

# Quasi-exactly solvable potentials on the line and orthogonal polynomials

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In this paper we show that a quasi-exactly solvable (normalizable or periodic) one-dimensional Hamiltonian satisfying very mild conditions defines a family of weakly orthogonal polynomials which obey a three-term recursion relation. In particular, we prove that (normalizable) exactly solvable one-dimensional systems are characterized by the fact that their associated polynomials satisfy a two-term recursion relation. We study the properties of the family of weakly orthogonal polynomials defined by an arbitrary one-dimensional quasi-exactly solvable Hamiltonian, showing in particular that its associated Stieltjes measure is supported on a finite set. From this we deduce that the corresponding moment problem is determined, and that the  $k$ th moment grows like the  $k$ th power of a constant as  $k$  tends to infinity. We also show that the moments satisfy a constant coefficient linear difference equation, and that this property actually characterizes weakly orthogonal polynomial systems. © 1996 American Institute of Physics. [S0022-2488(96)03108-8]

## I. INTRODUCTION

In a recent paper,<sup>1</sup> Bender and Dunne introduced a remarkable family of orthogonal polynomials associated to the one-dimensional Hamiltonian

$$H = -\partial_x^2 + \frac{(4s-1)(4s-3)}{4x^2} - (4s+4J-2)x^2 + x^6, \quad (1)$$

where  $J$  is a positive integer and  $s$  is a real parameter. If  $\psi_E(x)$  denotes an eigenfunction of  $H$  with energy  $E$ , the polynomials  $P_k(E)$  in question are proportional to the coefficients in the expansion of  $e^{x^4/4} x^{1/2-2s} \psi_E(x)$  in powers of  $x^2$ , namely

$$\psi_E(x) = e^{-x^4/4} x^{2s-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{P_k(E)}{\Gamma(k+2s)} \left(\frac{x^2}{4}\right)^k.$$

These polynomials are easily shown to satisfy a three-term recursion relation, from which it follows<sup>2</sup> that they are orthogonal with respect to a certain Stieltjes measure  $d\omega(E)$  ( $\omega$  being a function of bounded variation):

$$\int P_k(E) P_l(E) d\omega(E) = 0, \quad k \neq l. \quad (2)$$

The form of the coefficients of the recursion relation satisfied by the polynomial system  $\{P_k(E)\}_{k=0}^{\infty}$  implies that this system has several remarkable properties. First of all, the norm of the polynomials  $P_k$  with  $k \geq J$  vanishes. Thus the polynomials  $P_k$  form what is called a *weakly orthogonal polynomial system*.<sup>2</sup> To be precise, we shall use from now on the term *orthogonal polynomial system* for a family of orthogonal polynomials  $\{P_k(E)\}_{k=0}^{\infty}$  with  $\deg P_k = k$  for all  $k$ , and such that the norm of  $P_k$  does not vanish for any  $k$ .

Second, each  $P_k$  with  $k \geq J$  factors into the product of  $P_J$  and another polynomial, i.e.,

$$P_{J+m} = P_J Q_m, \quad m \geq 0,$$

where  $Q_m$  has degree  $m$ . Finally, the  $J$  simple real zeros of  $P_J$  are eigenvalues of  $H$  whose corresponding eigenfunctions, being the product of the factor

$$\mu(x) = e^{-x^{4/4} x^{2s-1/2}}$$

times a polynomial in  $x^2$ , are square integrable. The existence of these exactly computable eigenfunctions and eigenvalues of  $H$  had been deduced before<sup>3,4</sup> from the fact that  $H$  is *quasi-exactly solvable*, meaning that it is an element of the enveloping algebra of a certain realization of  $\mathfrak{sl}(2, \mathbb{R})$  in terms of first-order differential operators acting on a finite-dimensional subspace of the space of  $C^\infty$  functions (see the next section for more details). The above results strongly suggest<sup>5</sup> that there is a connection between quasi-exactly solvable Hamiltonians and certain families of weakly orthogonal polynomials. In this paper, we show in detail that this is indeed the case for all one-dimensional quasi-exactly solvable Hamiltonians, both normalizable and periodic, satisfying very general conditions.

The paper is organized as follows. Using the results on quasi-exact solvability reviewed in Sec. II, we explain in Sec. III how to construct the weakly orthogonal polynomial system associated to each of the normal forms of a one-dimensional quasi-exactly solvable Hamiltonian listed in Refs. 4 and 6. Like the polynomial system introduced in Ref. 1, this system always satisfies a three-term recursion relation, whose coefficients we explicitly compute. This allows us to prove that one-dimensional (normalizable) exactly solvable Hamiltonians are characterized by the fact that their associated polynomials satisfy a two-term recursion relation. In Sec. IV we show that the polynomials associated to an arbitrary one-dimensional quasi-exactly solvable Hamiltonian enjoy properties completely akin to those listed above for the Hamiltonian (1). We also study in this section the properties of the moment functional defined by the family of weakly orthogonal polynomials of a quasi-exactly solvable Hamiltonian, giving a rigorous proof of the fact that its associated Stieltjes measure is supported on a finite set,<sup>5</sup> so that the integral (2) reduces to a finite sum. From this we deduce that the associated (Hamburger or Stieltjes) moment problem is determined, and that the  $k$ th moment behaves like the  $k$ th power of a constant for large  $k$ , illustrating this statement with an explicit example for the Hamiltonian (1). We also show that the moments satisfy a constant coefficient linear difference equation, a property which in fact characterizes weakly orthogonal polynomial systems. The paper ends (Sec. V) with a brief review of these results, stressing the role played by weak orthogonality—as opposed to true orthogonality—in their derivation.

## II. QUASI-EXACTLY SOLVABLE POTENTIALS

For the reader's convenience, we present in this section a summary of the major results in the theory of quasi-exactly solvable systems that we shall need in the sequel. A one-dimensional Schrödinger operator (or Hamiltonian)  $H = -\partial_x^2 + V(x)$  is *quasi-exactly solvable* if there exists a finite-dimensional Lie algebra of first-order differential operators

$$\mathfrak{g} = \text{Span}\{\xi_a(x)\partial_x + \eta_a(x) \mid 1 \leq a \leq r\} = \text{Span}\{T_a(x) \mid 1 \leq a \leq r\}$$

such that

- (i)  $\mathfrak{g}$  leaves invariant a finite-dimensional module of smooth functions  $\mathfrak{N} \subset C^\infty(\mathbb{R})$ , i.e.,  $X \cdot f \in \mathfrak{N}$  for all  $f \in \mathfrak{N}$  and all  $X \in \mathfrak{g}$ . In other words,  $\mathfrak{g}$  admits a finite-dimensional representation in terms of smooth functions.
- (ii)  $H$  is in the universal enveloping algebra of  $\mathfrak{g}$ , i.e.,  $H$  can be expressed as a polynomial in the generators  $T_a$ ,  $1 \leq a \leq r$ , of  $\mathfrak{g}$ .

A Lie algebra of first-order differential operators satisfying (i) is called *quasi-exactly solvable*. A Hamiltonian  $H$  satisfying condition (ii) above for an arbitrary (not necessarily quasi-exactly solvable) Lie algebra  $\mathfrak{g}$  is said to be *Lie algebraic*.

If  $H$  is quasi-exactly solvable, it follows that the restriction of  $H$  to  $\mathfrak{N}$  is a finite-dimensional linear operator  $\mathfrak{N} \rightarrow \mathfrak{N}$ , and therefore the eigenfunctions of  $H$  lying in  $\mathfrak{N}$  and their corresponding eigenvalues can be exactly computed by purely algebraic methods (diagonalizing a square matrix of order  $\dim \mathfrak{N}$ ). We shall refer to these eigenfunctions of  $H$  lying in  $\mathfrak{N}$  as its *algebraic eigenfunctions* (although, of course, they need not be algebraic functions in the technical sense of the word). The functions in  $\mathfrak{N}$  need not *a priori* satisfy any boundary conditions (like square-integrability, periodicity, vanishing at the end points, etc.) coming from the physics of the problem, whose mathematical purpose is to guarantee that  $H$  is a self-adjoint operator. If they do, then the restriction of  $H$  to  $\mathfrak{N}$  is self-adjoint, and therefore  $H$  has exactly  $\dim \mathfrak{N}$  linearly independent algebraic eigenfunctions, whose corresponding  $\dim \mathfrak{N}$  *real* eigenvalues (counting multiplicities) are exactly (i.e., algebraically) computable. We shall say in this case that the quasi-exactly solvable potential  $H$  (or the potential  $V$ ) is *fully algebraic*. See Refs. 4 and 6 for an in-depth discussion of fully algebraic potentials under the boundary condition of square integrability on  $\mathbb{R}$ .

