

Quantum corrections to the string Bethe ansatz

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Abstract

One-loop corrections to the energy of semiclassical rotating strings contain both analytic and non-analytic terms in the 't Hooft coupling. Analytic contributions agree with the prediction from the string Bethe ansatz based on the classical S -matrix, but in order to include non-analytic contributions quantum corrections are required. We find a general expression for the first quantum correction to the string Bethe ansatz.

In recent years impressive precision tests of the AdS/CFT correspondence have been performed. Moreover, the ambitious prospect of solving the large N limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills and correspondingly, the sigma model that describes Type IIB string theory on $AdS_5 \times S^5$, looks now a step closer. The crucial ingredient for these developments has been the appearance of integrable structures on both sides of the correspondence. A major development was to reinterpret the planar dilatation operator of the gauge theory as the hamiltonian of a spin chain [1], which up to one-loop for the complete theory [2, 3] and several loops in restricted sectors [4]-[6] was shown to be integrable. Assuming integrability, a long range Bethe ansatz was then conjectured to describe all higher loop effects except for non-local processes in the chain [7]. On the string theory side, integrability arises because the classical string sigma model on $AdS_5 \times S^5$ admits a Lax representation [8, 9]. The integral equations satisfied by the spectral density for the Lax operator are the string analogue of the Bethe equations for the gauge theory dilatation operator in the thermodynamic limit [10]. Assuming that integrability survives after quantization, a discrete Bethe ansatz has been suggested [11]-[13] which should determine the quantum spectrum of the string sigma model.

The structure of the gauge and string long range Bethe ansätze is remarkably similar. In particular the tower of conserved charges is given by the same expressions in terms of the spectral parameter on both cases [7]. But the S -matrices differ by a phase [11],

$$S_{st}(p_j, p_k) = S_g(p_j, p_k) e^{i\theta(p_j, p_k)} , \quad (1)$$

given by

$$\theta(p_j, p_k) = 2 \sum_{r=2}^{\infty} c_r(\lambda) \left(\frac{\lambda}{16\pi^2} \right)^r \left(q_r(p_j) q_{r+1}(p_k) - q_{r+1}(p_j) q_r(p_k) \right) , \quad (2)$$

where λ is the 't Hooft coupling constant and $q_r(p)$ are the conserved charges of the integrable system. In order to recover the integrable structure of the classical string the coefficients in (2) must satisfy $c_r(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$. The phase $\theta(p_j, p_k)$ is central to reproduce [11] the well known behaviour for the masses of small strings in weakly curved $AdS_5 \times S^5$, $m^2 \sim \sqrt{\lambda}$ [14]. This provides a very strong and simple test of the quantum string Bethe ansatz.

Gauge and string theory sides of the AdS/CFT correspondence are accessible in opposite regimes of the coupling constant λ . Indeed, quantitative comparison between anomalous dimensions of gauge theory operators and string energies has only been possible on

configurations whose dynamics is governed by an effective coupling constant $1/\mathcal{J} \equiv \lambda/J^2$. This parameter can be kept small even if λ is large provided the associated configuration carries a large enough quantum number J [15]. Perfect agreement has been found in this way between the spectrum of long gauge operators and semiclassical string to order λ^2 in all cases analyzed [16, 10] [17], including also the first quantum corrections [18]-[21]. However, in spite of this impressive results, disagreement is known to start at order λ^3 [22, 23, 6]. This problem can be traced back to the dressing phase in (1). Understanding how $\theta(p_j, p_k)$ changes in going from strong to weak coupling is crucial to uncover how the gauge theory can rearrange itself in terms of a string theory. The objective of this note is to study the first quantum corrections to the limiting value of the phase (2), given by $c_r(\infty)=1$.

In the following we will compare the leading quantum correction to the energy of semiclassical strings derived from the string Bethe ansatz and from a one-loop world-sheet calculation. One-loop corrections can be obtained from the spectrum of quadratic fluctuations around a given classical solution,

$$E_1 = \sum_{n=-\infty}^{\infty} e(n), \quad (3)$$

where $e(n)$ is the sum over frequencies of bosonic and fermionic fluctuations with mode number n . Finding the spectrum of fluctuations is a difficult problem for generic semiclassical solutions, but it has been done for the simplest example: circular strings [24]-[27]. We will consider two cases: circular strings rotating in an S^3 section of S^5 ($SU(2)$ sector) and in an $AdS_3 \times S^1$ section of $AdS_5 \times S^5$ ($SL(2)$ sector).

