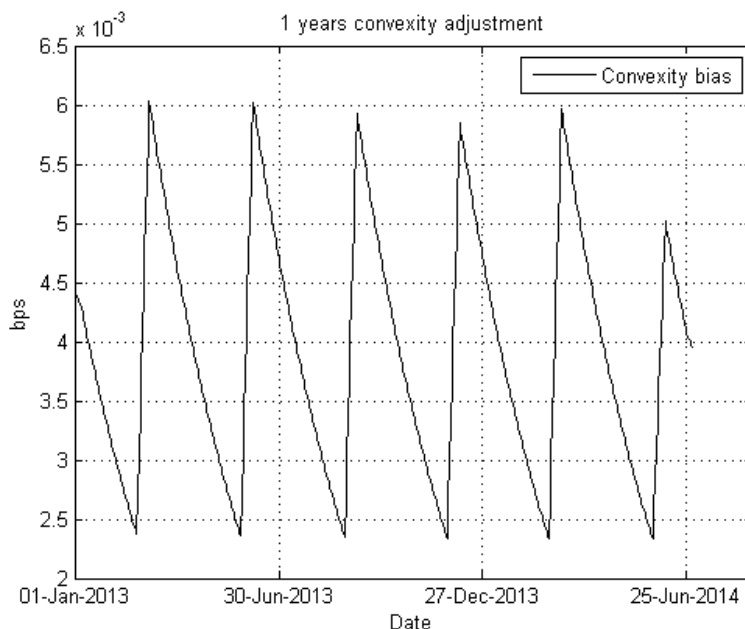


Figure 33: CIR model 1 year convexity bias

So the convexity bias for 1 year would be almost zero. The picks shown in Figure 33 are generated by the change of maturity that a future has with time, i.e, the time until the loan 3-months period begins.

CIR model results: 5 Year Convexity

To 5 years we get:



Figure 34: CIR model 5 year convexity adjustment

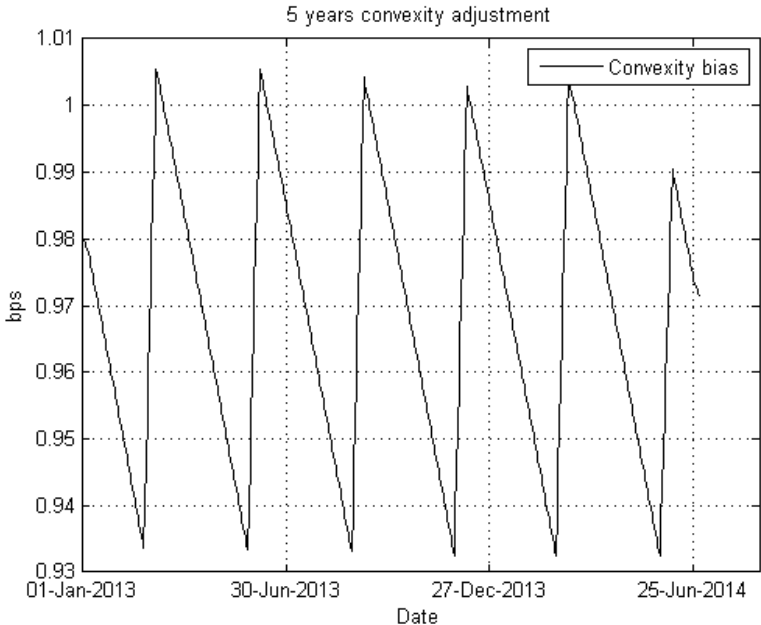


Figure 35: CIR model 5 year convexity bias

Here the convexity bias is around 1 basis point. The picks arise for the same reason as above.