It can be shown (cf. Ref. 7) that a quasi-exactly solvable Schrödinger operator  $H$  can be expressed as a polynomial of degree at most two in the generators  $T_a$ ,  $1 \leq a \leq r$ , of  $\mathfrak{g}$ . Moreover, a well known theorem<sup>3,8,7</sup> asserts that every quasi-exactly solvable Lie algebra of first-order differential operators  $\mathfrak{g}$  is related by a (local) change of variable

$$z = \zeta(x) \quad (3)$$

and a *gauge transformation* with gauge factor  $\mu(z) > 0$  to (a subalgebra of) one of the Lie algebras  $\mathfrak{g}^n = \mathfrak{h}^n \oplus \mathbb{R}$ , where  $\mathfrak{h}^n = \text{Span}\{J_-^n, J_0^n, J_+^n\} \approx \mathfrak{sl}(2, \mathbb{R})$ ,

$$J_-^n = \partial_z, \quad J_0^n = z \partial_z - \frac{n}{2}, \quad J_+^n = z^2 \partial_z - n z, \quad (4)$$

and  $n$  is a nonnegative integer. In other words, every element  $X(x) \in \mathfrak{g}$  is of the form

$$X(x) = \mu(z) \cdot J(z) \cdot \frac{1}{\mu(z)} \Big|_{z=\zeta(x)}, \quad J(z) \in \mathfrak{g}^n,$$

for some fixed  $n$ . This implies that the *gauge Hamiltonian*

$$H_{\text{gauge}}(z) = \frac{1}{\mu(z)} \cdot H(x) \cdot \mu(z) \Big|_{x=\zeta^{-1}(z)} \quad (5)$$

is also a polynomial of degree at most two in the generators  $J_\epsilon^n$ , i.e., (dropping the explicit  $n$  dependence in the generators  $J_\epsilon^n$ )

$$-H_{\text{gauge}} = \sum_{a,b} c_{ab} J_a J_b + \sum_a c_a J_a + c_*, \quad (6)$$

for some real constants  $c_*$ ,  $c_a$ , and  $c_{ab} = c_{ba}$  (the minus sign is for later convenience). The spectral problems of  $H$  and  $H_{\text{gauge}}$  are related in an obvious way: indeed, from (5) it follows that if  $\chi(z)$  is an eigenfunction of  $H_{\text{gauge}}$  with eigenvalue  $E$  then

$$\psi(x) = \mu(z) \chi(z) \Big|_{z=\zeta(x)} \quad (7)$$

will be an eigenfunction of  $H$  with the same eigenvalue (not taking into account the boundary conditions). Since the Lie algebra  $\mathfrak{g}^n$  admits as invariant module the space  $\mathcal{P}_n$  of real polynomials of degree at most  $n$  in  $z$ , if  $H$  is fully algebraic then  $H_{\text{gauge}}$  has  $n+1$  linearly independent algebraic eigenfunctions lying in  $\mathcal{P}_n$ . Hence  $H$  has  $n+1$  linearly independent algebraic eigenfunctions of the form (7), with  $\chi \in \mathcal{P}_n$  a polynomial of degree at most  $n$ .

From (4) and (6) it follows<sup>4</sup> that the gauge Hamiltonian is of the form

$$-H_{\text{gauge}} = P(z) \partial_z^2 + \left\{ Q(z) - \frac{n-1}{2} P'(z) \right\} \partial_z + \left\{ R - \frac{n}{2} Q'(z) + \frac{n(n-1)}{12} P''(z) \right\}, \quad (8)$$

where  $P$ ,  $Q$ , and  $R$  are polynomials of degrees 4, 2, and 0, respectively, given by

$$P(z) = c_{++} z^4 + 2c_{+0} z^3 + c_{00} z^2 + 2c_{0-} z + c_{--}, \quad (9)$$

$$Q(z) = c_{+} z^2 + c_0 z + c_{-}, \quad (10)$$

$$R = \frac{n(n+2)}{12} c_{00} + c_{*}. \quad (11)$$

Note that, due to the Casimir relation

$$J_0^2 - \frac{1}{2} (J_{+} J_{-} + J_{-} J_{+}) = \frac{n}{4} (n+2),$$

we have set, without loss of generality,  $c_{+-} = 0$ . There are also explicit formulas for the change of variables (3) and gauge factor  $\mu(z)$  needed to put the differential operator (8) in Schrödinger form, cf. Ref. 4. Indeed, assuming that  $P(z) > 0$  on an interval  $I$  then for  $z \in I$  we have

$$x = \zeta^{-1}(z) = \int^z \frac{dy}{\sqrt{P(y)}}, \quad \mu(z) = P(z)^{-n/4} \exp \left\{ \int^z \frac{Q(y)}{2P(y)} dy \right\} \quad (12)$$

and

$$V(x) = -R + \frac{-n(n+2)(PP'' - \frac{3}{4}P'^2) - 3(n+1)(QP' - 2PQ') + 3Q^2}{12P} \Big|_{z=\zeta(x)}, \quad (13)$$

where the primes denote derivatives with respect to  $z$ .

The canonical form (8) of the quasi-exactly solvable Hamiltonian  $H$  is not unique, since there is a residual symmetry group preserving the Lie algebra  $\mathfrak{h}^n$ , given by the adjoint action on  $\mathfrak{h}^n$  of the Lie group of transformations generated by  $\mathfrak{g}^n = \mathfrak{h}^n \oplus \mathbb{R}$ . More precisely, the elements of  $\mathfrak{g}^n$  are the infinitesimal generators of the standard  $GL(2, \mathbb{R})$  action on the space  $\mathcal{P}_n$ , given by

$$p(z) \in \mathcal{P}_n \mapsto \hat{p}(w) = (\gamma w + \delta)^n p \left( \frac{\alpha w + \beta}{\gamma w + \delta} \right), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{R}). \quad (14)$$

We shall denote, as is customary, by  $\rho_n$  this (irreducible) multiplier representation of  $GL(2, \mathbb{R})$  on  $\mathcal{P}_n$ . Note that the action (14) is just the composition of the projective transformation

$$z = \frac{\alpha w + \beta}{\gamma w + \delta}$$

and the gauge transformation with gauge factor  $\mu(w) = (\gamma w + \delta)^n$ . The adjoint action of  $GL(2, \mathbb{R})$  on  $\mathfrak{h}^n$  induced by (14) is given by

$$J(z) \mapsto \hat{J}(w) = (\gamma w + \delta)^n \cdot J\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right) \cdot (\gamma w + \delta)^{-n}. \quad (15)$$

A straightforward calculation<sup>4</sup> shows that the generators of  $\mathfrak{h}^n$  transform under the representation  $\rho_{2,-1}$ —where  $\rho_{n,i} = \rho_n \otimes \det^i$ ,  $\det: A \mapsto \det A$  being the standard determinantal representation— independently of  $n$ . As a consequence of all this, the transformed differential operator

$$\hat{H}_{\text{gauge}} = (\gamma w + \delta)^n \cdot H_{\text{gauge}}\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right) \cdot (\gamma w + \delta)^{-n} \quad (16)$$

is still of the form (8), with  $P$ ,  $Q$ , and  $R$  replaced by appropriate polynomials  $\hat{P}$ ,  $\hat{Q}$ , and  $\hat{R}$  of respective degrees 4, 2, and 0. It can be shown, cf. Ref. 4, that  $\hat{R} = R$  and

$$\hat{P}(w) = \frac{(\gamma w + \delta)^4}{\Delta^2} P\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right), \quad \hat{Q}(w) = \frac{(\gamma w + \delta)^2}{\Delta} Q\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right), \quad (17)$$

with

$$\Delta = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Hence the polynomials  $P$ ,  $Q$ , and  $R$  determining the differential operator  $H_{\text{gauge}}$  transform under the representations  $\rho_{4,-2}$ ,  $\rho_{2,-1}$ , and  $\rho_0$  of  $GL(2, \mathbb{R})$ . Furthermore, the algebraic eigenfunctions of  $H_{\text{gauge}}$  clearly transform under the representation  $\rho_n$ ; indeed, if  $\chi(z)$  is an eigenfunction of  $H_{\text{gauge}}$  with eigenvalue  $E$  then it follows from (16) that

$$\hat{\chi}(w) = (\gamma w + \delta)^n \cdot \chi\left(\frac{\alpha w + \beta}{\gamma w + \delta}\right) \quad (18)$$

is an eigenfunction of  $\hat{H}_{\text{gauge}}$  with the same eigenvalue.