In order to evaluate (3), it is useful to first expand $e(n)$ in terms of the effective coupling $1/\mathcal{J}$ and then perform the sum. But this approach involves some problems. Since we are working in a supersymmetric string theory, the sum (3) is finite as $n \rightarrow \infty$ (see (A.1) and (A.2) in the appendix). Expanding $e(n)$ for fixed n at large \mathcal{J} produces however divergences at high mode number [21, 28], indicating that this is not appropriate for the high energy tail of the spectrum. On the contrary, expanding $e(n)$ at fixed $x = n/\mathcal{J}$ is regular at large x but contains divergences at $x=0$. With this alternative expansion, the highest modes are well described but problems are now shifted to the lowest part of the spectrum. The way out is to combine both expansions [29]

$$e(n) = e_1(n) + e_2(n/\mathcal{J}), \quad (4)$$

where $e_1(n)$ and $e_2(n/\mathcal{J})$ denote the regular terms for the fixed n and n/\mathcal{J} expansions, respectively. The e_1 terms are defined using a simple zeta-function regularization, and e_2 by subtracting negative powers of x . The important relation (4) has been proved recently using a Cauchy integral representation of the frequency sum [30].

The contributions to the quantum corrected energy associated to e_1 and e_2 are rather different. From (A.1) and (A.2), it is clear that $\sum e_1$ contains only even powers of $1/\mathcal{J}$. Therefore it is analytic on the coupling constant λ , as it is the case for the classical energy [31]. Up to exponentially suppressed terms, $\sum e_2$ can be evaluated by substituting the sum over modes by an integral. The simplest example to analyze is that of a $SU(2)$ circular string with $k=2m$, leading to [29, 32, 30]

$$\int_{-\infty}^{\infty} \mathcal{J} dx e_2(x) = \frac{1}{\sqrt{\mathcal{J}^2 + m^2}} \left(m^2 + 2\mathcal{J}^2 \log \frac{\mathcal{J}^2}{\mathcal{J}^2 + m^2} - \frac{\mathcal{J}^2 - m^2}{2} \log \frac{\mathcal{J}^2 - m^2}{\mathcal{J}^2 + m^2} \right). \quad (5)$$

This expression expands in odd powers of $1/\mathcal{J}$, giving thus raise to non-analytic contributions in λ .¹ This is a generic pattern. For $SL(2)$ circular strings we have [29]

$$\int_{-\infty}^{\infty} \mathcal{J} dx e_2(x) = -\frac{(k-m)^3 m^3}{3\mathcal{J}^5} \left(1 - \frac{3k^2 - 8km}{2\mathcal{J}^2} + \dots \right), \quad (6)$$

where k and m are winding numbers that characterize the 2-spin circular strings.

From the point of view of the Bethe ansatz for quantum strings, there are two possible sources of contribution to E_1 : finite size effects and quantum corrections to the classical integrable structure. Finite size effects provide $1/J$ corrections to the leading order result, obtained using the thermodynamic limit of the Bethe ansatz. Since the classical energy is analytic in λ for large \mathcal{J} , this will also be the case for the finite size corrections. It was shown in [21] that the S -matrix (1), with coefficients $c_r(\lambda) = 1$ in the dressing phase, reproduces the contribution to E_1 from $\sum e_1$ at least up to order $1/\mathcal{J}^6$. This simple choice, trivially consistent with the classical value $c_r(\infty) = 1$, does not produce however non-analytic terms in λ . This implies that in order to include the contribution from $\sum e_2$, and assuming that integrability is maintained at the quantum level, it is necessary to introduce quantum corrections to the string Bethe ansatz.

The most general Bethe ansatz for a system with $SU(2|2)$ symmetry was analyzed in [33]. The dispersion relation turned to be fixed by the symmetry algebra. The only freedom left in the integrability structure was contained in an undetermined phase in the

¹Circular strings rotating on $S^3 \subset S^5$ are unstable due to tachyonic modes at low momentum. Since (5) is an effect of the highest tail of the spectrum, it should not be affected by that problem.

