UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

On the rigidity of real solvable Lie algebras of rank one

Sobre la rigidez de las álgebras de Lie resolubles reales de rango uno

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Universidad Complutense de Madrid

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Índice general

Abstract

The main objetive of this memoir is the study and classification of solvable rigid Lie algebras of rank one associated to certain types of eigenvalues spectra for a maximal torus of derivations on its nilradical. The employed techniques are the Chevalley cohomology of Lie algebras and the theory of deformations, as well as the Jacobi scheme. For the study of spectra in arbitrary dimensions, a specific code in the Mathematica® language has been developed. The principal results of this study are reunited in five research papers published in specialized journals. This structures the thesis into six chapters, that are structured as follows: In Chapter one the main structural results are recalled, mainly concerning the cohomological and geometrical analysis of rigidity. We also present a schematic description of the contents of these articles, as well as the symbolic computer package used for the computation of the cohomology groups with values in the adjoint module, illustrating it by an example. The chapter is closed with some conclusions as well as a description of potential prospective research lines, some of which are currently being analyzed. In Chapters 2-6, the five articles are reproduced (without the journal format).

Resumen.

El principal objetivo de esta memoria es el estudio y la clasificación de álgebras de Lie rígidas resolubles reales de rango uno asociadas a ciertos tipos de espectros de autovalores para un toro maximal sobre su nilradical. Las técnicas empleadas son la cohomología de Chevalley de álgebras de Lie y la teoría de deformaciones, así como el esquema de Jacobi. Para el estudio de los espectros en dimensión arbitraria, se ha desarrollado un código específico en el lenguaje Mathematica®. Los resultados de este estudio se encuentran reunidos en cinco artículos publicados en el transcurso de la investigación, lo cual estructura esta tesis en seis capítulos, que se disponen de la siguiente manera: en el primer capítulo se reúnen los resultados estructurales utilizados a lo largo de este trabajo, principalmente referidos a la cohomología y el estudio geométrico de la rigidez. A continuación, se presenta también una descripción esquemática del contenido de los artículos publicados. El capítulo se cierra con unas conclusiones, así como una descripción de potenciales líneas de investigación futuras, algunas de las cuales ya están siendo analizadas. En los capítulos 2-6, se reproducen los cinco artículos (sin el formato de las revistas).
Por último, en el apéndice A, se muestra el código simbólico empleado para el cálculo de los grupos de cohomología con valores en el módulo adjunto, ilustrando el procedimiento con un ejemplo.

Preámbulo

Las álgebras de Lie, que aparecen históricamente en el contexto de los grupos de transformaciones infinitesimales de las ecuaciones diferenciales [33], han demostrado ser una potente herramienta de investigación en el campo de la geometría y la física, ya que se utilizan en la teoría de campos moderna para analizar las propiedades de sus transformaciones, como el modelo quark de los hadrones, en los sistemas dinámicos clásicos o cuánticos, en esquemas de ruptura de la simetría de gauge, en los modelos de Higgs en el marco del método de reducción dimensional, entre otros. Actualmente, también se utilizan en teoría de control y robótica, en el desarrollo de inteligencia artificial, así como en problemas de reconocimiento facial, y un largo etcétera de aplicaciones.

Dentro de la teoría estructural de las álgebras de Lie, la noción de rigidez es importante, y puede expresarse como la estabilidad de su estructura cuando está sometida a variaciones infinitesimales, siendo este concepto análogo al del equilibrio estable en la mecánica de sólidos [65]. Estudiar la rigidez de un álgebra de Lie es analizar si se puede deformar en otra no isomorfa, o puede ser obtenida como una contracción, si bien ambos conceptos no son completamente equivalentes [39]. El concepto de contracción fue originalmente introducido por Segal, Inönü y Wigner en el contexto de la transición por medio de pasos al límite de las mecánicas relativista y no relativista, mientras que la definición formal de deformación se debe a Murray Gerstenhaber [44]. Ambas nociones fueron posteriormente combinadas por diversos autores, entre los que destacan Boyer, Levy-Nahas, Löhmus y Hermann [14, 53, 59, 60].

El concepto de rigidez es, formalmente, topológico. En esta acepción, un álgebra de Lie es rígida si su órbita por la acción del grupo general lineal es un abierto en la topología de Zariski. Posteriormente, Mumford estableció la equivalencia de esta condición en las topologías de Zariski y la topología usual euclídea [67]. Con la demostración de Nijenhuis y Richardson en 1964 [71] de que las álgebras cuyo segundo grupo de cohomología de Chevalley nulo son rígidas, el problema de la rigidez obtuvo un nuevo enfoque algebraico, pues el estudio lineal de las deformaciones se sitúa en el marco de los espacios de cohomología, proporcionando a su vez amplias clases de álgebras de Lie rígidas, tales como las semisimples, las subálgebras parabólicas y las subálgebras de Borel [83]. A su vez, una deformación es una serie formal que satisface la igualdad de Jacobi y cuya parte lineal es un cociclo que adopta valores en el módulo adjunto, lo que plantea el problema de decidir si para un 2-cociclo dado existe una deformación en la que aparezca como la parte lineal de una deformación. Dentro de la teoría de la rigidez de las álgebras resolubles, las álgebras de Lie nilpotentes juegan un papel fundamental.[1] En este contexto, Carles dió en 1984 [23] una respuesta negativa a una conjetura formulada por Vergne en los años setenta para álgebras de Lie nilpotentes de rango mayor o igual que uno, que asegura que no existen álgebras de Lie nilpotentes rígidas , a su vez relacionada con la conjetura formulada en 1993 por Grunewald y O'Halloran [52], en la cual se asegura que toda álgebra de Lie nilpotente es la contracción de otra álgebra (no necesariamente nilpotente) no isomorfa. Aunque las dos conjeturas no son estrictamente equivalentes, la veracidad de esta última implica la de la primera. Se conoce que la conjetura de Grunewald-O'Halloran es cierta para álgebras de Lie nilpotentes que admiten al menos una derivación semisimple, es decir para las de rango mayor o igual que uno, y que existen algunas álgebras de Lie nilpotentes de rango cero (llamadas característicamente nilpotentes) que son contracciones de otras. Dixmier y Morozov iniciaron, de manera independiente, el estudio sistemático de las álgebras de Lie nilpotentes, además de la clasificación completa de estas álgebras en dimensión seis, pusieron en evidencia

[1]Los primeros resultados de esta clase se deben a los trabajos pioneros de K. Umlauf en 1891.

que en dimensiones superiores existe una infinidad de álgebras de Lie nilpotentes complejas no isomorfas [34, 49, 66, 78]. Posteriormente, Vergne desarrolló su trabajo sobre las álgebras de Lie dentro del formalismo de la geometría algebraica. La clasificación de las álgebras de Lie nilpotentes ha sido sistemáticamente estudiada por Rhomdani, Ancochea, Goze, Hakimjanov, Magnin y Vergne en [47, 48, 50, 63, 84]. En 1955 Jacobson demostró que cualquier álgebra de Lie sobre un cuerpo de característica cero que tiene derivaciones no degeneradas es nilpotente [56] ; desafortunadamente, el recíproco de esta afirmación no es cierto, es decir, no toda álgebra de Lie nilpotente admite una derivación externa no degenerada (el primer contraejemplo fue construido por Dixmier y Lister en [35]).

El estudio formal de la clase de álgebras de Lie rígidas se debe fundamentalmente a Roger Carles, quién desarrolló también criterios propios de la geometría algebraica para analizar el caso de álgebras rígidas cuya cohomología de Chevalley es no nula. En particular, en [24] se demuestra que, como espacio lineal, toda álgebra de Lie $\mathfrak{g}$ cuyo radical es nilpotente puede descomponerse como una suma directa $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t} \oplus \mathfrak{n}$, donde $\mathfrak{s}$ es una subálgebra de Levi (semisimple), $\mathfrak{t}$ una subálgebra cuyos elementos son todos semisimples, y $\mathfrak{n}$ es el nilradical de $\mathfrak{g}$. En el caso de las álgebras de Lie resolubles rígidas, la descomposición se simplifica en $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, por lo que, en la búsqueda de álgebras de Lie resolubles rígidas, para el caso de un álgebra nilpotente que no tenga derivaciones externas, se descarta que sea el nilradical de otra álgebra rígida. La subálgebra $\mathfrak{t}$, a la que se denomina toro maximal, está formada por las derivaciones externas que son semisimples y su dimensión, llamado rango del álgebra, es un invariante decisivo para la clasificación de las álgebras resolubles rígidas. Como se deduce de la teoría estructural, el rango de un álgebra de Lie está acotada por la desigualdad siguiente [64, 80]:

$$\text{rank } \mathfrak{t} \leq \dim \mathfrak{n} - \dim \ [\mathfrak{n}, \mathfrak{n}] \tag{1}$$

condición que reduce de manera significativa el número de casos posibles.

Dentro de las álgebras de Lie rígidas se distinguen dos tipos principales, las que satisfacen $\dim H^2 [\mathfrak{g}, \mathfrak{g}] = 0$, denominadas cohomológicamente rígidas, y aquellas para las cuales la segunda clase de cohomología es no nula, llamadas geométricamente rígidas [74] [2]. De estas últimas, las primeras álgebras estudiadas geométricamente rígidas son del tipo $\mathrm{sl}\,(2, \mathbb{C}) \oplus V_i$ donde V_i es un $\mathrm{sl}\,(2, \mathbb{C})$-módulo irreducible. En 1972, Rauch demostró en [75] la rigidez de estas álgebras por medio de la inyectividad de la aplicación cuadrática definida por Rim sq : $H^2(\mathfrak{g}, \mathfrak{g}) \to H^3(\mathfrak{g}, \mathfrak{g})$ (véase [77]).
En 1985, Carles estudia en [23] la rigidez de las álgebras introducidas por Braztlavsky en [15], definidas por sus corchetes $[X_i, X_j] = C_{i,j}^{i+j} X_{i+j}$, donde demuestra que, independientemente del valor de las constantes de estructura, siempre admiten un toro maximal $\mathfrak{t} = \{T\}$ Tal que $[T, X_i] = iX_i$ con $1 \leq i \leq n$.
En 1988, Ancochea clasificó en [1] las álgebras de Lie rígidas resolubles de dimensión menor o igual que 8. En [47], este autor y Michel Goze estudian la rigidez de álgebras de Lie que tienen un nilradical filiforme, y su rango es de uno o dos. En 1992, estos autores determinan las propiedades del sistema de raíces de las álgebras de Lie resolubles rígidas complejas [9], lo que proporciona un nuevo criterio algebraico para determinar la rigidez de álgebras de Lie resolubles mediante técnicas del álgebra lineal.

Pese a las amplias clases de álgebras de Lie rígidas conocidas, el problema de rigidez está lejos de estar resuelto satisfactoriamente. Este trabajo se ha centrado en las álgebras de

[2] Originariamente, la noción de rigidez aparece en el contexto de transformaciones complejas (véase, por ejemplo, [58]).

Lie resolubles reales de rango uno, que constituyen, en cierto sentido, la columna vertebral de la rigidez en el caso resoluble. Este trabajo complementa y amplia diversos resultados obtenidos por Carles, Ancochea, Goze y Bratzlavsky, entre otros. Se comienza estudiando la rigidez de las álgebras de Lie descritas por Bratzlavsky, que poseen un espectro del toro maximal de derivaciones externas del tipo $\mathrm{Spec}(\mathfrak{t}) = \{1, 4, 5 \ldots, n + 2\}$. A continuación, se generaliza el estudio incluyendo un parámetro k arbitrario en el espectro, de forma que quede definido por $\mathrm{Spec}(\mathfrak{t}) = \{1, k, k + 1 \ldots, n + k - 2\}$. También se estudian las álgebras que presentan un toro $\mathrm{Spec}(\mathfrak{t}) = \{1, k, k + 1 \ldots, n + k - 3, n + 2k - 3\}$, las cuales presentan un nilradical que se contrae un álgebra de Lie filiforme del tipo $\mathcal{Q}_n$, y, por tanto aumentan el rango. Como caso genérico final, se introduce un nuevo parámetro en el espectro $\mathrm{Spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q + 1, q + 2, \ldots, N + q - k - 1)$, lo que implica que su sucesión característica sea menor. De todas estas álgebras, se demuestra su rigidez, ya sea algebraica o geométrica y se clasifican según su espectro y su dimensión, así como familias de ellas que también son rígidas. Del último espectro, en particular, se obtiene una respuesta positiva a una cuestión planteada en [3], referente a la existencia de álgebras de Lie resolubles rígidas de rango uno, cuyo nilradical tenga una sucesión característica descendiente arbitraria. De este hecho se colige que, en dimensiones arbitrarias, una clasificación de las álgebras rígidas resolubles es inviable, vista la obstrucción planteada por la clasificación de las álgebras de Lie metabelianas [43].

Finalmente, se presentan algunas cuestiones relevantes que constituyen la base de futuras investigaciones, y en las que se espera obtener resultados empleando los códigos simbólicos desarrollados a estos efectos. En primer lugar, se trataría de obtener nuevos criterios para la determinación de la rigidez geométrica, vista la insuficiencia de la aplicación de Rim. Para ello, se abordaría el problema desde el análisis sistematizado del esquema de Jacobi [27, 30]. Este enfoque es potencialmente de interés en el contexto de las álgebras no resolubles (no semisimples), en la que predomina la rigidez no cohomológica. Como segunda cuestión abierta, se plantea la posibilidad del estudio de las álgebras rígidas no racionales a partir de polinomios que codifiquen el tensor de estructura. Este punto es innovador, dado que la aproximación numérica de raíces reales puede generar errores computacionales, que podría ser evitados mediante un enfoque global. En este sentido, una cuestión abierta, y que fue postulada por Goze y Ancochea en 2001 [10, 48], es si toda álgebra de este tipo es necesariamente cohomológicamente rígida. Finalmente, los resultados y códigos simbólicos desarrollados pueden aplicarse al caso de rango superior, en combinación con otras técnicas para la generación de álgebras rígidas con toros multidimensionales.

Capítulo 1

Sobre la rigidez de las álgebras de Lie reales resolubles de rango uno

1.1 Preliminares.

En esta sección se dan, de forma general, en ocasiones sin dar demostraciones explícitas, pero con las referencias a trabajos donde se puedan encontrar las mismas, las nociones fundamentales de álgebras de Lie de dimensión finita que serán utilizadas a lo largo de este trabajo.

1.1.1 Nociones básicas.

Definición 1 *Sea el espacio vectorial V sobre el cuerpo $\mathbf{K}$ de dimensión finita, se dice que V es un álgebra de Lie si tiene definida una aplicación bilineal $V \times V \longrightarrow V$ que cumple las siguientes propiedades:*

1. *Antisimétrica:* $[X, Y] = -[Y, X]$, $\forall X, Y \in \mathfrak{g}$

2. *Condición de Jacobi:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, $\forall X, Y, Z \in \mathfrak{g}$.

En este trabajo, a no ser que se indique lo contrario, $\mathbb{K}$ se refiere al cuerpo de los números complejos. Cuando se haga referencia a los cuerpos de los números resales $\mathbb{R}$ o al de los racionales $\mathbb{Q}$, se especificará de manera explícita.

Ejemplos

1. El ejemplo más sencillo e inmediato es que cualquier espacio vectorial V puede ser dotado una estructura de álgebra de Lie definiendo sus corchetes como

$$[X, Y] = 0, \quad \forall X, Y \ \in V.$$

   A las álgebras así definidas se les llama álgebras de Lie abelianas.

2. Sea $\mathfrak{g}$ un espacio vectorial de dimensión dos en $\mathbb{K}$ y su base $\{X, Y\}$ con el corchete definido por

$$[X, Y] = \alpha X + \beta Y, \ \alpha, \ \beta \ \in \mathbb{K}$$

   este corchete con la propiedad antisimétrica dota a $\mathfrak{g}$ de estructura de álgebra de Lie.

3. Sea el espacio de las matrices cuadradas de dimensión n $M_n(k)$ sobre el cuerpo $\mathbb{K}$, la aplicación bilineal
$$[A, B] = AB - BA$$
cumple las propiedades 1 y 2. Este álgebra de Lie es denotada por $gl(n, \mathbb{K})$ se la denomina álgebra de Lie lineal.

Definición 2 *Sea $\mathfrak{g}$ un álgebra de Lie. Un subespacio vectorial $\mathfrak{h}$ de $\mathfrak{g}$ se denomina subálgebra de Lie de $\mathfrak{g}$ si $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.*

Definición 3 *Un subálgebra de Lie $\mathfrak{J}$ de $\mathfrak{g}$ es un ideal de $\mathfrak{g}$ si $[\mathfrak{g}, \mathfrak{J}] \subseteq \mathfrak{J}$, es decir si $[X, Y] \in \mathfrak{J}\ \forall\ X \in \mathfrak{J}$ e $Y \in \mathfrak{g}$.*

Definición 4 *Se define como centro de un álgebra de Lie $\mathfrak{g}$ al ideal:*
$$\mathrm{Z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0,\ \forall\ Y \in \mathfrak{g}\}.$$

Definición 5 *Sea $\mathfrak{g}$ un algebra de Lie y $\mathfrak{J}$ un ideal de $\mathfrak{g}$, se define el álgebra de Lie cociente $\mathfrak{g}/\mathfrak{J}$ como el espacio vectorial cociente dado por la relación de las clases de equivalencia:*
$$\left[\overline{X}, \overline{Y}\right] = \overline{[X, Y]}.$$

Definición 6 *Sea $\mathfrak{g}$ un álgebra de Lie, al conjunto de los elementos t de la forma $t = [X, Y]$, con $X,\ Y \in \mathfrak{g}$ y todas sus posibles combinaciones lineales se le denomina álgebra derivada de $\mathfrak{g}$ y se denota por $D(\mathfrak{g})$.*

Es inmediato ver que $D(\mathfrak{g})$ es un ideal de $\mathfrak{g}$.

Definición 7 *Sean $\mathfrak{g}_1$ y $\mathfrak{g}_2$ álgebras de Lie sobre el cuerpo* $\mathbf{K}$*. Un homomorfismo entre $\mathfrak{g}_1$ y $\mathfrak{g}_2$ es una aplicación lineal $\phi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ que cumple:*
$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$
Si ϕ es biyectiva se dice que $\mathfrak{g}_1$ y $\mathfrak{g}_2$ son isomorfas.

Ejemplos

1. Todas las álgebras de Lie abelianas de dimensión n son isomorfas.
2. Las álgebras de Lie no abelianas bidimensionales son isomorfas.

1.1.2 Álgebras resolubles.

Sea $\mathfrak{g}$ un álgebra de Lie, se denomina serie derivada de $\mathfrak{g}$ a la cadena de ideales
$$D^0\mathfrak{g} = \mathfrak{g} \supset D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset D^2\mathfrak{g} = [D^1\mathfrak{g}, D^1\mathfrak{g}] \supset \cdots \supset D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}] \supset \cdots \quad (1.1)$$
De igual modo, se denomina serie central descendente de $\mathfrak{g}$ a la cadena de ideales
$$C^0\mathfrak{g} = \mathfrak{g} \supset C^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset C^2\mathfrak{g} = [C^1\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset C^k\mathfrak{g} = [C^{k-1}, \mathfrak{g}] \supset \cdots \quad (1.2)$$

Definición 8 *Se dice que $\mathfrak{g}$ es un álgebra resoluble si existe algún k tal que $D^k\mathfrak{g} = 0$.*
De la misma manera, se dice que $\mathfrak{g}$ es nilpotente si existe algún k tal que $C^k\mathfrak{g} = 0$.

Si bien toda álgebra nilpotente es resoluble, el recíproco no es cierto.

Definición 9 *Sea $\mathfrak{g}$ un álgebra de Lie de dimensión n, se llama operador adjunto a la representación:*
$$\mathrm{ad} : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$$
$$X \longrightarrow \mathrm{ad}X$$
donde
$$\mathrm{ad}X(Y) = [X, Y].$$

1.1.3 Teorema de Engel.

Teorema 1 *Un álgebra de Lie* $\mathfrak{g}$ *es nilpotente si y sólo si* adX *es un endomorfismo nilpotente para todo* X *de* $\mathfrak{g}$.

Para demostración véase Humphreys ([55] capítulo 1.3, página 13).

1.1.4 Sucesión característica de un álgebra de Lie nilpotente.

Sea $\mathfrak{n}$ un álgebra de Lie nilpotente y $D(\mathfrak{n})$ su álgebra derivada. Sea la secuencia ordenada:

$$c(Y) = \{h_1, h_2, \cdots, \}$$

con $h_1, \geq h_2, \cdots \geq h_p$, donde h_i es la dimensión de la i-ésima caja de Jordan del operador nilpotente adY. Como Y es necesariamente autovector de adY entonces $h_1 = 1$. Sean Y_1 e Y_2 en $\mathfrak{n} - D(\mathfrak{n})$ y sus correspondientes secuencias $c(Y_1) = \{h_1, \cdots, h_{p_2} = 1\}$ y $c(Y_2) = \{k_1, \cdots, k_{p_2} = 1\}$, se tiene que $h_1 \geq h_2 \geq \cdots \geq h_{p_1}$ y $k_1 \geq k_2 \geq \cdots \geq k_{p_2}$ con $h_1 + h_2 + \cdots + h_{p_1} = k_1 + k_2 + \cdots + k_{p_2} = \dim \mathfrak{n}$. Se dice que $c(Y_1) \geq c(Y_2)$ si existe un i tal que $h_1 = k_1,\ h_2 = k_2, \ldots, h_{i-1} = k_{i-1}, h_i \geq k_i$. Esto define una relación de orden total sobre el conjunto de las secuencias $c(Y)$, por lo que tiene un máximo.

Definición 10 *La sucesión característica de un álgebra de Lie nilpotente se define por:*

$$c(\mathfrak{n}) = Sup \{c(Y) | Y \in \mathfrak{n} - D(\mathfrak{n})\}.$$

Esta secuencia es invariante por isomorfismos de $\mathfrak{n}$. Al vector que cumple $c(Y) = c(\mathfrak{n})$ se le denomina vector característico de $\mathfrak{n}$.

Proposición 1 *Si* $\mathfrak{n}$ *y* $\mathfrak{n}'$ *son dos álgebras de Lie nilpotentes tal que* $\mathfrak{n}'$ *es una perturbación de* $\mathfrak{n}$ *entonces:*

$$c(\mathfrak{n}') \geq \mathfrak{n}.$$

Que $\mathfrak{n}'$ sea una perturbación de $\mathfrak{n}$ es equivalente a decir que las constantes de estructura de $\mathfrak{n}'$ son infinitamente próximas a las de $\mathfrak{n}$.

1.1.5 Álgebras de Lie simples y semisimples.

Definición 11 *Un álgebra de Lie no abeliana* $\mathfrak{g}$ *se le llama simple si no tiene ideales propios. En el caso de que no tenga ideales abelianos distintos de cero, entonces se dice semisimple.*

Definición 12 *Al ideal maximal resoluble de un álgebra de Lie* $\mathfrak{g}$ *se le llama radical y se denota por* $rad(\mathfrak{g})$.

Dados dos ideales resolubles $\mathfrak{I}$ y $\mathfrak{J}$ de $\mathfrak{g}$, como $\mathfrak{I} + \mathfrak{J}$, que está generado por los elementos $X + Y$ con $X \in \mathfrak{I}$ e $Y \in \mathfrak{J}$, es un ideal también resoluble, sumando todos los ideales resolubles se obtiene el maximal.

Proposición 2 *El álgebra de Lie* $\mathfrak{g}$ *es semisimple si y solo si* $rad(\mathfrak{g}) = 0$.

Demostración. Si el $\mathrm{rad}(\mathfrak{g}) = 0$ y existe un ideal abeliano $\mathfrak{I}$ de $\mathfrak{g}$ entonces $\mathfrak{I} \subset \mathrm{rad}(\mathfrak{g})$ por lo tanto $\mathfrak{I} = 0$ y por lo tanto $\mathfrak{g}$ es semisimple. Por el contrario, si se supone que $\mathfrak{g}$ es un álgebra de Lie semisimple y sea $\mathfrak{r}$ su radical, entonces existe un $k \in \mathbb{N}$ tal que $D^k(\mathfrak{r}) \neq 0$ y $D^{k+1}(\mathfrak{r}) = 0$ entonces $D^k(\mathfrak{r})$ es un ideal abeliano de $\mathfrak{g}$ lo que es una contradicción con la suposición de partida.

Teorema 2 Teorema de la descomposición de Levi. *Sea $\mathfrak{g}$ una álgebra de Lie de dimensión finita y $\mathfrak{r}$ su radical, entonces existe una subálgebra semisimple $\mathfrak{s}$ tal que $\mathfrak{g}$ es suma semidirecta del radical y de esta subálgebra.*

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}.$$

Para demostración véase Jacobson en [57] capítulo III, página 91.

1.1.6 Cohomología de álgebras de Lie.

Existe un amplio estudio de la cohomología del álgebra de Lie considerando la cohomología en valores de un $\mathfrak{g}$-módulo, sin embargo, este trabajo se centra en la cohomología de Chevalley [22] que es la cohomología con valores en $\mathfrak{g}$, que es la utilizada a lo largo de esta memoria.

1.1.7 Derivaciones de álgebras Lie.

Definición 13 *Una derivación f del álgebra de Lie $\mathfrak{g}$ es una aplicación lineal*

$$f : \mathfrak{g} \longrightarrow \mathfrak{g}$$

tal que:

$$f\,[X, Y] = [f(X), Y] + [X, f(Y)]\,.$$

Proposición 3 *Para todo X de $\mathfrak{g}$ el endomorfismo* $\mathrm{ad}X$ *es una derivación.*

La demostración de esta proposición es inmediata, ya que es una consecuencia de la condición de Jacobi.

Definición 14 *La derivación f de $\mathfrak{g}$ tal que $f = \mathrm{ad}X$ para algún X de $\mathfrak{g}$ se le denomina derivación interna. En caso contrario, derivación externa.*

Teorema 3 *Si el álgebra de Lie $\mathfrak{g}$ es semisimple entonces todas sus derivaciones son internas.*

El recíproco no es cierto, pues existen álgebras de Lie que no son semisimples y cuyas derivaciones son internas.

Ejemplo:
Sea el álgebra de Lie de dimensión ocho cuya base es $B = \{X_1, X_2, \cdots, X_8\}$ que queda definida por los corchetes:

$$\begin{array}{lll}
[X_1, X_2] = 2X_2, & [X_1, X_3] = -2X_3, & [X_1, X_4] = 3X_4, \\
[X_1, X_5] = X_5, & [X_1, X_6] = -X_6, & [X_1, X_7] = -3X_7, \\
[X_2, X_3] = X_1, & [X_2, X_5] = 3X_4, & [X_2, X_6] = 2X_5, \\
[X_3, X_4] = X_5, & [X_3, X_5] = 2X_6, & [X_3, X_6] = 3X_7, \\
[X_4, X_8] = X_4, & [X_5, X_8] = X_5, & [X_6, X_8] = X_6, \\
[X_7, X_8] = X_7. & &
\end{array} \tag{1.3}$$

Sea $D(X_i) = \sum_{p=1}^{8} \alpha_i^p X_p$ una derivación, una base de $\mathrm{Der}(\mathfrak{g})$ es $\{\alpha_1^2, \alpha_1^3, \alpha_1^4, \alpha_1^5, \alpha_1^6, \alpha_1^7, \alpha_2^2, \alpha_4^4\}$ que corresponden a las ocho derivaciones internas. Como el ideal $\mathfrak{J}$=$\{X_4, X_5, X_6, X_7\}$ es abeliano, $\mathfrak{g}$ no es semisimple.

1.1.8 Espacio de p-cocadenas de un álgebra de Lie.

Sea $\mathfrak{g}$ un álgebra de Lie, una cocadena de dimensión p con valores en $\mathfrak{g}$ es una aplicación p-lineal alternada de $\mathfrak{g}^p$ en $\mathfrak{g}$ con $p \in \mathbb{N}$. Una 0-cocadena es una aplicación constante de $\mathfrak{g}$ a $\mathfrak{g}$. Se denota al espacio con $C^P(\mathfrak{g}, \mathfrak{g})$ de todas las p-cocadenas y

$$C^*(\mathfrak{g}, \mathfrak{g}) = \oplus_{p \geq 0} C^p(\mathfrak{g}, \mathfrak{g}).$$

Que tiene estructura de g-módulo con la operación:

$$(X\Phi)(X_1, X_2, \cdots, X_p) = [X, \Phi(X_1, X_2, \cdots, X_p)] - \sum_{1 \leq i \leq p} \Phi(X_1, \cdots, [X, X_i], \cdots, X_p)$$

para todo $X_1, X_2, \cdots, X_p \in \mathfrak{g}$.
Se define el operador coborde como el endomorfismo en el espacio $C^*(\mathfrak{g}, \mathfrak{g})$

$$\delta : \Phi \in C^p(\mathfrak{g}, \mathfrak{g}) \longrightarrow C^{p+1}(\mathfrak{g}, \mathfrak{g})$$

estableciendo

$$\delta\Phi(X) = X.\Phi \quad \text{si} \quad \Phi \in C^0(\mathfrak{g}, \mathfrak{g}) \tag{1.4}$$

$$\begin{aligned} \delta\Phi(X_1, ..., X_{p+1}) &= \textstyle\sum_{1 \leq s \leq p+1} (-1)^{s+1} \left[X_s, \Phi(X_1, ..., \widehat{X}_s, ..., X_{p+1})\right] \\ &+ \textstyle\sum_{1 \leq s \leq t \leq p+1} (-1)^{s+t} \Phi\left([X_s, X_t], X_1, ..., \widehat{X}_s, ..., \widehat{X}_t, ..., X_{p+1}\right) \end{aligned} \tag{1.5}$$

si $\Phi \in C^p(\mathfrak{g}, \mathfrak{g})$ con $p \geq 1$. El operador coborde cumple que $\delta \circ \delta = 0$.
Para definir el espacio de cohomología hay que definir primero los espacios de los cociclos y cobordes que se denotan de $Z^p(\mathfrak{g}, \mathfrak{g})$ y $B^p(\mathfrak{g}, \mathfrak{g})$, respectivamente. Se definen:

$$\begin{cases} Z^p(\mathfrak{g}, \mathfrak{g}) = Ker\delta \Big|_{C^p(\mathfrak{g}, \mathfrak{g})} & p \geq 1 \\ \\ B^p(\mathfrak{g}, \mathfrak{g}) = Im\delta \Big|_{C^p(\mathfrak{g}, \mathfrak{g})} & p \geq 1 \end{cases}$$

Es espacio de cohomología de grado p queda definido por:

$$H^p(\mathfrak{g}, \mathfrak{g}) = Z^p(\mathfrak{g}, \mathfrak{g}) / B^p(\mathfrak{g}, \mathfrak{g}),$$

para $p \geq 1$.
En el caso de $p = 0$, $B^0(\mathfrak{g}, \mathfrak{g}) = \{0\}$ y $H^0(\mathfrak{g}, \mathfrak{g}) = Z^0(\mathfrak{g}, \mathfrak{g})$. Este último espacio puede ser identificado con el espacio de todos los elementos $\mathfrak{g}$-invariantes, es decir

$$\{X \in \mathfrak{g} \text{ tal que } \mathrm{ad}Y(X) = 0 \quad, \forall Y \in \mathfrak{g}\}.$$

Entonces $Z^0(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g})$ (El centro de $\mathfrak{g}$).
El espacio $H^1(\mathfrak{g}, \mathfrak{g})$

$$Z^1(\mathfrak{g}, \mathfrak{g}) = \{f : \mathfrak{g} \longrightarrow \mathfrak{g} \text{ tal que } \delta f = 0\}.$$

Pero $\delta f(X, Y) = [f(X), Y] + [X, f(Y)] - f[X, Y]$. Entonces $Z^1(\mathfrak{g}, \mathfrak{g})$ es el álgebra de derivaciones de $\mathfrak{g}$:

$$Z^1(\mathfrak{g}, \mathfrak{g}) = Der\mathfrak{g}.$$

De la misma manera :

$$B^1(\mathfrak{g}, \mathfrak{g}) = \{\mathrm{ad}X, \ X \in \mathfrak{g}\}.$$

Entonces $B^1(\mathfrak{g}, \mathfrak{g})$ es el espacio de la derivaciones internas y $H^1(\mathfrak{g}, \mathfrak{g})$ puede ser interpretado como el conjunto de las derivaciones externas del álgebra de Lie $\mathfrak{g}$.

1.1.9 La secuencia espectral de Hochschild-Serre.

Sea I un ideal de $\mathfrak{g}$. Se consideran las cocadenas

$$\varphi : I^p \longrightarrow \mathfrak{g}$$

de I con valores en $\mathfrak{g}$. Para estas cadenas se puede definir, por restricción, el operador coborde δ. Como I es un ideal de $\mathfrak{g}$, $H^1(\mathfrak{g},\mathfrak{g})$ es un $\mathfrak{g}$-módulo. Por ello se puede considerar el espacio de cohomología $H^*(I,\mathfrak{g})$.

Definición 15 *Una p-cocadena φ de $C^p(I,\mathfrak{g})$ es $\mathfrak{g}$-invariante si cumple:*

$$\begin{aligned} X\varphi(X_1,...,X_p) &= [X,\varphi(X_1,...,X_p)] \\ -\sum_{1\leq i\leq p} \varphi(X_1,...,X_{i-1},[X,X_i]),...,X_p) &= 0. \end{aligned} \tag{1.6}$$

Se denota con $C^*(I,\mathfrak{g})^{\mathfrak{g}}$ el conjunto de las cocadenas de I que son $\mathfrak{g}$-invariantes y $H^*(I,\mathfrak{g})^{\mathfrak{g}}$ el correspondiente espacio de cohomología. Cada elemento $\overline{\varphi}$ de $H^p(I,\mathfrak{g})^{\mathfrak{g}}$ tiene un representante cuya restricción a I de la cocadena Ψ en $C^p(\mathfrak{g},\mathfrak{g})$ es tal que $d\psi \in \left(\mathfrak{g}/I,\mathfrak{g}^I\right)$ donde $\mathfrak{g}^I = \{X \in \mathfrak{g}/[X,Y]=0 \ \ \forall Y \in I\}$. Este elemento $d\psi$ no depende de la elección del representante $\overline{\varphi}$. Sea t_{p+1}el homomorfismo definido por:

$$t_{p+1} : H^p(I,\mathfrak{g})^{\mathfrak{g}} \longrightarrow H^{p+1}\left(\mathfrak{g}/I,\mathfrak{g}^I\right). \tag{1.7}$$

Se define la secuencia exacta:

$$0 \longrightarrow H^p\left(\mathfrak{g}/I,\mathfrak{g}^I\right) \xrightarrow{l_p} H^p(\mathfrak{g},\mathfrak{g}) \xrightarrow{r_p} H^p(I,\mathfrak{g})^{\mathfrak{g}} \tag{1.8}$$

$$\xrightarrow{r_{p+1}} H^{p+1}\left(\mathfrak{g}/I,\mathfrak{g}^I\right) \longrightarrow H^{p+1}(\mathfrak{g},\mathfrak{g}) \tag{1.9}$$

donde r_p es la restricción del homomorfismo l_p es definido considerando las cocadenas de $\mathfrak{g}/I$ en $\mathfrak{g}^I$ como cocadenas de $\mathfrak{g}$ en $\mathfrak{g}$.

Aplicación.

Supóngase que $codim(I) = 1$.
Entonces $\dim \mathfrak{g}/I = 1$ y $C^p(\mathfrak{g}/I,\mathfrak{g}) = 0$ por $p \geq 2$. Por esto

$$0 \longrightarrow 0 \longrightarrow H^2(\mathfrak{g},\mathfrak{g}) \longrightarrow H^2(I,\mathfrak{g})^{\mathfrak{g}} \longrightarrow 0 \tag{1.10}$$

y se tiene

$$H^2(\mathfrak{g},\mathfrak{g}) = H^2(I,\mathfrak{g})^{\mathfrak{g}}. \tag{1.11}$$

Ejemplo. Sea $\mathfrak{g}$ un álgebra de Lie que admite la descomposición:

$$\mathfrak{g} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n} \tag{1.12}$$

donde $\mathfrak{n}$ es el nilradical (el ideal nilpotente maximal) y $\mathfrak{t}$ un subalgebra abeliana cuyos elementos $ad_{\mathfrak{g}}$-semisimples. Supóngase que $\dim\mathfrak{t} = 1$, $\dim\mathfrak{g} = n$. Sea $\{X, X_1,...,X_{n-1}\}$ una base adaptada de $\mathfrak{g}$ tal que $X_1,...,X_n$ son los autovectores de ad$X$ con $X \neq 0 \in \mathfrak{t}$. Entonces se tiene $[X,X_i] = \lambda_i X_i$. Un caso particular interesante consiste en elegir $\mathfrak{n}$ un álgebra de Lie filiforme definida por:

$$[X_1,X_2] = X_3,\ [X_1,X_3] = X_4,\cdots,[X_1,X_{n-2}] = X_{n-1}. \tag{1.13}$$

Como $\{X_1, ..., X_{n-1}\}$ es una familia libre de autovectores propios adX, si los autovalores son $\lambda_1, ..., \lambda_{n-1}$, entonces $[X_i, X_j]$ es un autovector adX asociado a $\lambda_i + \lambda_j$. Sea f un 1-cocyclo de $C(I, \mathfrak{g})^{\mathfrak{g}}$. Entonces

$$f : I \longrightarrow \mathfrak{g} \tag{1.14}$$

y verifica

$$[X, f(X_i)] = f[X, X_i] = \lambda_i f(X_i). \tag{1.15}$$

Como $[adX, f] = 0$ entonces $f(X_i) = \mu_i X_i$ y f es una aplicación diagonal. Esto implica que

$$\delta f(X_i, X_j) = [f(X_i), X_j] + [X_i, f(X_j)] - f[X_i, X_j] = \tag{1.16}$$

$$= (\mu_i + \mu_j)[X_i, X_j] - f[X_i, X_j]. \tag{1.17}$$

Estas relaciones permiten calcular $B^2(I, \mathfrak{g})^{\mathfrak{g}}$.

1.1.10 Teorema de la factorización de Hochschild-Serre.

Una herramienta muy útil para calcular la dimensión de las clases de cohomología es el teorema de la factorización de Hochschild-Serre [54], simplifica los cálculos de manera significativa.

Teorema 4 *Sea $\mathfrak{g}$ un álgebra de Lie de dimensión finita sobre el cuerpo F de característica 0, M un $\mathfrak{g}$-módulo de dimensión finita y L un ideal de $\mathfrak{g}$ tal que $\mathfrak{g}/L$ es semisimple, entonces:*

$$H^p(\mathfrak{g}, M) \simeq \sum_{i+j=p} H^i(\mathfrak{g}/L, F) \otimes H^j(L, M)^{\mathfrak{g}}. \tag{1.18}$$

Aplicado a las álgebras de Lie que admiten la descomposición $\mathfrak{g} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}$ estudiadas en este trabajo, el teorema puede reformularse de la siguiente manera:

$$H^p(\mathfrak{g}, \mathfrak{g}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{g})^{\mathfrak{t}}, \tag{1.19}$$

Donde $H^b(\mathfrak{n}, \mathfrak{g})^{\mathfrak{t}}$ es el espacio de las clases de cociclos $\mathfrak{t}$-invariantes de $\mathfrak{n}$ con valores en $\mathfrak{g}$, definido por:

$$H^b(\mathfrak{n}, \mathfrak{g})^{\mathfrak{t}} = \left\{[\varphi] \in H^b(\mathfrak{n}, \mathfrak{g}) \mid (T.\varphi) = 0,\ T \in \mathfrak{t}\right\} \tag{1.20}$$

La condición de invarianza simplifica la aplicación coborde sobre las p-cocadenas, que quedan definidas por la siguiente expresión:

$$(T.\varphi)(X_1, X_2, \cdots, X_b) = [T, \varphi(X_1, X_", \cdots, X_b)] - \sum_{s=1}^{b} \varphi(X_1, \cdots, [T, X_s], \cdots, X_b), \tag{1.21}$$

El isomorfismo (1.19) implica que $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ si y solo si $H^b(\mathfrak{n}, \mathfrak{g})^{\mathfrak{g}} = 0$ para todo $0 \leq b \leq p$ puesto que $H^a(\mathfrak{t}, \mathbb{R}) = \bigwedge^a \mathfrak{t}$. En las álgebras de Lie completas, es decir $H^i(\mathfrak{n}, \mathfrak{g}) = 0$ con $0 \leq i \leq 1$, para hallar la dimensión de $H^2(\mathfrak{n}, \mathfrak{g})^{\mathfrak{g}}$, es suficiente calcular la dimensión de $H^2(\mathfrak{n}, \mathfrak{g})^{\mathfrak{g}}$.

1.1.11 Álgebras de Lie algebraicas.

Sea V un espacio vectorial complejo de dimensión finita. Se denota por $gl(V)$ al álgebra de Lie de todos los endomorfismos de V.

Definición 16 *Una subálgebra $\mathfrak{g}$ de $gl(V)$ se dice algebraica si es isomorfa a un álgebra de Lie de un grupo de automorfismos de Lie algebraico de V. Un álgebra de Lie $\mathfrak{g}$ es algebraica si es isomorfa a un álgebra de Lie algebraica lineal (es decir, a un subálgebra algebraica de $gl(V)$ para algún espacio vectorial V).*

A continuación se enumeran brevemente algunas propiedades de las álgebras de Lie algebraicas.

Proposición 4 *Sea $\mathfrak{g}$ un subálgebra de Lie de $gl(V)$. Entonces existe una subálgebra de Lie algebraica minimal $\mathfrak{g}'$ de $gl(V)$ que contiene a $\mathfrak{g}$. Además, todo ideal de $\mathfrak{g}$ es un ideal de $\mathfrak{g}'$, y $D^1\mathfrak{g} = D^1\mathfrak{g}'$.*

Proposición 5 *Si $\mathfrak{g}$ es un álgebra de Lie compleja, entonces $Der\mathfrak{g}$ es un álgebra de Lie algebraica.*

Se puede dar una caracterización del álgebra de Lie algebraica. Sea $\mathfrak{g}$ un álgebra de Lie lineal, es decir, un subálgebra de Lie de $gl(V)$, para cualquier $X \in \mathfrak{g}$, sea $\mathfrak{g}(X)$ la mayor subálgebra de $gl(V)$ que contiene a X. Los elementos de $\mathfrak{g}(X)$ se denominan réplicas de X.

Proposición 6 *El álgebra de Lie $\mathfrak{g}$ es algebraica si y solo si para todo $X \in \mathfrak{g}$, $\mathfrak{g}(X) \subset \mathfrak{g}$.*

El siguiente teorema se incluye porque va a ser fundamental en los capítulos que forman en este trabajo:

Teorema 5 *Sea $\mathfrak{g}$ un álgebra de Lie resoluble y $\mathfrak{n}$ su nilradical. Entonces $\mathfrak{g}$ es un producto semidirecto de $\mathfrak{n}$ y de un subálgebra abeliana cuyos elementos son semisimples (adX semisimple).*

Una consecuencia de este teorema es que si un álgebra de Lie $\mathfrak{g}$ contiene un elemento X tal que adX es diagonalizable, entonces $\mathfrak{g}$ es algebraica y admite una descomposición como la descrita en el anterior teorema.

1.1.12 Álgebras de Lie completas.

Las álgebras de Lie completas aparecen en 1951 en el contexto de la teoría de álgebras de Lie subinvariantes. En su teorema de la secuencia de derivación, Schenkman demostró que para álgebras de Lie resolubles sin centro la secuencia de álgebras de derivación $Der^n\mathfrak{g} = Der^{n-1}(Der\mathfrak{g})$ es finita, y que el último término, el cual también carece de centro, tiene sólo derivaciones internas [79]. Sin embargo, no se dio la definición formal de álgebra de Lie completa hasta 1962, cuando Jacobson la usó explícitamente. Este trabajo ha demostrado la importancia de estas álgebras, las cuales juegan un papel importante en las teoría de la rigidez. En 1974 Favre en [37] estudia en profundidad esas álgebras, detallando su descomposición y las propiedades de sus derivaciones.