In Refs. 6 and 4, the form invariance of the differential operator  $H_{\text{gauge}}$  under the  $GL(2, \mathbb{R})$  action (16) described above was exploited to place  $H_{\text{gauge}}$  in canonical form. Indeed, it can be shown that there are ten inequivalent real normal forms for a (nonzero) fourth-degree polynomial  $P$  (Ref. 9) transforming under the representation  $\rho_{4,-2}$  of  $GL(2, \mathbb{R})$ , each of which leads to a canonical form for  $H_{\text{gauge}}$ . Of these ten canonical forms, five correspond to *normalizable* Hamiltonians, whose algebraic eigenfunctions are square integrable (provided the coefficients  $c_{ab}$  and  $c_a$  satisfy certain inequalities), and the remaining are associated to Hamiltonians with periodic potentials. The five normal forms associated to normalizable Hamiltonians, which are characterized by the fact that  $P$  has at least one multiple root on the real projective line  $\mathbb{RP}$ , are given by

1.  $\nu(z^2 + 1)$ ,
  2.  $\nu(z^2 - 1)$ ,
  3.  $\nu z^2$ ,
  4.  $z$ ,
  5.  $1$ ,
- (19)

where  $\nu > 0$  is a real parameter. For example, the quasi-exactly solvable potential discussed in Ref. 1 corresponds to the fourth normalizable canonical form  $P(z) = z$ . The remaining normal forms, corresponding to periodic potentials, are

$$\begin{aligned} 6. & \quad \nu(1-z^2)(1-\kappa^2 z^2), \\ 7. & \quad \nu(1-z^2)(1-\kappa^2(1-z^2)), \\ 8. & \quad \nu(1+z^2)(1+(1-\kappa^2)z^2), \\ 9. & \quad \nu(1+z^2)^2, \\ 10. & \quad \nu(1-z^2), \end{aligned} \tag{20}$$

where  $\nu > 0$ ,  $0 < \kappa < 1$ .

### III. THE RECURSION RELATION

Let  $H = -\partial_x^2 + V(x)$  be a quasi-exactly solvable Hamiltonian. From the previous section, we know that there is a change of variable (3) and gauge factor  $\mu(z) > 0$  such that  $H(x) = \mu(z) \cdot H_{\text{gauge}}(z) \cdot [1/\mu(z)]|_{z=\zeta(x)}$ , with  $H_{\text{gauge}}$  given by (6) (and  $c_{+-} = 0$ ). Furthermore, if  $H$  is fully algebraic then it has  $n+1$  algebraic eigenfunctions of the form (7), with  $\chi(z) \in \mathcal{P}_n$  an eigenfunction of  $H_{\text{gauge}}$ . Let  $\chi_E(z)$  be an eigenfunction of  $H_{\text{gauge}}$  with eigenvalue  $E$  (not necessarily a polynomial in  $z$ ). Writing

$$\chi_E(z) = \sum_{k=0}^{\infty} P_k(E) \chi_k(z), \tag{21}$$

where

$$\chi_k(z) = \frac{z^k}{k!}, \quad k \geq 0,$$

and taking into account that

$$J_- \cdot \chi_k = \chi_{k-1}, \quad J_0 \cdot \chi_k = \left(k - \frac{n}{2}\right) \chi_k, \quad J_+ \cdot \chi_k = (k-n)(k+1) \chi_{k+1}, \tag{22}$$

cf. (4), we easily find that the coefficients  $P_k(E)$  satisfy the following five-term recursion relation:

$$\begin{aligned} -c_{--} P_{k+2} &= [(2k-n+1)c_{0-} + c_-] P_{k+1} + \left[ E + c_* + c_0 \left(k - \frac{n}{2}\right) + c_{00} \left(k - \frac{n}{2}\right)^2 \right] P_k + k(k-1-n) \\ &\quad \times [(2k-n-1)c_{+0} + c_+] P_{k-1} + k(k-1)(k-1-n)(k-2-n)c_{++} P_{k-2}, \quad k \geq 0. \end{aligned} \tag{23}$$

If  $c_{--} \neq 0$ , the general solution of the recursion relation (23) depends on the two arbitrary functions  $P_0(E)$  and  $P_1(E)$ . This simply reflects the fact that when  $c_{--} \neq 0$  the leading coefficient  $P(z)$  of  $H_{\text{gauge}}$  does not vanish at  $z=0$  [cf. (9)]; thus, the differential equation  $(H_{\text{gauge}} - E)\chi_E = 0$  has a regular point at the origin, and therefore it admits two linearly independent solutions (21) analytic at 0. If  $P_0(E)$  and  $P_1(E)$  are chosen to be polynomials in  $E$ , then (23) implies that all the coefficients  $P_k(E)$  are polynomials in  $E$ . However, the general recursion relation (23) suffers from two major drawbacks. In the first place, even if we choose  $P_0(E)$  and  $P_1(E)$  as polynomials of

degree 0 and 1 in  $E$ , respectively, (23) is incompatible with the desirable property that  $P_k(E)$  be of degree  $k$  in  $E$  for all  $k$ , unless  $c_{--}=0$ . Second, even in this case (23) will be in general a four-term recursion relation, implying that the polynomials  $P_k(E)$  may not be orthogonal with respect to any (nonzero) Stieltjes measure  $d\omega(E)$ . Indeed, it is well known,<sup>2,10</sup> that a necessary and sufficient condition for a family of polynomials  $\{P_k\}_{k=0}^\infty$  (with  $\deg P_k=k$ ) to form an orthogonal polynomial system is that  $P_k(E)$  satisfies a *three-term* recursion relation of the form

$$P_k = (A_k E + B_k) P_{k-1} + C_k P_{k-2}, \quad k \geq 1, \quad (24)$$

where the coefficients  $A_k, B_k, C_k$  are *independent* of  $E$ ,  $A_k \neq 0$ ,  $C_1=0$ , and  $C_k \neq 0$  for  $k > 1$ . If the coefficient  $C_k$  in (24) vanishes for some positive integer  $k$ , then this recursion relation only defines a *weakly* orthogonal polynomial system.<sup>11</sup> It is one of the main goals of this paper to show that both difficulties described above can always be overcome, provided (roughly speaking) that we expand the eigenfunction  $\chi_E$  with respect to an appropriate variable. This will be achieved by using the nonuniqueness of  $H_{\text{gauge}}$ , due to the  $GL(2)$  symmetry described in the previous section, to place  $H_{\text{gauge}}$  in a suitable canonical form.

From the form of the recursion relation (23), it follows that both difficulties described above disappear if

$$c_{--} = c_{++} = 0. \quad (25)$$

Indeed, if (25) holds then (23) reduces to the three-term recursion relation

$$\begin{aligned} -[(2k-n-1)c_{0-} + c_-]P_k &= \left[ E + c_* + c_0 \left( k - \frac{n}{2} - 1 \right) + c_{00} \left( k - \frac{n}{2} - 1 \right)^2 \right] P_{k-1} + (k-1) \\ &\quad \times (k-2-n)[(2k-n-3)c_{+0} + c_+]P_{k-2}, \quad k \geq 1, \end{aligned} \quad (26)$$

which uniquely determines all the functions  $P_k(E)$  in terms of  $P_0(E)$  provided that, for all positive integer values of  $k$ , the coefficient of the left-hand side of (26) does not vanish. If  $P_0(E)$  is taken as a constant, for instance if  $P_0(E)=1$ , then (26) implies that  $P_k(E)$  is a polynomial of degree  $k$  in  $E$  for all  $k \geq 0$ .

Let us see now that we can always arrange for (25) to be satisfied, by using the action (17) to transform  $P(z)$  into a normal form  $\hat{P}(w)$  for which (25) holds. Indeed, (25) simply states that the polynomial  $P(z)$  vanishes at  $z=0$  and  $z=\infty$ , when  $z$  is allowed to vary over the complex projective line  $\mathbb{CP}$ . Note that we need  $z$  to belong to the *complex* projective line at this stage so that  $P$  is guaranteed to have a root, which is essential for the argument that follows. Consequently, the  $GL(2, \mathbb{R})$  action described in the previous section will be replaced in what follows by a  $GL(2, \mathbb{C})$  action.

We can assume, first of all, that  $P(z)$  is one of the normal forms listed in Eqs. (19) and (20). We must distinguish three cases, characterized by the position of the roots of  $P$  in the complex projective line. Indeed, either  $P$  has two different roots  $z_1 \neq z_2$  in  $\mathbb{CP}$ , or it has four coincident roots. In the first case, either one of the roots is at infinity, or both roots are finite.

