

# Phase-space rotations and orbital Stokes parameters

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We introduce the orbital Stokes parameters as a linear combination of a beam's second-order moments. Similar to the ones describing the field polarization and associated with beam energy and its spin angular momentum, the orbital Stokes parameters are related to the total beam width and its orbital angular momentum. We derive the transformation laws for these parameters during beam propagation through first-order optical systems associated with phase-space rotations. The values of the orbital Stokes parameters for Gaussian modes and arbitrary fields expressed as their linear superposition are obtained. © 2009 Optical Society of America

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The growing applications of beams with orbital angular momentum (OAM) for light-matter interaction, metrology, communication, etc., require the development of methods to characterize such beams and to manipulate their OAM. It has been proposed to use an orbital Poincaré sphere to represent Gaussian modes [1,2], analogous to the one used for the description of polarization. This approach has been generalized to quantized fields and partially coherent beams [3]. Here we propose a simple way of mapping a beam on the orbital Poincaré sphere, based on the analysis of the beam's second-order moments, which can be measured easily, independent of the spatial coherence of the beam.

Optical scalar paraxial monochromatic beams are often characterized by the ten central second-order moments [4–6] of their Wigner distribution [7]. We recall that the Wigner distribution  $W(\mathbf{r}, \mathbf{q})$  of a partially coherent optical signal with cross-spectral density function  $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$  is defined as

$$W(\mathbf{r}, \mathbf{q}) = \int_{-\infty}^{\infty} \Gamma\left(\mathbf{r} + \frac{1}{2}\mathbf{r}', \mathbf{r} - \frac{1}{2}\mathbf{r}'\right) \exp(-i2\pi\mathbf{q}^t\mathbf{r}') d\mathbf{r}'. \quad (1)$$

In the above expression  $\mathbf{r}=[x, y]^t$  represents the transversal space coordinates  $x$  and  $y$  in the form of a column vector, and  $\mathbf{q}=[u, v]^t$  is a similar representation of the spatial-frequency variables  $u$  and  $v$ . For convenience, we use dimensionless space and spatial-frequency variables  $\mathbf{r}$  and  $\mathbf{q}$ , normalized in the  $(x, y)$  directions to  $w_{x,y}$  and  $w_{x,y}^{-1}$ , respectively. Moreover, without loss of generality, we choose the origin of the coordinate system such that it coincides with the center of gravity of  $W(\mathbf{r}, \mathbf{q})$ .

With  $E = \iint_{-\infty}^{\infty} W(\mathbf{r}, \mathbf{q}) d\mathbf{r} d\mathbf{q}$  denoting the total energy of the optical signal, the normalized second-order moments of the Wigner distribution are defined as [4]

$$\mathbf{M} = \frac{1}{E} \int \int_{-\infty}^{\infty} \begin{bmatrix} \mathbf{r} \\ \mathbf{r}^t, \mathbf{q}^t \end{bmatrix} W(\mathbf{r}, \mathbf{q}) d\mathbf{r} d\mathbf{q}, \quad (2)$$

and constitute a positive-definite, real symmetric  $4 \times 4$  moment matrix  $\mathbf{M}$ ,

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{\mathbf{r}\mathbf{r}} & \mathbf{M}_{\mathbf{r}\mathbf{q}} \\ \mathbf{M}_{\mathbf{r}\mathbf{q}}^t & \mathbf{M}_{\mathbf{q}\mathbf{q}} \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xu} & m_{xv} \\ m_{xy} & m_{yy} & m_{yu} & m_{yv} \\ m_{xu} & m_{yu} & m_{uu} & m_{uv} \\ m_{xv} & m_{yv} & m_{uv} & m_{vv} \end{bmatrix} = \mathbf{M}^t. \quad (3)$$

When an optical signal propagates through a lossless first-order optical system, its moments in the input and output planes are related as [4]  $\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t$ , where  $\mathbf{T}$  is the real symplectic  $4 \times 4$  ray transformation matrix, which relates the position vector  $\mathbf{r}_i$  and direction vector  $\mathbf{q}_i$  of an incoming ray to the position vector  $\mathbf{r}_o$  and direction vector  $\mathbf{q}_o$  of the outgoing one,

$$\begin{bmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{bmatrix}. \quad (4)$$

The two well-known independent invariants [4] for general first-order optical systems are (i)  $\det \mathbf{M}$  and (ii) the moment combination  $(m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) + 2(m_{xy}m_{uv} - m_{xv}m_{yu})$ ; the latter combination is known as the beam quality parameter [5].