CIR model results: 10 Year Convexity

In the 10 years case:

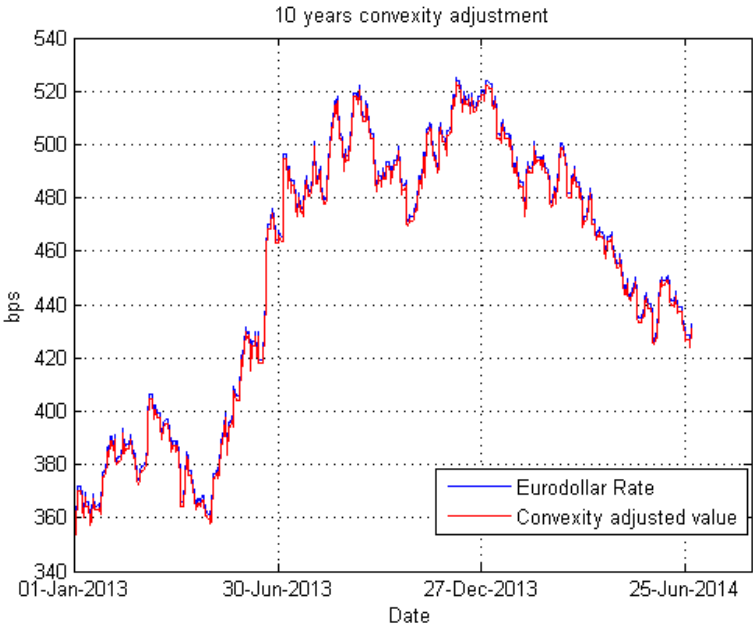


Figure 36: CIR model 10 year convexity adjustment

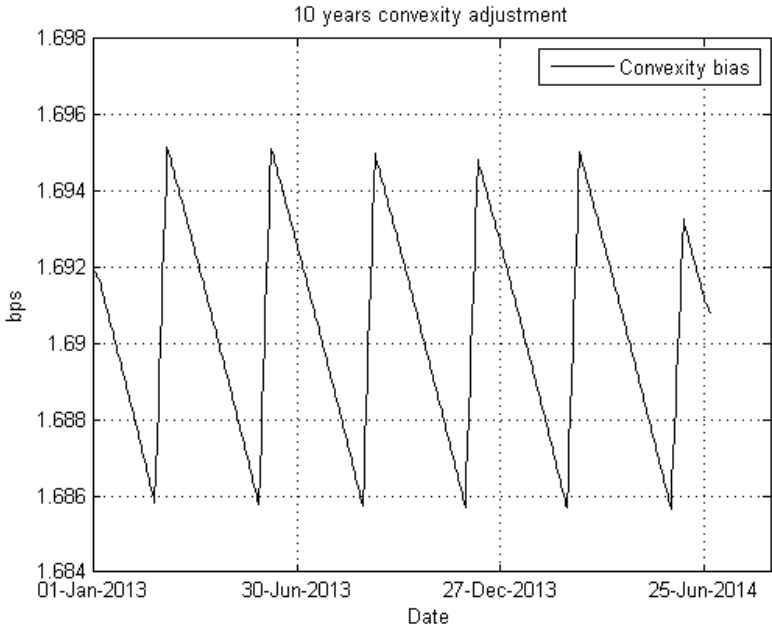


Figure 37: CIR model 10 year convexity bias

The convexity bias moves above 1.5 basis points. The table with the rest of the values is:

Tenor	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
Average Bias (bps)	0,00	0,06	0,28	0,62	0,96	1,24	1,44	1,57	1,65	1,69

3.5 Model Comparison

In this section we are going to compare the value of the different models. The different values are shown in the next Figure (see Figure 38):

