SECOND ORDER DIFFERENTIABILITY AND RELATED TOPICS IN THE TAKAGI CLASS

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Abstract. In this paper we study some properties concerning the second order differentiability of the functions belonging to the Takagi class as well as the size of the sets where these properties hold. In particular, we characterize the set of points where these functions have a Taylor expansion of order two. Moreover, we also characterize when they satisfy a Stepanov condition of order two at a point. In this context, the convexity of such functions is also investigated. Finally, some interesting examples are provided.

1. Introduction

The Takagi function is a classical example of a continuous nowhere differentiable function (see [16]). Throughout the years, it has been studied by a large number of authors, who have brought to light its special properties as well as its connection with different fields, including probability theory, number theory, and mathematical analysis. The surveys [2] and [14] contain a lot of information about the Takagi function, and they are highly recommended for readers who want to acquaint themselves with this function.

Among the different ways to define it, we choose the following one: we consider D the set of all dyadic numbers of the interval $[0,1]$, and we decompose it as the increasing union of the following sets

$$D_n = \left\{ \frac{k}{2^{n-1}} \in [0,1] : k \in \mathbf{Z} \right\}$$

with $n \geq 1$. Then, we define the Takagi function $T : [0,1] \to \mathbb{R}$ as

$$T(x) = \sum_{n=1}^{\infty} g_n(x)$$

where $g_n(x) = d(x, D_n)$ denotes the distance from the point x to the closed set D_n. In some sense, the Takagi function is more regular than Weierstrass function, since it may have infinite derivatives (see [1]). To find out more about the relationship between the Takagi function and the Weierstrass function we refer the reader to the recent paper [6].

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In 1930, a variant of the Takagi function was rediscovered by Van der Waerden (see [17]), where this author used base ten instead of base two, that is the reason why it is often called the Takagi-Van der Waerden function.

In the mid-1980s, Hata and Yamaguti introduced a new family of functions named the Takagi class (see [12]). It consists of all the functions T_w defined in the following way: for a sequence $w = (w_n)_n$ satisfying $(2^{-n}w_n)_n \in \ell^1$, we define the function $T_w : [0,1] \to \mathbb{R}$ as

$$T_w(x) = \sum_{n=1}^{\infty} w_n g_n(x).$$

In particular, they proved that the Takagi class is a closed subspace of the space of continuous functions with the sup norm. It should be noted that this class contains a number of interesting examples from the classical literature (see [7] or [4] for instance).

A few years later, Kôno carried out a deep study of the differentiability properties of the functions belonging to the Takagi class (see [13]). His work revealed that three qualitatively different cases may arise depending on the sequence of weights we are considering. More specifically, he proved that if $w \notin c_0$ then T_w is nowhere differentiable, if $w \in c_0 \setminus \ell^2$ then T_w is not differentiable a.e. although the range of the derivative is $\mathbb{R}$, and finally if $w \in \ell^2$ then T_w is absolutely continuous and consequently differentiable a.e. A wide generalization of these results can be found in [9].

The aim of this paper is to carry out a study of the second order differentiability and other related properties of the functions of the Takagi class. In this sense, this work lays bare a fourth qualitatively different case.

In section 2 we spell out what kind of second order properties will be investigated throughout the paper. For instance, having a second order Taylor expansion at a point or satisfying a Stepanov condition of order two. We also discuss why the study of such properties must be done with the assumption $w \in \ell_1$, which will constitute our framework.

Section 3 investigates piecewise convexity and we obtain the following main result:

Theorem 1.1. *The function T_w is piecewise convex if and only if there exists n_0 such that the sequence*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0}$$

is non positive and non decreasing.

In section 4 we characterize the existence of second order Taylor expansion at a point by means of a technical condition which involves the sequence $\beta = (\beta_n)_n$ defined as

$$\beta_n = 2^n\Big(- w_n + \sum_{j=n+1}^{\infty} w_j\Big).$$

As we shall see, such sequence will play a key role in the development of this work. Furthermore, we obtain that the function T_w has Taylor expansion of order two at $x \notin D$ if and only if $(T_w'^{+})'(x)$ exists.

Section 5 is devoted to obtain the following striking result:

Theorem 1.2. *The following statements are equivalent:*

(1) T_w *has a Taylor expansion of order two a.e.*
(2) $T_w'^{+}$ *has bounded variation.*
(3) $T_w'^{+}$ *is derivable a.e.*
(4) $\beta \in \ell^1$.
(5) T_w *is the difference of two convex functions of the Takagi class.*

Finally, in section 6 we develop other technical condition which allows us to characterize when the function T_w satisfies a Stepanov condition of order two at a point. Then, we invoke Calderon-Zygmund's result (see [5, Theorem 5]) in order to characterize when T_w satisfies a Stepanov condition of order two a.e.

Throughout the paper, we present some examples which show how the previous results allow us to construct functions T_w with prescribed properties. We finish this introduction by listing some of them.

Example 1.3. Let $w_n = r^{-n}$. If $r \geq 2$ then T_w is concave, and consequently, it satisfies a Stepanov condition of order two a.e. However, if $1 < r < 2$, the function T_w satisfies nowhere the Stepanov condition of order two.

Example 1.4. Let $w_n = \frac{(-1)^n}{n^2 2^n}$. Then T_w has Taylor expansion of order two (and satisfies the Stepanov condition of order two) a.e. However there exist points x such that T_w satisfies the Stepanov condition of order two at x but it does not have Taylor expansion of order two at x.

Example 1.5. Let $w_n = \frac{(-1)^n}{n 2^n}$. Then, T_w has Taylor expansion of order two at a null set of Hausdorff dimension one.

2. Preliminary results

In this section we shall provide a series of concepts concerning the second order differentiability of a function as well as some elementary properties of T_w in order to establish our framework.

When a function is not derivable everywhere, a natural way to study second order differentiability is by looking at the Taylor expansion of order two. Let us recall this concept.

Definition 2.1. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has Taylor expansion of order two at x, whenever there exist two numbers d_x and A_x such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - d_x h - A_x h^2}{h^2} = 0.$$

It is immediate that if f has Taylor expansion of order two at x then f is derivable at x and $d_x = f'(x)$. Hence we may rewrite the condition as

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h - A_x h^2}{h^2} = 0$$

or equivalently

$$A_x = \lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h^2}.$$

A weaker property is a Stepanov condition of order two.

Definition 2.2. We say that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Stepanov condition of order two at x provided that there exists a constant d_x such that

$$\limsup_{h\to 0} \left| \frac{f(x+h) - f(x) - d_x h}{h^2} \right| < +\infty.$$

Observe that f is necessarily derivable at x and $d_x = f'(x)$.

As we saw above, the Stepanov condition of order two is weaker than the existence of Taylor expansion of order two, although a celebrated result of Calderon and Zygmund, see [5], asserts that if a function satisfies a Stepanov condition of order two a.e. then it has Taylor expansion of order two a.e. (not necessarily at the same points as we will see below).

In our setting, as we require the existence of $T_w'(x)$ at some points, we have that $w \in c_0$ necessarily by Kôno's result, but we may affirm more. The derivative of a function T_w at $x \notin D$ is necessarily

$$T_w'(x) = \sum_{j=1}^{\infty} w_j g_j'(x). \tag{2.1}$$

For instance, this fact is an immediate consequence of Theorem 2.5 in [9], where a wider setting is considered. On the other hand, it is not hard to prove that if T_w is differentiable at $x \in D_{n+1} \setminus D_n$ then

$$T_w'(x) = \sum_{j=1}^{n-1} w_j g_j'(x).$$

Furthermore, we have that $w \in \ell^1$ is a necessary requirement in our frame. Before proving it, we introduce some notation.