scattering matrix. The most general form of the dressing phase of the S -matrix was studied in [34] for an integrable system with $GL(n)$ symmetry. It was found to be of the form (2), but with two uncorrelated conserved charges $q_r(p_j)$ and $q_s(p_k)$. In accordance with these results, it was proposed in [29] that agreement with the sum over frequencies (3), including non-analytic terms, could be restored by generalizing the phase (2) to

$$\theta(p_j, p_k) = 2 \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(\lambda) \left(\frac{\lambda}{16\pi^2} \right)^{\frac{r+s-1}{2}} \left(q_r(p_j)q_s(p_k) - q_s(p_j)q_r(p_k) \right), \quad (7)$$

with

$$c_{r,s} = \delta_{r+1,s} + \frac{1}{\sqrt{\lambda}} a_{r,s}. \quad (8)$$

The result of this note is a conjecture for the previous coefficients,

$$a_{r,s} = -8 \frac{(r-1)(s-1)}{(r+s-2)(s-r)}, \quad (9)$$

for odd $r+s$. Parity invariance implies that $a_{r,s} = 0$ for even $r+s$ [33]. The value of the coefficient for the lowest correction term, $a_{2,3}$, was already derived in [29]. We are assuming that the modified dressing phase (7) is common for all sectors of the correspondence, as it was the case in [12, 13]. In consonance, it is interesting to stress that $e_2(x)$, source of the non-analytical/quantum terms, receives contributions from bosonic and fermionic string fluctuations in all $AdS_5 \times S^5$ directions [30]. However, at the same time this fact renders remarkable that its effects can be accommodated in a Bethe ansatz formulation, which can be consistently truncated inside each sector. In the following we will provide evidence in favor of (9).

The above generalization of the dressing phase represents a quantum correction to the scattering matrix in infinite volume. Therefore the modifications that it produces on the energy can be analyzed using the thermodynamic limit of the Bethe ansatz. The integral Bethe equations that correspond to including a single correction term are

$$2\eta \oint_{\mathcal{C}} dy \frac{\rho(y)}{x-y} = 2\pi\eta k_i + \frac{x}{x^2 - g^2} \left[1 + g^2 \int_{\mathcal{C}} dy \frac{\rho(y)}{y} \left(\frac{1+\eta}{y} - \frac{1-\eta}{x} \right) - 2a_{r,s} \epsilon g^{r+s-1} \int_{\mathcal{C}} dy \rho(y) \left(\frac{1}{x^{r-1}y^{s-1}} - \frac{1}{x^{s-1}y^{r-1}} \right) \right], \quad x \in \mathcal{C}_i, \quad (10)$$

where $\eta = 1, -1$ for the $SU(2)$ and $SL(2)$ cases, respectively. For simplicity we have denoted the coupling constants that govern the thermodynamic scaling and the quantum corrections by g and ϵ , respectively,

$$g = \frac{1}{4\pi\mathcal{J}}, \quad \epsilon = \frac{1}{\sqrt{\lambda}}, \quad (11)$$

and $\mathcal{C} = \cup_i \mathcal{C}_i$ represents the set of curves in the complex x -plane where the Bethe roots condense. Circular strings correspond to solutions of the Bethe equations where all roots lie on a single connected curve. In this case there is just one mode number $k_i \equiv k$, and (10) can be rewritten as an algebraic equation for the resolvent,

$$G^2 - 2\pi k G - \eta G^{(1)} = g^2 (G^{(1)^2} - 2\pi k G^{(2)}) + g^2 (1 + \eta) (G^{(1)} Q_2 - G^{(2)} Q_1) - 2a_{r,s} \eta \epsilon g^{r+s-1} (G^{(r)} Q_s - G^{(s)} Q_r), \quad (12)$$

where

$$G(x) = \int_{\mathcal{C}} dy \frac{\rho(y)}{x-y}. \quad (13)$$

The infinite tower of conserved charges can be directly read from the expansion of the resolvent around $x = 0$,

$$G(x) = - \sum_{n=0}^{\infty} Q_{n+1} x^n, \quad Q_n = \int_{\mathcal{C}} dy \frac{\rho(y)}{y^n}. \quad (14)$$

The Virasoro constraints imply that the total momentum of the sigma model configuration must be a multiple of 2π : $P = Q_1 = -2\pi m$, with $m \in \mathbb{Z}$. The second conserved charge determines the energy of the circular string: $E = J(1 + 2g^2 Q_2)$. Finally, in (12) we have introduced the convenient notation

$$G^{(r)}(x) = - \sum_{n=r}^{\infty} Q_{n+1} x^{n-r}. \quad (15)$$

The simplest case to analyze is again $SU(2)$ with $k = 2m$, since for it $G^{(2)}$ cancels in equation (12) among the first and second terms of the rhs. The modification on the energy caused by the additional term in the dressing phase at leading order in ϵ is then easily derived,

$$\delta E = a_{r,s} g^{r+s} \frac{Q_{r+1} Q_s - Q_{s+1} Q_r}{\pi(1 + 2g^2 Q_2)}. \quad (16)$$

This expression can be evaluated at any order in g using the expansion of the resolvent, which for this case has the simple form

$$G(x) = 2\pi m + \frac{\sqrt{1 + (4\pi m g)^2} - \sqrt{1 + (4\pi m x)^2}}{2(x - g^2/x)}. \quad (17)$$