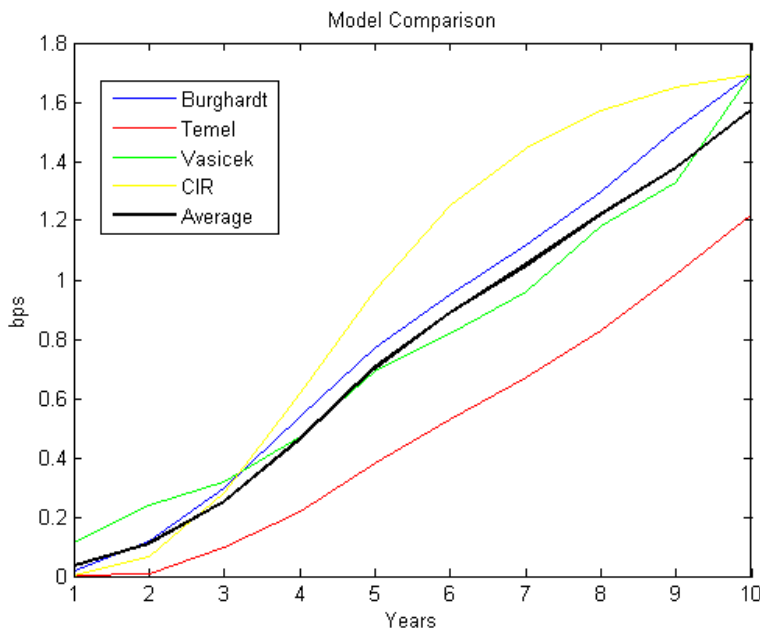


Figure 38: Model Comparison

We can see how the 4 models give similar values for the convexity bias in the same time period. The only method that has convexity values far smaller than the average between the 4 models is Campbell model. It could be due to the unrealistic hypothesis that this model makes. We also can observe a big difference between the shape of the curves. We have a almost linear curve in Burghardt and Campbell models whereas we have a quadratic function for Vasicek model and even a function with order greater than 2 for CIR model.

If we focuss on the average curve we can conclude that 0.5 basis point of convexity bias is reached in average with 4 years of maturity, 1 basis point with 7 years and 1.5 basis points with 10 years. This let us to have a clear idea of how big is the convexity bias in each part of the maturity curve. This fact could give us a great advantage in both the trading of Eurodollar futures and forward rate agreements. By knowing the difference between FRAs and Eurodollars we can know how changes in Eurodollar curve affect to FRAs curve or how changes in FRA curve should affect to Eurodollar curve.

4 Conclusion

On the one hand, we have studied how to hedge bond portfolios by using duration and convexity. We have seen that an investor can lose big amounts of money if he does not control well his portfolio risk. Duration and convexity provide a great tool to control risk portfolio. Although this tool is not perfect and has some limitations (as shown in 2.3).

On the other hand, we have studied the differences between two similar contracts: Eurodollar futures and Forward Rate Agreements (FRA). The non-linearity of the FRA payoff function makes these instruments very different from Eurodollar futures. To price a FRA correctly we have to introduce some models that let us to compute the convexity bias that should be subtracted from Eurodollar rates. We have use four models for this goal. The first model use no stochastic model on $r(t)$ so was much more simple mathematically. The second model used a very simplified stochastic model that allows us to derivate a simple formula to compute the convexity bias. This model is not very accurate but let us to understand which are the two variables that affect the most to convexity: maturity and volatility. Third and fourth models are similar. Both use stochastic models on $r(t)$ and both must be calibrated. For Vasicek model we used momentum method to calibrate it. For CIR model we used a numerical method based on minimizing a function by a genetic algorithm. This procedure can be extended to any stochastic model on $r(t)$ and it is a great alternative to momentum method. Finally we got the value of the convexity that let us to have a clear idea of how is the current relation between FRAs and Eurodollar futures. This relation is very important not only to price these assets properly but to get advantage in trading strategies.

An important observation is that convexity behaves not always in the same way. If we compare the computed convexity value with the value of the convexity in 1994 (see "The Convexity Bias in Eurodollar Futures", Burghardt) we can observe how the convexity has decreased with time in Figure 39.