### A. Case 1: $P$ has two different roots $z_1 \neq z_2 = \infty$

This case occurs when  $P$  is one of the first four normalizable canonical forms (19), or the fifth periodic canonical form (20). In this case, the translation  $w = z - z_1$  transforms  $P(z)$  into a polynomial  $\hat{P}(w)$  vanishing at zero and infinity. In the original  $z$  coordinate, by (18) this amounts to replacing (21) by

$$\chi_E(z) = \sum_{k=0}^{\infty} P_k(E) \frac{(z-z_1)^k}{k!}. \quad (27)$$

In other words, we expand  $\chi_E(z)$  as a power series around the point  $z=z_1$ , which is a singular point of the linear differential equation  $(H_{\text{gauge}}-E)\chi_E=0$  (if  $z_1$  is a simple root of  $P$ ,  $z_1$  is actually a regular singular point, whose indicial equation is easily seen to have 0 as a root). By (7), in the “physical” coordinate  $x$  (27) becomes

$$\psi_E(x) = \mu(\zeta(x)) \sum_{k=0}^{\infty} P_k(E) \frac{(\zeta(x)-z_1)^k}{k!} \quad (28)$$

### B. Case 2: $P$ has two different finite roots $z_1 \neq z_2$

This is the case when  $P$  is one of the first four periodic normal forms (20). The projective transformation  $w=(z-z_1)/(z-z_2)$  will again transform  $P(z)$  into a polynomial  $\hat{P}(w)$  vanishing at  $w=0, \infty$ . Going back to the original  $z$  coordinate, by (18) we just have to replace (21) by

$$\chi_E(z) = (z-z_2)^n \sum_{k=0}^{\infty} \frac{1}{k!} P_k(E) \left( \frac{z-z_1}{z-z_2} \right)^k, \quad (29)$$

apart from an inessential overall factor. In terms of the physical coordinate  $x$ , (29) can be written as

$$\psi_E(x) = \mu(\zeta(x)) (\zeta(x)-z_2)^n \sum_{k=0}^{\infty} \frac{1}{k!} P_k(E) \left( \frac{\zeta(x)-z_1}{\zeta(x)-z_2} \right)^k. \quad (30)$$

### C. Case 3: $P$ has a quadruple root

This corresponds to the fifth normalizable canonical form,  $P=1$ , which has a quadruple root at infinity. Note that  $P=1$  implies that the physical coordinate  $x$  can be taken as the canonical coordinate  $z$ . By (9), we have

$$c_{++}=c_{+0}=c_{00}=c_{0-}=0, \quad c_{--}=1.$$

Performing an additional translation, if necessary, we can also take without loss of generality  $c_{-}=Q(0)=0$  (notice that  $P$  is constant, and therefore does not change under translations). Thus Eq. (23) reduces in this case to

$$-P_{k+2} = \left[ E + c_* + c_0 \left( k - \frac{n}{2} \right) \right] P_k + k(k-1-n)c_+P_{k-1}, \quad k \geq 0. \quad (31)$$

Since  $P=1$  is the fifth normalizable case of Refs. 4 and 6,  $c_+$  must vanish if we want  $H$  to be *normalizable*, i.e., the algebraic eigenfunctions of  $H$  to be square integrable. Therefore, in this case (31) reduces to

$$-P_{k+2} = \left[ E + c_* + c_0 \left( k - \frac{n}{2} \right) \right] P_k, \quad k \geq 0,$$

which is equivalent to two two-term recursion relations for the even and odd coefficients  $P_j^0 = P_{2j}$  and  $P_j^1 = P_{2j+1}$ , namely

$$-P_{j+1}^\epsilon = \left[ E + c_* + c_0 \left( 2j + \epsilon - \frac{n}{2} \right) \right] P_j^\epsilon, \quad j \geq 0; \quad \epsilon = 0, 1. \quad (32)$$

Note that in this case the potential is  $V(x) = 1/4c_0^2x^2 - c_*$  (with  $c_0 < 0$ ), cf. Ref. 4.



To complete the discussion of cases 1 and 2, we still have to deal with an important technical issue; namely, we must find under what conditions the coefficient of  $P_k$  in (26) never vanishes for positive integer values of  $k$ . Let  $\hat{P}$  and  $\hat{Q}$  be the transforms of  $P$  and  $Q$  under the projective transformation  $z \mapsto w$  defined in the foregoing discussion of cases 1 and 2; note that, by construction,  $\hat{P}(w)$  vanishes at  $w=0, \infty$ . The coefficient of interest can be expressed as

$$(2k-n-1)\hat{c}_{0-} + \hat{c}_-, \quad k \geq 1, \quad (33)$$

where

$$\hat{c}_{0-} = \frac{1}{2}\hat{P}'(0), \quad \hat{c}_- = \hat{Q}(0).$$

From (17) it easily follows that

$$\hat{c}_{0-} = \frac{1}{2}P'(z_1), \quad \hat{c}_- = Q(z_1), \quad (34)$$

for case 1 ( $w=z-z_1$ ), and

$$\hat{c}_{0-} = \frac{P'(z_1)}{2(z_1-z_2)}, \quad \hat{c}_- = \frac{Q(z_1)}{z_1-z_2}, \quad (35)$$

for case 2 ( $w=(z-z_1)/(z-z_2)$ ). We shall now distinguish three subcases:

### 1. Case i. $z_1$ is a simple real root of $P$

This case occurs when  $P$  is one of the canonical forms 2, 4, 6, 7, or 10. Note that in this case the mapping  $z \mapsto w$  is real, and so are the coefficients  $\hat{c}_{0-}$ ,  $\hat{c}_-$ . From (12) and (34)–(35) it is immediate to deduce the asymptotic formulas

$$x \underset{z \rightarrow z_1}{\sim} |z-z_1|^{1/2}, \quad \mu(z) \underset{z \rightarrow z_1}{\sim} |z-z_1|^{1/4(\hat{c}_-/\hat{c}_{0-}-n)},$$

where we have dropped unessential constant multiplicative factors from the right-hand side, and have taken for convenience  $z_1$  as the lower limit of the integral giving  $x$  in terms of  $z$ . We saw in the previous section that when  $H$  is fully algebraic it has  $n+1$  linearly independent algebraic eigenfunctions of the form (7), where  $\chi \in \mathcal{P}_n$ . It follows that the polynomial factor  $\chi(z)$  cannot vanish at the origin for all the algebraic eigenfunctions of  $H$ . Hence there is at least one algebraic eigenfunction of  $H$  whose asymptotic behavior at  $x=0$  is given by

$$\psi(x) \underset{x \rightarrow 0}{\sim} |x|^{1/2(\hat{c}_-/\hat{c}_{0-}-n)}.$$

If all the algebraic eigenfunctions of  $H$  are regular at  $x=0$ , then we must have

$$\hat{c}_-/\hat{c}_{0-}-n \geq 0. \quad (36)$$

Since (33) can be written as

$$2\hat{c}_{0-} \left[ \frac{1}{2} \left( \frac{\hat{c}_-}{\hat{c}_{0-}} - n \right) + \left( k - \frac{1}{2} \right) \right],$$

it follows from (36) that the coefficient (33) cannot vanish in this case.

## 2. Case ii. $z_1$ is a simple complex root of $P$

In this case  $P$  is either the first or the eighth canonical form. Since  $z_1$  is not real, the mapping  $z \mapsto w$  is not real either, and the above asymptotic argument is not valid (the eigenfunctions of  $H$  need not be regular outside the real axis). For the first canonical form (19), we can take  $w = z - i$  and therefore

$$\hat{c}_{0-} = i\nu, \quad \hat{c}_- = c_- - c_+ + ic_0$$

from (34). Hence the coefficient (33) does not vanish in this case provided that the following conditions are satisfied:

$$c_- \neq c_+ \quad \text{or} \quad \frac{1}{2} \left( n+1 - \frac{c_0}{\nu} \right) \neq 1, 2, \dots \quad (37)$$

It is easily checked that the choice  $w = z + i$  leads exactly to the same conditions. For the eighth canonical form, we can take  $w = (z - i)/(z + i)$ , and therefore, from (35),

$$\hat{c}_{0-} = \frac{1}{2} \nu \kappa^2, \quad \hat{c}_- = \frac{c_0}{2} + \frac{i}{2} (c_+ - c_-).$$

Hence in this case the conditions for the coefficient (33) not to vanish are

$$c_- \neq c_+ \quad \text{or} \quad \frac{1}{2} \left( n+1 - \frac{c_0}{\nu \kappa^2} \right) \neq 1, 2, \dots \quad (38)$$

It is straightforward to check that the choice  $w = (z + i)/(z - i)$  yields the same conditions, while the other natural choice  $w = (\sqrt{1 - \kappa^2} z \mp i)/(\sqrt{1 - \kappa^2} z \pm i)$  only has the effect of replacing the first condition (38) by  $c_+ \neq (1 - \kappa^2)c_-$ .