Substituting in (16) the values of $a_{r,s}$ proposed in (9), we get the following contribution to the energy

$$\delta E = -\frac{m^6}{3 \mathcal{J}^5} + \frac{m^8}{3 \mathcal{J}^7} - \frac{49 m^{10}}{120 \mathcal{J}^9} + \frac{2 m^{12}}{5 \mathcal{J}^{11}} - \frac{5749 m^{14}}{13440 \mathcal{J}^{13}} + \dots \quad (18)$$

This series coincides with the expansion of the one-loop string calculation (5), at least up to the order that we have checked: $1/\mathcal{J}^{101}$! This matching is already a very strong indication, but however it is not concluding. There are $(r+s-3)/2$ possible terms in (7) contributing to the variation of the energy at order $1/\mathcal{J}^{r+s}$, and only one number to fit, the corresponding coefficient in the expansion of (5). Many different choices of $a_{r,s}$ could provide the same agreement.

To perform a more precise test, we consider next the $SL(2)$ case for general values of k and m . Equation (16) turns into

$$(1 + 2g^2 Q_2) \delta Q_2 + 2\pi k g^2 \delta Q_3 = 2a_{r,s} \epsilon g^{r+s-1} (Q_{r+1} Q_s - Q_{s+1} Q_r). \quad (19)$$

The second term on the lhs makes the explicit expression for δE lengthy and cumbersome although straightforward to derive recursively in the coupling constant g . Substituting again the values given in (9) for $a_{r,s}$, and with the help of Mathematica, we obtain

$$\delta E = -\frac{(k-m)^3 m^3}{3 \mathcal{J}^5} \left[1 - \frac{P_2}{2 \mathcal{J}^2} + \frac{P_4}{40 \mathcal{J}^4} - \frac{P_6}{80 \mathcal{J}^6} + \frac{P_8}{4480 \mathcal{J}^8} + \dots \right], \quad (20)$$

where P_n are homogeneous polynomials of degree n in the parameters m and k

$$\begin{aligned} P_2 &= 3k^2 - 8km, \\ P_4 &= 75k^4 - 455k^3m + 679k^2m^2 - 153km^3 + 29m^4, \\ P_6 &= 175k^6 - 1755k^5m + 5635k^4m^2 - 6843k^3m^3 + 2823k^2m^4 - 562km^5 + 2m^6, \\ P_8 &= 11025k^8 - 159565k^7m + 820785k^6m^2 - 1923509k^5m^3 + 2159033k^4m^4 \\ &\quad - 1141813k^3m^5 + 303665k^2m^6 - 31753km^7 + 2557m^8. \end{aligned} \quad (21)$$

As before, this is in perfect agreement with the contribution (6) to the sum over string fluctuations. We were only able to check the matching to order $1/\mathcal{J}^{13}$. However, contrary to the previous case, coincidence between the quantum corrected Bethe ansatz and the one-loop sigma model calculation completely fixes now the free coefficients in the phase (7). Moreover, the correction to the energy of circular strings at order $1/\mathcal{J}^{r+s}$ is a polynomial with $r+s-4$ coefficients while the modified dressing phase (7) contains only $(r+s-3)/2$ terms contributing to that order. Therefore the agreement that we have found is a very non-trivial check both for our conjectured form of $a_{r,s}$, and for the proposal [29, 33] that an extended dressing phase as in (7) can take care of the first quantum corrections to the string Bethe ansatz.

The quantum correction in the lowest term of the dressing phase, $r, s=2, 3$, reproduces the $1/\mathcal{J}^5$ contribution in (5) and (6). This lowest term is also present in the strong coupling limit of the dressing phase that distinguishes the gauge and string S -matrices (1). These two facts imply that the leading discrepancy between the gauge and string results for the classical energy must be proportional to the leading non-analytical contribution to the quantum corrected string energy [29]

$$-\frac{16}{3\sqrt{\lambda}}(E_{st} - E_g) = -\frac{(k-m)^3 m^3}{3\mathcal{J}^5} + \mathcal{O}(1/\mathcal{J}^7) \ , \quad (22)$$

where the proportionality factor determines $c_{2,3}(\lambda) = 1 - 16/3\sqrt{\lambda}$. Equation (22) indicates a direct relation between the quantum corrections to the string Bethe ansatz and the gauge/string discrepancy. The negative correction term opens the possibility that $c_{2,3}(\lambda)$ could interpolate smoothly between the strong coupling value, $c_{2,3}(\infty) = 1$, and zero at weak coupling, suggesting thus a solution of the puzzling three-loop discrepancy [29].