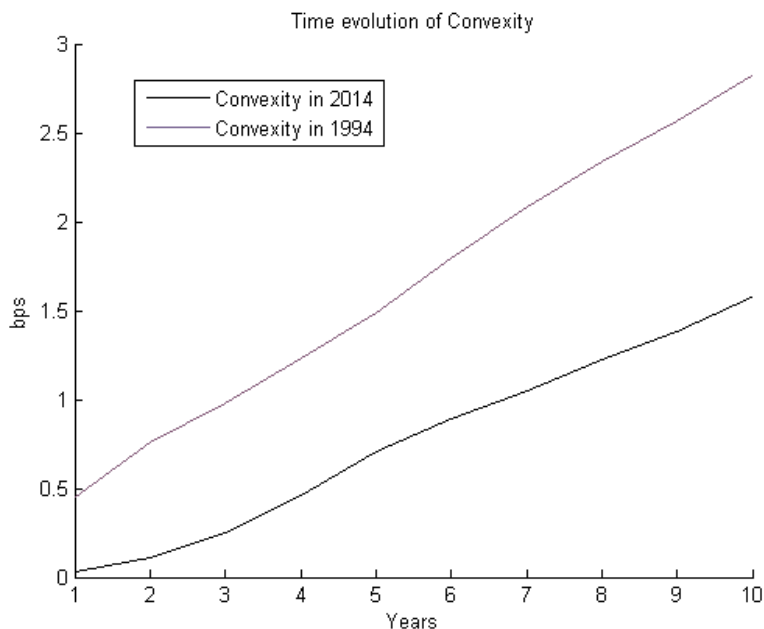


Figure 39: Convexity evolution

It occurs because all our models assume the hypothesis of a efficient market, i.e, a market where

the investors know all its possibilities and chose the best of its options. This convexity bias was not so known in 1990 and for this reason the market does not behaves as an efficient market with these financial instruments.

A possibility to go further in this work would be analysing the value of convexity bias with other models such as Hull and White or HJM models. Also we could study how this convexity bias behaves in different time periods and how big movements of volatility affect it.

5 Appendix

5.1 Nelson-Siegel-Svensson and Nelson-Siegel-Svensson Models

Nelson-Siegel (1987) and Nelson-Siegel-Svensson model (1994) are three-factor models of the yield curve that consist of using a specific regression models for the zero-coupon rate function. Nelson and Siegel have suggested to model the continuously zero-coupon rate $R(0, \theta)$ for maturity θ as

$$R(0, \theta) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\frac{\theta}{\tau_1}}}{\frac{\theta}{\tau_1}} \right] + \beta_2 \left[\frac{1 - e^{-\frac{\theta}{\tau_1}}}{\frac{\theta}{\tau_1}} - e^{-\frac{\theta}{\tau_1}} \right]$$

this form was later extended by Svensson as

$$R(0, \theta) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\frac{\theta}{\tau_1}}}{\frac{\theta}{\tau_1}} \right] + \beta_2 \left[\frac{1 - e^{-\frac{\theta}{\tau_1}}}{\frac{\theta}{\tau_1}} - e^{-\frac{\theta}{\tau_1}} \right] + \beta_3 \left[\frac{1 - e^{-\frac{\theta}{\tau_2}}}{\frac{\theta}{\tau_2}} - e^{-\frac{\theta}{\tau_2}} \right]$$

where β_0 is the limit of $R(0, \theta)$ as θ goes to infinity. In practice, β_0 should be regarded as a long-term interest rate. β_1 is the limit of $R(0, \theta) - \beta_0$ as θ goes to 0. In practice, β_1 should be regarded as long to short term spread. β_2 and β_3 are curvature parameters. τ_1 and τ_2 are scale parameters that measure the rate at which the short-term and medium-term components decay to zero.

As shown by Svensson, the extended form allows for more flexibility in yield-curve estimation, in particular at the short-term end of the curve.

5.2 Itô's Calculus

Brownian motion

A random process W_t , $t \in [0, T]$, is a (standard) Brownian motion if:

- The process begins at zero, $W_0 = 0$.
- W_t has stationary, independent increments.
- W_t is continuous in t .
- The increments $W_t - W_s$ have a normal distribution with mean zero and variance $|t - s|$, that is $W_t - W_s \sim \mathcal{N}(0, |t - s|)$.