Definición 17 *Un álgebra de Lie $\mathfrak{g}$ se dice completa si $H^0(\mathfrak{g},\mathfrak{g}) = H^1(\mathfrak{g},\mathfrak{g}) = \{0\}$.*

Sea $\mathfrak{n}$ un álgebra de Lie nilpotente y $\mathfrak{t} \subset Der(\mathfrak{n})$ un subálgebra toral maximal, es decir, un subálgebra abeliano cuyos elementos son *ad*-semisimples y que es maximal, denominada toro maximal [82]. Como en la teoría clásica [55], el álgebra $\mathfrak{t}$ induce una descomposición del espacio de raíces sobre $\mathfrak{n}$:

$$\mathfrak{n} = \sum_{\alpha \in \mathfrak{t}^*} \mathfrak{n}_\alpha$$

Siendo $\mathfrak{t}^* = Hom(\mathfrak{t}, \mathbb{C})$ y el espacio de raíces dado por

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \mid [h, X] = \alpha(h).X \quad \forall h \in \mathfrak{t}\}$$

Si $\mathfrak{n}_\alpha \neq 0$, la forma α es llamada raíz de $\mathfrak{n}$. Sea $\Phi = \{\alpha \mid \mathfrak{n}_a \neq 0\}$ el sistema de raíces de $\mathfrak{n}$ (asociado al toro $\mathfrak{t}$).

Proposición 7 *Sea* $\mathfrak{t} \subset Der(\mathfrak{n})$ *un toro* $\mathfrak{n}$*. entonces existe un sistema generador* $\{X_1, .., X_n\}$ *de* $\mathfrak{n}$ *y formas* $\{\alpha_1, .., \alpha_n\} \subset \mathfrak{t}^*$ *tal que* $[h, X_i] = \alpha_i(h) X_i, \ 1 \leq i \leq n$.

Demostración. La acción de $\mathfrak{t}$ sobre $\mathfrak{n}$ muestra que la aplicación $\rho : \mathfrak{t} \to Der(\mathfrak{n})$ es una representación de $\mathfrak{n}$, y como el espacio de raíces es irreducible, la representación es completamente reducible. Como el álgebra de conmutadores $C^1\mathfrak{n}$ es claramente $\mathfrak{t}$-submódulo de $\mathfrak{n}$, existe un submódulo complementario de $C^1\mathfrak{n}$. Esto debe generar el álgebra, como los generadores de un álgebra de Lie son extraídos del cociente por el álgebra derivada. Por lo tanto se puede encontrar una base $\{X_1, .., X_n\}$ del complementario $\mathfrak{m}$ que satisfaga las condiciones expuestas. La minimalidad del sistema generador se justifica fácilmente.

Definición 18 *La dimensión del toro maximal definido anteriormente es llamado el rango de* $\mathfrak{n}$ *y denota* $rank(\mathfrak{n})$.

En 1945, Mal'cev demostró que cualquiera dos toros maximales de derivaciones son conjugados por un automorfismo interno. De aquí se sigue que el rango es un invariante del álgebra de Lie.

Teorema 6 *Sea* $\mathfrak{g}$ *un álgebra de Lie resoluble y completa. Entonces*

1. $\mathfrak{g}$ *se descompone como* $\mathfrak{g} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$*, donde* $\mathfrak{n}$ *es el nilradical de* $\mathfrak{g}$ *y* $\mathfrak{t}$ *es el subálgebra toral maximal.*

2. *La restricción* $\mathrm{ad}_{\mathfrak{g}}\mathrm{h}|_{\mathfrak{n}}$ *es un subálgebra abeliana de* $Der(\mathfrak{n})$*, y un toro maximal sobre* $\mathfrak{n}$.

Este resultado, cuya demostración se encuentra en [64], permite referirse directamente a la descomposición, lo cual es de interés para el análisis estructural de tal álgebra de Lie.

Proposición 8 *Sea* $\mathfrak{g} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ *resoluble y completa. Entonces el centro del nilradical es igual a su centralizador en* $\mathfrak{g}$.

Demostración. La inclusión $Z(\mathfrak{n}) \subset C_{\mathfrak{g}}(\mathfrak{n})$ es evidente. Sea $Z \in C_{\mathfrak{g}}(\mathfrak{n})$. Por la descomposición existe $h \in \mathfrak{t}$, $X \in \mathfrak{n}$ tal que $Z = h + X$. Sea $\{X_1, .., X_n\}$ un conjunto minimal de generadores de $\mathfrak{n}$ y seas $V = \langle X_1, .., X_n \rangle$ el espacio generados por ellos. Entonces

$$[h + X, X_i] = \alpha_i(h) X_i + [X, X_i], \ \forall i$$

y como $V \cap C^1\mathfrak{n} = \{0\}$, $\alpha_i(h) X_i = 0$ para todo i. Esto implica $[h, Y] = 0$ para todo elemento de $\mathfrak{n}$, por lo tanto $h \in C_{\mathfrak{g}}(\mathfrak{n})$, y por ser abeliano $\mathfrak{t}$, $h \in Z(\mathfrak{g}) = \{0\}$.

Definición 19 *Un álgebra de Lie nilpotente* $\mathfrak{n}$ *es denominada de tipo* n *si admite un conjunto minimal de generadores de cardinal* n.

Definición 20 *Un álgebra de Lie nilpotente* $\mathfrak{n}$ *de tipo* n *es llamado de rango maximal si* $rank(\mathfrak{n}) = n$.

Para todo álgebra de Lie resoluble $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ donde $\mathfrak{t}$ es el toro maximal $\mathfrak{n}$. A menudo es conveniente llamar al rango de $\mathfrak{g}$ el rango de $\mathfrak{n}$.
El resultado central para álgebras de Lie resolubles de rango maximal es el siguiente:

Teorema 7 *Sea* $\mathfrak{n}$ *un álgebra de Lie nilpotente de rango maximal y de toro maximal* $\mathfrak{t}$ *. Entonces la suma semidirecta* $\mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ *es un álgebra de Lie resoluble completa.*

Este teorema, debido a Zhou y Meng, véase [69], tiene muchas consecuencias importantes para la teoría de álgebras de Lie completas:

Proposición 9 *Dos álgebras de Lie resolubles y completas de rango maximal son isomorfas si y solo si sus correspondientes nilradicales son isomorfos.*

Proposición 10 *Toda álgebra de Lie compleja completa es algebraica.*

Demostración. Como $\mathfrak{g}$ no es nilpotente existe un vector X distinto de cero tal que $ad_{\mathfrak{g}}X$ tiene una parte semisimple distinta de cero $(adX)_s$. Que es claramente una deriviación de $\mathfrak{g}$ (como consecuencia de la descomposición de Jordan). Pero toda derivación es interna, por lo que se puede encontrar $Y \neq 0$ con $ad\,(Y) = (adX)_s$. Como esta derivación es diagonalizable, el álgebra de Lie $\mathfrak{g}$ es algebraica.

Proposición 11 *Sea* $\mathfrak{g}$ *un álgebra de Lie compleja que satisfaga* $\dim Der\mathfrak{g} = \dim\mathfrak{g}$. *Entonces* $\mathfrak{g}$ *es algebraica. Además, si* $\dim\mathfrak{g} - \dim D(\mathfrak{g}) \geq 1$, *entonces* $\mathfrak{g}$ *es completa.*

De hecho, si el centro de $\mathfrak{g}$ es nulo, toda derivación es interna (porque $Der\mathfrak{g}$ y $\mathfrak{g}$ tienen la misma dimensión). Si $\mathfrak{g}$ es nilpotente, entonces es algebraica. Si no, la parte semisimple de una derivación interna no nilpotente es también un derivación interna y $\mathfrak{g}$ es algebraica. Supóngase que $Z(\mathfrak{g}) \neq \{0\}$. Como todo álgebra de Lie $\mathfrak{g}$ puede descomponerse como $\mathfrak{g} = \mathfrak{s} + \mathfrak{t} + \mathfrak{n}$ donde $\mathfrak{t}$ es un subálgebra nilpotente del radical que comnuta con $\mathfrak{s}$. Supóngase que el radical no es nilpotente, entonces $\mathfrak{t} \neq \{0\}$. Se quiere probar que la hipótesis implica que toda derivación es interna o que es distinta de cero en el centro y cero en el subálgebra derivada. Considérese que el conjunto A de endomorfismos sobre $\mathfrak{g}$ con valores en $Z(\mathfrak{g})$. Si $f \in A$, entonces $f \in Der\mathfrak{g}$ si y solo si $f(X) = 0$ para todo $X \in [\mathfrak{g}, \mathfrak{g}]$. Tal derivación no es interna. Considérese el ideal $I = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{n}$ de $\mathfrak{g}$. Se tiene

$$\dim \mathrm{ad}\mathfrak{g} = \dim \mathfrak{g} - \dim \mathrm{Z}(\mathfrak{g})$$

$$\dim A = \dim Z(\mathfrak{g}) \times \mathrm{codim} I$$

Entonces

$$\dim(\mathrm{ad}\mathfrak{g} + \mathrm{A}) = \dim \mathfrak{g} + (\mathrm{codimI} - 1) \times \dim \mathrm{Z}(\mathfrak{g}).$$

Como $\dim Der\mathfrak{g} = \dim \mathfrak{g}$, $Z(\mathfrak{g}\ ) \neq \{0\}$ implica $\mathrm{codim} I = 1$. de este modo $\dim \mathfrak{t} = 1$ y $Der\mathfrak{g} = ad\mathfrak{g} + A$. Sea $X \in \mathfrak{t}$. La parte semisimple de $\mathrm{ad}X$ puede escribirse $\mathrm{ad}Y$. Entonces $\mathfrak{g}$ es algebraica.

1.1.13 La variedad algebraica L^n.

En el estudio de la rigidez de las álgebras de Lie es fundamental la noción de variedad algebraica de álgebras de Lie, denotada por L^n. Si un álgebra de Lie $\mathfrak{g}$ se define como el par $(\mathbb{C}^n, \mu)$, donde μ es la aplicación bilineal antisimétrica correspondiente al corchete definido sobre $\mathfrak{g}$, el conjunto de todas las posibles μ de las álgebras de dimensión n se le denomina variedad algebraica de las álgebras de Lie L^n [31, 49]. Sobre el que se puede definir una métrica, la de Zariski o euclídea, y estudiarlo desde un punto de visa topológico. A continuación se exponen sus propiedades más importantes.

Leyes de álgebras de Lie sobre $\mathbb{C}^n$

Definición 21 *Una ley de álgebra de Lie sobre* $\mathbb{C}^n$ *es una aplicación lineal*

$$\mu : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$$

que cumple:

$$\mu(X, Y) = -\mu(Y, X), \quad \forall X, Y \in \mathbb{C}^n,$$

$$\mu(X, \mu(Y, Z)) + \mu(Y, \mu(Z, X)) + \mu(Z, \mu(X, Y)) = 0, \quad \forall X, Y, Z \in \mathbb{C}^n.$$

Un álgebra de Lie compleja $n-$dimensional puede expresarse como el par $\mathfrak{g} = (\mathbb{C}^n, \mu)$ donde μ es una ley de álgebra de Lie sobre $\mathbb{C}^n$, es espacio vectorial subyacente a $\mathfrak{g}$ es $\mathbb{C}^n$ y μ el corchete de $\mathfrak{g}$. Se denota por L^n el conjunto de las leyes de álgebra de Lie sobre $\mathbb{C}^n$. Que es un subconjuto del espacio vectorial de las aplicaciones bilineales alternadas sobre $\mathbb{C}^n$.

Definición 22 *Dos leyes μ y $\mu' \in L^n$ se las denomina isomorfas, si existe $f \in Gl(n, \mathbb{C})$ tal que*

$$\mu'(X,Y) = f * \mu(X,Y) = f^{-1}(\mu(f(X), f(Y)))$$

para todo $X, Y \in \mathbb{C}^n$. En este caso, las álgebras de Lie $\mathfrak{g} = (\mathbb{C}^n, \mu)$ y $\mathfrak{g}' = (\mathbb{C}^n, \mu')$ son isomorfas.

Se denota $\mathcal{O}(\mu)$ al conjunto de leyes isomorfas a μ y se le denomina órbita de μ.
Sea una base $\{e_1, e_2, \cdots, e_n\}$ de $\mathbb{C}^n$. Las constantes de estructura de $\mu \in L^n$ son los números complejos C_{ij}^k dados por

$$\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k.$$

Fijada la base se pueden identificar la ley μ con sus constantes de estructura, las cuales satisfacen:

$$(1) \begin{cases} C_{ij}^k = -C_{ji}^k \ , \quad 1 \leq i < j \leq n \ , \quad 1 \leq k \leq n \\ \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{jl}^s = 0 \ , \quad 1 \leq i < j < k \leq n \ , \quad 1 \leq s \leq n. \end{cases}$$

Entonces L^n aparece como una variedad algebraica dentro del espacio lineal de las aplicaciones bilineales alternadas sobre $\mathbb{C}^n$, isomorfo a $\mathbb{C}^{\frac{n^3-n^2}{2}}$.
Sea $\mu \in L^n$ y se considera el subgrupo de Lie G_μ de $Gl(n, \mathbb{C})$ definido por

$$G_\mu = \{f \in Gl(n, \mathbb{C}) \mid \quad f * \mu = \mu\}$$

Este álgebra de Lie es el álgebra de Lie de las derivaciones $Der(\mathfrak{g})$ del álgebra de Lie $\mathfrak{g}$ de la ley μ. Se denota por $Der(\mu)$ a la órbita $\mathcal{O}(\mu)$, es isomorfa al espacio homogéneo $Gl(n, \mathbb{C})/G_\mu$. Que es una variedad diferenciable $\mathcal{C}^\infty$, de dimensión

$$\dim \mathcal{O}(\mu) = n^2 - \dim Der(\mu).$$

Proposición 12 *La órbita $\mathcal{O}(\mu)$ de la ley μ es un espacio homogéneo de dimensión*

$$\dim \mathcal{O}(\mu) = n^2 - \dim Der(\mu).$$

El espacio tangente a $\mathcal{O}(\mu)$ en μ.

Se considera el punto μ' cercano to μ en $\mathcal{O}(\mu)$. Existe $f \in Gl(n, \mathbb{C})$ tal que $\mu' = f * \mu$. Supóngase que f es cercano a la identidad: $f = Id + \varepsilon g$, con $g \in gl(n)$. Entonces

$$\begin{aligned} \mu'(X,Y) &= \mu(X,Y) + \varepsilon[-g(\mu(X,Y)) + \mu(g(X),Y) + \mu(X,g(Y))] \\ &\quad + \varepsilon^2[\mu(g(X),g(Y)) - g(\mu(g(X),Y) + \mu(X,g(Y)) - g\mu(X,Y)] \end{aligned}$$

y

$$lim_{\varepsilon \to 0} \frac{\mu'(X,Y) - \mu(X,Y)}{\varepsilon} = \delta_\mu g(X,Y)$$

donde $\delta_\mu g$ es el operador coborde de la cocadena g por la comohología de Chevalley del álgebra de Lie $\mathfrak{g}$ de ley μ. Para simplificar la escritura, se denota $B^2(\mu, \mu)$ y $Z^2(\mu, \mu)$ así como $B^2(\mathfrak{g}, \mathfrak{g})$ and $Z^2(\mathfrak{g}, \mathfrak{g})$

Proposición 13 *El espacio tangente a la órbita $\mathcal{O}(\mu)$ en el punto μ es el espacio $B^2(\mu, \mu)$ de los 2-cociclos de la comohología de Chevalley de μ.*

1.1.14 Deformaciones y contracciones de las álgebras de Lie.

Tanto las deformaciones como las contracciones aparecen en el campo de la geometría, así como en la física, donde han resultado ser unas herramientas eficaces en el estudio y comprensión de las leyes naturales [60, 65, 68]. Las deformaciones y contracciones juegan un importante papel en la teoría de la estabilidad de las álgebras de Lie, pues un álgebra de Lie es rígida si no admite ninguna deformación que no sea trivial, también se puede enfocar la cuestión analiarando si un álgebra puede ser obtenida como contracción de otra (véase [50]).

Contracciones de álgebras de Lie.

Considérese la variedad algebraica L^n con la topología de Zariski.

Definición 23 *Un álgebra de Lie* $\mu_0 \in L^n$ *es una contracción de* $\mu \in L^n$ *si* $\mu_0 \in \overline{\mathcal{O}(\mu)}$.

Sea f_ε una familia de isomorfismos de $\mathbb{C}^n$. Con la ley μ y la familia f_ε se puede construir la familia de leyes $\mu_\varepsilon \in \mathcal{O}(\mu)$ estableciendo $\mu_\varepsilon = f_\varepsilon * \mu$. Supóngase que $\lim_{\varepsilon\to 0}\mu_\varepsilon$ existe. Entonces, en este caso, $\mu_0 = \lim_{\varepsilon\to 0}\mu_\varepsilon$ está en L^n y pertenece a $\overline{\mathcal{O}(\mu)}$. Entonces μ_0 es una contracción de μ.

Deformaciones de álgebras de Lie.

Definición 24 *Una deformación formal de una ley* $\mu_0 \in L^n$ *es una familia* μ_t *con* $t \in \mathbb{C}$

$$\mu_t = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_i$$

donde el φ_i *es una aplicación bilineal antisimétrica* $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ *tal que* μ_t *satisface la identidad de Jacobi* $\mu_t \circ \mu_t = 0$.

Si se desarrolla la última expresión se obtiene:

$$\mu_t \circ \mu_t = \mu_0 \circ \mu_0 + t\delta_{\mu_0}\varphi_1 + t^2(\varphi_1 \circ \varphi_1 + \delta_{\mu_0}\varphi_2) + t^3(\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 + \delta_{\mu_0}\varphi_3) + ...$$

y la ecuación $\mu_t \circ \mu_t = 0$ es equivalente al sistema infinito

$$(I)\ \left\{\begin{array}{l} \mu_0 \circ \mu_0 = 0 \\ \delta_{\mu_0}\varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = -\delta_{\mu_0}\varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -\delta_{\mu_0}\varphi_3 \\ \vdots \\ \varphi_p \circ \varphi_p + \sum_{1\le i\le p-1} \varphi_i \circ \varphi_{2p-i} + \varphi_{2p-i} \circ \varphi_i = -\delta_{\mu_0}\varphi_{2p} \\ \sum_{1\le i\le p} \varphi_i \circ \varphi_{2p+1-i} + \varphi_{2p+1-i} \circ \varphi_i = -\delta_{\mu_0}\varphi_{2p+1} \\ \vdots \end{array}\right. .$$

Por lo que el primer término φ_1 de la deformación μ_t de la ley de álgebra de Lie μ_0 pertenece a $Z^2(\mu_0, \mu_0)$. Este término se le denomina parte infinitesimal de la deformación μ_t de μ_0. Para más detalles véase [44, 61].

Definición 25 *Una deformación de* μ_0 *es llamada deformación lineal si es de longitud uno, es decir, si es del tipo* $\mu_0 + t\varphi_1$ *con* $\varphi_1 \in Z^2(\mu_0, \mu_0)$.

Para estas deformaciones se tiene que necesariamente $\varphi_1 \circ \varphi_1 = 0$ lo que implica que $\varphi_1 \in L^n$.

Proposición 14 *Si $H^3(\mu_0, \mu_0) = 0$ entonces todo $\varphi_1 \in Z^2(\mu_0, \mu_0)$ es la parte infinitesimal de una deformación de μ_0.*

De hecho, si $\varphi_1 \in Z^2(\mu_0, \mu_0)$ entonces $\varphi_1 \circ \varphi_1 \in Z^3(\mu_0, \mu_0)$. Si $H^3(\mu_0, \mu_0) = 0$, entonces existe $\varphi_2 \in C^2(\mu_0, \mu_0)$ tal que $\varphi_1 \circ \varphi_1 = \delta\varphi_2$. En este caso $\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 \in Z^3(\mu_0, \mu_0)$. Existe $\varphi_3 \in C^2(\mu_0, \mu_0)$ tal que

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = \delta\varphi_3.$$

Por lo tanto, se puede resolver, paso a paso, todas las ecuaciones del sistema (I).

Considérese dos deformaciones μ_t^1 y μ_t^2 de la ley μ_0. Son llamadas equivalentes si existe isomorfismo Φ_t de $\mathbb{C}^n$ de la siguiente manera

$$\Phi_t = Id + \sum_{i \geq 1} t^i g_i$$

con $g_i \in gl(n, \mathbb{C})$ tal que

$$\mu_t^2(X, Y) = \Phi_t^{-1}(\mu_t^1(\Phi_t(X), \Phi_t(Y))$$

para todo $X, Y \in \mathbb{C}^n$.

Definición 26 *Una deformación μ_t de μ_0 se denomina trivial si es equivalente a μ_0.*

Sea $\mu_t^1 = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_i$ y $\mu_t^2 = \mu_0 + \sum_{t=1}^{\infty} t^i \psi_i$ dos deformaciones equivalentes de μ_0. Es fácil comprobar que

$$\varphi_1 - \psi_1 \in B^2(\mu_0, \mu_0).$$

Por lo tanto se puede considerar que el conjunto de las partes infinitesimales de deformaciones está parametrizado por $H^2(\mu_0, \mu_0)$.

Supóngase que μ_t es una deformación de μ_0 para la cual $\varphi_t = 0$ para $t = 1, .., p$. Entonces $\delta_{\mu_0}\varphi_{p+1} = 0$. Si además $\varphi_{p+1} \in B^2(\mu_0, \mu_0)$, existe $g \in gl(n, \mathbb{C})$ tal que $\delta_{\mu_0} g = \varphi_{p+1}$. Considérese el isomorfismo $\Phi_t = Id + tg$. Entonces

$$\Phi_t^{-1}\mu_t(\Phi_t, \Phi_t) = \mu_0 + t^{p+2}\varphi_{t+2} + ...$$

y de nuevo $\varphi_{t+2} \in Z^2(\mu_0, \mu_0)$.

El siguiente teorema, debido a Nijenhuis y Richardson [71], es fundamental en la teoría de la rigidez de las álgebras de Lie.

Teorema 8 *Si $H^2(\mu_0, \mu_0) = 0$, entonces μ_0 es cohomológicamente rígida, es decir que toda deformación es equivalente a μ_0.*

Para ver más sobre la cohomología y caracterización de las álgebras de Lie ver [73, 83]

1.1.15 Rigidez de las álgebras de Lie.

Un álgebra de Lie $\mathfrak{g}$ de $\mathcal{L}_n$ es rígida si su órbita $\mathcal{O}(\mathfrak{g})$, el conjunto de las álgebras de Lie isomorfas a $\mathfrak{g}$, es un abierto en la topología de Zariski. Carles en [24, 27] demuestra que esto es equivalente a que $\mathcal{O}(\mathfrak{g})$ sea un abierto de la topología euclídea en el espacio de las constantes de estructura de $\mathfrak{g}$ $\mathbb{C}^N$ con $N = \frac{n^3 - n^2}{2}$.
Al concepto de rigidez, que es puramente topológico, se le puede dar un enfoque algebraico, pues Jacobson en [57] da una condición necesaria para que un álgebra de Lie sea rígida, es que

la segunda clase de comohología sea nula. Esta condición no es suficiente, puesto que existen álgebras de Lie rígidas con dim $(H^2(\mathfrak{g},\mathfrak{g})) \neq 0$.
Carles demostró en [23] que un álgebra de Lie es rígida si es isomorfa al álgebra de Lie de un grupo algebraico. Como la algebraicidad es equivalente a la descomponibilidad del álgebra, entonces un álgebra de Lie resoluble rígida $\mathfrak{g}$ admite una descomposición $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ donde $\mathfrak{t}$ es un toro maximal de derivaciones externas, es decir que es un subalgebra abeliana de $\mathfrak{g}$ en el que todos sus elementos son semisimples.
Ancochea y Goze introducen en [9] el sistema de raíces S de un álgebra de Lie rígida, que es el sistema de ecuaciones lineales formado por las expresiones [1]:

$$x_i + x_j = x_p, \quad \text{si la } X_p \text{ componente de } [X_i, X_j] \text{ es distinta de cero.} \tag{1.22}$$

y establecen que un álgebra de Lie no es rígida si

$$\text{rang}(S) \neq n - 1, \tag{1.23}$$

lo que es equivalente a que un álgebra de Lie no es rígida si

$$\text{rang}(S) \neq D^1(\mathfrak{g}) - 1. \tag{1.24}$$

Donde $D^1 = [\mathfrak{g}, \mathfrak{g}]$.

Definición 27 *Un álgebra de Lie $\mathfrak{g}$ se la denomina descomponible si se puede expresar como*

$$\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus} \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$$

donde $\mathfrak{s}$ subalgebra de Levi, $\mathfrak{n}$ es el nilradical y $\mathfrak{t}$ una subálgebra abeliana cuyos elementos son ad-semisimples y los cuales satisfacen $[\mathfrak{s} + \mathfrak{t}, \mathfrak{t}] = 0$. Esta descomposición es llamada descomposición normal de $\mathfrak{g}$.

Un resultado central en la teoría de la rigidez de las álgebras de Lie es el teorema de Carles [29]:

Teorema 9 *Toda álgebra de Lie compleja rígida $\mathfrak{g}$ es algebraica.*

Demostración. Supóngase que el radical $\mathfrak{r}$ no es nilpotente, ya que si lo fuese sería algebraica, pues toda álgebra de Lie que tiene un radical nilpotente es algebraica. Sea $\mathfrak{a}$ un ideal de codimensión uno de $\mathfrak{g}$. Considérese un subespacio de dimensión uno V de $\mathfrak{g}$ tal que

$$\mathfrak{g} \ = \mathfrak{a} \oplus V$$

Si $X \in V$, $X \neq 0$, entonces ad$X$ restringido a $\mathfrak{a}$ es una derivación f_0 de $\mathfrak{a}$. Entonces $\mathfrak{g}$ aparece como una extensión de la derivación de $\mathfrak{a}$. para cada derivación $f \in Der\mathfrak{a}$, se puede construir un álgebra de Lie $\mathfrak{g}^f$ definido por

$$\mathfrak{g}^{\ f} = \mathfrak{a} \oplus \mathbb{C}$$

y, si $X \in \mathbb{C}$ es un vector no trivial fijo

$$[X, Y] = f(Y)$$

para todo $Y \in \mathfrak{a}$. Si $f = f_0$, entonces $\mathfrak{g}^f = \mathfrak{g}$. El álgebra de Lie $\mathfrak{g}^f$ es isomorfo a $\mathfrak{g}$ si y solo si existe un atomorfismo h de $\mathfrak{a}$ tal que

$$f - \alpha(h^{-1} \circ f_0 \circ h) \in Int\mathfrak{a}$$

[1]Véase Favre [36] para el estudio general de pesos de álgebras de Lie nilpotentes.

donde $Int\mathfrak{a}$ es el conjunto de las derivaciones internas de $\mathfrak{a}$, por algún $\alpha \in \mathbb{C}$. Supóngase que $\mathfrak{g}$ es rígida. Entonces cada ideal de codimensión uno $\mathfrak{I}$ es un ideal característico, es decir que satisface

$$F(\mathfrak{I}) \subseteq \mathfrak{I},$$

para toda derivación F de $\mathfrak{g}$.
Otro enfoque de la noción de rigidez de álgebras de Lie es entenderla como la estabilidad del tensor de estructura que las define frente a alguna acción de grupo o una deformación, es decir, que cualquier variación en dicho tensor por medios geométricos o algebraicos el álgebra de Lie resultante es isomorfa a la inicial.

Teorema 10 *Toda álgebra de Lie rígida satisface alguna de las siguientes condiciones:*

1. *EL radical $\mathfrak{r}$ no es nilpontente y satisface $\dim Der(\mathfrak{g}) = \dim \mathfrak{g}$. Si además $codim_{\mathfrak{g}}[\mathfrak{g},\mathfrak{g}] > 1$, entonces $\mathfrak{g}$ es completa.*
2. *El radical es nilpotente y verifica una de las siguientes restricciones:*
   - *(a) $\mathfrak{g}$ es perfecta (es decir, $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$),*
   - *(b) $\mathfrak{g}$ es la suma directa de $\mathbb{K}$ y un álgebra estable perfecta, cuyas derivaciones son todas internas,*
   - *(c) $\mathfrak{g}$ no es perfecta, no tiene un factor abeliano distinto de cero y satisface $\mathfrak{t}_e(\mathfrak{g}) = 0$,*

   donde $\mathfrak{t}_e(\mathfrak{g})$ denota la dimensión de $Der(\mathfrak{g})$ generada por las derivaciones semisimples externas.

Definición 28 *Una deformación $\mathfrak{g}_t$ $(0 \leq t \leq 1)$ se la llama de tipo meseta si $\mathfrak{g}_0 \not\simeq \mathfrak{g}_1$ y $\mathfrak{g}_t \simeq \mathfrak{g}_1$ para todo $t \in (0,1]$.*

La relación que hay entre deformaciones y contracciones viene dada por el siguiente teorema.

Teorema 11 *Para toda contracción $\mathfrak{g} \rightsquigarrow \mathfrak{g}'$ existe una deformación de tipo meseta $\mathfrak{g}' \to \mathfrak{g}$ inversa a la contracción. Recíprocamente, para toda deformación de tipo meseta existe una contracción inversa a esta.*

Un estudio complementario de la relación entre deformaciones y contracciones se encuentra en el trabajo de Fialowski [38].

Esquema de Jacobi.

En este apartado se van a establecer las relaciones entre las constantes de estructura de la álgebras de Lie de dimensión n, estudiadas por Carles en [29], que están definidas por los corchetes:

$$[X_i, X_j] = C_{i,j}X_{i+j} \text{ con } 1 \leq i < j,\ i+j \leq n \text{ y } C_{2,3} \neq 0.$$

Se puede establecer, sin perdida de generalidad, que $C_{1,j} = 1$. Para pasar de la dimensión n a $n+1$ se añaden las constantes de estructura $C_i, n+1-i$ para $i = 1, 2, \ldots, [\frac{n}{2}]$, y como las anteriores $C_{1,n} = 1$. Teniendo en cuenta las nuevas expresiones de Jacobi en las que aparecen estas constantes, es decir, en las tripletas $i < j < k,\ i+j+k = n+1$:

$$C_{i,j+}C_{j,k} + C_{j,i+k}C_{i+k} + C_{k,i+j}C_{i,j} = 0$$

Para $n=5$ se tiene que:

$$C_{1,4}=1 \text{ y } C_{2,3}=1.$$

Para $n=6$ de la tripleta (X_1,X_2,X_3) se obtiene la ecuación $1-C_{2,4}=0$, por lo que las nuevas constantes de estructura son:

$$C_{1,5}=1 \text{ y } C_{2,4}=1.$$

Para $n=7$ la tripleta (X_1,X_2,X_4) la identidad de Jacobi genera la ecuación $C_{3,4}+C_{2,5}=0$. Se asigna el parámetro $t\in\mathbb{K}$ que da lugar a una familia continua de álgebras de Lie. Por ello:

$$C_{1,6}=1,\ C_{2,5}=t,\ \text{ y } C_{3,4}=1-t.$$

Para $n=8$ de la identidad de Jacobi sobre las tripletas (X_1,X_2,X_5), (X_1,X_3,X_4) se obtienen las constantes de estructura:

$$C_{1,7}=1,\quad C_{2,6}=2t-1,\ \text{ y } C_{3,5}=1-t.$$

Para $n=9$

$$C_{2,7}+C_{3,6}=2t-1,\ C_{C_{3,6}}+C_{4,5}=1-t. \text{ y } C_{3,6}-C_{4,5}=(1-t)\,C_{2,7}$$

Si se simplifica el sistema formado por las tres ecuaciones anteriores se obtiene:

$$C_{1,8}=1,\quad C_{2,7}=\frac{12}{3-t}-5,\quad C_{3,6}=4+2t-\frac{12}{3-t} \text{ y } C_{4,5}=-3-3t-\frac{12}{3-t}.$$

Para $n=10$ las ecuaciones que se obtienen son:

$$C_{2,8}+C_{3,7}=-5+\frac{12}{3-t},\ C_{3,7}+C_{4,6}=4+2t-\frac{12}{3-t},\ C_{4,6}=-3-3t+\frac{12}{3-t} \text{ y } tC_{3,7}=(1-t)C_{2,8}.$$

De donde se tiene:

$$C_{1,9}=1,\ C_{2,8}=-12-5t+\frac{36}{3-t},\ C_{3,7}=7+5t-\frac{24}{3-t},\ C_{4,6}=3-3t+-\frac{12}{3-t}.$$

Para $n=11$ la identidad de Jacobi genera:

$$C_{2,9}+C_{3,8}=-12+5t+\frac{36}{3-t},\ C_{3,8}+C_{4,7}=7+5t-\frac{24}{3-t}.\ C_{4,7}+C_{5,6}=-3-3t+-\frac{12}{3-t},$$

$$\left(4+2t-\frac{12}{3-t}\right)C_{2,9}+tC_{4,7}-C_{5,6}=0 \text{ y } \left(3+3t-\frac{12}{3-t}\right)C_{2,9}+tC_{4,7}-C_{5,6}=0.$$

De las que se obtienen.

$$C_{1,10}=1,\ C_{2,9}=\frac{3-7+t^2+5t^3}{4t-4t-2t^2},\ C_{3,8}=\frac{-5t^4+12t^3-22t^2+24t-9}{2t\,(t^2-5t+6)},$$

$$C_{4,7}=\frac{3\,(5t^4-16t^3-60t^2+148t+3)}{2(t-3)(t-2)t} \text{ y } C_{5,6}=-\frac{3(t-1)^3(7t-3)}{2(t-3)(t-2)t}.$$

Para $n=12$ de la identidad de Jacobi se obtiene:

$$C_{2,9}-C_{2,10}-C_{3,9}=0,\ C_{2,9}+C_{3,9}+C_{4,8}=-12-5t+\frac{36}{3-t},\ -C_{2,9}+C_{4,8}+C_{3,7}=19+10t-\frac{60}{3-t},$$

$$C_{2,9}+C_{5,7}=-22-13t+\frac{72}{3-t},\ \left(7-5t-\frac{24}{3-t}\right)C_{2,10}+\left(5-\frac{12}{3-t}\right)C_{3,9}=C_{5,7},$$

$$\left(-3-3t+\frac{12}{3-t}\right)C_{2,10}=(2t-1)\,C_{4,8} \text{ y } \left(-3-3t+\frac{12}{3-t}\right)C_{3,9}-(1-t)\,(C_{4,8}-C_{5,7})=0.$$

De donde se obtienen las constantes de estructura:

$$C_{1,11}=1,\ C_{2,10}=53+28t-\frac{168}{3-t}+4C_{2,9},\ C_{3,9}=-53-28t+\frac{168}{3-t}-3C_{2,9},$$

$$C_{4,8}=\frac{3\,(1-t)^2\,(3-8t+6t^2)}{(3-t)\,(2t-3t^2)} \text{ y } C_{5,7}=-22-13t+\frac{72}{3-t}-C_{2,9}$$

Estas condiciones con las de $n=11$ da dos posibles resultados $t=1$ o $t=\frac{9}{16}$ con lo que quedan sólo dos álgebras $\mathfrak{f}_{12}(1)$ y $\mathfrak{f}_{12}(\frac{9}{16})$.
De todo esto le sigue el lema:

Lema 1 *Toda álgebra de Lie con $C_{1,i}\neq 0$ $(1<j<n)$ y $C_{2,3}\neq 0$ con $n\geq 12$ es isomorfa a $\mathfrak{w}_n$ o a $\mathfrak{f}_n$.*

Demostración. El álgebra de Lie $\mathfrak{f}_n$ es igual $\mathfrak{f}_n(1)$ si $n\leq 12$. Las álgebras de dimensión $n+1$ que verifican las hipótesis y obtenidas a partir de $\mathfrak{f}_n$ satisfacen Jacobi por los $i<j<k,\ i+j+k=n+1$. Para $i=1$, se tiene que $C_{2,n-1}=1-C_{3,n-2}$ (si $j=2$)y para $j>2$:

$$C_{j,n+1-j}=(-1)^{p-j}C_{p,p+1} \text{ si } n=2p \text{ y } 0(\text{ si } n=2p+1).$$

Se tiene $C_{5,n-4}=C_{3,n-2}$ para $(2,3,n-4)$ de donde $C_{j,n+1-j}=$ para $j\geq 3$ en ambos casos se establece que $C_{2,n-1}=1$ y se obtiene $\mathfrak{f}_n+1$. Entonces se tiene que $\mathfrak{f}_n=\mathfrak{f}_n(1)$ para todo $n\geq 7$. Hay que destacar que para todo n, $\mathfrak{w}$ verifica las hipótesis del lema no es isomorfa a $\mathfrak{f}_n$ para $n\geq 7$ debido a $\mathfrak{w}_7\simeq\frac{9}{10}$ para $n\leq 12$ y por recurrencia para $n\geq 12$, se obtiene para $i+j+k=n+1$ las condiciones de Jacobi:

$$(i-j)C_{i+j,k}+(j-k)C_{j+k,i}+(k-i)C_{k+i,j}=0.$$

Si se establece $C_{j,n+1-j}=2j-n-1+a_j$ se obtiene para $i=1$ y se fija $C_{1,n}=1-n$:

$$(1-j)a_j+1=(n-j-1)a_j\ 1<j<\frac{n}{2} \text{ es decir}$$

$$a_j=(-1)^{l-j}\frac{(l-2)!(n-l-1)!}{(j-2)!(n-j-1)!}a_j \text{ para Âž} <j\leq l<\frac{n+1}{2}$$

Si $n=2p+1$, las condiciones de Jacobi hacen que $a_p=0$ para $(1,p,p+1)$ de donde $a_j=0\ \forall j$. Si $n=2p$ se obtiene $(p-3)a_p+a_2-(p-2)a_{p-1}=0$ para $(2,p-1,p)$ de lo que se deduce que $a_p=0$ ya que $a_j=0$ para todo j con a_2 y a_p en función de a_p, debido a la última expresión escrita arriba. Con la obtención de las dos álgebras para $n=12$ en el estudio precedente termina la demostración.
Unas concidciones suplementarias para el estudio del esquema de Jacobi pueden hallarse, por ejemplo, en [28, 29].

1.2 Descripción de la memoria.

En esta sección se hace una breve descripción de los capítulos 2-6. La motivación de esta memoria que aquí se expone es la de encontrar y describir álgebras de Lie rígidas de distintos tipos y diversas dimensiones. Determinar si dada un álgebra de Lie $\mathfrak{g}$ es rígida o no desde la definición de rigidez, es decir, un álgebra de Lie es rígida si su órbita, el conjunto de todas las álgebras isomorfas a $\mathfrak{g}$, es un conjunto abierto en la topología de Zariski en L_n, suele resultar laborioso y complicado. L_n es la variedad algebraica de las álgebras de Lie sobre el cuerpo $\mathbb{C}^n$. El que las topologías de Zariski y la euclídea sean equivalentes no hace de la definición una herramienta útil para valorar la rigidez de un álgebra. Nijenhuis and Richardson en [71] establecen que si un álgebra de Lie posee la segunda clase de cohomología de Chevalley nula entonces es rígida. Aunque no es condición necesaria, esto convierte el problema topológico en un problema algebraico. Que la dim $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ no sea una condición necesaria implica que existen álgebras de Lie que son rígidas pero que no tienen la segunda clase de cohomología nula. A estas álgebras se las denomina geométricamente rígidas. En el caso de que sí que presenten dim $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ se la denomina algebraicamente rígidas.
Carles en [23] demuestra que si un álgebra de Lie es rígida entonces es algebraica, si además es resoluble se deduce que $\mathfrak{g}$ se puede descomponer como $\mathfrak{g} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$, donde $\mathfrak{t}$ es un toro maximal de derivaciones externas y $\mathfrak{n}$ es el nilradical de $\mathfrak{g}$. Es por esto que este trabajo se centre en álgebras de Lie resolubles, de modo que primero se exigen unas condiciones determinadas a las constantes de estructuras del nilradical y cuando se han hallado éstas, quedan definidos los elementos del toro, aunque las álgebras que se detallan a lo largo de esta memoria tienen genéricamente rango uno, es decir, que el toro está generado por un único elemento.
Otra herramienta útil en la construcción de álgebras de Lie rígidas fue introducida por Ancochea en [1], donde demuestra que la condición que debe cumplir un sistema de ecuaciones lineales para que el álgebra sea rígida es que el rango de S sea igual a $n - 1$, siendo S el sistema que está formado por las ecuaciones $x_i + x_j = x_{i+j}$ si la componente X_{i+j} es distinta de cero en $[X_i, X_j]$, y los x_l son los valores propios del toro cuando actúa sobre el nilradical.

1.2.1 Capítulo 2.

Nuevos ejemplos de álgebras de Lie resolubles reales de rango uno geométricamente rígidas.

En este capítulo, cuyo contenido puede verse en [7], se estudia la rigidez geométrica y algebraica de las álgebras de Lie descritas por Bratzlavsky, denominadas $\mathfrak{B}_{4,n}$ que posen un espectro de los autovalores del toro maximal sobre el nilradical $\mathfrak{n}$ de dimensión n Spec $= \{1, 4, 5 \ldots, n+2\}$, así como las álgebras de Lie a las que les caracteriza este espectro pero que tienen unas constantes de estructuras distintas a las de Bratzlavsky, para esto, se analiza qué constantes de estructura pueden ser distintas de cero y que respondan al espectro antes definido.

Álgebras of Lie resolubles de rango uno $\mathfrak{Br}_{4,n}$.

Estas álgebras se pueden escribir como el producto semidirecto $\mathfrak{Br}_{4,n} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ donde $\mathfrak{t} = \{T\}$ es el toro maximal de derivaciones y $\mathfrak{n} = \{X_1, X_2, \ldots, X_n\}$ es un álgebra nilpotente de dimensión n denominado nilradical. Estas álgebras están definidas por los corchetes:

$$\begin{array}{ll} [T, X_1] = X_1, & [T, X_i] = (i+2)X_i, \text{ con } 2 \leq i \leq n, \\ [X_1, X_i] = X_{i+1}, \ 2 \leq i \leq n-1, & [X_2, X_i] = X_{i+4}, \ 3 \leq i \leq n-4. \end{array} \tag{1.25}$$

El resto de los corchetes son nulos.
En este apartado se demuestran las siguientes proposiciones:

1. *Para $n = 7, 8$ la dimensión de $H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n})$ es cero, por lo tanto $\mathfrak{Br}_{4,n}$ es cohomológicamente rígida.*
2. *Para $n = 9, 10$ $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 1$.*
3. *Para $n \geq 11$ $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 2$.*
4. *La aplicación bilineal de Rim Sq: $H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) \to H^3(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n})$ satisface las siguientes propiedades para $\varphi, \ \psi \in H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n})$:*
   (a) *Para $n \leq 14$ la aplicación de Rim Sq es idénticamente nula $[Sq(\varphi)] = [Sq(\psi)] = 0$.*
   (b) *Para $n \geq 15$ Sq es distinta de cero con $\mathrm{Ker}(Sq) = \langle [\varphi] \rangle$ y $Sq([\psi]) \neq 0$.*

Álgebras de Lie resolubles con toro maximal $\mathfrak{t}$.

En esta sección se trata de hallar la existencia de álgebras de Lie resolubles de dimensión $n+1$ que admitan la descomposición $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{g}_n$, donde t es el toro maximal de derivaciones definido en la sección anterior y que tiene un espectro de valores propios sobre el nilradical

$$\mathrm{spec}(\mathfrak{t}) = \{1, 4, 5 \cdots, n+2\} \tag{1.26}$$

El nilradical $\mathfrak{n}$ queda definido por los siguientes corchetes:

$$\begin{array}{ll} [X_1, X_i] = C_{1,i}^{i+1} X_{1+i}, & 2 \leq i \leq n-1, \\ [X_i, X_j] = C_{i,j}^{i+j+2} X_{i+j+2}, & 2 \leq i < j, \ i+j \leq n-2. \end{array} \tag{1.27}$$

Sin pérdida de generalidad, se puede establecer que las constantes de estructura tengan el siguiente valor $C_{1,i}^{i+1} = 1$. A continuación se resuelve el sistema de ecuaciones lineales formado por las identidades de Jacobi con las tripletas $\{X_1, X_i, X_j\}$ con lo que las constantes de estructura han de cumplir:

$$C_{i,j}^{i+j+2} - C_{i,j+1}^{i+j+3} - C_{i+1,j}^{i+j+3} = 0 \tag{1.28}$$

A continuación se resuelve este sistema y se sustituye en el resto de las identidades de Jacobi de las tripletas $\{X_i, X_j, X_l\}$, $2 \leq i < j < l \leq n$. Al resolver este sistema cuadrático se obtiene, en función de la dimensión, los siguientes resultados:

1. Para $n = 7, 8$ no existe ninguna condición de Jacobi cuadrática y la solución depende de un único parámetro. Si a dicho parḿetro le asignamos el valor uno se obtiene que $\mathfrak{g}_n \simeq \mathfrak{Br}_{4,n}$ y como se ha demostrado en la sección anterior estas álgebras son algebraicamente rígidas.