An orthogonal ray transformation matrix  $\mathbf{T}_o = (\mathbf{T}_o^{-1})^t$  corresponds to rotations in phase space. The orthogonality of the symplectic matrix  $\mathbf{T}_o$  leads to the additional relations  $\mathbf{A}=\mathbf{D}$  and  $\mathbf{B}=-\mathbf{C}$ , and the combined properties of symplecticity and orthogonality can be elegantly expressed by the unitarity of the  $2 \times 2$  matrix  $\mathbf{U}=\mathbf{A}+i\mathbf{B}$  [8]:  $\mathbf{U}^{-1}=(\mathbf{U}^*)^t=\mathbf{U}^\dagger$ . We easily verify that the input-output relation for a phase-space rotator can be expressed in the form

$$\mathbf{r}_o - i\mathbf{q}_o = \mathbf{U}(\mathbf{r}_i - i\mathbf{q}_i), \quad (5)$$

which is a useful alternative for Eq. (4). Basic systems that perform a phase-space rotation are a rotator  $\mathcal{R}(\alpha)$ , a gyrator  $\mathcal{G}(\vartheta)$ , and a separable fractional

Fourier transformer (FT)  $\mathcal{F}(\gamma_x, \gamma_y)$ , with unitary matrices

$$\mathbf{U}_r(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}, \quad \mathbf{U}_g(\vartheta) = \begin{bmatrix} \cos \vartheta & i \sin \vartheta \\ i \sin \vartheta & \cos \vartheta \end{bmatrix},$$

$$\mathbf{U}_f(\gamma_x, \gamma_y) = \begin{bmatrix} \exp(i\gamma_x) & 0 \\ 0 & \exp(i\gamma_y) \end{bmatrix}, \quad (6)$$

respectively. We remark that all these systems have indeed a rotating character: the rotator performs a rotation in the  $(x, y)$  and  $(u, v)$  planes, the gyrator in the  $(x, v)$  and  $(y, u)$  planes, and the separable fractional FT in the  $(x, u)$  and  $(y, v)$  planes (possibly with different angles). A separable fractional FT can be represented as a cascade of a symmetric and an antisymmetric one,  $\mathcal{F}(\mu, \mu)$  and  $\mathcal{F}(\gamma, -\gamma)$ . It has been shown that it is the antisymmetric fractional FT, the gyrator, and the rotator [with  $\det \mathbf{U}_f(\gamma, -\gamma) = \det \mathbf{U}_g(\vartheta) = \det \mathbf{U}_r(\alpha) = 1$ ] that describe movements on the orbital Poincaré sphere [2,9].

Similar to the coherence matrix used for the description of polarization and the definition of the associated Stokes parameters, we introduce the Hermitian matrix

$$\begin{aligned} \mathbf{M}' &= \frac{1}{E} \iint (\mathbf{r} - i\mathbf{q})(\mathbf{r} - i\mathbf{q})^\dagger W(\mathbf{r}, \mathbf{q}) d\mathbf{r} d\mathbf{q} \\ &= \mathbf{M}_{rr} + \mathbf{M}_{qq} + i(\mathbf{M}_{rq} - \mathbf{M}_{rq}^\dagger) \\ &= \begin{bmatrix} Q_0 + Q_1 & Q_2 + iQ_3 \\ Q_2 - iQ_3 & Q_0 - Q_1 \end{bmatrix}, \end{aligned} \quad (7)$$

where the four real second-order moment combinations

$$Q_0 = \frac{1}{2}[(m_{xx} + m_{uu}) + (m_{yy} + m_{vv})], \quad (8a)$$

$$Q_1 = \frac{1}{2}[(m_{xx} + m_{uu}) - (m_{yy} + m_{vv})], \quad (8b)$$

$$Q_2 = m_{xy} + m_{uv}, \quad (8c)$$

$$Q_3 = m_{xv} - m_{yu} \quad (8d)$$

correspond to the orbital Stokes parameters. The parameters  $Q_j$  ( $j=0, 1, 2, 3$ ) are the expectation values of the Hermitian operators [8,10,11] associated with the symmetric and antisymmetric fractional FT, the gyrator, and the rotator, respectively. Moreover, the matrix  $\mathbf{M}'$  resembles the one introduced in [3], which is based on the operator approach.

