THE NEUMANN PROBLEM IN GRAPH LIPSCHITZ DOMAINS IN THE PLANE

MARÍA JESÚS CARRO, VIRGINIA NAIBO, AND CARMEN ORTIZ-CARABALLO

ABSTRACT. We study new aspects of the solvability of the classical Neumann boundary value problem in a graph Lipschitz domain in the plane. When the domain is the upper half-plane, the boundary data is assumed to belong to weighted Lebesgue or weighted Lorentz spaces; we show that the solvability of the Neumann problem in these settings may be characterized in terms of Muckenhoupt weights and related weights, respectively. For a general graph Lipschitz domain Ω, as proved in an unpublished work by E. Fabes and C. Kenig, there exists $\varepsilon_\Omega > 0$ such that the Neumann problem is solvable with data in $L^p(\partial\Omega)$ for $1 < p < 2+\varepsilon_\Omega$; we review the proof of this result and show that the Neumann problem is solvable at the endpoint $2 + \varepsilon_\Omega$ with data in the Lorentz space $L^{2+\varepsilon_\Omega,1}(\partial\Omega)$. We present examples of our results in Schwarz-Christoffel Lipschitz domains and related domains.

1. INTRODUCTION AND MAIN RESULTS

Let Ω be a domain in $\mathbb{R}^n$ and consider the classical *Neumann boundary value problem* in Ω :

$$\Delta v = 0 \text{ on } \Omega \qquad \text{and} \qquad \nabla v \cdot \mathbf{n} = g \text{ on } \partial\Omega, \tag{1.1}$$

where Δ is the Laplacian operator and $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$. For bounded domains, one must also assume that $\int_{\partial\Omega} g\, d\sigma = 0$, where σ denotes surface measure on $\partial\Omega$. The solvability of (1.1) with data g in $L^p(\partial\Omega)$ has been extensively studied for different types of domains Ω. Fabes–Jodeit–Riviere [8] considered the Neumann problem in a bounded domain $\Omega \subset \mathbb{R}^n$ with C^1 boundary and $n \geq 3$ and proved that (1.1) is solvable in $L^p(\partial\Omega)$ for any $1 < p < \infty$. Later on, Jerison–Kenig [14] proved that the Neumann problem is solvable in $L^2(\partial\Omega)$ for bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ and $n \geq 3$. This result was extended by Dahlberg–Kenig [5], who showed that the Neumann problem in bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ is solvable in $L^p(\partial\Omega)$ for $1 < p < 2 + \varepsilon_\Omega$, where ε_Ω depends only on Ω and the range for p is sharp (see also Kenig [17]). An analogous result for Lipschitz domains Ω that are graphs was also proved in Dahlberg–Kenig [5] in dimensions $n \geq 3$ while the corresponding case for $n = 2$ was obtained by E. Fabes and C. Kenig (unpublished, see Jerison–Kenig [14, Remark (d), p. 207] and Kenig [15, p. 270]). In all of these results, the equality $\nabla v \cdot \mathbf{n} = g$ is interpreted in terms of non-tangential convergence; moreover, it is shown that $\|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$, with constants independent of g, where $\mathcal{M}_\alpha$ is the non-tangential maximal operator corresponding to cones with aperture α.

In this article, we treat the solvability of the Neumann problem in a graph Lipschitz domain $\Omega \subset \mathbb{C}$ with boundary data in Lebesgue, Lorentz and Lorentz-related spaces; in particular, we

2010 *Mathematics Subject Classification.* Primary: 35J05, 46E30, 42B30. Secondary: 42B99.

Key words and phrases. Neumann problem, graph Lipschitz domain, non-tangential maximal operator, Muckenhoupt weights, Lorentz spaces, Hardy spaces.

First and third authors were partially supported by MTM2016-75196-P (MINECO / FEDER, UE) and grant PID2020-113048GB-I00 funded by MCIN/AEI/ 10.13039/501100011033. Second author was partially supported by the National Science Foundation under grant DMS 1500381 and the Simons Foundation under grant 705953. Third author was partially supported by Project GR18108 (Junta de Extremadura-Spain and the European Union-European Regional Development Funds).

obtain new results regarding the solvability of (1.1) corresponding to the endpoint $p = 2 + \varepsilon_\Omega$. Before stating our results, we describe with more details the setting for the Neumann problem in a graph Lipschitz domain.

Let Λ be a curve in the complex plane given parametrically by $\xi(x) = x + i\gamma(x)$ for $x \in \mathbb{R}$, where γ is a real-valued Lipschitz function with constant L, and consider the Lipschitz domain

$$\Omega = \{z_1 + iz_2 \in \mathbb{C} : z_2 > \gamma(z_1)\}. \tag{1.2}$$

For the rest of this manuscript, Ω will be a graph Lipschitz domain as defined in (1.2) with Λ and L as given above; note that $\partial\Omega = \Lambda$.

Given $0 < \alpha < \arctan(1/L)$, define the non-tangential maximal operator $\mathcal{M}_\alpha$ as

$$\mathcal{M}_\alpha(F)(\xi) = \sup_{z \in \Gamma_\alpha(\xi)} |F(z)|, \quad \xi \in \Lambda,$$

where F is a complex-valued function defined in Ω and

$$\Gamma_\alpha(\xi) = \{z_1 + iz_2 \in \mathbb{C} : z_2 > \mathrm{Im}(\xi) \text{ and } |\mathrm{Re}(\xi) - z_1| < \tan(\alpha)|z_2 - \mathrm{Im}(\xi)|\}.$$

Definition 1.1. For a locally integrable function g in Λ, a function v defined on Ω is *a solution of the Neumann problem* (1.1) *in* Ω *with datum* g if v is harmonic in Ω and $\nabla v \cdot \mathbf{n}$ converges non-tangentially to g almost everywhere in Λ with respect to arc length measure, that is, there exists $0 < \alpha < \arctan(1/L)$ such that $\lim_{z \in \Gamma_\alpha(\xi), z \to \xi} \nabla v(z) \cdot \mathbf{n}(\xi) = g(\xi)$ for almost every $\xi \in \Lambda$.

In the sequel, we will use the notation $z \lhd \xi$ to mean that $z \to \xi$ with $z \in \Gamma_\alpha(\xi)$ for some $0 < \alpha < \arctan(1/L)$.

Definition 1.2. If X is a Banach space of measurable functions defined on Λ, we say that the *Neumann problem in* Ω *is solvable in* X if there exist a Banach space Y of measurable functions defined on Λ and $0 < \alpha < \arctan(1/L)$ such that for every $g \in X$ there exists a solution v_g of the Neumann problem in Ω with datum g and

$$\|\mathcal{M}_\alpha(\nabla v_g)\|_Y \lesssim \|g\|_X,$$

where the implicit constant is independent of g and, $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms in X and Y, respectively.

Our first stated result, Theorem 1.3 below, is in the context of the upper half-plane $\mathbb{R}^2_+$ and weighted Lebesgue, weighted Lorentz and weighted Lorentz-related spaces in the real line. We then use this result to address the Neumann problem in Ω with data in Lebesgue and Lorentz spaces in Λ as presented in Theorems 1.4 and 1.5 below.

In the following, $A_p(\mathbb{R})$ denotes the Muckenhoupt class of weights in $\mathbb{R}$, which characterizes the boundedness of the Hardy-Littlewood maximal operator on the weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ if $1 < p < \infty$ and from $L^1(\mathbb{R}, w)$ to the Lorentz space $L^{1,\infty}(\mathbb{R}, w)$ if $p = 1$, while $A_p^{\mathcal{R}}(\mathbb{R})$ stands for the class of weights w in $\mathbb{R}$ that characterizes the boundedness of the Hardy-Littlewood maximal operator from the weighted Lorentz space $L^{p,1}(\mathbb{R}, w)$ to the weighted Lorentz space $L^{p,\infty}(\mathbb{R}, w)$; it holds that $A_1(\mathbb{R}) = A_1^{\mathcal{R}}(\mathbb{R})$ and $A_p(\mathbb{R}) \subsetneq A_p^{\mathcal{R}}(\mathbb{R})$ if $1 < p < \infty$. For a weight w in $\mathbb{R}$, the notation $\mathcal{L}^{p,1}(\mathbb{R}, w)$ (respectively, $\mathcal{L}^{p,\infty}(\mathbb{R}, w)$) is used for the space of functions f such that $fw^{-1} \in L^{p,1}(\mathbb{R}, w)$ (respectively, $L^{p,\infty}(\mathbb{R}, w)$). For $1 \leq p \leq \infty$, p' denotes the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. We direct the reader to Section 2 for detailed definitions, related results and references.

Theorem 1.3 (Solvability of the Neumann problem in the upper half-plane)**.** *The Neumann problem in* $\mathbb{R}^2_+$ *is solvable in the following settings:*

(a) $X = Y = L^p(\mathbb{R}, w)$, $1 < p < \infty$ *and* $w \in A_p(\mathbb{R})$;
(b) $X = L^{p,1}(\mathbb{R}, w)$, $Y = L^{p,\infty}(\mathbb{R}, w)$, $1 \leq p < \infty$ *and* $w \in A_p^{\mathcal{R}}(\mathbb{R})$;
(c) $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$, $Y = \mathcal{L}^{p,\infty}(\mathbb{R}, w)$, $1 < p < \infty$, $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$.

Moreover, the converses of (a) *and* (b) *hold:*

(e) *if* $1 < p < \infty$, w *is a weight in* $\mathbb{R}$ *and the Neumann problem is solvable in* $X = L^p(\mathbb{R}, w)$ *with* $Y = L^p(\mathbb{R}, w)$, *then* $w \in A_p(\mathbb{R})$;
(f) *if* $1 \leq p < \infty$, w *is a weight in* $\mathbb{R}$ *and the Neumann problem is solvable in* $X = L^{p,1}(\mathbb{R}, w)$ *with* $Y = L^{p,\infty}(\mathbb{R}, w)$, *then* $w \in A_p^{\mathcal{R}}(\mathbb{R})$.

In the what follows, $\Phi : \mathbb{R}^2_+ \longrightarrow \Omega$ is a conformal mapping that extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$. It turns out that $\Phi'(x)$ exists for almost every $x \in \mathbb{R}$ and is locally integrable; moreover, $|\Phi'| \in A_2(\mathbb{R})$. Additional details and references are included in Section 4.1. We define p_Φ such that the conjugate exponent p'_Φ is given by

$$p'_\Phi = \inf\{q \in [1, \infty) : |\Phi'| \in A_q(\mathbb{R})\}$$

and we note that $p_\Phi > 2$ since $|\Phi'| \in A_2(\mathbb{R})$.

The result stated next, Theorem 1.4, relates to the unplublished work by E. Fabes and C. Kenig mentioned in the introduction (see Jerison–Kenig [14, Remark (d), p. 207] and Kenig [15, p. 270]): given a Lipschitz domain Ω contained in $\mathbb{R}^2$, there exists $\varepsilon_\Omega > 0$ such that if $1 < p < 2 + \varepsilon_\Omega$, then the Neumann problem is solvable in $L^p(\Lambda)$. Our next result shows that such $2 + \varepsilon_\Omega$ can be taken to be equal to p_Φ.

Theorem 1.4 (Solvability of the Neumann problem in $L^p(\Lambda)$)**.** *If* $1 < p < p_\Phi$, *then the Neumann problem in* Ω *is solvable in* $X = L^p(\Lambda)$ *with* $Y = L^p(\Lambda)$; *this is, for each* $g \in L^p(\Lambda)$ *there exists a solution* v_g *of the Neumann problem in* Ω *with datum* g *such that*

$$\|\mathcal{M}_\alpha(\nabla v_g)\|_{L^p(\Lambda)} \lesssim \|g\|_{L^p(\Lambda)}, \tag{1.3}$$

where $0 < \alpha < \arctan(1/L)$ *and the implicit constant is independent of* g.

Assuming $p_\Phi < \infty$, the condition $1 < p < p_\Phi$ in Theorem 1.4 is equivalent to $|\Phi'| \in A_{p'}(\mathbb{R})$ and we have that $|\Phi'| \notin A_{p'_\Phi}(\mathbb{R})$. The next theorem addresses the solvability of the Neumann problem in Ω for the endpoint $p = p_\Phi$ assuming $|\Phi'|$ belongs to the larger class $A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$ (see (2.4)).

Theorem 1.5 (Solvability of the Neumann problem in $L^{p_\Phi,1}(\Lambda)$)**.** *If* $p_\Phi < \infty$ *and* $|\Phi'| \in A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$, *the Neumann problem in* Ω *is solvable in* $X = L^{p_\Phi,1}(\Lambda)$ *with* $Y = L^{p_\Phi,\infty}(\Lambda)$; *this is, for each* $g \in L^{p_\Phi,1}(\Lambda)$ *there exists a solution* v_g *of the Neumann problem in* Ω *with datum* g *such that*

$$\|\mathcal{M}_\alpha(\nabla v_g)\|_{L^{p_\Phi,\infty}(\Lambda)} \lesssim \|g\|_{L^{p_\Phi,1}(\Lambda)}, \tag{1.4}$$

where $0 < \alpha < \arctan(1/L)$ *and the implicit constant is independent of* g.

Remark 1.6. A statement of Theorem 1.5 for p such that $p_\Phi < p < \infty$ instead of p_Φ does not make sense. Indeed, the assumption $|\Phi'| \in A^{\mathcal{R}}_{p'}(\mathbb{R})$ for some $p_\Phi < p < \infty$ implies that $|\Phi'| \in A_{p'+\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$ (see (2.4)), which contradicts the fact that, by the definition of p_Φ, $|\Phi'| \notin A_q(\mathbb{R})$ for $1 \leq q \leq p'_\Phi$.

The proofs of Theorems 1.4 and 1.5 make use of Theorem 1.3 and are inspired by tools and techniques from Kenig [15, 16]. More precisely, the existence of a solution of the Neumann problem in Ω with data in $L^p(\Lambda)$ for $1 < p < p_\Phi$ (respectively, $L^{p_\Phi,1}(\Lambda)$) follows from the existence of a solution of the Neumann problem in $\mathbb{R}^2_+$ with data in $L^p(\mathbb{R}, |\Phi'|^{1-p})$ (respectively, $\mathcal{L}^{p_\Phi,1}(\mathbb{R}, |\Phi'|)$).

Likewise, the estimates associated to the non-tangential maximal operator for the Newman problem in Ω are obtained from corresponding estimates for the Neumann problem in $\mathbb{R}^2_+$ coupled with the theory of Hardy spaces developed in Kenig [15] and new results presented here about Hardy and weak Hardy spaces (Lemmas 4.1, 4.2 and 4.4).

The organization of the article is as follows. In Section 2, we present notation and preliminaries associated to the proof of Theorem 1.3 (solvability of the Neumann problem in the upper half-plane), which is carried out in Section 3. The solvability of the Neumann problem in a general Lipschitz domain Ω is addressed in Section 4: In Section 4.1, we include corresponding notation and preliminaries; in Section 4.2, we treat the solvability of the Neumann problem in $L^p(\Lambda)$ for $1 < p < p_\Phi$ (Theorem 1.4); Section 4.3 is devoted to the solvability of the Neumann problem in $L^{p_\Phi,1}(\Lambda)$ (Theorem 1.5); finally, in Section 4.4, we give the proofs of several lemmas related to weighted Hardy spaces used in the proofs of the main results. Section 5 contains examples of our results in domains Ω where Λ is a polygonal curve with a finite or a countably infinite number of vertices (Schwarz-Christoffel Lipschitz domains and related domains); we also include a comparison regarding the solvability of the Neumann problem and the corresponding Dirichlet problem in a given domain Ω. For the purpose of self-containment, Appendix A includes several lemmas from Kenig [16] that are used throughout the proofs in Section 4.

2. Notation and preliminaries

In this section, we introduce notation and definitions associated to the results on the solvability of the Neumann problem in the upper half-plane; we also state and prove some lemmas that will be used in the proof of Theorem 1.3.

Let w be a weight on $\mathbb{R}$, i.e. a non-negative locally integrable function defined in $\mathbb{R}$, and $1 \le p \le \infty$. We will denote by $L^p(\mathbb{R}, w)$ the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R},w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)\, dx \right)^{\frac{1}{p}} < \infty,$$

with the corresponding changes for $p = \infty$. For $1 \le p < \infty$, the Lorentz spaces $L^{p,1}(\mathbb{R}, w)$ and $L^{p,\infty}(\mathbb{R}, w)$ are defined as the classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that, respectively,

$$\|f\|_{L^{p,1}(\mathbb{R},w)} = \int_0^\infty \left(\lambda_f^w(x)\right)^{\frac{1}{p}} dx = \frac{1}{p} \int_0^\infty f_w^*(x)\, x^{\frac{1}{p}-1} dx < \infty,$$
$$\|f\|_{L^{p,\infty}(\mathbb{R},w)} = \sup_{x>0} x \left(\lambda_f^w(x)\right)^{\frac{1}{p}} = \sup_{x>0} x^{\frac{1}{p}} f_w^*(x) < \infty,$$

where λ_f^w is given by $\lambda_f^w(x) = w(\{t \in \mathbb{R} : |f(t)| > x\})$, with $w(A) = \int_A w(x)dx$ for A a measurable subset of $\mathbb{R}$, and f_w^* is the decreasing rearrangement of f with respect to w, that is $f_w^*(x) = \inf\{t > 0 : \lambda_f^w(t) \le x\}$. We note that $L^{1,1}(\mathbb{R}, w) = L^1(\mathbb{R}, w)$. Finally, for $1 < p < \infty$, we will use the notations $\mathcal{L}^{p,1}(\mathbb{R}, w)$ and $\mathcal{L}^{p,\infty}(\mathbb{R}, w)$ for the spaces of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{L}^{p,1}(\mathbb{R},w)} = \|fw^{-1}\|_{L^{p,1}(\mathbb{R},w)} < \infty,$$
$$\|f\|_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} = \|fw^{-1}\|_{L^{p,\infty}(\mathbb{R},w)} < \infty.$$

Note that the fact that $L^{p,1}(\mathbb{R}, w) \subset L^p(\mathbb{R}, w)$ continuously implies that $\mathcal{L}^{p,1}(\mathbb{R}, w) \subset L^p(\mathbb{R}, w^{1-p})$ continuoulsy.

When $w \equiv 1$, we will drop w in the notations introduced above; for instance, $L^p(\mathbb{R})$ means the space $L^p(\mathbb{R}, w)$ when $w \equiv 1$. The notation $C_c(\mathbb{R})$ is used for the class of continuos functions in

$\mathbb{R}$ that have compact support and $L^1_c(\mathbb{R})$ denotes the subspace of compactly supported functions of $L^1(\mathbb{R})$.

The Hardy-Littlewood maximal operator will be denoted by $\mathcal{M}_{hl}$; it is defined for locally integrable functions f on $\mathbb{R}$ by

$$\mathcal{M}_{hl}f(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(x)|\,dx,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ that contain x and, for a measurable set $A \subset \mathbb{R}$, $|A|$ denotes the Lebesgue measure of A.