We will denote by $\mathcal{F}_n$ the family of all the connected components of $[0,1] \setminus D_n$. Moreover, if $I \in \mathcal{F}_n$ we will denote $I = (a_n, b_n)$ and c_n will be the midpoint of I, in other words, $c_n = \frac{1}{2}(a_n + b_n)$.

Lemma 2.3. *If T_w satisfies a Stepanov condition of order two at some point x, then $w \in \ell^1$.*

Proof. We will prove that there exists $K > 0$ such that $2^n|w_n| \leq K$ eventually. Let $M > 0$ satisfying

$$|T_w(x+h) - T_w(x) - T_w'(x)h| \leq Mh^2 \tag{2.2}$$

for $|h|$ small enough. Firstly, assume that $x \notin D$. For n big enough, if $x \in (a_n, b_n) \in \mathcal{F}_n$, we take $h_n = 2(c_n - x)$. Now, we make the following observations:

(1) $|h_n| < 2^{-n+1}$,
(2) $g_k(x+h_n) - g_k(x) - g_k'(x)h_n = 0$ for every $k < n$,
(3) $g_k(x+h_n) = g_k(x)$ for every $k \geq n$,

and consequently

$$|T_w(x+h_n) - T_w(x) - T_w'(x)h_n| = \left| - h_n \sum_{k=n}^{\infty} w_k g_k'(x) \right|.$$

By (2.2), if n is big enough then

$$|w_n| = \left| \sum_{k=n}^{\infty} w_k g_k'(x) - \sum_{k=n+1}^{\infty} w_k g_k'(x) \right| \leq M|h_n| + M|h_{n+1}| < \frac{3M}{2^n}.$$

Finally, if $x \in D$, we take n big enough such that $h_n = 2^{1-n}$ satisfies the inequality (2.2). Then, changing g_k' by $g_k'^{+}$, the proof follows in a similar way as above. □

For this reason, we will assume that $w \in \ell^1$. We have that T_w is $\|w\|_1$-Lipschitz when $w \in \ell^1$ and it characterizes the Lipschitz character of T_w. Thanks to Rademacher's theorem we already know that T_w is differentiable a.e. provided that $w \in \ell^1$. In addition, it is not difficult to see that the lateral derivatives of T_w are given by

$$T_w'^{+} = \sum_{j=1}^{\infty} w_j g_j'^{+} \quad \text{and} \quad T_w'^{-} = \sum_{j=1}^{\infty} w_j g_j'^{-}.$$

In particular, if T_w is differentiable at $x \in D_{n+1} \setminus D_n$ then

$$T_w'^{+}(x) = \sum_{j=1}^{n-1} w_j g_j'(x) - \Big(w_n - \sum_{j=n+1}^{\infty} w_j \Big) \tag{2.3}$$

and

$$T_w'^{-}(x) = \sum_{j=1}^{n-1} w_j g_j'(x) + \Big(w_n - \sum_{j=n+1}^{\infty} w_j \Big), \tag{2.4}$$

which implies

$$w_n = \sum_{j=n+1}^{\infty} w_j.$$

Remark 2.4. *The previous result also holds true when $w \in c_0$.*

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $w \in \ell^1$. Then, $T_w'^+$ and $T_w'^-$ are continuous at every $x \notin D$. Moreover $T_w'^+$, respectively $T_w'^-$, is right, respectively left, continuous at every $x \in D$.*

For every $x \notin D$ we denote

$$x = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{2^j} = \sum_{j=1}^{n} \frac{\varepsilon_j}{2^j} + \sum_{j=n+1}^{\infty} \frac{\varepsilon_j}{2^j} := \hat{x}_n + \widetilde{x}_n, \quad \varepsilon_j \in \{0,1\}.$$

When necessary, to avoid confusion, we write $\varepsilon_n(x)$ instead of ε_n. The next lemma, whose proof is immediate, shows the relation between this binary expansion and the values of g_n and g_n'.

Lemma 2.6. *Let $x \notin D$. For every n, we have*

(1) *$g_n(x) = \widetilde{x}_n$ provided that $\varepsilon_n = 0$ and $g_n(x) = 2^{-n} - \widetilde{x}_n$ otherwise.*
(2) *$g_n'(x) = 1 - 2\varepsilon_n$.*

Although the following result is proved in [13], we present an elementary proof since it will be useful later.

Lemma 2.7. *Let $w \in \ell^1$. The following statements are equivalent:*

(1) *$w_n = M2^{-n}$ for every n and $M \in \mathbb{R}$.*
(2) *T_w is a polynomial and in such case $T_w(x) = Mx(1-x)$.*
(3) *T_w is derivable at every $x \in D$.*

Proof. Firstly, (1) ⇒ (2) follows from the following elementary differential equation:

$$T_w'(x) = M \sum_{n=1}^{\infty} \frac{1 - 2\varepsilon_n}{2^n} = M(1-2x)$$

for every x. Secondly, (2) ⇒ (3) is immediate. Finally, (3) ⇒ (1) is a consequence of the fact that (3), in view of (2.3) and (2.4), implies $w_n = \sum_{j=n+1}^{\infty} w_j$ for every n. □

Another way to see order two differentiability at $x \notin D$ is to consider the derivative of $T_w'^+ : [0,1] \to \mathbb{R}$. This function is defined everywhere and it is continuous at every $x \notin D$ by Lemma 2.5. Hence, for every $x \notin D$ we may study the existence of

$$\left(T_w'^+\right)'(x) = \lim_{y \to x} \frac{T_w'^+(y) - T_w'^+(x)}{y - x}.$$

Observe also that the existence of $(T_w'^+)'(x)$ is equivalent to the existence of the limit

$$L_x = \lim_{y \notin D, y \to x} \frac{T_w'(y) - T_w'(x)}{y - x}.$$

Indeed $L_x = (T_w'^+)'(x)$. The nontrivial implication follows since $(T_w'^+)'(x)$ is right continuous everywhere by Lemma 2.5.

For an arbitrary function the existence of A_x and L_x are independent conditions, since there exist derivable functions which have order two Taylor expansion at points where they are not twice derivable; and although if a function has second derivative at x, then it has Taylor expansion of order two at x, it is not clear that a similar result holds if we only require that the right derivative is derivable at x. The reason is that the Mean Value Theorem is not true for the right derivative. However, for absolutely continuous functions we have the following:

Proposition 2.8. *Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function. If $A = \{x \in (0, 1); f'(x)$ exists$\}$, $x \in A$ and there exists*

$$\lim_{\substack{y \to x \\ y \in A}} \frac{f'(y) - f'(x)}{y - x} = L,$$

then

$$\lim_{y \to x} \frac{f(y) - f(x) - f'(x)(y - x)}{(y - x)^2} = \frac{L}{2}.$$

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(y) - f'(x) - L(y - x)| < \varepsilon |y - x|$$

if $y \in A$ and $|y - x| < \delta$. The absolute continuity of f implies

$$\left| \frac{f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} L(y - x)^2}{(y - x)^2} \right|$$
$$= \left| \frac{\int_x^y (f'(t) - f'(x) - L(t - x)) \, dt}{(y - x)^2} \right| \leq \varepsilon.$$

□

3. Convexity

Since T_w is absolutely continuous, we have that T_w is convex if and only if $T_w'^+$ is non decreasing (see [15] for instance). Thus, we obtain the first result of this section.

Lemma 3.1. *If there exists n_0 such that*

$$w_n \leq \sum_{j=n+1}^{\infty} w_j \leq 0 \tag{C}$$

for every $n \geq n_0$, then T_w is piecewise convex.