Diffusion process

Let dt be a infinitesimal time interval for a random process. The random process in continuous time can be written as

$$dX(t) = \mu dt + \sigma dW(t)$$

with $dW(t) \equiv \mathcal{N}(0, \sqrt{(dt)})$ and $X(0) = x_0$. The process X is said to follow an arithmetic Brownian motion (with drift μ) and $dW(t)$ is called a Wiener increment. Naturally

$$\begin{aligned}\mathbb{E}[dW] &= 0 \\ \text{Var}[dW] &= dt\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}[dX] &= \mu dt \\ \text{Var}[dX] &= \sigma^2 dt\end{aligned}$$

More generally, a variable X is said to follow a diffusion process (or Itô process) if

$$dX = \mu(t, X)dt + \sigma(t, X)dW$$

where $\mu(t, X)$ is the drift function and $\sigma(t, X)$ is the volatility for an increment in X . The process X is said to follow a geometric Brownian motion if

$$dX = \mu(t, X)Xd t + \sigma(t, X)XdW$$

Of particular interest in finance is the case where $\mu(t, X) = \mu X$ and $\sigma(t, X) = \sigma X$. This process is used to describe the return dynamics of a wide range of assets.

Itô's Lemma

Let X be a general diffusion process of the form

$$dX = \mu(t, X)dt + \sigma(t, X)dW$$

. For a function f at least twice differentiable on X and once differentiable on t , df follows a diffusion process of the form:

$$df(t, X) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X} \mu(t, X) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} \sigma^2(t, X) \right] dt + \frac{\partial f}{\partial X} \sigma(t, X) dW$$

For a better understanding we present the following example.

Example 5.1 *A stock price process S follows the random motion*

$$\frac{dS}{S} = \mu dt + \sigma dW$$

We are interested in the process followed by $d(\log S)$. Set $f(t, S) = \log(S)$. Then

$\frac{\partial f}{\partial S} = \frac{1}{S}$; $\frac{\partial f}{\partial t} = 0$; $\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$ Applying Itô's lemma for the geometric Brownian motion, we obtain

$$df = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW$$

This allows us to derive an explicit formulation for the evolution of a stock price. Integrating between 0 and T the above expression

$$\int_0^T d(\log S) = \left(\mu - \frac{\sigma^2}{2} \right) \int_0^T dt + \sigma \int_0^T dW$$

we get

$$\log(S_T) - \log(S_0) = \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma(W(T) - W(0))$$

Taking exponentials and noting that $W(0) = 0$, we have an expression for S_T as a function of S_0 for a geometric Brownian motion:

$$S_T = S_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma W(T) \right]$$

Itô's lemma for the two-dimensional geometric Brownian motions:

$$\begin{aligned} \frac{dX}{X} &= \mu_X dt + \sigma_X dW_X \\ \frac{dY}{Y} &= \mu_Y dt + \sigma_Y dW_Y \end{aligned}$$

is given by:

$$\begin{aligned} df(t, X, Y) &= \left(\frac{\partial f}{\partial t} + \frac{f}{X} \mu_X X + \frac{\partial f}{\partial Y} \mu_Y Y + \frac{\partial^2 f}{\partial X \partial Y} \rho \sigma_X \sigma_Y XY + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} \sigma_X^2 X^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y^2} \sigma_Y^2 Y^2 \right) dt \\ &+ \frac{\partial f}{\partial X} \sigma_X X dW_X + \frac{\partial f}{\partial Y} \sigma_Y Y dW_Y \end{aligned}$$

This can be seen by simply applying Taylor's theorem to a function of three variables.

5.3 Stochastic models

We are interested in using stochastic models to construct a model for the structure of interest rates. A single-factor model is a stochastic model in which only one factor, generally the instantaneous interest rate r , is used to describe the whole structure of interest rates.