For $\varepsilon > 0$, we use the notation $\mathcal{H}_\varepsilon$ for the truncated Hilbert transform, that is,

$$\mathcal{H}_\varepsilon f(x) = \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy,$$

and denote by $\mathcal{H}$ and $\mathcal{H}^*$ the Hilbert transform and the maximal Hilbert transform, respectively; that is,

$$\mathcal{H}f(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy \quad \text{and} \quad \mathcal{H}^*f(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy\right|.$$

We next define the classes of weights to be considered in $\mathbb{R}$: the well-known Muckenhoupt class $A_p(\mathbb{R})$ and the larger class $A_p^{\mathcal{R}}(\mathbb{R})$. If $1 < p < \infty$, w belongs to $A_p(\mathbb{R})$ if

$$\|w\|_{A_p} = \sup_{I\subset\mathbb{R}} \left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}dx\right)^{p-1} < \infty, \tag{2.1}$$

where the supremum is taken over all intervals contained in $\mathbb{R}$. Note that the above condition can be rewritten as

$$\sup_{I\subset\mathbb{R}} \frac{1}{|I|}\|\chi_I\|_{L^p(\mathbb{R},w)}\|\chi_I w^{-1}\|_{L^{p'}(\mathbb{R},w)} < \infty.$$

We say that $w \in A_1(\mathbb{R})$ if

$$\|w\|_{A_1} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \frac{\mathcal{M}_{hl}w(x)}{w(x)} < \infty.$$

We recall that if $f \in L^1_c(\mathbb{R})$ is such that $\mathcal{M}_{hl}f(x) < \infty$ for almost every $x \in \mathbb{R}$ and $0 < \delta < 1$, then $(\mathcal{M}_{hl}f)^\delta \in A_1(\mathbb{R})$; and, for $1 < p < \infty$, $w \in A_p(\mathbb{R})$ if and only if $w^{1-p'} \in A_{p'}(\mathbb{R})$. Moreover, $A_p(\mathbb{R}) \subset A_q(\mathbb{R})$ if $p < q$.

We set $A_\infty(\mathbb{R}) = \cup_{p\geq 1} A_p(\mathbb{R})$ and recall that weights in $A_\infty(\mathbb{R})$ satisfy the reverse Hölder's inequality: if $w \in A_\infty(\mathbb{R})$, there exists $\varepsilon > 0$ such that

$$\left(\frac{1}{|I|}\int_I w(x)^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}} \lesssim \frac{1}{|I|}\int_I w(x)\,dx, \tag{2.2}$$

for all intervals $I \subset \mathbb{R}$.

For $1 \leq p < \infty$, $w \in A_p^{\mathcal{R}}(\mathbb{R})$ if

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{E\subset I} \frac{|E|}{|I|}\left(\frac{w(I)}{w(E)}\right)^{1/p} < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ and all measurable sets $E \subset I$. This condition is equivalent to

$$\sup_{I\subset\mathbb{R}} \frac{1}{|I|}\|\chi_I\|_{L^{p,1}(\mathbb{R},w)}\|\chi_I w^{-1}\|_{L^{p',\infty}(\mathbb{R},w)} < \infty, \tag{2.3}$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$. We have $A_1(\mathbb{R}) = A_1^{\mathcal{R}}(\mathbb{R})$ and, for $1 < p < \infty$,

$$A_p(\mathbb{R}) \subsetneq A_p^{\mathcal{R}}(\mathbb{R}) \subsetneq \cap_{\varepsilon>0} A_{p+\varepsilon}(\mathbb{R}). \tag{2.4}$$

We refer the reader to Carro–Ortiz-Caraballo [3, p. 2019] for examples regarding the strict inclusions.

The following theorem summarizes known results regarding boundedness properties of $\mathcal{M}_{hl}$, $\mathcal{H}$ and $\mathcal{H}^*$ in weighted spaces that will be used throughout the article.

Theorem 2.1. *Let $1 < p < \infty$.*

(a) $\mathcal{M}_{hl}$, $\mathcal{H}$ and $\mathcal{H}^*$ are bounded from $L^p(\mathbb{R}, w)$ to $L^p(\mathbb{R}, w)$ if and only if $w \in A_p(\mathbb{R})$ and $\mathcal{M}_{hl}$, $\mathcal{H}$ and $\mathcal{H}^*$ are bounded from $L^1(\mathbb{R}, w)$ to $L^{1,\infty}(\mathbb{R}, w)$ if and only if $w \in A_1(\mathbb{R})$; see Muckenhoupt [21] for $\mathcal{M}_{hl}$ and Hunt–Muckenhoupt–Wheeden [13] for $\mathcal{H}$ and $\mathcal{H}^*$.

(b) $\mathcal{M}_{hl}$ is bounded from $L^{p,\infty}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ if and only if $w \in A_p(\mathbb{R})$; see Chung–Hunt–Kurtz [4].

(c) $\mathcal{M}_{hl}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ if and only if $w \in A_p^{\mathcal{R}}(\mathbb{R})$; see [4] and Kerman–Torchinsky [18].

(d) $\mathcal{H}^*$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ for $w \in A_p^{\mathcal{R}}(\mathbb{R})$; see [4].

(e) $\mathcal{H}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ if and only if $w \in A_p^{\mathcal{R}}(\mathbb{R})$.

For part (e) of Theorem 2.1, the fact that $\mathcal{H}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ if $w \in A_p^{\mathcal{R}}(\mathbb{R})$ follows from part (d) of Theorem 2.1. The converse is a consequence of Agora–Carro–Soria [1, Theorem 1.1], which gives that if $\mathcal{H}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ then $\mathcal{M}_{hl}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$; part (c) of Theorem 2.1 then implies that $w \in A_p^{\mathcal{R}}(\mathbb{R})$.

We next state and prove two lemmas that will be used in Section 3. Lemma 2.2 is concerned with dense subsets of $L^p(\mathbb{R}, w)$, $L^{p,1}(\mathbb{R}, w)$ and $\mathcal{L}^{p,1}(\mathbb{R}, w)$ for general weights while Lemma 2.3 presents estimates for weights in the classes defined above.

Lemma 2.2. *Let w be a weight on $\mathbb{R}$ and $1 \le p < \infty$.*

(a) $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R}, w)$.

(b) $C_c(\mathbb{R}) \cap L^{p,1}(\mathbb{R}, w)$ is dense in $L^{p,1}(\mathbb{R}, w)$.

(c) $L_c^1(\mathbb{R}) \cap \mathcal{L}^{p,1}(\mathbb{R}, w)$ is dense in $\mathcal{L}^{p,1}(\mathbb{R}, w)$.

Proof. We first prove (a) and (b) together and then we prove (c).

Proof of (a) and (b): Since finite linear combinations of characteristic functions of measurable sets E such that $w(E)$ is finite are dense in $L^p(\mathbb{R}, w)$ and $L^{p,1}(\mathbb{R}, w)$ (see Grafakos [12, Theorem 1.4.13]), it is enough to show that if E is such a set and $\varepsilon > 0$, then there exists $h \in C_c(\mathbb{R})$ (respectively, $h \in C_c(\mathbb{R}) \cap L^{p,1}(\mathbb{R}, w)$) such that $\|\chi_E - h\|_{L^p(\mathbb{R},w)} < \varepsilon$ (respectively, $\|\chi_E - h\|_{L^{p,1}(\mathbb{R},w)} < \varepsilon$). In turn, we may assume that E is bounded and contained in an open interval I sufficiently large (since $\chi_{E_k} \to \chi_E$ in $L^p(\mathbb{R}, w)$ and $L^{p,1}(\mathbb{R}, w)$ with $E_k = E \cap (-k, k)$).

Since $w\chi_I \in L^1(\mathbb{R})$, there exists $\delta > 0$ such that if $A \subset \mathbb{R}$ is measurable and $|A| < \delta$ then $w(A \cap I) < \varepsilon^p$. Let U and K be open and closed sets in $\mathbb{R}$, respectively, such that $K \subset E \subset U \subset I$ and $|U \setminus K| < \delta$ and let $h \in C_c(\mathbb{R})$ be a function with values in $[0, 1]$ such that $|\chi_E(x) - h(x)| \le \chi_{U\setminus K}(x)$. Then, we have, for $X = L^p(\mathbb{R}, w)$ or $X = L^{p,1}(\mathbb{R}, w)$,

$$\|\chi_E - h\|_X \le \|\chi_{U\setminus K}\|_X = w(U \setminus K)^{1/p} < \varepsilon,$$

as desired.

Proof of (c): Let $f \in \mathcal{L}^{p,1}(\mathbb{R}, w)$, then $f = gw$ with $g \in L^{p,1}(\mathbb{R}, w)$. Since $C_c(\mathbb{R}) \cap L^{p,1}(\mathbb{R}, w)$ is dense in $L^{p,1}(\mathbb{R}, w)$, there exists $\{g_m\}_{m\in\mathbb{N}} \subset C_c(\mathbb{R}) \cap L^{p,1}(\mathbb{R}, w)$ such that $g_m \to g$ in $L^{p,1}(\mathbb{R}, w)$. Set $f_m = g_m w$; then $f_m \in L^1_c(\mathbb{R}) \cap \mathcal{L}^{p,1}(\mathbb{R}, w)$ and we have,

$$\|f_m - f\|_{\mathcal{L}^{p,1}(\mathbb{R},w)} = \|g_m - g\|_{L^{p,1}(\mathbb{R},w)} \to 0,$$

which proves the desired result. □

Lemma 2.3. *Let w be a weight in $\mathbb{R}$.*
(a) If $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, then

$$\int_{\mathbb{R}} \frac{w(x)}{(t + |x_0 - x|)^p}\, dx \lesssim \frac{1}{t^p} \int_{|x_0-x|<t} w(x)\, dx, \quad \forall x_0 \in \mathbb{R}, t > 0.$$

(b) If $1 < p < \infty$ and $w \in A_p^{\mathcal{R}}(\mathbb{R})$, then

$$\|w^{-1}(\cdot)(1 + |\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)} < \infty.$$

Proof. We first prove (a) and then (b).

Proof of (a): For $x \in \mathbb{R}$, we have that $|y - x_0| < t$ implies $|y - x| < t + |x - x_0|$. Then,

$$\mathcal{M}_{hl}(\chi_{(x_0-t,x_0+t)})(x) \geq \frac{1}{2(t + |x - x_0|)} \int_{|y-x|<t+|x-x_0|} \chi_{(x_0-t,x_0+t)}(y)\, dy = \frac{t}{t + |x - x_0|}.$$

Since $\mathcal{M}_{hl}$ is bounded on $L^p(\mathbb{R}, w)$ for $w \in A_p(\mathbb{R})$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{w(x)}{(t + |x_0 - x|)^p}\, dx &\leq \frac{1}{t^p} \|\mathcal{M}_{hl}(\chi_{x_0-t,x_0+t})\|^p_{L^p(\mathbb{R},w)} \\ &\lesssim \frac{1}{t^p} \|\chi_{(x_0-t,x_0+t)}\|^p_{L^p(\mathbb{R},w)} = \frac{1}{t^p} \int_{|x_0-x|<t} w(x)\, dx,\end{aligned}$$

as desired.

Proof of (b): Let $I_0 = [0, 1)$ and $I_k = [2^{k-1}, 2^k)$ for $k \in \mathbb{N}$. We have

$$\begin{aligned}\|w^{-1}(\cdot)(1 + |\cdot|)^{-1}\|^{p'}_{L^{p',\infty}(\mathbb{R},w)} &= \sup_{y>0} y^{p'} \int_{\{x\in\mathbb{R}: w(x)(1+|x|)<\frac{1}{y}\}} w(x)dx \\ &= \sup_{y>0} y^{p'} \sum_{k=0}^{\infty} \int_{\{x\in\mathbb{R}:|x|\in I_k \text{ and } w(x)(1+|x|)<\frac{1}{y}\}} w(x)dx \\ &\leq \sup_{y>0} y^{p'} \sum_{k=0}^{\infty} \int_{\{x\in\mathbb{R}:|x|\in I_k \text{ and } w(x)<\frac{2}{2^k y}\}} w(x)dx \\ &\leq 2^{p'} \sum_{k=0}^{\infty} 2^{-kp'} \sup_{z>0} z^{p'} \int_{\{x\in\mathbb{R}:|x|\leq 2^k \text{ and } w^{-1}(x)>z\}} w(x)dx \\ &= 2^{p'} \sum_{k=0}^{\infty} 2^{-kp'} \|\chi_{(-2^k,2^k)} w^{-1}\|^{p'}_{L^{p',\infty}(\mathbb{R},w)}.\end{aligned}$$

Since $w \in A_p^{\mathcal{R}}(\mathbb{R})$, (2.3) gives

$$\|\chi_{(-2^k,2^k)} w^{-1}\|^{p'}_{L^{p',\infty}(w)} \lesssim \frac{2^{kp'}}{w((-2^k, 2^k))^{p'/p}},$$

and therefore

$$\|w^{-1}(\cdot)(1+|x|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)}^{p'} \lesssim \sum_{k=0}^{\infty} \frac{1}{w((-2^k,2^k))^{p'/p}}. \tag{2.5}$$

Taking into account that $w \in A_\infty(\mathbb{R})$, the reverse Hölder inequality (2.2) implies that there exists $\varepsilon > 0$ such that

$$\left(\frac{1}{2^k}\int_{-2^k}^{2^k} w(x)^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}} \lesssim \frac{1}{2^k}\int_{-2^k}^{2^k} w(x)\,dx, \quad \forall k \in \mathbb{N}_0.$$

As a consequence, we have

$$w((-2^k,2^k)) \gtrsim 2^{\frac{k\varepsilon}{1+\varepsilon}} \left(\int_{-1}^{1} w(x)^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}},$$

which implies that the series on the right-hand side of (2.5) converges and the desired result follows. □

We end this section by stating a particular case of Lerner [20, Theorem 4.2] regarding operators associated to sparse families of cubes in $\mathbb{R}^n$, which will be used in the proof of Lemma 3.2 regarding the solvability of the Neumann problem in the upper half-plane in the space $\mathcal{L}^{p,1}(w)$ with $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$.

Given $0 < \eta < 1$, a family $\mathcal{S}$ of cubes contained in $\mathbb{R}^n$ is said to be η-sparse if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$ and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. A family of cubes is sparse if it is η-sparse for some $0 < \eta < 1$. For a cube $Q \subset \mathbb{R}^n$ and a constance $c > 0$, $c\,Q$ will denote the cube with the same center as Q and side length c times the side length of Q.

Theorem 2.4 (Theorem 4.2 in Lerner [20])**.** *Let T be a sublinear operator defined on functions of $\mathbb{R}^n$ such that T and $\mathcal{M}_T$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, where*

$$\mathcal{M}_T(f)(x) = \sup_{Q:x\in Q} \operatorname*{ess\,sup}_{\xi\in Q} |T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|,$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$ that contain x. Then, for every $f \in L^1_c(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$ such that

$$|T(f)(x)| \leq C\mathcal{A}_{\mathcal{S}}(|f|)(x), \qquad a.e.\, x \in \mathbb{R}^n,$$

where $C = C_n(\|T\|_{L^1\to L^{1,\infty}} + \|\mathcal{M}_T\|_{L^1\to L^{1,\infty}})$ and

$$\mathcal{A}_{\mathcal{S}}(f)(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q f(y)\,dy\,\chi_Q(x).$$

3. Solvability of the Neumann problem in the upper half-plane

In this section we will prove Theorem 1.3, which states that the Neumann problem in the upper half-plane is solvable in $L^p(\mathbb{R},w)$ with $w \in A_p(\mathbb{R})$, in $L^{p,1}(\mathbb{R},w)$ with $w \in A_p^{\mathcal{R}}(\mathbb{R})$ and in $\mathcal{L}^{p,1}(\mathbb{R},w)$ with $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$.

In Section 3.1, after presenting some preliminaries, we state and prove Lemmas 3.1 and 3.2 to be used in the proof of Theorem 1.3. Lemma 3.1 states that u_f as given in equation (3.1) is a well-defined harmonic function for f in any of the above function spaces as well as a result regarding convergence of sequences of such harmonic functions. Lemma 3.2 shows that the non-tangential maximal operator is bounded appropriately in dense subsets of the function spaces of interest to Theorem 1.3. The proof of Theorem 1.3 is completed in Section 3.2

3.1. **Proof of Theorem 1.3: preliminaries and some lemmas.** Let $f \in L^1(\mathbb{R})$ and define

$$u_f(y_1, y_2) := \frac{1}{\pi} \int_{\mathbb{R}} \log \left(\frac{\sqrt{(y_1-x)^2+y_2^2}}{(1+|x|)} \right) f(x)\, dx, \quad (y_1, y_2) \in \mathbb{R}^2_+. \tag{3.1}$$

Note that, as proved in Armitage [2, Lemma 1 and p. 103], for every compact set $K \subset \mathbb{R}^2_+$ there is a constant $C_K > 0$ such that

$$(1+|x|) \left| \log \left(\frac{\sqrt{(y_1-x)^2+y_2^2}}{(1+|x|)} \right) \right| \leq C_K, \quad \forall x \in \mathbb{R}, (y_1, y_2) \in K. \tag{3.2}$$

In particular, u_f is well-defined on $\mathbb{R}^2_+$.

The estimate (3.2) and the Dominated Convergence Theorem imply that u_f is continuous in $\mathbb{R}^2_+$. Applying Fubini's theorem, the mean value property follows at every point of $\mathbb{R}^2_+$ from the fact that $\log\left(\sqrt{(y_1-x)^2+y_2^2}\right)$ is harmonic in $\mathbb{R}^2_+$ as a function of (y_1, y_2) for each $x \in \mathbb{R}$. As a consequence, u_f is harmonic in $\mathbb{R}^2_+$. Moreover, formula (3.3) given below implies that $\frac{\partial u_f}{\partial y_2}$ converges non-tangentially to f for any $0 < \alpha < \pi/2$ (see, for instance, Stein [23, p. 197, Theorem 1]). We then conclude that u_f is a solution of the Neumann problem in $\mathbb{R}^2_+$ with datum f.