Proof. It is enough to prove that T_w is convex while restricted to $I_{n_0} \in \mathcal{F}_{n_0}$. If $x, y \in I_{n_0}$ and $x < y$ then there exists $m \geq n_0$ such that

$\varepsilon_j(x) = \varepsilon_j(y)$ for $j < m$, $\varepsilon_m(x) = 0$ and $\varepsilon_m(y) = 1$. Hence

$$T_w'^{+}(y) - T_w'^{+}(x) = 2 \sum_{n \geq m} w_n \left(\varepsilon_n(x) - \varepsilon_n(y)\right)$$

$$= -2w_m + 2 \sum_{n>m} w_n \left(\varepsilon_n(x) - \varepsilon_n(y)\right) \geq 2 \left(-w_m + \sum_{n>m} w_n \right) \geq 0.$$

□

Lemma 3.2. *If T_w is piecewise convex, then there exists n_0 such that w fulfils the condition* (C) *for every $n \geq n_0$.*

Proof. Let n be big enough such that T_w is convex while restricted to $[x_n, y_n]$ for some $x_n, y_n \in D_n$ consecutive. As usual, let c_n be the midpoint of such interval. We have that $T_w'^{-}(c_n) \leq T_w'^{+}(c_n)$. As

$$T_w'^{+}(c_n) = \sum_{j=1}^{n-1} w_j g_j'(c_n) - w_n + \sum_{j=n+1}^{\infty} w_j \quad \text{and}$$

$$T_w'^{-}(c_n) = \sum_{j=1}^{n-1} w_j g_j'(c_n) + w_n - \sum_{j=n+1}^{\infty} w_j,$$

we get that $w_n \leq \sum_{j=n+1}^{\infty} w_j$. Furthermore, we have that $T_w'^{+}(x_n) \leq T_w'^{-}(y_n)$. Now we observe that

$$T_w'^{+}(x_n) = \sum_{j=1}^{n-1} w_j g_j'^{+}(x_n) + \sum_{j=n}^{\infty} w_j \quad \text{and}$$

$$T_w'^{-}(y_n) = \sum_{j=1}^{n-1} w_j g_j'^{-}(y_n) - \sum_{j=n}^{\infty} w_j.$$

This implies $\sum_{j=n}^{\infty} w_j \leq 0$ since $g_j'^{+}(x_n) = g_j'^{-}(y_n)$ for every $1 \leq j \leq n-1$. □

Lemma 3.3. *A sequence w satisfies the condition* (C) *for every $n \geq n_0$ if and only if the sequence*

$$\left(2^n \sum_{k=n}^{\infty} w_k \right)_{n \geq n_0}$$

is non positive and non decreasing.

Proof. We have that w fulfils the condition (C) for every $n \geq n_0$ if and only if

$$\sum_{j=n}^{\infty} w_j \leq 2 \sum_{j=n+1}^{\infty} w_j$$

and equivalently

$$2^n \sum_{j=n}^{\infty} w_j \leq 2^{n+1} \sum_{j=n+1}^{\infty} w_j$$

for every $n \geq n_0$. □

Remark 3.4. *In particular, if a sequence w satisfies the condition* (C) *for every $n \geq n_0$, then $(2^n w_n)_n$ converges since it can be written as difference of two convergent sequences.*

Lemmas 3.1, 3.2 and 3.3 give us the proof of Theorem 1.1. Similarly, regarding the concavity, we have the following:

Theorem 3.5. *The function T_w is piecewise concave if and only if there exists n_0 such that the sequence*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0}$$

is non negative and non increasing.

Example 3.6. If $w_n = r^{-n}$ with $r \geq 2$ then T_w is concave. For instance, if $r = 3$ then we have

$$T_w'(x) = \sum_{n=1}^{\infty} \frac{1 - 2\varepsilon_n}{3^n} = \frac{1}{2} - 2 \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n}, \quad \varepsilon_n \in \{0, 1\}$$

for every $x \notin D$. Notice that $T_w'([0,1] \setminus D)$ is an affine copy of the classical ternary Cantor set. Consequently $T_w'([0,1] \setminus D)$ is a null set and, as a matter of fact it has Hausdorff dimension $\log_3 2$.

Example 3.7. If $w_n = (n+1)(n2^n)^{-1}$ then T_w is concave.

4. Taylor expansion of order two

The aim of this section is to prove that the existence of Taylor expansion of order two and the differentiability of the lateral derivatives are properties that agree in our frame and we may characterize them in terms of the sequence w. Before proceeding, we observe the following reduction of the problem:

Lemma 4.1. *Let $x \notin D$ and $A_x \in \mathbb{R}$. Then,*

$$\lim_{h \to 0} \frac{T_w(x+h) - T_w(x) - T_w'(x)h}{h^2} = A_x$$

if and only if

$$\lim_{h \to 0} \frac{T_v(x+h) - T_v(x) - T_v'(x)h}{h^2} = 0$$

where $v = (v_n)_n$ is defined by $v_n = w_n + A_x 2^{-n}$.

Proof. The proof is an immediate consequence of Lemma 2.7 and the fact that the function $P(y) = A_x(y - y^2)$ satisfies $P''(y) = -2A_x$. □

We deduce some consequences of the existence of Taylor expansion of order two.

Lemma 4.2. *If T_w has Taylor expansion of order two at some point x, then $A_x = -\lim_n 2^n w_n$.*

Proof. Since $w_n = -A_x 2^{-n} + v_n$, it is enough to prove that $\lim_n v_n 2^n = 0$. Firstly, assume that $x \notin D$. For every n, we consider $x \in (a_n, b_n) \in \mathcal{F}_n$. By formula (2.1) we have

$$\lim_{h\to 0} \frac{T_v(x+h) - T_v(x) - T_v'(x)h}{h^2} = 0.$$

Since

$$\left|\frac{T_v(b_n) - T_v(a_n) - T_v'(x)(b_n - a_n)}{(b_n - a_n)^2}\right| \leq \left|\frac{T_v(b_n) - T_v(x) - T_v'(x)(b_n - x)}{(b_n - x)^2}\right| + \left|\frac{T_v(a_n) - T_v(x) - T_v'(x)(a_n - x)}{(a_n - x)^2}\right|$$

we deduce

$$\lim_n \frac{T_v(b_n) - T_v(a_n) - T_v'(x)(b_n - a_n)}{(b_n - a_n)^2} = 0.$$

As we have

$$\frac{T_v(b_n) - T_v(a_n) - T_v'(x)(b_n - a_n)}{(b_n - a_n)^2} = 2^{2(n-1)} \sum_{j=n}^{\infty} v_j(-g_j'(x)(b_n - a_n)) = -2^{n-1} \sum_{j=n}^{\infty} v_j g_j'(x)$$

we deduce that

$$\lim_n 2^n \sum_{j=n}^{\infty} v_j g_j'(x) = 0,$$

and consequently

$$\lim_n 2^n v_n g_n'(x) = \lim_n 2^n \sum_{j=n}^{\infty} v_j g_j'(x) - \frac{1}{2} \lim_n 2^{n+1} \sum_{j=n+1}^{\infty} v_j g_j'(x) = 0,$$

which implies that $\lim_n 2^n v_n = 0$.

Finally, when $x \in D$ it is enough to observe that

$$\frac{T_v(x + 2^{-n}) - T_v(x) - T_v'(x)2^{-n}}{2^{-2n}} = -2^n \sum_{j=n+1}^{\infty} v_j$$

provided that n is big enough. □

Remark 4.3. *The existence of the limit $L := \lim_n 2^n w_n$ is equivalent to the following statements:*

(1) $w_n = \frac{L}{2^n} + v_n$ *with* $\lim_n 2^n v_n = 0$.
(2) $\lim_n 2^n \sum_{k=n+1}^{\infty} w_k = L$.