## 3. Case iii. $z_1$ is a multiple root of $P$

This case takes place when  $P$  is either the third or the ninth canonical form, and in both cases (33) reduces to  $\hat{c}_-$ . For the third canonical form (19), if  $c_- \neq 0$  then we take  $w = z$ , and therefore  $\hat{c}_- = c_- \neq 0$ . If  $c_- = 0$ , then  $c_+ \neq 0$  if all the algebraic eigenfunctions of  $H$  are square integrable (see Ref. 4). Hence, taking  $w = 1/z$ , we get  $\hat{P}(w) = \nu w^2$  and  $\hat{Q} = -(c_+ + c_0 w)$ , so that  $\hat{c}_{0-} = 0$  and  $\hat{c}_- = -c_+ \neq 0$ . Hence the coefficient (33) cannot vanish in this case. Finally, if  $P$  is the ninth canonical form (20) then  $w = (z - i)/(z + i)$  and

$$\hat{c}_- = \frac{c_0}{2} + \frac{i}{2} (c_+ - c_-).$$

Hence (33) will not vanish if

$$c_+ \neq c_- \quad \text{or} \quad c_0 \neq 0. \quad (39)$$

Note that when (39) does not hold  $V$  reduces to a constant potential:

$$V = \frac{c_-^2}{4\nu} - \frac{5}{12} n(n+2) - c_*.$$

In summary, the previous analysis shows that the critical coefficient (33) cannot vanish for any positive integer  $k$  provided that  $V$  is fully algebraic, that all its algebraic eigenfunctions are

regular [or square integrable, for the third normalizable canonical form (19)], and that conditions (37), (38), and (39) are satisfied when  $P$  is one of the normal forms 1, 8, or 9, respectively. If (33) does not vanish, defining new polynomials  $\hat{P}_k$  by

$$P_k = \begin{cases} \frac{(-1)^k}{(2\hat{c}_{0-})^k} \frac{\hat{P}_k}{\Gamma\left(\frac{\hat{c}_{-}}{2\hat{c}_{0-}} + k - \frac{n}{2} + \frac{1}{2}\right)}, & \text{if } \hat{c}_{0-} \neq 0; \\ \frac{(-1)^k}{\hat{c}_{-}^k} \hat{P}_k, & \text{if } \hat{c}_{0-} = 0 \end{cases} \quad (40)$$

the recursion relation (26) can be written in the more standard form

$$\begin{aligned} \hat{P}_{k+1} = & \left[ E + c_* + \hat{c}_0 \left( k - \frac{n}{2} \right) + \hat{c}_{00} \left( k - \frac{n}{2} \right)^2 \right] \hat{P}_k - k(k-n-1) [\hat{c}_{+0}(2k-n-1) + \hat{c}_{+}] \\ & \times [\hat{c}_{0-}(2k-n-1) + \hat{c}_{-}] \hat{P}_{k-1}, \quad k \geq 0. \end{aligned} \quad (41)$$

We have thus proved the main theorem in this section.

**Theorem 1:** Let  $V$  be a fully algebraic one-dimensional quasi-exactly solvable potential whose algebraic eigenfunctions are all regular [or normalizable, if  $V$  corresponds to the third or fifth canonical forms in (19)]. Assume, furthermore, that conditions (37), (38), or (39) are satisfied, if  $V$  is obtained from the first, eighth, or ninth canonical forms (19)–(20), respectively. Then  $V$  defines a family of weakly orthogonal polynomials  $\{\hat{P}_k\}_{k=0}^\infty$  satisfying a three-term recursion relation (41) [or (32), if  $V$  corresponds to the fifth canonical form]. The polynomials  $\hat{P}_k$  are defined by (40) and (28), if  $V$  is associated to one of the canonical forms 1–4 or 10, or by (40) and (30), if  $V$  corresponds to one of the normal forms 6–9. Finally, the potential  $V$  associated to the fifth canonical form defines two families of weakly orthogonal polynomials  $P_j^0 = P_{2j}$  and  $P_j^1 = P_{2j+1}$  through (28).

We shall say that a quasi-exactly solvable potential  $V$  is *exactly solvable* if it is independent of the “spin” parameter  $n$ . This implies that  $V$  has  $n$  algebraic eigenvalues and eigenfunctions for arbitrary  $n \in \mathbb{N}$ , so that we can algebraically compute an infinite number of eigenvalues of  $V$  (leaving aside the boundary conditions). All exactly solvable normalizable one-dimensional potentials have been classified; see Ref. 4 for a complete list. The quintessential example of exactly solvable one-dimensional potential is the harmonic oscillator potential, which corresponds to the fifth canonical form (19). We have seen in the previous section that in this case there are two families of orthogonal polynomials [the odd and even coefficients in (28)], each of which satisfies a two-term recursion relation (32). We shall now show that, as conjectured in Ref. 1, the latter property actually characterizes exactly solvable normalizable potentials:

**Theorem 2:** The weakly orthogonal polynomial system associated to an exactly solvable normalizable potential satisfies a two-term recursion relation.

*Proof:* The proof is a simple case-by-case analysis using the classification of exactly solvable normalizable potentials given in Ref. 4. Indeed, for the first normalizable canonical form  $P(z) = \nu(z^2 + 1)$  we have  $w = z \mp i$ , and therefore  $\hat{P}(w) = P(w \pm i) = \nu w(w \pm 2i)$ , so that  $\hat{c}_{+0} = 0$ . Since  $\hat{Q}(w) = Q(w \pm i)$ , we also have  $\hat{c}_{+} = \hat{Q}''(0)/2 = Q''(\pm i)/2 = c_{+}$ . But the exactly solvable potentials associated to this normal form are characterized by the vanishing of  $c_{+}$ , Ref. 4, so that  $\hat{c}_{+} = \hat{c}_{+0} = 0$ , and (41) is a two-term recursion relation. Similarly, for the second normalizable canonical form,  $P(z) = \nu(z^2 - 1)$  and, for instance,  $w = z \mp 1$ . Proceeding as before we obtain that  $\hat{c}_{+0} = 0$  and  $\hat{c}_{+} = c_{+}$ . Since exactly solvable potentials are again those satisfying the condition  $c_{+} = 0$ , (41) reduces to a two-term recursion relation.

The third normalizable canonical form has  $P(z) = \nu z^2$ , and therefore  $c_{+0} = c_{0-} = 0$ . The exactly solvable potentials are characterized by the vanishing of the coefficients  $c_{+}$  or  $c_{-}$ , but not

both simultaneously. In the former case we can take  $w=z$ , while in the latter  $w$  is proportional to  $1/z$  [see the foregoing discussion on the vanishing of the critical coefficient (33)]. In either case, the coefficient of  $P_{k-1}$  in (41) vanishes identically.

The fourth normalizable canonical form is given by  $P(z)=z$ , so that  $w=z$  and  $\hat{c}_{+0}=c_{+0}=0$ , and its exactly solvable potentials are defined by the vanishing of the coefficient  $c_{+}=\hat{c}_{+}=0$ , so that (41) is two term. Finally, for the fifth normalizable canonical form  $P(z)=1$  all normalizable potentials are automatically exactly solvable (they are translates of the harmonic oscillator), and we have already seen that its associated orthogonal polynomials satisfy the two-term recursion relations (32). Q.E.D.

#### IV. THE ORTHOGONAL POLYNOMIALS

We shall study in this section the properties of the family of weakly orthogonal polynomials associated to a quasi-exactly solvable one-dimensional Hamiltonian in the manner described in the previous section. Since, as we shall see, these properties can be established directly from the recursion relation (41) or (32), these polynomials have basically the same properties as those studied by Bender and Dunne in Ref. 1.

We have seen in the previous section that the polynomials  $\hat{P}(E)$  defined by a quasi-exactly solvable one-dimensional Hamiltonian satisfy a three-term recursion relation of the form

$$\hat{P}_{k+1}=(E-b_k)\hat{P}_k-a_k\hat{P}_{k-1}, \quad k \geq 0, \quad (42)$$

with  $a_0=0$  and

$$a_{n+1}=0. \quad (43)$$

For the fifth canonical form, the polynomials  $(-1)^k P_k^0$  and  $(-1)^k P_k^1$  also satisfy a recursion relation of the form (42), with  $a_k=0$  for all  $k \geq 0$ . Note that the coefficients  $a_k, b_k$  in (42) are guaranteed to be real only for the canonical forms 2–7 and 10 (for which  $P$  has a real root). As remarked in the previous section, the vanishing of  $a_k$  for a positive integer value of  $k$  means that the polynomials  $\hat{P}_k$  are only weakly orthogonal. In particular, many classical results, based on the fact that  $a_k > 0$  (or sometimes  $a_k \geq 0$ ) for  $k \geq 1$  cannot be applied in our case.