We next estimate $\mathcal{M}_\alpha(\nabla u_f)$ with $0 < \alpha < \pi/2$. We have

$$\frac{\partial u_f}{\partial y_2}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} f(x) \frac{y_2}{(x-y_1)^2 + y_2^2}\, dx = -(P_{y_2} * f)(y_1), \tag{3.3}$$

where P_{y_2} denotes the Poisson kernel, that is, $P_{y_2}(x) = y_2^{-1} P(y_2^{-1} x)$ with $P(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ and $x \in \mathbb{R}$. Note that differentiation under the integral sign is possible since, if $K \subset \mathbb{R}^2_+$ is a compact set, we have

$$\left| \frac{y_2}{(x-y_1)^2 + y_2^2} \right| \leq \frac{1}{\sqrt{(x-y_1)^2 + y_2^2}} \leq \frac{C_K}{1+|x|}, \quad \forall (y_1, y_2) \in K.$$

Hence, it holds that

$$\mathcal{M}_\alpha(\tfrac{\partial u_f}{\partial y_2})(x) = \sup\nolimits_{(y_1+iy_2)\in\Gamma_\alpha(x)} |P_{y_2} * f(y_1)| \lesssim \mathcal{M}_{hl} f(x); \tag{3.4}$$

see, for instance, Stein [23, p. 197, Theorem 1] for the last inequality. We also have

$$\frac{\partial u_f}{\partial y_1}(y_1, y_2) = -\frac{1}{\pi} \int_{\mathbb{R}} f(x) \frac{(x-y_1)}{(x-y_1)^2 + y_2^2}\, dx = -(Q_{y_2} * f)(y_1), \tag{3.5}$$

where Q_{y_2} is the conjugate Poisson kernel, that is $Q_{y_2}(x) = y_2^{-1} Q(y_2^{-1} x)$ with $Q(x) = \frac{1}{\pi}\frac{x}{1+x^2}$ and $x \in \mathbb{R}$. Note that differentiation under the integral sign is possible since, if $K \subset \mathbb{R}^2_+$ is a compact set, we have that

$$\left| \frac{(x-y_1)}{(x-y_1)^2 + y_2^2} \right| \leq \frac{1}{\sqrt{(x-y_1)^2 + y_2^2}} \leq \frac{C_K}{1+|x|}, \quad \forall (y_1, y_2) \in K.$$

If $0 < q < \infty$, since $(Q_{y_2} * f)(y_1)$ is harmonic in $\mathbb{R}^2_+$, we obtain

$$\mathcal{M}_\alpha(\tfrac{\partial u_f}{\partial y_1})(x) \lesssim (\mathcal{M}_{hl}(\sup_{y_2>0} |(Q_{y_2} * f)|^q)(x))^{1/q}; \tag{3.6}$$

see Fefferman–Stein [9, p. 170]. In addition, $Q_{y_2} * f = Q_{y_2} * f - \mathcal{H}_{y_2} f + \mathcal{H}_{y_2} f = \psi_{y_2} * f + \mathcal{H}_{y_2} f$, where $\psi_{y_2}(z) = {y_2}^{-1}\psi({y_2}^{-1}z)$, for an appropriate function ψ, is an approximation to the identity. It then follows that

$$\sup_{y_2>0} |(Q_{y_2} * f)(x)|^q \lesssim (\mathcal{M}_{hl} f(x) + \mathcal{H}^* f(x))^q, \tag{3.7}$$

where we have used that $\sup_{y_2>0} |(\psi_{y_2} * f)(x)| \lesssim \mathcal{M}_{hl} f(x)$ (see, for instance, Stein [23, p.62, Theorem 2]). The estimates (3.6) and (3.7) imply

$$\mathcal{M}_\alpha(\tfrac{\partial u_f}{\partial y_1})(x) \lesssim (\mathcal{M}_{hl}((\mathcal{M}_{hl} f + \mathcal{H}^* f)^q)(x))^{1/q}. \tag{3.8}$$

The estimates (3.4) and (3.8) then lead to

$$\mathcal{M}_\alpha(\nabla u_f)(x) \lesssim (\mathcal{M}_{hl}((\mathcal{M}_{hl} f + \mathcal{H}^* f)^q)(x))^{1/q} + \mathcal{M}_{hl} f(x), \quad x \in \mathbb{R}, f \in L^1(\mathbb{R}).$$

Denoting $\mathcal{T}_q(f) = (\mathcal{M}_{hl}((\mathcal{M}_{hl} f + \mathcal{H}^* f)^q))^{1/q}$ for $0 < q < \infty$, we then have

$$\mathcal{M}_\alpha(\nabla u_f)(x) \lesssim \mathcal{T}_q(f)(x) + \mathcal{M}_{hl} f(x), \quad x \in \mathbb{R}, f \in L^1(\mathbb{R}), \tag{3.9}$$

We are now ready to state and prove Lemmas 3.1 and 3.2, which will be used in the proof of Theorem 1.3

Lemma 3.1. *For $1 < p < \infty$, let $X = L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$ or $X = L^{p,1}(\mathbb{R}, w)$ with $w \in A_p^{\mathcal{R}}(\mathbb{R})$ or $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$ with $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$; for $p = 1$, let $X = L^1(\mathbb{R}, w)$ with $w \in A_1(\mathbb{R})$. If $f \in X$, then u_f as given by (3.1) is well-defined and is harmonic in $\mathbb{R}^2_+$. Moreover, for any sequence $\{f_m\}_{m\in\mathbb{N}} \subset X$ converging to f in X, it holds that u_{f_m} converges to u_f uniformly on compact sets of $\mathbb{R}^2_+$.*

Proof. In view of (3.2), u_f is well-defined for any function f on $\mathbb{R}$ that satisfies $\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\, dx < \infty$. We will show that

$$\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\, dx \lesssim \|f\|_X, \quad \forall f \in X. \tag{3.10}$$

<u>Case $X = L^p(\mathbb{R}, w)$, $w \in A_p(\mathbb{R})$, $1 < p < \infty$:</u> By Hölder's inequality, we have

$$\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\, dx \lesssim \|f\|_{L^p(\mathbb{R},w)} \left(\int_{\mathbb{R}} \frac{w(x)^{1-p'}}{(1+|x|)^{p'}}\, dx\right)^{1/p'}.$$

Since $w \in A_p(\mathbb{R})$ then $w^{1-p'} \in A_{p'}(\mathbb{R})$; Lemma 2.3 then implies that the integral on the right hand side is finite.

<u>Case $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$, $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$, $1 < p < \infty$:</u> We have

$$\begin{aligned}\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\, dx &\lesssim \|f w^{-1}\|_{L^{p,1}(\mathbb{R},w)} \|(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)} \\ &= \|f\|_{\mathcal{L}^{p,1}(\mathbb{R},w)} \|(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)}.\end{aligned}$$

We next show that $\|(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)}$ is finite. Denoting $I_\lambda = (-\frac{1-\lambda}{\lambda}, \frac{1-\lambda}{\lambda})$ and $E_\lambda = (-(1-\lambda), 1-\lambda)$ for $0<\lambda<1$, we have

$$\begin{aligned}\|(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)} &= \sup_{0<\lambda<\infty} \lambda \left(w(\{x\in\mathbb{R} : (1+|x|)^{-1} > \lambda\})\right)^{1/p'} \\ &= \sup_{0<\lambda<1} \lambda w(I_\lambda)^{1/p'} = \sup_{0<\lambda<1} w(E_\lambda)^{1/p'} \frac{|E_\lambda|}{|I_\lambda|} \left(\frac{w(I_\lambda)}{w(E_\lambda)}\right)^{1/p'} \\ &\leq w(E_0)^{1/p'} \|w\|_{A^{\mathcal{R}}_{p'}} < \infty.\end{aligned}$$

Case $X = L^{p,1}(\mathbb{R},w)$, $w\in A_p^{\mathcal{R}}(\mathbb{R})$, $1<p<\infty$: We have

$$\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\,dx \lesssim \|f\|_{L^{p,1}(\mathbb{R},w)} \|w^{-1}(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)}.$$

Lemma 2.3 gives that $\|w^{-1}(1+|\cdot|)^{-1}\|_{L^{p',\infty}(\mathbb{R},w)}$ is finite.

Case $X = L^1(\mathbb{R},w)$, $w\in A_1(\mathbb{R})$: Since $w\in A_1(\mathbb{R})$,

$$w(x) \gtrsim \mathcal{M}_{hl} w(x) \gtrsim \frac{1}{1+|x|}\int_{-1}^{1} w(y)\,dy, \quad \text{a.e. } x\in\mathbb{R}.$$

Therefore, noting that $\int_{-1}^1 w(y)\,dy > 0$, we obtain

$$\frac{1}{1+|x|} \lesssim w(x), \quad \text{a.e. } x\in\mathbb{R};$$

multiplying on both sides by $|f(x)|$ and integrating over $\mathbb{R}$ gives the desired inequality.

It then follows that u_f is well-defined for any f in the spaces X considered. The fact that u_f is harmonic in $\mathbb{R}^2_+$ for any $f\in X$ follows as described in the paragraph following (3.2).

We next prove the convergence result. Given $f\in X$, let $\{f_m\}_{m\in\mathbb{N}}\subset X$ be a sequence that converges to f in X. If $K\subset\mathbb{R}^2_+$ is a compact set, using (3.2) and (3.10), it follows that

$$\sup_{(y_1,y_2)\in K} |u_{f_m}(y_1,y_2) - u_f(y_1,y_2)| \lesssim \|f_m - f\|_X,$$

where the implicit constant depends only on w, p and K. As a consequence, $u_{f_m}\to u_f$ uniformly on K. □

Lemma 3.2. *If X and Y are given as in the statement of Theorem 1.3, it holds that*

$$\|\mathcal{M}_\alpha(\nabla u_f)\|_Y \lesssim \|f\|_X, \quad \forall f\in L^1_c(X)\cap X,$$

where u_f is as given in (3.1) and $0<\alpha<\pi/2$.

Proof. We analyze each pair of spaces separately by using the estimate (3.9).

Case $X = Y = L^p(\mathbb{R},w)$, $w\in A_p(\mathbb{R})$, $1<p<\infty$: Since the Hardy–Littelwood maximal operator $\mathcal{M}_{hl}$ and the maximal Hilbert transform $\mathcal{H}^*$ are bounded from $L^p(\mathbb{R},w)$ to $L^p(\mathbb{R},w)$, it then follows that $\mathcal{T}_1$ is bounded in the same manner. This fact and (3.9) give the desired result.

Case $X = L^{p,1}(\mathbb{R}, w)$, $Y = L^{p,\infty}(\mathbb{R}, w)$, $w \in A_p^{\mathcal{R}}(\mathbb{R})$, $1 \le p < \infty$: Let r be such that $1 \le p < r$; then $w \in A_p^{\mathcal{R}}(\mathbb{R})$ implies $w \in A_r(\mathbb{R})$ and it holds that

$$\begin{aligned}\|\mathcal{T}_{p/r}(f)\|_{L^{p,\infty}(\mathbb{R},w)} &= \|\mathcal{M}_{hl}((\mathcal{M}_{hl}f + \mathcal{H}^*f)^{p/r})\|_{L^{r,\infty}(\mathbb{R},w)}^{r/p} \\ &\lesssim \|(\mathcal{M}_{hl}f + \mathcal{H}^*f)^{p/r}\|_{L^{r,\infty}(\mathbb{R},w)}^{r/p} \\ &= \|\mathcal{M}_{hl}f + \mathcal{H}^*f\|_{L^{p,\infty}(\mathbb{R},w)} \\ &\lesssim \|f\|_{L^{p,1}(\mathbb{R},w)},\end{aligned}$$

where in the first inequality we have used that $\mathcal{M}_{hl}$ is bounded on $L^{r,\infty}(\mathbb{R}, w)$ for $w \in A_r(\mathbb{R})$ and in the last inequality we have used that $\mathcal{M}_{hl}$ and $\mathcal{H}^*$ are bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ for $w \in A_p^{\mathcal{R}}(\mathbb{R})$ (see Theorem 2.1).

The estimate for $\mathcal{T}_{p/r}$, the fact that $\mathcal{M}_{hl}$ is bounded from $L^{p,1}(\mathbb{R}, w)$ to $L^{p,\infty}(\mathbb{R}, w)$ for $w \in A_p^{\mathcal{R}}(\mathbb{R})$ and (3.9) lead to the desired result.

Case $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$, $Y = \mathcal{L}^{p,\infty}(\mathbb{R}, w)$, $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$, $1 < p < \infty$: We will show the desired result through an application of Theorem 2.4. Recall that u_f is well-defined for $f \in L^1(\mathbb{R})$ and that the estimate (3.9) holds for functions in $L^1(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, consider the sublinear operator T defined by $T(f)(x) = \mathcal{M}_\alpha(\nabla u_f)(x)$.

We will show that T and $\mathcal{M}_T$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. Assuming these facts for the moment, Theorem 2.4 gives that, for each compactly supported integrable function in $\mathbb{R}$ and, in particular, for $f \in L_c^1(\mathbb{R}) \cap \mathcal{L}^{p,1}(\mathbb{R}, w)$, there exists a sparse family $\mathcal{S}$ of intervals in $\mathbb{R}$ such that

$$\mathcal{M}_\alpha(\nabla u_f)(x) \lesssim \mathcal{A}_{\mathcal{S}}(|f|)(x) = \sum_{I \in \mathcal{S}} \left(\frac{1}{|I|} \int_I |f(y)| dy \right) \chi_I(x),$$

where the implicit constant is independent of f and $\mathcal{S}$. Since $\mathcal{S}$ is sparse, there exists a disjoint family of measurable sets $\{E_I\}_{I \in \mathcal{S}}$ such that $E_I \subset I$ and $|E_I| \sim |I|$; note that this and the fact that $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$ imply $w(E_I) \sim w(I)$.

Denote by $\mathcal{M}_w$ the maximal operator defined as $\mathcal{M}_w(h)(x) = \sup_{x \in I} \frac{1}{w(I)} \int_I |h(y)| w(y) dy$, where the supremum is taken over all intervals $I \subset \mathbb{R}$ containing x. It then follows that

$$\begin{aligned}\|\mathcal{M}_\alpha(\nabla u)\|_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} &\lesssim \left\| A_{\mathcal{S}}(|f|) w^{-1} \right\|_{L^{p,\infty}(\mathbb{R},w)} \\ &= \sup_{\|h\|_{L^{p',1}(\mathbb{R},w)}=1} \sum_{I \in \mathcal{S}} \left(\frac{1}{|I|} \int_I |f(y)| dy \right) \int_I |h(y)| dy \\ &\sim \sup_{\|h\|_{L^{p',1}(\mathbb{R},w)}=1} \sum_{I \in \mathcal{S}} \left(\frac{1}{w(I)} \int_I \frac{|f(y)|}{w(y)} w(y) dy \right) \left(\frac{1}{|I|} \int_I |h(y)| dy \right) w(E_I) \\ &\lesssim \sup_{\|h\|_{L^{p',1}(\mathbb{R},w)}=1} \int_{\mathbb{R}} \mathcal{M}_w \left(f w^{-1} \right)(x) \mathcal{M}_{hl} h(x) w(x) dx \\ &\lesssim \sup_{\|h\|_{L^{p',1}(\mathbb{R},w)}=1} \|\mathcal{M}_w(f w^{-1})\|_{L^{p,1}(\mathbb{R},w)} \|\mathcal{M}_{hl} h\|_{L^{p',\infty}(\mathbb{R},w)} \\ &\lesssim \|f w^{-1}\|_{L^{p,1}(\mathbb{R},w)} = \|f\|_{\mathcal{L}^{p,1}(\mathbb{R},w)},\end{aligned}$$

where in the last inequality we have used that $\|\mathcal{M}_{hl} h\|_{L^{p',\infty}(\mathbb{R},w)} \lesssim \|h\|_{L^{p',1}(\mathbb{R},w)}$ since $w \in A_{p'}^{\mathcal{R}}(\mathbb{R})$ and the fact that $\mathcal{M}_w$ is bounded on $L^{p,1}(\mathbb{R}, w)$ (see M. de Guzmán [6, p. 114]). This proves the desired result.

We are left to show the facts that T and $\mathcal{M}_T$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. In view of (3.9), it is enough to prove that $\mathcal{T}_q$, $\mathcal{M}_{\mathcal{T}_q}$ and $\mathcal{M}_{\mathcal{M}_{hl}}$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ for some $0<q<\infty$. Fix $0<q<1$.

Boundedness of $\mathcal{T}_q$: Since $\mathcal{M}_{hl}$ is bounded on $L^{1/q,\infty}(\mathbb{R})$ and both $\mathcal{M}_{hl}$ and $\mathcal{H}^*$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, we have

$$\|\mathcal{T}_q(f)\|_{L^{1,\infty}(\mathbb{R})} = \|\mathcal{M}_{hl}((\mathcal{M}_{hl}f+\mathcal{H}^*f)^q)\|^{1/q}_{L^{1/q,\infty}(\mathbb{R})} \lesssim \|\mathcal{M}_{hl}f+\mathcal{H}^*f\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})};$$

therefore, $\mathcal{T}_q$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$.

Boundedness of $\mathcal{M}_{\mathcal{M}_{hl}}$: Since $\mathcal{M}_{hl}(f\chi_{\mathbb{R}\setminus 3I})(\xi)\sim \mathcal{M}_{hl}(f\chi_{\mathbb{R}\setminus 3I})(x)$ for $\xi, x\in I$, we have that

$$\mathcal{M}_{\mathcal{M}_{hl}}(f)(x)\lesssim \mathcal{M}_{hl}(f)(x)$$

and therefore $\mathcal{M}_{\mathcal{M}_{hl}}$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$.

Boundedness of $\mathcal{M}_{\mathcal{T}_q}$: Observe that, since $(\mathcal{M}_{hl}f)^q\in A_1(\mathbb{R})$, then

$$\begin{aligned}\mathcal{T}_q(f)(x) &\lesssim (\mathcal{M}_{hl}(\mathcal{M}_{hl}f(x))^q)^{1/q} + (\mathcal{M}_{hl}(\mathcal{H}^*f(x))^q)^{1/q}\\ &\sim \mathcal{M}_{hl}f(x) + (\mathcal{M}_{hl}(\mathcal{H}^*f(x))^q)^{1/q};\end{aligned}$$

this implies that

$$\mathcal{M}_{\mathcal{T}_q}(f)(x)\lesssim \mathcal{M}_{\mathcal{M}_{hl}}(f)(x)+\mathcal{M}_{\widetilde{\mathcal{T}}_q}(f)(x),$$

where $\widetilde{\mathcal{T}}_q(f) := (\mathcal{M}_{hl}(\mathcal{H}^*f)^q)^{1/q}$. Then the boundeness of $\mathcal{M}_{\mathcal{T}_q}$ from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ follows from that of $\mathcal{M}_{\mathcal{M}_{hl}}$ and $\mathcal{M}_{\widetilde{\mathcal{T}}_q}$.

We next analyze the operator $\mathcal{M}_{\widetilde{\mathcal{T}}_q}$ and show that

$$\mathcal{M}_{\widetilde{\mathcal{T}}_q}f(x)\le \mathcal{M}_{\mathcal{M}_{hl}}(f)(x)+\mathcal{M}_{\mathcal{H}^*}f(x)+\widetilde{\mathcal{M}}_{\mathcal{H}^*}f(x), \tag{3.11}$$

where $\widetilde{\mathcal{M}}_{\mathcal{H}^*}f(x)$ is defined as $\mathcal{M}_{\mathcal{H}^*}f(x)$ using $\frac{5}{2}$ instead of 3. We have

$$\begin{aligned}\mathcal{M}_{\widetilde{\mathcal{T}}_q}(f)(x) &= \sup_{I:x\in I}\operatorname*{ess\,sup}_{\xi\in I}\sup_{R:\xi\in R}\left(\frac{1}{|R|}\int_R |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)|^q\,dy\right)^{1/q}\\ &= \sup_{I:x\in I}\sup_{R:R\cap I\neq\emptyset}\left(\frac{1}{|R|}\int_R |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)|^q\,dy\right)^{1/q},\end{aligned}$$

where I and R are intervals in $\mathbb{R}$.