Regarding the case $x \in D$, we have the following result:

Proposition 4.4. *Let $x \in D_{n+1} \setminus D_n$ and assume that $L := \lim_n 2^n w_n$ exists. Then, T_w has Taylor expansion of order two at x if and only if T_w is derivable at x.*

Proof. By Lemma 2.7 and Remark 4.3, we may assume without loss of generality that $\lim_m 2^m w_m = 0$. Given $\varepsilon > 0$ there exists m_0 such that $|w_m| < \varepsilon 2^{-m}$ for every $m \geq m_0$. For h small enough there exists $m \geq \max\{m_0, n\}$ satisfying that $2^{-(m+1)} < |h| \leq 2^{-m}$. Thus, we get

$$\frac{T_w(x+h) - T_w(x) - T_w'(x)h}{h^2} = \frac{1}{h^2} \sum_{j=m+1}^{\infty} w_j(g_j(x+h) - h)$$
$$\leq \frac{1}{|h|} \sum_{j=m+1}^{\infty} |w_j| \leq 2\varepsilon.$$

□

Now, we introduce some notation in order to characterize at which points $x \notin D$ the function T_w has Taylor expansion of order 2.

Let us take $x \notin D$ with base two expansion $x = \sum_n \varepsilon_n 2^{-n}$ as usual. For every n, we denote $k_n := k_n(x)$ as the nonnegative integer satisfying

$$\varepsilon_n = \cdots = \varepsilon_{n+k_n} \neq \varepsilon_{n+k_n+1}$$

and we consider

$$\mathcal{N} := \mathcal{N}_x = \{n \in \mathbb{N} : k_n = 0\},$$

that is the set of indices n such that $\varepsilon_n \neq \varepsilon_{n+1}$.

With respect to the sequence $w = (w_n)_n$, if T_w has Taylor expansion of order two at some point then, as we saw above in Lemma 4.2, we necessarily have that $\lim_n 2^n w_n$ exists. This fact allows us to define the sequence $\beta = (\beta_n)_n$ as

$$\beta_n = 2^n \Big(-w_n + \sum_{j=n+1}^{\infty} w_j \Big),$$

and it is immediate that this sequence converges to 0.

The following lemma will enable us to simplify the proof of the next theorem.

Lemma 4.5. *Let $x \notin D$ and $n \in \mathcal{N}_x$. If $h_n = g_n'(x) 2^{-(n+k_{n+1})}$ then the following statements hold true:*

1. $g_j(x+h_n) - g_j(x) - g_j'(x)h_n = 0$ *for every* $j < n$.
2. $g_n(x+h_n) + g_j(x+h_n) = 2^{-n} = g_n(x) + g_j(x)$ *for every j such that* $n+1 \leq j \leq n+k_{n+1}$.
3. $g_j(x+h_n) - g_j(x) - g_j'(x)h_n = -(g_n(x+h_n) - g_n(x) - g_n'(x)h_n)$ *for every j such that* $n+1 \leq j \leq n+k_{n+1}$.
4. $-2|h_n| < g_n(x+h_n) - g_n(x) - g_n'(x)h_n < -|h_n|$.

(5) $g_j(x+h_n) = g_j(x)$ *for every* $j > n + k_{n+1}$.

Proof. For every n let $(a_n, b_n) \in \mathcal{F}_n$ such that $x \in (a_n, b_n)$. (1) It follows since $x, x+h_n \in (a_n, b_n)$, and consequently g_j is linear between them for every $j < n$.

(2) If $g_n'(x) = 1$, that is $\varepsilon_n = 0$, then

$$\begin{aligned} & g_n(x+h_n) + g_j(x+h_n) = (2^{-n} - \widetilde{x}_{n+k_{n+1}}) + \widetilde{x}_{n+k_{n+1}} \\ & = 2^{-n} = \widetilde{x}_n + (2^{-j} - \widetilde{x}_j) = g_n(x) + g_j(x) \end{aligned}$$

by Lemma 2.6. We leave the other case to the reader.

(3) This is an immediate consequence of (2) since $g_n'(x) = -g_j'(x)$.

(4) It is enough to check that if $\varepsilon_n = 0$ then

$$g_n(x+h_n) - g_n(x) - g_n'(x)h_n = -2\widetilde{x}_{n+k_{n+1}},$$

and

$$g_n(x+h_n) - g_n(x) - g_n'(x)h_n = 2(\widetilde{x}_{n+k_{n+1}} - |h_n|)$$

provided that $\varepsilon_n = 1$.

(5) It follows since g_j is h_n-periodic for every $j > n + k_{n+1}$. □

We proceed to prove the main theorem of this section.

Theorem 4.6. *For every* $x \notin D$*, the following statements are equivalent:*

(1) $(T_w'^{+})'(x)$ *exists.*
(2) T_w *has a Taylor expansion of order two at* x*.*
(3) $\lim_{n \in \mathcal{N}_x} 2^{k_{n+1}} \beta_n = 0$ *and* $\lim_n 2^n w_n$ *exists.*

Proof. (1) ⇒ (2) This is a consequence of Proposition 2.8 since T_w is absolutely continuous.

(2) ⇒ (3) We have that $\lim_n 2^n w_n$ exists by Lemma 4.2, and we may assume that $A_x = \lim_n 2^n w_n = 0$ by Lemma 4.1. Now, let us consider $n \in \mathcal{N}_x$ and $k_{n+1} > 0$. For $h_n = g_n'(x) 2^{-(n+k_{n+1})}$, we have that

$$\begin{aligned} & \frac{T_w(x+h_n) - T_w(x) - T_w'(x)h_n}{h_n^2} \\ & = 2^{2(n+k_{n+1})} \sum_{j=n}^{n+k_{n+1}} w_j (g_j(x+h_n) - g_j(x) - h_n g_j'(x)) \\ & - 2^{n+k_{n+1}} \sum_{j=n+k_{n+1}+1}^{\infty} w_j g_j'(x) \\ & = 2^{2(n+k_{n+1})} (g_n(x+h_n) - g_n(x) - h_n g_n'(x)) \Big(w_n - \sum_{j=n+1}^{n+k_{n+1}} w_j \Big) \\ & - 2^{n+k_{n+1}} \sum_{j=n+k_{n+1}+1}^{\infty} w_j g_j'(x) \end{aligned}$$

by Lemma 4.5 (1), (3) and (5).

If we define

$$s_n = -2^{n+k_{n+1}}(g_n(x+h_n) - g_n(x) - h_n g_n'(x)),$$

we have that $s_n \in (1,2)$ by Lemma 4.5 (4). Hence, taking limits (with $k_{n+1} > 0$) we have

$$\begin{aligned} 0 &= \lim_{n\in\mathcal{N}_x} 2^n 2^{k_{n+1}} s_n \Big(-w_n + \sum_{j=n+1}^{n+k_{n+1}} w_j \Big) \\ &= \lim_{n\in\mathcal{N}_x} s_n 2^{k_{n+1}}\beta_n - \lim_{n\in\mathcal{N}_x} s_n 2^{n+k_{n+1}} \sum_{j=n+k_{n+1}+1}^{\infty} w_j = \lim_{n\in\mathcal{N}_x} s_n 2^{k_{n+1}}\beta_n. \end{aligned}$$

Therefore $\lim_{n\in\mathcal{N}_x} 2^{k_{n+1}}\beta_n = 0$ since if $k_{n+1} = 0$ we have that $2^{k_{n+1}}\beta_n = \beta_n$ and $\lim_n \beta_n = 0$.