A general single-factor model follows the following equation:

$$dr = \mu(r, t)dt + \sigma(r, t)dW \quad (28)$$

A bond price P can be expressed as a function of an underlying variable, r in this case, and time. If the bond matures at T , then

$$P = P(r, t, T)$$

By Itô's lemma, we can get an expression for the bond return. This expression allows us to identify the drift a and the volatility b of the bond return:

$$\frac{dP}{P} = a(r, t, T)dt + b(r, t, T)dW \quad (29)$$

where a and b are:

$$a(r, t, T) \equiv \frac{\frac{\partial P}{\partial r} \mu(r, t) + \frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r, t)}{P} \quad (30)$$

and

$$b(r, t, T) \equiv \frac{\frac{\partial P}{\partial r} \sigma(r, t)}{P} \quad (31)$$

Note that b is negative for most bonds.

Consider now two bonds, "Bond 1" and "Bond 2", priced at P_1 and P_2 , maturing at T_1 and T_2 , with drifts a_1 and a_2 and volatilities b_1 and b_2 . We form a self-financing portfolio comprised of:

1. Bonds 1 worth V_1
2. Bonds 2 worth V_2
3. an amount $(V_1 + V_2)$ borrowed at the riskless short rate r .

The instantaneous change in the portfolio value π is

$$d\pi = V_1 \frac{dV_1}{V_1} + V_2 \frac{dV_2}{V_2} - (V_1 + V_2)r dt = V_1(a_1 - r)dt + V_2(a_2 - r)dt + (V_1 b_1 + V_2 b_2)dW$$

If we choose a portfolio such that

$$V_1 = -V_2 \frac{b_2}{b_1}$$

then the stochastic term in dW disappears. $d\pi$ is therefore deterministic when portfolio weights are chosen this way. Moreover, because the portfolio is riskless and self-financed, it can only earn an instantaneous rate of zero. We obtain:

$$-\frac{V_2 b_2}{b_1} (a_1 - r)dt + V_2 (a_2 - r)dt = 0$$

or

$$\frac{a_1 - r}{b_1} = \frac{a_2 - r}{b_2} \quad (32)$$

Equation (32) states that the expected bond return in excess of the risk-free rate per unit of volatility is the same for all interest-rate-sensitive securities. We can call this ratio λ . Note that because λ is the same for bonds of all maturities, it does not depend on T . We reformulate equation (32) as

$$\frac{a_i - r}{b_i} = \lambda(r, t) \quad (33)$$

where a_i and b_i are the expected return and volatility of any bond i . $\lambda(r, t)$ can be viewed as the market price of the risk attached to long bonds. A negative λ means that long bond expected returns are higher than short rates.

We can now combine equation (33) with equations (30) and (31) to get the general pricing equation for interest-rate-sensitive securities:

$$\frac{\partial P}{\partial r} [\mu(r, t) - \lambda(r, t) \sigma(r, t)] + \frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r, t) = rP \quad (34)$$

The boundary condition for a zero-coupon bond with a face value of 1 is

$$P(r, T, T) = 1 \quad (35)$$

The solution of (34) subject to boundary condition (35) is

$$P(t, T) = \tilde{\mathbb{E}}_t \left[\exp \left(- \int_t^T r(s) ds - \int_t^T \lambda(r, s) dW(s) - \frac{1}{2} \int_t^T \lambda^2(r, s) ds \right) \right] \quad (36)$$

To prove (36) , consider the expression

$$X(u, W(u)) = \exp \left(- \int_t^u r(s)ds - \int_t^u \lambda(r, s)dW(s) - \frac{1}{2} \int_t^u \lambda^2(r, s)ds \right)$$

By Itô's lemma, the differential of $X(u, W(u))$ is

$$dX(u, W(u)) = \frac{\partial X}{\partial u} du + \frac{\partial X}{\partial W} dW(u) + \frac{1}{2} \frac{\partial^2 X}{\partial W^2} du$$