Consider first the case where $|R|>|I|/10$, with $R\cap I\neq\emptyset$. Since $3I\subset 3(R\cup I)$, we have $\mathbb{R}\setminus 3I = (\mathbb{R}\setminus 3(R\cup I))\cup(3(R\cup I)\setminus 3I)$, $f\chi_{\mathbb{R}\setminus 3I} = f_{\mathbb{R}\setminus(3(R\cup I))}+f\chi_{3(R\cup I)\setminus 3I}$ and

$$\sup_{I:x\in I}\operatorname*{ess\,sup}_{\xi\in I}\sup_{\substack{R:\xi\in R\\ |R|>|I|/10}}\left(\frac{1}{|R|}\int_R |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)|^q\,dy\right)^{1/q}\lesssim A+B, \tag{3.12}$$

where

$$A=\sup_{I:x\in I}\sup_{R:R\cap I\neq\emptyset}\left(\frac{1}{|R|}\int_R |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3(R\cup I)})(y)|^q\,dy\right)^{1/q},$$

$$B=\sup_{I:x\in I}\operatorname*{ess\,sup}_{\xi\in I}\sup_{\substack{R:\xi\in R\\ |R|>|I|/10}}\left(\frac{1}{|R|}\int_R |\mathcal{H}^*(f\chi_{3(R\cup I)\setminus 3I})(y)|^q\,dy\right)^{1/q}.$$

For the term A, we have, with Q denoting an interval in $\mathbb{R}$,

$$\begin{aligned}
A &\le \sup_{I:x\in I}\ \sup_{R:R\cap I\neq\emptyset}\ \operatorname*{ess\,sup}_{y\in R\cup I} |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus(3(R\cup I))})(y)| \\
&\le \sup_{Q:x\in Q}\ \operatorname*{ess\,sup}_{y\in Q} |\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3Q})(y)| \\
&= \mathcal{M}_{\mathcal{H}^*}(f)(x).
\end{aligned} \tag{3.13}$$

For the term B, setting $h = \mathcal{H}^*(f\chi_{3(R\cup I)\setminus 3I})$, recalling that $0<q<1$ and using that $\mathcal{H}^*$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, we first note that

$$\frac{1}{|R|}\int_R |h(y)|^q\,dy = \frac{1}{|R|}\int_0^{|R|}(h^*(t)t)^q t^{-q}\,dt \lesssim \frac{1}{|R|^q}\|h\|^q_{L^{1,\infty}} \lesssim \frac{1}{|R|^q}\|f\chi_{3(R\cup I)\setminus 3I}\|^q_{L^1},$$

where h^* denotes the decreasing rearrangement of h with respect to Lebesgue measure. Using that the hypothesis $|R|>|I|/10$ implies that $|3(R\cup I)|\sim|R|$, we obtain

$$\begin{aligned}
B &\lesssim \sup_{I:x\in I}\ \operatorname*{ess\,sup}_{\xi\in I}\ \sup_{R:\xi\in R}\frac{1}{|3(R\cup I)|}\int_{3(R\cup I)}|f(y)\chi_{\mathbb{R}\setminus 3I}(y)|\,dy \\
&\le \sup_{I:x\in I}\ \operatorname*{ess\,sup}_{\xi\in I}\mathcal{M}_{hl}(f\chi_{\mathbb{R}\setminus 3I})(\xi) \\
&= \mathcal{M}_{\mathcal{M}_{hl}}(f)(x).
\end{aligned} \tag{3.14}$$

We next consider the case where $|R|\le|I|/10$, with $R\cap I\neq\emptyset$; these conditions imply that $R\cup I\subset\frac{6}{5}I$. We have

$$\begin{aligned}
\sup_{I:x\in I}\ \sup_{\substack{R:R\cap I\neq\emptyset\\ |R|\le|I|/10}}\left(\frac{1}{|R|}\int_R|\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)|^q\,dy\right)^{1/q} &\lesssim \sup_{I:x\in I}\ \sup_{\substack{R:R\cap I\neq\emptyset\\ |R|\le|I|/10}}\ \sup_{y\in R\cup I}|\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)| \\
&\le \sup_{I:x\in\frac{6}{5}I}\ \sup_{y\in\frac{6}{5}I}|\mathcal{H}^*(f\chi_{\mathbb{R}\setminus 3I})(y)| \\
&= \sup_{I:x\in Q}\ \sup_{y\in Q}|\mathcal{H}^*(f\chi_{\mathbb{R}\setminus\frac{5}{2}Q})(y)| \\
&= \widetilde{\mathcal{M}}_{\mathcal{H}^*}f(x).
\end{aligned} \tag{3.15}$$

Putting the estimates (3.12)-(3.15) together we conclude (3.11).

We are left to show that $\mathcal{M}_{\mathcal{H}^*}$ and $\widetilde{\mathcal{M}}_{\mathcal{H}^*}$ are bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. We will prove this for $\mathcal{M}_{\mathcal{H}^*}$ noting that the differences with the proof for $\widetilde{\mathcal{M}}_{\mathcal{H}^*}$ are obvious. We have

$$\mathcal{M}_{\mathcal{H}^*}f(x) = \sup_{I:x\in I}\ \operatorname*{ess\,sup}_{\xi\in I}\ \sup_{\varepsilon>0}|\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus 3I})(\xi)|.$$

Fix an interval I such that $x,\xi\in I$ and consider the interval I_x centered at x and length $4|I|$; note that $3I\subset I_x\subset 5I$. We have

$$\begin{aligned}
|\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus 3I})(\xi)| \le{}& |\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(\xi)-\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(x)| + |\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(x)| \\
&+ |\mathcal{H}_\varepsilon(f\chi_{I_x\setminus 3I})(\xi)| =: C+D+E.
\end{aligned}$$

For the term D, we obtain

$$D = |\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(x)| = \left|\int_{\{|y-x|>\max(\varepsilon,2|I|)\}}\frac{f(y)}{x-y}dy\right| \le \mathcal{H}^*f(x).$$

For the term E, recalling that $\xi \in I$, it follows that

$$
\begin{aligned}
E = |\mathcal{H}_\varepsilon(f\chi_{I_x\setminus 3I})(\xi)| &\le \left| \int_{\{|y-\xi|>\varepsilon, |y-x|\le 2|I|, |\xi-y|>|I|\}} \frac{f(y)}{\xi-y} dy \right| \\
&\lesssim \frac{1}{|I_x|}\int_{I_x} |f(y)| dy \le \mathcal{M}_{hl} f(x),
\end{aligned}
$$

where in the last inequality it was used that $|I| \sim |I_x|$.

For the term C, we get

$$
\begin{aligned}
C =& |\mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(\xi) - \mathcal{H}_\varepsilon(f\chi_{\mathbb{R}\setminus I_x})(x)| \\
\lesssim& \left| \int_{\{|x-y|>\max(2|I|,\varepsilon)\}} f(y) \left(\frac{1}{\xi-y} - \frac{1}{x-y} \right) dy \right| \\
&+ \left| \int_{\{|x-y|>2|I|, |\xi-y|>\varepsilon\}} \frac{f(y)}{\xi-y}\, dy - \int_{\{|x-y|>\max(2|I|,\varepsilon)\}} \frac{f(y)}{\xi-y}\, dy \right| \\
=:& C_1 + C_2.
\end{aligned}
$$

Note that $|x-y| > 2|I|$ and $x, \xi \in I$ imply $|\xi - y| > |I|$ and $|x-y| \sim |\xi - y|$.

For C_1, we have

$$
\begin{aligned}
C_1 &\lesssim \int_{\{|y-x|>\max(2|I|,\varepsilon)\}} |f(y)| \left(\frac{|x-\xi|}{|\xi-y||x-y|} \right) dy \\
&\lesssim \int_{\mathbb{R}} |f(y)| \left(\frac{|I|}{|x-y|^2 + |I|^2} \right) dy \lesssim \mathcal{M}_{hl} f(x).
\end{aligned}
$$

Regarding the term C_2, note that, since $|x-y| > 2|I|$ implies $|\xi - y| > |I|$ for $x, \xi \in I$, then $C_2 = 0$ if $\varepsilon < |I|$. Assume then that $\varepsilon \ge |I|$.

$$
\begin{aligned}
C_2 &= \left| \int_{\{|x-y|>2|I|, |\xi-y|>\varepsilon\}} \frac{f(y)}{\xi-y}\, dy - \int_{\{|x-y|>\max(2|I|,\varepsilon)\}} \frac{f(y)}{\xi-y}\, dy \right| \\
&= \left| \int_{\{|x-y|>2|I|, |\xi-y|>\varepsilon\}\setminus\{|x-y|>\max(2|I|,\varepsilon)\}} \frac{f(y)}{\xi-y}\, dy - \int_{\{|x-y|>\max(2|I|,\varepsilon)\}\setminus\{|x-y|>2|I|, |\xi-y|>\varepsilon\}} \frac{f(y)}{\xi-y}\, dy \right| \\
&= \left| \int_{\{\max(2|I|,\varepsilon)\ge|x-y|>2|I|, |\xi-y|>\varepsilon\}} \frac{f(y)}{\xi-y}\, dy - \int_{\{|x-y|>\max(2|I|,\varepsilon), |\xi-y|\le\varepsilon\}} \frac{f(y)}{\xi-y}\, dy \right| \\
&\lesssim \left| \int_{\{\max(2|I|,\varepsilon)\ge|x-y|>2|I|, |\xi-y|>\varepsilon\}} \frac{f(y)}{\xi-y}\, dy \right| + \left| \int_{\{|x-y|>\max(2|I|,\varepsilon), |\xi-y|\le\varepsilon\}} \frac{f(y)}{\xi-y}\, dy \right| \\
&= C_2^{(1)} + C_2^{(2)}.
\end{aligned}
$$

If $\varepsilon < 2|I|$, then $C_2^{(1)} = 0$ and, using that $|x-y| \sim |\xi - y|$, we have

$$
C_2^{(2)} \lesssim \frac{1}{|I|} \int_{|x-y|\lesssim|I|} |f(y)|\, dy \lesssim \mathcal{M}_{hl} f(x).
$$

If $\varepsilon > 2|I|$, we obtain

$$
C_2^{(1)} \lesssim \frac{1}{\varepsilon} \int_{|x-y|<\varepsilon} |f(y)|\, dy \lesssim \mathcal{M}_{hl} f(x),
$$

and, using again that $|x-y| \sim |\xi - y|$,

$$C_2^{(2)} \lesssim \frac{1}{\varepsilon} \int_{|x-y|\lesssim \varepsilon} |f(y)|\, dy \lesssim \mathcal{M}_{hl} f(x).$$

The estimates above allow then to conclude that

$$\mathcal{M}_{\mathcal{H}^*} f(x) \lesssim \mathcal{M}_{hl} f(x) + \mathcal{H}^* f(x),$$

and, consequently, $\mathcal{M}_{\mathcal{H}^*}$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, as desired.

□

3.2. **Completion of the proof of Theorem 1.3.** We first prove items (a)-(c) and then items (e) and (f)

Proofs of (a)-(c): For $f \in X$, where X is any of the spaces in items (a)-(c), let u_f be the harmonic function in $\mathbb{R}^2_+$ given in (3.1). We will prove that

$$\|\mathcal{M}_\alpha(\nabla u_f)\|_Y \lesssim \|f\|_X, \quad \forall f \in X, \tag{3.16}$$

where $0 < \alpha < \pi/2$, and that $\nabla u_f \cdot \mathbf{n} = -\frac{\partial u_f}{\partial y_2}$ converges non-tangentially to f almost everyewhere with respect to Lebesgue measure in $\mathbb{R}$.

Proof of (3.16)*:* For $s \in \mathbb{N}$, $0 < \alpha < \pi/2$ and $x \in \mathbb{R}$ define $\Gamma^s_\alpha(x) = \Gamma_\alpha(x) \cap \{(y_1 + iy_2) \in \mathbb{C} : 1/s < y_2 < s\}$. Given $f \in X$, consider $\{f_m\}_{m\in\mathbb{N}} \subset L^1_c(\mathbb{R}) \cap X$ such that $\{f_m\}_{m\in\mathbb{N}}$ converges to f in X; such sequence exists since $L^1_c(\mathbb{R}) \cap X$ is dense in X by Lemma 2.2. Since $\{u_{f_m}\}_{m\in N}$ converges uniformly to u_f on compact sets of $\mathbb{R}^2_+$ by Lemma 3.1 and u_{f_m} is harmonic for each m, then $\{\nabla u_{f_m}\}_{m\in\mathbb{N}}$ converges uniformly to ∇u_f on compact sets of $\mathbb{R}^2_+$. This and the fact that

$$\begin{aligned}\Bigg| \sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} &|\nabla u_f(y_1,y_2)| - \sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_{f_m}(y_1,y_2)|\Bigg| \\ &\le \sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_f(y_1,y_2) - \nabla u_{f_m}(y_1,y_2)|, \quad \forall x \in \mathbb{R},\end{aligned}$$

imply that

$$\lim_{m\to\infty} \sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_{f_m}(y_1,y_2)| = \sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_f(y_1,y_2)|, \quad \forall x \in \mathbb{R}.$$

It then follows that

$$\begin{aligned}\|\sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_f(y_1,y_2)|\|_Y &\le \liminf_{m\to\infty} \|\sup_{y_1+iy_2\in\Gamma^s_\alpha(x)} |\nabla u_{f_m}(y_1,y_2)|\|_Y \\ &\le \liminf_{m\to\infty} \|\mathcal{M}_\alpha(\nabla u_{f_m})\|_Y \\ &\lesssim \liminf_{m\to\infty} \|f_m\|_X = \|f\|_X,\end{aligned}$$

where in the last inequality we have used that (3.16) holds for functions in $L^1_c(\mathbb{R}) \cap X$ by Lemma 3.2. Since $\{\sup_{(y_1,y_2)\in\Gamma^s_\alpha(x)} |\nabla u_f(y_1,y_2)|\}_{s\in\mathbb{N}}$ is an increasing sequence that converges to $\mathcal{M}_\alpha(\nabla u)$ as $s \to \infty$ and the Monotone Convergence Theorem holds in Y, the desired estimate follows.

Non-tangential convergence: We will prove the non-tangential convergence for $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$ with $w \in A^{\mathcal{R}}_{p'}$ and $1 < p < \infty$ using a standard argument. The result for the other cases follows in a similar fashion.

In view of (3.3), the non-tangential convergence holds for functions in $L^1(\mathbb{R})$ for any $0 < \alpha < \pi/2$; that is, $\nabla u_f \cdot \mathbf{n} = -\frac{u_f}{\partial y_2}$ converges non-tangentially to f for any $0 < \alpha < \pi/2$ and $f \in L^1(\mathbb{R})$.

Fix $f \in \mathcal{L}^{p,1}(\mathbb{R}, w)$ and let $\{f_m\}_{m\in\mathbb{N}} \subset L^1_c(\mathbb{R}) \cap X$ be such that $f_m \to f$ in $\mathcal{L}^{p,1}(\mathbb{R}, w)$. Since w is positive almost everwhere with respect to Lebesgue measure in $\mathbb{R}$, the result will follow if we prove that

$$w(\{x \in \mathbb{R} : \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_f}{\partial y_2}(y_1, y_2) + f(x)| > \varepsilon w(x)\}) = 0, \quad \forall \varepsilon > 0.$$

Using that $\frac{\partial u_{f_m}}{\partial y_2}$ converges non-tangentially to $-f_m$, (3.16) with $X = \mathcal{L}^{p,1}(\mathbb{R}, w)$ and $Y = \mathcal{L}^{p,\infty}(\mathbb{R}, w)$, and that $\|h\|_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} \leq \|h\|_{\mathcal{L}^{p,1}(\mathbb{R},w)}$, it follows that, for $0 < \alpha < \pi/2$,

$$\begin{aligned}
&w(\{x \in \mathbb{R} : \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_f}{\partial y_2}(y_1, y_2) + f(x)| > \varepsilon w(x)\}) \\
&\quad \leq w(\{x \in \mathbb{R} : \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_f}{\partial y_2}(y_1, y_2) - \tfrac{\partial u_{f_m}}{\partial y_2}(y_1, y_2)| > \tfrac{\varepsilon}{3}\, w(x)\}) \\
&\qquad + w(\{x \in \mathbb{R} : \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_{f_m}}{\partial y_2}(y_1, y_2) + f_m(x)| > \tfrac{\varepsilon}{3}\, w(x)\}) \\
&\qquad + w(\{x \in \mathbb{R} : | - f_m(x) + f(x)| > \tfrac{\varepsilon}{3}\, w(x)\}) \\
&\quad \leq (\tfrac{3}{\varepsilon})^p \| \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_f}{\partial y_2}(y_1, y_2) - \tfrac{\partial u_{f_m}}{\partial y_2}(y_1, y_2)| \|^p_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} \\
&\qquad + (\tfrac{3}{\varepsilon})^p \|f_m - f\|^p_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} \\
&\quad \leq (\tfrac{3}{\varepsilon})^p \|\mathcal{M}_\alpha(\tfrac{\partial u_{f-f_m}}{\partial y_2})\|^p_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} + (\tfrac{3}{\varepsilon})^p \|f_m - f\|^p_{\mathcal{L}^{p,\infty}(\mathbb{R},w)} \\
&\quad \lesssim (\tfrac{3}{\varepsilon})^p \|f_m - f\|^p_{\mathcal{L}^{p,1}(\mathbb{R},w)}.
\end{aligned}$$

Since $\|f_m - f\|_{\mathcal{L}^{p,1}(\mathbb{R},w)} \to 0$ as $m \to \infty$, we obtain the desired result.

The proofs for $X = L^p(\mathbb{R}, w)$ and $X = L^{p,1}(\mathbb{R}, w)$ follow similarly by showing that

$$w(\{x \in \mathbb{R} : \limsup_{(y_1,y_2)\lhd(x,0)} |\tfrac{\partial u_f}{\partial y_2}(y_1, y_2) + f(x)| > \varepsilon\}) = 0, \quad \forall \varepsilon > 0.$$

Proofs of (e) and (f): We will prove (e); the proof of (f) follows similarly with obvious changes. Let $1 < p < \infty$ and w be a weight in $\mathbb{R}$ such that the Neumann problem in $\mathbb{R}^2_+$ is solvable in $X = L^p(\mathbb{R}, w)$ with $Y = L^p(\mathbb{R}, w)$. We have to prove that $w \in A_p(\mathbb{R})$. By hypothesis, we have that

$$\|\mathcal{M}_\alpha(\nabla u_f)\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)} \quad \forall f \in L^p(\mathbb{R}, w).$$

This estimate and (3.5) give

$$\|Q_{y_2} * f\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)} \quad \forall f \in C_c(\mathbb{R})$$

for all $y_2 > 0$ and with the implicit constant independent of y_2. Since $\lim_{y_2\to 0} Q_{y_2} * f = \mathcal{H}f$ pointwise for $f \in C_c(\mathbb{R})$, Fatou's lemma and the fact that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R}, w)$ (Lemma 2.2) imply that

$$\|\mathcal{H}f\|_{L^p(\mathbb{R},w)} \lesssim \|f\|_{L^p(\mathbb{R},w)} \quad \forall f \in L^p(\mathbb{R}, w)$$

By Theorem 2.1, we conclude that $w \in A_p(\mathbb{R})$. □

We end this section with some remarks that will be used in Section 4:

Remark 3.3.

(a) The proof of Theorem 1.3 of the non-tangential convergence shows that

$$\lim_{\substack{(y_1,y_2)\in\Gamma_\alpha(x,0)\\(y_1,y_2)\to(x,0)}} \frac{\partial u_f}{\partial y_2}(y_1,y_2) = -f(x), \quad \text{a.e } x\in\mathbb{R},$$

for all $0<\alpha<\pi/2$, not just for some α as required in the definition of a solution of the Neumann problem given in Section 1.

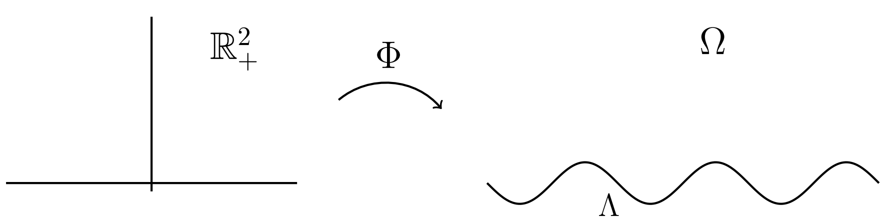
(b) Formula (3.5) implies that $\frac{\partial u_f}{\partial y_1}$ converges non-tangentially to $-\mathcal{H}f$ for $f\in L^1_c(\mathbb{R})$ for any $0<\alpha<\pi/2$. A similar argument to the one used in the non-tangential convergence section in the proof of Theorem 1.3 shows that $\frac{\partial u_f}{\partial y_1}$ converges non-tangentially for every $f\in X$, where X is as in the statement of Theorem 1.3.

4. Solvability of the Neumann problem in Ω

In this section, we study the solvability of the Neumann problem in Ω in $L^p(\Lambda)$ for $1<p<p_\Phi$ and in $L^{p_\Phi,1}(\Lambda)$. Inspired by ideas in Kenig [15, 16], we will obtain such results through arguments that invoke the results on the solvability of the Neumann problem in the upper half-plane given by Theorem 1.3.

In Section 4.1, we introduce notation and present preliminaries. In Section 4.2, we treat the solvability of the Neumann problem in $L^p(\Lambda)$ for $1<p<p_\Phi$ (Theorem 1.4). Section 4.3 is devoted to the solvability of the Neumann problem in $L^{p_\Phi,1}(\Lambda)$ (Theorem 1.5). Finally, in Section 4.4, we give the proofs of Lemma 4.1, used in the proof of Theorem 1.4, and of Lemmas 4.2 and 4.4, used in the proof of Theorem 1.5.