By Favard's theorem, Ref. 2, there is a *moment functional*, that is a linear functional  $\mathcal{L}$  acting in the space  $\mathbb{C}[E]$  of (complex) univariate polynomials, such that the polynomials  $\hat{P}_k$  are orthogonal under  $\mathcal{L}$ :

$$\mathcal{L}(\hat{P}_k \hat{P}_l) = \gamma_k \delta_{kl}, \quad k, l \in \mathbb{N}. \quad (44)$$

The functional  $\mathcal{L}$  is unique if we impose the normalization condition  $\mathcal{L}(\hat{P}_0)=\mathcal{L}(1)=1$ . It is also known (Boas's theorem<sup>2</sup>) that there is a (not necessarily unique) function of bounded variation  $\omega$  such that

$$\mathcal{L}(p) = \int_{-\infty}^{\infty} p(E) d\omega(E) \quad (45)$$

for an arbitrary polynomial  $p$ . The coefficient  $\gamma_k = \mathcal{L}(\hat{P}_k^2)$ , which therefore plays the role of the square of the norm of  $\hat{P}_k$ , can be computed by multiplying (42) by  $\hat{P}_{k-1}$  and taking  $\mathcal{L}$  of both sides, obtaining

$$0 = \gamma_k - a_k \gamma_{k-1}, \quad k \geq 1.$$

Taking into account that  $\gamma_0 = \mathcal{L}(1) = 1$  we get

$$\gamma_k = \prod_{j=1}^k a_j, \quad k \geq 1. \quad (46)$$

In particular, from this formula follows one of the key properties of the weakly orthogonal polynomial system associated to a one-dimensional quasi-exactly solvable Hamiltonian. Namely, from (43) we have

$$\gamma_k = 0, \quad k \geq n+1,$$

so that *all the polynomials  $\hat{P}_k$  with  $k \geq n+1$  have zero norm*. From this formula it also follows that the “squared norms”  $\gamma_k$  will be positive for  $k \leq n$  if and only if  $a_k > 0$  for  $1 \leq k \leq n$ . It can be shown by a straightforward computation that this is always the case when  $P$  is one of canonical forms 2–4 in (19), assuming that all the eigenfunctions of  $H$  are square integrable and that  $H$  is *not* exactly solvable. Note also that when  $H$  is normalizable [canonical forms 1–5 in (19)] and exactly solvable then  $a_k = 0$  for all  $k \geq 0$ . Hence the square norms of all the polynomials  $\hat{P}_k$  vanish, from which it easily follows from (42) that  $\mathcal{L} = \mathcal{R}(E - b_0)$ .

Other important properties of the polynomials  $\hat{P}_k$  concern their zeros. Classically,<sup>2</sup> it can be shown that if  $a_k > 0$  for all  $k \in \mathbb{N}$  then the zeros of the polynomials  $\hat{P}_k$  satisfying a three-term recursion relation (42) are real and simple. In our case the condition  $a_k > 0$  for all  $k \in \mathbb{N}$  can never hold on account of (43). However, if  $H$  is fully algebraic it can still be proved that all the zeros of  $\hat{P}_{n+1}$  are real and simple. Indeed, by hypothesis  $H$  is self-adjoint on the space  $\mathfrak{N}$  of functions of the form (7), with  $\chi \in \mathcal{R}_n$ . Hence  $H$  has  $n+1$  linearly independent algebraic eigenfunctions lying in  $\mathfrak{N}$ , whose corresponding eigenvalues are real (by self-adjointness) and distinct ( $H$  being a one-dimensional Sturm–Liouville operator). Let us denote by  $E_0 < E_1 < \dots < E_n$  these  $n+1$  real eigenvalues of  $H$  on  $\mathfrak{N}$ , and by  $\psi_l(x) \equiv \psi_{E_l}(x)$  the eigenfunction corresponding to the eigenvalue  $E_l$ . Then (7) and either (28) or (30) imply that  $P_k(E_l) = 0$ , or equivalently  $\hat{P}_k(E_l) = 0$ , for  $k \geq n+1$  and  $0 \leq l \leq n$ . In particular, since  $\hat{P}_{n+1}$  is of degree  $n+1$  and all the eigenvalues  $E_l$  are different, it follows that

$$\hat{P}_{n+1}(E) = \prod_{l=0}^n (E - E_l), \quad (47)$$

where we have used (42) and the fact that  $\hat{P}_0 = 1$ . In other words,  $\hat{P}_{n+1}$  has  $n+1$  simple real zeros at the  $n+1$  algebraic eigenvalues of  $H$ . Furthermore, from the fact that  $\hat{P}_k$  vanishes at  $E_l$  for  $k \geq n+1$  we conclude that there exist monic polynomials  $Q_k$  of degree  $k$  such that

$$\hat{P}_{k+n+1} = Q_k \hat{P}_{n+1}, \quad k \geq 0. \quad (48)$$

This is the so called *factorization property* of the polynomial system  $\{\hat{P}_k\}_{k \in \mathbb{N}}$ , cf. Ref. 1. Note that the vanishing of  $\hat{P}_k(E_l)$  for all  $k \geq n+1$  is consistent with the recursion relation on account of (43). In fact, when  $a_k$  is positive for  $k \geq 1$  and  $b_k$  is real for  $k \geq 0$ , (47) follows directly from the recursion relation by Lemma 3, without using the fact that the polynomials  $\hat{P}_k$  are associated to a fully algebraic quasi-exactly solvable one-dimensional Hamiltonian. The vanishing of  $\hat{P}_k(E_l)$  for  $k > n+1$  is then an immediate consequence of  $\hat{P}_{n+1}(E_l) = 0$ , the recursion relation (42) and (43).

From the previous equation and (42) it follows that the polynomials  $Q_k$  also satisfy a three-term recursion, namely

$$Q_{k+1} = (E - b_{k+n+1})Q_k - a_{k+n+1}Q_{k-1}, \quad k \geq 0,$$

and are therefore orthogonal with respect to an appropriate moment functional  $\mathcal{L}_Q$  (in general different from  $\mathcal{L}$ ).

It was heuristically argued in Ref. 5 that

$$\mathcal{L} = \sum_{j=0}^n \omega_j \delta(E - E_j) \quad (49)$$

on  $\mathbb{C}[E]$ , where the coefficients  $\omega_j$  are defined by

$$\sum_{l=0}^n \hat{P}_k(E_l) \omega_l = \delta_{k0}, \quad k=0,1,\dots,n. \quad (50)$$

Equivalently, the discrete Stieltjes measure  $d\hat{\omega}(E)$  defined by the function

$$\hat{\omega}(E) = \sum_{j=0}^n \omega_j \theta(E - E_j), \quad (51)$$

where  $\theta(t)$  is Heaviside's step function, satisfies (45). Note that the linear system (50) uniquely defines the  $n+1$  constants  $\omega_j$ , since by (27) or (29) its coefficient matrix is the matrix of the change of basis  $\{c_k(z-z_1)^k/k!\}_{k=0}^n$  or  $\{c_k(z-z_1)^k(z-z_2)^{n-k}/k!\}_{k=0}^n$  to  $\{\chi_{E_l}\}_{l=0}^n$  in  $\mathbb{C} \otimes \mathcal{P}_n$ ,  $c_k$  being the coefficient of  $\hat{P}_k$  in (40). It is not difficult to show rigorously that (44) is satisfied. Indeed, by the uniqueness of  $\mathcal{L}$  this is equivalent to showing that if  $\mathcal{L}_0 = \sum_{j=0}^n \omega_j \delta(E - E_j)$  then

$$\mathcal{L}_0(\hat{P}_k \hat{P}_l) = 0, \quad k \neq l, \quad (52)$$

and that

$$\mathcal{L}_0(\hat{P}_0) = \mathcal{L}_0(1) = 1,$$

since  $\mathcal{L}(\hat{P}_k^2)$  and  $\mathcal{L}_0(\hat{P}_k^2)$  must coincide if (52) holds due to the recursion relation (42) and (46). From the definition of  $\omega_j$  we deduce that the last equation, together with (52) for  $k=0$  and  $l=1,\dots,n$ , are satisfied. Suppose now that (52) holds for  $k=0,1,\dots,K$  ( $K \leq n-1$ ) and  $k < l \leq n$ . Multiplying (42) by  $\hat{P}_l$  and taking  $\mathcal{L}_0$  of both sides we obtain

$$\mathcal{L}_0(\hat{P}_{K+1} \hat{P}_l) = \mathcal{L}_0((E - b_K) \hat{P}_K \hat{P}_l) - a_K \mathcal{L}_0(\hat{P}_{K-1} \hat{P}_l) = \mathcal{L}_0(E \hat{P}_K \hat{P}_l)$$

if  $K+1 < l \leq n$ , by the induction hypothesis. But, using again (42),

$$\mathcal{L}_0(E \hat{P}_K \hat{P}_l) = \mathcal{L}_0(\hat{P}_K \cdot E \hat{P}_l) = \mathcal{L}_0(\hat{P}_K \hat{P}_{l+1}) + b_l \mathcal{L}_0(\hat{P}_K \hat{P}_l) + a_l \mathcal{L}_0(\hat{P}_K \hat{P}_{l-1}) = 0,$$

by the induction hypothesis (since  $l > K+1$  implies  $l-1 > K$ ). Hence (52) is true for  $0 \leq k, l \leq n$ . Finally, (52) is trivially true when  $k$  or  $l$  are greater than  $n$  by the factorization property (48) and (47).

We shall next show that all the coefficients  $\omega_j$  are positive if  $b_k$  is real for all  $0 \leq k \leq n$  and  $a_k > 0$  for  $1 \leq k \leq n$ . [Several instances of this result were checked numerically in Ref. 5 for the orthogonal polynomials associated to the Hamiltonian (1).] The proof is based on the following simple lemma:

**Lemma 3:** If  $a_k > 0$  for  $k=1,2,\dots,n$  and  $b_k$  is real for  $k=0,1,\dots,n$  then  $\mathcal{L}$  is positive-definite on  $\mathcal{P}_{2n}$ . In other words, if  $p \in \mathcal{P}_{2n}$  is a real polynomial of degree at most  $2n$ ,  $p \neq 0$  and  $p(E) \geq 0$  for all  $E \in \mathbb{R}$  then  $\mathcal{L}(p) > 0$ .