But

$$\begin{aligned} \frac{\partial X}{\partial u} &= -r(u)X(u) - \frac{1}{2} \lambda^2(r, u)X(u) \\ \frac{\partial X}{\partial W} &= -\lambda(r, u)X(u) \end{aligned}$$

and

$$\frac{\partial^2 X}{\partial W^2} = \lambda^2(r, u)X(u)$$

Therefore

$$dX(u, W(u)) = -r(u)X(u)du - \lambda(r, u)X(u)dW(u) \quad (37)$$

From (29), we also know that

$$dP(r, u, T) = \left[\frac{\partial P}{\partial r} \mu(r, u) + \frac{\partial P}{\partial u} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2(r, u) \right] du + \frac{\partial P}{\partial r} \sigma(r, u)dW(u) \quad (38)$$

From Itô's lemma, (37) and (38), and using short form, we get

$$\begin{aligned} d[P(r, u, T)X(u)] &= XdP + PdX + dXdP = \\ &= \left[\frac{\partial P}{\partial r} \mu + \frac{\partial P}{\partial u} + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \sigma^2 - rP - \lambda \sigma \frac{\partial P}{\partial r} \right] Xdu + \left(\frac{\partial P}{\partial r} \sigma X - \lambda X P \right) dW \end{aligned} \quad (39)$$

By (34), the term in du disappears. We integrate from t to T , the remaining expression in (39)

$$\int_t^T d[P(r, u, T)X(u)] = \int_t^T \left(\frac{\partial P}{\partial r} \sigma X - \lambda X P \right) dW(u)$$

Taking expectations on both sides, the right-hand side disappears. We then obtain

$$\tilde{\mathbb{E}}_t [P(r, T, T)X(T) - P(r, t, T)X(t)] = 0$$

Because $P(r, T, T) = 1$ and $X(t) = 1$, equation (40) follows:

$$P(r, t, T) = \tilde{\mathbb{E}}_t [X(T)] = \tilde{\mathbb{E}}_t \left[\exp \left(- \int_t^T r(s)ds - \int_t^T \lambda(r, s)dW(s) - \frac{1}{2} \int_t^T \lambda^2(r, s)ds \right) \right] \quad (40)$$

Equation (40) is the solution of our general model. To get a closed-form formula for zero-coupon bonds, we need to specialize this model by specifying the $\mu(r, t)$, $\sigma(r, t)$, and $\lambda(r, t)$ functions. We study several models that postulate specific forms for these functions.

5.4 Vasicek Model

In this case, the stochastic differential equation for the short rate is:

$$dr = k(\theta - r)dt + \sigma dW.$$

Here, k is the speed of mean reversion ($k > 0$) and θ is the long-term target for r . It is clear to see that if $r < \theta$, then the drift is positive. Instead, if $r > \theta$, the drift is negative. A part from that, this model supposes that market price of risk is constant.

In terms of our general model (28), we have that $\mu(r, t) = k(\theta - r)$, $\sigma(r, t) = \sigma$ and $\lambda(r, t) = \lambda$. Then, the general pricing equation (34) is now:

$$\frac{\partial P}{\partial r}[k(\theta - r) - \lambda\sigma] + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} = rP, \quad (41)$$

subject to $P(r, T, T) = 1$ for a zero-coupon bond with face value 1. Let $\tau \equiv T - t$. We will try to find a solution of the form:

$$P(r, t, T) = A(\tau) \exp(-rB(\tau)). \quad (42)$$

Looking at (42), the derivatives of P can be calculated on the following way:

$$\begin{aligned} \frac{\partial P}{\partial r} &= -AB \exp(-rB) \\ \frac{\partial P}{\partial t} &= AB^2 \exp(-rB) \\ \frac{\partial^2 P}{\partial r^2} &= rAB' \exp(-rB) - A' \exp(-rB) \end{aligned}$$