4.1. **Definitions and preliminaries.** Let γ, Λ and Ω be as in Section 1 (see (1.2)). Since Ω is simply connected, it is conformally equivalent to $\mathbb{R}^2_+$. Let $\Phi:\mathbb{R}^2_+\longrightarrow\Omega$ be a conformal mapping such that $\Phi(\infty)=\infty$ and $\Phi(i)=iy_0$ for some $y_0>\gamma(0)$. Such mapping extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$ and $\Phi(x)$, $x\in\mathbb{R}$, is absolutely continuous when restricted to any finite interval; in particular, $\Phi'(x)$ exists for almost every $x\in\mathbb{R}$ and is locally integrable. Moreover, $\Phi'(x)\neq 0$ for almost every $x\in\mathbb{R}$, $\lim_{z\lhd x}\Phi'(z)=\Phi'(x)$ for almost every $x\in\mathbb{R}$ and $|\Phi'|\in A_2(\mathbb{R})$. If $\Phi'(x)$ exists and is not zero, then it is a vector tangent to Λ at $\Phi(x)$. We refer the reader to Kenig [16, Theorems 1.1 and 1.10] for the proof of those properties and additional ones.



The arch length measure in Λ will be denoted by ds. Given a weight ν in Λ (i.e. a non-negative locally integrable (with respect to ds) function defined on Λ), we will consider the spaces $L^p(\Lambda,\nu)$, $L^{p,1}(\Lambda,\nu)$, $L^{p,\infty}(\Lambda,\nu)$, whose definitions are analogous to the corresponding ones in Section 2. When $\nu\equiv 1$, we will just write $L^p(\Lambda)$, $L^{p,1}(\Lambda)$, $L^{p,\infty}(\Lambda)$. For $1<p<\infty$, $A_p(\Lambda)$ means the class of Muckenhoupt weigths defined on Λ, whose definition is analogous to (2.1) with the supremum taken over all intervals contained in Λ ($J\subset\Lambda$ is an interval if $b(J)$ is an interval in $\mathbb{R}$, where $b:\Lambda\to\mathbb{R}$ is defined by $b(x+i\gamma(x))=x$ for $x\in\mathbb{R}$); we set $A_\infty(\Lambda)=\cup_{p>1}A_p(\Lambda)$. Finally, given a weight ν in Λ, we use $\Phi(\nu)$ to refer to the measure in $\mathbb{R}$ given by $\Phi(\nu)(U) = \nu(\Phi(U))$ for a measurable set $U \subset \mathbb{R}$, where $\nu(A) = \int_A \nu(\xi) ds(\xi)$ for a measurable set $A \subset \Lambda$.

Let $\nu \in A_\infty(\Lambda)$. Given $0 < p < \infty$ and $0 < \alpha < \arctan(1/L)$, we define the Hardy space $H^p(\Omega, \nu)$ and the weak Hardy space $H^{p,\infty}(\Omega, \nu)$ as follows:

$$H^p(\Omega, \nu) := \{h : \Omega \to \mathbb{C} : h \text{ is analytic in } \Omega \text{ and } \|\mathcal{M}_\alpha(h)\|_{L^p(\Lambda,\nu)} < \infty\},$$
$$H^{p,\infty}(\Omega, \nu) := \{h : \Omega \to \mathbb{C} : h \text{ is analytic in } \Omega \text{ and } \|\mathcal{M}_\alpha(h)\|_{L^{p,\infty}(\Lambda,\nu)} < \infty\},$$

and set $\|h\|_{H^p(\Omega,\nu)} = \|\mathcal{M}_\alpha(h)\|_{L^p(\Lambda,\nu)}$ and $\|h\|_{H^{p,\infty}(\Omega,\nu)} = \|\mathcal{M}_\alpha(h)\|_{L^{p,\infty}(\Lambda,\nu)}$. By Lemma A.4, the definitions of $H^p(\Omega, \nu)$ and $H^{p,\infty}(\Omega, \nu)$ are independent of α and any other chosen value gives an equivalent norm. In particular, we denote by $H^p(\Omega, ds)$ and $H^{p,\infty}(\Omega, ds)$ the corresponding Hardy and weak Hardy spaces in Ω with respect to arc length measure in Λ, and by $H^p(\mathbb{R}^2_+, dx)$ and $H^{p,\infty}(\mathbb{R}^2_+, dx)$ the corresponding Hardy and weak Hardy spaces in $\mathbb{R}^2_+$ with respect to Lebesgue measure in $\mathbb{R}$.

Given a measurable function g defined in Λ, let $\mathcal{T}(g)$ be given by

$$\mathcal{T}(g)(x) = |\Phi'(x)| g(\Phi(x)), \quad x \in \mathbb{R}. \tag{4.1}$$

Consider $1 \leq p < \infty$. We note that $\mathcal{T}$ is a bijection from $L^p(\Lambda)$ onto $L^p(\mathbb{R}, |\Phi'|^{1-p})$ and

$$\begin{aligned}\|\mathcal{T}(g)\|^p_{L^p(\mathbb{R},|\Phi'|^{1-p})} &= \int_{\mathbb{R}} |\mathcal{T}(g)(x)|^p |\Phi'(x)|^{1-p} dx = \int_{\mathbb{R}} |g(\Phi(x))|^p |\Phi'(x)| dx \\ &= \int_\Lambda |g(\xi)|^p ds(\xi) = \|g\|^p_{L^p(\Lambda)}.\end{aligned}$$

Moreover, it holds that $\mathcal{T}$ is also an isometry from $L^{p,1}(\Lambda)$ onto $\mathcal{L}^{p,1}(\mathbb{R}, |\Phi'|)$; the equality in norms follows from the fact that

$$|\Phi'|\big(\{x \in \mathbb{R} : |\mathcal{T}(g)(x)||\Phi'(x)|^{-1} > \lambda\}\big) = \int_{\{x\in\mathbb{R}:|g(\Phi(x))|>\lambda\}} |\Phi'(x)| dx = \int_{\{\xi\in\Lambda:|g(\xi)|>\lambda\}} ds(\xi).$$

By Theorem 1.3, the Neumann problem in the upper half-plane is solvable in $L^p(\mathbb{R}, |\Phi'|^{1-p})$ if $|\Phi'|^{1-p} \in A_p(\mathbb{R})$, or equivalently, $|\Phi'| \in A_{p'}$. Define p_Φ such that

$$p'_\Phi = \inf\{q \in [1, \infty) : |\Phi'| \in A_q(\mathbb{R})\}.$$

Note that $p'_\Phi < 2$ since $|\Phi'| \in A_2(\mathbb{R})$ and, therefore, $p_\Phi > 2$; moreover, (2.4) implies that

$$p'_\Phi = \inf\{q \in [1, \infty) : |\Phi'| \in A^{\mathcal{R}}_q(\mathbb{R})\}. \tag{4.2}$$

We remark that if $p'_\Phi > 1$ then $|\Phi'| \notin A_{p'_\Phi}(\mathbb{R})$; however, $|\Phi'|$ might belong to $A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$.

4.2. **The Neumann problem in Ω : solvability in $L^p(\Lambda)$, $1 < p < p_\Phi$.** In this section, we prove Theorem 1.4, which states that the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < p_\Phi$.

The following lemma, whose proof we defer to Section 4.4, will be used in the proof of Theorem 1.4.

Lemma 4.1. *Let h be an analytic function in $\mathbb{R}^2_+$ and $1 < p < p_\Phi$. Then $(h \circ \Phi^{-1})(\Phi^{-1})' \in H^p(\Omega, ds)$ if and only if $h \circ \Phi^{-1} \in H^p(\Omega, |(\Phi^{-1})'|^p)$, with equivalence in norms.*

Proof of Theorem 1.4. Since $1 < p < p_\Phi$, we have $|\Phi'|^{1-p} \in A_p(\mathbb{R})$. Given $g \in L^p(\Lambda)$, let $\mathcal{T}(g)$ be as defined in (4.1); as observed in Section 4.1, $\mathcal{T}(g) \in L^p(\mathbb{R}, |\Phi'|^{1-p})$ and $\|\mathcal{T}(g)\|_{L^p(\mathbb{R},|\Phi'|^{1-p})} = \|g\|_{L^p(\Lambda)}$. Without loss of generality, assume g is real-valued. We first show that (1.1) has a solution for such g and then we prove (1.3).

Solution of (1.1): Let $u_{\mathcal{T}(g)}$ be the solution of the Neumann problem in the upper half-plane with datum $\mathcal{T}(g)$ as given in Theorem 1.3; note that $u_{\mathcal{T}(g)}$ is real-valued since $\mathcal{T}(g)$ is real-valued. Define $v_g = u_{\mathcal{T}(g)} \circ \Phi^{-1}$; observe that, since Φ is analytic, then v_g is harmonic in Ω.

In order to prove that v_g is a solution of the Neumann problem in Ω with datum g, we have to show that $\lim_{z \lhd \xi} \nabla v_g(z) \cdot \mathbf{n}(\xi) = g(\xi)$ for almost every $\xi \in \Lambda$ with respect to ds, where $\mathbf{n}(\xi)$ is the outward unit normal vector to Λ at ξ. Let $\xi \in \Lambda$ be such that $\Phi'(\Phi^{-1}(\xi))$ exists and is non-zero; with $z = z_1 + iz_2$, where $z_1, z_2 \in \mathbb{R}$, and recalling that $(\Phi^{-1})' = \frac{\partial \mathrm{Re}(\Phi^{-1})}{\partial z_1} - i\frac{\partial \mathrm{Re}(\Phi^{-1})}{\partial z_2} = \frac{\partial \mathrm{Im}(\Phi^{-1})}{\partial z_2} + i\frac{\partial \mathrm{Im}(\Phi^{-1})}{\partial z_1}$, we have

$$\begin{aligned}
\nabla v_g(z) \cdot \mathbf{n}(\xi) &= \mathrm{Re}\left(\left(\tfrac{\partial v_g}{\partial z_1}(z) - i\tfrac{\partial v_g}{\partial z_2}(z)\right)\mathbf{n}(\xi)\right) \\
&= \mathrm{Re}\left(\left(\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_1}(\Phi^{-1}(z)) - i\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_2}(\Phi^{-1}(z))\right)(\Phi^{-1})'(z)\,\mathbf{n}(\xi)\right) \\
&= \mathrm{Re}\left(\left(\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_1}(\Phi^{-1}(z)) - i\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_2}(\Phi^{-1}(z))\right)\frac{1}{\Phi'(\Phi^{-1}(z))}\,\mathbf{n}(\xi)\right) \qquad (4.3)\\
&= |\Phi'(\Phi^{-1}(z))|^{-1}\mathrm{Re}\left(\left(\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_1}(\Phi^{-1}(z)) - i\tfrac{\partial u_{\mathcal{T}(g)}}{\partial y_2}(\Phi^{-1}(z))\right)\frac{|\Phi'(\Phi^{-1}(z))|}{\Phi'(\Phi^{-1}(z))}\,\mathbf{n}(\xi)\right).
\end{aligned}$$

Above, $\mathbf{n}(\xi)$ is considered as a vector on the left hand side while $\mathbf{n}(\xi)$ is regarded as a complex number on the right hand; therefore, on the left hand side, we have the dot product of $\nabla v_g(z)$ with $\mathbf{n}(\xi)$ and, on the right hand, we are multiplying by $\mathbf{n}(\xi)$ as a complex number.

As $z \lhd \xi$, $|\Phi'(\Phi^{-1}(z))|$ converges to $|\Phi'(\Phi^{-1}(\xi))|$ and $\frac{\Phi'(\Phi^{-1}(z))}{|\Phi'(\Phi^{-1}(z))|}$ converges to $\frac{\Phi'(\Phi^{-1}(\xi))}{|\Phi'(\Phi^{-1}(\xi))|}$, which is a unit vector tangent to Λ at ξ; this is a consequence of the properties of Φ stated in Section 4.1 and Lemma A.1 (Lemma A.1 holds for almost every $\xi \in \Lambda$; assume ξ is a point so that the thesis of the lemma holds). We then have that $\frac{|\Phi'(\Phi^{-1}(z))|}{\Phi'(\Phi^{-1}(z))}\,\mathbf{n}(\xi)$ converges to $-i$ as $z \lhd \xi$. Moreover, by Theorem 1.3, Lemma A.1 and Remark 3.3(a), $\frac{\partial u_{\mathcal{T}(g)}}{\partial y_2}(\Phi^{-1}(z))$ converges to $-\mathcal{T}(g)(\Phi^{-1}(\xi))$ as $z \lhd \xi$, and by Remark 3.3(b) and Lemma A.1, $\frac{\partial u_{\mathcal{T}(g)}}{\partial y_1}(\Phi^{-1}(z))$ converges as $z \lhd \xi$. We conclude that

$$\lim_{z \lhd \xi} \nabla v_g(z) \cdot \mathbf{n}(\xi) = |\Phi'(\Phi^{-1}(\xi))|^{-1}\mathcal{T}(g)(\Phi^{-1}(\xi)) = g(\xi).$$

Proof of (1.3): If $0 < \alpha < \pi/2$, Theorem 1.3 gives

$$\|\mathcal{M}_\alpha(\nabla u_{\mathcal{T}(g)})\|_{L^p(\mathbb{R},|\Phi'|^{1-p})} \lesssim \|f\|_{L^p(\mathbb{R},|\Phi'|^{1-p})} = \|g\|_{L^p(\Lambda)}. \qquad (4.4)$$

Define

$$F(y_1,y_2) = \frac{\partial u_{\mathcal{T}(g)}}{\partial_{y_1}}(y_1,y_2) - i\frac{u_{\mathcal{T}(g)}}{\partial_{y_2}}(y_1,y_2) \quad \text{and} \quad G(z_1,z_2) = \frac{\partial v_g}{\partial_{z_1}}(z_1,z_2) - i\frac{\partial v_g}{\partial_{z_2}}(z_1,z_2).$$

Since $u_{\mathcal{T}(g)}$ and v_g are harmonic, the Cauchy-Riemann equations imply that F and G are analytic in $\mathbb{R}^2_+$ and Ω, respectively. Moreover, (4.4) implies that $F \in H^p(\mathbb{R}^2_+, |\Phi'|^{1-p})$. We will prove that $G \in H^p(\Omega, ds)$ with $\|G\|_{H^p(\Omega,ds)} \sim \|F\|_{H^p(\mathbb{R}^2_+,|\Phi'|^{1-p})}$. This along with (4.4), and the fact that $|G| = |\nabla v_g|$, implies (1.3).

Note that $G = (F \circ \Phi^{-1})(\Phi^{-1})'$. By Lemma 4.1, we have

$$G \in H^p(\Omega, ds) \quad \Longleftrightarrow \quad F \circ \Phi^{-1} \in H^p(\Omega, |(\Phi^{-1})'|^p) \qquad (4.5)$$

and

$$\|G\|_{H^p(\Omega,ds)} \sim \|F \circ \Phi^{-1}\|_{H^p(\Omega,|(\Phi^{-1})'|^p)}. \qquad (4.6)$$

We next observe that $|(\Phi^{-1})'|^p \in A_\infty(\Lambda)$. Indeed, this follows by applying Lemma A.8 with $\Omega_1 = \mathbb{R}^2_+$, $\Omega_2 = \Omega$, $\sigma_{1,2} = \Phi^{-1}$ and $\nu = |\Phi'|^{1-p}$, which belongs to $A_p(\mathbb{R})$, and noting that, for a measurable set $U \subset \Lambda$, we have

$$\Phi^{-1}(\nu)(U) = \nu(\Phi^{-1}(U)) = \int_{\Phi^{-1}(U)} |\Phi'(x)|^{1-p}\, dx = \int_U |\Phi'(\Phi^{-1}(\xi))|^{-p}\, d\xi = \int_U |(\Phi^{-1})'(\xi)|^p\, d\xi.$$

Applying Theorem A.5 with $\nu = |(\Phi^{-1})'|^p$ we have that, with equivalent norms,

$$F \circ \Phi^{-1} \in H^p(\Omega, |(\Phi^{-1})'|^p) \quad \Longleftrightarrow \quad F \in H^p(\mathbb{R}^2_+, |\Phi'|^{1-p}),$$

since for a measurable set $U \subset \mathbb{R}$,

$$\Phi(\nu)(U) = \nu(\Phi(U)) = \int_{\Phi(U)} |(\Phi^{-1})'(\xi)|^p d\xi = \int_U |(\Phi^{-1})'(\Phi(x))|^p |\Phi'(x)| dx = \int_U |\Phi'(x)|^{1-p} dx.$$

It then follows that $\|G\|_{H^p(\Omega,ds)} \sim \|F\|_{H^p(\mathbb{R}^2_+,|\Phi'(x)|^{1-p})}$, as desired.

□

4.3. **The Neumann problem in Ω : solvability in $L^{p_\Phi,1}(\Lambda)$.** In this section, we prove Theorem 1.5, which states that the Neumann problem in Ω is solvable in $L^{p_\Phi,1}(\Lambda)$ when $|\Phi'| \in A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$, assuming that $p_\Phi < \infty$.

The following results, whose proof we defer to Section 4.4, will be used in the proof of Theorem 1.5.

Lemma 4.2. *Let $0 < p < \infty$ and $\nu \in A_\infty(\Lambda)$. Then $h \in H^{p,\infty}(\Omega, \nu)$ if and only if $h \circ \Phi \in H^{p,\infty}(\mathbb{R}^2_+, \Phi(\nu))$, with equivalence in norms.*

Noting that when $d\nu = ds$ we have $d\Phi(\nu) = |\Phi'(x)|dx$, Lemma 4.2 gives the following result.

Corollary 4.3. *Let $0 < p < \infty$. Then $h \in H^{p,\infty}(\Omega, ds)$ if and only if $h \circ \Phi \in H^{p,\infty}(\mathbb{R}^2_+, |\Phi'|)$, with equivalence in norms.*

Lemma 4.4. *Let $1 < p < p_\Phi$. Then $h \in H^p(\mathbb{R}^2_+, |\Phi'|^{1-p})$ if and only if $h/\Phi' \in H^p(\mathbb{R}^2_+, |\Phi'|)$.*

Proof of Theorem 1.5. Given $g \in L^{p_\Phi,1}(\Lambda)$, let $\mathcal{T}(g)$ be as defined in (4.1); as observed in Section 4.1, $\mathcal{T}(g) \in \mathcal{L}^{p_\Phi,1}(\mathbb{R}, |\Phi'|)$ and $\|\mathcal{T}(g)\|_{\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)} = \|g\|_{L^{p_\Phi,1}(\Lambda)}$. Without loss of generality, assumed that g is real-valued. Let $u_{\mathcal{T}(g)}$ be the solution of the Neumann problem in $\mathbb{R}^2_+$ with datum $\mathcal{T}(g)$ as given in Theorem 1.3 and set $v_g = u_{\mathcal{T}(g)} \circ \Phi^{-1}$. The same argument used in the proof of Theorem 1.4 shows that v_g is a solution of the Neumann problem in Ω with datum g.

We next prove (1.4). Let F and G be defined as in the proof of Theorem 1.4. Since $|F| = |\nabla u_{\mathcal{T}(g)}|$ and $\|\mathcal{T}(g)\|_{\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)} = \|g\|_{L^{p_\Phi,1}(\Lambda)}$, Theorem 1.3 gives that

$$\|\mathcal{M}_\alpha(F)\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \lesssim \|g\|_{L^{p_\Phi,1}(\Lambda)}, \tag{4.7}$$

where $0 < \alpha < \pi/2$. We will prove that

$$G \in H^{p_\Phi,\infty}(\Omega, ds) \quad \text{and} \quad \|G\|_{H^{p_\Phi,\infty}(\Omega,ds)} \lesssim \|\mathcal{M}_\alpha(F)\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)}. \tag{4.8}$$

Since $\|G\|_{H^{p_\Phi,\infty}(\Omega,ds)} \sim \|\mathcal{M}_\beta(\nabla v_g)\|_{L^{p_\Phi,\infty}(\Lambda)}$ for any $0 < \beta < \arctan(1/L)$ because $|G| = |\nabla v_g|$, (1.4) follows from (4.7) and (4.8).