*Proof:* A polynomial  $p \in \mathcal{P}_{2n}$  which is non-negative for all real values of  $E$  must be of the form  $q^2 + r^2$ , where  $q, r \in \mathcal{P}_n$  are real polynomials. Write  $q = \sum_{k=0}^n q_k \hat{P}_k$ ; then all the coefficients  $q_k$  are real, since  $\hat{P}_k$  is a real polynomial for  $0 \leq k \leq n$  by the hypotheses. Using the orthogonality of the polynomials  $\hat{P}_k$  we obtain  $\mathcal{L}(q^2) = \sum_{k=0}^n q_k^2 \gamma_k$ . Similarly, if  $r = \sum_{k=0}^n p_k \hat{P}_k$  then  $\mathcal{L}(r) = \sum_{k=0}^n r_k^2 \gamma_k$ , and  $\mathcal{L}(p) = \sum_{k=0}^n (q_k^2 + r_k^2) \gamma_k$ . Since  $\gamma_k > 0$  for  $k=0,1,\dots,n$  by (46) and the

hypothesis on the coefficients  $a_k$ , it follows that  $\mathcal{L}(p) \geq 0$ , and  $\mathcal{L}(p) = 0$  if and only if  $q_k = r_k = 0$  for  $k = 0, 1, \dots, n$ , that is if and only if  $p = 0$ . Q.E.D.

*Proposition 4:* If  $a_k > 0$  for  $k = 1, 2, \dots, n$  and  $b_k$  is real for  $k = 0, 1, \dots, n$  then  $\omega_k > 0$  for all  $k = 0, 1, \dots, n$ .

*Proof:* Apply the previous lemma to the polynomials  $\Pi_{0 \leq j \neq k \leq n} (E - E_j)^2 \in \mathcal{P}_{2n}$  for  $k = 0, 1, \dots, n$ . Q.E.D.

Note that the hypotheses of the previous proposition are satisfied when  $P$  is one of the canonical forms 2, 3, or 4, provided that all the eigenfunctions of  $H$  are square-integrable and that  $H$  is *not* exactly solvable. In particular, it is satisfied by the Hamiltonian (1).

The (Hamburger) *moment problem* for the moment functional (49) associated to the weakly orthogonal polynomials defined by a quasi-exactly solvable one-dimensional Hamiltonian consists in determining whether there is a *distribution function* (i.e., a nondecreasing function of bounded variation)  $\omega$  such that  $\mathcal{L}$  can be represented by (45) for an arbitrary polynomial  $p$ . We have already shown that this problem has a solution (51), since (51) is clearly nondecreasing and of bounded variation. We shall next show that this solution is unique (up to an additive constant), so that the moment problem associated to the weakly orthogonal polynomial system  $\{\hat{P}_k\}_{k \in \mathbb{N}}$  is always determined.<sup>12</sup> Essentially, this is due to the fact that the *spectrum*

$$\sigma(\hat{\omega}) = \{E \in \mathbb{R} : \hat{\omega}(E + \delta) - \hat{\omega}(E - \delta) > 0, \forall \delta > 0\}$$

of the distribution function (51) is the finite set  $\{E_l\}_{l=0}^n$ .<sup>13</sup> According to a well known result in the classical theory of orthogonal polynomials,<sup>2</sup> a distribution function  $\omega$  defines a positive-definite functional on  $\mathbb{C}[E]$  through integration with respect to the Stieltjes measure  $d\omega(E)$  if and only if the spectrum of  $\omega$  is infinite. Since  $\mathcal{L}$  is not positive-definite ( $\mathcal{L}(\hat{P}_{n+1}^2) = \gamma_{n+1} = 0$ ), any solution  $\omega$  of (45) must have a finite spectrum, and will thus be of the form

$$\omega(E) = \sum_{k=0}^{\tilde{n}} \tilde{\omega}_k \theta(E - \tilde{E}_k) + C$$

for some constant  $C$ , up to an immaterial redefinition of  $\omega$  in  $\sigma(\omega)$ . If  $I$  is a compact interval containing  $\sigma(\hat{\omega}) \cup \sigma(\omega)$ , then

$$\mathcal{L}(p) = \int_I p(E) d\hat{\omega}(E) = \int_I p(E) d\omega(E), \quad \forall p \in \mathbb{C}[E].$$

Since  $I$  is compact, a well known theorem (cf. Ref. 2) shows that  $\hat{\omega}$  and  $\omega$  differ by a constant at all points in which both  $\hat{\omega}$  and  $\omega$  are continuous. But this easily implies that  $E_k = \tilde{E}_k$  and  $\omega_k = \tilde{\omega}_k$  for  $k = 0, 1, \dots, n = \tilde{n}$ , whence  $\omega = \hat{\omega} + C$ , as stated. Note that the same argument shows that the moment problem in any interval containing  $[E_0, E_n]$  is determined; in particular, the (Stieltjes) moment problem in  $[E_0, \infty)$  is also determined. In this respect, the weakly orthogonal polynomials associated to a quasi-exactly solvable one-dimensional Hamiltonian behave in exactly the same way as the classical orthogonal polynomials, whose moment problem is also determined.<sup>2</sup>

The *moments* of the moment functional  $\mathcal{L}$  are by definition the numbers

$$\mu_k = \mathcal{L}(E^k) = \int_{-\infty}^{\infty} E^k d\hat{\omega}(E) = \sum_{l=0}^n \omega_l E_l^k, \quad k \in \mathbb{N}. \quad (53)$$

If the hypotheses of Proposition 4 hold, all the moments are real. From (53) we see that the module of the  $k$ th moment  $\mu_k$  does not grow factorially as  $k$  tends to infinity, as argued in Ref. 1, but instead it diverges like the  $k$ th power of a constant.<sup>14</sup>

We shall next show that if the coefficient  $a_k$  satisfies the condition

$$a_k \neq 0, \quad 1 \leq k \leq n, \quad (54)$$

which guarantees that the polynomials  $\hat{P}_k$  have nonzero norm for  $k \leq n$ , then *the moments  $\mu_k$  with  $k \geq n+1$  satisfy a constant coefficient difference equation of order  $n+1$* . To this end, recall first of all that the bilinear form  $\langle p, q \rangle = \mathcal{L}(pq)$  defined by  $\mathcal{L}$  in  $\mathbb{C}[E]$ , when restricted to the subspace  $\mathbb{C} \otimes \mathcal{P}_l$ , is represented in the basis  $\{E^k\}_{0 \leq k \leq l}$  by the symmetric matrix  $(\mu_{i+j})_{0 \leq i, j \leq l}$ , whose determinant we shall denote by  $\Delta_l$ . On the other hand, the matrix of the bilinear form  $\langle \cdot, \cdot \rangle$  in the basis  $\{\hat{P}_k\}_{0 \leq k \leq l}$  is clearly  $\text{diag}(1, \gamma_1, \dots, \gamma_l)$ ; therefore, by (46) and the hypothesis on the coefficients  $a_k$ , we conclude that  $\Delta_n \neq 0$  and

$$\Delta_k = 0, \quad k \geq n+1. \quad (55)$$

In particular, since  $\Delta_n \neq 0$  but  $\Delta_{n+1} = 0$ , the last column of  $\Delta_{n+1}$  must be a linear combination of the remaining columns, so that

$$\mu_k = \sum_{i=1}^{n+1} c_i \mu_{k-i}, \quad n+1 \leq k \leq 2(n+1), \quad (56)$$

for some (in general complex) constants  $c_1, \dots, c_{n+1}$ . An easy induction argument using (55) then shows that the above relation is actually valid with the *same* constant coefficients  $c_i$  for all  $k \geq n+1$ , as claimed. In fact, it is not hard to see that  $c_i$  in (56) is minus the coefficient of  $E^{n+1-i}$  in  $\hat{P}_{n+1}$ . Indeed, write  $\hat{P}_{n+1} = E^{n+1} - p_n$ , with

$$p_n = \sum_{i=1}^{n+1} \tilde{c}_i E^{n+1-i},$$

and let  $Q_k = E^k - q_{k-1}$ , so that  $q_{-1} = 0$  and  $\deg q_{k-1} \leq k-1$  for  $k \geq 1$ . From (48) it follows that

$$\hat{P}_k = E^k - E^{k-n-1} p_n - q_{k-n-2} \hat{P}_{n+1}, \quad k \geq n+1,$$

which by (44) implies that

$$\mu_k = \mathcal{L}(E^k) = \mathcal{L}(E^{k-n-1} p_n) = \sum_{i=1}^{n+1} \tilde{c}_i \mu_{k-i}, \quad n+1 \leq k \leq 2(n+1).$$

Comparing with (56) and taking into account the linear independence of the columns of  $\Delta_n$  we immediately obtain that  $\tilde{c}_i = c_i$  for  $i = 1, 2, \dots, n+1$ , as stated.