Using Eq. (5), we get the similarity transformation

$$\mathbf{M}'_o = \mathbf{U}\mathbf{M}'_i\mathbf{U}^\dagger = \mathbf{U}\mathbf{M}'_i\mathbf{U}^{-1} \quad (9)$$

for phase-space rotations, which results in the invariance of the eigenvalues of the matrix  $\mathbf{M}'$ . The characteristic equation with which the eigenvalues of  $\mathbf{M}'$  can be determined reads

$$\det(\mathbf{M}' - \nu\mathbf{I}) = 0 = \nu^2 - 2Q_0\nu + Q_0^2 - Q^2 = (\nu - Q_0)^2 - Q^2,$$

where we have introduced  $Q = \sqrt{Q_1^2 + Q_2^2 + Q_3^2}$ . The eigenvalues  $\nu$  are real and we can write  $\nu_{1,2} = Q_0 \pm Q$ . Since the eigenvalues are invariant, we immediately get that  $Q_0$  and  $Q$  are invariant; note that  $2Q_0$ , the trace of  $\mathbf{M}'$ , is also the trace of  $\mathbf{M}$ . From the invariance of  $Q$  we conclude that the three-dimensional vector  $(Q_1, Q_2, Q_3)$  lives on a sphere with radius  $Q$ , i.e., the Poincaré sphere. In the special case that the phase-space rotator is a symmetric fractional FT, with diagonal matrix  $\mathbf{U} = \exp(i\mu)\mathbf{I}$ , the matrix  $\mathbf{M}'$  itself is invariant, and so is the complete vector  $(Q_1, Q_2, Q_3)$ .

A phase-space rotator may change only the position of the vector  $(Q_1, Q_2, Q_3)$  on the Poincaré sphere, but does not change the invariants  $Q_0$  and  $Q$ . Moreover, from Eq. (9), we easily derive that for a separable fractional FT  $\mathcal{F}(\gamma_x, \gamma_y)$ ,  $Q_1$  is an invariant and  $Q_2 + iQ_3$  undergoes a (counterclockwise) rotator-type transformation:

$$(Q_2 + iQ_3)_o = \exp[i(\gamma_x - \gamma_y)](Q_2 + iQ_3)_i. \quad (10)$$

Similar properties hold for a gyrator  $\mathcal{G}(\vartheta)$ , for which  $Q_2$  is an invariant and  $(Q_3 + iQ_1)_o = \exp(i2\vartheta)(Q_3 + iQ_1)_i$ , and for a rotator  $\mathcal{R}(-\alpha)$ , for which  $Q_3$  is an invariant and  $(Q_1 + iQ_2)_o = \exp(i2\alpha)(Q_1 + iQ_2)_i$ .

To represent an optical signal on the orbital Poincaré sphere, we first need to calculate its Stokes parameters. With  $(Q_1, Q_2, Q_3) = Q(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ , a point on this sphere (with radius  $Q$ ) can be described by the spherical coordinates  $\theta \in [0, \pi]$  and  $\varphi \in (-\pi, \pi]$ , where the angles  $\theta$  and  $\varphi$  indicate the colatitude of a parallel and the longitude of a meridian on the sphere, respectively. A rotation through a positive angle  $\Delta\varphi$  along the equator, i.e., in the  $(Q_1, Q_2)$  plane for which  $\theta = \frac{1}{2}\pi$ , corresponds to a counterclockwise rotation of the signal produced by a rotator  $\mathcal{R}(-\alpha)$  with  $-2\alpha = \Delta\varphi$ . A movement in the  $(Q_2, Q_3)$  plane (with  $\varphi = \pm \frac{1}{2}\pi$ ) corresponds to an antisymmetric fractional FT  $\mathcal{F}(\gamma, -\gamma)$  with  $2\gamma = \frac{1}{2}\pi - \Delta\theta$ , while a rotation in the  $(Q_3, Q_1)$  plane, i.e., along the main meridian (with  $\varphi = 0, \pi$ ), is described by a gyrator  $\mathcal{G}(\vartheta)$  with  $2\vartheta = \Delta\theta$ .