(3) $\Rightarrow$ (1) We may assume without loss of generality that $\lim_n 2^n w_n = 0$. Let $2^{-(n+1)} < h \le 2^{-n}$, we may assume that n is big enough such that $\varepsilon_j = 0$ for some $j < n$. If $\varepsilon_n = 0$ then

$$\left| \frac{T_w'^{+}(x+h) - T_w'^{+}(x)}{h} \right| = \left| \sum_{j=n}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \le 2^{n+2} \sum_{j=n}^{\infty} |w_j|$$

which goes to 0 when n goes to $+\infty$. If $\varepsilon_n = 1$, let $m(n)$ be the maximum of $\{j < n : \varepsilon_j = 0\}$. We have that $m(n) \in \mathcal{N}_x$ and $m(n) + 1 + k_{m(n)+1} \ge n$. Hence, by Lemma 4.5. (2) and (3) we have

$$\begin{aligned} &\left| \frac{T_w'^{+}(x+h) - T_w'^{+}(x)}{h} \right| = \left| \sum_{j=m(n)}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \\ &\le \left| \sum_{j=m(n)}^{n} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| + \left| \sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \\ &= \frac{2}{h} \left| -w_{m(n)} + \sum_{j=m(n)+1}^{n} w_j \right| + \left| \sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \\ &\le 2^{n+2} \left| -w_{m(n)} + \sum_{j=m(n)+1}^{\infty} w_j \right| + \left| \sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \\ &\le 2^{m(n)+k_{m(n)+1}+3} \left| -w_{m(n)} + \sum_{j=m(n)+1}^{\infty} w_j \right| + \left| \sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \\ &= 2^{k_{m(n)+1}+3} |\beta_{m(n)}| + \left| \sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h} \right| \end{aligned}$$

which goes to zero when n goes to $+\infty$. The proof for $h < 0$ is similar. □

Remark 4.7. *Observe that* $L_x := (T_w'^{+})'(x) = 2A_x$.

Theorem 4.6 implies that unless $\beta_n = 0$ eventually, there always exist points $x \notin D$ where T_w does not have Taylor expansion of order two. Remember also that β_n eventually null means that $w_n = \frac{M}{2^n}$ for some constant M eventually. Nevertheless, the existence of the limit $L = \lim_n 2^n w_n$ guarantees that there are many points where T_w has Taylor expansion of order two. Let us introduce the following subsets of $[0,1]$:

$$\mathcal{A}_N = \{x \notin D : k_n(x) \leq N \text{ for every } n\}.$$

Then we define

$$\mathcal{A} = \bigcup_{N=1}^{\infty} \mathcal{A}_N. \tag{4.1}$$

We have that $(k_n)_n$ is bounded if and only if $x \in \mathcal{A}$. It is not difficult to see that $\mathcal{A}$ is a Lebesgue null set with Hausdorff dimension 1 (see [10] for instance).

As a consequence of Theorem 4.6, we have the following result.

Corollary 4.8. *If w satisfies that $\lim_n 2^n w_n$ exists, then T_w has Taylor expansion of order two at every $x \in \mathcal{A}$.*

5. Characterizing how big is the set where T_w has Taylor expansion of order two

If $T_w'^+$ has bounded variation then $T_w'^+$ is derivable almost everywhere (see [11] for instance), and consequently L_x and A_x exist almost everywhere. Of course this situation happens when T_w is concave or convex, and more generally when T_w is difference of two convex functions. For the functions in the Takagi class, the converse is also true in the following strong way: $T_w'^+$ has bounded variation if and only if T_w is the difference of two convex functions belonging to the Takagi class too (see Corollary 5.9 below).

Our next goal is characterize when $T_w'^+$ has bounded variation in terms of w. We start introducing some notation: Every interval in $\mathcal{F}_n$ has an extreme that belongs to $D_n \setminus D_{n-1}$ meanwhile the other one belongs to some D_k with $k < n$. We denote by $\mathcal{F}_{n,k} \subset \mathcal{F}_n$ the family of intervals with one extreme belonging to $D_k \setminus D_{k-1}$. $\mathcal{F}_{n,1}$ consists of two intervals, and if $1 < k < n$ then $\mathcal{F}_{n,k}$ consists of 2^{k-1} intervals.

Although $T_w'^+$ is not defined at $x = 1$, we make the convention that $T_w'^+(1) = \sum_{k=1}^{\infty} w_k$, which would be the value of $T_w'^+$ if we periodically extend the function T_w to the whole real line.

Lemma 5.1. *For the partition $\mathcal{P}_n = D_n$ we have*

$$V(T_w'^+, D_n) := \sum_{I=(a,b)\in\mathcal{F}_n} \left|T_w'^+(b) - T_w'^+(a)\right| = 2^{n-1}|w_{n-1}|$$

$$+ 2|w_1 + \cdots + w_{n-1}| + \sum_{k=2}^{n-1} 2^{k-1}| - w_{k-1} + w_k + \cdots + w_{n-1}|.$$

Proof. Let $I = (a, b) \in \mathcal{F}_n$ and assume that $I \in \mathcal{F}_{n,k}$, $k \geq 2$. Firstly, we consider the case $a \in D_n \setminus D_{n-1}$ and $b \in D_k \setminus D_{k-1}$. We have:

$$|T_w'^+(b) - T_w'^+(a)| = \Big| \sum_{j=k-1}^{n-1} w_j(g_j'^+(b) - g_j'^+(a))\Big|$$

since $g_j'^+(b) = g_j'^+(a) = 1$ for every $j \geq n$, and $a, b \in J$ for some interval $J \in \mathcal{F}_{k-1}$, which implies that $g_j'^+(a) = g_j'^+(b)$ provided that $j < k-1$. For the rest of the indices we have that $g_{k-1}'^+(b) = -1$ and $g_{k-1}'^+(a) = 1$, meanwhile $g_j'^+(b) = 1$ and $g_j'^+(a) = -1$ for every $k \leq j < n$. Hence

$$|T_w'^+(b) - T_w'^+(a)| = 2| - w_{k-1} + w_k + \cdots + w_{n-1}|.$$

Alternatively, if $b \in D_n \setminus D_{n-1}$ and $a \in D_k \setminus D_{k-1}$, then $g_j'^+(b) = g_j'^+(a) = 1$ for every $j \geq n$ and $g_j'^+(a) = g_j'^+(b)$ for every $j < k-1$ as above, but now $g_{k-1}'^+(b) = g_{k-1}'^+(a) = -1$ and $g_j'^+(b) = g_j'^+(a) = 1$ for every $k \leq j < n-1$, hence

$$|T_w'^+(b) - T_w'^+(a)| = |w_{n-1}(g_{n-1}'^+(b) - g_{n-1}'^+(a))| = | - 2w_{n-1}|.$$

When $k = 1$, we have

$$|T_w'^+(2^{-n+1}) - T_w'^+(0)| = | - 2w_{n-1}|$$

and

$$|T_w'^+(1) - T_w'^+(1 - 2^{-n+1})| = 2|w_1 + \cdots + w_{n-1}|.$$

We obtain the result adding all items. ☐

Theorem 5.2. *The function $T_w'^+$ has bounded variation if and only if*

$$\limsup_n \left(2^{n-1}|w_{n-1}| + \sum_{k=2}^{n-1} 2^{k-1}|-w_{k-1} + w_k + \cdots + w_{n-1}|\right) < +\infty. \tag{5.1}$$

Proof. The necessity of this condition follows easily from Lemma 5.1. Conversely, we first observe that condition (5.1) implies $w \in \ell^1$. Moreover, it is immediate that

$$V(T_w'^+) = \sup_{P\subset D} V(T_w'^+, P)$$

since D is dense and $T_w'^+$ is continuous at every $x \notin D$. Given a partition $P \subset D$, we have that $P \subset D_n$ for some n, and hence the result follows from Lemma 5.1 since $2|w_1 + \cdots + w_{n-1}| \leq 2\|w\|_1$. ☐