2. Para $9 \leq n \leq 15$ las soluciones de los sistemas son multiparamétricas. En cada una de las dimensiones las álgebras de Lie que se obtienen al asignar valores cualesquiera a dichos parámetros no son isomorfas entre sí. Por lo tanto estas álgebras de Lie no son rígidas.

3. Para $n = 16$ el resultado que se obtiene del sistema de ecuaciones cuadráticas depende de un único parámetro, que genera una familia parametrizada de álgebras de Lie, los distintos miembros de esta familia pueden ser deformados los unos en los otros por lo que las álgebras obtenidas para distintos valores del parámetro no son rígidas.
Además, también se obtienen tres álgebras de Lie aisladas, que corresponden a las raíces de un polinomio de grado tres. Las álgebras de Lie obtenidas a cada solución cumplen $\dim H^2 = 1$ y no son rígidas pues pueden ser deformadas en la familia parametrizada.

4. Para $n = 17$ las soluciones del sistema, como en el caso anterior, depende un único parámetro que genera una familia de álgebras de Lie parametrizada, cuyos miembros no son isomorfos dos a dos. Además, aparecen cuatro álgebras aisladas correspondientes a las raíces de dos polinomios de segundo grado. De nuevo, estas álgebras se pueden deformar en la familia parametrizada, por lo que no son rgídas.

5. Para $n = 18$ aparecen 17 soluciones de las cuales cinco son biparamétricas, doce son álgebras aisladas que son rígidas, cinco de ellas son álegras de Lie racionales entre las que se encuentra la ya estudiada $\mathfrak{Br}_{4,18}$, las álgebras de Lie resolubles asociadas al resto de soluciones presentan $H^2(\mathfrak{g}, \mathfrak{g}) = 0$.

6. Para valores $n \geq 19$ debido a las restricciones impuestas por las nuevas identidades de Jacobi el sistema de ecuaciones sólo admite cuatro o cinco soluciones dependiendo de la paridad de n. En el caso impar se obtienen dos familias parametrizadas y en el par sólo una familia parametrizada. Además aparecen tres soluciones aisladas, una de ellas es isomorfa a $\mathfrak{Br}_{4,19}$ y las otras dos son algebraicamente rígidas.

Rigidez del las Álgebras de Lie resolubles $\mathfrak{Br}_{4,n}$ con $n \geq 19$.

En esta sección se enuncia y se demuestra el siguiente teorema:
Teorema 1. *Para $n \geq 19$ las álgebras de Lie resolubles $\mathfrak{Br}_{4,n} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{f}_{4,n}$ son geométricamente rígidas y cumplen* $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 0$.

1.2.2 Capítulo 3

Algunas características de las álgebras de Lie resolubles reales de rango uno cohomológicamente rígidas con un nilradical que se contrae al modelo de álgebra de Lie filiforme $\mathcal{Q}_n$.

En este capítulo, cuyo contenido ha aparecido en [18], que amplia el anterior, se estudian las álgebras de Lie resolubles $\mathfrak{r}$ de rango uno que poseen un toro maximal de derivaciones $\mathfrak{t}$ cuyo espectro viene dado por los autovalores sobre el nilradical $\mathfrak{n}$, con $\mathfrak{r} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_n^k$ y $\mathrm{spec}(\mathfrak{t}) = \{1, k, k+1, \cdots, n+k-3, n+2k-3\}$.
Estos nilradicales se contraen en el llamado álgebra filiforme del modelo $\mathcal{Q}_n$, que es un álgebra nilpotente que cumple con $[\mathfrak{n}, \mathfrak{n}] \neq 0$ [84]. Se comienza estudiando en profundidad los nilradicales con un k bajo y se obtienen las álgebras de Lie resolubles que son cohomológicamente rígidas, se analizan los patrones que presentan dichas álgebras y se extrapolan los resultados para obtener familias infinitas de álgebras de Lie rígidas.
Con este trabajo se complementa y amplía el estudio ya realizado de álgebras de Lie rígidas por Ancochea, Carles, Goze y Campoamor-Stursberg, siguiendo las mismas técnicas y motivaciones que en los trabajos [8, 9, 26, 47] y es continuación del anterior, donde se estudian con detalle las álgebras que poseen los espectros $\mathrm{spec}(\mathfrak{t}) = \{1, 2, 3, \cdots, n\}$, $\mathrm{spec}(\mathfrak{t}) = \{1, 3, 4, \cdots, n+1\}$ y $\mathrm{spec}(\mathfrak{t}) = \{1, 4, 5, \cdots, n+2\}$.

Álgebras de Lie rígidas con un nilradical filiforme.

En esta sección se da la caracterización de las álgebras de Lie que tienen un nilarcical filiforme, es decir que el índice nilradical sea igual a $n-1$, esto implica que $\dim(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) = 2$, como la dimensión del toro está acotada superiormente por la inecuación:

$$\mathrm{rank}(\mathfrak{t}) \leq \dim\mathfrak{n} - \dim[\mathfrak{n}, \mathfrak{n}] \tag{1.29}$$

las álgebras de Lie con nilradical filiforme han de tener como mucho rango 2. En [84] se ha demostrado que existen dos clases de isomorfismo con rango 2, las llamadas L_n y Q_n. Las de rango uno han sido estudiadas en [50] y se demuestra que existen tres tipos, denominadas A_n^k, B_n^k y C_n^k. Como las álgebras de Lie de rango uno con nilradical filiforme no son isomorfas, el estudio de su rigidez se va a hacer por un lado la de las álgebras A_n^k y por otro las de B_n^k. Hay que señalar que de las álgebras tipo A_n^k se han obtenido una gran cantidad de álgebras rígidas para los valores de $k = 2, 3$ y 4 en [47], [25] y en el capítulo precedente, así como la completa clasificación de álgebras de Lie rígidas resolubles en dimensiones bajas [48]. Este trabajo se centra en las álgebras del tipo B_n^k para cualquier entero $k \geq 2$ y con la condición de que la constante de estructura $C_{2,3}^{3+k}$ sea distinta de 0.

Álgebras de Lie cohomológicamente rígidas de rango uno con un espectro de autovalores dado.

En esta sección se estudian algunas características genéricas de las álgebras de Lie resolubles reales denominadas $\mathfrak{r}_n^k = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_n^k$ de rango uno, donde $\mathfrak{t}$ es el toro maximal de derivaciones que tiene el siguiente espectro de autovalores sobre el nilradical

$$\mathrm{spect} = \{1, k, k+1, k+2, \cdots, n+k-3, n+2k-3\}$$

Sea $\{T, X_1, X_2, \cdots X_n\}$ la base de $\mathfrak{r}_n^k$, la acción del toro sobre el nilradical queda definida por su espectro, por lo que se tiene que:

$$[T, X_1] = X_1, \; [T, X_i] = (i+k-2)X_i, \; 2 \leq i \leq n-2, \; [T, X_n] = (n+2k-3)X_n. \tag{1.30}$$

El espectro también establece el resto de los corchetes entre los elementos del nilradical, que son:

$$\begin{array}{lr} [X_1, X_i] = C_{1,i}^{i+1} X_{i+1} & 2, \leq i \leq n-2, \\ [X_i, X_j] = C_{i,j}^{i+j-2} X_{i+j-2}, & 2 \leq i < j,\ i+j \leq n+1-k, \\ [X_i, X_{n+1-i}] = C_{i,n+1-i}^{n} X_n, & 2 \leq i \leq \frac{n}{2}. \end{array} \tag{1.31}$$

Al resolver el sistema de ecuaciones formado por las igualdades de Jacobi, imponiendo que $[X_2, X_{n-1}] \neq 0$ se obtienen las distintas álgebras en función de los valores de n y k que se detalla someramente a continuación.

Para que la descripción de todas las constantes de estructura no sea un proceso tedioso por exhaustivo se define la *secuencia diagonal*:

$$(\alpha_1, \alpha_2, \cdots, \alpha_p) = \left(C_{2,3}^{3+k}, C_{3,4}^{5+k}, \cdots, C_{p,p+1}^{n-1-\frac{1+(-1)^k}{2}} \right),\ p = \left[\frac{n-k-1-\frac{1+(-1)^k}{2}}{2} \right] \tag{1.32}$$

y el resto de las constantes de estructura quedan determinadas a partir de éstas.

En los casos $k \geq 5$.

Los resultados de estos casos se dan por separado porque presentan singularidades con respecto a los casos en los que los valores superiores de k.

4.1 En el caso $k = 2$.
En los casos en que $n = 8, 10$ aparecen algunas álgebras rígidas racionales, además hay que destacar que existen también unas álgebras complejas conjugadas y un par de racionales, todas ellas cohomológicamente rígidas. Si $n \geq 14 + 2l$ las conciciones de Jacobi obligan a que la *secuencia diagonal* sea $(0, 0, \cdots, 0)$ que corresponde al tipo Q_n, por lo que no hay álgebras de Lie de rango uno con el espectro antes definido.

4.2 $k = 3$.
Las primeras álgebras rígidas que aparecen cuando la dim $\mathfrak{n} = 14$, también existen con $n = 8$, con $n = 10, 12$ no aparecen ninguna cohomológicamente rígida, en $n = 16$ hay dos con la clase de cohomología nula y a partir de $n = 18 + 2l$ las identidades de Jacobi obligan a que *secuencia diagonal* $= \left(1, \overbrace{0, 0 \cdots, 0}^{5+l} \right)$ que corresponden a las del tipo B_n, y tienen la seguda clase de cohomología nula.

4.3 $k = 4$.
Este caso es similar al de $k = 2$, en él existen álgebras cohomológicamente rígidas a partir de $n = 8$, también aparecen las irracionales y complejas conjugadas y cuando $n \leq 20 + 2l$ con $l \leq 0$ las únicas álgebras existentes son las que tienen una *secuencia diagonal*$= (0, \cdots, 0)$ por lo que, de nuevo, no existen álgebras de Lie de rango uno con el espectro propuesto.

4.4 $k = 5$.
En este caso no aparece ninguna álgebra rígida para valores de $n = 12, 14$ y 16, para $n = 20$ existen dos álgebras que presentan una dim $H^2(\mathfrak{r}_{20}^5, \mathfrak{r}_{20}^5) = 1$, además no son geométricamente rígidas.
Para $n = 20$ y 22 existen álgebras que son cohomológicamente rígidas, en el último caso una de ellas es irracional.
En dimensiones mayores aparece una única álgebra que tiene como *secuencia diagonal*

$(1,0,0\cdots,0)$ y su cohomología depende de su dimensión. Hay que distinguir dos casos:

$$\dim H^2\left(\mathfrak{r}^5_{24+4l},\mathfrak{r}^5_{20+4l}\right)=1$$
$$\dim H^2\left(\mathfrak{r}^5_{26+4l},\mathfrak{r}^5_{26+4l}\right)=0$$

En ambos casos $l\geq 0$ y en el primero aunque no estas álgebras no presentan un segundo grupo de cohomología nulo son geométricamente rígidas.
Esta sección se cierra con una proposición, que resume las conclusiones sacadas de los cálculos realizados en la misma y se presenta un lista con las álgebras de Lie cohomológicamente rígidas con $n\leq 30$ y $k\geq 6$.

Proposición 3.
Sea $\mathfrak{n}^k_n$ un álgebra de Lie nilpotente que tiene las características exigidas anteriormente, entonces se cumplen los siguientes enunciados:

1. *Si $k\geq 6$ y par y $n\geq 4k$ entonces $\mathfrak{n}^k_n$ es isomorfa a Q_n.*
2. *Si $k\geq 7$ e impar y $n\geq 4k$ entonces $\mathfrak{n}^k_n$ es isomorfa a Q_n o al álgebra definida por los corchetes*:

$$\begin{array}{ll}[X_1,X_j]=X_{1+j}, & 2\leq j\leq n-2,\\ [X_2,X_j]=X_{j+k}, & 3\leq j\leq n-1-k,\\ [X_j,X_{n+1-j}], & (-1)^jX_n, 2\leq i\leq \frac{n}{2}.\end{array} \tag{1.33}$$

Álgebras cohomológicamente rígidas $n\geq 30$.

Tal y como se ha visto en al anterior sección, para valores impares de k y superiores a 5 comienzan a aparecer un número considerable de álgebras de Lie cohomológicamente rígidas que merece la pena estudiarse por separado. Como las soluciones irracionales presentan un patrón bastante complicado, en esta sección se presenta una lista de las álgebras cohomológicamente rígidas racionales entre los valores $30\leq n\leq 40$.

Algunas series de álgebras $H^2\left(\mathfrak{r}^k_n,\mathfrak{r}^k_n\right)=0$.

En esta sección se analizan y describen los patrones de las álgebras referidas en las listas de las dos secciones anteriores en función de los valores del parámetro k y de la dimensión n. Así mismo se demuestra su rígidez cohomológica.

1.2.3 Capítulo 4

Construcción algorítmica de álgebras de Lie rígidas resolubles determinadas por funciones generatrices.

En este capítulo, cuyo contenido ha sido publicado en [19], se extienden los trabajos de Ancochea y Goze en [47, 48] y Carles en [29], donde se estudian las álgebras de rango uno que tienen un espectro del toro sobre el nilradical $\text{spect}(\mathfrak{g}) = \{1, , k, k+1, \cdots, n+k-2\}$, es, a su vez, la continuación del primer capítulo.
Primero se introduce la noción de saturación de un nilradical, $[X_i, X_j] \neq 0$ si $i+j \in \text{spect }(\mathfrak{g})$, se impone esta condición a las álgebras introducidas por Carles en [26] y estudiadas por Ancochea y Goze en [48] y se encuentra que para valores de $n \geq 12$ sólo existen dos álgebras de Lie que cumplen estas condiciones. Una corresponde a la denominada de Bratzlavsky $\mathfrak{Br}_2$, que es geométricamente rígida y presenta dim $H^2(\mathfrak{Br}_2, \mathfrak{Br}_2) = 1$, y la otra, la denominada $\mathfrak{w}_n$ que queda totalmente determinada por medio de su secuencia diagonal a través de una función que la genera dependiendo de los valores de n y de la posición que ocupe la constante de estructura en dicha secuencia diagonal. Este álgebra es cohomológicamente rígida.
A continuación se estudian los casos en los que $k \geq 3$, para los que existen las $\mathfrak{w}_{k,n}$ que son cohomológicamente rígidas si $n \geq 3k+6$.
También se estudian y se detallan las álgebras que con el espectro descrito no son saturadas, las denominadas $\mathfrak{L}_k, n$ que están definidas por los corchetes:

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, \quad 2 \leq i \leq n-1, \\ [X_2, X_i] &= X_{i+k} \quad 3 \leq i \leq n-k. \end{aligned} \tag{1.34}$$

y el resto de los corchetes nulos. Estas álgebras de Lie aparecen siempre y cuando $n \geq k+3$ y se estudia si para dimensiones altas son las únicas que existen junto a las $\mathfrak{w}_{k,n}$ con las condiciones de que las constantes de estructuras $C_{1,i}^{i+1}$ y $C_{2,i}^{i+k}$ sean no nulas. Esto es cierto sólo para los valores pares de k.
Si k y n son impares aparecen las dos álgebras para $n \geq 3k+8$, sin embargo k es impar y n par se obtiene la $\mathfrak{n}_{k,n}$ y una familia uniparmétrica, por lo que concluimos que sólo las álgebras de Lie $\mathfrak{w}_{k,n}$ son rígidas, pues la llamada $\mathfrak{L}_{k,n}$ no lo es.
En la penúltima sección se estudia el caso en el $k = 4$ y en que las álgebras tienen nulas las constantes de estructuras $C_{2,i}^{k+i}$, aparecen en altas dimensiones una serie de álgebras cohomológicamente rígidas, que se dan detalladamente. Por último, se detalla el algoritmo utilizado para encontrar álgebras de Lie cohomológicamente rígidas con la restricción de la constante de estructura $C_{2,3}^{k+3} = 0$, a continuación se expone un ejemplo de la utilización de dicho algoritmo para $k = 4$ y los resultados obtenidos.

1.2.4 Capítulo 5

Estudio computacional de álgebras de Lie rígidas resolubles con un espectro de autovalores dado.

El contenido de este capítulo, que es una ampliación del trabajo realizado en los anteriores, se puede consultar en [20], siendo, en particular, una continuación de la última sección del capítulo precedente, en el que se han estudiado las álgebras de Lie resolubles de rango uno con el espectro spec $(\mathfrak{t}) = \{1, k, k+1, \cdots, n+k-2\}$ con $k = 4$ y con un nilradical no saturado. En éste se amplia el estudio para valores de $k \geq 5$ y se hace un pormenorizado análisis de todas las álgebras de Lie resolubles de rango uno con el espectro detallado que aparecen para los distintos valores de k y en las distintas dimensiones del nilradical, exceptuando las no racionales o imaginarias.
El caso en las que las álgebras no poseen un nilradical saturado es mucho más complejo que los casos precedentes. Existe gran variedad de álgebras rígidas con nilradicales distintos en función de la paridad de k y de n, ya que el cómo se definan las constantes de estructura del nilradical es determinante en su comohología. Hay que destacar, como se ha evidenciado en el transcurso de los cálculos realizados, que la constante de estructura $C_{2,k}^n$ juega un papel primordial al definir el nilradical, donde k es el mayor valor de j donde el el corchete $[X_2, X_j]$ puede ser distinto de cero. Se distinguen los siguientes casos:
Si se exigen las que las constantes de estructura $C_{2,n-k}^n = 0$ se obtiene a partir de $k \geq 3$ una única álgebra cohomológicamente rígida, para valores de $k \geq 8$ la mínima dimensión en la que aparecen estas álgebras es $n \geq 3k + 4$. Las constantes de estructura de este álgebra se pueden expresar por medio de funciones factoriales. El resto de las soluciones de las ecuaciones cuadráticas de las expresiones de Jacobi dan familias parametrizadas que se pueden deformar de forma no trivial. Por lo que en este caso se puede afirmar que estas álgebras están completamente clasificadas. De hecho, para que las álgebras que poseen el espectro dado sean cohomológicamente rígidas han de cumplir que $C_{2,n-k}^n = 0$. En dimensiones más bajas también aparecen varias álgebras rígidas, cohomológicamente o geométricamente.
En el caso de que $C_{2,n-k}^n \neq 0$ aparecen álgebras de Lie aisladas, geométricamente rígidas, para ciertos valores de k y de n en las que sus constantes de estructura no presentan ningún patrón que permita obtener una función que las genere, lo que hace que su completa clasificación sea una tarea complicada. Dado que sus estructuras son distintas y que con el aumento de valores de k y n da lugar a nuevas álgebras. Es de esperar que el número de éstas álgebras sea infinito, por ello, para facilitar la resolución de las ecuaciones de Jacobi, se ha recurrido a la relación que presentan sus constantes de estructura:

$$\frac{C_{2,i+s}^{k+i+s}}{C_{2,i+s+1}^{k+i+s}}, \quad \text{con } s \geq 0. \tag{1.35}$$

La dimensión de la segunda clase de cohomología varía en función de los valores de k y n y la aplicación de Rim no aporta ninguna información concluyente respecto a su rigidez, por lo que su determinación hay que resolverla de manera topológica. Los autores están trabajando actualmente en una ampliación del método para simplificar la determinación de la rigidez.
Si se añade la condición $C_{2,k-1}^{n-1} = 0$, se obtienen álgebras de Lie cohomológicamente rígidas para ciertos valores de k y de n.

1.2.5 Capítulo 6

Construcción de álgebras de Lie rígidas resolubles de rango uno cuyo nilradical tiene un índice de nilpotencia decreciente.

En este capítulo, cuyos contenidos se pueden ver en [21], se estudian álgebras de Lie resolubles de rango uno que presentan un espectro sustancialmente diferente a los anteriores, pues se introduce un parámetro nuevo, q, en él. Se determina, haciendo uso de las ecuaciones de Maurer-Cartan, las propiedades estructurales de unas álgebras de Lie nilpotentes de dimensión N, denominadas $\mathfrak{n}_{N,k}^{0}$, con un índice de nilpotencia $N-k$ y cuya secuencia característica viene dada por $c(\mathfrak{n}_{N,k}^{0}) = (N-k, 1^{k})$. A continuación se analiza el rango de estas álgebras y se concluye que para valores de $k \geq 1$ y $N \geq 2k+1$ la dimensión del toro maximal es dos. Posteriormente, se plantea la cuestión de si se puede obtener una deformación no trivial de estas álgebras que sea isomorfa a otra álgebra de Lie nilpotente que tenga rango uno. La respuesta es afirmativa, se muestra un ejemplo, el álgebra $\mathfrak{r}_{q+4,q}$, definida por los corchetes:

$$\begin{aligned}
&[X_1, X_j] = X_{j+1},\ k+1 \leq j \leq N-1,\\
&[X_2, X_j] = X_{j+2},\ k+1 \leq j \leq N-2,\\
&\quad\vdots\\
&[X_k, X_j] = X_{j+k},\ k+1 \leq j \leq N-k.
\end{aligned} \tag{1.36}$$

que ha sido determinada en [46] para una dimensión fija y donde se demuestra su rigidez chomológica. El álgebra que se obtiene por la deformación $\mathfrak{n}_{N,k}^{0} + \epsilon\varphi$ con

$$\varphi\left(X_{k+1}, X_j\right) = X_{q+j},\ k+1 \leq j \leq N-q, \tag{1.37}$$

tiene rango uno y muestra un espectro que es una combinación lineal de los autovalores de los elemntos del toro del álgebra inicial, dado por:

$$\mathrm{spec}(\mathfrak{t}) = \left(1, 2, \ldots, k, q, q+1, q+2, \ldots, N+q-k-1\right). \tag{1.38}$$

En el siguiente apartado se demuestra la rigidez cohomológica de las álgebras de Lie resolubles $\mathfrak{r}_{2,q,N} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_{2,q,N}$, para ello se utiliza, como en los anteriores capítulos, el teorema de la factorización de Hochschild-Serre. A continuación, en el siguiente apartado, se extiende el resultado anterior a cualquier valor de k, donde se demuestra que las álgebras de Lie $\mathfrak{r}_{k,q,N} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_{k,q,N}$ tienen también la segunda clase de cohomología nula.
El capítulo concluye dando un ejemplo de un álgebra de Lie resoluble de rango uno que es geométricamente rígida y que la dimensión de su segunda clase de cohomología es uno, se da el cociclo no integrable aislado y se muestra que la aplicación de Rim es inyectiva.

1.3 Conclusiones.

El objetivo primordial de esta memoria ha sido el iniciar un estudio sistemático de las álgebras de Lie resolubles reales de rango uno a partir del espectro de autovalores asociado a un toro maximal, mediante métodos computacionales. Analizando las clasificaciones en baja dimensión, se ha observado que varias clases de isomorfía pertenecen a espectros del mismo tipo, a partir de los cuales se ha buscado su generalización a dimensión arbitraria. Una de las dificultades principales en este análisis, al ser extrapolado a dimensiones arbitrarias, es la determinación de funciones generatrices que permitan caracterizar los nilradicales de las álgebras rígidas. Debe tenerse en cuenta que, con la imposibilidad de una clasificación de álgebras de Lie nilpotentes a partir de dimensión nueve [43], la separación de clases de isomorfía o la reducción de parámetros redundantes es un problema de cierta complejidad, tanto desde el punto de vista formal como computacional.

El estudio computacional realizado, aunque restringido al caso de rango uno, que supone el más importante en el contexto del problema de rigidez, han permitido obtener nuevas jerarquías de álgebras resolubles rígidas que generalizan resultados en baja dimensión, así como complementan otros enfoques, tales como el estudio de la rigidez basado en el teorema del rango y la teoría de conjuntos interna de Nelson [1, 46, 48, 70] o la construcción a partir del esquema de Jacobi y su cohomología, desarrollado por Roger Carles. Entre los resultados secundarios de este análisis, han aparecido ciertas patologías que no habían sido observadas con anterioridad, al estar restringidos muchas de las construcciones explícitas a dimensiones $n \leq 50$. En particular, se ha observado, para los diferentes espectros analizados, un patrón de comportamiento genérico para las álgebras de Lie resolubles con un toro especificado, que puede subdividirse en tres etapas principales:

1. En dimensiones bajas, las soluciones son cohomológicamente rígidas por colapso dimensional.

2. En dimensiones intermedias aparecen familias dependientes de uno o varios parámetros, así como algunas soluciones aisladas. Mientras las familias no son rígidas, algunas de las soluciones aisladas proporcionan álgebras rígidas, bien desde el punto de vista geométrico o algebraico.

3. A partir de cierta dimensión, el esquema de Jacobi no admite más que una única solución, o varias soluciones aisladas, todas ellas generalmente rígidas. En algunos casos, hay una alternancia de rigidez algebraica/geométrica dependiendo de la paridad de la dimensión.

Debe observarse que, si bien las familias parametrizadas no proporcionan álgebras de Lie rígidas aisladas, si puede darse el caso de que toda deformación pertenezca a la propia familia, dando lugar a la noción de rigidez de familias, que no se ha abordado en este trabajo, pero de cuya existencia se sabe a través del análisis de las componentes de la variedad de leyes de álgebras de Lie [81]. Las álgebras de Lie cohomológicamentes rígidas se han clasificado y detallado a lo largo del trabajo. En el caso de las geométricamente rígidas, la cuestión consiste en estudiar propiedades topológicas de anillos locales, y hasta el momento no existe ningún procedimiento computacional para codificar adecuadamente esta información, al menos en el contexto que nos ocupa. Aunque se han proporcionado nuevas clases de este tipo, el estudio no es sistemático y una posible extensión de los resultados de esta memoria consistiría en desarrollar un procedimiento algorítmico para estudiar el esquema de Jacobi [29, 30]. En este contexto, debe observarse que los criterios de suficiencia de rigidez geométrica, como el establecido por Rauch en 1972 [75], resulta inaplicable en muchos casos, al ser trivial la aplicación cuadrática de Rim, lo que obliga a buscar nuevas condiciones de tipo no cohomológica que permitan deducir la rigidez.

A diferencia de los métodos computacionales usuales [12, 32, 51], que calculan la cohomología de manera directa, en esta memoria se ha empleado una simplificación estructural. La novedad del código simbólico desarrollado para analizar la integrabilidad de cociclos pertenecientes a $Z^2(\mathfrak{g}, \mathfrak{g})$, ya sea por medio del estudio de la aplicación de Rim, o bien analizando si dichos cociclos pueden aparecer en la parte lineal de una deformación del álgebra estudiada, consiste en reducir el cálculo de la cohomología a una subclase especial, mediante la implementación de la secuencia espectral de Hochschild-Serre, que reduce significativamente el tiempo de cálculo y la memoria empleada. Esto ha permitido estudiar dimensiones de magnitud considerable, donde un enfoque directo es inviable por cuestiones de la capacidad computacional. En un futuro, se pretende generalizar el código, mediante la citada secuencia de Hochschild-Serre, para abordar el cálculo de los grupos de cohomología de orden $n > 3$, con el fin de obtener nuevas condiciones de obstrucción para la integrabilidad de cociclos.

Otra de las cuestiones que se pretende estudiar en el futuro, asimismo mediante un refinamiento de los códigos computacionales, es la conjetura referente a la inexistencia de álgebras de Lie resolubles de rango uno cuyo espectro de autovalores contenga el cero. Entre las jerarquías de álgebras rígidas conocidas, no se ha observado aún ningún ejemplo que contenga el cero como autovalor asociado a un elemento del nilradical, y puede justificarse que, para rango del álgebra $r > 1$, esta patología permite generalmente encontrar deformaciones no triviales y no equivalentes. El problema fundamental consiste en decidir la conjetura para rango uno; si bien existen indicios que sugieren que el cero no puede formar parte del espectro, no se han hallado aún argumentos convincentes que permitan demostrar la afirmación. Confiamos que los métodos computacionales, que permiten el análisis de espectros de dimensiones arbitrariamente altas, permitan detectar alguna característica que posibilite avanzar en la resolución del problema.

Como nuevo problema que ha aflorado en el transcurso de la redacción de este trabajo, mencionamos la existencia de álgebras de Lie rígidas cuyas constantes de estructura no son racionales, ya sean irracionales o complejas. Este fenómeno esta en consonancia con los pocos trabajos que han sido dedicados a álgebras rígidas de este tipo, como los realizados por Ancochea y Goze [1, 8, 10]. Según se desprende del análisis pormenorizado realizado en esta memoria, la existencia de soluciones no racionales constitute una patología usual en el rango de dimensiones intermedias, debido a la estructura de las condiciones de Jacobi. Para dimensiones muy altas, dichas condiciones restringen fuertemente el número de soluciones, y todas las soluciones observadas admiten un representante racional. El problema del estudio de las álgebras de Lie rígidas no racionales ha sido poco tratado, y entre los problemas a ser abordados en este contexto, el más relevante es tratar de encontrar un criterio que permita analizarlas en conjunto. Debe observarse que, al ser las constantes de estructura no racionales, los programas de cálculo simbólico emplean aproximaciones numéricas, lo que puede generar errores en el cálculo de las cohomologías. El enfoque debe por tanto diferir, posiblemente empleando polinomios formales que codifiquen las constantes de estructura. Esta cuestión se está analizando actualmente a partir de los ejemplos obtenidos en esta memoria. Otra cuestión de interés respecto a este tipo de álgebras es su comportamiento topológico; más concretamente, determinar si este tipo de álgebras de Lie resolubles rígidas no racionales forma un abierto en la variedad de leyes de álgebras de Lie, así como su densidad o clausura. Para esto hay que utilizar otro tipo de técnicas no cohomológicas, que varían desde los anillos de valoración a la teoría de invariantes, ver [16].

Formalmente, los códigos desarrollados pueden adaptarse y ampliarse al análisis de espectros que contengan autovalores múltiples, donde el problema está en la separación de las clases de isomorfía, y donde sólo se conocen pocos casos de estas álgebras, que aparecen en bajas dimensiones (cuya demostración o bien es inmediata a partir del teorema del rango [49], o se sigue a través de métodos del análisis no estándar [5], específicamente la teoría de conjuntos de Nelson [6, 61, 70]. Asimismo, los procedimientos desarrollados en esta memoria son válidos para álgebras resolubles de rango $r > 1$, donde pueden combinarse con otros métodos propuestos en la literatura para generar álgebras de rango mayor a partir de una amalgama de álgebras de rango uno [4].

Finalmente, observamos que para la clase de álgebras de Lie (indescomponibles) con descomposición de Levi no trivial, no existen actualmente muchos resultados concernientes a la rigidez, salvo para el caso con factor de Levi $\mathfrak{so}(3)$ y $\mathfrak{sl}(2,\mathbb{R})$ [17, 41]. Debe destacarse que, históricamente, esta última está álgebra de Lie relacionada con el descubrimiento de la rigidez geométrica [74]. En este sentido, cabe completar el análisis estudiando la rigidez en dimensión diez, cuya clasificación ha sido recientemente abordada por Bandara y Thompson [11]. Aunque las técnicas difieren considerablemente, los métodos computacionales pueden ser relevantes para determinar el tipo de representaciones características asociadas a sumas semidirectas cohomológicamente rígidas, y utilizando los resultados conocidos sobre las álgebras simples de rango uno, tratar de obtener algunos resultados anĺogos para las álgebras con factor de Levi de rango mayor.
En este sentido, actualmente se está analizando la estructura de las deformaciones de las álgebras no resolubles del tipo

$$\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R\mathbb{R}^{(\dim R)}$$

siendo la representación característica R la suma directa de tres representaciones irreducibles V_k de $\mathfrak{sl}(2,\mathbb{R})$, distintas de la trivial. Dependiendo de las diferentes paridades de la dimensión (o el peso máximo) de las representaciones V_k, se presentan esencialmente tres casos:

- La representación característica es incompatible con la estructura de álgebras nilpotentes no abelianas.
- Existen nilradicales aislados compatibles con la representación característica.
- Aparecen familias parametrizadas de soluciones.

Mientras la rigidez de las soluciones del primer tipo, correspondientes a álgebras inhomogéas, es fácil de estudiar mediante las leyes de ramificación [62], los restantes casos requieren un análisis cohomológico o geométrico más detallado, sin que se haya obtenido aún un criterio que permita clasificarlas a partir de la terna de pesos máximos de las componentes irreducibles de la representación R. Un estudio sistemático y detallado de este problema está actualmente en curso.

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Capítulo 2

New examples de rank one solvable real rigid Lie algebras possessing a nonvanishing Chevalley cohomology

2.1 Introduction

This chapter develops the results obtained in the article *New examples de rank one solvable real rigid Lie algebras possessing a nonvanishing Chevalley cohomology*, published in the journal Applied Mathematics and Computation. In which the solvable Lie algebras of rank one that present a spectrum $\text{spect}(\mathfrak{t}) = \{1, 4, 5 \ldots, n+2\}$ are studied in the field of rigidity. It is shown that for algebras having a dimension greater than or equal to 18 there are three classes of isomorphism, of which two are cohomologically rigid and the other is geometrically rigid.
The concept of rigidity, within the context of real and complex Lie algebras, and beyond its importance in the geometrical analysis of the affine variety $\mathcal{L}^n$ of n-dimensional Lie algebras, more specifically in the study and classification of its irreducible components [11, 19], has been proved to be also a valuable tool in the structural theory, as it allows to analyze those Lie algebras that are obtained either by a deformation or a limiting process from other Lie algebras [16, 18, 26, 28]. This idea has been particularly useful in applications, where rigidity of Lie algebras is deeply related to the stability of physical models [8, 13, 31]. Basing on this motivation, the approach to rigidity via the Chevalley cohomology of Lie algebras [17, 23, 30] has shown that large types fall within this class, such as semisimple and parabolic Lie algebras. For the special case of solvable Lie algebras, specific methods like the theory of roots have been developed in order to characterize rigid algebras by means of the eigenvalue spectrum of generators of maximal tori of derivations ([1, 7, 15, 21] and references therein). Currently, a satisfactory structure theory for rigid Lie algebras only exists for the complex case [9], due to the difficulties in characterizing the real forms of solvable Lie algebras, as well as the failure in the real case of important results like the Mal'cev conjugacy or the Carles' algebraicity theorems [9, 24]. A truly systematic study of real rigid Lie algebras, on the other hand, is relatively recent [2, 3, 4].

Although the first examples of the existence of rigid Lie algebras with nonvanishing cohomology $H^2(\mathfrak{g}, \mathfrak{g})$ were indicated by Nijenhuis and Richardson in their analysis of semidirect sums of semisimple and solvable Lie algebras [28], for the specific case of solvable Lie algebras, only a few types presenting this pathology are known and have been studied in detail [10, 20], although it has been pointed out recently that non-cohomologically rigid Lie algebras are quite common, and exist for any rank greater or equal than one and dimensions greater than twelve [5]. In this work we analyze solvable Lie algebras of rank one possessing a maximal torus of derivations $\mathfrak{t}$, the spectrum of which is given by the eigenvalues $\mathrm{spec}(\mathfrak{t}) = (1, 4, 5, \cdots, n+2)$ on its n-dimensional nilradical $\mathfrak{n}$. It results that, for values $n \geq 19$, there are three rigid Lie algebras in each dimension, two of which are cohomologically rigid, while the third always has a two-dimensional cohomology space $H^2(\mathfrak{g}, \mathfrak{g})$. This extends the results of [12] and [20], where the eigenvalue spectra $\mathrm{spec}(\mathfrak{t}) = (1, 2, 3, \cdots n)$ and $\mathrm{spec}(\mathfrak{t}) = (1, 3, 4, \cdots n)$ respectively were analyzed.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and defined over the fields $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

2.1.1 Rigid Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra formed by its derivations.

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus of derivations is an Abelian subalgebra of $Der(\mathfrak{g})$ the generators of which are semisimple.*

By semi-simplicity of the torus generators, the maps $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ are simultaneously diagonalizable over $\mathbb{C}$. Maximal tori of complex Lie algebras are known to be conjugate [24], hence their common dimension is a scalar invariant $\mathrm{r}(\mathfrak{g})$ of the Lie algebra, called the rank of $\mathfrak{g}$.

We denote by $\mathcal{L}^n$ the variety of n-dimensional Lie algebras $\mathfrak{g} = \left(\mathbb{K}^n, [\, , \,]_{\mathfrak{g}}\right)$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The group $GL\,(n, \mathbb{K})$ acts on $\mathcal{L}^n$ by means of changes of basis:

$$(f \star \mathfrak{g})\,(X, Y) = f\left(\left[f^{-1}\,(X)\,, f^{-1}\,(Y)\right]_{\mathfrak{g}}\right), \;\; f \in GL\,(n, \mathbb{K})\,, \;\; X, Y \in \mathfrak{g}. \tag{2.1}$$

For any $\mathfrak{g}$, the orbit $\mathcal{O}\,(\mathfrak{g})$ is given by all Lie algebras that are isomorphic to $\mathfrak{g}$. It can be shown that $\mathcal{O}\,(\mathfrak{g})$ identifies with the homogeneous space $GL\,(n, \mathbb{K})\,/\mathrm{Aut}(\mathfrak{g})$. It follows in particular that $\dim \mathcal{O}\,(\mathfrak{g}) = n^2 - \dim \mathrm{Der}(\mathfrak{g})$.

Definition 2 *A Lie algebra $\mathfrak{g}$ is called rigid if the orbit $\mathcal{O}\,(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

We observe that an equivalent definition is obtained requiring the orbit $\mathcal{O}\,(\mathfrak{g})$ to be open with respect to the Zariski topology of $\mathcal{L}^n$ [12]. Rigid algebras $\mathfrak{g}$ are of importance within the study of irreducible components of the variety $\mathcal{L}^n$, as their orbit closure $\overline{\mathcal{O}\,(\mathfrak{g})}$ determines one such component. Irreducible components of the variety $\mathcal{L}^n$ have been completely determined for the values $n \leq 7$ in [11].

2.1.2 Rigidity and cohomology of Lie algebras

To analyze whether a Lie algebra is rigid using the topological definition is far from being a practical procedure, a fact that has motivated the search for complementary criteria that ensure rigidity. The most extended method is based on the Chevalley cohomology of Lie algebras [30], specifically on the computation of the cohomology spaces $H^p\,(\mathfrak{g}, \mathfrak{g})$ for $p \leq 3$. In particular, the spaces $H^0\,(\mathfrak{g}, \mathfrak{g})$ and $H^1\,(\mathfrak{g}, \mathfrak{g})$ correspond to the centre $Z(\mathfrak{g})$ and the outer derivations $\mathrm{Der}(\mathfrak{g})/\mathrm{IDer}(\mathfrak{g})$ of $\mathfrak{g}$ respectively.

In this context, it is useful to apply the Hochschild-Serre factorization theorem [23], that simplifies considerably the determination of the cohomology classes. For our purposes is is enough to consider the case of solvable Lie algebras $\mathfrak{r} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}$, where $\mathfrak{t}$ is Abelian and the operators $\mathrm{ad}_{\mathfrak{r}}T$ ($T \in \mathfrak{t}$) are diagonal. The adjoint cohomology $H^p(\mathfrak{r}, \mathfrak{r})$ satisfies the isomorphism

$$H^p(\mathfrak{r}, \mathfrak{r}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}}, \tag{2.2}$$

where

$$H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = \left\{[\varphi] \in H^b(\mathfrak{n}, \mathfrak{r}) \mid (T.\varphi) = 0,\ T \in \mathfrak{t}\right\} \tag{2.3}$$

denotes the space of $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$, the invariance condition being given by

$$(T.\varphi)(Z_1, \cdots, Z_b) = [T, \varphi(Z_1, \cdots, Z_b)] - \sum_{s=1}^{b} \varphi(Z_1, \cdots, [T, Z_s], \cdots, Z_b). \tag{2.4}$$

We observe that, as a consequence of the identities $H^a(\mathfrak{t}, \mathbb{R}) = \bigwedge^a \mathfrak{t}$, the vanishing of the spaces $H^p(\mathfrak{r}, \mathfrak{r}) = 0$ is equivalent to the condition $H^b(\mathfrak{n}, \mathfrak{r}) = 0$ for $0 \leq b \leq p$.

Proposition 1 *Let $\mathfrak{g}$ be a Lie algebra. If the condition $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$ holds, then $\mathfrak{g}$ is rigid. Conversely, if the point in the variety $\mathcal{L}^n$ corresponding to the Lie algebra $\mathfrak{g}$ is nonsingular, then $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$.*

Lie algebras satisfying this theorem are called cohomologically rigid [26]. A condition on the regularity of a point in $\mathcal{L}^n$, sometimes useful as a negative test for the rigidity of Lie algebras, was given in [27]:

Proposition 2 *Let $\mathfrak{g}$ be a Lie algebra such that $H^3(\mathfrak{g}, \mathfrak{g}) = 0$ holds. Then the point in $\mathcal{L}^n$ corresponding to $\mathfrak{g}$ is nonsingular.*

Proposition (1), proved in [28], also pointed out that the nullity of the cohomology space $H^2(\mathfrak{g}, \mathfrak{g})$ is merely a sufficient, albeit not necessary condition for rigidity. In general, geometrically rigid Lie algebras, i.e., satisfying $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$, are lesser known than cohomologically rigid algebras, their determination being quite more complicated, although they are probably predominant in higher dimensions. Criteria to guarantee that a Lie algebra is rigid if the point in the variety $\mathcal{L}^n$ corresponding to it is singular were systematically studied in [27], by means of the quadratic Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \rightarrow H^3(\mathfrak{g}, \mathfrak{g})$ introduced in [29]. This map, that can be generalized to higher cohomology spaces (see e.g. [17], chapter 1) and inherits an interpretation in a more wide context, is formally defined as

$$\mathrm{Sq}(\psi)(X_i, X_j, X_k) := \psi(\psi(X_i, X_j), X_k) + \psi(\psi(X_j, X_k), X_i) + \psi(\psi(X_k, X_i), X_j). \tag{2.5}$$

It actually constitutes a first measure on the obstructions to the integrability of infinitesimal deformations of the type $\mathfrak{g}_\varepsilon = \mathfrak{g} + \varepsilon\varphi$, where $\varphi \in Z^2(\mathfrak{g}, \mathfrak{g})$ [18]. The criterion proved in [27] can be formulated as follows:

Proposition 3 *If the Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \rightarrow H^3(\mathfrak{g}, \mathfrak{g})$ is injective, then $\mathfrak{g}$ is a rigid Lie algebra. Moreover, if $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$, the point in $\mathcal{L}^n$ corresponding to $\mathfrak{g}$ is singular.*

Solvable rigid Lie algebras $\mathfrak{r}$ have a rather definite structure [9], as they satisfy the decomposition

$$\mathfrak{r} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n} \tag{2.6}$$

into the semidirect sum of a maximal torus $\mathfrak{t}$ and the nilradical $\mathfrak{n}$ (i.e., the maximal nilpotent ideal). The rank of $\mathfrak{r}$ is upper bounded by

$$\dim \mathfrak{n} - \dim [\mathfrak{n}, \mathfrak{n}] \geq \dim \mathfrak{t}. \tag{2.7}$$

If moreover $\mathfrak{r}$ is solvable and non-nilpotent, the inequality

$$(-1)^p \dim H^0(\mathfrak{r}, \mathfrak{r}) + (-1)^{p+1} \dim H^1(\mathfrak{r}, \mathfrak{r}) + \cdots + (-1)^{2p} \dim H^p(\mathfrak{r}, \mathfrak{r}) \geq 0 \tag{2.8}$$

holds for positive integers $p \geq 0$. It provides sometimes relevant information, as shown with the following examples:

- For $p = 2$ the inequality (2.8) implies

$$\dim H^0(\mathfrak{r}, \mathfrak{r}) - \dim H^1(\mathfrak{r}, \mathfrak{r}) + \dim H^2(\mathfrak{r}, \mathfrak{r}) \geq 0. \tag{2.9}$$

  If $Z(\mathfrak{r}) = 0$ and $\mathfrak{r}$ is cohomologically rigid, then it is complete, i.e., $H^0(\mathfrak{r}, \mathfrak{r}) = H^1(\mathfrak{r}, \mathfrak{r}) = 0$ hold.