Regarding the physical meaning of the orbital Stokes parameters, we conclude that  $Q_0$  characterizes the entire signal width,  $Q_3$  defines the longitudinal component of the OAM of a paraxial beam, and  $Q$  describes the vorticity (asymmetry) of the beam. Thus  $Q$  measures the capacity of the beam to carry a longitudinal OAM component: if  $Q_3 = 0$  but  $Q \neq 0$ , the beam can be transformed by an appropriate phase-space rotator to another one with  $Q_3 = Q$ .

Let us consider the components  $Q_j$  of some commonly used beams. It is well known that for the Hermite–Gaussian modes  $\mathcal{H}_{m,n}(\mathbf{r}) = \mathcal{H}_m(x)\mathcal{H}_n(y)$ , where (in properly normalized coordinates)  $\mathcal{H}_n(x) = 2^{1/4}(2^n n!)^{-1/2} H_n(\sqrt{2}x) \exp(-\pi x^2)$  and  $H_n(\cdot)$  denote the Hermite polynomials, only the second-order moments that appear on the main diagonal are nonvanishing:  $m_{xx} = m_{uu} = m + \frac{1}{2}$  and  $m_{yy} = m_{vv} = n + \frac{1}{2}$ . This

leads to  $Q_0=m+n+1$ ,  $Q_1=m-n$ ,  $Q_2=Q_3=0$ , and  $Q=|m-n|$ ; in particular, for a symmetric mode  $\mathcal{H}_{m,n}$  (with  $m=n$ ) we have  $Q=Q_1=Q_2=Q_3=0$ . We recall that  $\mathcal{H}_{m,n}$  is an eigenfunction of the separable fractional FT  $\mathcal{F}(\mu+\gamma, \mu-\gamma)$  with eigenvalue  $\exp[-i(Q_0\mu+Q_1\gamma)]$ ; therefore, the Stokes parameters define the Gouy phase—in particular its dynamic and geometric parts—accumulated by coherent Gaussian beams [12].

If we apply a general phase-space rotator with ray transformation matrix  $\mathbf{T}_o$ , say, to the Hermite–Gaussian mode  $\mathcal{H}_{m,n}$ , another structurally stable mode  $\mathcal{H}_{m,n}^{\mathbf{T}_o}$  arises [9]. This mode is actually an eigenfunction of the phase-space rotator with ray transformation matrix  $\mathbf{T}_o\mathbf{T}_f(\mu+\gamma, \mu-\gamma)\mathbf{T}_o^{-1}$ ; its eigenvalue reads  $\exp[-i(Q_0\mu+Q_1\gamma)]$ , again, and is defined by the  $Q_j$  components of the generator mode  $\mathcal{H}_{m,n}$ . Applying, in particular, a gyrator  $\mathcal{G}(\mp\pi/4)$  to the Hermite–Gaussian mode  $\mathcal{H}_{m,n}(\mathbf{r})$ , leads to the Laguerre–Gaussian mode

$$\begin{aligned} \mathcal{L}_p^{\pm l}(r, \phi) = & 2^{1/2} \left[ \frac{(\min\{m, n\})!}{(\max\{m, n\})!} \right]^{1/2} (\sqrt{2\pi}r)^{|m-n|} \\ & \times \exp[\pm i(m-n)\phi] L_{\min\{m, n\}}^{|m-n|}(2\pi r^2) \\ & \times \exp(-\pi r^2), \end{aligned} \quad (11)$$

where  $x=r\cos\phi$  and  $y=r\sin\phi$ , where  $p$  and  $l$  are related to  $m$  and  $n$  through  $p=\min\{m, n\}$  and  $l=m-n$ , and where  $L_p^{(l)}(\cdot)$  denote the generalized Laguerre polynomials. In this case,  $Q_0=m+n+1$  and  $Q_2=0$  keep their values, and we get  $Q_1=0$  and  $Q_3=\pm(m-n)$ . Note that the Laguerre–Gaussian beam is an eigenfunction of the symmetric fractional FT  $\mathcal{F}(\mu, u)$  and the rotator  $\mathcal{R}(\alpha)$ , with eigenvalues  $\exp(-iQ_0\mu)$  and  $\exp(-iQ_3\alpha)$ , respectively.