As we saw above, if L_x exists for some point x, then $L := \lim_n 2^n w_n = \lim_n 2^n \sum_{j>n} w_j$ exists too. Furthermore, as

$$\left|\sum_{k=1}^{n} 2^k\Big| - w_k + \sum_{j=k+1}^{\infty} w_j\Big| - \sum_{k=1}^{n} 2^k\Big| - w_k + \sum_{j=k+1}^{n} w_j\Big|\right|$$
$$\leq \sum_{k=1}^{n} 2^k\Big| \sum_{j=n+1}^{\infty} w_j\Big| \leq \Big|2^{n+1} \sum_{j=n+1}^{\infty} w_j\Big|$$

we have that

$$\limsup_n \sum_{k=2}^{n-1} 2^{k-1}| - w_{k-1} + w_k + \cdots + w_{n-1}| < +\infty$$

if and only if the series

$$\sum_{k=1}^{\infty} 2^k\Big| - w_k + \sum_{j=k+1}^{\infty} w_j\Big|$$

converges, which is equivalent to $\beta \in \ell^1$. Additionally, it is easy to see that the sequence $(2^n w_n)_n$ converges provided that $\beta \in \ell^1$. Therefore, we may rewrite condition (5.1) of Theorem 5.2 in the following way:

Corollary 5.3. *The function $T_w'^+$ has bounded variation if and only if $\beta \in \ell^1$.*

Theorem 5.4. *The function T_w has Taylor expansion of order two a.e. if and only if $\beta \in \ell^1$.*

Proof. If $\beta \in \ell^1$ then $T_w'^+$ has bounded variation, and consequently T_w has Taylor expansion of order 2 a.e. Conversely, assume for the sake of contradiction that $\beta \notin \ell^1$. For every n, we consider the set

$$\mathcal{B}_n = \{x : k_n(x) \geq r_n,\ \varepsilon_{n-1}(x) \neq \varepsilon_n(x)\}$$

where $r_n = \log_2 \frac{1}{|\beta_{n-1}|}$. We consider

$$\mathcal{B} = \bigcap_{n=1}^{\infty} \left(\bigcup_{j=n}^{\infty} \mathcal{B}_j \right) = \limsup_n \mathcal{B}_n.$$

We are under the hypothesis of Theorem 6.4 in [3] since $\sum 2^{-r_n}$ diverges (the required conditions hold exactly as in Example 6.6 of [3]), hence the set $\mathcal{B}$ has measure one.

Finally, we observe that if $x \in \mathcal{B}$ then $k_n(x) \geq r_n$ and $n-1 \in \mathcal{N}_x$ for infinitely many indices n, and consequently

$$\limsup_{n \in \mathcal{N}_x} 2^{k_{n+1}} |\beta_n| \geq 1.$$

The result follows from Theorem 4.6. □

In view of the previous results, we characterize when $\beta \in \ell^1$ in terms of a more manageable condition. Let us define the sequence $\alpha = (\alpha_n)_n$ as

$$\alpha_n = 2^{n+1} w_{n+1} - 2^n w_n.$$

Lemma 5.5. *We have that $\beta \in \ell_1$ if and only if $\alpha \in \ell_1$.*

Proof. Since

$$\begin{aligned} 2\beta_n - \beta_{n+1} &= 2^{n+1}\Big(- w_n + \sum_{j=n+1}^{\infty} w_j + w_{n+1} - \sum_{j=n+2}^{\infty} w_j \Big) \\ &= 2^{n+1}\big(- w_n + 2w_{n+1}\big) = 2\alpha_n \end{aligned} \tag{5.2}$$

for every n, we obtain that $\alpha \in \ell_1$ provided that $\beta \in \ell_1$.

Now, assume that $\alpha \in \ell_1$. Then, for $1 \le p < q$ we get

$$2^q w_q - 2^p w_p = \sum_{n=p}^{q-1} (-2^n w_n + 2^{n+1} w_{n+1}) = \sum_{n=p}^{q-1} \alpha_n = \sum_{n=1}^{q-1} \alpha_n - \sum_{n=1}^{p-1} \alpha_n$$

which implies that $(2^n w_n)_n$ is a Cauchy sequence and therefore that it converges. It follows from Remark 4.3 that $\beta \in c_0$. Finally, using the equality (5.2) we get

$$2\sum_{n=1}^{N} |\beta_n| \le \sum_{n=1}^{N} |\beta_{n+1}| + 2\sum_{n=1}^{N} |\alpha_n| = \sum_{n=2}^{N+1} |\beta_n| + 2\sum_{n=1}^{N} |\alpha_n|$$

and therefore

$$\sum_{n=1}^{N} |\beta_n| \le |\beta_{N+1}| - |\beta_1| + 2\sum_{n=1}^{N} |\alpha_n|.$$

which implies that $\beta \in \ell^1$. □

In order to see that if $T_w'^{+}$ has bounded variation, then T_w is the difference of two convex functions that belong to the Takagi class we start with a couple of lemmas.

Lemma 5.6. *A sequence w is the difference of two sequences satisfying the condition* (C) *if and only if the sequence*

$$\Big(2^n \sum_{k=n}^{\infty} w_k \Big)_n$$

is the difference of two non decreasing and non positive sequences.

Proof. If w is the difference of two sequences satisfying the condition (C), then the condition holds by Lemma 3.3. Conversely, if

$$2^n \sum_{k=n}^{\infty} w_k = u_n - v_n$$

with $u = (u_n)_n$ and $v = (v_n)_n$ non positive and non decreasing, then

$$w_n = (2^{-n}u_n - 2^{-(n+1)}u_{n+1}) - (2^{-n}v_n - 2^{-(n+1)}v_{n+1}),$$

and both $(2^{-n}u_n - 2^{-(n+1)}u_{n+1})_n$ and $(2^{-n}v_n - 2^{-(n+1)}v_{n+1})_n$ satisfy the condition (C) by Lemma 3.3 again. □

Lemma 5.7. *A sequence (c_n) is the difference of two non decreasing and non positive sequences $(a_n)_n$ and $(b_n)_n$, if and only if*

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| < +\infty. \tag{5.3}$$

Proof. If $c_n = a_n - b_n$, with $(a_n)_n$ and $(b_n)_n$ non decreasing and non positive, then

$$|c_n - c_{n+1}| \leq |a_n - a_{n+1}| + |b_n - b_{n+1}| = a_{n+1} - a_n + b_{n+1} - b_n,$$

hence

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| \leq -a_1 + \lim_n a_n - b_1 + \lim_n b_n < +\infty.$$

Conversely, if (5.3) holds, then $(c_n)_n$ is convergent. Let $\gamma > 0$ and we define

$$b_m = \gamma + |\lim_n c_n| + \sum_{n=m}^{\infty} |c_n - c_{n+1}| \qquad \text{and} \qquad a_m = b_m + c_m.$$

It is clear that $c_m = a_m - b_m = -b_m - (-a_m)$, and that $(b_m)_m$ is non negative and non increasing. Moreover,

$$\begin{aligned} a_m &= c_m + \gamma + |\lim_n c_n| + \sum_{n=m}^{\infty} |c_n - c_{n+1}| \\ &= \sum_{n=m}^{\infty} (c_n - c_{n+1}) + \lim_n c_n + \gamma + |\lim_n c_n| + \sum_{n=m}^{\infty} |c_n - c_{n+1}| \geq \gamma > 0 \end{aligned}$$

and

$$a_m - a_{m+1} = c_m - c_{m+1} + |c_m - c_{m+1}| \geq 0,$$

which gives us the result. □

Proposition 5.8. *The sequence w may be written as $v - u$ with u, v satisfying the condition* (C) *if and only if $\beta \in \ell^1$.*

Proof. We have that $w = u - v$ with u, v satisfying the condition (C) if and only if the sequence

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_n$$

is the difference of two non decreasing and non positive sequences by Lemma 5.6, which is the case if and only if

$$\sum_{n=1}^{\infty} 2^n \Big| w_n - \sum_{k=n+1}^{\infty} w_k \Big| = \sum_{n=1}^{\infty} \Big| 2^n \sum_{k=n}^{\infty} w_k - 2^{n+1} \sum_{k=n+1}^{\infty} w_k \Big| < +\infty$$

by Lemma 5.7. □

The next result follows immediately from Corollary 5.3.