- For $p = 3$ we obtain

$$-\dim H^0(\mathfrak{r}, \mathfrak{r}) + \dim H^1(\mathfrak{r}, \mathfrak{r}) - \dim H^2(\mathfrak{r}, \mathfrak{r}) + \dim H^3(\mathfrak{r}, \mathfrak{r}) \geq 0. \tag{2.10}$$

  In this case, if $\mathfrak{r}$ is complete and satisfies $H^3(\mathfrak{r}, \mathfrak{r}) = 0$, then it is cohomologically rigid.

This illustrates that complete Lie algebras play a relevant role in the rigidity problem, a fact that has motivated their structural study ([7, 14, 25] and references therein). In this frame, it is worthy to be mentioned here that it is currently unknown whether there can exist rank one solvable rigid Lie algebras $\mathfrak{r}$ with nontrivial centre $Z(\mathfrak{r}) \neq 0$ and satisfying the condition $H^1(\mathfrak{r}, \mathfrak{r}) \neq 0$.

2.2 The rank one solvable Lie algebras $\mathfrak{Br}_{4,n}$

The first explicit example of a rigid series of solvable Lie algebras with non-vanishing cohomology, the so-called Bratzlavsky algebra $\mathfrak{Br}_n$, was described in [10], considering the class of rank one solvable Lie algebras $\mathfrak{Br}_n$ admitting a maximal torus $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = \{1, 2, 3, \cdots, n\}$ and such that the commutator $[X_2, X_3]$ does not vanish [6]. In this frame, solvable Lie algebras $\mathfrak{r}$ with a torus $\mathfrak{t}$ having as spectrum $\mathrm{spec}(\mathfrak{t}) = \{1, 3, 4, \cdots, n+1\}$, also corresponding to a type introduced in [6], were analyzed for rigidity in [20], albeit their cohomological behavior and the Rim map were not explicitly considered.

In this section we consider the next case in the series, the Lie algebras $\mathfrak{Br}_{4,n}$ of rank one and satisfying the decomposition $\mathfrak{Br}_{4,n} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{f}_{4,n}$, where the commutators are given by

$$\begin{array}{ll} [X_1, X_i] = X_{i+1},\ 2 \leq i \leq n-1; & [X_2, X_i] = X_{i+4},\ 3 \leq i \leq n-4; \\ [T, X_1] = X_1; & [T, X_i] = (2+i)\, X_i, \quad 2 \leq i \leq n. \end{array} \tag{2.11}$$

over a basis $\{X_1, \cdots, X_n, T\}$, with T denoting the generator of the maximal torus $\mathfrak{t}$.

Proposition 4 *The Lie algebra $\mathfrak{Br}_{4,n}$ satisfies following identities*

1. *For $n = 7, 8$, $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 0$ and $\mathfrak{Br}_{4,n}$ is cohomologically rigid.*

2. *For $n = 9, 10$, $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 1$.*

3. For $n \geq 11$*,* $\dim H^2\left(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}\right) = 2$*.*

We prove the assertion using the Hochschild-Serre factorization theorem, that enables to reduce the computation of the cohomology to that of $\mathfrak{t}$-invariant cocycles [23].

A generic 1-cochain $f \in C^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)$ has the form

$$f\left(X_i\right) = \sum_{l=1}^{n} a_i^l X_l + b_i T,\ 1 \leq i \leq n. \tag{2.12}$$

Using the invariance condition (6.13), it follows that $f \in C^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}}$ must satisfy the identities

$$f\left(X_i\right) = a_i^i X_i,\ 1 \leq i \leq n. \tag{2.13}$$

Applying now the coboundary operator d, it follows that the only nontrivial entries of df are the following:

$$\begin{array}{r}df\left(X_1, X_j\right) = \left(a_1^1 + a_j^j - a_{1+j}^{1+j}\right) X_{1+j},\quad 2 \leq j \leq n-1 \\ df\left(X_i, X_j\right) = \left(a_i^i + a_j^j - a_{i+j+2}^{i+j+2}\right) X_{i+j+2},\quad 2 \leq i,\ i+j \leq n-2\end{array} \tag{2.14}$$

A basis of 1-cocycles $Z^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}}$ is thus determined by

$$f_1\left(X_1\right) = X_1,\quad f_1\left(X_j\right) = \left(j+2\right) X_j,\ 2 \leq j \leq n, \tag{2.15}$$

which is immediately seen to correspond to the adjoint representation of the torus generator T. Therefore $Z^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}} = B^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}}$, so that $H^1\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}} = H^1\left(\widehat{\mathfrak{Br}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right) = 0$ holds. It further follows that

$$\dim B^2\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}} = n-1, \tag{2.16}$$

where a basis of $B^2\left(\widehat{\mathfrak{f}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)^{\mathfrak{t}}$ is given by the image by d of the cochains $f_i\left(X_j\right) = a_i^j X_j$ for $2 \leq i \leq n$.

Let now $\varphi \in C^2\left(\widehat{\mathfrak{f}}_{n+2s}, \widehat{\mathfrak{Br}}_{n,s}\right)$ be a generic 2-cochain

$$\varphi\left(X_i, X_j\right) = \sum_{k=1}^{n} \alpha_{ij}^k X_k + \beta_{ij} T. \tag{2.17}$$

The invariance condition (6.13) imposes the following constraints:

$$\begin{array}{r}\varphi\left(X_1, X_j\right) = \alpha_{1,j}^{1+j} X_{1+j},\quad 2 \leq j \leq n-1; \\ \varphi\left(X_i, X_j\right) = \alpha_{i,j}^{i+j+2} X_{i+j},\quad 2 \leq i < j,\ i+j \leq n-2.\end{array} \tag{2.18}$$

For the coboundary operator d we easily deduce that

$$\begin{array}{ll}d\varphi\left(X_i, X_j, X_k\right) = 0, & 3 \leq i < j,\ k \neq i+j+2, \\ d\varphi\left(X_i, X_j, T\right) = 0, & 1 \leq i < j \leq n.\end{array} \tag{2.19}$$

The coboundary condition for the remaining triples, after a long but routine computation, implies that $d\varphi$ is subjected to the following restrictions

$$\begin{array}{rlll}d\varphi\left(X_1, X_2, X_k\right) = & \left(\alpha_{1,k+4}^{k+5} - \alpha_{1,k}^{k+1} + \alpha_{2,k}^{k+4} - \alpha_{3,k}^{k+5} - \alpha_{2,k+1}^{k+5}\right) X_{5+k}, & 3 \leq k \leq n-5 \\ d\varphi\left(X_1, X_j, X_k\right) = & \left(\alpha_{j,k}^{j+k+2} - \alpha_{j+1,k}^{3+j+k} - \alpha_{j,k+1}^{3+j+k}\right) X_{j+k+3}, & 2 < j < k \leq n-j-3 \\ d\varphi\left(X_2, X_j, X_k\right) = & \left(\alpha_{j,k}^{j+k+2} - \alpha_{j+4,k}^{6+j+k} - \alpha_{j,4+k}^{6+j+k}\right) X_{6+j+k}, & 3 \leq j < k \leq n-j-6\end{array} \tag{2.20}$$

Thus invariant 2-cocycles correspond to solutions of the linear system

$$\begin{array}{ll} \alpha_{1,k+4}^{k+5}-\alpha_{1,k}^{k+1}+\alpha_{2,k}^{k+4}-\alpha_{3,k}^{k+5}-\alpha_{2,k+1}^{k+5}=0, & 3\leq k\leq n-5 \\ \alpha_{j,k}^{j+k+2}-\alpha_{j+1,k}^{3+j+k}-\alpha_{j,k+1}^{3+j+k}=0, & 2<j<k\leq n-j-3 \\ \alpha_{j,k}^{j+k+2}-\alpha_{j+4,k}^{6+j+k}-\alpha_{j,4+k}^{6+j+k}=0, & 3\leq j<k\leq n-j-6 \end{array} \tag{2.21}$$

Manipulation of these equations further leads to the the following identities

$$\alpha_{j,n-j-2}^{n}=-\frac{1}{2}\alpha_{j+1,j+2}^{2j+5}-\sum_{p=3}^{n-3-2j}\alpha_{j+1,j+p}^{2j+p+3},\ 4\leq j\leq\frac{n-6-\frac{(-1)^n-1}{2}}{2}. \tag{2.22}$$

We conclude that all coefficients can be expressed in terms of

$$\alpha_{1,j}^{j+1},\ 2\leq j\leq n-1;\ \alpha_{2,3}^{7},\ \alpha_{2,5}^{9},\ \alpha_{2,7}^{11}, \tag{2.23}$$

which are not subjected to constraints among them, showing that

$$Z^2\left(\hat{\mathfrak{f}}_{4,n},\widehat{\mathfrak{Br}}_{4,n}\right)^{\mathrm{t}}=n+1$$

for $n\geq 9$. Observe that for $n=7,8$, the two last parameters are skipped, showing that any cocycle is a coboundary, hence the cohomology space vanishes. For n$\geq$ 9 it follows that

$$H^2\left(\hat{\mathfrak{f}}_{4,n},\widehat{\mathfrak{Br}}_{4,n}\right)^{\mathrm{t}}=H^2\left(\widehat{\mathfrak{Br}}_{4,n},\widehat{\mathfrak{Br}}_{4,n}\right)\leq 2. \tag{2.24}$$

As class representatives for the cohomology, we can take the classes $[\varphi]$ and $[\psi]$, defined as

$$\varphi\left(X_2,X_3\right)=X_7;\quad \varphi\left(X_2,X_{4+j}\right)=(j+1)\,X_{8+j};\quad \varphi\left(X_3,X_{4+j}\right)=-X_{4+j},\quad 0\leq j\leq n-8 \tag{2.25}$$

and

$$\psi=\psi_0+\psi_1, \tag{2.26}$$

respectively, with $z_0=\frac{1}{4}\left(2n+(-1)^{(1+n)}-19\right)$, the components being given by

$$\psi_0\left(X_2,X_{5+j}\right)=(j+1)\,X_{9+k},\ \psi_0\left(X_3,X_{4+j}\right)=-X_{9+k},\ 0\leq j\leq n-9 \tag{2.27}$$

and

$$\psi_1\left(X_{2+j},X_{7+2s+k-j}\right)=\frac{(-1)^{s+j+1}\,3\times 2^s}{(s+2-j)!}\prod_{l=0}^{s+j-1}(k+l+1)\,X_{11+2s+k}, \tag{2.28}$$

where

$$0\leq s\leq z_0,\ 0\leq j\leq s+1,\ 0\leq k\leq n-11-2s. \tag{2.29}$$

We observe that, alternatively, the cocycle ψ_1 can be expressed as follows in terms of the Gamma function:

$$\psi_1\left(X_{2+j},X_{7+2s+k-j}\right)=\frac{(-1)^{s+j+1}\,3\times 2^s}{(s+2-j)!}\frac{(1+k)\,\Gamma\left[3+s+3-j\right]}{\Gamma\left[2+k\right]}X_{11+2s+k}. \tag{2.30}$$

It follows in particular from (2.30) that this cocycle can only exist for values $n\geq 12$, showing that the identity $\dim H^2\left(\widehat{\mathfrak{Br}}_{4,n},\widehat{\mathfrak{Br}}_{4,n}\right)=1$ holds for $9\leq n\leq 11$, whereas $\dim H^2\left(\widehat{\mathfrak{Br}}_{4,n},\widehat{\mathfrak{Br}}_{4,n}\right)=2$ for values $n\geq 12$.

In order to obtain information concerning the (infinitesimal) integrability of the cocycle classes $[\varphi]$ and $[\psi]$, we now compute the Rim map Sq.

Proposition 5 *The Rim map* $Sq : H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) \rightarrow H^3(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n})$ *satisfies the following properties*

1. *For* $n \leq 14$*,* Sq *vanishes identically:* $[Sq(\varphi)] = [Sq(\psi)] = [0]$.

2. *For* $n \geq 15$*,* Sq *is nonzero with* $\ker Sq = \langle [\varphi] \rangle$ *and* $Sq([\psi]) \neq [0]$.

Proof. We compute the Rim map for the cohomology space representatives separately. For $n \leq 12$, it is routine to verify that $Sq(\varphi) = 0$. From $n \geq 13$ onwards, the image of φ gives the 3-cochain defined by

$$Sq(\varphi)(X_2, X_3, X_j) = 5X_{j+9}, \; 4 \leq j \leq n-9. \tag{2.31}$$

From the structure of 3-coboundaries (see equation (4.40)), it follows that $Sq(\psi_1) = d\phi$, where ϕ is the 2-cochain given by

$$\phi(X_2, X_{7+j}) = \frac{5}{8}(1+j)(2+j)X_{9+j}, \; \phi(X_3, X_{6+j}) = -\frac{5}{4}(1+j)X_{9+j}, \; \phi(X_4, X_{5+j}) = \frac{5}{4}X_{9+j}, \tag{2.32}$$

where $0 \leq j \leq n-9$. As a consequence, the identity $Sq([\varphi]) = 0$ holds for any $n \geq 13$. Concerning the image of the cocycle ψ, for $n \leq 14$ it is immediate that $Sq(\psi) = 0$. This shows, jointly with (2.32), that the Rim map vanishes identically for any $n \leq 14$. For the value $n = 15$, we have that

$$Sq(\psi)(X_2, X_3, X_6) = -Sq(\psi)(X_2, X_4, X_5) = -X_{15}.$$

A short computation shows that $Sq(\psi)$ cannot be obtained as a linear combination of 3-coboundaries, showing that the class $Sq([\psi])$ is not zero. Considering finally the values $n \geq 16$, it turns out that the image of ψ implies, among other relations, the following one:

$$Sq(\psi)(X_3, X_4, X_5) = 2X_{16} \tag{2.33}$$

Now, as follows immediately from equation (4.40), any 3-coboundary $\theta = d\varphi$ satisfies $\theta(X_i, X_j, X_k) = 0$ for $3 \leq i < j < k$, from which we deduce that the class of $Sq(\psi)$ in $H^3\left(\widehat{\mathfrak{Br}}_{4,n}, \widehat{\mathfrak{Br}}_{4,n}\right)$ cannot be zero. We hence conclude that the Rim map does not vanish for $n \geq 16$. ■

As a consequence of this result, the cohomology will not provide conclusive information concerning the rigidity of the Lie algebras $\mathfrak{Br}_{4,n}$ for values $n \geq 9$. In the following section, a different approach is considered in order to determine the rigidity of these Lie algebras from a certain dimension onwards.

2.3 Solvable Lie algebras with maximal torus $\mathfrak{t}$

Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{g}_n$ be a solvable Lie algebra of dimension $n+1$ and possessing a maximal torus $\mathfrak{t}$, the eigenvalues of which are given by

$$\text{spec}(\mathfrak{t}) = \{1, 4, 5, \cdots, n+2\}. \tag{2.34}$$

If $\{X_1, \cdots, X_n\}$ is a basis of eigenvectors for the linear operator $ad(T)$, it follows at once that only non-vanishing commutators in the nilradical $\mathfrak{g}_n$ adopt the form

$$[X_1, X_j] = C_{1,j}^{j+1} X_{j+1}, \; 2 \leq j \leq n-1; \quad [X_i, X_j] = C_{i,j}^{i+j+2} X_{i+j+2}, 2 \leq i < j; \; i+j \leq n-2. \tag{2.35}$$

Without loss of generality, if $C_{1,j}^{j+1} \neq 0$ holds, and rescaling the basis if necessary, we can suppose that $C_{1,j}^{j+1} = 1$ for $2 \leq j \leq n-1$. The Jacobi identities for the triples $\{X_1, X_i, X_j\}$ imply that the structure constants $C_{i,j}^k$ must fulfill the following linear equations:

$$C_{i,j}^{i+j+2} - C_{i+1,j}^{i+j+3} - C_{i,j+1}^{i+j+3} = 0, \ 2 \leq i < j \leq n-2. \tag{2.36}$$

We fix the notation $\alpha_{i-2} = C_{i,i+1}^{2i+3}$ for any $2 \leq i \leq \left[\frac{n-3}{2}\right]$. Given any scalars a, n, the Pochhammer symbol $(a)_n$ is defined in terms of the Gamma function Γ as

$$(a)_n = \prod_{k=0}^{n-1} (a+k) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.37}$$

Further let ε_p be the stepwise function defined by

$$\varepsilon_p = \begin{cases} 0, & x < 2p+1 \\ 1, & x \geq 2p+1 \end{cases} \tag{2.38}$$

Now, solving successively the system (4.10), we find the expressions

$$C_{2+i,4+i+j}^{8+2i+j} = \alpha_i - j\alpha_{i+1} + \sum_{k=2}^{n-8-2i} (-1)^k |A(k,j)| \alpha_{k+i} \varepsilon_{k-1}, \ 0 \leq i \leq \left[\frac{n-7}{2}\right], \ 0 \leq j \leq n-8-i, \tag{2.39}$$

where $A(k,j)$ is defined in terms of the Pochhammer symbol as

$$A(k,j) = (k-1-j)_k \tag{2.40}$$

and ε_{k-1} is the stepwise function of (2.38). After this first simplification, the non-vanishing brackets of $\mathfrak{g}_n$ are:

$$\begin{aligned} &[X_1, X_j] = X_{j+1}, \ 1 \leq j \leq n-1, \\ &[X_{2+j}, X_{j+3}] = \alpha_j X_{2j+7}, \ 0 \leq j \leq \left[\tfrac{n-3}{2}\right] - 2, \\ &[X_{2+i}, X_{4+i+j}] = \left(\alpha_i - j\alpha_{i+1} + \textstyle\sum_{k=2}^{n-8-2i} (-1)^k |A(k,j)| \alpha_{k+i} \varepsilon_{k-1}\right) X_{8+2i+j}, \end{aligned} \tag{2.41}$$

It remains to solve the Jacobi identities for the triples $\{X_i, X_j, X_k\}$ for $2 \leq i < j < k$. To this extent, basing on formula (2.39), it is convenient to introduce the following coefficients

$$\lambda_{i,-1} = C_{2+i,3+i}^{2i+7}, \ \lambda_{i,j} = C_{2+i,4+i+j}^{8+2i+j}, \ i \geq 2, \ j \geq 0, \tag{2.42}$$

that are added to the parameters α_k defined before. Then the Jacobi conditions for the triples $\{X_i, X_{i+a}, X_{i+b}\}$ with $i \geq 2, \ 0 < a < b$ lead to the equations

$$\lambda_{i+a-2,b-a-2}\lambda_{i-2,i+a+b} + \lambda_{i-2,a-2}\lambda_{i+b-2,i+a-b} - \lambda_{i-2,b-2}\lambda_{i+1-2,i+b-a} = 0, \tag{2.43}$$

which, after replacing the $\lambda_{i,j}$ by the expressions in (2.39), can be seen to be homogeneous polynomials of degree 2 in the parameters α_k. For example, taking $a = 1$ and $b = 2$ we obtain the polynomials

$$P(j) = \sum_{k=0}^{\left[\frac{j+3}{2}\right]} \frac{(-1)^{k+1}}{(k+1)!} \frac{\Gamma(j+5-k)}{\Gamma(j+4-2k)} \alpha_j \alpha_{j+k} + \sum_{k=1}^{\left[\frac{j+3}{2}\right]} (-1)^{k+1} (j+3) \frac{\Gamma(j+3-k)}{\Gamma(j+4-2k)} \alpha_{j-1} \alpha_{j+k}. \tag{2.44}$$

It should be observed that the number of linearly independent equations of type (2.43) never exceeds $n - 12$.

Let $\mathfrak{g}_S$ be a Lie algebra with brackets (5.19) that satisfies the Jacobi identities for some set of parameters $S = \left(\alpha_0, \cdots, \alpha_{\left[\frac{n-3}{2}\right]-2}\right)$ and let $\mathcal{A}$ be the set of Lie algebras isomorphic to $\mathfrak{g}_S$. Using symbolic computation packages specifically developed for the computation of cohomologies (see e.g. [22]), the analysis of the solutions to the Jacobi conditions (2.43) for the different dimensions leads to the following cases:

1. For $n = 7, 8$, there is only one parameter α_0 and no quadratic Jacobi conditions. If $\alpha_0 \neq 0$, then it can be normalized to 1, and the resulting Lie algebra $\mathfrak{g} \simeq \mathfrak{Br}_{4,n}$ is easily seen to satisfy the condition $H^2(\mathfrak{g}, \mathfrak{g}) = 0$.

2. For $n = 9, 10$, there are no quadratic Jacobi conditions, and for all values of α_0 and α_1 we have a Lie algebra such that for any two distinct values of the parameters, the corresponding Lie algebras are not isomorphic.

3. For $n = 11, 12$, there are no quadratic Jacobi conditions, and three parameters α_0,α_1 and α_2 are given. Lie algebras corresponding to different values of the parameters are non-isomorphic.

4. For $n = 13, 14$, there is one quadratic condition

$$-5\alpha_1^2 + \alpha_1(6\alpha_2 - \alpha_3) + 2\alpha_0(2\alpha_2 - \alpha_3) = 0, \tag{2.45}$$

from which a dependence relation of α_3 in terms of α_0, α_1 and α_2 is obtained. Lie algebras corresponding to different values of the remaining parameters (α_1, α_2) are pairwise non-isomorphic.

5. For $n = 15$, the are two quadratic Jacobi constraints, the previous one (2.45) as well as

$$6\alpha_1\alpha_2 - 15\alpha_2^2 - 4\alpha_0\alpha_4 + 3\alpha_1\alpha_3 + 10\alpha_2\alpha_3 + (2\alpha_0 - \alpha_1 - \alpha_2)\alpha_4 = 0. \tag{2.46}$$

Again, after elimination of two parameters from these conditions, we obtain a continuous parameterized family of Lie algebras, and no rigid algebras within the family exist.

6. For $n = 16$, the Jacobi conditions (2.43) imply the quadratic system

$$\begin{aligned} -5\alpha_1^2 + \alpha_1(6\alpha_2 - \alpha_3) + 2\alpha_0(2\alpha_2 - \alpha_3) = 0, \\ 6\alpha_1\alpha_2 - 15\alpha_2^2 - 4\alpha_0\alpha_4 + 3\alpha_1\alpha_3 + 10\alpha_2\alpha_3 + (2\alpha_0 - \alpha_1 - \alpha_2)\alpha_4 = 0, \\ -12\alpha_1\alpha_2 + 4\alpha_0(2\alpha_3 - \alpha_4) + (19\alpha_1 + 30\alpha_2)\alpha_3 - (23\alpha_1 + 18\alpha_2)\alpha_4 = 0. \end{aligned} \tag{2.47}$$

A first type of solution can be described in terms of a parameter $\alpha_1 = t$, leading to a parameterized family, the members of which can be deformed onto each other. Three additional isolated solutions are given by the values

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, -\tfrac{2}{3}, -\tfrac{5}{3}, a, \tfrac{30-62a-45a^2}{3a-8}\right),$$

where a is a root of the polynomial

$$729a^3 + 432a^2 - 795a + 250 = 0.$$

All the solvable Lie algebras associated to these solutions satisfy $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 1$, and are not rigid, as they can deformed into the parameterized family.

7. For $n=17$, the situation is similar. The equations (2.43) reduce to four, the three given in (2.47), to which the equation

$$\left(15\alpha_3-7\alpha_1+28\alpha_2\right)\alpha_3-35\alpha_3^2+\left(4\alpha_0-7\alpha_1\right)\alpha_4+\left(3\alpha_1-2\alpha_0-\alpha_3\right)\alpha_5=0 \qquad (2.48)$$

is added. Solutions are described in terms of a parameter $\alpha_1=t$, leading to a parameterized family the members of which are pairwise non-isomorphic. Further we have four isolated solutions corresponding to the values

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=\left(1,-\frac{2}{3},-\frac{b+7a+45}{20},-\frac{5}{3},a,b\right),$$

where a,b are the common roots of the polynomials

$$\begin{aligned} 9b^2+114ab+562b+357a^2+2754b+4665 &= 0, && (2.49)\\ 378a^2+54ab-4270a+8388b-9580 &= 0. && (2.50)\end{aligned}$$

The corresponding Lie algebras are not rigid, as they allow nontrivial deformations into the parameterized family.

8. For $n=18$, the Jacobi conditions (2.43) can be reduced to the five quadratic equations

$$\begin{aligned}
&-5\alpha_1^2+4\alpha_2\alpha_0+6\alpha_1\alpha_2-2\alpha_0\alpha_3-\alpha_1\alpha_3=0,\\
&-15\alpha_2^2+6\alpha_1\alpha_2+3\alpha_1\alpha_3+10\alpha_2\alpha_3+2\alpha_0\alpha_4-4\alpha_1\alpha_3-\alpha_1\alpha_4-\alpha_2\alpha_4=0,\\
&-35\alpha_3^2-7\alpha_1\alpha_3+28\alpha_2\alpha_3+4\alpha_0\alpha_4-7\alpha_1\alpha_4+15\alpha_3\alpha_4-2\alpha_0\alpha_5+3\alpha_1\alpha_5-\alpha_3\alpha_5=0,\\
&-12\alpha_1\alpha_2+8\alpha_0\alpha_3+19\alpha_1\alpha_3+30\alpha_2\alpha_3-4\alpha_0\alpha_4-23\alpha_1\alpha_4-18\alpha_2\alpha_4=0,\\
&21\alpha_1\alpha_3-49\alpha_2\alpha_3-12\alpha_0\alpha_4-4\alpha_1\alpha_4+20\alpha_2\alpha_4+55\alpha_3\alpha_4\\
&+6\alpha_0\alpha_5+16\alpha_1\alpha_5-15\alpha_2\alpha_5-22\alpha_3\alpha_5=0
\end{aligned}$$

Seventeen solutions to these equations are found, five of them corresponding to parameterized families depending on either the parameter pairs $\{\alpha_1,\alpha_2\}$, $\{\alpha_2,\alpha_3\}$ or $\{\alpha_4,\alpha_5\}$. In addition, twelve isolated solutions that give rise to rigid Lie algebras exist, five of them corresponding to rational Lie algebras, and determined by the values of the parameters:

$$\begin{aligned}
(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (0,0,0,5,14,35),\\
(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= (1,0,0,0,0,0),\\
(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= \left(-1,1,\tfrac{5}{3},\tfrac{5}{3},\tfrac{5}{3},\tfrac{7}{3}\right)\\
(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= \left(1,\tfrac{2}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},0\right),\\
(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) &= \left(1,\tfrac{1}{6},\tfrac{1}{33},\tfrac{5}{858},\tfrac{1}{858},\tfrac{7}{29172}\right)
\end{aligned}$$

The second of these solutions corresponds to the Lie algebra $\mathfrak{Br}_{4,18}$, while the other four have a vanishing second cohomology space, showing their algebraic rigidity. The seven remaining solutions, associated to the values

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=\left(a,1,\frac{5+b+2ab}{2\left(3+2a\right)},b,\frac{30ab^2+15b^2+16a^2b+50ab+126b-30}{8a^2+18ab+58a+9b+114},\alpha_5\right) \qquad (2.51)$$

where

$$\alpha_5=\tfrac{128a^3b\left(a+2b+3b^2\right)+8a^2\left(16b^3-568b^2+76b-30\right)}{(3+2a)(2a+b-3)(8a^2+18ab+58a+9b+114)}-\tfrac{4a\left(54b^3+985b^2+239b-15\right)-6\left(24b^3+763b^2-265b-105\right)}{(3+2a)(2a+b-3)(8a^2+18ab+58a+9b+114)},$$

are associated to the seven nonrational common roots of the polynomials

$$\begin{aligned} P_1(a,b) = & \; 32a^4b(10+b) - 8a^3b(3b^2+47b-120) - 4a^2(69b^3+346b^2+140b+150) \\ & -6a(43b^3+501b^2+395b+75) - 9(7b^3+134b^2+525-450) \\ P_2(a,b) = & \; 64a^5b(20+b) + 16a^4b(b-1)(220+173b) \\ & -8a^3(123b^4-4113b^3+3746b^2-185b+300) \\ & +4a^2(2385b^4+33923b^3-20989b^2-1330b-300) \\ & +9(791b^4+16498b^3-16013b^2+750) \\ & +12870b+6a(3209b^4+39045b^3-32541b^2+5695b+1350) \end{aligned}$$

Computing the corresponding cohomology space with the two parameters a, b subjected to the previous quadratic relations, a quite laborious computation shows that for any nonzero solution of the type (2.51), the corresponding solvable Lie algebra satisfies the condition $\dim H^2(\mathfrak{g},\mathfrak{g}) = 0$, showing that these Lie algebras are also algebraically rigid.

9. For values $n \geq 19$, the Jacobi conditions imply further restrictions on the parameters, and the system (2.43) only admits five or four solutions, depending on the parity of n. If n is odd, we obtain two parameterized families

$$\left(\alpha_0,\alpha_1,\cdots,\alpha_{\left[\frac{n-3}{2}\right]-2}\right) = \left(0,\cdots,0,\alpha_{\left[\frac{n-3}{2}\right]-4},\alpha_{\left[\frac{n-3}{2}\right]-3},21\alpha_{\left[\frac{n-3}{2}\right]-3}-70\alpha_{\left[\frac{n-3}{2}\right]-4}\right) \quad (2.52)$$

$$\left(\alpha_0,\alpha_1,\cdots,\alpha_{\left[\frac{n-3}{2}\right]-2}\right) = \left(0,\cdots,0,\alpha_{\left[\frac{n-3}{2}\right]-3},\alpha_{\left[\frac{n-3}{2}\right]-2}\right), \quad (2.53)$$

while for even n only the second family exists. In addition, exactly three isolated solutions are found, two of them arising from the constraint $\alpha_0 \neq 0$, which we can set equal to one. The first of such solutions is given by $\alpha_i = 0$ for $1 \leq i \leq \left[\frac{n-3}{2}\right]-2$, thus the corresponding Lie algebra is isomorphic to $\mathfrak{Br}_{4,n}$. The second Lie algebra $\mathfrak{K}_n$ is obtained from the values

$$\alpha_i = \frac{105}{32}\frac{\sqrt{\pi}\,\Gamma(i+3)}{4^k\,\Gamma\left(\frac{9}{2}+i\right)},\; 1 \leq i \leq \left[\frac{n-3}{2}\right]-2. \quad (2.54)$$

The complete commutators of the nilpotent part $\mathfrak{g}_n$ of $\mathfrak{K}_n$ are obtained using equation (2.39). The solvable Lie algebra $\mathfrak{K}_n$ satisfies $\dim H^2(\mathfrak{K}_n,\mathfrak{K}_n) = 0$. The third isolated solution $\mathfrak{M}_n$ is different depending on the parity. For odd $n = 2k+1$, it is given by the values

$$(\alpha_0,\alpha_1,\cdots,\alpha_{k-3}) = \left(0,\cdots,0,5,\frac{(k-2)(k-5)}{2},\frac{(k-2)!}{4!(k-6)!},\frac{(k-1)(k-2)k!}{144(k-4)!}\right),$$

while for even $n = 2k$, it is determined by

$$(\alpha_0,\alpha_1,\cdots,\alpha_{k-4}) = \left(0,\cdots,0,5,\frac{(k-2)(k-5)}{2},\frac{(k-2)!}{4!(k-6)!}\right)$$

In both cases, the brackets of the nilpotent part $\mathfrak{m}_n$ of $\mathfrak{M}_n$ are obtained from equation (2.39), and the condition $\dim H^2(\mathfrak{M}_n,\mathfrak{M}_n) = 0$ holds.

This analysis shows that, from the value $n = 19$ onwards, two series of algebraically rigid Lie algebras exists. The proof of the rigidity of $\mathfrak{Br}_{4,n}$ for these dimensions is essentially the same as for the case of the classical Bratzlavsky series (see [10]), and is essentially based on the fact that for these dimensions, only a finite number of isomorphism classes of solvable algebras subjected to the constraint $[X_2, X_3] \neq 0$ arise from the Jacobi conditions.

Theorem 1 *For $n \geq 19$, the solvable Lie algebras $\mathfrak{Br}_{4,n} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{f}_{4,n}$ are geometrically rigid and satisfy* $\dim H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n}) = 2$.

Proof. Endowing the base field $\mathbb{K}$ with a valuation, it follows that the metric topology on the variety $\mathcal{L}^n$ is finer than the Zariski topology. With respect to the former topology, we consider the two following open sets: the open set of complete Lie algebras and the set of rank one Lie algebras $\mathfrak{g}$ possessing a semisimple element T such that the spectrum $ad_{\mathfrak{g}}T$ is contained in the ball $B_{\varepsilon}^{n+1}(\mathbb{K})$ of radius $\varepsilon \to 0$ and centered at the point $(0, 1, 4, \cdots, n+2)$. Lie algebras lying in this open set are semidirect sums of a maximal torus $\mathfrak{t}$ and its nilradical $\mathfrak{n}$. As a consequence of the finiteness of conjugacy classes for tori for Lie algebras of a given dimension (see e.g. Proposition 1 in [6]) , it follows that the weights of $\mathfrak{t}$ are given by the sequence $(1, 4, \cdots, n+2)$. As the torus is fixed, the additional constraints $C_{1j}^{j+1} \neq 0$ and $C_{23}^{7} \neq 0$ imposed on the nilradical $\mathfrak{n}$ imply that the open set only intersects the orbits of $\mathfrak{K}_n$ and $\mathfrak{Br}_{4,n}$, $\mathfrak{M}_n$ and the parameterized families being excluded as $C_{23}^{7} = 0$ holds for these algebras. ■

It should be remarked that the rigidity of $\mathfrak{Br}_{4,n}$ can also be proved in terms of the properties of the so-called canonical deformation, within the approach to deformations by means of the affine scheme defined by the Jacobi identities [12]. In this context, an arduous computation shows that the cohomology classes of $\mathfrak{Br}_{4,n}$ are integrable up to order 13, being non-integrable for orders $d \geq 14$ onwards.

Final remarks

Rank one solvable Lie algebras with a maximal torus $\mathfrak{t}$, the eigenvalues of which are given by $(1, 4, \cdots, n+2)$ have been analyzed in the context of rigidity. It has been shown that for Lie algebras satisfying the additional constraint $[X_2, X_3] \neq 0$, the number of isomorphism classes is finite for values $n \geq 18$, whereas the case $[X_2, X_3] = 0$ always leads to parameterized families, as well as to the exception of an algebraically rigid algebra that appears for dimensions greater than 18. In low dimensions, isolated algebraically rigid Lie algebras have also been found. The Lie algebra $\mathfrak{Br}_{4,n}$ has been shown to be geometrically rigid for values $n \geq 18$, possessing a two-dimensional Chevalley cohomology space $H^2(\mathfrak{Br}_{4,n}, \mathfrak{Br}_{4,n})$, and such that the Rim map is not injective.

Basing on the results obtained in this work, as well as those in [12, 20], it seems reasonable to conjecture that within the class of rank one solvable Lie algebras possessing a maximal torus of derivations $\mathfrak{t}$ with eigenvalue spectrum given by $\mathrm{spec}(\mathfrak{t}) = (1, 2k, 2k+1, \cdots, n+2k-2)$, there always exists a series $\mathfrak{G}_n$ of rigid algebras satisfying the condition $\dim H^2(\mathfrak{G}_n, \mathfrak{G}_n) = k$, at least from some dimension onwards. For the case of spectra $\mathrm{spec}(\mathfrak{t}) = (1, 2k+1, 2k+2, \cdots, n+2k-1)$, the rigidity analysis is more delicate, as possibly the dimension of the cohomology space depends on the parity of the dimension, thus leading to two subcases with structurally different properties. For both possibilities, it is expected that new types of rank one cohomologically rigid solvable Lie algebras will also arise. In this context, a formalization of the whole class of rigid Lie algebras with a torus $\mathfrak{t}$ having the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, 2+k, 3+k, \cdots, n+k)$ for any $k \geq 2$ would constitute a further step towards the characterization of rank one rigid Lie algebras, as well as an important tool in the study of tori possessing eigenvalues with multiplicities. Work in this direction is currently in progress.

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Capítulo 3

Some features of rank one real solvable cohomologically rigid Lie algebras with a nilradical contracting onto the model filiform Lie algebra Q_n.

3.1 Introduction

This chapter presents and develops the results obtained in the article *Some features of rank one real solvable cohomologically rigid Lie algebras with a nilradical contracting onto the model filiform Lie algebra Q_n.*, in which the rigidity of solvable Lie algebras of rank one with a contracting nilradical in the Qn filiform model is studied.
In a wide sense, the notion of rigidity of Lie algebras can be understood as the stability of its defining structure tensor with respect to some action of a group or a deformation, in the sense that, whatever the alteration of the structure tensor defined by geometric or algebraic means, the resulting Lie algebra is isomorphic to the starting one. Although originally developed in the context of varieties or functional spaces, the rigidity concept has turned out to be very important in the Lie algebraic frame, not only for classification purposes, but also for the geometric study of the variety determined by the structure tensor of Lie algebras in a given dimension [13], its irreducible components and various limiting processes on the variety [36]. The development of Lie algebra cohomology [28], as well as its implications in the theory of deformations and contractions [22, 23, 36] motivated the search for rigidity criteria beyond the well-known case of semisimple algebras, that follows as a consequence of the classical Whitehead lemmata [40]. The rigidity of Lie algebras has been analyzed from various perspectives, ranging from the geometrical and topological point of view in terms of the so-called Jacobi schemes [17] to the pure algebraic approach, based on the Chevalley cohomology of Lie algebras [28] or the root theory of solvable Lie algebras, based on study of the eigenvalue spectrum of generators of maximal tori of derivations of nilpotent algebras, that generalizes naturally the approach of roots systems of semisimple Lie algebras [2, 20]. In this sense, large classes of rigid Lie algebras in the cohomological sense have been classified (see e.g. [7, 8, 16, 24, 27, 32, 33] and references therein), albeit it is well known that the rigidity notion goes far beyond the cohomological criteria [34, 35], even allowing the existence of rigid Lie algebras that can be purely nonrational and nonreal [25].

In this work we consider solvable Lie algebras $\mathfrak{r}$ of rank one possessing a maximal torus of derivations $\mathfrak{t}$, the spectrum of which is given by the eigenvalues $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3)$ on its n-dimensional nilradical $\mathfrak{n}$. These nilradicals are known to contract onto the so-called Q_n-model filiform algebra, a nilpotent Lie algebra of maximal nilpotence index with the additional property of satisfying the condition $[[\mathfrak{n},\mathfrak{n}],[\mathfrak{n},\mathfrak{n}]] \neq 0$ [39]. Analyzing in detail the cases for low values of k, we concentrate on the resulting solvable Lie algebras that are rigid in the cohomological sense. Basing on certain patterns observed for these low values, we extrapolate the result to obtain various infinite series of rigid Lie algebras.

This work complements and extends the existing analysis of rank one solvable rigid Lie algebras, following the same motivations and techniques used in the works [1, 2, 9, 16, 24], where the eigenvalue spectra $\mathrm{spec}(\mathfrak{t}) = (1,2,3,\cdots,n)$, $\mathrm{spec}(\mathfrak{t}) = (1,3,4,\cdots,n+1)$ and $\mathrm{spec}(\mathfrak{t}) = (1,4,5,\cdots,n+2)$ were studied in detail.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and defined over the real field $\mathbb{K} = \mathbb{R}$.

3.1.1 Solvable real (rigid) Lie algebras

Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra formed by its derivations, i.e., by the linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D[X,Y] = [D(X), D(Y)], \quad X, Y \in \mathfrak{g}.$$

A derivation D will be called inner if there exists an element $X \in \mathfrak{g}$ such that $D(Y) = \mathrm{ad}(X)(Y) := [X,Y]$ for all $Y \in \mathfrak{g}$. Otherwise it will be called an outer derivation.

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus of derivations is an Abelian subalgebra of $Der(\mathfrak{g})$ the generators of which are semisimple.*

Due to the semi-simplicity and abelianity, the maps $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ are simultaneously diagonalizable over the complex field $\mathbb{C}$. For complex Lie algebras, maximal tori $\mathfrak{t}$ are known to be conjugate to each other, while for the real case the maximal tori are divided into a finite number of equivalence classes [31]. In both cases, the dimension of a maximal torus is a scalar invariant $\mathrm{r}(\mathfrak{g})$ of the Lie algebra, called the rank of $\mathfrak{g}$.
One important structural result (see e.g. [37] and references therein) states that any real or complex solvable Lie algebra $\mathfrak{r}$ admits a decomposition

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n} \tag{3.1}$$

satisfying the relations

$$[\mathfrak{t},\mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{n}, \tag{3.2}$$

where $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{r}$ (called the nilradical) and $\overrightarrow{\oplus}$ denotes the semidirect sum determined by the complementary linear space $\mathfrak{t}$ formed by linearly nil-independent outer derivations of $\mathfrak{n}$. It can be further shown that the dimension of $\mathfrak{t}$ satisfies the following inequality

$$\dim \mathfrak{n} - \dim [\mathfrak{n},\mathfrak{n}] \geq \dim \mathfrak{t}, \tag{3.3}$$

providing an upper bound for the rank of a solvable Lie algebra.

3.1.2 Cohomologically rigid Lie algebras

Let $\mathcal{L}^n_{\mathbb{R}}$ denote the variety of n-dimensional real Lie algebras $\mathfrak{g} = \left(\mathbb{R}^n, [\,,\,]_{\mathfrak{g}}\right)$ over $\mathbb{R}$. The general linear group $GL(n,\mathbb{R})$ acts naturally on $\mathcal{L}^n$ by means of:

$$(f \star \mathfrak{g})(X,Y) = f\left(\left[f^{-1}(X), f^{-1}(Y)\right]_{\mathfrak{g}}\right),\ f \in GL(n,\mathbb{K}),\ X, Y \in \mathfrak{g}. \tag{3.4}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is thus given by all algebras isomorphic to $\mathfrak{g}$. The space $\mathcal{O}(\mathfrak{g})$ can further be identified with the homogeneous space $GL(n,\mathbb{R})/\mathrm{Aut}(\mathfrak{g})$, from which the identity $\dim \mathcal{O}(\mathfrak{g}) = n^2 - \dim \mathrm{Der}(\mathfrak{g})$ follows at once.

Definition 2 *A Lie algebra $\mathfrak{g}$ is called rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

An equivalent definition can be obtained using the openess of the orbit $\mathcal{O}(\mathfrak{g})$ with respect to the Zariski topology of $\mathcal{L}^n$ [13, 14]. Although rigidity is primarily a topological notion, sufficient conditions to ensure that a Lie algebra is rigid can be obtained by purely algebraic means, using the Chevalley cohomology [28]. This approach involves the computation of the cohomology spaces $H^p(\mathfrak{g},\mathfrak{g})$ for $p \leq 3$. The spaces $H^0(\mathfrak{g},\mathfrak{g})$ and $H^1(\mathfrak{g},\mathfrak{g})$ are identified with the centre $Z(\mathfrak{g})$ and the outer derivations $\mathrm{Der}(\mathfrak{g})/\mathrm{IDer}(\mathfrak{g})$ of $\mathfrak{g}$ respectively [40]. In order to illustrate how the rigidity problem leads naturally to cohomological methods, we recall the notion of contraction of Lie algebras (see e.g. [28, 12, 21, 41] and references therein).

Let $\mathfrak{g}$ be a Lie algebra and $\Phi_\varepsilon \in Aut(\mathfrak{g})$ a family of automorphisms of $\mathfrak{g}$, where $\varepsilon \in [0,1)$. For any $X, Y \in \mathfrak{g}$ we define

$$[X,Y]_{\Phi_\varepsilon} := \Phi_\varepsilon^{-1}\left[\Phi_\varepsilon(X), \Phi_\varepsilon(Y)\right], \tag{3.5}$$

which obviously are the brackets of the Lie algebra over the transformed basis. Now suppose that the limit

$$[X,Y]_0 := \lim_{\varepsilon\to 0} \Phi_\varepsilon^{-1}\left[\Phi_\varepsilon(X), \Phi_\varepsilon(Y)\right] \tag{3.6}$$

exists for any $X, Y \in \mathfrak{g}$. Then equation (3.6) defines a Lie algebra $\mathfrak{g}'$ called the contraction of $\mathfrak{g}$ (by Φ_ε), non-trivial if $\mathfrak{g}$ and $\mathfrak{g}'$ are non-isomorphic, and trivial otherwise. A contraction for which there exists some basis $\{Y_1, .., Y_n\}$ such that the contraction matrix A_Φ is diagonal, i.e., adopts the form

$$(A_\Phi)_{ij} = \delta_{ij}\varepsilon^{n_j}, \quad n_j \in \mathbb{R}, \varepsilon > 0,$$

is further known as a generalized Inönü-Wigner contraction [41].