The observation (22) does not extend to higher orders in the coupling constant. The mismatch between gauge and string classical energies at order λ^5 does not turn out to be proportional to the $1/\mathcal{J}^7$ term in the expansion of the integral (6). Coincidence would have suggested that a single coefficient running appropriately from strong to weak coupling, perhaps $c_{r,r+1}$, could have been enough to cure the discrepancy. The fact that this is not the case is thus consistent with the need to include all the terms, with r and s uncorrelated, in order to take into account the complete quantum corrections. For $s \neq r+1$ we have $c_{r,s}(\infty) = c_{r,s}(0) = 0$, and $c_{r,s}(\lambda)$ different from zero otherwise. Although this situation is more involved than a simple monotonic running implying only the coefficients $c_{r,r+1}$, it is clear from the simple form of the $a_{r,s}$ in (9) that all the terms in the series are related. It would be extremely important to have a deeper understanding of the structure behind the extended dressing phase in order to clarify the nature of the gauge/string discrepancy.

The structure of the dressing phase should play a central role in the construction of a quantum sigma model for the string on $AdS_5 \times S^5$. Recently a parallel approach to the problem has been provided by the study of closely related sigma models with a known quantum S -matrix [35]-[37]. The classical limit of these models reproduces the thermodynamic Bethe ansatz for the string in [10]. In addition, the sigma model considered in [35] contains non-analytic terms in λ associated to quantum effects, although it misses finite size corrections as it is defined on a plane.² The study of these models could also

²For an analysis of corrections to integrable quantum field theories on a cylinder in the AdS/CFT

illuminate the series in the dressing phase.

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A Appendix

For completeness we include in this appendix the expressions of the sum over string frequencies of mode number n . For the $SU(2)$ case, with $k = 2m$, we have [24, 25, 18]

$$e(n) = \sqrt{1 + \frac{(n + \sqrt{n^2 - 4m^2})^2}{4(\mathcal{J}^2 + m^2)}} + \sqrt{1 + \frac{n^2 - 2m^2}{\mathcal{J}^2 + m^2}} + 2\sqrt{1 + \frac{n^2}{\mathcal{J}^2 + m^2}} \quad (\text{A.1})$$

$$- 4\sqrt{1 + \frac{n^2 - m^2}{\mathcal{J}^2 + m^2}}.$$

The first contribution comes from bosonic fluctuations along the S^3 where the classical string is rotating. The second and third terms correspond to bosonic fluctuations on the remaining S^5 and AdS_5 directions, respectively. And the last term comes from the fermionic fluctuations.

For the general $SL(2)$ case, we have the following more involved expressions [26]

$$e(n) = \frac{1}{4\kappa}(\omega_1 + \omega_2 - \omega_3 - \omega_4) + \frac{1}{\kappa}\sqrt{n^2 + \kappa^2} + \frac{2}{\kappa}\sqrt{n^2 + \mathcal{J}^2 - m^2} \quad (\text{A.2})$$

$$- \frac{2}{\kappa}\sqrt{(n - \gamma)^2 + \frac{1}{2}(\kappa^2 + \mathcal{J}^2 - m^2)} - \frac{2}{\kappa}\sqrt{(n + \gamma)^2 + \frac{1}{2}(\kappa^2 + \mathcal{J}^2 - m^2)}.$$

The first term describes the contribution from the bosonic fluctuations along the AdS_3 section where the circular string is rotating. The second and third terms come from the bosonic fluctuations along the remaining AdS_5 and S^5 directions, respectively. And the last two terms correspond to the fermionic fluctuations. The frequencies ω_i are solutions of the quartic equation

$$(\omega^2 - n^2)^2 + 4r^2\kappa^2\omega^2 - 4(1 + r^2)(\omega\sqrt{\kappa^2 + k^2} - kn)^2 = 0, \quad (\text{A.3})$$

context see [38].

ordered in decreasing magnitude. The remaining quantities that appear in (A.2) are defined through

$$r^2 = -\frac{\mathcal{J}m}{k\sqrt{\kappa^2 + k^2}}, \quad (\text{A.4})$$

$$\gamma = \frac{\kappa m}{\sqrt{\kappa^2 + k^2}} \frac{\kappa^2 - \mathcal{J}^2 + k^2}{\kappa^2 - \mathcal{J}^2 + m^2}.$$

Finally, the parameter κ can be determined from

$$(\kappa^2 - \mathcal{J}^2 - m^2)\sqrt{\kappa^2 + k^2} + 2\mathcal{J}km = 0. \quad (\text{A.5})$$

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