Note that the fact that the moments satisfy a constant coefficient recursion relation (56) (with  $k \geq n+1$ ) actually characterizes weakly orthogonal polynomial systems. Indeed, (56) simply expresses the fact that the  $(n+2)$ th column of  $\Delta_l$  for  $l \geq n+1$  is a linear combination of the first  $n+1$  columns. Hence (56) implies (55), and since  $\Delta_{n+1} = \prod_{j=1}^{n+1} \gamma_j$  this means that  $\gamma_k = 0$  for some  $k \leq n+1$ , so that  $a_k = 0$  for some  $k \leq n+1$  by (46).

Consider, for example, the Hamiltonian (1) studied in Ref. 1, which corresponds to the fourth canonical form with

$$n = J-1, \quad c_+ = -16, \quad c_0 = c_* = 0, \quad c_- = 2s + \frac{1}{2}(n-1). \quad (57)$$

The coefficients of the corresponding recursion relation (42) are easily found to be

$$b_k = 0, \quad a_k = 16k(J-k)(k+2s-1), \quad k \geq 0. \quad (58)$$

Since we can take  $s \geq 1/2$  without loss of generality, we see that  $a_k > 0$  for  $1 \leq k \leq n$ , so that (54) is satisfied. Furthermore, since  $b_k$  vanishes for all  $k \geq 0$  the polynomials  $\hat{P}_k$  have parity  $(-1)^k$ , and



therefore all the odd moments vanish (the corresponding moment functional is said to be *symmetric*). For  $J=3$  (that is,  $n=2$ ), according to the foregoing observations we know that the moments satisfy a third-order recursion relation of the form (56), whose coefficients are minus the coefficients of  $E^2$ ,  $E$ , and 1 in  $\hat{P}_3$ . From (42) (with  $\hat{P}_0=1$ ) we obtain

$$\hat{P}_1=E, \quad \hat{P}_2=E^2-64s, \quad \hat{P}_3(E)=E^3-32(4s+1)E, \quad (59)$$

so that  $c_1=c_3=0$ —as expected, since the moment functional is symmetric—and  $c_2=32(4s+1)$ . Therefore the even moments satisfy the first-order recursion relation

$$\mu_{2j}=32(4s+1)\mu_{2j-2}, \quad j \geq 2, \quad (60)$$

and since  $\mu_2=\gamma_1=a_1=64s$ , from (60) we obtain

$$\mu_{2j}=32^{j-1}(4s+1)^{j-1} \cdot 64s, \quad j \geq 1. \quad (61)$$

Thus, in this case  $\mu_{2j}$  has a pure power growth. The same result can be obtained using (53). Indeed, from (59) we have

$$E_0=-\lambda \equiv -\sqrt{32(4s+1)}, \quad E_1=0, \quad E_2=\lambda,$$

and therefore

$$\omega_0=\frac{s}{4s+1}, \quad \omega_1=\frac{2s+1}{4s+1}, \quad \omega_2=\omega_0$$

from (50) and (59). Thus

$$\mu_k=\frac{s}{4s+1} [(-\lambda)^k+\lambda^k],$$

which yields  $\mu_{2j+1}=0$  for  $j \geq 0$  and (61).

## V. CONCLUSIONS

We have shown in this paper how every quasi-exactly solvable one-dimensional Hamiltonian satisfying conditions (37)–(39) defines a weakly orthogonal polynomial system  $\{\hat{P}_k\}_{k=0}^\infty$  through the three-term recursion relation (41) (with initial condition  $\hat{P}_0=1$ ). It is important, in this context, to emphasize the *weak* orthogonality of the polynomials  $\hat{P}_k$ , i.e., the fact that the norm of  $\hat{P}_k$  may vanish—and in fact *does* vanish for  $k \geq n+1$ ,  $n$  being the “spin” parameter present in the Hamiltonian. As explained in Sec. IV, this is an inevitable consequence of the vanishing of the coefficient of  $\hat{P}_{k-1}$  in the recursion relation (41) for  $k=n+1$ , which is made possible by the fact that the parameter  $n$  is a non-negative integer. The latter fact, however, is an intrinsic property of one-dimensional quasi-exactly solvable (as opposed to merely Lie-algebraic) Hamiltonians; indeed, it is a key factor in the explanation of the partial integrability of a quasi-exactly solvable Hamiltonian outlined in Sec. II. To better illustrate this point, consider the Hamiltonian (1), which is Lie-algebraic for all real values of the parameter  $J$ . Indeed,  $H$  can be written in the form (5)–(6), with  $\zeta(x)=x^2/4$ ,  $\mu(z)=e^{-4z^2}z^{s-1/4}$ ,  $c_{++}=c_{+0}=c_{00}=c_{+-}=c_{--}=0$ ,  $c_{0-}=1/2$ , and the remaining coefficients given by (57), where now  $n$  is to be regarded as an arbitrary real parameter. When  $n$  is not a non-negative integer, the generators (4) do not leave invariant any finite-dimensional polynomial module  $\mathcal{P}_n$ , so that  $H$  is in general nonintegrable—there is no special reason for  $H$  to have algebraically computable eigenfunctions of the form (7), with  $\chi$  a polynomial. However, even when  $n$  is not a non-negative integer, the Lie-algebraic nature of  $H$  and

conditions (25) imply that the polynomials  $\hat{P}_k$  defined by (7), (21), and (40) still satisfy a three-term recursion relation (42), with the coefficients given by (58). In other words, what makes  $H$  quasi-exactly solvable is not merely the fact that its associated polynomials satisfy a three-term recursion relation (42) (which implies their orthogonality with respect to some Stieltjes measure), but the fact that the coefficient  $a_k$  in this recursion relation vanishes for some positive integer value of  $k$ , so that the associated polynomials  $\hat{P}_k$  can only be weakly orthogonal.

As we saw in Sec. IV, the Stieltjes measure with respect to which the polynomials  $\hat{P}_k$  associated to a quasi-exactly solvable Hamiltonian  $H$  are orthogonal is supported in the set of algebraic eigenvalues of  $H$ , which is a finite set. For this reason, the polynomials  $\hat{P}_k$  are *discrete polynomials*. Although the classical (Hermite, Legendre, Laguerre, Tchebycheff, etc.) polynomials of Mathematical Physics are orthogonal with respect to a continuous measure, discrete (Charlier, Hahn, Krawtchouk, Meixner, Tchebycheff, etc.) polynomials have also been studied in the mathematical literature of orthogonal polynomials, cf. Ref. 2. Note that a discrete polynomial system is truly—as opposed to weakly—orthogonal if and only if the supporting set of its Stieltjes measure is infinite. Some of the discrete polynomials cited above, like the Hahn, Krawtchouk, or discrete Tchebycheff polynomials, are in fact weakly orthogonal. In general, weakly orthogonal polynomials arise naturally, for instance, in the theory of approximate polynomial curve fitting.<sup>15</sup> More recently,<sup>16</sup> the study of second-order finite difference eigenvalue equations with infinitely many polynomial solutions has led to an interesting connection between a nonstandard finite-dimensional representation of  $\mathfrak{sl}(2)$  and certain families of weakly orthogonal discrete polynomials (Hahn polynomials and analytically continued Hahn polynomials).

Let us stress, in closing, that the present paper deals only with one-dimensional quasi-exactly solvable Hamiltonians. It is an interesting open problem to generalize these results to quasi-exactly solvable multi-dimensional systems, a possibility already considered in Ref. 5, where a heuristic (but inconclusive, in our opinion) argument was advanced suggesting that all quasi-exactly solvable systems give rise to weakly orthogonal polynomials. In the two-dimensional case, at least, the classification of quasi-exactly solvable Lie algebras of first-order differential operators in two variables presented in Refs. 7 and 17 could be used as a starting point for an analysis along the present lines.

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<sup>9</sup>We assume that  $P$  is positive in some interval of the real line, so that the first integral in (12) is real.

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<sup>11</sup>Note, however, that a weakly orthogonal polynomial system need *not* satisfy a three-term recursion relation.

<sup>12</sup>The moment problem in  $(-\infty, \infty)$  is never determined unless we impose the condition that  $\omega$  is nondecreasing. Indeed, there are nonzero functions of bounded variation on  $(-\infty, \infty)$  whose moment functional (45) vanishes identically.

<sup>13</sup>A classical theorem, Ref. 2, asserts that the moment problem of a positive-definite moment functional with bounded spectrum is determined; this theorem is not directly applicable in our case, however, since  $\mathcal{L}$  is not positive-definite.

<sup>14</sup>When the coefficient  $a_k$  is positive for all  $k \geq 1$ , several classical criteria due to Carleman relate the growth rate of the moments with the determinacy of the moment problem. Again, these results are not relevant here because of (43).

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