Corollary 5.9. *The function T_w is the difference of two piecewise convex functions of the Takagi class if and only if $T_w'^+$ has bounded variation.*

Collecting the results that we have presented above we obtain Theorem 1.2 and we finish this section observing the following dichotomy.

Lemma 5.10. *The set $\mathcal{T}$ of points x such that T_w has Taylor expansion of order two at x is either a null set or it has measure one.*

Proof. Let $I_1, I_2 \in \mathcal{F}_n$, and let G be the translation that satisfies $G(I_1) = I_2$. It is immediate that T_w has Taylor expansion of order two at $x \in I_1$ if and only if T_w has Taylor expansion of order two at $G(x)$, since this property does not depend on the first n digits of the binary expansion of x, and x and $G(x)$ have the same digits ε_k for every $k > n$. From this fact we deduce that $\mathcal{L}(I \cap \mathcal{T}) = 2^{1-n}\mathcal{L}(\mathcal{T}) = \mathcal{L}(I)\mathcal{L}(\mathcal{T})$ for every $I \in \mathcal{F}_n$.

We may split every interval $J \subset [0,1]$ as a disjoint union of a countable number of intervals belonging to different $\mathcal{F}_n$ plus a null set (points belonging to D). Hence $\mathcal{L}(J \cap \mathcal{T}) = \mathcal{L}(J)\mathcal{L}(\mathcal{T})$ for every interval J, and it is well known that this implies our result. □

The next proposition follows immediately from the previous lemma, Theorem 5.4 and Corollary 4.8.

Proposition 5.11. *If $\lim_n 2^n w_n$ exists, then the set of points where T_w has Taylor expansion of order two is a null set with Hausdorff dimension one if and only if $\beta \notin \ell^1$.*

In view of the results obtained above, we conclude that two mutually exclusive alternatives arise when $\lim_n 2^n w_n$ exists:

(1) $T_w'^+$ has bounded variation and equivalently T_w has Taylor expansion of order two a.e., or
(2) $T_w'^+$ does not have bounded variation and equivalently T_w has Taylor expansion of order two at a null set of Hausdorff dimension one.

Example 5.12. For the sequence w defined by $w_n = \frac{(-1)^n}{n2^n}$, T_w has Taylor expansion of order two at a null set of Hausdorff dimension one.

Example 5.13. For the sequence w defined by $w_n = \frac{(-1)^n}{n^2 2^n}$, T_w has Taylor expansion of order two a.e. Observe that it is neither convex nor concave, although it can be split as the difference of two concave functions, namely $u_n = \frac{1}{n^2 2^n}$ if n is even and $u_n = 0$ for n odd; and $v_n = \frac{1}{n^2 2^n}$ if n is odd and $v_n = 0$ for n even.

6. The Stepanov condition

Before getting started, we must point out that the proof of Lemma 2.3 provides us a stronger result:

Lemma 6.1. *If T_w satisfies a Stepanov condition of order two at some point x, then there exists $M > 0$ such that $2^n |w_n| \leq M$ for every n.*

We will begin by studying the case $x \in D$ which is the easiest one. Before proceeding, we highlight the following facts that will enable us to simplify the proofs of the results.

Assume that $x \in D_{n+1} \setminus D_n$. This implies that x is the midpoint of some connected component of $\mathcal{F}_n$ and we may write T_w around x as

$$T_w(z) = L(z) + \sum_{j=n}^{\infty} w_j g_j(z)$$

with L a linear function. Let $a := w_n - \sum_{j=n+1}^{\infty} w_j$. If $a \neq 0$ then T_w is not derivable at x, hence it cannot fulfil the Stepanov condition at this point. Therefore, we obtain that $a = 0$ is a necessary condition in order that T_w satisfies a Stepanov condition of order two.

Proposition 6.2. *Let $x \in D_{n+1} \setminus D_n$. Then, T_w satisfies a Stepanov condition of order two at x if and only if T_w derivable at x and there exists $M > 0$ such that $2^n |w_n| \leq M$ for every n.*

Proof. It is enough to prove the sufficiency part. We may assume that $n = 1$, that is $x = \frac{1}{2}$, and by (2.3) and (2.4) we have $T_w'(x) = w_1 - \sum_{j=2}^{\infty} w_j = 0$, hence we have to estimate $T_w(x+h) - T_w(x)$. Assume that $h > 0$, we have

$$\begin{aligned} T_w(x+h) - T_w(x) &= w_1(g_1(x+h) - g_1(x)) + \sum_{j=2}^{\infty} w_j g_j(x+h) \\ &= -w_1 h + \sum_{j=2}^{\infty} w_j g_j(x+h) = \sum_{j=2}^{\infty} w_j (g_j(x+h) - h). \end{aligned}$$

In particular, if $2^{-(m+1)} < h \leq 2^{-m}$, then

$$|T_w(x+h) - T_w(x)| = \left| \sum_{j=2}^{\infty} w_j (g_j(x+h) - h) \right| \leq \sum_{j=m+1}^{\infty} |w_j| h \leq 2Mh^2.$$

The proof for $h < 0$ is identical since $T_w(x+h) = T_w(x-h)$. □

Now, we consider the situation $x \notin D$.

Lemma 6.3. *If T_w satisfies a Stepanov condition of order 2 at $x \notin D$, then there exists $M > 0$ such that $2^{k_{n+1}}|\beta_n| \leq M$ for every $n \in \mathcal{N}_x$.*

Proof. Let $K > 0$ satisfy that $|w_n| \leq K2^{-n}$ for every n, and $|T_w(x + h) - T_w(x) - T_w'(x)h| \leq Kh^2$ for $|h|$ small. Let $n \in \mathcal{N}_x$. If we take $h_n = g_n'(x)2^{-(n+k_{n+1})}$ with n big enough, then

$$K2^{-2(n+k_{n+1})} \geq |T_w(x + h_n) - T_w(x) - T_w'(x)h_n|$$
$$= \left| \sum_{j=n}^{n+k_{n+1}} w_j(g_j(x + h_n) - g_j(x) - g_j'(x)h_n) - h_n \sum_{j=n+1+k_{n+1}}^{\infty} w_j g_j'(x) \right|.$$

Hence we obtain

$$\left| \sum_{j=n}^{n+k_{n+1}} w_j(g_j(x + h_n) - g_j(x) - g_j'(x)h_n) \right|$$
$$\leq K2^{-2(n+k_{n+1})} + \left| h_n \sum_{j=n+1+k_{n+1}}^{\infty} w_j g_j'(x) \right| \leq 2K2^{-2(n+k_{n+1})}.$$