Let us consider a coherent signal  $f(x, y)$  represented as a superposition of Hermite–Gaussian modes, which constitute a complete orthonormal set,

$$f(x, y) = \sum_{m, n=0}^{\infty} a_{m, n} \mathcal{H}_{m, n}(x, y), \quad \sum_{m, n=0}^{\infty} |a_{m, n}|^2 = 1. \quad (12)$$

From the formulas [13] for the second-order moments of  $f(x, y)$ , expressed in terms of  $m$ ,  $n$ , and  $a_{m, n}$ , we obtain the expressions for the orbital Stokes parameters

$$Q_0 = \sum_{m, n=0}^{\infty} |a_{m, n}|^2 (m+n+1), \quad (13)$$

$$Q_1 = \sum_{m, n=0}^{\infty} |a_{m, n}|^2 (m-n), \quad (14)$$

$$Q_2 + iQ_3 = 2 \sum_{m, n=0}^{\infty} a_{m, n+1} a_{m+1, n}^* \sqrt{(m+1)(n+1)}. \quad (15)$$

We observe that  $Q_2=Q_3=0$  if in the signal decomposition, Eq. (12), for each  $m$  and  $n$  at least one of the modes  $\mathcal{H}_{m, n+1}$  and  $\mathcal{H}_{m+1, n}$  vanishes; this is, in particu-

lar, the case for the superposition  $\sum_{n=0}^{\infty} a_{nn} \mathcal{H}_{n, n}$ , for which, moreover,  $Q_1=0$ . Furthermore, for a real signal we have  $Q_3=0$ . Note that the invariants  $Q_0$  and  $Q$  reach their minimum values  $Q_0=1$  and  $Q=0$  for the fundamental Gaussian mode  $\mathcal{H}_{0,0}$ .

In many image processing tasks, as, for example, in rotationally invariant pattern recognition, an image decomposition in circular harmonics is used [14]. The two-dimensional signal, represented in polar coordinates, is thus expressed as  $f(r, \phi) = \sum_{l=-\infty}^{\infty} f_l(r) \times \exp(il\phi)$ . We now write  $f_l(r) = \sum_{p=0}^{\infty} b_{l, p} \mathcal{L}_p^{(l)}(r, 0)$ . If the circular harmonic  $f_l(r) \exp(il\phi)$  forms the input of the gyrator  $\mathcal{G}(\pi/4)$ , the signal at the output consists of a sum of Hermite–Gaussian modes  $\mathcal{H}_{m, n}(r \cos \phi, r \sin \phi)$ , where the summation is over all modes with  $m-n=l$ . We readily see from Eqs. (14) and (15) that  $Q_1=l \sum_{m-n=l} |a_{m, n}|^2$  and  $Q_2=Q_3=0$ . For the circular harmonic  $f_l(r) \exp(il\phi)$  itself, we have  $Q_1=Q_2=0$  and  $Q_3=l \sum_{m-n=l} |a_{m, n}|^2$ , similar to a single Laguerre–Gaussian mode  $\mathcal{L}_p^{\pm l}(r, \phi)$ .

We have introduced the orbital Stokes parameters as linear combinations of the normalized second-order moments. The invariants and the rotation-type relations of these parameters under phase-space rotations have been found. The orbital Stokes parameters permit one to characterize the spatial structure of a light beam and its capacity to carry orbital angular momentum, they allow one to map the beam on the Poincaré sphere, and they are useful for the determination of the Gouy phase accumulated during the propagation of Gaussian modes through first-order optical systems. They describe coherent as well as partially coherent light, and they can be measured easily.

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