In order to prove (4.8), we first observe that by Corollary 4.3 and using that $F = (G \circ \Phi)\Phi'$, we have that $G \in H^{p_\Phi,\infty}(\Omega, ds)$ if and only if $F/\Phi' \in H^{p_\Phi,\infty}(\mathbb{R}^2_+, |\Phi'|)$ and

$$\|G\|_{H^{p_\Phi,\infty}(\Omega,ds)} \sim \|\tfrac{F}{\Phi'}\|_{H^{p_\Phi,\infty}(\mathbb{R}^2_+,|\Phi'|)}. \tag{4.9}$$

We will next prove that $F/\Phi' \in H^{p_\Phi,\infty}(\mathbb{R}^2_+, |\Phi'|)$.

Let $\{f_m\}_{m\in\mathbb{N}} \subset \mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|) \cap L^1_c(\mathbb{R})$ be a sequence of real-valued functions that converges to $\mathcal{T}(g)$ in $\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)$ (see Lemma 2.2). Let F_m be associated to f_m in the same way F is associated to $\mathcal{T}(g)$ as defined in the proof of Theorem 1.4, that is,

$$F_m(y_1,y_2) = \frac{\partial u_{f_m}}{\partial_{y_1}}(y_1,y_2) - i\frac{\partial u_{f_m}}{\partial_{y_2}}(y_1,y_2),$$

where u_{f_m} is the solution of the Neumann problem in the upper half-plane with datum f_m as given in Theorem 1.3. Applying Theorem 1.3, we have

$$\|\mathcal{M}_\alpha(F_m)\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \lesssim \|f_m\|_{\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)}. \tag{4.10}$$

Moreover, by Theorem 1.3 and Remark 3.3, F_m and F converge non-tangentially to functions h_m and h defined on $\mathbb{R}$, respectively. Estimates (4.7) and (4.10) imply that h_m/Φ' and h/Φ' belong to $L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)$.

On the other hand, since $\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|) \subset L^{p_\phi}(\mathbb{R},|\Phi'|^{1-p_\Phi})$, f_m has compact support and $|\Phi'|$ is locally integrable, we have that $f_m \in L^p(\mathbb{R},|\Phi'|^{1-p})$ for $0 < p \le p_\Phi$. Fix $p=2$ and recall that $|\Phi'| \in A_2(\mathbb{R})$; Theorem 1.3 then gives that $F_m \in H^2(\mathbb{R}^2_+,|\Phi'|^{-1})$ and therefore $F_m/\Phi' \in H^2(\mathbb{R}^2_+,|\Phi'|)$ by Lemma 4.4. As a consequence, $|F_m(\cdot,y_2)/\Phi'(\cdot,y_2)| \in L^2(\mathbb{R},|\Phi'|)$ uniformly in $y_2>0$ by Theorem A.7.

Next, noting that $|F_m/\Phi'|$ is subharmonic in $\mathbb{R}^2_+$ (since F_m/Φ' is holomorfic in $\mathbb{R}^2_+$) and that $\lim_{y_2\to 0}|F_m(y_1,y_2)/\Phi'(y_1,y_2)| = |h_m(y_1)/\Phi'(y_1)|$ for almost every $y_1\in\mathbb{R}$, Lemma A.12 gives that

$$\left|\frac{F_m(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \le \left(P_{y_2} * \left|\tfrac{h_m}{\Phi'}\right|\right)(y_1), \quad y_1\in\mathbb{R}, y_2>0.$$

As a consequence, it holds that

$$\mathcal{M}_\alpha(\tfrac{F_m}{\Phi'})(x) \le \sup_{y_1+iy_2\in\Gamma_\alpha(x)} \left(\left(P_{y_2} * \left|\tfrac{h_m}{\Phi'}\right|\right)(y_1)\right) \lesssim \mathcal{M}_{hl}(\tfrac{h_m}{\Phi'})(x).$$

This estimate and the facts that $\mathcal{M}_{hl}$ is bounded on $L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)$ by Theorem 2.1 (recall that $|\Phi'|\in A_{p_\Phi}(\mathbb{R})$) and $h_m/\Phi' \in L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)$ imply that $F_m/\Phi' \in H^{p_\Phi,\infty}(\mathbb{R}^2_+,|\Phi'|)$ and

$$\|\tfrac{F_m}{\Phi'}\|_{H^{p_\Phi,\infty}(\mathbb{R}^2_+,|\Phi'|)} \sim \|\mathcal{M}_\alpha(\tfrac{F_m}{\Phi'})\|_{L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \lesssim \|\tfrac{h_m}{\Phi'}\|_{L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} = \|h_m\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)}.$$

We will prove that as m goes to infinity, the above inequality gives

$$\|\tfrac{F}{\Phi'}\|_{H^{p_\Phi,\infty}(\mathbb{R}^2_+,|\Phi'|)} \lesssim \|h\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)}. \tag{4.11}$$

Assuming (4.11) and using that $|h(x)| \le \mathcal{M}_\alpha(F)(x)$ for almost every $x\in\mathbb{R}$ we obtain

$$\|\tfrac{F}{\Phi'}\|_{H^{p_\Phi,\infty}(\mathbb{R}^2_+,|\Phi'|)} \lesssim \|\mathcal{M}_\alpha(F)\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)},$$

which along with (4.9), gives (4.8) as desired.

We are left to prove (4.11). Note first that $\{F_m\}_{m\in\mathbb{N}}$ converges to F uniformly on compact sets of $\mathbb{R}^2_+$. Indeed, since $\{f_m\}_{m\in\mathbb{N}}$ converges to $\mathcal{T}(g)$ in $\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)$, Lemma 3.1 gives that $\{u_{f_m}\}_{m\in\mathbb{N}}$ converges uniformly to $u_{\mathcal{T}(g)}$ on compact sets of $\mathbb{R}^2_+$ and, since u_{f_m} is harmonic for each m, then $\{\nabla u_{f_m}\}_{m\in\mathbb{N}}$ converges uniformly to $\nabla u_{\mathcal{T}(g)}$ on compact sets of $\mathbb{R}^2_+$. It then follows that $\{F_m\}_{m\in\mathbb{N}}$ converges to F uniformly on compact sets of $\mathbb{R}^2_+$. In addition, we have that $\{h_m\}_{m\in\mathbb{N}}$ converges to h in $\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)$; this follows from the fact that $\{f_m\}_{m\in\mathbb{N}}$ converges to $\mathcal{T}(g)$ in $\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)$ and the inequalities

$$\|h_m-h\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \le \|\mathcal{M}_\alpha(F_m-F)\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \lesssim \|f_m-\mathcal{T}(g)\|_{\mathcal{L}^{p_\Phi,1}(\mathbb{R},|\Phi'|)}.$$

(Note that F_m-F is associated to $f_m-\mathcal{T}(g)$ in the same way that F_m and F are associated to f_m and $\mathcal{T}(g)$, respectively.)

We next proceed similarly to the argument used in the proof of (3.16). Define $\Gamma_\alpha^s(x) = \Gamma_\alpha(x) \cap \{(y_1 + iy_2) \in \mathbb{C} : 1/s < y_2 < s\}$ for $x \in \mathbb{R}$ and $s \in \mathbb{N}$; let $C_s(x) > 0$ be such that $|\Phi'(y_1, y_2)| > C_s(x)$ for all $y_1 + iy_2 \in \Gamma_\alpha^s(x)$ ($C_s(x)$ exists since Φ' is continuous on $\mathbb{R}_+^2$ and Φ' does not vanish in $\mathbb{R}_+^2$). We then observe that

$$\begin{aligned}\left| \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F(y_1,y_2)}{\Phi'(y_1,y_2)}\right| - \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F_m(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \right| &\le \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F(y_1,y_2)-F_m(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \\ &\le \tfrac{1}{C_s(x)} \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} |F(y_1,y_2) - F_m(y_1,y_2)| \quad \forall x \in \mathbb{R}.\end{aligned}$$

As a consequence, we obtain

$$\lim_{m\to\infty} \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F_m(y_1,y_2)}{\Phi'(y_1,y_2)}\right| = \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \quad \forall x \in \mathbb{R},$$

and it follows that

$$\begin{aligned}\left\| \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \right\|_{L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} &\le \liminf_{m\in\mathbb{N}} \left\| \sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F_m(y_1,y_2)}{\Phi'(y_1,y_2)}\right| \right\|_{L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \\ &\le \liminf_{m\to\infty} \|\mathcal{M}_\alpha(\tfrac{F_m}{\Phi'})\|_{L^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \\ &\le \liminf_{m\to\infty} \|h_m\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)} \\ &\lesssim \|h\|_{\mathcal{L}^{p_\Phi,\infty}(\mathbb{R},|\Phi'|)}.\end{aligned}$$

Since $\left\{\sup_{y_1+iy_2\in\Gamma_\alpha^s(x)} \left|\tfrac{F(y_1,y_2)}{\Phi'(y_1,y_2)}\right|\right\}_{s\in\mathbb{N}}$ is an increasing sequence that converges to $\mathcal{M}_\alpha(F/\Phi')(x)$ for $x \in \mathbb{R}$ and the Monotone Convergence Theorem holds in $L^{p_\Phi,\infty}(\mathbb{R}, |\Phi'|)$, we obtain (4.11). $\square$

4.4. **Proofs of Lemmas 4.1, 4.2 and 4.4.**

Proof of Lemma 4.1. As shown in the proof of Theorem 1.4, $|(\Phi^{-1})'|^p$ is a weight in $A_\infty(\Lambda)$ for $1 < p < p_\Phi$. Then, by Theorem A.7, $h \circ \Phi^{-1} \in H^p(\Omega, |(\Phi^{-1})'|^p)$ if and only if $(h \circ \Phi^{-1})k^{1/p} \in H^p(\Omega, ds)$, where k is an analytic function in Ω, k is different from zero in Ω, k has a non-tangential limit satisfying $|k(\xi)| = |(\Phi^{-1})'(\xi)|^p$ for almost every $\xi \in \Lambda$ with respect to ds, and there exists $m \ge 0$ such that $(k \circ \Phi)(z)\Phi'(z)/(i + z)^m \in H^1(\mathbb{R}_+^2, dx)$. Moreover, $\|h \circ \Phi^{-1}\|_{H^p(\Omega,|(\Phi^{-1})'|^p)} \sim \|(h \circ \Phi^{-1})k^{1/p}\|_{H^p(\Omega,ds)}$. In view of this, the result will follow if we show that we can take $k = ((\Phi^{-1})')^p$.

We are left to prove that $(k\circ\Phi)(z)\Phi'(z)/(i+z)^m = (\Phi'(z))^{1-p}/(i+z)^m \in H^1(\mathbb{R}_+^2, dx)$ for some $m \ge 0$ (k verifies the other required properties in view of the properties of Φ stated in Section 4.1).

Case $p = 2$: Using $m = 2$ and Kenig [16, Theorem 1.10, Part (iii)], we have $(\Phi'(z))^{-1}/(i+z)^2 \in H^1(\mathbb{R}_+^2, dx)$. This fact will be used in the next two cases.

Case $1 < p < 2$: We have

$$\int_{\mathbb{R}} \frac{|(\Phi'(x+it))^{1-p}|}{|(i+(x+it))^m|}\,dx = \int_{\mathbb{R}} \left(\frac{|\Phi'(x+it)|^{-1}}{|(i+(x+it))^2|}\right)^{p-1} \frac{1}{|(i+(x+it))^{m-2(p-1)}|}\,dx.$$

Since $1 < p < 2$, then $1/(p-1) > 1$ and applying Hölder's inequality, we obtain

$$\int_{\mathbb{R}} \frac{|(\Phi'(x+it))^{1-p}|}{|(i+(x+it))^m|}\,dx$$
$$\leq \left(\int_{\mathbb{R}} \frac{|\Phi'(x+it)|^{-1}}{|(i+(x+it))^2|}\,dx\right)^{p-1} \left(\int_{\mathbb{R}} \frac{1}{|(i+(x+it))^{\frac{m-2(p-1)}{2-p}}|}\,dx\right)^{2-p}.$$

The first integral on the right-hand side is bounded uniformly in $t > 0$ because $(\Phi'(z))^{-1}/(i+z)^2 \in H^1(\mathbb{R}^2_+, dx)$ while the second integral on the right hand side is bounded uniformly in $t > 0$ for $m > p$; as a consequence, $(\Phi'(z))^{1-p}/(i+z)^m \in H^1(\mathbb{R}^2_+, dx)$.

Case $2 < p < p_\Phi$: Define $m = 2(p-1)$; we have $(\Phi'(z))^{-1}/(i+z)^{m/(p-1)} \in H^1(\mathbb{R}^2_+, dx)$ or, equivalently, $(\Phi'(z))^{1-p}/(i+z)^m \in H^{\frac{1}{p-1}}(\mathbb{R}^2_+, dx)$. This last function converges non-tangentially to $(\Phi'(x))^{1-p}/(i+x)^m$ and, using that $m > p$, $|\Phi'(x)|^{1-p} \in A_p(\mathbb{R})$ and Lemma 2.3, we have

$$\int_{\mathbb{R}} \left|\frac{(\Phi'(x))^{1-p}}{(i+x)^m}\right| dx = \int_{\mathbb{R}} \frac{|\Phi'(x)|^{1-p}}{(1+x^2)^{\frac{p}{2}}} \frac{1}{(1+x^2)^{\frac{m-p}{2}}}\,dx \leq \int_{\mathbb{R}} \frac{|\Phi'(x)|^{1-p}}{(1+x^2)^{\frac{p}{2}}}\,dx < \infty.$$

As a consequence, $(\Phi'(z))^{1-p}/(i+z)^m$ is a function in $H^{\frac{1}{p-1}}(\mathbb{R}^2_+, dx)$ whose non-tangential limit is in $L^1(\mathbb{R})$; this implies that $(\Phi'(z))^{1-p}/(i+z)^m \in H^1(\mathbb{R}^2_+, dx)$ by Lemma A.11. $\square$

Proof of Lemma 4.2. The proof of this lemma follows the same ideas as the ones for Theorem A.5; we include it here for self-containment.

Assume that $h \circ \Phi \in H^{p,\infty}(\mathbb{R}^2_+, \Phi(\nu))$. By Lemma A.1, there exist $0 < \alpha < \arctan(1/L)$ and $0 < \beta < \pi/2$ such that $\Phi^{-1}(\Gamma_\alpha(\xi)) \subset \Gamma_\beta(\Phi^{-1}(\xi))$ for almost every $\xi \in \Lambda$ with respect to ds. This implies that $\mathcal{M}_\alpha(h)(\xi) \leq \mathcal{M}_\beta(h \circ \Phi)(\Phi^{-1}(\xi))$ for almost every $\xi \in \Lambda$ with respect to ds; as a consequence, if $\lambda > 0$, it holds that

$$\begin{aligned}
\nu(\{\xi \in \Lambda : \mathcal{M}_\alpha(h)(\xi) > \lambda\}) &\leq \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h \circ \Phi)(\Phi^{-1}(\xi)) > \lambda\}) \\
&= \nu(\Phi(\{x \in \mathbb{R} : \mathcal{M}_\beta(h \circ \Phi)(x) > \lambda\})) \\
&= \Phi(\nu)(\{x \in \mathbb{R} : \mathcal{M}_\beta(h \circ \Phi)(x) > \lambda\}).
\end{aligned}$$

It then follows that $\|\mathcal{M}_\alpha(h)\|_{L^{p,\infty}(\Lambda,\nu)} \leq \|\mathcal{M}_\beta(h \circ \Phi)\|_{L^{p,\infty}(\mathbb{R},\Phi(\nu))}$ and, therefore, $h \in H^{p,\infty}(\Omega, \nu)$ and $\|h\|_{H^{p,\infty}(\Omega,\nu)} \lesssim \|h \circ \Phi\|_{H^{p,\infty}(\mathbb{R}^2_+,\Phi(\nu))}$.

Suppose now that $h \in H^{p,\infty}(\Omega, \nu)$ and $L < 1$. By Lemma A.1, there exist $0 < \alpha < \pi/2$ and $0 < \beta < \arctan(1/L)$ such that $\Phi(\Gamma_\alpha(x)) \subset \Gamma_\beta(\Phi(x)) \subset \Omega$ for almost every $x \in \mathbb{R}$ with respect to the Lebesgue measure. This implies that $\mathcal{M}_\alpha(h \circ \Phi)(x) \leq \mathcal{M}_\beta(h)(\Phi(x))$ for almost every $x \in \mathbb{R}$ with respect to the Lebesgue measure and therefore, for $\lambda > 0$,

$$\begin{aligned}
\Phi(\nu)(\{x \in \mathbb{R} : \mathcal{M}_\alpha(h \circ \Phi)(x) > \lambda\}) &\leq \Phi(\nu)(\{x \in \mathbb{R} : \mathcal{M}_\beta(h)(\Phi(x)) > \lambda\}) \\
&= \nu(\Phi(\{x \in \mathbb{R} : \mathcal{M}_\beta(h)(\Phi(x)) > \lambda\})) \\
&= \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h)(\xi) > \lambda\}).
\end{aligned}$$

This shows that $\|\mathcal{M}_\alpha(h \circ \Phi)\|_{L^{p,\infty}(\mathbb{R},\Phi(\nu))} \leq \|\mathcal{M}_\beta(h)\|_{L^{p,\infty}(\Lambda,\nu)}$; then $h \circ \Phi \in H^{p,\infty}(\mathbb{R}^2_+, \Phi(\nu))$ and $\|h \circ \Phi\|_{H^{p,\infty}(\mathbb{R}^2_+,\Phi(\nu))} \lesssim \|h\|_{H^{p,\infty}(\Omega,\nu)}$.