Deformations of Lie algebras, deeply related to contractions and cohomology [22, 23], originally arise from the study of local geometric properties of the variety $\mathcal{L}^n_{\mathbb{R}}$ when considered as a transformation space. In this context, a formal deformation $\mathfrak{g}_t$ of a Lie algebra $\mathfrak{g} = (V, \mu)$ is given by the deformed commutator:

$$[X,Y]_\varepsilon := [X,Y] + \psi_m(X,Y)\varepsilon^m,$$

where t is a parameter and $\psi_m : V \times V \to V$ is a skew-symmetric bilinear map. Imposing the Jacobi identity (up to quadratic order of t) to the deformed commutator, we obtain:

$$\begin{aligned}\Big[X_i, [X_j, X_k]_\varepsilon\Big]_\varepsilon + \Big[X_k, [X_i, X_j]_\varepsilon\Big]_\varepsilon + [X_j, [X_k, X_i]_\varepsilon]_\varepsilon &= \varepsilon\, d\psi_1(X_i, X_j, X_k) + \\ &\varepsilon^2\left(\frac{1}{2}[\psi_1,\psi_1] + d\psi_2\right)(X_i, X_j, X_k) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}d\psi_l(X_i, X_j, X_k) := &[X_i, \psi_l(X_j, X_k)] + [X_k, \psi_l(X_i, X_j)] + [X_j, \psi_l(X_k, X_i)] + \\ &+\psi_l(X_i, [X_j, X_k]) + \psi_l(X_k, [X_i, X_j]) + \psi_l(X_j, [X_k, X_i]), \quad l = 1,2\end{aligned} \tag{3.8}$$

and

$$\frac{1}{2}[\psi_1,\psi_1](X_i, X_j, X_k) := \psi_1(\psi_1(X_i, X_j), X_k) + \psi_1(\psi_1(X_j, X_k), X_i) + \psi_1(\psi_1(X_k, X_i), X_j). \tag{3.9}$$

In order to satisfy the Jacobi identity, the conditions

$$d\psi_1(X_i, X_j, X_k) = 0, \tag{3.10}$$

$$\frac{1}{2}[\psi_1,\psi_1](X_i, X_j, X_k) + d\psi_2(X_i, X_j, X_k) = 0 \tag{3.11}$$

must be fulfilled. The expression (5.8) satisfied by ψ_1 characterizes it as a 2-cocycle in the second cohomology space $H^2(\mathfrak{g},\mathfrak{g})$, while equation (5.9) implies that the deformation has to satisfy an integrability condition.[1] The latter is given by the condition

$$\frac{1}{2}\left[\varphi,\varphi\right]\left(X_i,X_j,X_k\right)=0, \tag{3.12}$$

for all X_i, X_j, X_k in $\mathfrak{g}$. Hence, if φ is an integrable cocyle, the linear deformation given by

$$[X,Y]_\varepsilon := [X,Y]+\varepsilon\,\varphi(X,Y)$$

satisfies the Jacobi identity and defines a Lie algebra. In particular, nullity of $H^2(\mathfrak{g},\mathfrak{g})$ implies that any deformation is isomorphic to $\mathfrak{g}$. This is one possible formulation of the Nijenhuis-Richardson criterion, from which ample classes of Lie algebras such as semisimple and parabolic algebras are easily shown to be rigid [32, 33, 35, 38].

Proposition 1 *Let* $\mathfrak{g}$ *be a Lie algebra. If the condition* $\dim H^2(\mathfrak{g},\mathfrak{g})=0$ *holds, then* $\mathfrak{g}$ *is rigid.*

Lie algebras satisfying $H^2(\mathfrak{g},\mathfrak{g})=0$ are usually called cohomologically rigid. However, as was also shown in [32], the nullity of the cohomology space $H^2(\mathfrak{g},\mathfrak{g})$ is not a necessary condition for rigidity, i.e., there exist Lie algebras satisfying $H^2(\mathfrak{g},\mathfrak{g})\neq 0$ but possessing an open orbit. Such Lie algebras, known as geometrically rigid, are lesser known and harder to obtain than cohomologically rigid algebras. Cohomologically rigid Lie algebras (over $\mathbb{R}$ and $\mathbb{C}$) are dominant in low dimensions and certain types of maximal tori [4, 5, 6, 27], although for dimensions $n\geq 12$ continuous series of geometrically (solvable) rigid Lie algebras have been found [8, 9, 11, 15, 16, 24, 32]. In this latter case, the analysis of the cohomology space $H^3(\mathfrak{g},\mathfrak{g})$, based on the so-called Rim map defined in equation (5.7), has been shown to be of crucial importance [34].

The main tool in the cohomological approach is the Hochschild-Serre factorization theorem [28], a procedure that simplifies the explicit computation of the cohomology classes. For the purpose of this work it suffices to restrict ourselves to the case of solvable real Lie algebras $\mathfrak{r}=\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}$ such that $\mathfrak{t}$ is Abelian and the operators $\mathrm{ad}_{\mathfrak{r}}T$ $(T\in\mathfrak{t})$ are diagonal. The symbol $\overrightarrow{\oplus}$ indicates that the sum is semidirect, with $\mathfrak{t}$ acting on $\mathfrak{n}$ by derivations. In these conditions (see e.g. [40]), the cohomology space $H^p\left(\mathfrak{r},\mathfrak{r}\right)$ satisfies the isomorphism

$$H^p\left(\mathfrak{r},\mathfrak{r}\right)\simeq\sum_{a+b=p}H^a\left(\mathfrak{t},\mathbb{R}\right)\otimes H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}}, \tag{3.13}$$

where the space of $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$ is defined as

$$H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}}=\left\{[\varphi]\in H^b\left(\mathfrak{n},\mathfrak{r}\right)\;\middle|\;(T.\varphi)=0,\;T\in\mathfrak{t}\right\}. \tag{3.14}$$

The invariance condition of a b-cochain φ is given by

$$\left(T.\varphi\right)\left(Z_1,\cdots,Z_b\right)=\left[T,\varphi\left(Z_1,\cdots,Z_b\right)\right]-\sum_{s=1}^{b}\varphi\left(Z_1,\cdots,\left[T,Z_s\right],\cdots,Z_b\right). \tag{3.15}$$

Using the well known fact that $H^a\left(\mathfrak{t},\mathbb{R}\right)=\bigwedge^a\mathfrak{t}$, it is straightforward to verify that $H^p\left(\mathfrak{r},\mathfrak{r}\right)=0$ holds if and only if $H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}}=0$ for $0\leq b\leq p$.

We observe that if a complex Lie algebra $\mathfrak{g}'$ admits the real form $\mathfrak{g}$, i.e., such that $\mathfrak{g}'\simeq\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$, then it follows from the Chevalley cohomology that [5]

$$\dim_{\mathbb{R}}H^2(\mathfrak{g},\mathfrak{g})=\dim_{\mathbb{C}}H^2(\mathfrak{g}',\mathfrak{g}'). \tag{3.16}$$

[1]Further constraints are obtained if the deformed bracket is developed up to higher order. See e.g. [28].

In this sense, a real Lie algebra $\mathfrak{g}$ is algebraically rigid if

$$\dim_{\mathbb{R}} H^2(\mathfrak{g},\mathfrak{g}) = 0. \tag{3.17}$$

Clearly, in these conditions the complexified algebra $\mathfrak{g}' = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex algebraically rigid Lie algebra. Conversely, real forms $\mathfrak{g}_i$ of a complex algebraically rigid Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_i \otimes \mathbb{C}$ are algebraically rigid. It should however be observed that there exist algebraically rigid purely complex Lie algebras that do not admit real forms [25]. A negative test for a solvable Lie algebra $\mathfrak{r}$ to be algebraically rigid is further given by the following relation (see e.g. [14, 19]):

$$\dim \mathrm{Der}(\mathfrak{r}) \leq \dim \mathfrak{r} + \dim H^2(\mathfrak{r},\mathfrak{r}). \tag{3.18}$$

It follows that if $\dim \mathrm{Der}(\mathfrak{r}) > \dim \mathfrak{r}$, then $\mathfrak{r}$ cannot be cohomologically rigid. In particular, if a real (or complex) Lie algebra satisfying the decomposition (6.6) is rigid, an important structural result due to R. Carles implies that $\mathfrak{t}$ must be a maximal external torus of derivations of $\mathfrak{n}$ [14].

3.2 Rigid Lie algebras with filiform nilradical

For studying nilpotent Lie algebras, we recall an extremely useful invariant called the so-called characteristic sequence $c(\mathfrak{n})$. Given a nilpotent Lie algebra $\mathfrak{n}$, for a non-zero element $X \in \mathfrak{n} \setminus [\mathfrak{n},\mathfrak{n}]$ we consider the decreasing sequence of dimensions of the Jordan blocks of the linear operator $\mathrm{ad}(X)$.

$$c(X) = (c_1(X), c_2(X), \cdots, c_k(X), 1), \quad c_i(X) \geq c_{i+1}(X) \geq 1. \tag{3.19}$$

Definition 3 *The characteristic sequence of a nilpotent Lie algebra* $\mathfrak{n}$ *is defined as*

$$c(\mathfrak{n}) = \sup \{c(X) \mid X \in \mathfrak{n} \setminus [\mathfrak{n},\mathfrak{n}]\} \tag{3.20}$$

In particular, if a nilpotent Lie algebra has characteristic sequence $c(\mathfrak{n}) = (n-1, 1)$, then there exists a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{n}$ such that

$$[X_1, X_j] = X_{j+1},\ 2 \leq j \leq n-1.$$

Such Lie algebras are commonly called filiform [39]. The remaining commutators $[X_i, X_j]$ for $2 \leq i, j$ are related by means of the Jacobi condition

$$[X_1,[X_i,X_j]] + [X_j,[X_1,X_i]] + [X_i,[X_j,X_1]] = 0. \tag{3.21}$$

We observe that, as a consequence of the upper bound (6.8), a filiform Lie algebra has a maximal torus of dimension at most two. It was shown in [39] that there exist only two isomorphism classes (called the model filiform algebras) possessing rank two, and given respectively by

- L_n $(n \geq 3)$:

$$[X_1, X_j] = X_{j+1},\ 2 \leq j \leq n-1 \tag{3.22}$$

- Q_n $(n = 2q \geq 6)$:

$$[X_1, X_i] = X_{i+1},\ \ [X_j, X_{n+1-j}] = (-1)^j X_n, \quad 2 \leq i \leq n-2,\ 2 \leq j \leq q. \tag{3.23}$$

If $\mathfrak{t}_L$ and $\mathfrak{t}_Q$ denote a maximal torus of L_n and Q_n respectively, it can be easily that the eigenvalue spectrum is given by

$$\text{spec}(\mathfrak{t}_L) = (\lambda, \mu, \lambda + \mu, 2\lambda + \mu, \cdots, (n-3)\lambda + \mu, (n-2)\lambda + \mu)\,, \tag{3.24}$$
$$\text{spec}(\mathfrak{t}_Q) = (\lambda, \mu, \lambda + \mu, 2\lambda + \mu, \cdots, (n-3)\lambda + \mu, (n-3)\lambda + 2\mu)\,, \tag{3.25}$$

where λ, μ are integers. The remaining filiform Lie algebras are all of rank one, and generic properties of their tori have been studied in [26]. In particular, it was shown there that any n-dimensional filiform Lie algebra of rank one is isomorphic to one of the following Lie algebras:[2]

1. $A_n^k(\lambda_1, \cdots, \lambda_{l-1})\,,\ l = \left[\frac{n+1-k}{2}\right],\ 2 \leq k \leq n-3:$

$$\begin{array}{ll} [X_1, X_i] = X_{i+1}, & 2 \leq i \leq n-1 \\ [X_i, X_{i+1}] = \lambda_{i-1} X_{2i+k-1}, & 2 \leq i \leq l \\ [X_i, X_j] = a_{ij} X_{i+j+k-2}, & 2 \leq i < j,\ i+j+k \leq n \end{array}$$

2. $B_n^k(\lambda_1, \cdots, \lambda_{l-1})\,,\ n = 2m,\ l = \left[\frac{n-k}{2}\right],\ 2 \leq k \leq n-3:$

$$\begin{array}{ll} [X_1, X_i] = X_{i+1}, & 2 \leq i \leq n-2 \\ [X_i, X_{n+1-i}] = (-1)^i X_n, & 2 \leq i \leq n-1 \\ [X_i, X_{i+1}] = \lambda_{i-1} X_{2i+k-1}, & 2 \leq i \leq l \\ [Y_i, Y_j] = a_{ij} X_{i+j+k-2}, & 2 \leq i < j,\ i+j+k \leq n-2 \end{array}$$

The parameters $(\lambda_1, \cdots, \lambda_{l-1})$ are non-simultaneously vanishing and satisfy the Jacobi relations. Moreover, the constants $a_{i,j}$ satisfy the conditions

$$a_{i,j} = a_{i,j+1} + a_{i+1,j},\ a_{i,i+1} = \lambda_{i-1}.$$

Now let $\mathfrak{n}$ be isomorphic to either A_n^k or B_n^k and consider the endomorphism $F : \mathfrak{n} \to \mathfrak{n}$ defined by

$$X_1' = X_1,\ X_j' = \varepsilon\, X_j,\ X_n' = \varepsilon^2\, X_n,\ 2 \leq j \leq n-1. \tag{3.26}$$

Computing the brackets over the transformed basis we obtain

$$\left[X_1', X_j'\right] = X_{j+1}',\ \left[X_i', X_{i+1}'\right] = \varepsilon \lambda_{i-1} X_{2i+k-1}',\ \left[X_i', X_j'\right] = a_{ij} \varepsilon X_{i+j+k-2}'$$

for both cases, and

$$\left[X_i', X_{n+1-i}'\right] = (-1)^i X_n'$$

for the B_n^k series. For the limit $\varepsilon \to 0$, it follows that A_n^k contracts onto the model filiform Lie algebra L_n, while B_n^k contracts onto Q_n. In this context, it is convenient to separate the rigidity analysis of rank one solvable Lie algebras with filiform nilradical into the two cases, depending whether the nilradical belongs to the A_n^k or B_n^k, although some conclusions can be obtained simultaneously for both types (see [24]). We mention that the so-called Bratzlavsky series [11], the first of the solvable geometrically rigid Lie algebras to be described in detail [15], has a nilradical of type A_n^k. Other large classes of rigid Lie algebras based on this nilradical have been obtained for the values $k = 2, 3, 4$, covered in full detail (see [15, 9, 24] and references therein), as well as in the (full) classification of solvable rigid algebras in low dimensions [27].

[2] In [26] a third type C_n^k was given, that must be discarded as it actually has rank two and leads to the model algebra Q_{2n}.

In the sequel, we are principally interested on rank one algebraically rigid Lie algebras with nilradical of type B_n^k. In this context, it is well known that for any integer $k \geq 2$, the $n = \left(k+4+\frac{1-(-1)^k}{2}\right)$ -dimensional real nilpotent Lie algebra $\mathfrak{n}_n^k$ defined over a basis $\{X_1, \cdots, X_n\}$ by the commutators

$$\begin{array}{ll} [X_1, X_j] = X_{j+1}, & 2 \leq j \leq n-2, \\ [X_i, X_{n+1-i}] = (-1)^i X_n, & 2 \leq i \leq \frac{n}{2}, \\ [X_2, X_3] = X_{3+k}, & [X_2, X_4] = \frac{1-(-1)^k}{2} X_{4+k}. \end{array} \tag{3.27}$$

is algebraically rigid (this is immediate from the results in [24]). A short computation shows that the spectrum corresponding to these algebras is given by

$$\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3), \tag{3.28}$$

and actually corresponds to the lowest possible dimension for which the condition $C_{23}^{3+k} \neq 0$ is satisfied. It may seem surprising that merely adding one or two additional commutators (depending on the parity of k) we obtain a rigid Lie algebra of rank one. This fact suggests that the number of (cohomologically) rigid Lie algebras of rank one and possessing a torus $\mathfrak{t}$ with spectrum (3.28) must be very ample. We will see that large families of rigid Lie algebras with vanishing cohomology emerge from this eigenvalue spectrum, with a range of dimensions for which such algebras exist determined by the value and parity of k.

3.3 Cohomologically rigid rank one Lie algebras with eigenvalue spectrum (3.28)

In this section we analyze some generic features of real solvable Lie algebras $\mathfrak{r}_n^k = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_n^k$ of rank one, the torus $\mathfrak{t}$ of which has eigenvalues $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3)$ and such that the nilradical contracts $\mathfrak{n}_n^k$ onto the model filiform algebra Q_n. Let $\{T, X_1, X_2, ..., X_n\}$ be a basis of $\mathfrak{r}_n^k$. The action of T over the nilradical is given by

$$[T, X_1] = X_1,\ [T, X_j] = (k+j-2)X_j,\ [T, X_n] = (n+2k-3)X_n, \quad (2 \leq j \leq n-1). \tag{3.29}$$

As follows from this action, for any $i, j \geq 1$ the commutator $[X_i, X_j]$ corresponds to an element of $\mathfrak{n}_n^k$ of eigenvalue $2k+i+j-4$, from which we conclude that the commutators adopt the generic form

$$\begin{aligned} [X_1, X_j] &= C_{1,j}^{j+1} X_{j+1}, 2 \leq j \leq n-2, \\ [X_i, X_j] &= C_{i,j}^{i+j+k-2} X_{i+j+k-2},\ i+j < n+1, \\ [X_i, X_{n+1-i}] &= C_{i,n+1-i}^{n} X_n,\ 2 \leq i \leq \frac{n}{2}. \end{aligned} \tag{3.30}$$

In particular, as we require that $[X_2, X_{n-1}] \neq 0$ holds, it can be immediately verified by the Jacobi condition that $C_{i,n+1-i}^n = (-1)^i C_{2,n-1}^n$ holds for all $i \geq 2$. Without loss of generality we can assume that $C_{2,n-1}^n = 1$, so that the generic structure is given by

$$\begin{array}{ll} [X_1, X_j] = X_{j+1}, & 2 \leq j \leq n-2, \\ [X_i, X_j] = C_{i,j}^{i+j+k-2} X_{i+j+k-2}, & i+j < n+1, \\ [X_i, X_{n+1-i}] = (-1)^i X_n, & 2 \leq i \leq \frac{n}{2}. \end{array} \tag{3.31}$$

The Lie algebras of type (3.31) are therefore determined by the solutions to the polynomial equations defined by Jacobi condition for $1 \leq i \leq j \leq \ell \leq n-1$. The Jacobi condition applied to the triple $\{X_1, X_i, X_j\}$ with $i < j$ leads to the linear system

$$C_{i,j}^{i+j+k-2} - C_{i+1,j}^{i+j+k-1} - C_{i,j+1}^{i+j+k-1} = 0,\ 2 \leq i < j. \tag{3.32}$$

In particular, for $j = i+1$ we get the relations

$$C_{i,i+1}^{2i+k-1} - C_{i,i+2}^{2i+k} = 0,\ i \geq 2. \tag{3.33}$$

Introducing the notation $\alpha_{i-1} = C_{i,i+1}^{2i+k-1}$ for $2 \leq i \leq \left[\frac{n-k}{2}\right]$, a cumbersome but routine computation shows that the system (3.32) admits the solution

$$C_{i,j}^{i+j+k-2} = \sum_{p=1}^{\left[\frac{j+1-i}{2}\right]} (-1)^{p+1} \frac{\Gamma(j+1-i-p)}{(p-1)!\,\Gamma(j+2-2p-i)}\,\alpha_{i+p-2}, \tag{3.34}$$

where $\Gamma(z)$ denotes the Gamma function. With this reduction, any Lie algebra of type (3.31) will be determined by the values α_s for $1 \leq s \leq \left[\frac{n-k}{2}\right]$. It still remains to evaluate the Jacobi conditions for the triples $\{X_i, X_j, X_l\}$ for $i, j, l \geq 2$. Expanding these relations we obtain the equations

$$C_{j,l}^{j+l+\kappa-2} C_{i,j+l+\kappa-2}^{i+j+l+2\kappa-4} - C_{i,j}^{i+j+\kappa-2} C_{i+j+\kappa-2,l}^{i+j+l+2\kappa-4} - C_{i,l}^{i+l+\kappa-2} C_{j,i+l+\kappa-2}^{i+j+l+2\kappa-4} = 0. \tag{3.35}$$

We observe that whenever the numerical relation

$$i + j + l = n + 3 - \kappa$$

is satisfied, the corresponding equation in (3.35) is linear, because it involves commutators of the type $[X_a, X_{n+1-a}] = (-1)^a X_n$. Now, replacing the $C_{j,l}^{j+l+\kappa-2}$ by the expression determined by equation (3.34), the system (3.35) reduces to a set of linear or homogeneous quadratic polynomials in the variables α_s, so that the nilradical $\mathfrak{n}_n^k$ is completely specified (using the previous identities) by the “diagonal"sequence

$$(\alpha_1, \dots, \alpha_s) = \left(C_{2,3}^{k+3}, \cdots C_{p_0,p_0+1}^{n-1-\frac{1+(-1)^k}{2}} \right),\ p_0 = \left[\frac{n-k-1-\frac{1+(-1)^k}{2}}{2} \right]. \tag{3.36}$$

Although for low values of k and n these equations can still be solved by hand, for increasing dimensions $k+6 \leq n \leq 4k$ the use of computer methods becomes a necessity, in order to obtain the solutions for the quadratic equations determined by (3.35), as well as to determine the cohomology $H^2\left(\mathfrak{r}_n^k, \mathfrak{r}_n^k\right)$. It is therefore unavoidable, from a certain dimension onwards, to use computer packages in order to determine precise (even partial) classifications of rigid Lie algebras.

The computation of the cohomologies has been done using the symbolic computation package *SuperLie* (see [29, 30] and references therein) as well as specific codes for rank one Lie algebras developed by the authors, that have also been adapted to solve the quadratic systems given by the Jacobi conditions. Both programs have been executed on different platforms in order to double-check the cohomologies.

3.4 A case study: $k \leq 5$

For low values $k = 2, 3, 4, 5$, the analysis of the Jacobi conditions (3.35) can still be developed directly, due to the low dimension of the resulting algebras and the relative simplicity of the Jacobi conditions. It is therefore instructive to consider these values in detail, even if most of the rigid algebras that arise in these cases have already been considered in the literature (see for instance the references in [5, 16, 24]). The dimensions analyzed are $n \geq k+4$ or $k+5$ depending on the parity of k, as the series given in (3.27) describes the lowest-dimensional case for which $C_{2,3}^{k+3} \neq 0$ is satisfied.

3.4.1 $k = 2$

Let $\mathfrak{r}_n^2 = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_n^2$. For this value, the corresponding eigenvalue spectrum is given by

$$\operatorname{spec}(\mathfrak{t}) = (1, 2, 3, \cdots, n-1, n+1) . \tag{3.37}$$

We assume that $n > k + 4$, as $n = k + 4 = 6$ is already covered by (3.27).

1. $\dim \mathfrak{n} = 8$. There is only one solution, corresponding to $(\alpha_1, \alpha_2) = (1, -2)$. A short computation shows that $H^2(\mathfrak{r}_8^2, \mathfrak{r}_8^2) = 0$.

2. $\dim \mathfrak{n} = 10$.The Jacobi conditions (3.35) reduce to the equations

$$2\alpha_1 - \alpha_2 - \alpha_3 = 0,\ 2\alpha_2^2 - 2\alpha_1\alpha_3 - \alpha_2\alpha_3 = 0.$$

   There are only two solutions corresponding to the triples $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$ and $(1, -1, 3)$. Both Lie algebras are algebraically rigid.

3. $\dim \mathfrak{n} = 12$. In this case, the Jacobi conditions lead to the system

$$2\alpha_1 - 3\alpha_2 + \alpha_4 = 0,\ 2\alpha_2^2 - 2\alpha_1\alpha_3 - \alpha_2\alpha_3 = 0,\ 4\alpha_2\alpha_3 - 6\alpha_3^2 - (2\alpha_1 - \alpha_2 - \alpha_3)\,\alpha_4 = 0.$$

   It admits four independent solutions, two of which are complex conjugate and two real non-rational solutions given by

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, \frac{2}{3}\left(1 \pm \mathrm{i}\sqrt{2}\right), \pm\mathrm{i}\sqrt{2}, \pm 2\mathrm{i}\sqrt{2}\right),$$
$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, \pm\sqrt{\frac{2}{5}}, \frac{1}{15}\left(10 \mp \sqrt{10}\right), \frac{1}{5}\left(-10 \pm 3\sqrt{10}\right)\right).$$

   All these Lie algebras can be verified to have a vanishing second cohomology space, hence they are algebraically rigid.

4. For $n = 2q \geq 14$, the Jacobi conditions have only the trivial solution $\alpha_i = 0$, implying that there are no rank one solvable Lie algebras with eigenvalues (3.37).

We remark that the existence of complex nonreal rigid Lie algebras was announced in [25], where the first examples were are given.

3.4.2 $k = 3$

If $n = 8$, the Lie algebra is rigid by (3.27).

1. For $\dim \mathfrak{n} = 10$ the Jacobi conditions are trivially satisfied for any values of the variables α_1 and α_2, hence the Lie algebra is not rigid.

2. $\dim \mathfrak{n} = 12$. The Jacobi conditions reduce to the equations:

$$2\alpha_2 + \alpha_3 = 0,\ 4\alpha_2^2 - 3\alpha_1\alpha_3 - 3\alpha_2\alpha_3 = 0,$$

   admitting the two independent solutions $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$ and $(1, -\frac{3}{5}, \frac{6}{5})$. Both resulting solvable Lie algebras are cohomologically rigid.

3. $\dim \mathfrak{n} = 14$. The quadratic equations are obtained:

$$2\alpha_2 - \alpha_3 - \alpha_4 = 0,\ 4\alpha_2^2 - 3\alpha_1\alpha_3 - 3\alpha_2\alpha_3 = 0,\ 10\alpha_3^2 + 3\alpha_1\alpha_4 - 5\alpha_2\alpha_3 - 2\alpha_2\alpha_4 - 4\alpha_3\alpha_4 = 0.$$

   There are four solutions, corresponding to the values

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1,0,0,0), (1,3,3,3), \left(1, \frac{3}{4}, \frac{3}{7}, \frac{15}{14}\right) \left(1, -\frac{3}{7}, \frac{3}{7}, -\frac{9}{7}\right).$$

   All the resulting solvable Lie algebras are verified to have a second vanishing cohomology space, from which their cohomological rigidity follows.

4. For $\dim \mathfrak{n} = 16$ the Jacobi conditions lead to the five independent quadratic equations

$$\begin{aligned}
&2\alpha_2 - 3\alpha_3 + \alpha_5 = 0,\\
&4\alpha_2^2 - 3\alpha_1\alpha_3 - 3\alpha_2\alpha_3 = 0,\\
&10\alpha_3^2 + 3\alpha_1\alpha_4 - 5\alpha_2\alpha_3 - 2\alpha_2\alpha_4 - 4\alpha_3\alpha_4 = 0,\\
&5\alpha_3^2 - 4\alpha_2\alpha_4 - 6\alpha_3\alpha_4 + 2\alpha_2\alpha_5 + \alpha_3\alpha_5 = 0,\\
&20\alpha_4^2 - 3\alpha_1\alpha_5 + 5\alpha_2\alpha_5 - 6\alpha_2\alpha_4 - 21\alpha_3\alpha_4 - 5\alpha_4\alpha_5 = 0.
\end{aligned}$$

   This system admits only two nontrivial solutions

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1,0,0,0,0), \left(1, \frac{3}{5}, \frac{3}{10}, 0, -\frac{3}{10}\right).$$

   Again, the computation shows that the second cohomology space vanishes in both cases.

5. For $\dim \mathfrak{n} = 18 + 2\ell$ with $\ell \geq 0$ the only nontrivial solution to the Jacobi conditions is given by $(\alpha_1, \cdots, \alpha_{6+\ell}) = \left(1, 0^{5+\ell}\right)$. This series is actually the analogue of the so-called Bratzlavsky series (see e.g. [11, 15, 24]) for nilradicals of type B_n^k.

Proposition 2 *For any $n = 18 + 2\ell$ with $\ell \geq 0$, the solvable Lie algebra $\mathfrak{r}_n^3$ with nilradical corresponding to the solution $(\alpha_1, \cdots, \alpha_{6+\ell}) = \left(1, 0^{5+\ell}\right)$ satisfies $\dim H^2\left(\mathfrak{r}_n^3, \mathfrak{r}_n^3\right) = 0$.*

One point that deserves a comment is the fact that all the rigid rank one solvable Lie algebras obtained for $k = 3$ are rational, in contrast to the case of $k \neq 3$, where there always exist nonrational and even complex solutions.

3.4.3 $k = 4$

Again, the case $n = 8$ belongs to the family (3.27) and leads to a rigid Lie algebra.

1. If $n = 10$, there is only one nontrivial solution to the Jacobi conditions, given by $(\alpha_1, \alpha_2) = (1, -2)$. The corresponding solvable Lie algebra has vanishing cohomology.

2. For $\dim \mathfrak{n} = 12$ the Jacobi conditions are trivially satisfied for any values of the variables α_1 and α_2, hence the Lie algebra is not rigid.

3. For $\dim \mathfrak{n} = 14$ we obtain parameterized families, and no solutions with vanishing cohomology exist.

4. For $\dim \mathfrak{n} = 16$ we obtain four nontrivial solutions to the Jacobi conditions, two real and two complex ones, respectively:

$$
\begin{array}{ll}
(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = & \\
\left(1, \frac{2}{105}\left(11 \mp \sqrt{226} + \sqrt{-1963+188\sqrt{226}}\right), \right. & \frac{1}{63}\left(10+\sqrt{226} \mp \sqrt{577+62\sqrt{226}}\right), \\
\frac{2}{63}\left(-10-\sqrt{226} \pm \sqrt{577+62\sqrt{226}}\right) & \left., \frac{1}{9}\left(10+\sqrt{226} \mp \sqrt{-1057+122\sqrt{226}}\right)\right) \\
(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = & \\
\left(1, \frac{2}{105}\left(11 + \sqrt{226} \pm i\sqrt{1963+188\sqrt{226}}\right)\right. & , \frac{1}{63}\left(10-\sqrt{226} \pm i\sqrt{577+62\sqrt{226}}\right), \\
\frac{2}{63}\left(-10+\sqrt{226} \mp i\sqrt{577+62\sqrt{226}}\right), & \left.\frac{1}{9}\left(10-\sqrt{226} \mp \sqrt{1057+122\sqrt{226}}\right)\right)
\end{array}
$$

The corresponding Lie algebras (complex and real) all satisfy the Richardson criterion and are cohomologically rigid.

5. For $\dim \mathfrak{n} = 18$ the Jacobi conditions admit two nontrivial solutions

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \left(1.\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},0\right),\ \left(1,-1,-\frac{5}{3},-\frac{5}{3},-\frac{5}{3},-\frac{7}{3}\right).$$

Also in this case, the corresponding solvable Lie algebras satisfy $H^2\left(\mathfrak{r}_{18}^4, \mathfrak{r}_{18}^4\right) = 0$.

6. Finally, for any $n \geq 20$ the only solution to the system of Jacobi equations is given by $\alpha_i = 0$, corresponding to the rank two model algebra Q_n. Hence no rank one solvable algebras with the given eigenvalues exist.

3.4.4 $k = 5$

As the last case to be explicitly analyzed for all dimensions we consider $k = 5$. By (3.27) we already know that the nilradical in $n = 10$ leads to a rigid Lie algebra. We will see that for this value, a same type of solution leads to solvable Lie algebras that are either cohomologically or geometrically rigid, depending on the dimension.

1. For $\dim \mathfrak{n} = 12, 14$ and 16, we obtain parameterized families of nilpotent algebras, that can be deformed into each other. No cohomologically rigid Lie algebras exist.

2. If $\dim \mathfrak{n} = 18$, the equations (3.35) admits a two-parameter family of solutions, as well as two isolated solutions

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \left(0,0,1,\frac{9}{5},3\right),\ \left(0,0,0,1,0,\frac{5}{24}\right).$$

The solvable Lie algebras corresponding to these values are not cohomologically rigid, as they both satisfy $\dim H^2\left(\mathfrak{r}_{18}^5, \mathfrak{r}_{18}^5\right) = 1$. Moreover, they can be deformed into the family, so that they are not geometrically rigid.

3. The Jacobi conditions for $\dim \mathfrak{n} = 20$ admit the two nontrivial solutions

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) = (1,0,0,0,0),\ \left(1,\frac{5}{8},\frac{5}{24},0,0,\frac{5}{24}\right).$$

The corresponding solvable Lie algebras $\mathfrak{r}_{20}^5$ satisfy in both cases $H^2\left(\mathfrak{r}_{18}^4, \mathfrak{r}_{18}^4\right) = 0$.

4. For $\dim \mathfrak{n} = 22$ we find two nontrivial solutions to the equations (3.35)

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) =$$
$$\left(1, \tfrac{5}{14}, \tfrac{5}{42}\left(5 \pm \sqrt{22}\right), \tfrac{5}{126}\left(17 \mp 4\sqrt{22}\right), \tfrac{5}{126}\left(17 \mp 4\sqrt{22}\right), \tfrac{5}{126}\left(19 \mp 5\sqrt{22}\right)\right)$$
$$(1, 0, 0, 0, 0, 0)$$

In both cases the solutions lead to cohomologically rigid Lie algebras.

5. For $\dim \mathfrak{n} = 24 + 4l \ (l \geq 0)$, the only nontrivial solution is given by

$$(\alpha_1, \dots, \alpha_{7+2l}) = (1, 0, \dots, 0).$$

If we compute the cohomology of the associated solvable Lie algebra $\mathfrak{r}^5_{24+4l}$, we find that it does not vanish, but satisfies $\dim H^2\left(\mathfrak{r}^5_{24+4l}, \mathfrak{r}^5_{24+4l}\right) = 1$. However, this Lie algebra is always geometrically rigid, as can be shown with a topological argument (see for instance the argument in [15], that is also valid for this case).

6. For $\dim \mathfrak{n} = 26 + 4l \ (l \geq 0)$, the only nontrivial solution is again

$$(\alpha_1, \dots, \alpha_{8+2l}) = (1, 0, \dots, 0),$$

but, in contrast to the previous case, we have that $\dim H^2\left(\mathfrak{r}^5_{26+4l}, \mathfrak{r}^5_{26+4l}\right) = 0$, showing that the Lie algebras $\mathfrak{r}^5_{26+4l}$ are cohomologically rigid.

As a consequence of this analysis we conclude that, depending on the parity of k, for high dimensions $n \geq 4k$ we either obtain only the trivial solution for the equations determined by the Jacobi conditions (hence the inexistence of rank one Lie algebras with the prescribed eigenvalue spectrum) or the solution $(1, 0, \dots, 0)$, the cohomological behaviour of which depends on the value of n. As a general fact, we can establish the following

Proposition 3 *Let $\mathfrak{n}^k_n$ be a nilpotent Lie algebra of type (3.31) satisfying the quadratic equations (3.35). Then following relations hold:*

1. *If $k \geq 6$ is even and $n \geq 4k$, then $\mathfrak{n}^k_n$ is isomorphic to Q_n.*
2. *If $k \geq 7$ is odd and $n \geq 4k$, then $\mathfrak{n}^k_n$ is either isomorphic to Q_n or to the Lie algebra with brackets*

$$\begin{array}{ll} [X_1, X_j] = X_{j+1}, & 2 \leq j \leq n-2, \\ [X_2, X_j] = X_{j+k}, & 3 \leq j \leq n-1-k, \\ [X_i, X_{n+1-i}] = (-1)^i X_n, & 2 \leq i \leq \frac{n}{2}. \end{array} \tag{3.38}$$

Clearly the Jacobi conditions always admit the zero solution $(\alpha_1, \cdots, \alpha_\ell) = (0, \cdots, 0)$ with $\ell = \left[\frac{n-k}{2}\right]$ corresponding to a nilpotent Lie algebra isomorphic to Q_n. For any fixed dimension n the number of Jacobi conditions is given by $\frac{1}{6}n(n-1)(n-2)$, while the number of parameters equals ℓ. Now, while for the change of dimension $n \to n+2$ the number of parameters α_i increases by one unity, the number of new Jacobi conditions that appear is given by n^2, implying that the number of independent equations increases quadratically. Therefore, from a certain dimension onwards, the number of independent Jacobi conditions exceeds that of parameters, implying that only the trivial solution is possible. The only exception to this rule is given by the case of odd k and $(\alpha_1, \cdots, \alpha_\ell) = (1, 0, \cdots, 0)$, where the coefficients $C^{k+3}_{2,3} = C^{q+k}_{2,q}$ for $4 \leq q \leq n-1-k-q$ always constitute a solution to the Jacobi conditions. The corresponding nilradical is the analogue, contracting onto Q_n, of the nilradical $\mathfrak{f}_n$ of the so-called Bratzlavsky series contracting onto L_n [8, 11].

Technically, the systematized analysis of nilpotent Lie algebras admitting a one-dimensional maximal torus of derivations with eigenvalues $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3)$ can be continued beyond $k = 5$. However, even with the help of computer packages, we encounter dimensions where the solutions can only be obtained numerically, as the Jacobi conditions cannot be solved in closed form. The lowest value and dimension for which this occurs is given by $k = 6$ and $d = 22$ respectively. Supposed that $\mathfrak{n}_{22}^6$ is a Lie algebra of type (3.31) and that $C_{2,3}^9 \neq 0$ is satisfied, the Jacobi conditions, besides two rational solutions (see Table 1) are the solutions of the following system of linear and quadratic equations equations:[3]

$$
\begin{array}{l}
2\alpha_3 - 3\alpha_4 + \alpha_6 = 0,\ 2 - 9\alpha_2 + 11\alpha_3 - 4\alpha_4 + \alpha_6 - \alpha_7 = 0\\
7\alpha_2^2 - \alpha_3(6 + 15\alpha_2) + \alpha_4(9 + 10\alpha_2) - \alpha_5(2 + \alpha_2) = 0,\\
8\alpha_3^2 - \alpha_4(7\alpha_2 + 21\alpha_3) + \alpha_5(14\alpha_2 + 20\alpha_3) - \alpha_6(7\alpha_2 + 5\alpha_3) = 0,\\
588\alpha_4^2 + \alpha_2(48\alpha_3 - 54\alpha_4 + 335\alpha_5 - 246\alpha_6 + 21\alpha_7) + 12\alpha_5 + 51\alpha_6\\
-14\alpha_7 - \alpha_3(546\alpha_4 - 330\alpha_5 + 99\alpha_6) - \alpha_4(36 + 490\alpha_5 - 147\alpha_6 + 7\alpha_7) = 0\\
\alpha_2(16\alpha_3 - 39\alpha_4 + 86\alpha_5 - 47\alpha_6) + \alpha_3(-77\alpha_3 + 110\alpha_5 - 33\alpha_6 - 77) - 12\alpha_4 + 18\alpha_5 - 4\alpha_6 = 0.
\end{array}
$$

The solutions to these equations cannot be found in closed form, and require a numerical analysis. There are twelve solutions, divided into six real and six complex solutions. The real solutions (with a six digits approximation) are given by

$$
\begin{array}{rcl}
(\alpha_2, \cdots, \alpha_7)_1 &=& (-0{,}672116, -0{,}689923, -2{,}3539, -3{,}77899, -5{,}68185, 4{,}19364)\,,\\
(\alpha_2, \cdots, \alpha_7)_2 &=& (-0{,}53422, 0{,}401517, -0{,}507803, 0{,}647213, -2{,}32645, 10{,}9294)\,,\\
(\alpha_2, \cdots, \alpha_7)_3 &=& (-0{,}268464, -0{,}425877, -0{,}212938, 0, 0{,}212938, 0{,}796227)\,,\\
(\alpha_2, \cdots, \alpha_7)_4 &=& (0{,}763277, 0{,}27263, -0{,}00432786, -0{,}271782, -0{,}558244, -2{,}41149)\,,\\
(\alpha_2, \cdots, \alpha_7)_5 &=& (0{,}485855, 0{,}259869, 0{,}129934, 0, -0{,}129934, -0{,}163816)\,,\\
(\alpha_2, \cdots, \alpha_7)_6 &=& (0{,}13456, -0{,}142263, -0{,}143553, -0{,}101978, -0{,}146134, -0{,}347857)\,.
\end{array}
$$

In these conditions, computing the cohomology with these values can lead to error due to the approximation, due to the approximate values, hence an indirect approach is required. A quite laborious one consists in solving first the linear equations of the system and looking for a reduced set of polynomial equations equivalent to the system. Then, computing formally the cohomology with the diagonal entries $(1, \alpha_2, \cdots, \alpha_7)$ and introducing succesively the constraints on the α_i, a cumbersome computation allows us to establish that for the previous solutions the cohomology space $H^2(\mathfrak{r}_{22}^6, \mathfrak{r}_{22}^6)$ actually vanishes, so that these six nonrational solutions indeed provide cohomologically rigid Lie algebras.[4] As can be expected, the computing time for solving such cases for higher dimensions and values of k, even numerically, increases exponentially. This makes the separation of the solutions a difficult task, added to the fact that for certain values of k and n, several dozens of solutions may exist. Therefore, in the following we focus only on the isolated solutions to the Jacobi conditions (3.35), specially emphasizing those solutions $(\alpha_1, \cdots, \alpha_s)$ that are rational.

3.5 Cohomologically rigid algebras in dimension $n \leq 30$

In this section, we present the nilradicals of type (3.31) that lead to real cohomologically rigid Lie algebras for values $k \geq 6$ and dimensions $n \leq 30$. For the reasons mentioned above, we restrict ourselves to the real solutions of the equations (3.35) that can be described in closed form, i.e., that do not require numerical analysis for their description. The nilradicals are given

[3] As $\alpha_1 = C_{2,3}^9 \neq 0$, we can suppose that it equals 1.

[4] Actually, the direct computation with a nine digits approximation of the values α_i give a vanishing cohomology.

in terms of their fundamental parameters $(\alpha_1, \dots, \alpha_s)$ (see equation (3.36)) in tabular form, specifying the dimension, the value of k and the number of fundamental parameters.

Table 3.1: Cohomological rigid Lie algebras in dimensions $n \leq 30$ for $k \geq 6$.

$\dim \mathfrak{n}$	k	s	$(\alpha_1, \dots, \alpha_s)$
22	6	7	$\left(0,0,0,1,\frac{74}{35},3,-1\right)$
22	6	7	$\left(0,1,\frac{3}{2},\frac{9}{5},\frac{5}{2},\frac{15}{2}\right)$
22	6	7	$\left(1,\frac{(5\pm\alpha)}{46},\frac{(-21\pm5\alpha)}{253},\frac{(-21\pm5\alpha)}{506},0,\frac{(21\mp5\alpha)}{506},\frac{(80\mp7\alpha)}{253}\right),\ \alpha=\sqrt{301}$
24	7	7	$(0,0,0,0,1,-1,0)$
24	7	7	$\left(0,0,0,1,\frac{10}{7},\frac{4}{7},-4\right)$
26	7	8	$\left(0,0,0,1,2,\frac{20}{7},4,7\right)$
26	7	8	$\left(0,0,0,0,1,\frac{295}{126},3,-1\right)$
26	7	8	$\left(0,0,1,\frac{27}{22},\frac{12}{11},\frac{10}{11},\frac{9}{11},\frac{21}{22}\right)$
26	7	8	$\left(\frac{353\pm12\alpha}{161},1,\frac{16\mp3\alpha}{11},\frac{138\mp30\alpha}{121},\frac{69\pm15\alpha}{121},0,\frac{-69\pm15\alpha}{121},\frac{-169\pm42\alpha}{121}\right),\ \alpha=\sqrt{26}$
26	8	8	$\left(1,\frac{25\pm\alpha}{120},\frac{15\pm3\alpha}{520},\frac{5\pm\alpha}{120},0,0,\frac{5\pm\alpha}{120},\frac{-95\pm7\alpha}{780}\right),\ \alpha=\sqrt{1585}$
28	8	9	$\left(0,0,\frac{61}{44},\frac{31}{22},\frac{15}{11},\frac{16}{11},\frac{91}{44},7\right)$
28	8	9	$(0,0,0,0,0,1,-1,0,0)$
28	8	9	$\left(0,0,0,\frac{42}{25},\frac{64}{35},\frac{268}{175},\frac{7}{25},-\frac{308}{25}\right)$
28	8	9	$\left(0,0,0,0,1,\frac{265}{168},\frac{71}{168},-4,1\right)$
30	8	10	$\left(0,0,0,0,\frac{17}{7},\frac{1135}{294},\frac{37}{7},\frac{46}{7},-6\right)$
30	8	10	$\left(0,0,0,\frac{11}{20},1,\frac{5}{4},\frac{10}{7},\frac{7}{4},\frac{14}{5},\frac{21}{2}\right)$
30	8	10	$\left(0,0,0,0,0,1,\frac{387}{154},3,-1,0\right)$
30	9	9	$\left(0,0,1,\frac{23}{26},\frac{5}{11},\frac{25}{286},-\frac{25}{143},-\frac{119}{286},-\frac{126}{143}\right)$
30	9	9	$\left(0,0,0,0,1,\frac{20}{27},-\frac{40}{27},-\frac{115}{27},5\right)$
30	9	9	$(0,0,0,0,0,1,-2,1,0)$
30	9	9	$\left(0,0,0,1,\frac{64}{55},\frac{5}{11},-\frac{10}{11},-\frac{37}{11},-\frac{112}{11}\right)$
30	9	9	$\left(1,\frac{3(4720\pm27\alpha}{28859},\frac{3(196169\pm5665\alpha}{750334},\frac{3(527147\pm16664\alpha}{9754342},\frac{3(527147\pm16664\alpha}{9754342},\right.$ $\left.0,0,\frac{3(527147\pm16664\alpha}{9754342},\frac{3(2194126\pm76151\alpha}{9754342}\right),\ \alpha=\sqrt{1147}$

Just as observed for the value $k = 5$, for odd values of k and dimensions $n \geq 30$ there begin to appear many solutions having nonvanishing cohomology, that must be analyzed separately (unless they are the only nontrivial solution). The nonrational solutions also have a quite complicated form, so that in order to find some pattern that lead to series of cohomologically rigid Lie algebras, it is in practive convenient to separate the rational solutions. Table 2 contains the rational solutions in dimensions $32 \leq n \leq 40$.