Furthermore, if we denote $r_n = -2^{n+k_{n+1}}(g_n(x+h_n)-g_n(x)-g_n'(x)h_n)$, we have that $r_n \in (1, 2)$ and

$$2^{2(n+k_{n+1})} \left| \sum_{j=n}^{n+k_{n+1}} w_j(g_j(x + h_n) - g_j(x) - g_j'(x)h_n) \right|$$
$$= r_n 2^{n+k_{n+1}} \left| - w_n + \sum_{j=n+1}^{n+k_{n+1}} w_j \right|$$
$$\geq r_n 2^{k_{n+1}} |\beta_n| - r_n 2^{n+k_{n+1}} \sum_{j=n+k_{n+1}+1}^{\infty} |w_j|$$
$$\geq 2^{k_{n+1}} |\beta_n| - 2^{n+k_{n+1}+1} \sum_{j=n+k_{n+1}+1}^{\infty} |w_j|,$$

where the equality follows by Lemma 4.5. Finally, we conclude

$$2^{k_{n+1}} |\beta_n| \leq 2K + 2^{n+k_{n+1}+1} \sum_{j=n+k_{n+1}+1}^{\infty} |w_j| \leq 4K.$$

□

Lemma 6.4. *Let $x \notin D$. If there exists $M > 0$ such that*

1. *$2^n|w_n| \leq M$ for every n, and*
2. *$2^{k_{n+1}}\beta_n \leq M$ for every $n \in \mathcal{N}_x$*

then T_w satisfies a Stepanov condition of order 2 at x.

Proof. Let $2^{-(n+1)} < h \leq 2^{-n}$, we may assume that n is big enough such that $\varepsilon_{j_1} = 0$ and $\varepsilon_{j_2} = 1$ for some $j_1, j_2 < n$. If $\varepsilon_n = 0$ then

$$\begin{aligned}
&\left|\frac{T_w(x+h) - T_w(x) - hT_w'(x)}{h^2}\right| \\
&= \frac{1}{h^2}\left|\sum_{j=n}^{\infty} w_j(g_j(x+h) - g_j(x) - hg_j'(x))\right| \\
&\leq \frac{2}{h}\sum_{j=n}^{\infty}|w_j| \leq 2^{n+2}\sum_{j=n}^{\infty}\frac{M}{2^j} = 8M
\end{aligned}$$

since $x, x+h \in (a_n, b_n)$ and consequently g_j is linear on $[x, x+h]$ for every $j < n$.

If $\varepsilon_n = 1$, let us denote $m(n) = \max\{j < n : \varepsilon_j = 0\}$. We have that $m(n) \in \mathcal{N}_x$ and $m(n) + 1 + k_{m(n)+1} \geq n$. We get

$$|T_w(x+h) - T_w(x) - hT_w'(x)| = \left|\sum_{j=m(n)}^{\infty} w_j(g_j(x+h) - g_j(x) - hg_j'(x))\right|$$

since $x \in (a_{m(n)}, c_{m(n)})$ which implies that $x + h \in (a_{m(n)}, b_{m(n)})$. In order to estimate the right side of the formula, similarly to that we did in Lemma 4.5, we observe

$$g_j(x+h) - g_j(x) - hg_j'(x) = -(g_{m(n)}(x+h) - g_{m(n)}(x) - hg_{m(n)}'(x))$$

for every $m(n) < j \leq n$. Hence we have

$$\begin{aligned}
&\left|\sum_{j=m(n)}^{\infty} w_j(g_j(x+h) - g_j(x) - hg_j'(x))\right| \\
&\leq |g_{m(n)}(x+h) - g_{m(n)}(x) - hg_{m(n)}'(x)|\left|w_{m(n)} - \sum_{j=m(n)+1}^{n} w_j\right| \\
&+ \left|\sum_{j=n+1}^{\infty} w_j(g_j(x+h) - g_j(x) - hg_j'(x))\right| \\
&\leq 2h\left|w_{m(n)} - \sum_{j=m(n)+1}^{n} w_j\right| + 2h\sum_{j=n+1}^{\infty}|w_j| \\
&\leq \frac{2h}{2^{m(n)}}|\beta_{m(n)}| + 4h\sum_{j=n+1}^{\infty}|w_j| \leq 8h^2 2^{k_{m(n)+1}}|\beta_{m(n)}| + 8Mh^2 \leq 16Mh^2
\end{aligned}$$

which implies

$$\limsup_{h\to 0^+}\frac{|T_w(x+h) - T_w(x) - hT_w'(x)|}{h^2} < +\infty.$$

The case $h < 0$ follows from the symmetry of T_w. □

Lemmas 6.1, 6.3, and 6.4 give us the following result:

Theorem 6.5. *The function T_w satisfies a Stepanov condition of order 2 at $x \notin D$ if and only if there exists $M > 0$ such that*

(1) $2^n|w_n| \leq M$ *for every* n, *and*
(2) $2^{k_{n+1}}\beta_n \leq M$ *for every* $n \in \mathcal{N}_x$.

With respect to the set of points where T_w satisfies a Stepanov condition of order two, we have the following result as an immediate consequence of Theorem 6.5.

Corollary 6.6. *If there exists $M > 0$ such that $2^n|w_n| \leq M$ for every n, then T_w satisfies a Stepanov condition of order two at every $x \in \mathcal{A}$, which is defined by* (4.1).

In addition, Calderon-Zygmund's result (see [5, Theorem 5]) gives us the following:

Theorem 6.7. *The following statements are equivalent:*

(1) T_w *has Taylor expansion of order two a.e.*
(2) T_w *satisfies a Stepanov condition of order two a.e.*
(3) $\beta \in \ell^1$.

Proceeding as in Lemma 5.10, we deduce the next result analogous to Proposition 5.11.

Proposition 6.8. *If there exists $M > 0$ such that $2^n|w_n| \leq M$ for every n, then the set of points where T_w satisfies a Stepanov condition of order two is a null set with Hausdorff dimension one if and only if $\beta \notin \ell^1$.*

Lastly, we present some interesting examples:

Example 6.9. As we saw above, for the sequence w defined by $w_n = \frac{(-1)^n}{n^2 2^n}$, T_w has Taylor expansion of order two (and satisfies the Stepanov condition of order two) a.e. However there exist points x such that T_w satisfies the Stepanov condition of order two at x but it does not have Taylor expansion of order two at x

Proof. It is enough to consider x defined in the following way: $\varepsilon_n = 1$ if $n = n_m$ for some m, and $\varepsilon_n = 0$ otherwise, where we define recursively the increasing sequence $(n_m)_m$ by $n_1 = 1$, and $n_{m+1} = n_m + [2\log_2 n_m] + 2$.

The sequence w satisfies $\lim_n 2^n w_n = 0$ and $\beta \in \ell^1$. The point x satisfies that if $n \geq 2$ then $n \in \mathcal{N}_x$ if and only if either $n = n_m$ or $n = n_m - 1$ for some m. Moreover $k_{n_m} = 0$ and $k_{n_m+1} = [2\log_2 n_m]$. Hence $\lim_m 2^{k_{n_m}}\beta_{n_m-1} = 0$, but

$$2^{k_{n_m+1}}|\beta_{n_m}| = 2^{[2\log_2 n_m]}|\beta_{n_m}| \geq \frac{1}{2}n_m^2|\beta_{n_m}| \geq \frac{1}{2}n_m^2 2^{n_m}|w_{n_m}| = \frac{1}{2}$$

which is bounded but does not converge to 0. $\square$

Example 6.10. Let us consider the sequence w defined by $w_n = \frac{(-1)^n}{2^n}$. Then, T_w has nowhere Taylor expansion of order two but it satisfies the Stepanov condition of order two at every point of the set $\mathcal{A}$.

Example 6.11. For the sequence $w_n = r^{-n}$ with $1 < r < 2$, the function T_w satisfies nowhere the Stepanov condition of order two.

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