Assume next that $h \in H^{p,\infty}(\Omega, \nu)$ and $L \geq 1$. Using the notation introduced in Definition A.2 and Lemma A.3, we have $\Phi_0(z) = z$, $\Phi_1(z) = \Phi$, $\Omega_0 = \mathbb{R}^2_+$ and $\Omega_1 = \Omega$. Let $\varepsilon > 0$, $\alpha > 0$ and

$\beta > 0$ be as given by Lemma A.3. Set $\lambda_1 = 1$ and let $0 \le \lambda_2 < \lambda_1$ be such that $\lambda_1 - \lambda_2 < \varepsilon$; then Lemma A.3 implies that, for a. e. $\xi \in \Lambda_{\lambda_2}$ (w.r.t. ds_{λ_2}), it holds that

$$\sigma_{\lambda_1,\lambda_2}(\Gamma_\alpha(\xi)) \subset \Gamma_\beta(\sigma_{\lambda_1,\lambda_2}(\xi)) \subset \Omega_{\lambda_1} = \Omega.$$

As a consequence, for almost every $\xi \in \Lambda_{\lambda_2}$ (with respect to arc length measure ds_{λ_2})

$$\mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_2})(\xi) \le \mathcal{M}_\beta(h)(\sigma_{\lambda_1,\lambda_2}(\xi)).$$

We then obtain that

$$\begin{aligned} \sigma_{\lambda_1,\lambda_2}(\nu)(\{\xi \in \Lambda_{\lambda_2} : \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_2})(\xi) > \lambda\}) &\le \sigma_{\lambda_1,\lambda_2}(\nu)(\{\xi \in \Lambda_{\lambda_2} : \mathcal{M}_\beta(h)(\sigma_{\lambda_1,\lambda_2}(\xi)) > \lambda\}) \\ &= \nu(\sigma_{\lambda_1,\lambda_2}(\{\xi \in \Lambda_{\lambda_2} : \mathcal{M}_\beta(h)(\sigma_{\lambda_1,\lambda_2}(\xi)) > \lambda\})) \\ &= \nu(\{\xi \in \Lambda_{\lambda_1} : \mathcal{M}_\beta(h)(\xi) > \lambda\}) \\ &= \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h)(\xi) > \lambda\}). \end{aligned} \tag{4.12}$$

We next choose $0 \le \lambda_3 < \lambda_2$ such that $\lambda_2 - \lambda_3 < \varepsilon$; applying Lemma A.3 we obtain that, for almost every $\xi \in \Lambda_{\lambda_3}$ (with respect to arc length measure ds_{λ_3}),

$$\sigma_{\lambda_2,\lambda_3}(\Gamma_\alpha(\xi)) \subset \Gamma_\beta(\sigma_{\lambda_2,\lambda_3}(\xi)) \subset \Omega_{\lambda_2}$$

and therefore, for almost every $\xi \in \Lambda_{\lambda_3}$ (with respect to arc length measure ds_{λ_3}),

$$\mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_3})(\xi) = \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_2} \circ \sigma_{\lambda_2,\lambda_3})(\xi) \le \mathcal{M}_\beta(h \circ \sigma_{\lambda_1,\lambda_2})(\sigma_{\lambda_2,\lambda_3}(\xi)).$$

This gives that

$$\begin{aligned} &\sigma_{\lambda_1,\lambda_3}(\nu)(\{\xi \in \Lambda_{\lambda_3} : \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_3})(\xi) > \lambda\}) \\ &\quad = \sigma_{\lambda_2,\lambda_3}(\sigma_{\lambda_1,\lambda_2}(\nu))(\{\xi \in \Lambda_{\lambda_3} : \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_3})(\xi) > \lambda\}) \\ &\quad \le \sigma_{\lambda_2,\lambda_3}(\sigma_{\lambda_1,\lambda_2}(\nu))(\{\xi \in \Lambda_{\lambda_3} : \mathcal{M}_\beta(h \circ \sigma_{\lambda_1,\lambda_2})(\sigma_{\lambda_2,\lambda_3}(\xi)) > \lambda\}) \\ &\quad = \sigma_{\lambda_1,\lambda_2}(\nu)(\sigma_{\lambda_2,\lambda_3}(\{\xi \in \Lambda_{\lambda_3} : \mathcal{M}_\beta(h \circ \sigma_{\lambda_1,\lambda_2})(\sigma_{\lambda_2,\lambda_3}(\xi)) > \lambda\})) \\ &\quad = \sigma_{\lambda_1,\lambda_2}(\nu)(\{\xi \in \Lambda_{\lambda_2} : \mathcal{M}_\beta(h \circ \sigma_{\lambda_1,\lambda_2})(\xi) > \lambda\}) \\ &\quad \sim \sigma_{\lambda_1,\lambda_2}(\nu)(\{\xi \in \Lambda_{\lambda_2} : \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_2})(\xi) > \lambda\}) \\ &\quad \le \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h)(\xi) > \lambda\}), \end{aligned}$$

where in the last inequality we have used (4.12) and in the second to the last line we have applied Lemma A.4 noting that $\sigma_{\lambda_1,\lambda_2}(\nu) \in A_\infty(\Lambda_2)$ by Lemma A.8. Continuing in this way, after a finite number of steps m, we can choose $\lambda_m = 0$ and obtain that

$$\sigma_{\lambda_1,\lambda_m}(\nu)(\{\xi \in \Lambda_{\lambda_m} : \mathcal{M}_\alpha(h \circ \sigma_{\lambda_1,\lambda_m})(\xi) > \lambda\}) \lesssim \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h)(\xi) > \lambda\}).$$

Since $\sigma_{\lambda_1,\lambda_m} = \Phi$ and $\Lambda_{\lambda_m} = \mathbb{R}$, we have

$$\Phi(\nu)(\{\xi \in \mathbb{R} : \mathcal{M}_\alpha(h \circ \Phi)(\xi) > \lambda\}) \le \nu(\{\xi \in \Lambda : \mathcal{M}_\beta(h)(\xi) > \lambda\}).$$

This shows that $\|\mathcal{M}_\alpha(h \circ \Phi)\|_{L^{p,\infty}(\mathbb{R},\Phi(\nu))} \le \|\mathcal{M}_\beta(h)\|_{L^{p,\infty}(\Lambda,\nu)}$; then $h \circ \Phi \in H^{p,\infty}(\mathbb{R}^2_+, \Phi(\nu))$ and $\|h \circ \Phi\|_{H^{p,\infty}(\mathbb{R}^2_+,\Phi(\nu))} \lesssim \|h\|_{H^{p,\infty}(\Omega,\nu)}$.

□

Proof of Lemma 4.4. We will use Theorem A.10 with $\Omega_1 = \Omega_2 = \mathbb{R}^2_+$, $\Lambda_1 = \Lambda_2 = \mathbb{R}$, $\sigma_{1,2} = \mathrm{Id}$, $\nu = |\Phi'|^{1-p}$ and $\mu = |\Phi'|$. By such result, $h \in H^p(\mathbb{R}^2_+, |\Phi'|^{1-p})$ if and only if there exists an analytic function k on $\mathbb{R}^2_+$ such that $h\, k^{1/p} \in H^p(\mathbb{R}^2_+, |\Phi'|)$ and k satisfies the following properties: k does not vanish in $\mathbb{R}^2_+$, k has non-tangential limit almost everywhere with respect to Lebesgue measure in $\mathbb{R}$ that equals $|\Phi'(x)|^{-p}$ for almost every $x \in \mathbb{R}$, and there exists $m \ge 0$ such that

$k(z)b(z)/(i+z)^m \in H^1(\mathbb{R}^2_+, dx)$ for some b analytic in $\mathbb{R}^2_+$ that does not vanish in $\mathbb{R}^2_+$, has non-tangential limit almost everywhere with respect to Lebesgue measure in $\mathbb{R}$ that coincides with $|\Phi'(x)|$ for almost every $x \in \mathbb{R}$ and $b(z)/(i+z)^{\widetilde{m}} \in H^1(\mathbb{R}^2_+, dx)$ for some $\widetilde{m} \geq 0$.

Set $k(z) = (\Phi'(z))^{-p}$, $b(z) = \Phi'(z)$; then all the requirements above are satisfied, in particular $k(z)b(z)/(i+z)^m = (\Phi'(z))^{1-p}/(i+z)^m \in H^1(\mathbb{R}^2_+, dx)$ with m as given in the proof of Lemma 4.1 and $b(z)/(i+z)^2 = \Phi'(z)/(i+z)^2 \in H^1(\mathbb{R}^2_+, dx)$ by Kenig [16, Theorem 1.10]. Since $h\, k^{1/p} = h/\Phi'$, the desired result follows. □

5. Examples: Schwarz-Christoffel Lipschitz domains and related domains

In this section, we will give examples of our results in domains Ω where the boundary Λ is a polygonal curve, that is, Λ is a continuous curve that consists of a finite or countably infinite number of line segments. In Section 5.1 we treat the case where Λ is a polygonal curve with a finite number of vertices while in Section 5.2 we present examples where Λ is a polygonal curve with a countably infinite number of vertices. In addition, we present in Section 5.3 a comparison regarding the solvability of the Neumann problem and the Dirichlet problem in a general domain Ω of the type (1.2).

5.1. **Λ is a polygonal curve with a finite number of vertices.** In this case, Ω is defined by a collection of vertices $w_1, \ldots, w_n \in \mathbb{C} \cup \{\infty\}$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$. The vertices are given in *counterclockwise order* with respect to the interior of Ω; that is, locally, Ω is to the left as one moves along the segment from w_k to w_{k+1}. We will call Ω a *polygon*.

The interior angle at vertex w_k is the angle swept from the outgoing side at w_k to the incoming side at w_k; the definition of the interior angle is applied on the Riemann sphere if $w_k = \infty$. If $w_k \in \mathbb{C}$, we have $0 < \alpha_k \leq 2$; if $w_k = \infty$, we have $-2 \leq \alpha_k \leq 0$. We will always assume that the polygon makes a total turn of 2π, which means that $\sum_{k=1}^n (1-\alpha_k) = 2$ or, equivalently,

$$\sum_{k=1}^{n} \alpha_k = n - 2. \tag{5.1}$$

We have the following result regarding conformal mappings from $\mathbb{R}^2_+$ onto a polygon Ω.

Theorem 5.1. *(Theorem 2.1 in Driscoll–Trefethen [7]) Suppose Ω is a polygon with vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots \alpha_n\pi$ in counterclockwise order. If Φ is a conformal mapping from $\mathbb{R}^2_+$ onto Ω with $\Phi(\infty) = \omega_n$ then*

$$\Phi(z) = A + B \int_{[z_0,z]} (\xi - x_1)^{\alpha_1 - 1} \cdots (\xi - x_{n-1})^{\alpha_{n-1}-1}\, d\xi, \tag{5.2}$$

where $A, B \in \mathbb{C}$, z_0 is a suitably chosen point in $\mathbb{R}^2_+$ or its boundary, $[z_0, z]$ denotes the straight line segment from z_0 to z, $x_1, \ldots, x_{n-1}$ are real numbers such that $x_1 < \cdots < x_{n-1}$ and $\Phi(x_k) = w_k$ for $k = 1, \ldots, n-1$.

A mapping of the form given on the right-hand side of (5.2) is called a *Schwarz-Christoffel transformation.*

For the rest of this subsection, let Ω be a polygon (that is the graph of a function defined in $\mathbb{R}$) with vertices $w_1, \ldots, w_{n-1} \in \mathbb{C}$, $w_n = \infty$ and interior angles $\alpha_1\pi, \ldots \alpha_n\pi$ in counterclockwise order, where $0 < \alpha_1, \ldots, \alpha_{n-1} < 2$, $-2 < \alpha_n < 0$ and (5.1) holds; see Figure 1. Let $\Phi : \mathbb{R}^2_+ \to \Omega$ be a Schwarz-Christoffel transformation as given in Theorem 5.1. We then have that

$$|\Phi'(x)| = B|x - x_1|^{\alpha_1 - 1}|x - x_2|^{\alpha_2 - 1} \cdots |x - x_{n-1}|^{\alpha_{n-1}-1}.$$

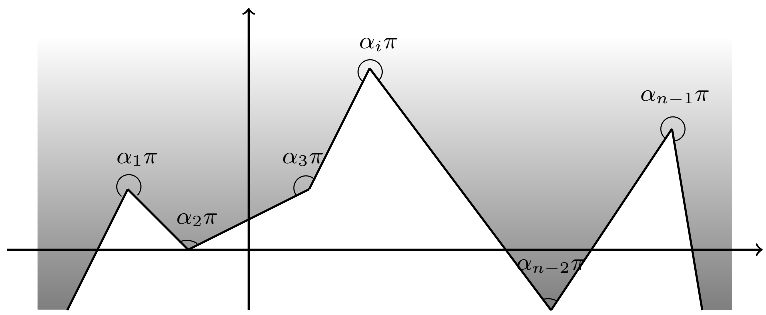


FIGURE 1. Unbounded polygon with $w_1, \dots, w_{n-1} \in \mathbb{C}$ and $w_n = \infty$

It was proved in Carro–Ortiz-Caraballo [3, Lemma 2.12] that

$$|\Phi'| \in A_q^{\mathcal{R}}(\mathbb{R}) \text{ with } q = \max\{1, \alpha_1, \dots, \alpha_{n-1}, -\alpha_n\} \quad \text{and} \quad |\Phi'| \notin A_p^{\mathcal{R}}(\mathbb{R}) \text{ for } p < q, \tag{5.3}$$

where the second condition is void if $q = 1$. As a consequence, we have that

$$p'_\Phi = \max\{1, \alpha_1, \dots, \alpha_{n-1}, -\alpha_n\}.$$

In view of (5.3), (4.2) and Theorems 1.4 and 1.5 we obtain the following result.

Corollary 5.2. *Let Ω be a polygon (that is the graph of a function defined in $\mathbb{R}$) with vertices $w_1, \dots, w_{n-1} \in \mathbb{C}$, $w_n = \infty$ and interior angles $\alpha_1\pi, \dots \alpha_n\pi$ in counterclockwise order, where $0 < \alpha_1, \dots, \alpha_{n-1} < 2$, $-2 < \alpha_n < 0$ and (5.1) holds. Then we have that*

(a) $p_\Phi = \frac{\max\{1,\alpha_1,\dots,\alpha_{n-1},-\alpha_n\}}{\max\{1,\alpha_1,\dots,\alpha_{n-1},-\alpha_n\}-1}$;
(b) if $1 < p < p_\Phi$, the Neumann problem in Ω is solvable in $X = L^p(\Lambda)$ with $Y = L^p(\Lambda)$;
(c) if $p_\Phi < \infty$, the Neumann problem in Ω is solvable in $X = L^{p_\Phi,1}(\Lambda)$ with $Y = L^{p_\Phi,\infty}(\Lambda)$.

We next present examples of domains that satisfy the hypothesis of Corollary 5.2.

Example 1. For $0 < \alpha < 2$, let Ω be a cone with aperture $\alpha\pi > 0$ as in Figures 2 or 3. Then $n = 2$, $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$ and, by Theorem 5.1, if Φ is a conformal mapping from $\mathbb{R}^2_+$ onto Ω with $\Phi(\infty) = \infty$, then $|\Phi'(x)| = B\,|x - x_1|^{\alpha-1}$ for $x \in \mathbb{R}$ and for some constants $B \in \mathbb{C}$ and $x_1 \in \mathbb{R}$. We observe two different behaviors depending on α :

Case $0 < \alpha \leq 1$: In view of (5.3), $p'_\Phi = 1$, $p_\Phi = \infty$ and $|\Phi'| \in A_1(\mathbb{R})$. By Corollary 5.2, the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < \infty$.

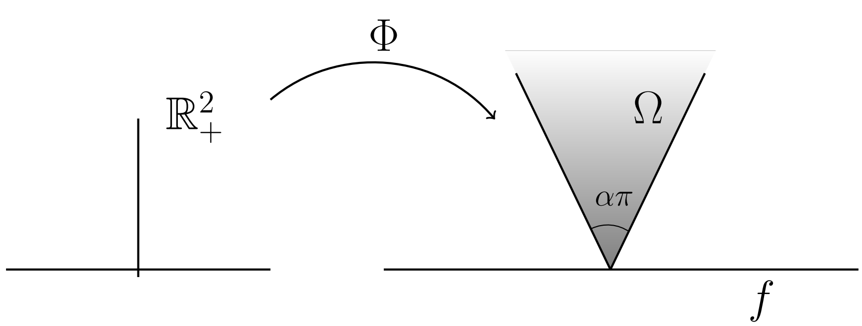


FIGURE 2. Cone with aperture $\alpha\pi$ with $0 < \alpha \leq 1$

Case $1 < \alpha < 2$: In view of (5.3), $p'_\Phi = \alpha$, $p_\Phi = \alpha/(\alpha-1)$ and $|\Phi'| \in A_\alpha^{\mathcal{R}}(\mathbb{R})$. Applying Corollary 5.2, the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < \alpha/(\alpha-1)$ and in $L^{\frac{\alpha}{\alpha-1},1}(\Lambda)$.

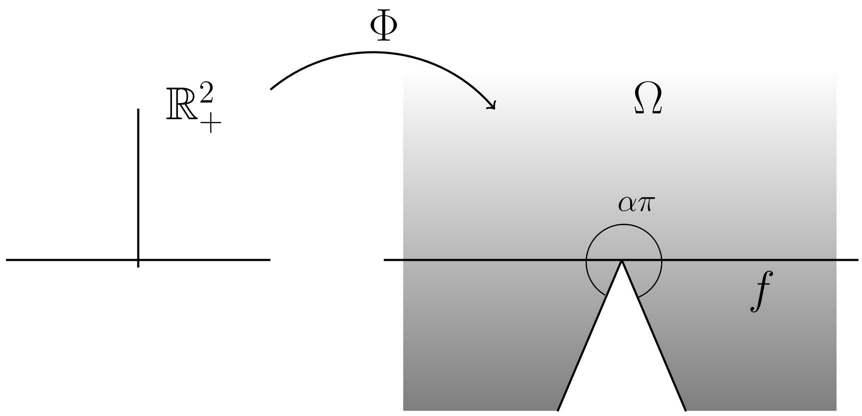


FIGURE 3. Cone with aperture $\alpha\pi$ with $1 < \alpha < 2$

Example 2. Let Ω be the two-angle domain shown in Figure 4 with $n = 3$, $\alpha_1 = 3/2$; $\alpha_2 = 1/2$ and $\alpha_3 = -1$. By Theorem 5.1, a conformal mapping from $\mathbb{R}^2_+$ onto Ω such that $\Phi(\infty) = \infty$ satisfies $\Phi'(x) = |x - x_1|^{1/2}|x - x_2|^{-1/2}$ for $x \in \mathbb{R}$ and some constants $B \in \mathbb{C}$ and $x_1, x_2 \in \mathbb{R}$. In view of (5.3), we obtain that $p'_\Phi = 3/2$, $p_\Phi = 3$ and $|\Phi'| \in A^{\mathcal{R}}_{3/2}(\mathbb{R})$. As a consequence of Corollary 5.2, the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < 3$ and in $X = L^{3,1}(\Lambda)$.

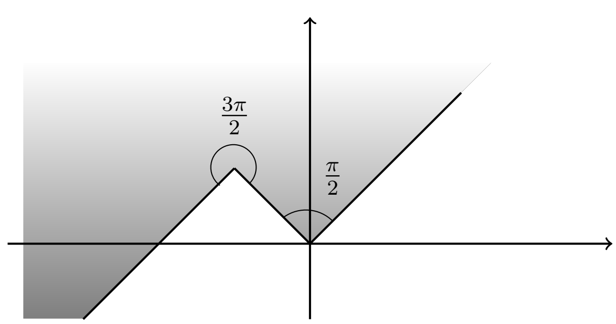


FIGURE 4. Two-angle domain

5.2. Λ **is a polygonal curve with a countably infinite number of vertices.** In this section, we present examples associated to domains with a boundary given by a polygonal curve with a countably infinite number of vertices.

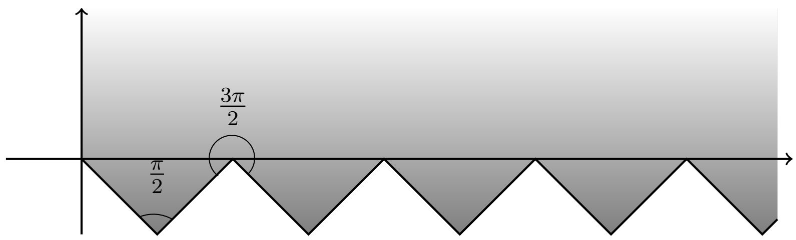


FIGURE 5. Infinite rotated staircase with alternating interior angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Example 1. Let Ω to be the "infinite rotated staircase" with interior angles alternately equal to $\pi/2$ and $3\pi/2$ as shown in Figure 5.

The formula of a conformal mapping Φ from $\mathbb{R}^2_+$ onto Ω is given by (see Riera–Carrasco–Preiss [22, p. 3])

$$\Phi(z) = A + B \int_0^z \sqrt{\tan \xi}\, d\xi,$$

from which it follows that $|\Phi'(x)| = |B\sqrt{\tan x}|$ for $x \in \mathbb{R}$. As proved in Carro–Ortiz-Caraballo [3, p. 2026-2027], $p'_\Phi = 3/2$ and $|\Phi'| \in A^{\mathcal{R}}_{3/2}(\mathbb{R})$. As a consequence of Theorems 1.4 and 1.5, the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < 3$ and in $L^{3,1}(\Lambda)$.