Table 3.2: Rational cohomological rigid Lie algebras in dimensions $30 < d \leq 40$.

$\dim \mathfrak{n}$	k	s	$(\alpha_1, \ldots, \alpha_s)$
32	9	10	$\left(0, 0, 0, 0, 0, 1, \frac{56}{33}, \frac{10}{33}, -4, 1\right)$
32	9	10	$(0, 0, 0, 0, 0, 0, 1, -1, 0, 0)$
32	9	10	$\left(0, 0, 0, 0, 1, 2, \frac{7}{3}, \frac{5}{3}, -1, 12\right)$
32	9	10	$\left(0, 0, 0, 1, \frac{22}{13}, \frac{25}{13}, \frac{25}{13}, \frac{25}{13}, \frac{28}{13}, \frac{42}{13}\right)$
32	10	10	$\left(0, 0, 1, 0, 0, \frac{40}{37}, \frac{1359}{518}, \frac{120}{37}, \frac{34}{37}, -36\right)$
34	9	11	$\left(0, 0, 0, 0, 0, 1, \frac{14}{5}, \frac{791}{165}, \frac{32}{5}, \frac{31}{5}, -6\right)$
34	9	11	$\left(0, 0, 0, 0, 1, \frac{30}{13}, \frac{45}{13}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}\right)$
34	9	11	$\left(0, 0, 0, 1, \frac{8}{5}, \frac{22}{13}, \frac{20}{13}, \frac{35}{26}, \frac{16}{13}, \frac{84}{65}, \frac{24}{13}\right)$
34	9	11	$\left(0, 0, 0, 0, 0, 0, 1, \frac{413}{156}, 3, -1, 0\right)$
34	10	11	$\left(0, 0, 0, 0, 1, \frac{10}{7}, \frac{41}{56}, -\frac{82}{63}, -\frac{283}{63}, -\frac{62}{7}, \frac{130}{7}\right)$
34	10	11	$(0, 0, 0, 0, 0, 0, 1, -2, 1, 0, 0)$
34	10	11	$\left(0, 0, 0, 0, 0, 1, \frac{137}{65}, -\frac{274}{65}, -\frac{688}{165}, 5, -1\right)$
34	10	11	$\left(0, 0, 0, 1, \frac{118}{91}, \frac{85}{91}, \frac{25}{91}, -\frac{50}{91}, -\frac{164}{91}, -\frac{66}{13}, -30\right)$
34	11	10	$\left(1, 0, 0, 0, \frac{737}{988}, \frac{737}{546}, \frac{31691}{20748}, \frac{8107}{6916}, -\frac{3685}{20748}, -\frac{47168}{5187}\right)$
36	10	12	$(0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0)$
36	10	12	$\left(0, 0, 0, 0, 1, \frac{188}{91}, \frac{243}{91}, \frac{109}{39}, \frac{695}{273}, \frac{162}{91}, -\frac{114}{91}, -\frac{462}{13}\right)$
36	10	12	$\left(0, 0, 0, 0, 0, 0, 1, \frac{70}{39}, \frac{8}{39}, -4, 1, 0\right)$
36	10	12	$\left(0, 0, 0, 0, 0, 1, \frac{151}{66}, \frac{3073}{1089}, \frac{1910}{1089}, -\frac{71}{33}, -\frac{773}{66}, 7\right)$
36	10	12	$\left(0, 0, 0, 1, \frac{7}{4}, \frac{52}{26}, \frac{107}{52}, \frac{105}{52}, \frac{109}{52}, \frac{33}{13}, \frac{111}{26}, \frac{231}{13}\right)$
36	12	11	$\left(0, 0, 1, 0, 0, \frac{343}{933}, \frac{266}{933}, -\frac{532}{933}, -\frac{483}{311}, -\frac{151}{933}, \frac{45374}{933}\right)$
38	10	13	$\left(0, 0, 0, 0, 1, \frac{13}{6}, 3, \frac{7}{2}, \frac{35}{9}, \frac{9}{2}, 6, 11, \frac{99}{2}\right)$
38	10	13	$\left(0, 0, 0, 0, 0, 1, \frac{135}{49}, \frac{33}{7}, \frac{1532}{231}, \frac{423}{9}, \frac{543}{49}, \frac{660}{49}, -\frac{1188}{49}\right)$
38	10	13	$\left(0, 0, 0, 0, 0, 0, 0, 1, \frac{124}{45}, 3, -1, 0, 0\right)$
38	10	13	$\left(0, 0, 0, 0, 0, 0, 1, \frac{378}{121}, \frac{8890}{1573}, \frac{892}{121}, \frac{711}{121}, -6, 1\right)$
38	11	12	$\left(0, 0, 0, 0, 1, \frac{23}{14}, \frac{10}{7}, \frac{1}{2}, -1, -\frac{47}{14}, -\frac{57}{7}, -\frac{165}{7}\right)$
38	11	12	$\left(0, 0, 0, 0, 0, 0, 1, \frac{259}{286}, -\frac{259}{143}, -\frac{1171}{286}, 5, -1\right)$
38	11	12	$(0, 0, 0, 0, 0, 0, 0, 1, -2, 1, 0, 0)$
38	11	12	$\left(0, 0, 0, 0, 0, 1, \frac{5}{3}, \frac{28}{33}, -\frac{56}{33}, -\frac{181}{33}, -\frac{23}{3}, \frac{55}{3}\right)$
38	11	12	$\left(0, 0, 0, 1, \frac{43}{34}, \frac{65}{68}, \frac{1}{2}, \frac{7}{68}, -\frac{7}{34}, -\frac{33}{68}, -\frac{15}{17}, -\frac{33}{17}\right)$
40	11	13	$(0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0)$
40	11	13	$\left(0, 0, 0, 0, 1, \frac{35}{17}, \frac{91}{34}, \frac{49}{17}, \frac{49}{17}, \frac{49}{17}, \frac{105}{34}, \frac{66}{17}, \frac{231}{34}\right)$
40	11	13	$\left(0, 0, 0, 0, 0, 0, 0, 1, \frac{516}{275}, \frac{34}{275}, -4, 1, 0\right)$
40	11	13	$\left(0, 0, 0, 0, 0, 0, 1, \frac{28}{11}, \frac{5166}{1573}, \frac{2842}{1573}, -\frac{35}{11}, -\frac{126}{11}, 7\right)$
40	11	13	$\left(0, 0, 0, 0, 0, 1, \frac{17}{7}, \frac{7}{2}, \frac{42}{11}, \frac{35}{11}, 1, -\frac{11}{2}, -33\right)$

3.6 Some series of rigid Lie algebras with $H^2(\mathfrak{r}_n^k, \mathfrak{r}_n^k) = 0$

As can be inferred from the tables, there are certain patterns in the (rational) solutions that suggest that the corresponding Lie algebras can be described for arbitrary values of k (for even or odd values respectively) while preserving the condition that the second cohomology space vanishes. The simplest of these series appears for even k, and has quite a low dimension, namely $k+2$, corresponding to the solution $(\alpha_1, \alpha_2) = (1, -2)$ of the Jacobi equations (3.35).

Proposition 4 *Let $k \geq 2$ be an even integer and $\mathfrak{n}_{k+6}^k$ be the nilpotent Lie algebra with brackets*

$$\begin{array}{ll} [X_1, X_j] = X_{j+1}, & 2 \leq j \leq n-2 \\ [X_i, X_{n+1-i}] = (-1)^i X_n, & 2 \leq i \leq \frac{n}{2} \\ [X_2, X_3] = X_{j+k}, & j = 3, 4 \\ [X_2, X_5] = 3X_{5+k}, & [X_3, X_4] = -2X_{4+k}. \end{array} \tag{3.39}$$

Then the solvable Lie algebra $\mathfrak{r}_{k+6}^k = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_{k+6}^k$ satisfies $H^2\left(\mathfrak{r}_{k+6}^k, \mathfrak{r}_{k+6}^k\right) = 0$.

Besides this family, exclusive for even values k, there are other series that have very similar structure for both parities of k. The following result enumerates those series that begin for values of k not exceeding twelve.

Proposition 5 *The following Lie algebras $\mathfrak{n}_n^k$ of type (3.31) satisfying the quadratic equations (3.35) lead to rank one cohomologically rigid Lie algebras $\mathfrak{r}_n^k$ with eigenvalue spectrum (3.28):*

1. *For odd $k \geq 7$ and dimension $n = 4k-4$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-7}{2}}\right) = \left(0^{k-3}, 1, -1, 0^{\frac{k-5}{2}}\right)$$

2. *For odd $k = 2q+1 \geq 7$ and dimension $n = 4k-2$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-5}{2}}\right) = \left(0^{k-3}, 1, \varphi(q), 3, -1, 0^{q-3}\right)$$

where

$$\varphi(q) = \frac{3\,(2q-1)\,(10q^2 - 11q + 2)}{2q(2q+1)(4q-3)}, \quad q \geq 3.$$

3. *For odd $k = 2q+1 \geq 9$ and dimension $n = 4k-4$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-7}{2}}\right) = \left(0^{k-4}, 1, \varphi_1(q), \varphi_2(q), -4, 1, 0^{q-4}\right)$$

where

$$\varphi_1(q) = \frac{(q-1)\,(2q-1)(11q-12)}{q(2q+1)(4q-5)}, \; \varphi_2(q) = \frac{3\,(4 - 19q + 15q^2 - 2q^3)}{q(2q+1)(4q-5)}, \quad q \geq 3.$$

4. *For odd $k = 2q \geq 11$ and dimension $n = 4k-6$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-9}{2}}\right) = \left(0^{k-5}, 1, \phi_1(q), \phi_2(q), \phi_3(q), 5, -1, 0^{q-5}\right)$$

where

$$\phi_1(q) = \frac{(2q-3)\,(20 - 37q + 14q^2)}{2q(2q+1)(4q-7)}, \quad \phi_2(q) = \frac{(3-2q)(40-74q+28q^2)}{2q(2q+1)(4q-7)},$$
$$\phi_3(q) = \frac{(60 - 221q + 16q^2 + 52q^3)}{2q(1+2q)(7-4q)}, \qquad q \geq 5.$$

5. *For even $k \geq 8$ and dimension $n = 4k - 4$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-6}{2}}\right) = \left(0^{k-3}, 1, -1, 0^{\frac{k-4}{2}}\right)$$

6. *For even $k = 2q \geq 6$ and dimension $n = 4k - 2$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-4}{2}}\right) = \left(0^{k-3}, 1, \psi(q), 3, -1, 0^{q-3}\right)$$

where

$$\psi(q) = \frac{3\,(q-1)\,(10q^2 - 21q + 10)}{q(2q-1)(4q-3)}, \quad q \geq 3.$$

7. *For even $k = 2q \geq 8$ and dimension $n = 4k - 4$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-6}{2}}\right) = \left(0^{k-4}, 1, \psi_1(q), \psi_2(q), -4, 1, 0^{q-4}\right)$$

where

$$\psi_1(q) = \frac{(q-1)\,(2q-3)(22q-35)}{2q(2q-1)(4q-7)}, \; \psi_2(q) = \frac{3\,(35 - 71q + 36q^2 - 4q^3)}{2q(2q-11)(4q-7)}, \quad q \geq 4.$$

8. *For even $k = 2q \geq 10$ and dimension $n = 4k - 6$:*

$$\left(\alpha_1, \cdots, \alpha_{\frac{3k-8}{2}}\right) = \left(0^{k-5}, 1, \psi_1(q), \psi_2(q), \psi_3(q), 5, -1, 0^{q-5}\right)$$

where

$$\begin{aligned} \psi_1(q) &= \frac{(q-2)\,(42 - 51q + 14q^2)}{q(2q-1)(4q-9)}, \quad \psi_2(q) = \tfrac{(2-q)(84-102q+28q^2)}{q(2q-1)(4q-9)}, \\ \psi_3(q) &= \frac{(84 - 99q - 31q^2 + 26q^3)}{q(1-2q)(4q-9)}, \qquad\qquad q \geq 5. \end{aligned}$$

The proof of the rigidity of these Lie algebras, as well as those of Proposition 5, is essentially the same, based on the Hochschild-Serre factorization theorem. In the following we give give a justification valid for all these cases. The first step consists in showing that for all these solvable algebras the derivations are all inner. To this extent, any 1-cochain $f \in C^1(\mathfrak{n}_k^n, \mathfrak{r}_k^n)$ can be written as

$$f(X_i) = \sum_{l=1}^{n} a_i^l X_l + b_i T, \; 1 \leq i \leq n. \tag{3.40}$$

The invariance condition for cochains, as a consequence of the diagonal action of the torus generator T on the nilradical, implies the following constraints

$$f(X_i) = a_i^i X_i, \; 1 \leq i \leq n. \tag{3.41}$$

If we now compute the coboundary operator for these cochains, we find that the only nonzero terms are the following:

$$\begin{aligned} df(X_1, X_j) = &\left(a_1^1 + a_j^j - a_{1+j}^{1+j}\right) X_{1+j}, \; 2 \leq j \leq n-2, \\ df(X_i, X_j) = &\left(a_i^i + a_j^j - a_{i+j+k-2}^{i+j+k-2}\right) C_{i,j}^{i+j+k-2} X_{i+j+k-2}, \; 2 \leq i \leq k, \; i < j \leq n-i-1, \\ df(X_i, X_{n-i+1}) = &\left(a_i^i + a_{n-i+1}^{n-i+1} - a_n^n\right) X_n, \; 2 \leq i \leq n-1. \end{aligned} \tag{3.42}$$

Solving the resulting coefficient system allows us to find a basis of 1-cocycles $Z^1\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}}$, that can be chosen as

$$f_1(X_1) = X_1,\ f_1(X_i) = (i+k-2)X_i,\ 2 \leq i \leq n-1,\ f_1(X_n) = (n+2k-3)X_n. \quad (3.43)$$

As the identity $[T, X_\ell] - f_1(X_\ell) = 0$ holds for $1 \leq \ell \leq n$, we can identify the 1-cocycle with the adjoint operator $\mathrm{ad}(T)$ of the torus generator, from which we conclude that $Z^1\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}} = B^1\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}}$, and further that $H^1\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}} = H^1\left(\mathfrak{r}_k^n, \mathfrak{r}_k^n\right) = 0$. The remaining cochains $f_i(X_j) = a_i^j X_j$ with $2 \leq i \leq n$ generate the space $B^2\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}}$ and are actually linearly independent, thus $\dim B^2\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}} = n-1$.
Consider now a generic 2-cochain, that we can write as

$$\varphi(X_i, X_j) = \sum_{k=1}^{n} \theta_{i,j}^k X_k + \eta_{i,j} T \quad (3.44)$$

If we evaluate the condition (6.13), we are led to the following constraints:

$$\begin{aligned} \varphi(X_1, X_j) &= \theta_{1,j}^{j+1} X_{j+1},\ 2 \leq j \leq n-2 \\ \varphi(X_i, X_j) &= \theta_{i,j}^{i+j+k-2} X_{i+j+k-2},\ 2 \leq i \leq j,\ i+j \leq n-k+1 \\ \varphi(X_i, X_{n-i+1}) &= \theta_{i,n-i+1}^{n} X_n,\ 2 \leq i \leq n-1. \end{aligned} \quad (3.45)$$

Using the identities (3.32) and (3.33), the analysis of 2-cocycles reduces to evaluate the condition $d\varphi(X_a, X_b, X_c) = 0$ for the triples $\{X_1, X_i, X_j\}$ and $\{X_2, X_i, X_j\}$. For the first triple we get

$$\begin{aligned} & d\varphi(X_1, X_i, X_j) = \\ & \left(\theta_{i,j}^{i+j+k-2} - \theta_{i,j+1}^{i+j+k-1} - \theta_{i+1,j}^{i+j+k-1}\right) X_{i+j+k-1},\ 2 \leq i+1 < j, i+j \leq 2k-3, \\ & d\varphi(X_1, X_i, X_{n-i}) = \\ & \left((-1)^i \theta_{1,i}^{i+1} + (-1)^{i+1} \theta_{1,n-i}^{n-i+1} - \theta_{i,n-i+1}^{n} - \theta_{i+1,n-i}^{n}\right) X_n,\ 2 \leq i \leq \tfrac{n}{2}, \\ & d\varphi(X_1, X_i, X_j) = \\ & \left(-C_{i+1,j}^{i+j+k-1} \theta_{1,i}^{i+1} - C_{i,j+1}^{i+j+k-1} \theta_{i,j}^{i+j+k-1} + \theta_{i,j}^{i+j+k-2} - \theta_{i+1,j}^{i+j+k-1} - \theta_{i,j+1}^{i+j+k-1}\right) X_{i+j+k-1}. \end{aligned} \quad (3.46)$$

where in the last identity $2 \leq i,\ i < j,\ 2k-2 \leq i+j \leq n-k+1, i < k$.
For the triple $\{X_2, X_i, X_j\}$, the restrictions obtained are

$$\begin{aligned} d\varphi(X_2, X_i, X_j) = & \left(C_{2,i}^{i+k} \theta_{j,i+k}^{i+j+2k-2} - C_{2,j}^{j+k} \theta_{i,j+k}^{i+j+2k-2} + C_{2,i+j+k-2} \theta_{i,j}^{i+j+k-2} +\right) X_{i+j+2k-2} + \\ & + \left(C_{i,j}^{i+j+k-2} \theta_{2,i+j+k-2}^{i+j+2k-2} - C_{i,j+k}^{i+j+2k-2} \theta_{2,j+k}^{j+k} - C_{j,i+k}^{i+j+2k-2} \theta_{2,i}^{i+k}\right) X_{i+j+2k-2} \end{aligned} \quad (3.47)$$

for $3 \leq i,\ i < j,\ i+j \leq n-2k+1, i < k$, as well as the equations

$$\begin{aligned} d\varphi(X_2, X_i, X_{n-k-i+1}) = & \left(\theta_{i,n-k-i+1}^{n-1} - (-1)^i \theta_{2,n-k-i+1}^{n-k-i+3} + (-1)^{i+k} \theta_{2,i}^{i+k} - C_{2,i}^{i+k} \theta_{i+k,n-k-i+1}^{n}\right. \\ & \left. - C_{2,n-k-i+1}^{n-i+1} \theta_{i,n-i+1}^{n} + C_{i,n-k-i+1}^{n-1} \theta_{2,n-1}^{n}\right) X_n,\ 3 \leq i \leq \tfrac{n}{2}. \end{aligned} \quad (3.48)$$

Now let ℓ_0 be the minimal index such that $\alpha_{\ell_0-1} = C_{\ell_0,\ell_0+1}^{k+2\ell_0-1} \neq 0$. Rescaling we can suppose that it has the value 1. The system of coefficients obtained from (3.46)-(3.48) allows us, after some heavy algebraic manipulation, to obtain the solutions as

$$\begin{aligned} \theta_{i,j}^{i+j+k-2} &= C_{i,j}^{i+j+k-2} \left(-\sum_{t=2}^{i-1} \theta_{1,t}^{t+1} + \sum_{t=1}^{2l+1-j} \theta_{1,2l+1-t}^{2l+2-t} - \sum_{t=1}^{j-2l-1} \theta_{1,2l+t}^{2l+1+t} + \right. \\ & \left. \sum_{t=1}^{i+j-2l-3} \theta_{1,2l+t+k}^{2l+t+k+1} + (-1)^{l+1} \theta_{2,2l+1}^{2l+k+1}\right), 2 \leq i < j, i+j \leq n-k+1 \quad (3.49) \\ \theta_{i,n-i+1}^{n} &= (-1)^i \left(-\sum_{t=1}^{i-2} \left(\theta_{1,t+1}^{t+2} + \theta_{1,n-1-t}^{n-t}\right) + \theta_{2,n-1}^{n}\right) X_n,\ 2 \leq i \leq \frac{n}{2}. \end{aligned}$$

As a consequence, any 2-cocycle can be expressed as a linear combination of $\theta_{1,i}^{i+1}$ with $2 \leq i \leq n-2$, as well as $\theta_{2,2l+1}^{2k+k-1}$ and $\theta_{2,n-1}^{n}$. The corresponding cocycles are linearly independent, from which we conclude that $\dim Z^2\left(\mathfrak{n}_k^n, \mathfrak{r}_k^n\right)^{\mathfrak{t}} = n-1$, showing that $\dim H^2(\mathfrak{r}_k^n, \mathfrak{r}_k^n) = 0$.

At this point, it is natural to ask whether the remaning Lie algebras covered in Tables 1 and 2 also belong to some family dependent on k. It is likely that many other series appear, although, as follows from the preceding analysis of the series in Proposition 6, the dimension of the first term in each family increases, and dimensions beyond $n = 100$ are required to recognize the coefficient pattern for the diagonal sequence (3.36). It is hoped to find a more systematic description in some future work.

3.7 Conclusions

We have studied some general properties of rank one solvable Lie algebras $\mathfrak{r}_k^n$ with an eigenvalue spectrum given by $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3)$, corresponding to algebras possessing a nilradical that contracts onto the model filiform algebra Q_n. This can be seen as a complementary study to that developed in [24], where rank one algebras the nilradical of which contracts onto L_n were considered. We have particularly focused on those Lie algebras that have a vanishing Chevalley cohomology $\dim H^2(\mathfrak{r}_k^n, \mathfrak{r}_k^n)$. The detailed analysis for low values of k has allows us to extrapolate the results and obtain some series of cohomologically rigid algebras defined for any k (within a parity).

An interesting observation emerges from the inspection of the rigid families enumerated in Proposition 5. The regularities observed there lead us to conjecture the existence, in the dimensions range $4k-6 \leq n \leq 4k-2$, of cohomologically rigid Lie algebras $\mathfrak{r}_n^k$, the nilradical of which is of type (3.31) and satisfies the equations (3.35) for the values

$$(\alpha_1, \cdots, \alpha_M) = \left(0^{k-\ell_0}, 1, \phi_1(q), \cdots, \phi_{\ell_0-2}(q), (-1)^{\ell_0-1}\ell_0, (-1)^{\ell_0}, 0^{q-\ell_0}\right),$$

where $M = \frac{3k+1-2\ell_0}{2}$ for $k = 2q+1$, $M = \frac{3k+2-2\ell_0}{2}$ for $k = 2q$ and $q \geq \ell_0$ and $\phi_s(q)$ some combinatorial function of q. However, to confirm this hypothesis further detailed analysis of the solutions of equations (3.35) for higher values of k and dimensions n are necessary, in order to establish precisely what values of ℓ_0 actually appear, the exact values for the coefficients $\phi_s(q)$ and whether the resulting solvable Lie algebras, if existing, have vanishing Chevalley cohomology for all admissible values of k. It may further be asked whether a global description of these Lie algebras is feasible by means of a generating function that covers simultaneously the case of even and odd values of k. Work in this direction is currently in progress.

It shall be observed that, in addition to the cohomologically rigid solutions found, there are also geometrically rigid solvable rank one Lie algebras with eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-3, n+2k-3)$ and cohomology space $H^2(\mathfrak{r}, \mathfrak{r})$ of arbitrary dimension $d > 1$, as can be expected from the analogue analysis of rigid algebras with a nilradical of type A_n^k, where rigid examples with higher cohomology have been found [9]. However, for geometrically rigid Lie algebras, the cohomology is merely an accessory tool, and their detailed analysis requires other methods like the Jacobi schemes [17, 18]. The detailed analysis of rigid Lie algebras of this class with the eigenvalues spectrum considered in this work will be presented elsewhere.

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Capítulo 4

Algorithmic construction of solvable rigid Lie algebras determined by generating functions

4.1 Introduction

This chapter presents and develops the results obtained in the article *Algorithmic construction of solvable rigid Lie algebras determined by generating functions* published in Linear and Multilinear Algebra, where the concept of saturation of the nilradical is introduced and Lie algebras that have a saturated nilradical and that present a certain spectrum with respect to the maximal torus of derivations are sought. An algorithm to carry out this search is also presented and a sequence of generating functions that completely determine the nilradical is given. The interest in studying the rigidity concept for Lie algebras emerged naturally from the work of Gerstenhaber in the mid 1960's, that moreover showed striking similarities with the deformation theory of complex analytic structures [21, 28]. Both approaches were solidly justified by the powerful formalism of cohomological theories [15, 26], an effective and reliable tool that turned out to be central to formalize the notion of perturbations and deformations of algebraic and geometric structures [29]. In the particular case of Lie algebras, the rigidity was quickly recognized to be useful in the context of physical applications [8, 25], as well as in the topological analysis of orbits of Lie algebras through the action of the general linear group $GL(n, \mathbb{K})$ [19, 20]. While the rigidity of semisimple and parabolic Lie algebras follows essentially from their structural theory [31], the situation is somewhat different when we deal with solvable Lie algebras, where some special techniques like the weight systems and their associated linear systems [2, 10, 18, 22, 23] had to be developed. This made it possible to classify rigid Lie algebras in low dimensions, as well as to find various infinite series in arbitrary dimensions (see e.g. [3, 12, 13, 22, 23] and references therein). It was however soon realized that the cohomological approach is not sufficient to characterize rigidity, in the sense that a rigid algebra can possess a non-vanishing Chevalley cohomology, stimulating the construction of the first examples with this pathology and giving rise to a geometric and topological formulation of rigidity [28, 29, 30]. For complex Lie algebras a general structure theory has been established [11, 11, 23], with some of its results being adaptable to the real case [4, 16]. With the introduction of computer methods specifically developed for the computation of cohomologies [1, 17, 24], calculations have simplified considerably, making possible the analysis of large series in arbitrary dimensions and under different constraints [5, 6].

This work refines and extends some results originally obtained in [23, 22] and [23] for rank-one solvable rigid Lie algebras that admit a maximal torus $\mathfrak{t}$ such that the eigenvalues are given by the sequence $\operatorname{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-2)$, and can be seen as a continuation of some recent work related to these tori [7]. More specifically, we introduce the notion of saturation of a nilradical with respect to the eigenvalue spectrum of $\mathfrak{t}$. In these conditions, it is shown how to construct algorithmically the structure tensor by means of a generating function along with a factor sequence. This idea is used to construct a series of rigid solvable Lie algebras for dimensions $n \geq 3k+6$ and arbitrary values $k \geq 2$ having a vanishing Chevalley cohomology. This generalizes naturally the algebras $\mathfrak{w}_n$ considered in [13]. Basing on further restrictions imposed on the generating function, an algorithmic prescription to obtain rigid Lie algebras is proposed, from which a new series of algebraically rigid Lie algebras for the value $k=4$ and dimensions $n \geq 18$ is extracted.

Unless otherwise stated, Lie algebras considered in this work are of finite dimension with real or complex coefficients.

4.1.1 Rigidity of Lie algebras

Given a Lie algebra $\mathfrak{g}$, we denote by $\mathrm{Der}(\mathfrak{g})$ the Lie algebra formed by its derivations, that is, by those endomorphisms $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz condition $D([X,Y]) - [D(X),Y] + [X, D(Y)] = 0$ for all $X, Y \in \mathfrak{g}$. A derivation D is inner if for some $X \in \mathfrak{g}$ we have $D(Y) = [X,Y]$ for all $Y \in \mathfrak{g}$. In the contrary case, we say that D is an outer derivation.

Definition 1 *Let $\mathfrak{g}$ be an n-dimensional Lie algebra. A (maximal) Abelian subalgebra of $Der(\mathfrak{g})$ such that its generators are diagonalizable will be called a (maximal) torus of derivations.*

By diagonalizability we mean that the maps $f \otimes_{\mathbb{R}} \mathrm{Id}$ can be diagonalized simultaneously as complex endomorphisms. Due to the conjugacy of maximal tori over $\mathbb{C}$, the dimension remains unchanged and is called the rank $\mathrm{r}(\mathfrak{g})$ of $\mathfrak{g}$.

As is well-known, the general linear group $GL(n, \mathbb{K})$ defines an action on the variety $\mathcal{L}^n$of n-dimensional Lie algebras over $\mathbb{K}$, given by

$$\left(f^{-1} \circ \mathfrak{g}\right)(X,Y) = f^{-1}\left([f(X), f(Y)]\right), \; X, Y \in \mathfrak{g}, \; f \in GL(n,\mathbb{K}). \tag{4.1}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ corresponds naturally to all Lie algebras equivalent to $\mathfrak{g}$ by a change of basis, where the following dimension formula holds: $\dim \mathcal{O}(\mathfrak{g}) = n^2 - \dim \mathrm{Der}(\mathfrak{g})$

Definition 2 *We say that a Lie algebra $\mathfrak{g}$ is rigid if $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

Rigidity can also be formulated in terms of the Zariski topology of $\mathcal{L}^n$ [14]. The adherence $\overline{\mathcal{O}(\mathfrak{g})}$ constitutes an irreducible component of $\mathcal{L}^n$, a fact that emphasises the relevance of the rigidity for the topological analysis of $\mathcal{L}^n$ [13, 19]

Although proven to be insufficient to completely characterize rigidity [28], the cohomological approach has turned out to be an effective tool, as cohomologies can be systematically determined using symbolic computation packages (see e.g [24]). Specifically, Lie algebraic deformations are described using the Chevalley cohomology of Lie algebras [21, 15, 20, 21, 29], with special emphasis on the cohomology spaces $H^s(\mathfrak{g},\mathfrak{g})$ for the values $s \leq 3$. In this context, $H^0(\mathfrak{g},\mathfrak{g})$ and $H^1(\mathfrak{g},\mathfrak{g})$ correspond to the centre $Z(\mathfrak{g})$ and the space of outer derivations respectively.

The main tool for practical computations is the Hochschild-Serre factorization theorem [26]. As we are dealing only with solvable Lie algebras given by the semi-direct product $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$, where $\mathfrak{t}$ is a torus the generators $T \in \mathfrak{t}$ of which act diagonally on $\mathfrak{n}$, we have that the spaces $H^s(\mathfrak{r},\mathfrak{r})$ are determined by the relation

$$H^s(\mathfrak{r},\mathfrak{r}) \simeq \sum_{p+q=s} H^p(\mathfrak{t},\mathbb{R}) \otimes H^q(\mathfrak{n},\mathfrak{r})^{\mathfrak{t}}, \tag{4.2}$$

with $H^q(\mathfrak{n},\mathfrak{r})^{\mathfrak{t}}$ the space formed by all $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$:

$$H^s(\mathfrak{n},\mathfrak{r})^{\mathfrak{t}} = \{[\varphi] \in H^s(\mathfrak{n},\mathfrak{r}) \mid (T.\varphi) = 0,\ \forall T \in \mathfrak{t}\}. \tag{4.3}$$

The $\mathfrak{t}$-invariance of a cocycle φ is defined as

$$(T.\varphi)(Z_1,\cdots,Z_b) = [T,\varphi(Z_1,\cdots,Z_b)] - \sum_{s=1}^{b} \varphi(Z_1,\cdots,[T,Z_s],\cdots,Z_b). \tag{4.4}$$

It is straightforward to verify that the conditions $H^s(\mathfrak{r},\mathfrak{r}) = 0$ and $H^b(\mathfrak{n},\mathfrak{r})^{\mathfrak{t}} = 0$ for $0 \leq q \leq s$ are equivalent, because of the identification of $H^p(\mathfrak{t},\mathbb{R})$ with $\bigwedge^p \mathfrak{t}$. The Nijenhuis-Richardson criterion, proved in [29], states that a vanishing cohomology ensures rigidity:

Proposition 1 *If the Lie algebra* $\mathfrak{g}$ *satisfies* $\dim H^2(\mathfrak{g},\mathfrak{g}) = 0$*, then it is rigid.*

If a Lie algebra satisfies the Nijenhuis-Richardson criterion, we shall say that it is cohomologically (or algebraically) rigid. Semisimple Lie algebras and Borel subalgebras, as well parabolic Lie algebras, are cohomologically rigid [11, 29]. Rigid algebras satisfying $H^2(\mathfrak{g},\mathfrak{g}) \neq 0$ are called geometrically rigid, and have been constructed for both the solvable and non-solvable cases [22, 6, 7, 23, 28, 30]. In contrast, for low dimensions, rigid Lie algebras (over $\mathbb{R}$ and $\mathbb{C}$) are always cohomologically rigid. This also holds for certain types of eigenvalue spectra of tori (see [4, 5, 9, 22, 23] and references therein).

In the context of the classification of real or complex rigid Lie algebras, the following identification between the Chevalley cohomology spaces of a complex Lie algebra $\mathfrak{g}'$ and a real form $\mathfrak{g}$ is given:

$$\dim_{\mathbb{R}} H^2(\mathfrak{g},\mathfrak{g}) = \dim_{\mathbb{C}} H^2(\mathfrak{g}',\mathfrak{g}'), \tag{4.5}$$

allowing us to translate the notion of cohomological rigidity to the real field. Thus the complexification $\mathfrak{g}' = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of a real cohomologically rigid Lie algebra satisfies the same property. The converse also holds. In spite of this fact, there are important differences between the real and complex cases, as nonreal rigid Lie algebras can be found for dimensions $n \geq 13$ (see e.g. [22]).

4.2 Solvable real Lie algebras with prescribed maximal torus

Solvable (real) Lie algebras $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ with a maximal torus such that the eigenvalues are given by the sequence $\mathrm{spec}(\mathfrak{t}) = (1, k, k, \cdots, n+k-2)$ correspond to the solvable extensions of the so-called filiform Lie algebras [10, 22, 32]. It is within this class, for the value $k = 2$, that the first examples of solvable geometrically rigid Lie algebras were found [23]. Further series of geometrically rigid algebras for values $k \geq 3$ have also been obtained [7, 22].

Among all rigid algebras that admit a maximal torus $\mathfrak{t}$ and the preceding eigenvalues, there are two families parameterized by k that stand out for their structural plainness. Over a basis $\{X_1, \cdots, X_n, T\}$ of the Lie algebra, where T generates $\mathfrak{t}$, the families are defined as:

1. $\mathfrak{r}_{k,k+3} = \langle T \rangle \overrightarrow{\oplus} \mathfrak{n}_{k,k+3} : (k \geq 2)$

$$\begin{array}{ll} [X_1, X_i] = X_{i+1},\ 2 \leq i \leq k+2; & [X_2, X_3] = X_{k+3}, \\ [T, X_1] = X_1; & [T, X_i] = (k-2+i)\, X_i, \quad 2 \leq i \leq k+3. \end{array} \tag{4.6}$$

2. $\mathfrak{r}_{k,k+4} = \langle T \rangle \overrightarrow{\oplus} \mathfrak{n}_{k,k+4} : (k \geq 2)$

$$\begin{array}{ll} [X_1, X_i] = X_{i+1},\ 2 \leq i \leq k+3; & [X_2, X_3] = X_{k+3}, \quad [X_2, X_4] = X_{k+4}, \\ [T, X_1] = X_1 & [T, X_i] = (k-2+i)\, X_i, \quad 2 \leq i \leq k+4. \end{array} \tag{4.7}$$

These Lie algebras were already identified and shown in [22] to be rigid. In fact, using the Hochschild-Serre factorization theorem it can be easily proved that they are indeed cohomologically rigid (see e.g. [2, 23]).

Proposition 2 *For any $k \geq 2$ the Lie algebras $\mathfrak{r}_{k,k+3+\varepsilon}$ ($\varepsilon = 0, 1$) satisfy $H^2(\mathfrak{r}_{k,k+3+\varepsilon}, \mathfrak{r}_{k,k+3+\varepsilon}) = 0$.*

The main particularity of these Lie algebras, besides their structural plainness, is given by the fact that for any sum of eigenvalues of the torus $\mathfrak{t}$ that gives rise to a value that still belongs to spectrum, the commutator of the corresponding eigenvectors is nonzero. This observation suggests a kind of “saturation"notion that will be made precise in the following section, and that will enable us to determine new hierarchies of rigid Lie algebras governed by a generating function.

4.2.1 Saturation of nilradicals

Consider a solvable (real) Lie algebra $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$, where $\mathfrak{t}$ denotes the torus and $\mathfrak{n}$ the nilradical. If $(\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of the torus over a given basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{n}$, the Jacobi identities imply that the commutator $[X_i, X_j]$, if it is nonzero, must be an eigenvector with corresponding eigenvalue $\lambda_i + \lambda_j$. In this context, we define the “saturation.ºf a nilradical whenever any sum of eigenvalues in the spectrum of the torus that is itself an eigenvalue leads to a nonzero commutator of the corresponding eigenvectors. More precisely:

Definition 3 *Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ have rank one with eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (\lambda_1, \cdots, \lambda_n)$. The nilradical $\mathfrak{n}$ is said to be saturated with respect to the torus $\mathfrak{t}$ if $[X_i, X_j] \neq 0$ whenever $\lambda_i + \lambda_j \in \mathrm{spec}(\mathfrak{t})$.*

The Lie algebras $\mathfrak{r}_{k,k+3}$ and $\mathfrak{r}_{k,k+4}$ correspond to the minimal case where the saturation property is satisfied, as there is only one (respectively two) sum(s) of eigenvalues that give rise to an eigenvalue of the torus. A question that arises naturally is if for arbitrary values of k and dimensions $n \geq k+4$ we can always find (cohomologically) rigid Lie algebras with the given torus and such that the nilradical is saturated in the above sense.

To this extent, let us inspect the commutators of saturated nilradicals more closely. Suppose that $\mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ has rank-one and that the dimension satisfies $n+1 \geq k+5$. Further let the eigenvalues of $\mathfrak{t}$ be given by

$$\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \cdots, n+k-2)\,. \tag{4.8}$$

If $\{X_1, \cdots, X_n\}$ is a basis formed by eigenvectors with respect to $ad\,(T)$, the Jacobi conditions imply that the only nonzero commutators of $\mathfrak{n}$ are

$$\begin{array}{ll} [X_1, X_j] = C_{1,j}^{j+1} X_{j+1}, & 2 \leq j \leq n-1 \\ [X_i, X_j] = C_{i,j}^{i+j+k-2} X_{i+j+k-2}, & 2 \leq i < j;\ i+j \leq n+2-k. \end{array} \tag{4.9}$$

As we are assuming that the nilradical is saturated with respect to $\mathfrak{t}$, we have that $C_{1,j}^{j+1} \neq 0$ holds for any j, so that by a change of basis we can set $C_{1,j}^{j+1} = 1$ for $j = 2, \cdots, n-1$.

Evaluating the Jacobi identity for $\{X_1, X_i, X_j\}$ provides a first generic relation between the structure constants $C_{i,j}^k$ of $\mathfrak{n}$:

$$C_{i,j}^{i+j+k-2} - C_{i+1,j}^{i+j+k-1} - C_{i,j+1}^{i+j+k-1} = 0,\ 2 \leq i < j \leq n-2. \tag{4.10}$$

In particular, considering the triples $\{X_1, X_i, X_{i+1}\}$ it follows that

$$C_{i,i+1}^{2i+k-1} - C_{i,i+2}^{2i+k} = 0,\ 2 \leq i \leq \left[\frac{n-k}{2}\right], \tag{4.11}$$

while the triples $\{X_1, X_i, X_{i+s}\}$ with $s \geq 2$ give rise to the constraints

$$C_{i,i+s}^{2i+k+s-2} - C_{i,1+i+s}^{2i+k+s-1} - C_{1+i,i+s}^{2i+k+s-1} = 0,\ 2 \leq i \leq \left[\frac{n-k}{2}\right]. \tag{4.12}$$

As these equations are linear in their arguments, the system can be solved by recursion for the unknowns $C_{i,j}^{i+j+k-2}$ satisfying $|j-i| > 1$ with respect to the structure constants $\left\{C_{2,3}^{k+3}, C_{3,4}^{k+5}, \cdots, C_{\rho,\rho+1}^{2\rho+k-2}\right\}$, where $\rho = \left[\frac{n+1-k}{2}\right]$, leading to equations of the type

$$C_{i,j}^{i+j+k-2} = \sum_{\ell=2}^{\rho} \lambda_{i,j}^{\ell} C_{\ell,\ell+1}^{2\ell+k-2}. \tag{4.13}$$

These coefficients $\lambda_{i,s}$ can be computed explicitly,[1] but they are inessential for the argumentation that follows. For practical reasons, it is convenient to introduce the parameters $\alpha_i = C_{i,i+1}^{2i+k-1}$ for $2 \leq i \leq \rho$ and set

$$\Delta = \{\alpha_2, \cdots, \alpha_\rho\} = \left\{C_{2,3}^{k+3}, C_{3,4}^{k+5}, \cdots, C_{\rho,\rho+1}^{2\rho+k-2}\right\}. \tag{4.14}$$

This formal simplification shows that the non-vanishing brackets of $\mathfrak{n}$ have the form

$$\begin{array}{l} [X_1, X_j] = X_{j+1},\ 1 \leq j \leq n-1, \\ [X_i, X_{i+1}] = \alpha_i X_{2i+k-1},\ 2 \leq i \leq \rho, \\ [X_i, X_{i+s}] = \sum_{\ell=2}^{\rho} \lambda_{i,s}^{\ell} \alpha_\ell X_{2i+k+s-2}, \quad i \geq 2,\ 2i+k+s-2 \leq n. \end{array} \tag{4.15}$$

Now the Jacobi identities that remain to be evaluated, corresponding to $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$, will lead to quadratic polynomials in the variables of Δ, the common solutions of which determine the nilradical. While for low dimensions these equations will generally depend on one or more parameters, from which no rigidity can be inferred, for sufficiently high dimensions n, the number of quadratic relations will exceed (considerably) that of the variables in Δ, leading eventually to isolated nontrivial solutions. This argumentation to analyze the possible rigidity of Lie algebras has been successfully used at various occasions (see e.g. [22, 7, 23, 3, 22]).