Then, we can rewrite (41) as:

$$r(kAB + AB' - A) + \left[(\lambda\sigma - k\theta)B + \frac{1}{2}\sigma^2 B^2 \right] A - A' = 0. \quad (43)$$

Before solving (43), it is necessary to note that, because $P(r, t, T) = 1$, we have

$$A(0)\exp(-rB(0)) = 1 \quad (44)$$

for all t . Then, since (43) needs to hold for any value of r , we can solve two ordinary differential equations to obtain A and B :

$$kB + B' = 1,$$

subject to $B(0) = 0$, and

$$\left[(\lambda\sigma - k\theta)B + \frac{1}{2}\sigma^2 B^2 \right] A - A' = 0,$$

subject to $A(0) = 1$. Note that the choice of boundary conditions is motivated by (44). The solution to this differential equations is:

$$\begin{aligned} B(x) &= \frac{1 - e^{-\kappa x}}{\kappa} \\ A(x) &= \exp \left[(B(x) - x) \left(\mu^* - \frac{\sigma^2}{2\kappa^2} \right) - \frac{\sigma^2 B(x)^2}{4\kappa} \right] \\ \mu^* &= \mu - \frac{\lambda\sigma}{\kappa} \end{aligned}$$

5.5 Cox, Ingersoll and Ross model

In models like Merton or Vasicek, the short rate can become negative if the realization of the Wiener process dW is negative enough. The Cox-Ingersoll-Ross process has no such drawback as the volatility of the short rate is proportional to the square root of this rate. The stochastic process for the short rate is

$$dr = K(\theta - r)dt + \sigma\sqrt{r}dW$$

As in the Vasicek model, K is the speed of the mean-reversion ($K > 0$) and θ is the long-term target for r . In reference to the general model, $\mu(r, t) = K(\theta - r)$ and $\sigma(r, t) = \sigma\sqrt{r}$. Cox, Ingersoll, and Ross also derive from a general equilibrium model a specific formulation for the market price of risk:

$$\lambda(r, t) = \frac{\lambda\sqrt{r}}{\sigma}$$

Under the above specifications, the general pricing equation (29) is

$$\frac{\partial P}{\partial r} [K(\theta - r) - \lambda r] + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} = rP \quad (45)$$

subject to $P(r, t, T) = 1$ for a zero-coupon bond paying 1 at maturity. We start searching for a solution of the form

$$P(r, t, T) = A(\tau)\exp[-B(\tau)r]$$

Here again, partial differential equation (45) can be tackled by solving two separable ordinary differential equations.

$$\begin{aligned} \frac{\partial P}{\partial r} &= -AB\exp(-rB) \\ \frac{\partial^2 P}{\partial r^2} &= AB^2\exp(-rB) \\ \frac{\partial P}{\partial t} &= rAB'\exp(-rB) - A'\exp(-rB) \end{aligned}$$

The equation (45) can be rewritten as

$$r \left(\frac{1}{2}\sigma^2 AB^2 + (K + \lambda)AB + AB' - A \right) - K\theta AB - A' = 0$$

Because this equation needs to hold for all values of r , the values of A and B can be obtained by solving two separate ordinary differential equations:

$$\frac{1}{2}\sigma^2 B^2 + (K + \lambda)B + B' = 1 \quad A' = -K\theta AB$$

The result is:

$$A(\tau) = \left[\frac{\alpha \exp\left[\left(\frac{K+\lambda+\alpha}{2}\right)\tau\right]}{\left(\frac{K+\lambda+\alpha}{2}\right) [\exp(\alpha\tau) - 1] + \alpha} \right]^\beta$$

and

$$B(\tau) = \frac{\exp(\alpha\tau) - 1}{\left(\frac{K+\lambda+\alpha}{2}\right) [\exp(\alpha\tau) - 1] + \alpha}$$

with

$$\begin{aligned} \alpha &\equiv \sqrt{2\sigma^2 + (K + \lambda)^2} \\ \beta &\equiv \frac{2K\theta}{\sigma^2} \end{aligned}$$

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