Example 2. We next consider a domain Ω given by a numerable polygon with double symmetry as presented in Kolesnikov–Kopaneva [19, p. 33]. In this case, Ω satisfies the following conditions: $T(\Omega) = \Omega$ for $T(z) = z + 2\pi$, Ω is symmetric with respect to the line $z = \pi + iy$, $y \in \mathbb{R}$; the part of Λ from a point ξ to $\xi + 2\pi$ consists of a finite number of segments and rays. See Figure 6.

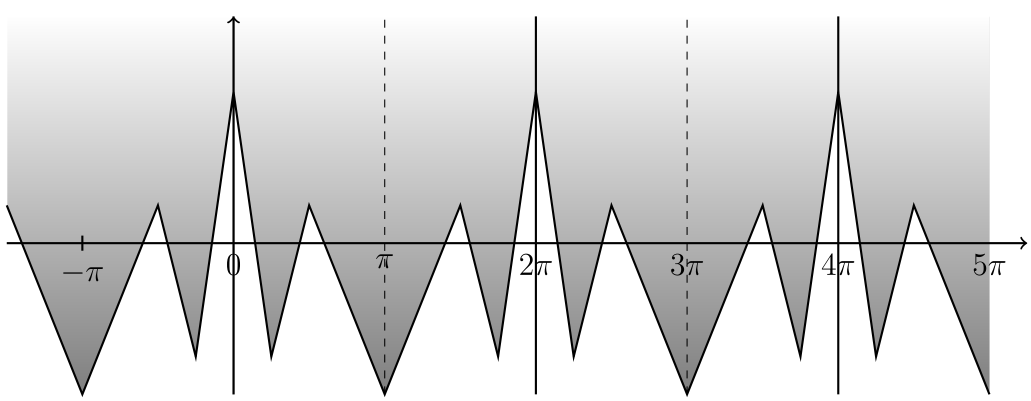


FIGURE 6. Numerable polygon with double symmetry

Let n be the number of vertices in the strip $\{z \in \mathbb{C} : \text{Re}(z) \in [0, \pi]\}$. Denote these vertices by $w_1, \cdots, w_n$ with their real parts in increasing order; without loss of generality assume $\text{Re}(w_1) = 0$ and $\text{Re}(w_n) = \pi$ (the angle at these vertices could be π.)

By Kolesnikov–Kopaneva [19, Theorem 2], a conformal mapping in this case is given by

$$\Phi(z) = A + B \int_0^z (1 - \cos \xi)^{\frac{\alpha_1 - 1}{2}} (1 + \cos \xi)^{\frac{\alpha_n - 1}{2}} \prod_{k=2}^{n-1} (\cos x_k - \cos \xi)^{\alpha_k - 1} d\xi,$$

where $x_k \in \mathbb{R}$ and $\Phi(\cos(x_k)) = w_k$ for $k = 2, \ldots, n-1$, $\Phi(1) = w_1$, $\Phi(-1) = w_n$, $0 < \alpha_k < 2$ for $k = 1, \ldots, n$, and $\alpha_k \pi$ is the interior angle at w_k for $k = 1, \ldots, n$. We have

$$|\Phi'(x)| \quad = \quad B\Big|(1-\cos x)^{\frac{\alpha_1 - 1}{2}}(1+\cos x)^{\frac{\alpha_n - 1}{2}}\prod_{k=2}^{n-1}(\cos x_k - \cos x)^{\alpha_k - 1}\Big|, \quad x \in \mathbb{R}.$$

Using (5.3), it was proved in Carro–Ortiz-Caraballo [3, p. 2027] that $p'_\Phi = \max\{1, \alpha_1, \ldots, \alpha_n, -(n-2) + \sum_{k=1}^n \alpha_k\}$ and $|\Phi'| \in A^{\mathcal{R}}_{p'_\Phi}$. As a consequence, Theorems 1.4 and 1.5 imply that the Neumann problem in Ω is solvable in $L^p(\Lambda)$ for $1 < p < p_\Phi$ and in $L^{p_\Phi,1}(\Lambda)$.

5.3. **A comparison on the solvability of the the Neumann problem and the Dirichlet problem.** In this section, we consider a general domain Ω as given by (1.2) and compare the solvability of the corresponding Neumann problem and Dirichlet problem.

The Dirichlet problem in Ω, given by

$$\Delta v = 0 \text{ on } \Omega \qquad \text{and} \qquad v = g \text{ on } \Lambda, \tag{5.4}$$

is said to be solvable in a Banach space X of measurable functions defined on Λ if for each $g \in X$ there is a harmonic function v in Ω such that $v = g$ on Λ in the sense of non-tangential convergence and there exists a Banach space Y of measurable functions defined on Λ such that $\|\mathcal{M}_\alpha(v)\|_Y \lesssim \|g\|_X$, with the implicit constant independent of g. It turns out that the Dirichlet problem is solvable in $L^p(\Lambda)$ for $p'_\Phi < p < \infty$ and in $L^{p'_\Phi,1}(\Lambda)$ if $|\Phi'| \in A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$; see Carro–Ortiz-Caraballo [3]. These facts along with Theorems 1.4 and 1.5 imply the following corollary:

Corollary 5.3. *The Neumann problem in Ω and the Dirichlet problem in Ω are both solvable in $L^p(\Lambda)$ for $p'_\Phi < p < p_\Phi$. Moreover, if $p_\Phi < \infty$ and $|\Phi'| \in A^{\mathcal{R}}_{p'_\Phi}(\mathbb{R})$, then the Neumann problem in Ω is solvable in $L^{p_\Phi,1}(\Lambda)$ and the Dirichlet problem in Ω is solvable in $L^{p'_\Phi,1}(\Lambda)$.*

In particular, Corollary 5.3 gives the following results for the domains Ω considered in Sections 5.1 and 5.2.

- In the cone with aperture $\alpha\pi$ as in Figure 2 ($0 < \alpha < 1$), the Neumann problem and the Dirichlet problem are both solvable in $L^p(\Lambda)$ for $1 < p < \infty$.
- In the cone with aperture $\alpha\pi$ as in Figure 3 ($1 < \alpha < 2$), the Neumann problem and the Dirichlet problem are both solvable in $L^p(\Lambda)$ for $\alpha < p < \alpha/(\alpha-1)$. Moreover, the Neumann problem is solvable in $L^{\frac{\alpha}{\alpha-1},1}(\Lambda)$ and the Dirichlet problem is solvable in $L^{\alpha,1}(\Lambda)$.
- In the two-angle domain as in Figure 4 or in the infinite rotated stair case as in Figure 5, the Neumann problem and the Dirichlet problem are both solvable in $L^p(\Lambda)$ for $3/2 < p < 3$. Moreover, the Neumann problem is solvable in $L^{3,1}(\Lambda)$ and the Dirichlet problem is solvable in $L^{\frac{3}{2},1}(\Lambda)$.
- In a numerable polygon with double symmetry as in Figure 6, the Neumann problem and the Dirichlet problem are both solvable in $L^p(\Lambda)$ for $p'_\Phi < p < p_\Phi$ with $p'_\Phi = \max\{1, \alpha_1, \ldots, \alpha_n, -(n-2) + \sum_{k=1}^n \alpha_k\}$. Moreover, the Neumann problem is solvable in $L^{p_\Phi,1}(\Lambda)$ and the Dirichlet problem in Ω is solvable in $L^{p'_\Phi,1}(\Lambda)$.

APPENDIX A.

In this appendix, we state a number of known results used in the proofs of Theorems 1.4 and 1.5.

For the following statements, Ω, Λ, Φ and L are as defined in Section 1 (see (1.2)).

Lemma A.1 (Lemma 1.13 in Kenig [16]). *There exist* $0 < \alpha < \arctan(1/L)$ *and* $0 < \beta < \pi/2$, *which depend only on* L, *such that for almost every* $\xi \in \Lambda$ *with respect to* ds, $\Phi^{-1}(\Gamma_\alpha(\xi)) \subset \Gamma_\beta(\Phi^{-1}(\xi))$. *If* $L < 1$, *there exist* $0 < \alpha < \pi/2$ *and* $0 < \beta < \arctan(1/L)$ *such that for almost every* $x \in \mathbb{R}$ *with respect to Lebesgue measure,* $\Phi(\Gamma_\alpha(x)) \subset \Gamma_\beta(\Phi(x)) \subset \Omega$.

Definition A.2 (Definition 1.14 in Kenig [16]). Given $0 \leq \lambda \leq 1$, define Φ_λ such that $\Phi'_\lambda(z) = (\Phi'(z))^\lambda$. Note that Φ'_λ is never zero and $|\arg(\Phi'_\lambda(z))| \leq \lambda \arctan(L) < \pi/2$; then Φ_λ is one-to-one. We define $\Omega_\lambda = \Phi_\lambda(\mathbb{R}^2_+)$ and note that Ω_λ is a Lipschitz domain of the same type as Ω and Φ_λ is a conformal mapping from $\mathbb{R}^2_+$ onto Ω_λ that satisfies the same properties as Φ. Set $\Lambda_\lambda = \partial\Omega_\lambda$ and denote its arc length measure by ds_λ. Also observe that Λ_λ can be parametrized by $\xi_\lambda(x) = x + i\gamma_\lambda(x)$ with $\|\gamma'_\lambda\|_{L^\infty} \leq \tan(\lambda \arctan(L))$.

Lemma A.3 (Lemma 1.15 in Kenig [16]). *For* $0 \leq \lambda_1 \leq 1$ *and* $0 \leq \lambda_2 \leq 1$ *and using the notation introduced in Definition A.2, consider* $\sigma_{\lambda_1,\lambda_2} : \Omega_{\lambda_2} \to \Omega_{\lambda_1}$ *given by* $\sigma_{\lambda_1,\lambda_2} = \Phi_{\lambda_1} \circ (\Phi_{\lambda_2})^{-1}$. *There exist* $\varepsilon > 0$, $0 < \alpha < \arctan(1/L)$ *and* $0 < \beta < \arctan(1/L)$, *depending only on* L, *such that if* $\lambda_2 < \lambda_1$ *and* $\lambda_1 - \lambda_2 < \varepsilon$ *then for almost every* $z \in \Lambda_{\lambda_2}$ *with respect to* ds_{λ_2}, *it holds that*

$$\sigma_{\lambda_1,\lambda_2}(\Gamma_\alpha(z)) \subset \Gamma_\beta(\sigma_{\lambda_1,\lambda_2}(z)) \subset \Omega_{\lambda_1}.$$

Lemma A.4 (Lemma 2.3 in Kenig [16]). *If* $0 < \alpha < \arctan(1/L)$, $0 < \beta < \arctan(1/L)$ *and* $\nu \in A_\infty(\Lambda)$ *then*

$$\nu(\{\xi \in \Lambda : \mathcal{M}_\alpha h(\xi) > \lambda\}) \sim \nu(\{\xi \in \Lambda : \mathcal{M}_\beta h(\xi) > \lambda\}), \quad \forall \lambda > 0,$$

where the implicit constant depends only on ν, L, α *and* β.

Theorem A.5 (Theorem 2.8 in Kenig [16]). *Let* $0 < p < \infty$ *and* $\nu \in A_\infty(\Lambda)$. *Then* $h \in H^p(\Omega, \nu)$ *if and only if* $h \circ \Phi \in H^p(\mathbb{R}^2_+, \Phi(\nu))$, *with equivalence of norms.*

Definition A.6 (Definition 2.10 in Kenig [16]). Let $\nu \in A_\infty(\Lambda)$. The space $AE(\nu)$ is the class of functions h that are analytic and different from 0 on Ω, have a non-tangential limit almost everywhere in Λ with respect to ds that satisfies $|h(\xi)| = \nu(\xi)$ for almost every $\xi \in \Lambda$, and there exists $m \geq 0$ such that $(h \circ \Phi)(z)\Phi'(z)/(i + z)^m \in H^1(\mathbb{R}^2_+, dx)$.

Theorem A.7 (Theorem 2.13 in Kenig [16]). *Let* $0 < p < \infty$ *and* $\nu \in A_\infty(\Lambda)$. *If* h *is analytic in* Ω, *the following conditions are equivalent:*

(a) $h \in H^p(\Omega, \nu)$,
(b) $\sup_{t>0}(\int_\Lambda |h(\xi + it)|^p d\nu(\xi))^{1/p} < \infty$,
(c) for any $k \in AE(\nu)$, $h\, k^{1/p} \in H^p(\Omega, ds)$.

Moreover, the corresponding norms are equivalent.

For the next two results, consider curves Λ_1 and Λ_2 in the complex plane, given parametrically by $\xi_1(x) = x + i\gamma_1(x)$ and $\xi_2(x) = x + i\gamma_2(x)$ for $x \in \mathbb{R}$, respectively, where γ_1 and γ_2 are Lipchitz functions. Denote $\Omega_j = \{z \in \mathbb{C} : z = x + iy \text{ and } y > \gamma_j(x)\}$ for $j = 1, 2$. Consider a conformal mapping $\sigma_{1,2} : \Omega_2 \to \Omega_1$ such that $\sigma_{1,2}(\infty) = \infty$ (then $\sigma_{1,2}$ extends as a homeomorphism from $\overline{\Omega_2}$ onto $\overline{\Omega_1}$). If ν is a measure on Λ_1, then $\sigma_{1,2}(\nu)$ denotes the measure on Λ_2 defined by $\sigma_{1,2}(\nu)(U) = \nu(\sigma_{1,2}(U))$ for any measurable set $U \subset \Lambda_2$.

Lemma A.8 (Lemma 1.16 in Kenig [16]). *If* $\nu \in A_\infty(\Lambda_1)$, *then* $\sigma_{1,2}(\nu) \in A_\infty(\Lambda_2)$.

Definition A.9 (Definition 2.11 in Kenig [16])**.** Consider the corresponding conformal mappings $\Phi_1 : \mathbb{R}^2_+ \to \Omega_1$ and $\Phi_2 : \mathbb{R}^2_+ \to \Omega_2$, and let $\nu \in A_\infty(\Lambda_1)$ and $\mu \in A_\infty(\Lambda_2)$. The space $AE(\nu, \mu)$ is the class of functions h that are analytic and different from 0 on Ω_2, have a non-tangential limit almost everywhere in Λ_2 with respect to arc length measure ds_2 that satisfies $|h(\xi)| = (d\sigma_{1,2}(\nu)/d\mu)(\xi)$ for almost every $\xi \in \Lambda_2$, and there exists $m \geq 0$ and $k \in AE(\mu)$ such that $(h(\Phi_2(z))k(\Phi_2(z))\Phi_2'(z))/(i+z)^m \in H^1(\mathbb{R}^2_+, dx)$.

Theorem A.10 (Corollary 2.18 in Kenig [16])**.** *Let $0 < p < \infty$, $\nu \in A_\infty(\Lambda_1)$, $\mu \in A_\infty(\Lambda_2)$ and $k \in AE(\nu, \mu)$. Then $h \in H^p(\Omega_1, \nu)$ if and only if $(h \circ \sigma_{1,2})\, k^{1/p} \in H^p(\Omega_2, \mu)$, with equivalent norms.*

We end this appendix by stating two lemmas in the setting of $\mathbb{R}^2_+$.

Lemma A.11 (Corollary 4.3 in Garnett [11])**.** *Let $0 < p, r < \infty$ and $h \in H^p(\mathbb{R}^2_+, dx)$. If the non-tangential limit of h belongs to $L^r(\mathbb{R})$, then $h \in H^r(\mathbb{R}^2_+, dx)$.*

Lemma A.12 (Lemma I.10 in García-Cuerva [10])**.** *Let $1 < q < \infty$, $w \in A_q(\mathbb{R})$ and $s(x,t)$ be a non-negative subharmonic function on $\mathbb{R}^2_+$ which is in $L^q(\mathbb{R}, w)$ uniformly in $t > 0$. Then s has a least harmonic majorant which is the Poison integral of some function $s_0 \in L^q(\mathbb{R}, w)$. Moreover, s_0 is the pointwise limite of $s(\cdot, t)$ as $t \to 0$ if such limit exits.*

References

[1] E. Agora, M.J. Carro, and J. Soria. Characterization of the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces. *J. Fourier Anal. Appl.*, 19(4):712–730, 2013.

[2] D. Armitage. The Neumann problem for a function harmonic in $\mathbf{R}^n \times (0, \infty)$. *Arch. Rational Mech. Anal.*, 63(1):89–105, 1976.

[3] M.J. Carro and C. Ortiz-Caraballo. On the Dirichlet problem on Lorentz and Orlicz spaces with applications to Schwarz-Christoffel domains. *J. Differential Equations*, 265(5):2013–2033, 2018.

[4] H. Chung, R. Hunt, and D. Kurtz. The Hardy-Littlewood maximal function on $L(p,\, q)$ spaces with weights. *Indiana Univ. Math. J.*, 31(1):109–120, 1982.

[5] B. Dahlberg and C. Kenig. Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains. *Ann. of Math. (2)*, 125(3):437–465, 1987.

[6] M. de Guzmán. *Real variable methods in Fourier analysis*, volume 75 of *Notas de Matemática [Mathematical Notes]*. North-Holland Publishing Co., Amsterdam-New York, 1981.

[7] T. Driscoll and L. Trefethen. *Schwarz-Christoffel mapping*, volume 8 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.

[8] E. B. Fabes, M. Jodeit, Jr., and N. M. Rivière. Potential techniques for boundary value problems on C^1-domains. *Acta Math.*, 141(3-4):165–186, 1978.

[9] C. Fefferman and E. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.

[10] J. García-Cuerva. Weighted H^p spaces. *Dissertationes Math. (Rozprawy Mat.)*, 162:63, 1979.

[11] J. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[12] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[13] R. Hunt, B. Muckenhoupt, and R. Wheeden. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.*, 176:227–251, 1973.

[14] D. Jerison and C. Kenig. The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):203–207, 1981.

[15] C. Kenig. Weighted Hardy spaces on Lipschitz domains. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 263–274. Amer. Math. Soc., Providence, R.I., 1979.

[16] C. Kenig. Weighted H^p spaces on Lipschitz domains. *Amer. J. Math.*, 102(1):129–163, 1980.

[17] C. Kenig. Elliptic boundary value problems on Lipschitz domains. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 131–183. Princeton Univ. Press, Princeton, NJ, 1986.
[18] R. Kerman and A. Torchinsky. Integral inequalities with weights for the Hardy maximal function. *Studia Math.*, 71(3):277–284, 1981/82.
[19] I. Kolesnikov and L. Kopaneva. Conformal mapping onto a numerable polygon with double symmetry. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (12):37–47, 2014.
[20] A. Lerner. On pointwise estimates involving sparse operators. *New York J. Math.*, 22:341–349, 2016.
[21] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.
[22] G. Riera, H. Carrasco, and R. Preiss. The Schwarz-Christoffel conformal mapping for "polygons" with infinitely many sides. *Int. J. Math. Math. Sci.*, pages Art. ID 350326, 20, 2008.
[23] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

MARÍA JESÚS CARRO, DEPARTMENT OF ANALYSIS AND APPLIED MATHEMATICS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE LAS CIENCIAS 3, 28040 MADRID, SPAIN.
Email address: mjcarro@ucm.es

VIRGINIA NAIBO, DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY. 138 CARDWELL HALL, 1228 N. 17TH STREET, MANHATTAN, KS 66506, USA.
Email address: vnaibo@ksu.edu

CARMEN ORTIZ-CARABALLO, DEPARTMENT OF MATHEMATICS, UNIVERSIDAD DE EXTREMADURA, ESCUELA POLITÉCNICA, AVDA. UNIVERSIDAD, S/N, 10003 CÁCERES, SPAIN
Email address: carortiz@unex.es