If we require in addition that the nilradical $\mathfrak{n}$ is saturated with respect to the torus $\mathfrak{t}$, then $\alpha_i \neq 0$ for all i, as well as the remaining structure constants in (5.17); thus in particular

$$\sum_{\ell=2}^{\rho} \lambda_{i,j}^{\ell} \alpha_\ell \neq 0,\ 2 \leq i < j. \tag{4.16}$$

For $0 \leq s \leq n-k$ we define the successive "diagonals"

$$\Delta_s := \left\{C_{i,i+1+s}^{2i+k+s-1}, \quad 2 \leq i \leq \left[\frac{n+1-k-s}{2}\right]\right\}, \tag{4.17}$$

[1] An explicit expression for such a formula can be found in [7].

where $\Delta = \Delta_0$. As no entry in these diagonals vanishes, the factor sequence given by

$$\phi_{i,s} = C_{i,i+2+s}^{2i+k+s} \,/\, C_{i,i+1+s}^{2i+k+s-1}, \tag{4.18}$$

for the values $2 \leq i \leq \left[\frac{n+1-k-s}{2}\right]$ is well defined. For $s = 0$ we clearly have $\phi_{i,0} = 1$, as $C_{i,i+2}^{2i+k} = C_{i,i+1}^{2i+k-1}$. Fixing the value of i and solving these factors for $C_{i,i+2+s}^{2i+k+s}$ we arrive at the expression

$$C_{i,i+2+s}^{2i+k+s} = \prod_{q=1}^{s} \phi_{i,q} C_{i,i+1}^{2i+k-1} = \alpha_i \prod_{q=1}^{s} \phi_{i,q}. \tag{4.19}$$

We see that with this reformulation, the linear combinations in (5.17) have been replaced by a functional multiple of α_i. This shows that the structure tensor of the saturated nilradical is completely determined by the main diagonal Δ_0 and the factor sequences $\phi_{i,s}$. In this sense, the main diagonal can be seen as a generating function of the structure tensor of $\mathfrak{n}$.

4.2.2 The Lie algebra $\mathfrak{w}_n$

Although it has structurally not been explicitly analyzed, the Lie algebra $\mathfrak{w}_n$ considered in [23] for the value $k = 2$ fulfills all of the preceding conditions, and thus serves as a nontrivial example of a rigid algebra with saturated nilradical. We briefly recall that for rank one solvable Lie algebras $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ with eigenvalues $(1, 2, \cdots, n)$ for the torus, it was found that from dimension $n \geq 12$ onwards only two Lie algebras exist for which the conditions $C_{1,j}^{j+1} \neq 0$ and $C_{2,3}^{5} \neq 0$ are simultaneously satisfied. One of these solutions corresponds to the geometrically rigid Bratzlavsky algebra $\mathfrak{Br}_{2,n}$ with $\dim H^2(\mathfrak{Br}_{2,n}, \mathfrak{Br}_{2,n}) = 1$ (see [23] for the details), while the other algebra found, denoted by $\mathfrak{w}_n$, is completely determined by the parameters α_i in (5.18). For the lowest dimensions $12 \leq n \leq 20$, the explicit values of these parameters are given by

n	Δ	ρ
12	$\left\{1, \frac{1}{10}, \frac{1}{70}, \frac{1}{420}\right\}$	4
13, 14	$\left\{1, \frac{1}{10}, \frac{1}{70}, \frac{1}{420}, \frac{1}{2310}\right\}$	5
15, 16	$\left\{1, \frac{1}{10}, \frac{1}{70}, \frac{1}{420}, \frac{1}{2310}, \frac{1}{12012}\right\}$	6
17, 18	$\left\{1, \frac{1}{10}, \frac{1}{70}, \frac{1}{420}, \frac{1}{2310}, \frac{1}{12012}, \frac{1}{60060}\right\}$	7
19, 20	$\left\{1, \frac{1}{10}, \frac{1}{70}, \frac{1}{420}, \frac{1}{2310}, \frac{1}{12012}, \frac{1}{60060}, \frac{1}{291720}\right\}$	8

(4.20)

Analyzing these sequences recursively, we can find the following generating function yielding the values of Δ for arbitrary n:

$$\alpha_{\ell+1} = \varphi(2, \ell) = \frac{4^{1-\ell}\,\Gamma\left(\frac{5}{2}\right)\Gamma(\ell)}{\Gamma\left(\frac{3+2\ell}{2}\right)},\ \ell \geq 1. \tag{4.21}$$

Reproducing the analysis of the Jacobi conditions in [23], a long but otherwise routine computation provides the structure constants of $\mathfrak{w}_n$ explicitly:

$$C_{i,i+1+s}^{2i+1+s} = \frac{(1+s)\Gamma(2i)\Gamma(i+s)}{\Gamma(i)\Gamma(2i+s)}\,\varphi(2, i-1),\ 2 \leq i \leq \rho,\ 0 \leq s+i \leq n-1, \tag{4.22}$$

where $\rho = \left[\frac{n-3}{2}\right]$. It follows at once from this equation that for any indices $i \geq 2$ and $s \geq 1$ we have the proportions

$$\frac{C_{i,i+2+s}^{2i+2+s}}{C_{i,i+1+s}^{2i+1+s}} = \frac{(2+s)(i+s)}{(1+s)(2i+s)}, \tag{4.23}$$

a fact that allows us to systematically reconstruct the structure tensor starting from the main diagonal Δ, following the prescription in (4.19). It is further immediate from this that the nilradical of $\mathfrak{w}_n$ is saturated with respect to the torus $\mathfrak{t}$. As this Lie algebra is along with $\mathfrak{Br}_{2,n}$ the only solution to the Jacobi conditions for $n > 11$, the rigidity of $\mathfrak{w}_n$ can alternatively be proved using topological methods (see [23], Lemma 5.1). It turns out that $\mathfrak{w}_n$ is actually cohomologically rigid.

Lemma 1 *For any $n \geq 12$, the solvable Lie algebra $\mathfrak{w}_n$ satisfies $H^2(\mathfrak{w}_n, \mathfrak{w}_n) = 0$.*

A proof can be found in [3]. We further observe that the Lie algebras $\mathfrak{w}_5$ and $\mathfrak{w}_6$ are actually isomorphic to $\mathfrak{r}_{2,5}$ and $\mathfrak{r}_{2,6}$, respectively. In the following paragraph, $\mathfrak{w}_n$ will be generalized to a family of (cohomologically) rigid Lie algebras defined for any value $k \geq 3$.

4.2.3 The series $\mathfrak{w}_{k,n}$ for $k \geq 3$

The preceding general remarks on saturated nilradicals and the example of $\mathfrak{w}_n$ somehow indicate that for any $k \geq 3$, an analogue of the Lie algebra $\mathfrak{w}_n$ should exist, preserving its two main properties, namely the saturation of the nilradical and the cohomological rigidity, at least from a sufficiently high dimension onwards. In this section we will construct effectively such a series using the idea of a generating function $\varphi(k, \ell)$ as in (4.19), from which the structure tensor of the nilradical will be deduced.

For any $k \geq 3$ let $\mathfrak{t}_k$ be a torus with eigenvalue spectrum $(1, k, k+1, \cdots, n+k-2)$ and consider a basis $\{X_1, \cdots, X_n\}$ of a nilpotent Lie algebra $\mathfrak{n}_{k,n}$ admitting $\mathfrak{t}_k$ as a maximal external torus of derivations. We further consider the function

$$\varphi(k, \ell) = \frac{4^{1-\ell}\Gamma\left(\frac{2k+1}{2}\right)\Gamma(k+\ell-2)}{\Gamma(k-1)\Gamma(\frac{2k+2\ell-1}{2})}, \quad 1 \leq \ell \leq \rho - 1 = \left[\frac{n+1-k}{2}\right] - 1. \qquad (4.24)$$

Such a function is a direct extrapolation of the diagonal function in (4.21) to higher values of the parameter k. Taking into account the displacement in the indices, the entries in the main diagonal Δ are hence given by $\alpha_i = \varphi(k, i-1)$ for $2 \leq i \leq \rho$. With this choice, it follows that the table of commutators (see equation (5.19)) of these algebras possesses the following generic structure[2]

[2]Here we have chosen even dimension $n = 2\rho + 2$. For odd dimension the last column has only one nonzero entry. Observe that this parity-conditioned pattern is already present for the Lie algebras $\mathfrak{r}_{k,k+3}$ and $\mathfrak{r}_{k,k+4}$.

	X_1	X_2	X_3	X_4	$\dots$		X_ρ	$X_{\rho+1}$	$\cdots$	X_n
X_1	0	0	0	0			0	0	$\cdots$	0
X_2	X_3	0	0	0			0	0	$\cdots$	0
X_3	X_4	$\alpha_2 X_{k+3}$	0	0			0	0	$\cdots$	0
X_4	X_5	$\alpha_2 X_{k+4}$	$\alpha_3 X_{k+5}$	0			0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\alpha_4 X_{k+7}$			0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		0	$\vdots$		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$	0	$\vdots$		$\vdots$
X_j	X_{j+1}	$C_{2,j}^{j+k} X_{j+k}$	$C_{3,j}^{j+k+1} X_{j+k+1}$	$C_{4,j}^{j+k+2} X_{j+k+2}$	$\cdots$	$\cdots$	$\alpha_\rho X_{n-1}$	0	$\cdots$	0
X_{j+1}	X_{j+1}	$C_{2,j+1}^{j+k+1} X_{j+k+1}$	$C_{3,j+1}^{j+k+2} X_{j+k+2}$	$C_{4,j}^{j+k+2} X_{j+k+3}$	$\cdots$	$\cdots$	$\alpha_\rho X_n$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\iddots$	0	$\vdots$		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$		0	$\vdots$		$\vdots$
X_{n-k-2}	X_{n-k-1}	$C_{2,n-k-2}^{n-2} X_{n-2}$	$C_{3,n-k-2}^{n-1} X_{n-1}$	$C_{4,n-k-2}^{n} X_n$	$\cdots$	$\cdots$	0	0	$\cdots$	0
X_{n-k-1}	X_{n-k}	$C_{2,n-k-1}^{n-1} X_{n-1}$	$C_{3,n-k-1}^{n} X_n$	0	$\cdots$	$\cdots$	0	0	$\cdots$	0
X_{n-k}	X_{n-k+1}	$C_{2,n-k}^{n} X_n$	0	0	$\cdots$	$\cdots$	0	0	$\cdots$	0
X_{n-k+1}	X_{n-k+2}	0	0	0	$\cdots$	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$		$\vdots$
X_{n-k}	X_{n-k+1}	0	0	0	$\cdots$	$\cdots$	0	0	$\cdots$	0
X_{n-1}	X_n	0	0	0	$\cdots$	$\cdots$	0	0	$\cdots$	0
X_n	0	0	0	0	$\cdots$	$\cdots$	0	0	$\cdots$	0

If all the possible commutators are different from zero, the nilradical will be saturated with respect to the torus. For $0 \leq s \leq n-k$ let Δ_s be the secondary diagonals as defined in (4.17). Now, as a natural generalization of (4.23), we define the following factor sequence:

$$\phi_{i,s} = \frac{C_{i,i+2+s}^{2i+k+s}}{C_{i,i+1+s}^{2i+k+s-1}} = \frac{(2+s)(i+s+k-2)}{(1+s)(2i+s+2k-4)}, \tag{4.25}$$

for the values $2 \leq \ell \leq \left[\frac{n+1-k-s}{2}\right]$. For $s=0$ this implies that

$$\phi_{i,0} = 1 \Rightarrow C_{i,i+2}^{2i+k} = C_{i,i+1}^{2i+k-1} = \varphi(k, i-1), \quad 2 \leq i \leq \left[\frac{n-k}{2}\right] \tag{4.26}$$

in agreement with equation (5.15). Now, iterating the procedure for $s \geq 1$ and simplifying the resulting expression we arrive at

$$C_{i,i+2+s}^{2i+k+s} = \prod_{q=2}^{s-1} \phi_{i,q}\, C_{i,i+1}^{2i+k+s-1} = \prod_{q=2}^{s-1} \phi_{i,q}\, \varphi(k, i-1). \tag{4.27}$$

The product on the right side of this equation admits the following expression using the Gamma function:

$$\prod_{q=2}^{s-1} \phi_{i,q} = \frac{(1+s)\Gamma(k+i+s-2)\Gamma(2k+2i-4)}{\Gamma(k+i-2)\Gamma(2k+2i+s-4)}. \tag{4.28}$$

We conclude that the structure constants are given by

$$C_{i,i+1+s}^{2i+k+s-1} = \frac{(1+s)\Gamma(2i+2k-4)\Gamma(i+s+k-2)}{\Gamma(i+k-2)\Gamma(2i+2k+s-4)}\, \varphi\left(k, i-1\right), \tag{4.29}$$

where $2 \leq i \leq \rho$, $1 \leq s \leq n-k$. It remains to check that all Jacobi identities are identically satisfied. For notational convenience let us write

$$\Phi\left(k, i-1, j-i-1\right) = C_{i,j}^{i+j+k-2} = \frac{(j-i)\Gamma(2i+2k-4)\Gamma(j+k-3)}{\Gamma(i+k-2)\Gamma(i+j+2k-5)}\, \varphi\left(k, i-1\right). \tag{4.30}$$

If we now develop the Jacobi identity for the elements $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$ in terms of the functions in (4.30) we obtain the products

$$\begin{aligned}&\Phi(k,i-1,j-i-1)\Phi(k,\ell-1,i+j+k-\ell-3)+\Phi(k,\ell-1,i-\ell-1)\Phi(k,j-1,i+\ell+k-j-3)\\ &+\Phi(k,j-1,\ell-j-1)\Phi(k,i-1,j+\ell+k-i-3)=\frac{4^{2k-1}\Gamma(i+k-3)\Gamma(j+k-3)\Gamma(\ell+k-3)}{\pi\,\Gamma(k-1)^2\Gamma(i+j+\ell+3k-7)}\times\\ &\Gamma\left(\frac{2k+1}{2}\right)^2\{(j-i)(i+j+k-\ell-2)+(i-\ell)(i+\ell+k-j-2)+(\ell-j)(j+\ell+k-i-2\}\equiv 0,\end{aligned} \quad (4.31)$$

as the last term in the expansion vanishes identically. This shows that the generating function $\varphi(k, \ell, s)$ determines a saturated nilpotent Lie algebra $\mathfrak{n}_{k,n}$ possessing $\mathfrak{t}_{k,n}$ as an external torus of derivations.

The solvable Lie algebras $\mathfrak{w}_{k,n}$ generalizing $\mathfrak{w}_n$ for values $k \geq 3$ are thus given by the structure tensor

$$\begin{array}{l}[T,X_1]=X_1,\ [T,X_j]=(k+j-2)X_j,\ 2\leq j\leq n;\\ [X_1,X_j]=X_{j+1},\ 1\leq j\leq n-1,\\ [X_i,X_{i+1}]=\varphi(k,i-1)X_{2i+k-1},\ 2\leq i\leq \rho,\\ [X_i,X_{i+s}]=\Phi\left(k,i-1,j-i-1\right)X_{i+j+k-2},\quad i\geq 2,\ 2i+k+s-2\leq n.\end{array} \quad (4.32)$$

In particular, $\mathfrak{w}_{2,n} \simeq \mathfrak{w}_n$ for any $n \geq 12$.

Proposition 3 *For any $k \geq 2$ and $n \geq 3k+6$, the solvable Lie algebra $\mathfrak{w}_{k,n} = \mathfrak{t}_k \overrightarrow{\oplus} \mathfrak{n}_{k,n}$ is cohomologically rigid with a nilradical saturated with respect to the torus.*

Demostración 1 *Applying the Hochschild-Serre factorization theorem, it follows that a 1-cochain $f \in C^1(\mathfrak{n}_{k,n}, \mathfrak{w}_{k,n})$*

$$f(X_i) = \sum_{l=1}^{n} a_i^l X_l + b_i T,\ 1 \leq i \leq n, \quad (4.33)$$

is invariant with respect to the torus when it adopts the form

$$f(X_i) = a_i^i X_i,\ 1 \leq i \leq n. \quad (4.34)$$

Using the exterior derivative d we obtain the image of invariant 1-cochains, given by

$$\begin{array}{ll} df(X_1,X_i)=\left(a_1^1+a_i^j-a_{1+i}^{1+i}\right)X_{1+i}, & 2\leq i\leq k+3\\ df(X_i,X_j)=\left(a_i^i+a_j^j-a_{i+j+k-2}^{i+j+k-2}\right)X_{i+j+k-2}, & 2\leq i\leq n+2-k,\ i+j\leq n,\end{array} \quad (4.35)$$

from which it follows at once that the cocycle

$$\theta(X_1) = X_1,\quad \theta(X_j) = (j+k-2)X_j,\ 2 \leq j \leq n \quad (4.36)$$

generates $\ker d$. *Hence*

$$\dim B^2(\mathfrak{n}_{k,n}, \mathfrak{w}_{k,n})^{\mathfrak{t}} = n-1, \quad (4.37)$$

and a basis is determined by the images of $f_j(X_j) = a_j^j X_j$ for $2 \leq i \leq n$. Similarly, for a generic 2-cochain $\varphi \in C^2(\mathfrak{n}_{k,n}, \mathfrak{w}_{k,n})$ defined as

$$\varphi(X_i, X_j) = \sum_{k=1}^{n} \alpha_{ij}^k X_k + \beta_{ij} T, \quad (4.38)$$

the invariance condition (6.13) is satisfied whenever the constraints

$$\begin{array}{ll}\varphi\left(X_1,X_j\right)=\alpha_{1,j}^{1+j}X_{1+j}, & 2\leq j\leq n-1;\\ \varphi\left(X_i,X_j\right)=\alpha_{i,j}^{i+j+k-2}X_{i+j+k-2}, & 2\leq i<j\leq n-k\end{array} \tag{4.39}$$

are fulfilled. While it is immediate that

$$d\varphi\left(T,X_i,X_j\right)=0,\ 1\leq i<j\leq n, \tag{4.40}$$

for the generators $\{X_1,X_i,X_j\}$ *we are led to the equations:*

$$\begin{array}{rl}d\varphi\left(X_1,X_j,X_\ell\right)= & \left(\alpha_{j,\ell}^{j+\ell+k-2}-\alpha_{1,\ell}^{1+\ell}C_{j,1+\ell}^{j+\ell+k-1}+\alpha_{1,j}^{1+j}C_{\ell,1+j}^{j+\ell+k-1}-\alpha_{j+1,\ell}^{j+\ell+k-1}-\alpha_{j,\ell+1}^{j+\ell+k-1}\right.\\ & \left.-\alpha_{1,j+\ell+k-1}^{j+\ell+k}\alpha_{j,\ell}^{j+\ell+k-2}\right)X_{j+\ell+k-1},\end{array} \tag{4.41}$$

where the structure constants C_{pq}^r *are given by (4.29). On the other hand, for the triples* $\{X_i,X_j,X_\ell\}$ *we get*

$$\begin{array}{rl}d\varphi\left(X_i,X_j,X_\ell\right)= & \left(\alpha_{j,\ell}^{j+\ell+k-2}C_{i,j+\ell+k-2}^{i+j+\ell+2k-4}-\alpha_{i,\ell}^{i+\ell+k-2}C_{j,i+\ell+k-2}^{i+j+\ell+2k-4}+\alpha_{i,j}^{i+j+k-2}C_{\ell,i+jl+k-2}^{i+j+\ell+2k-4}-C_{i,j}^{i+j+k-2}\times\right.\\ & \left.-\alpha_{\ell,i+j+k-2}^{i+j+\ell+2k-4}-\alpha_{j,i+\ell+k-2}^{i+j+\ell+2k-4}C_{i,\ell}^{i+\ell+k-2}+\alpha_{i,j+\ell+k-2}^{i+j+\ell+2k-4}C_{j,\ell}^{j+\ell+k-2}\right)X_{i+j+\ell+2k-4}.\end{array} \tag{4.42}$$

The system formed by (4.41) and (4.42), although cumbersome due to the saturation of the nilradical, can be explicitly solved by recursion. In fact, solving successively the equations $d\varphi\left(X_1,X_\ell,X_m\right)=0$ *for* $\ell\geq 2$ *and* $m\geq\ell+1$ *allows us to express the cocycles* $\alpha_{\ell+1,m}^{\ell+m+k-1}$ *in terms of the cocycles* $\left\{\alpha_{1,j}^{j+1}\right\}$ *with* $2\leq j\leq n-1$ *and* $\left\{\alpha_{2,\ell}^{\ell+k}\right\}$ *with* $3\leq\ell\leq n-k$*, showing that* $\dim Z^2\left(\mathfrak{n}_{k,n},\mathfrak{w}_{k,n}\right)^{\mathfrak{t}}\leq 2n-k-4$*. The equations (4.42) provide further relations that enable us to write the cocycles* $\alpha_{2,q}^{q+k}$ *with* $q\geq 4$ *in terms of* $\left\{\alpha_{1,j}^{j+1}\right\}$ *with* $2\leq j\leq n-1$ *and* $\left\{\alpha_{2,\ell}^{\ell+k}\right\}$ *with* $3\leq\ell\leq q-1$*, taking into account that* $n\geq 3k+6$*. Only the cocycle* $\alpha_{2,3}^{k+3}$ *remains unconstrained after these operations, from which we infer that invariant cocycles are linear combinations of* $\alpha_{2,3}^{k+3}$ *and* $\alpha_{1,j}^{j+1}$ *for* $2\leq j\leq n-1$*, hence* $\dim Z^2\left(\mathfrak{n}_{k,n},\mathfrak{w}_{k,n}\right)^{\mathfrak{t}}=n-1$*. Taking into account equation (4.37) we finally get*

$$\dim H^2\left(\mathfrak{n}_{k,n},\mathfrak{w}_{k,n}\right)^{\mathfrak{t}}=0,$$

and hence the algebra has vanishing Chevalley cohomology.

4.2.4 Case with maximal torus $\mathfrak{t}_{n,k}$ and $C_{2,3}^{k+3}\neq 0$ for dimensions $n\geq 3k+8$

As mentioned, the rigidity of $\mathfrak{w}_{k,n}$ can alternatively be justified using the affine scheme determined by the Jacobi conditions [14], as done originally in [23]. This requires however an extended analysis that is not restricted to the case of saturated nilradicals. In this paragraph we briefly sum up the general analysis in high dimensions for values $k\geq 4$.

From the spectrum $\mathrm{spec}(\mathfrak{t})=(1,k,k,\cdots,n+k-2)$ of the torus $\mathfrak{t}_{n,k}$ it is easily inferred that the algebra $\mathfrak{L}_{k,n}$ defined by the brackets

$$\begin{array}{l}[X_1,X_i]=X_{i+1},\ 1\leq i\leq n-1\\ [X_2,X_i]=X_{i+k},\ 3\leq i\leq n-k\end{array} \tag{4.43}$$

also satisfies the Jacobi identities for any $n\geq k+3$. Moreover, the rank of these Lie algebras is one, thus they admit $\mathfrak{t}_{k,n}$ as a maximal torus. It is natural to ask whether, from a certain dimension onwards, the algebras $\mathfrak{L}_{k,n}$ and $\mathfrak{n}_{k,n}$ are the only solutions to the Jacobi conditions (4.10)-(4.12) subjected to the conditions $C_{1,j}^{j+1}\neq 0$ and $C_{2,3}^{k+3}\neq 0$. While this assumption holds for even values of k, for odd values of k the result is conditioned by the parity of the dimension. For odd values of n, there are merely two solutions to the Jacobi conditions for $n \geq 3k+8$, while for even dimensions we get $\mathfrak{n}_{k,n}$ and a one-parameter family. The latter indicates that in this case, only one rigid Lie algebra is obtained, namely $\mathfrak{w}_{k,n}$.

Proposition 4 *For odd $k \geq 3$ and even $n \geq 3k+7$ the solvable Lie algebra $\mathfrak{t}_{k,n}\overrightarrow{\oplus}\mathfrak{L}_{k,n}$ is not rigid.*

Proof. It is immediate to verify that the 2-cochain θ defined by

$$\theta\left(X_j, X_{n+2-k-j}\right) = (-1)^j X_n, \quad 2 \leq j \leq \frac{n-1-k}{2} \tag{4.44}$$

is a cocycle with nonvanishing cohomology class.[3] Hence the deformation $\mathfrak{t}_{k,n}\overrightarrow{\oplus}\mathfrak{L}_{k,n}+\varepsilon\theta$ defines a Lie algebra $\mathcal{Q}(\varepsilon)_{k,n}$ with commutators

$$\begin{array}{l} [X_1, X_i] = X_{i+1},\ 1 \leq i \leq n-1 \\ [X_2, X_j] = X_{j+k},\ 3 \leq i \leq n-k-1 \\ [X_j, X_{n+2-k-j}] = \varepsilon\,(-1)^j X_n, \quad 2 \leq j \leq \frac{n-1-k}{2}. \end{array} \tag{4.45}$$

For $\varepsilon \neq 0$ the Lie algebra $\mathcal{Q}(\varepsilon)_{k,n}$ is clearly non-isomorphic to $\mathcal{Q}(0)_{k,n} = \mathfrak{L}_{k,n}$. Although for any value of ε we have

$$D^1\mathcal{Q}(\varepsilon)_{k,n} = [\mathcal{Q}(\varepsilon)_{k,n}, \mathcal{Q}(\varepsilon)_{k,n}] = \langle X_3, \cdots, X_n\rangle\,, \tag{4.46}$$

the cocycle implies that for $\varepsilon \neq 0$

$$D^2\mathcal{Q}(\varepsilon)_{k,n} = \left[D^1\mathcal{Q}(\varepsilon)_{k,n}, D^1\mathcal{Q}(\varepsilon)_{k,n}\right] = \langle X_n\rangle\,, \tag{4.47}$$

while $D^2\mathcal{Q}(0)_{k,n} = 0$. ■

In the remaining cases corresponding to either even values of k and arbitrary dimensions or odd values of k and n, the Lie algebras $\mathfrak{t}_{k,n}\overrightarrow{\oplus}\mathfrak{L}_{k,n}$ are rigid with a non-vanishing cohomology, enlarging the well-studied Bratzlavsky series discussed in the literature [11, 23]. For these Lie algebras it can further be shown that the dimension of the cohomology space $H^2\left(\mathfrak{t}_{k,n}\overrightarrow{\oplus}\mathfrak{L}_{k,n}, \mathfrak{t}_{k,n}\overrightarrow{\oplus}\mathfrak{L}_{k,n}\right)$ increases with the value of k. Using the previously mentioned argument of Lemma 5.1 in [23], the rigidity of these algebras as well as that of $\mathfrak{w}_{k,n}$ can be easily proved.

4.3 An algorithm for finding cohomologically rigid algebras with $C_{2,3}^{k+3} = 0$

As already observed for another type of nilradical (see e.g. [7, 9]), for values $k \geq 4$ the complexity of the Jacobi conditions requires the use of symbolic computation packages, and even in these conditions, an exhaustive and complete analysis may not always be possible. In order to circumvent these computational difficulties, some additional constraints on the structure tensor may be imposed, which allow a reduction in the number of possible solutions. In this sense, it is convenient to introduce such constraints on the main diagonal Δ, and inspect whether the corresponding equations of the Jacobi scheme admit conditioned solutions that can be adequately separated.

Still preserving the notations for the generating function in the main diagonal previously considered, but skipping the assumption on the saturation of the nilradical, a possible algorithmic procedure to find additional (cohomologically) rigid Lie algebras of rank one with eigenvalues $(1, k, k, \cdots, n+k-2)$ for fixed n and k, not already covered by the preceding discussion, can be established as follows:

[3]This follows also observing that $\mathfrak{L}_{k,n}$ has no subalgebra of Heisenberg type.

1. Set $C_{1,j}^{j+1} = 1$ for $2 \leq j \leq n-1$.

2. Start with $q = \rho - 1$.

3. In the main diagonal Δ, set $\alpha_{q+1} = 1$ and $\alpha_\ell = 0$ for $2 \leq \ell \leq q$.

4. If the linear equations (4.10) are incompatible, then replace q by $q-1$ and go to step 3.

5. If (4.10) is satisfied, evaluate the system $\mathcal{S}$ formed by the Jacobi identities for the generators $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$.

6. If the system $\mathcal{S}$ admits no solution, then replace q by $q-1$ and go to step 3.

7. If the system $\mathcal{S}$ admits a solution depending on one or more parameters α_i, then replace q by $q-1$ and go to step 3.

8. If the system $\mathcal{S}$ admits an isolated solution, take $\mathfrak{n}$ as the corresponding nilpotent Lie algebra.

9. Compute $H^2\left(\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}, \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}\right)$.

While the procedure is valid to localize cohomologically rigid algebras, for isolated solutions with non-vanishing cohomology it is not conclusive, and further analysis is required to decide on their possible rigidity. Currently no satisfactory computational solution to overcome this difficulty has been found.

As a specific example of the implementation of the preceding algorithm, we sum up the results obtained for the computer analysis of the case $k = 4$, covering all cases subjected to the constraint $C_{23}^6 = 0$, corresponding to the values $q \geq 2$ in the prescription above:

1. For $9 \leq n \leq 17$ we obtain families parameterized by either two or three parameters, so no rigid Lie algebras can exist.

2. for $n = 2p$ with $p \geq 9$ the Jacobi conditions allow two types of solutions, one corresponding to a parameterized family and one isolated.

3. for $n = 2p+1$ with $p \geq 9$ the Jacobi conditions allow three types of solutions, two corresponding to a parameterized family and one isolated.

The isolated solutions, that we again denote by $\mathfrak{n}$, are given in Table 4.1 in terms of the main diagonal Δ, with the remaining commutators being computed from equation (4.12). Only the solutions possessing a vanishing cohomology have been enumerated.

Table 4.1: Cohomologically rigid Lie algebras with $k = 4$ and $C_{23}^6 = 0$ for dimensions $18 \leq n \leq 33$.

$\dim \mathfrak{n}$	$\Delta = (\alpha_1, \dots, \alpha_s)$	$\dim \mathfrak{n}$	$\Delta = (\alpha_1, \dots, \alpha_\rho)$
18	$\left(0^3, 1, \frac{14}{5}, 7\right)$	19	$\left(0^3, 1, \frac{14}{5}, 7, 35\right)$
20	$(0^4, 1, 4, 14)$	21	$(0^4, 1, 4, 14, 98)$
22	$\left(0^5, 1, \frac{27}{5}, \frac{126}{5}\right)$	23	$\left(0^5, 1, \frac{27}{5}, \frac{126}{5}, \frac{1176}{5}\right)$
24	$(0^6, 1, 7, 42)$	25	$(0^6, 1, 7, 42, 504)$
26	$\left(0^7, 1, \frac{44}{5}, 66\right)$	27	$\left(0^7, 1, \frac{44}{5}, 66, 990\right)$
28	$\left(0^8, 1, \frac{54}{5}, 99\right)$	29	$\left(0^8, 1, \frac{54}{5}, 99, 1815\right)$
30	$(0^9, 1, 13, 143)$	31	$(0^9, 1, 13, 143, 3146)$
32	$\left(0^{10}, 1, \frac{77}{5}, \frac{1001}{5}\right)$	33	$\left(0^{10}, 1, \frac{77}{5}, \frac{1001}{5}, \frac{26026}{5}\right)$

We observe that while for even dimensions there are exactly three nonzero entries in the diagonal, in the odd case there are four, with the first three coinciding with those of the immediately preceding dimension. The detailed inspection of each of these sequences allows us to derive a generating function that provides their values in arbitrary dimensions. As an illustration, we derive the sequence of the last entry in the odd dimensional case, the other being deduced in analogous way. Let the sequence $c_9 \rightarrow c_{10} \rightarrow \ c_{11} \rightarrow \cdots$ given by

$$35 \rightarrow 98 \rightarrow \frac{1176}{5} \rightarrow 504 \rightarrow 990 \rightarrow 1815 \rightarrow 3146 \rightarrow \frac{26026}{5} \rightarrow \cdots$$

Comparing the successive factors c_{s+1}/c_s, it is not difficult to establish that these proportions are given by

$$\frac{c_{s+1}}{c_s} = \frac{(s-1)(s-2)}{(s-4)(s-5)}, \quad s \geq 9. \tag{4.48}$$

Expressing each c_q as a function of the preceding terms, we find that for any $q \geq 10$ the identity

$$c_q = \prod_{s=9}^{q-1} \frac{(s-1)(s-2)}{(s-4)(s-5)} c_9 = \frac{(s-2)\,(s-3)^2\,(s-4)^2\,(s-5)}{25200} c_9 \tag{4.49}$$

holds, where $c_9 = 35$. Therefore, $c_q = \frac{1}{6!}\,(s-2)\,(s-3)^2\,(s-4)^2\,(s-5)$. The same argument is valid for the remaining sequences, and we conclude that the main diagonal Δ_n is given respectively by

$$\Delta_{2s} = \left(0^{s-6}, 1, \psi_1(s), \psi_2(s)\right), \quad \Delta_{2s+1} = \left(0^{s-6}, 1, \psi_1(s), \psi_2(s), \psi_3(s)\right), \tag{4.50}$$

where

$$\psi_1(q) = \tfrac{1}{10}\,(s-2)\,(s-5)\,, \qquad \psi_2(q) = \tfrac{1}{5!}\,(s-2)\,(s-3)\,(s-4)\,(s-5)\,,$$
$$\psi_3(q) = \tfrac{1}{6!}\,(s-2)\,(s-3)^2\,(s-4)^2\,(s-5)\,. \tag{4.51}$$

We observe that these nilradicals are far from being saturated, as most of the entries in Δ vanish. Due to this circumstance, rigid Lie algebras of this type will appear only in relatively high dimensions, a fact that may explain why they have not been systematically described yet.

Final remarks

Motivated by previous results in the literature [23, 22, 23], it has been shown that for any $k \geq 2$ and any dimension $n \geq 3k+6$ there exists a cohomologically rigid solvable real Lie algebra $\mathfrak{w}_{k,n}$ of rank one admitting the eigenvalues spectrum $(1, k, k, \cdots, n+k-2)$, with the additional property that the nilradical $\mathfrak{n}_{k,n}$ of $\mathfrak{w}_{k,n}$ is saturated with respect to the torus. This series can be explicitly constructed as a natural generalization of the Lie algebra $\mathfrak{w}_n$ found in [23] by introducing a generating function and a factor sequence that completely characterize the nilradical. Considering also the geometrically rigid algebras that appear jointly with the series $\mathfrak{n}_{k,n}$ as only solutions to the Jacobi conditions for sufficiently high dimensions, the analysis of the case $C_{1,j}^{j+1} \neq 0$ and $C_{2,3}^{k+3} \neq 0$ can be considered complete. While a saturated nilradical is certainly not a sufficient condition to ensure the rigidity of a solvable Lie algebra, we can ask whether for solvable rigid algebras that have a nilradical saturated with respect to the maximal torus $\mathfrak{t}$ the cohomology space H^2 is zero. The ansatz based on a generating function can be extrapolated to an algorithmic procedure that enables one to find cohomologically rigid algebras with the same eigenvalue spectrum, even if the nilradical is no longer saturated. Application of the algorithm has led to a new series of rigid algebras for the value $k = 4$, showing that the rigidity analysis for the tori with spectrum $(1, k, k, \cdots, n+k-2)$ is by no means exhausted. There are still various problems that deserve detailed inspection, from which we enumerate the following:

1. For the case $C_{1,j}^{j+1} = 1$ and $C_{2,3}^{k+3} \neq 0$, obtain the precise identification of all (cohomologically) rigid algebras possessing a nilradical of dimensions $3k+6 \leq n \leq 3k+8$ with $k \geq 4$, just before the number of solutions to the Jacobi conditions stabilizes.

2. A characterization of all isolated (cohomologically) rigid algebras with a non-rational or a purely complex nilradical, which are known to exist for values $k \geq 3$.[4] Can such algebras be described in terms of a (complex) generating function?

3. A systematic analysis of the case $C_{1,j}^{j+1} = 1$ and $C_{2,3}^{k+3} = 0$ for all values $k \geq 5$.

The completion of this program, as well as other secondary questions related to rigid Lie algebras of rank one, will certainly only be possible using symbolic computation methods. Although effective computational procedures have been developed to compute the various cohomologies [1, 24], for geometrically rigid Lie algebras, where the cohomology spaces do not provide a conclusive answer in all cases [11], a computerization of the method based on the Jacobi scheme and the analysis of the nilpotence properties of the parameters (see [3, 14]) would mean a significant impulsion in the study of the rigidity problem. Advances in this direction will hopefully be reported in the near future.

[4] Examples for $k = 3$ can be found in [3, 22], while examples with $k = 4$ are found in [7].

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Capítulo 5

Computer-aided analysis of solvable rigid Lie algebras with a given eigenvalue spectrum

5.1 Introduction

This chapter is dedicated to exposing and developing the results of the article *Computer-aided analysis of solvable rigid Lie algebras with a given eigenvalue spectrum* published in Axioms, in which, with the help of Mathematic program codes, the rigidity of solvable Lie algebras of rank one is studied.
Emerging originally in the context of the deformation theory of complex analytic structures [1, 2], the notion of rigidity for Lie algebras has become increasingly important not only for classification purposes, but also for a geometrical comprehension of the variety $\mathcal{L}^n$ of n-dimensional Lie algebra laws, their irreducible components and the characteristics of points in $\mathcal{L}^n$ [3, 4, 5]. In the mid 1960's, several rigidity criteria based on the Chevalley cohomology were found [6, 7], showing that besides the topological approach to rigidity, a purely algebraic formalism was feasible. Although these approaches are not equivalent, as rigidity does not impose nullity of the Chevalley cohomology [8], this ansatz was crucial for finding large classes of rigid Lie algebras, beyond the well known semisimple and parabolic Lie algebras [9]. First examples of rigid algebras with nontrivial cohomology were found in [6], while in [10] certain obstructions for the integrability of cocycles were inspected in detail, in terms of the so-called Rim quadratic map [11], that provides a sufficiency criterion to ensure the geometrical rigidity. For the case of solvable Lie algebras, the weight systems of maximal tori of derivations (see e.g. [12, 13, 14]) have been shown to be an effective tool to study the rigidity without recourse to cohomological methods, leading to the first systematic procedures or algorithms to construct rigid Lie algebras [15, 16, 17]. In this context, even techniques of Nonstandard Analysis have been shown to be relevant [18, 19]. On the other hand, low dimensional rigid Lie algebras and several hierarchies of solvable rigid Lie algebras in arbitrary dimensions have been classified taking into account the structural properties of their maximal tori of derivations ([20, 21, 22, 23] and references therein), although this approach only solves partially the rigidity problem.[1] However, this procedure has allowed a better comprehension of the nilpotent Lie algebras in terms of the eigenvalue spectra of their torus. So, in [25] the maximal tori of nilpotent Lie algebras with maximal nilpotence index, called filiform, were studied in detail. It follows from this study that among the relevant classes of solvable rigid Lie algebras of rank one, those having a filiform nilradical play a central role, as other classes can be obtained from these by means of special constructions and

[1]It is worthy to be mentioned that this approach has also been used for the analysis of rigid superalgebras [24].

extensions [26, 27]. With the introduction of symbolic computer packages, the computation of cohomologies has been largely simplified [28, 29], allowing not only to obtain wide classes of cohomologically rigid algebras, but also to determine the integrability obstructions with precision and to find new classes of geometrically rigid Lie algebras [14, 30]. The work of R. Carles on the structure theory of rigid Lie algebras and the Jacobi schemes (see [31, 32, 22, 33, 34] and references therein) offered new approaches to the analysis of geometrical properties of the variety $\mathcal{L}^n$, although their practical computational implementation is still a question that has not yet been solved satisfactorily.

In this work we follow the analysis begun in [35], analyzing a question left open in that work, and concerning the cohomological rigidity of rank one solvable rigid Lie algebras with a filiform nilradical, an eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$ for its torus and the additional constraint $C_{2,3}^{k+3} = 0$. Although for low values of $k \leq 4$ various results have been obtained in the literature (see [20, 35, 36]), the general case $k \geq 5$ has not been studied in detail due to computational difficulties and the high number of solutions for the quadratic equations determined by the Jacobi conditions. It is found that, for each k and from a certain dimension m_0 onwards, only one cohomologically rigid Lie algebra exists, while for dimensions $n < m_0$ several isolated rigid algebras appear. In order to analyze these special solutions, the structure of certain types of factor sequences are studied, allowing to determine different types of rigid algebras according to specific structural properties of the associated nilradical.

Unless otherwise stated, any Lie algebra in this work is finite-dimensional and defined over the field of real numbers $\mathbb{R}$.

5.1.1 Generalities

Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{Der}(\mathfrak{g})$ its algebra of derivations, that is, the set of linear maps $D : \mathfrak{g} \to \mathfrak{g}$ such that the Leibniz condition

$$D[X,Y] = [D(X), Y] + [X, D(Y)], \quad X, Y \in \mathfrak{g}.$$

is satisfied. For derivations coinciding with the adjoint action, i.e., $D(Y) = \mathrm{ad}(X)(Y) := [X, Y]$ for some $X \in \mathfrak{g}$, we say that the derivation D is inner, and outer otherwise.

Definition 4 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus (of derivations) is an Abelian subalgebra $\mathfrak{t}$ of $Der(\mathfrak{g})$ such that its elements are semisimple.*

The semisimplicity of a set of outer derivations essentially means that the linear operators $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ are simultaneously diagonalizable over the field $\mathbb{C}$ of complex numbers. Maximal tori $\mathfrak{t}$ in the complex case belong to the same conjugacy class [37], while over the real field this class splits into a finite number of classes. The dimension of a maximal torus is easily seen to define a scalar invariant $\mathrm{r}(\mathfrak{g})$ called the rank of the Lie algebra of $\mathfrak{g}$.

From the structure theory of Lie algebras it is known that any (real or complex) solvable Lie algebra $\mathfrak{r}$ decomposes as the semidirect sum of the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{r}$, called the nilradical, and a linear space $\mathfrak{t}$ formed by linearly nil-independent outer derivations of $\mathfrak{n}$:

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}. \tag{5.1}$$

Further, the following relations hold

$$\begin{gathered} [\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \ [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \ [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \\ \dim \mathfrak{n} - \dim [\mathfrak{n}, \mathfrak{n}] \geq \dim \mathfrak{t}. \end{gathered} \tag{5.2}$$

The latter inequality in particular determines an upper bound for the rank of a solvable Lie algebra [37, 38].

5.1.2 Rigid Lie Algebras and Chevalley cohomology

(Real) Lie algebras of a given dimension n can be seen as points of an algebraic subset $\mathcal{L}^n$ of $\mathbb{R}^{n^3}$ determined by quadratic relations

$$\begin{aligned} C_{ij}^k + C_{ji}^k =& 0, \quad 1 \leq i,j,k \leq n \\ C_{ij}^l C_{lk}^m + C_{jk}^l C_{li}^m + C_{ki}^l C_{lj}^m =& 0, \quad 1 \leq i,j,k,l,m \leq n. \end{aligned} \tag{5.3}$$

The variety is indeed analytic and among its salient topological properties, it is locally piecewise connected, a fact that enables us to study the properties of $\mathcal{L}^n$ by means of deformations [39]. The general linear group $GL(n,\mathbb{R})$ induces a natural action on $\mathcal{L}^n$ given by

$$(f \star \mathfrak{g})(X,Y) = f\left(\left[f^{-1}(X), f^{-1}(Y)\right]_{\mathfrak{g}}\right), \quad f \in GL(n,\mathbb{R}),\ X,Y \in \mathfrak{g}. \tag{5.4}$$

Hence the orbit $\mathcal{O}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ corresponds to Lie algebras isomorphic to $\mathfrak{g}$. As the isotropy group of any point in an orbit corresponds to the automorphism group $\mathrm{Aut}(\mathfrak{g})$, $\mathcal{O}(\mathfrak{g})$ can be seen as the homogeneous space $GL(n,\mathbb{R})/\mathrm{Aut}(\mathfrak{g})$. Using this fact, the identity $\dim \mathcal{O}(\mathfrak{g}) = n^2 - \dim \mathrm{Der}(\mathfrak{g})$ follows immediately. As a submanifold of $\mathbb{R}^{n^3}$, we can consider the closure $\overline{\mathcal{O}}(\mathfrak{g})$ of any orbit with respect to the Euclidean topology.

Definition 5 *A Lie algebra $\mathfrak{g}$ is called rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

It can be shown that an equivalent rigidity notion is obtained by considering the natural Zariski topology of algebraic subsets [40]. It is in this context that the relation with the cohomology of Lie algebras emerges naturally [8]. For a generic point μ in $\mathcal{L}^n$,[2] it was shown in [10] that the following properties hold:

1. The tangent Zariski space to $\mathcal{L}^n$ at the point μ coincides with the space of 2-cocycles $Z^2(\mathfrak{g},\mathfrak{g})$ of the Chevalley cohomology.
2. The tangent space to the orbit $\mathcal{O}(\mathfrak{g})$ coincides with the space of 2-coboundaries $B^2(\mathfrak{g},\mathfrak{g})$ of the Chevalley cohomology.

These properties allow us to analyze the rigidity by means of purely algebraic methods. In fact, only the spaces $H^p(\mathfrak{g},\mathfrak{g})$ for $p \leq 3$ have to be considered, from which a further subdivision of rigid Lie algebras into cohomologically (or algebraically) rigid and geometrically rigid Lie algebras will be deduced. This division actually makes reference to the geometry of singularities in the variety $\mathcal{L}^n$ [5, 10]. The space $H^0(\mathfrak{g},\mathfrak{g})$ corresponds to the centre $Z(\mathfrak{g})$ of the Lie algebra, while $H^1(\mathfrak{g},\mathfrak{g})$ is identified with the space of outer derivations [8].

The notion of deformation in the variety $\mathcal{L}^n$, although expressed analytically in terms of formal series, is essentially a geometric concept relating to paths on $\mathcal{L}^n$ [41, 42, 43, 44]. We define a formal deformation $\mathfrak{g}_t$ of a Lie algebra $\mathfrak{g} = (\mathbb{R}^n, \mu) \in \mathcal{L}^n$ through a deformed commutator

$$[X,Y]_\varepsilon := [X,Y] + \psi_m(X,Y)\varepsilon^m,$$

with ε a real parameter and $\psi_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a skew-symmetric bilinear map. Developing the Jacobi identity up to the quadratic order of ε, we are led to the identity

$$\begin{aligned} &\left[X_i, [X_j,X_k]_\varepsilon\right]_\varepsilon + \left[X_k, [X_i,X_j]_\varepsilon\right]_\varepsilon + [X_j,[X_k,X_i]_\varepsilon]_\varepsilon = \\ &\varepsilon\, d\psi_1(X_i,X_j,X_k) + \varepsilon^2\left(\frac{1}{2}[\psi_1,\psi_1] + d\psi_2\right)(X_i,X_j,X_k) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{5.5}$$

[2] A point of $\mathcal{L}^n$ thus corresponds to a Lie algebra $\mathfrak{g}$ over a given basis.

with $d\psi_l$ $(l=1,2)$ and $[\psi_1,\psi_1]$ defined as

$$\begin{aligned} d\psi_l(X_i,X_j,X_k) := [X_i,\psi_l(X_j,X_k)] + [X_k,\psi_l(X_i,X_j)] + [X_j,\psi_l(X_k,X_i)] + \\ +\psi_l(X_i,[X_j,X_k]) + \psi_l(X_k,[X_i,X_j]) + \psi_l(X_j,[X_k,X_i]) \end{aligned} \tag{5.6}$$

and

$$\frac{1}{2}\left[\psi_1,\psi_1\right](X_i,X_j,X_k) := \psi_1\left(\psi_1(X_i,X_j),X_k\right)+\psi_1\left(\psi_1(X_j,X_k),X_i\right)+\psi_1\left(\psi_1(X_k,X_i),X_j\right), \tag{5.7}$$

respectively. In order to fulfill the Jacobi identity, the constraints

$$d\psi_1(X_i,X_j,X_k) = 0, \tag{5.8}$$

$$\frac{1}{2}\left[\psi_1,\psi_1\right](X_i,X_j,X_k) + d\psi_2(X_i,X_j,X_k) = 0 \tag{5.9}$$

must be satisfied. Equation (5.8) shows that ψ_1 is a 2-cocycle in $H^2(\mathfrak{g},\mathfrak{g})$, while (5.9) expresses an integrability condition (see e.g. [8])). We say that a cocycle φ is integrable whenever the linearly deformed commutator deformation

$$[X,Y]_{\varepsilon} := [X,Y] + \varepsilon\,\varphi(X,Y)$$

satisfies the Jacobi identity and thus defines a Lie algebra. If the space $H^2(\mathfrak{g},\mathfrak{g})$ reduces to zero, than any deformation of $\mathfrak{g}$ is trivial, implying that the Lie algebra is rigid [6, 7, 9, 41, 45]. Algebras such that $H^2(\mathfrak{g},\mathfrak{g}) = 0$ are called cohomologically rigid, and encompass the class of semisimple, Borel subalgebras and parabolic Lie algebras, among others. As shown in [6], another type of rigidity can appear for $H^2(\mathfrak{g},\mathfrak{g}) \neq 0$, whenever the cocycles are non-integrable. This constitutes the geometrical rigidity, quite harder to be established, and that often requires techniques from Algebraic Geometry [5, 10, 11, 31]. We merely indicate that a sufficiency criterion for the geometrical rigidity is given if the Rim map $Sq : H^2(\mathfrak{g},\mathfrak{g}) \to H^3(\mathfrak{g},\mathfrak{g})$ is injective, where

$$Sq\left(\psi\right) = \frac{1}{2}\left[\psi,\psi\right].$$

Within the frame of cohomologically rigid Lie algebras, the Hochschild–Serre factorization plays a central role, as it allows us to restrict the computations to a specific class of cocycles characterized by an invariance property [46]. In the case of solvable real Lie algebras $\mathfrak{r} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}$, where $\mathfrak{t}$ is an Abelian algebra of derivations and the adjoint operators $\mathrm{ad}_{\mathfrak{r}}T$ $(T \in \mathfrak{t})$ are diagonal, this decomposition is given by

$$H^p\left(\mathfrak{r},\mathfrak{r}\right) \simeq \sum_{a+b=p} H^a\left(\mathfrak{t},\mathbb{R}\right) \otimes H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}}, \tag{5.10}$$

where the invariance condition of a b-cochain φ is defined as

$$\left(T.\varphi\right)\left(Z_1,\ldots,Z_b\right) = \left[T,\varphi\left(Z_1,\ldots,Z_b\right)\right] - \sum_{s=1}^{b}\varphi\left(Z_1,\ldots,\left[T,Z_s\right],\ldots,Z_b\right). \tag{5.11}$$

and the space of invariant cocycles as

$$H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}} = \left\{[\varphi] \in H^b\left(\mathfrak{n},\mathfrak{r}\right) \;|\; (T.\varphi) = 0,\ T \in \mathfrak{t}\right\}. \tag{5.12}$$

Applying Künneth-type formulae, it is straightforward to verify that for the values $0 \leq b \leq p$, the condition $H^p\left(\mathfrak{r},\mathfrak{r}\right) = 0$ is equivalent to $H^b\left(\mathfrak{n},\mathfrak{r}\right)^{\mathfrak{t}} = 0$.

For Lie algebras that satisfy $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, the cohomology further satisfies the following inequality

$$\sum_{k=0}^{p} (-1)^{p+k} \dim H^k(\mathfrak{g}, \mathfrak{g}) \geq 0,$$

from which the condition $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g} + \dim H^2(\mathfrak{g}, \mathfrak{g})$ is easily deduced [31].

In particular, for a rank one solvable Lie algebra $\mathfrak{r}$ and $p = 3$ we have

$$\dim H^3(\mathfrak{r}, \mathfrak{r}) - \dim H^2(\mathfrak{r}, \mathfrak{r}) + \dim H^1(\mathfrak{r}, \mathfrak{r}) - \dim Z(\mathfrak{r}) \geq 0$$

It is worthy to be mentioned that it is currently unknown whether there exist rank-one solvable rigid algebras with $Z(\mathfrak{r}) \neq 0$ and $H^1(\mathfrak{r}, \mathfrak{r}) \neq 0$.

5.2 Rigid Lie Algebras with fixed eigenvalue spectrum

Consider a real solvable Lie algebra $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ with a maximal one-dimensional torus of derivations $\mathfrak{t}$ and a filiform nilradical $\mathfrak{n}$. As the torus acts diagonally on the latter, the action can be described in terms of the eigenvalue spectrum $(\lambda_1, \ldots, \lambda_n)$ of $\mathfrak{t}$ over a given basis $\{X_1, \ldots, X_n\}$ of the nilradical $\mathfrak{n}$. The Jacobi identity implies that for any non-vanishing commutator $[X_i, X_j]$, it corresponds to an eigenvector with eigenvalue $\lambda_i + \lambda_j$. In [35] the notion of saturation of nilradicals was introduced and analyzed. In the preceding conditions, we recall that $\mathfrak{n}$ is said to be saturated (w.r.t $\mathfrak{t}$) if the commutator $[X_i, X_j]$ is nonzero whenever the sum of the corresponding eigenvalues is an eigenvalue of the torus, i.e., $\lambda_i + \lambda_j \in \mathrm{spec}(\mathfrak{t})$.

In the following, let us suppose that $n \geq k + 4$ and that the eigenvalue spectrum of $\mathfrak{t}$ is given by

$$\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)\,, \tag{5.13}$$

and let $\{X_1, \ldots, X_n\}$ be an adapted basis of eigenvectors for the adjoint operator $ad\,(T)$. From the Jacobi identity it follows at once that the only nonvanishing commutators in the nilradical $\mathfrak{n}$ are given by

$$\begin{array}{ll} [X_1, X_j] = C_{1,j}^{j+1} X_{j+1}, & 2 \leq j \leq n-1 \\ [X_i, X_j] = C_{i,j}^{i+j+k-2} X_{i+j+k-2}, & 2 \leq i < j;\ i+j+k-2 \leq n. \end{array} \tag{5.14}$$

As we are assuming that the nilradical is filiform, we can suppose without loss of generality that $C_{1,j}^{j+1} = 1$ for $2 \leq j \leq n-1$ [25]. Now the detailed analysis of the Jacobi identity for different triples (i, j, k) of indices (see [35] for details) provides the following constraints on the structure constants $C_{i,j}^k$ of $\mathfrak{n}$:

$$\begin{array}{rl} (1, i, j): & C_{i,j}^{i+j+k-2} - C_{i+1,j}^{i+j+k-1} - C_{i,j+1}^{i+j+k-1} = 0, \quad 2 \leq i < j \leq n-k-4 \\ (1, i, i+1): & C_{i,i+1}^{2i+k-1} - C_{i,i+2}^{2i+k} = 0, \quad 2 \leq i \leq \left[\dfrac{n-k}{2}\right], \\ (1, i, i+s): & C_{i,i+s}^{2i+k+s-2} - C_{i,1+i+s}^{2i+k+s-1} - C_{1+i,i+s}^{2i+k+s-1} = 0, \quad 2i+s \leq n-k+1. \end{array} \tag{5.15}$$

These equations determine a linear system that can be solved recursively for the coefficients $C_{i,j}^{i+j+k-2}$ with $|j - i| > 1$ with respect to the set of diagonal entries

$$\Delta = \left\{ C_{2,3}^{k+3}, C_{3,4}^{k+5}, \ldots, C_{\rho,\rho+1}^{2\rho+k-1} \right\}, \quad \rho = \left[\frac{n+1-k}{2}\right] \tag{5.16}$$

Using these elements as basis for the solution, the remaining structure constants can be obtained as sums

$$C_{i,j}^{i+j+k-2} = \sum_{\ell=2}^{\rho} \lambda_{i,j}^{\ell} C_{\ell,\ell+1}^{2\ell+k-1} \tag{5.17}$$

for appropriate coefficients $\lambda_{i,s}$.[3] For practical reasons, it is convenient to introduce the parameters $\alpha_i = C_{i,i+1}^{2i+k-1}$ for $2 \leq i \leq \rho$ and set

$$\Delta = \{\alpha_2, \ldots, \alpha_\rho\} = \left\{ C_{2,3}^{k+3}, C_{3,4}^{k+5}, \ldots, C_{\rho,\rho+1}^{2\rho+k-1} \right\}. \tag{5.18}$$

This formal simplification shows that the non-vanishing brackets of $\mathfrak{n}$ have the form

$$\begin{array}{l} [X_1, X_j] = X_{j+1},\ 1 \leq j \leq n-1, \\ [X_i, X_{i+1}] = \alpha_i X_{2i+k-1},\ 2 \leq i \leq \rho, \\ [X_i, X_{i+s}] = \sum_{\ell=2}^{\rho} \lambda_{i,s}^{\ell} \alpha_\ell X_{2i+k+s-2}, \quad i \geq 2,\ 2i+k+s-2 \leq n. \end{array} \tag{5.19}$$

The Jacobi identities that correspond to the triples $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$ have still to be evaluated. The result is the quadratic system

$$C_{i,j+\ell+k-2}^{i+j+2k+\ell-4} C_{j,\ell}^{j+\ell+k-2} + C_{i,j}^{i+j+k-2} C_{\ell,i+j+k-2}^{i+j+\ell+2k-4} + C_{i,\ell}^{i+\ell+k-2} C_{j,i+\ell+k-2}^{i+j+\ell+2k-4} = 0, \tag{5.20}$$

which by virtue of equation (5.17) reduces to a quadratic system with respect to the variables $\{\alpha_2, \ldots, \alpha_\rho\}$ of Δ. Any solution of this system determines a nilradical with $\mathfrak{t}$ as torus. For low values of the dimension the solutions are generically parameterized, so that no rigid algebras will emerge. As a general rule, when n is sufficiently high in comparison with the number of equations, isolated solutions occur. Once such isolated solutions have been detected, their rigidity is analyzed either using cohomology or the Jacobi scheme (see [5, 20, 30, 32] for details).

5.3 Rigid algebras with spectrum (5.13) and $C_{2,3}^{k+3} = 0$

The case with $C_{2,3}^{k+3} \neq 0$, studied in detail in [35], was shown to lead to a small number of possibilities that can be analyzed without major computational difficulties, resulting in particular in the existence of some a continuous series of cohomologically rigid Lie algebras. Imposing the constraint $C_{2,3}^{k+3} = 0$, the detailed analysis complicates considerably, leading to a large hierarchy of rigid Lie algebras that satisfy the quadratic equations (5.20). In previous work the cases $k = 2, 3, 4$ have been considered (see [20, 35] and references therein). To handle the case general case $k \geq 5$, an algorithmic procedure was proposed in [35] to systematize the determination of such algebras, that we briefly recall there (see [35] for details). The assumptions are the same as above, i.e., $\mathfrak{r}$ is a rank one Lie algebra with an eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$ for fixed values of k and n.

1. `Set` $C_{1,j}^{j+1} = 1$ `for` $2 \leq j \leq n-1$.
2. `Start with` $q = \rho - 1$.
3. `In the main diagonal` Δ, `set` $\alpha_{q+1} = 1$ `and` $\alpha_\ell = 0$ for $2 \leq \ell \leq q$.
4. `If the linear equations (5.15) are incompatible, then replace` q `by` $q-1$ `and go to step 3.`
5. `If (5.15) is satisfied, evaluate the system` $\mathcal{S}$ `formed by the Jacobi conditions for the triples` $\{X_i, X_j, X_\ell\}$ `with` $2 \leq i < j < \ell$.
6. `If the system` $\mathcal{S}$ `admits no solution, then replace` q `by` $q-1$ `and go to step 3.`

[3] Explicit formulae for these coefficients can be found in [30].

7. `If the system` $\mathcal{S}$ `admits a solution depending on one or more parameters` α_i, `then replace` q `by` $q-1$ `and go to step 3.`
8. `If the system` $\mathcal{S}$ `admits an isolated solution, let` $\mathfrak{n}$ `be the corresponding nilpotent Lie algebra.`
9. `Compute` $H^2\left(\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}, \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}\right)$.

The algorithm is conclusive for the algebraically rigid case. In the case where the cohomology $H^2(\mathfrak{r},\mathfrak{r})$ does not vanish, the rigidity can still be inferred if the solution is isolated. Otherwise, in the presence of more than one solution, alternative approaches like the Jacobi scheme have to be applied [5].

Inserting the constraint $C_{2,3}^{k+3}=0$ into the system (5.17), it is easily seen that among the various linear combinations of structure constants, , the following proportions are obtained:

$$C_{2,4}^{k+4}=0,\quad C_{3,4}^{k+3}=\alpha_3=C_{2,5}^{k+5},\quad C_{3,5}^{k+6}=-C_{2,5}^{k+5},\quad C_{2,6}^{k+6}=2C_{2,5}^{k+5}. \tag{5.21}$$

A particularity of this case that distinguishes it from the saturated case treated in [35] is that one of more entries in the diagonal Δ may be zero, so that additional assumptions must be made. The key observation is that the structure constants $C_{2,p}^{k+p}$ for odd values of $7\leq p\leq n-k$ are not constrained, so that they be used as auxiliary or alternative parameters for solving the quadratic system (5.20). For this reason, for certain cases it is convenient to replace the parameters α_i by the corresponding linear combinations of the structure constants $C_{2,p}^{k+p}$.

Schematically, the commutator table of the nilradical $\mathfrak{n}$ has the following shape (where only the brackets $[X_i,X_j]$ with $j<i$ are shown; non-displayed commutators are either zero or obtained by antisymmetry):

	X_1	X_2	X_3	X_4	$\cdots$	$\cdots$	X_ρ	$X_{\rho+s}$
X_1	0							
X_2	X_3	0						
X_3	X_4	0	0					
X_4	X_5	0	α_3X_{k+5}	0				
X_5	X_6	$C_{2,5}^{k+5}X_{j+k}$	$\vdots$	α_4X_{k+7}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$		
X_j	X_{j+1}	$C_{2,j}^{j+k}X_{j+k}$	$C_{3,j}^{j+k+1}X_{j+k+1}$	$C_{4,j}^{j+k+2}X_{j+k+2}$	$\cdots$	$\cdots$	$\alpha_\rho X_{n-1}$	0
X_{j+1}	X_{j+1}	$C_{2,j+1}^{j+k+1}X_{j+k+1}$	$C_{3,j+1}^{j+k+2}X_{j+k+2}$	$C_{4,j}^{j+k+2}X_{j+k+3}$	$\cdots$	$\cdots$	$\alpha_\rho X_n$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\iddots$	0	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$		0	$\vdots$
X_{n-k-2}	X_{n-k-1}	$C_{2,n-k-2}^{n-2}X_{n-2}$	$C_{3,n-k-2}^{n-1}X_{n-1}$	$C_{4,n-k-2}^{n}X_n$	$\cdots$	$\cdots$	0	0
X_{n-k-1}	X_{n-k}	$C_{2,n-k-1}^{n-1}X_{n-1}$	$C_{3,n-k-1}^{n}X_n$	0	$\cdots$	$\cdots$	0	0
X_{n-k}	X_{n-k+1}	$C_{2,n-k}^{n}X_n$	0	0	$\cdots$	$\cdots$	0	0
X_{n-k+1}	X_{n-k+2}	0	0	0	$\cdots$	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$
X_{n-1}	X_n	0	0	0	$\cdots$	$\cdots$	0	0
X_n	0	0	0	0	$\cdots$	$\cdots$	0	0

where $1\leq s\leq n-\rho$.

In order to illustrate the algorithmic procedure, we consider the detailed analysis of the eigenvalue spectrum for the value $k=8$. The computer analysis for the quadratic Jacobi conditions (5.20) leads to the following results:

1. For $n\leq 26$ we obtain parameterized families with two or more parameters, so that no cohomologically rigid solution is obtained.

2. For $n = 27$, there are seven solutions, but none of the isolated ones has a vanishing cohomology.

3. For $n = 28$ there are six parameterized families and four isolated solutions. Among these, only two have a vanishing cohomology. The main diagonal is given by

$$\begin{aligned} \Delta &= \left\{0^2, 1, \frac{24}{11}, 3, \frac{40}{11}, \frac{51}{11}, \frac{84}{11}, \frac{322}{11}\right\}, \\ \Delta &= \left\{0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}\right\} \end{aligned} \tag{5.22}$$

4. For $n = 29$ there are nine solutions, four of which are isolated, and only two of these have vanishing cohomology. The diagonal is given by

$$\begin{aligned} \Delta &= \left\{0^2, 1, 0^2, \frac{55}{7}, 55, 616, 7535\right\}, \\ \Delta &= \left\{0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}, \frac{180}{143}\right\} \end{aligned}$$

5. For $n = 30$ there are five solutions, from which two are isolated with diagonals given respectively by

$$\begin{aligned} \Delta &= \left\{0^3, 1, \frac{22}{13}, \frac{25}{13}, \frac{25}{13}, \frac{25}{13}, \frac{28}{13}, \frac{42}{13}\right\}, \\ \Delta &= \left\{0^4, 1, 15, 50, \frac{725}{6}, 285, 846\right\} \end{aligned}$$

   Albeit the associated solvable Lie algebras $\mathfrak{r}$ satisfy in both cases $\dim H^2(\mathfrak{r}, \mathfrak{r}) = 1$, they are geometrically rigid, as can be shown by topological arguments (see e.g. [30]).

6. For $n = 31$ there are six solutions, from which two are isolated with diagonals given respectively by

$$\begin{aligned} \Delta &= \left\{0^3, 1, \frac{22}{13}, \frac{25}{13}, \frac{25}{13}, \frac{25}{13}, \frac{28}{13}, \frac{42}{13}, \frac{150}{13}\right\}, \\ \Delta &= \left\{0^4, 1, 15, 50, \frac{725}{6}, 285, 846, 5880\right\} \end{aligned}$$

   As before, $\dim H^2(\mathfrak{r}, \mathfrak{r}) = 1$ for the associated solvable Lie algebras $\mathfrak{r}$. Also in this case, these Lie algebras are geometrically rigid.

7. For $n = 32$ we find four solutions, one isolated. It is cohomologically rigid with diagonal

$$\Delta = \left\{0^4, 1, \frac{30}{13}, \frac{45}{43}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}\right\}$$

8. For $n = 33$ we find five solutions, one isolated. It is cohomologically rigid with diagonal

$$\Delta = \left\{0^4, 1, \frac{30}{13}, \frac{45}{43}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}, \frac{990}{13}\right\}.$$

9. For $n \geq 34$, we obtain at most four solutions, three of them parameterized and one isolated with vanishing cohomology. The precise structure of the resulting diagonal and the associated nilradical $\mathfrak{n}_0$ for this series will be described in the following paragraph.

As follows from this analysis, for given values of k and n there appear nilradicals $\mathfrak{n}$ with a diagonal Δ having quite a different structure. Comparing for instance the two nilradicals $\mathfrak{n}_1$ and $\mathfrak{n}_2$ associated to the diagonals in (5.22), the commutator table has the following form:[4]

$\mathfrak{n}_1$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_3	0								
X_3	X_4	0	0							
X_4	X_5	0	0	0						
X_5	X_6	0	0	X_{15}	0					
X_6	X_7	0	$-X_{15}$	X_{16}	$\frac{24}{11}X_{17}$	0				
X_7	X_8	X_{15}	$-2X_{16}$	$\frac{-13}{11}X_{17}$	$\frac{24}{11}X_{18}$	$3X_{19}$	0			
X_8	X_9	$3X_{16}$	$-\frac{9}{11}X_{17}$	$-\frac{37}{11}X_{18}$	$\frac{-9}{11}X_{19}$	$3X_{20}$	$\frac{40}{11}X_{21}$	0		
X_9	X_{10}	$\frac{42}{11}X_{17}$	$\frac{28}{11}X_{18}$	$\frac{-28}{11}X_{19}$	$\frac{-42}{11}X_{20}$	$\frac{-7}{11}X_{21}$	$\frac{40}{11}X_{22}$	$\frac{51}{11}X_{23}$	0	
X_{10}	X_{11}	$\frac{14}{11}X_{18}$	$\frac{56}{11}X_{19}$	$\frac{14}{11}X_{20}$	$\frac{-35}{11}X_{21}$	$\frac{-47}{11}X_{22}$	$-X_{23}$	$\frac{51}{11}X_{24}$	$\frac{84}{11}X_{25}$	0
X_{11}	X_{12}	$\frac{-42}{11}X_{19}$	$\frac{42}{11}X_{20}$	$\frac{49}{11}X_{21}$	$\frac{12}{11}X_{22}$	$\frac{-36}{11}X_{23}$	$\frac{-62}{11}X_{24}$	$-3X_{25}$	$\frac{84}{11}X_{26}$	$\frac{322}{11}X_{27}$
X_{12}	X_{13}	$\frac{-84}{11}X_{20}$	$\frac{-7}{11}X_{21}$	$\frac{37}{11}X_{22}$	$\frac{48}{11}X_{23}$	$\frac{26}{11}X_{24}$	$\frac{-25}{11}X_{25}$	$\frac{-117}{11}X_{26}$	$\frac{-238}{11}X_{27}$	$\frac{322}{11}X_{28}$
X_{13}	X_{14}	$-7X_{21}$	$-4X_{22}$	$-X_{23}$	$2X_{24}$	$5X_{25}$	$8X_{26}$	$11X_{27}$	$\frac{-560}{11}X_{28}$	0
X_{14}	X_{15}	$-3X_{22}$	$-3X_{23}$	$-3X_{24}$	$-3X_{25}$	$-3X_{26}$	$-3X_{27}$	$\frac{681}{11}X_{28}$	0	0
X_{15}	X_{16}	0	0	0	0	0	$\frac{-714}{11}X_{28}$	0	0	0
X_{16}	X_{17}	0	0	0	0	$\frac{714}{11}X_{28}$	0	0	0	0
X_{17}	X_{18}	0	0	0	$\frac{-714}{11}X_{28}$	0	0	0	0	0
X_{18}	X_{19}	0	0	$\frac{714}{11}X_{28}$	0	0	0	0	0	0
X_{19}	X_{20}	0	$\frac{-714}{11}X_{28}$	0	0	0	0	0	0	0
X_{20}	X_{21}	$\frac{714}{11}X_{28}$	0	0	0	0	0	0	0	0
X_{21}	X_{22}	0								
X_{22}	X_{23}	0								
X_{23}	X_{24}	0								
X_{24}	X_{25}	0								
X_{25}	X_{26}	0								
X_{26}	X_{27}	0								
X_{27}	X_{28}	0								
X_{28}	0	0								

$\mathfrak{n}_2$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_3	0								
X_3	X_4	0	0							
X_4	X_5	0	0	0						
X_5	X_6	0	0	X_{15}	0					
X_6	X_7	0	$-X_{15}$	X_{16}	$\frac{15}{13}X_{17}$	0				
X_7	X_8	X_{15}	$-2X_{16}$	$\frac{-2}{13}X_{17}$	$\frac{15}{13}X_{18}$	$\frac{135}{143}X_{19}$	0			
X_8	X_9	$3X_{16}$	$\frac{-24}{13}X_{17}$	$\frac{-17}{13}X_{18}$	$\frac{30}{143}X_{19}$	$\frac{135}{143}X_{20}$	$\frac{100}{143}X_{21}$	0		
X_9	X_{10}	$\frac{63}{13}X_{17}$	$\frac{-7}{13}X_{18}$	$\frac{-217}{143}X_{19}$	$\frac{105}{143}X_{20}$	$\frac{35}{143}X_{21}$	$\frac{100}{143}X_{22}$	$\frac{75}{143}X_{23}$	0	
X_{10}	X_{11}	$\frac{70}{13}X_{18}$	$\frac{140}{143}X_{19}$	$\frac{-112}{143}X_{20}$	$\frac{-140}{153}X_{21}$	$\frac{-5}{11}X_{22}$	$\frac{25}{143}X_{23}$	$\frac{75}{143}X_{24}$	$\frac{63}{143}X_{25}$	0
X_{11}	X_{12}	$\frac{630}{143}X_{19}$	$\frac{252}{143}X_{20}$	$\frac{28}{143}X_{21}$	$\frac{-75}{143}X_{22}$	$\frac{-90}{143}X_{23}$	$\frac{-50}{143}X_{24}$	$\frac{12}{143}X_{25}$	$\frac{63}{143}X_{26}$	$\frac{70}{143}X_{27}$
X_{12}	X_{13}	$\frac{378}{143}X_{20}$	$\frac{224}{143}X_{21}$	$\frac{103}{143}X_{22}$	$\frac{15}{143}X_{23}$	$\frac{-40}{143}X_{24}$	$\frac{-62}{143}X_{25}$	$\frac{-51}{143}X_{26}$	$\frac{-7}{143}X_{27}$	$\frac{70}{143}X_{28}$
X_{13}	X_{14}	$\frac{14}{13}X_{21}$	$\frac{11}{13}X_{22}$	$\frac{8}{13}X_{22}$	$\frac{5}{13}X_{23}$	$\frac{2}{13}X_{25}$	$\frac{-1}{13}X_{26}$	$\frac{-4}{13}X_{27}$	$\frac{-7}{13}X_{28}$	0
X_{14}	X_{15}	$\frac{3}{13}X_{22}$	$\frac{3}{13}X_{23}$	$\frac{3}{13}X_{24}$	$\frac{3}{13}X_{25}$	$\frac{3}{13}X_{26}$	$\frac{3}{13}X_{27}$	$\frac{3}{13}X_{28}$	0	0
X_{15}	X_{16}	0	0	0	0	0	0	0	0	0
X_{16}	X_{17}									
X_{17}	X_{18}									
X_{18}	X_{19}									
X_{19}	X_{20}									
X_{20}	X_{21}									
X_{21}	X_{22}									
X_{22}	X_{23}									
X_{23}	X_{24}									
X_{24}	X_{25}									
X_{25}	X_{26}									
X_{26}	X_{27}									
X_{27}	X_{28}									
X_{28}	0									

We observe that the main difference between the nilradicals, that are far from being saturated, lies in the fact that from the second column onwards, we either obtain a number of consecutive nonvanishing commutators or a sequence with some zeroes. This suggests to consider the values of the second column as an additional criterion to properly separate the isolated solutions to the equations (5.20). We will inspect the precise description of the cohomologically rigid solutions of the equations (5.20) according to this principle.

5.3.1 Algebras with $C^n_{2,n-k} = 0$

As follows from the equations (5.14), the last index for which the commutator $[X_2, X_j]$ is defined is $j = n-k$. For computational reasons, we analyze the solutions of system (5.20) in dependence of the value $C^n_{2,n-k}$. Imposing that the latter is zero, for any $k \geq 3$ there are isolated solutions,

[4] As before, the non-displayed commutators are either zero or obtained by antisymmetry.

and for $k \geq 8$ the minimal dimension for which such a solution appears is given by $n \geq 3k+4$. With this constraint, there is always an isolated solution for the Jacobi equations (5.20), leading to a new continuous family of cohomologically rigid Lie algebras with a resulting nilradical $\mathfrak{n}_0$ of dimension $n = 3k + 2s + 4$ or $n = 3k + 2s + 5$ ($s \geq -1$), depending on the parity of the dimension. In order to describe the generating functions of the diagonal (5.16) properly, it is convenient to separate the analysis according to the parity of both the dimension n and the index k.

k even, n odd

We define the auxiliar parameter t given by $t = \frac{n-1}{2}$. In this case, the first $t - \frac{3k}{2}$ entries of the main diagonal are zero, and we normalize the first non-vanishing value. Following this normalized value, the k remaining entries in the diagonal are different from zero

$$\Delta = \{\overbrace{0, 0, \ldots, 0}^{t-\frac{3k}{2}}, 1, \phi_1(k,n), \phi_2(k,n), \ldots, \phi_{k-1}(k,n)\}, \tag{5.23}$$

where the generating functions are defined as

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!} \prod_{i=0}^{\ell-1} \frac{(1+2i-3k+n)\,(n-1-2i-k)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-1. \tag{5.24}$$

k even, n even

Here $t = \frac{n}{2}$. As before, the first $t - \frac{3k}{2}$ entries of Δ are zero, so that

$$\Delta = \{\overbrace{0, 0, \ldots, 0}^{t-\frac{3k}{2}}, 1, \phi_1(k,n), \phi_2(k,n), \ldots, \phi_{k-2}(k,n)\}. \tag{5.25}$$

The generating functions are given by

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!} \prod_{i=0}^{\ell-1} \frac{(1+2i-3k+n)\,(n-2i-k)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-2. \tag{5.26}$$

k odd, n odd

The auxiliary parameter is given by $t = \frac{n-1}{2}$, and the diagonal Δ is given by

$$\Delta = \{\overbrace{0, 0, \ldots, 0}^{t-\frac{3k-1}{2}}, 1, \phi_1(k,n), \phi_2(k,n), \ldots, \phi_{k-2}(k,n)\}, \tag{5.27}$$

with generating functions

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!} \prod_{i=0}^{\ell-1} \frac{(n-2i-k)\,(2+2i-3k+n)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-2. \tag{5.28}$$

k odd, n even

The auxiliary parameter is given by $t = \frac{n-2}{2}$, and the diagonal Δ is given by

$$\Delta = \{\overbrace{0, 0, \ldots, 0}^{t-\frac{3k-1}{2}}, 1, \phi_1(k,n), \phi_2(k,n), \ldots, \phi_{k-1}(k,n)\}, \tag{5.29}$$

with generating function

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!} \prod_{i=0}^{\ell-1} \frac{(n-2i-k-1)(1+2i-3k+n)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-1. \tag{5.30}$$

As a particularity of this case, we observe that albeit the isolated solutions for $k=8$ and dimensions $n=30,31$ turn out to be rigid, they are not cohomologically rigid, as in both cases the identity $\dim H^2\left(\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}, \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}\right)=1$ holds. This anomaly has already been observed for other series of cohomologically rigid algebras (see [20, 22, 35] and references therein), and is directly related to the number of quadratic Jacobi equations to be solved. From these special dimension onwards, the solutions tend to stabilize, and cohomologically rigid Lie algebras are obtained.

In the following table, we enumerate the lowest-dimensional solutions for the continuous series for $k \geq 5$ and $n \leq 33$. [5]

Table 5.1: Cohomologically rigid Lie algebras with $C^n_{2,n-k}=0$

n	k	Δ	n	k	Δ
21	5	$\left(0^3, 1, \frac{16}{7}, 4, 8\right)$	29	5	$\left(0^7, 1, \frac{48}{7}, \frac{1188}{35}, \frac{1320}{7}\right)$
22	5	$\left(0^3, 1, \frac{16}{7}, 4, 8, 35\right)$	29	6	$\left(0^5, 1, \frac{11}{3}, \frac{55}{6}, 22, 66, 462\right)$
23	5	$\left(0^4, 1, \frac{45}{14}, \frac{54}{7}, 21\right)$	29	7	$\left(0^4, 1, \frac{5}{2}, \frac{25}{6}, \frac{25}{4}, 10, 21\right)$
24	5	$\left(0^4, 1, \frac{45}{14}, \frac{54}{7}, 21, 126\right)$	29	8	$\left(0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}, \frac{180}{143}\right)$
24	6	$\left(0^3, 1, 2, \frac{20}{7}, 4, 7\right)$	30	5	$\left(0^7, 1, \frac{48}{7}, \frac{1188}{35}, \frac{1320}{7}, \frac{16335}{7}\right)$
25	5	$\left(0^5, 1, \frac{30}{7}, \frac{27}{2}, 48\right)$	30	6	$\left(0^6, 1, \frac{14}{2}, \frac{44}{3}, 44, 165\right)$
25	6	$\left(0^3, 1, 2, \frac{20}{7}, 4, 7, 28\right)$	30	7	$\left(0^4, 1, \frac{5}{2}, \frac{25}{6}, \frac{25}{4}, 10, 21, 105\right)$
25	7	$\left(0^2, 1, \frac{27}{22}, \frac{12}{11}, \frac{10}{11}, \frac{9}{11}, \frac{21}{22}\right)$	31	5	$\left(0^8, 1, \frac{117}{14}, \frac{351}{7}, \frac{4719}{14}\right)$
26	5	$\left(0^5, 1, \frac{30}{7}, \frac{27}{2}, 48, 378\right)$	31	6	$\left(0^6, 1, \frac{14}{2}, \frac{44}{3}, 44, 165, 1452\right)$
26	6	$\left(0^4, 1, \frac{25}{9}, \frac{75}{14}, 10, \frac{70}{13}\right)$	31	7	$\left(0^5, 1, \frac{36}{11}, 7, \frac{40}{3}, 27, 72\right)$
26	7	$\left(0^2, 1, \frac{27}{22}, \frac{12}{11}, \frac{10}{11}, \frac{9}{11}, \frac{21}{22}, \frac{28}{11}\right)$	32	5	$\left(0^8, 1, \frac{117}{14}, \frac{351}{7}, \frac{4719}{14}, \frac{70785}{14}\right)$
27	5	$\left(0^6, 1, \frac{11}{2}, 22, 99\right)$	32	6	$\left(0^7, 1, \frac{52}{9}, \frac{156}{7}, \frac{572}{7}, \frac{7865}{21}\right)$
27	6	$\left(0^4, 1, \frac{25}{9}, \frac{75}{14}, 10, \frac{70}{13}, 126\right)$	32	7	$\left(0^5, 1, \frac{36}{11}, 7, \frac{40}{3}, 27, 72, 462\right)$
27	7	$\left(0^3, 1, \frac{20}{11}, \frac{25}{11}, \frac{200}{77}, \frac{35}{11}, \frac{56}{11}\right)$	32	8	$\left(0^4, 1, \frac{30}{13}, \frac{45}{13}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}\right)$
28	5	$\left(0^6, 1, \frac{11}{2}, 22, 99, 990\right)$	33	5	$\left(0^9, 1, 10, \frac{143}{2}, 572\right)$
28	6	$\left(0^5, 1, \frac{11}{3}, \frac{55}{6}, 22, 66\right)$	33	6	$\left(0^8, 1, 7, \frac{65}{2}, 143, \frac{1573}{2}\right)$
28	7	$\left(0^3, 1, \frac{20}{11}, \frac{25}{11}, \frac{200}{77}, \frac{35}{11}, \frac{56}{11}, \frac{210}{11}\right)$	33	7	$\left(0^6, 1, \frac{91}{22}, \frac{364}{33}, 26, 65, \frac{429}{2}\right)$
28	8	$\left(0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}\right)$	33	8	$\left(0^4, 1, \frac{30}{13}, \frac{45}{13}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}, \frac{990}{13}\right)$

For any k, there exists an integer $m(k)$ such that for $n \geq m(k)$, the only isolated solution for equations (5.20) is given by $\mathfrak{n}_0$, the remaining depending on two or more parameters. For values $k \geq 5$, $m(k)$ is given by[6]

$$m(k) = \begin{cases} 21 & k=5 \\ 4k & k \ \text{even} \\ 4k-1 & k \ \text{odd} \end{cases} \tag{5.31}$$

In each case, several other types of (cohomologically or geometrically) rigid algebras appear in dimensions lower than $m(k)$, the analysis of which requires the imposition of additional constraints. The characteristic fact of the series determined by $\mathfrak{n}_0$ is that it is the only cohomologically rigid solution for which the condition $C^n_{2,n-k}=0$ is satisfied.

[5]The values $k=3,4$ have already been considered in [35].

[6]The value for $k=2$ was given in [22], for $k=3$ in [20] and for $k=4$ in [35].

Concerning the commutators of the resulting nilradical $\mathfrak{n}_0$, the structure constants have the following structure

$$\begin{cases} C_{1,j}^{j+1} = 1 & 2 \leq j \leq n-1 \\ C_{2,j}^{k+j} \neq 0 & 2s+7 \leq j \leq k+s+6 \\ C_{i,j}^{i+j+k-2} \neq 0 & s+9 \leq i+j \leq n-k-2 \\ C_{i,j}^{i+j+k-2} = 0 & 2 \leq i < j,\ j > k+2s+6 \end{cases} \tag{5.32}$$

where, in particular

$$C_{2,k+2s+6}^{2k+2s+6} = C_{3,k+2s+6}^{2k+2s+7} = \cdots = C_{n-2k-2s-5,k+2s+6}^{n-1} = C_{n-2k-2s-4,k+2s+6}^{n}. \tag{5.33}$$

All the remaining values for the constants are recovered using equations (5.17).

5.4 Algebras with $C_{2,n-k}^{n} \neq 0$

Nilradicals satisfying this condition can only appear for dimensions $n < m(k)$. If the commutator $[X_2, X_{n-k}]$ does not vanish, the distinction of the solutions is computationally simplified if we consider the factor sequence (whenever the denominator is nonzero)

$$\frac{C_{2,i}^{k+i}}{C_{2,i+1}^{k+i+1}}, \quad 4 \leq i \leq n-k-1. \tag{5.34}$$

A relevant structural difference with respect to the previous case is that the diagonal (5.16) may have one or more zeroes after the first nonvanishing entry, implying that we do not obtain continuous series of algebras, but only solutions that exist in a certain range of dimensions, as expected. For the same values of k and n, several cohomologically rigid Lie algebras may exist, implying that a generic generating function that describes this type of solutions cannot be found in general. It must also be observed that not all values of k provide cohomologically rigid solutions. The computational analysis of the case $C_{2,n-k}^{n} \neq 0$ shows that it depends essentially on the rest class of k mod 4. We thus introduce the auxiliary parameter p such that $k = 4p$, $k = 4p+1$, $k = 4p+2$ and $k = 4p+3$. Within the range $k \leq 30$ and $n \leq 200$, the analysis of solutions for the constraint $C_{2,n-k}^{n} \neq 0$ leads to three main subcases.

5.4.1 Subcase 1: $C_{2,n-k-1}^{n-1} \neq 0$, $C_{s+r+6,s+r+7}^{2s+2r+k+11} = 0$, $0 \leq r \leq p-1$

For values $k \leq 7$ there is only one solution. For values $k \geq 8$, isolated solutions of this type appear with the diagonal (5.16) adopting the form

$$\Delta(\mathfrak{g}) = \{\overbrace{0,\ldots,0}^{s+3}, 1, \overbrace{0,\ldots,0}^{p}, C_{s+p+6,s+p+7}^{2s+2p+k+11}, \ldots, C_{3p+s+6,3p+s+7}^{6p+2s+k+11}\} \tag{5.35}$$

for even values of n, while

$$\Delta(\mathfrak{g}) = \{\overbrace{0,\ldots,0}^{s+3}, 1, \overbrace{0,\ldots,0}^{p}, C_{s+p+6,s+p+7}^{2s+2p+k+11}, \ldots, C_{3p+s+5,3p+s+6}^{6p+2s+k+9}\} \tag{5.36}$$

if n is an odd number. In both cases, the $3+s$ first entries as well as the p zeroes following the normalized entry are a consequence of the unique solution of the linear system of equations

$$\frac{C_{2,i+s}^{k+i+s}}{C_{2,i+s+1}^{k+i+s+1}} = \frac{i-8-2s}{i-5-s}, \tag{5.37}$$

where $9+2s \leq i \leq 2p+2s+10$. Taking into account the possible values of p and the parity of the dimension of the niltradical $\mathfrak{n}$, the following six possibilities are given:

1. For odd values of n and $k = 4p$, there exists an $n + 1 = (10 + 2s + 10)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq 2p - 4$.
2. For odd values of n and $k = 4p + 1$, there exists an $n + 1 = (10 + 2s + 12)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq 2p - 3$.
3. For odd values of n and either $k = 4p+2$ or $k = 4p+3$, there do not exist cohomologically rigid solutions.
4. For even values of n and either $k = 4p$ or $k = 4p + 3$, there do not exist cohomologically rigid solutions.
5. For even values of n and $k = 4p + 1$, there exists an $n + 1 = (10 + 2s + 13)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq 2p - 3$.
6. For even values of n and $k = 4p + 2$, there exists an $n + 1 = (10 + 2s + 15)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq 2p - 2$.

The following table enumerates the diagonals Δ associated to the lowest-dimensional cohomologically rigid solutions:

Table 5.2: Lowest dimensional solutions for Subcase 1.

$\dim \mathfrak{n}$	k	Δ
24	6	$\{0,0,0,1,0,0,-\frac{36}{7},-24,-189\}$
29	8	$\{0,0,0,1,0,0,\frac{55}{7},55,616,7535\}$
31	9	$\left\{0,0,0,1,0,0,\frac{100}{21},\frac{45}{2},96,\frac{2650}{7}\right\}$
32	9	$\left\{0,0,0,1,0,0,\frac{100}{21},\frac{45}{2},96,\frac{2650}{7},\frac{20340}{7}\right\}$
33	9	$\{0,0,0,0,1,0,0,\frac{65}{6},65,351,3900\}$
34	9	$\{0,0,0,0,1,0,0,\frac{65}{6},65,351,3900,\frac{124605}{2}\}$
34	10	$\{0,0,0,1,0,0,\frac{13}{4},\frac{143}{12},\frac{65}{2},78,\frac{871}{4}\}$
36	10	$\{0,0,0,0,1,0,0,\frac{245}{33},35,126,525,\frac{7421}{3}\}$
38	10	$\{0,0,0,0,0,1,0,0,\frac{490}{33},\frac{945}{11},378,1925,22275\}$
39	12	$\{0,0,0,1,0,0,0,-\frac{675}{187},-\frac{302}{17},-\frac{1075}{17},-\frac{3375}{17},-\frac{12345}{17},-\frac{100380}{17}\}$
41	12	$\{0,0,0,0,1,0,0,0,-\frac{980}{99},-\frac{672}{11},-280,-\frac{4400}{3},-\frac{79450}{9},-\frac{1005760}{9}\}$
43	12	$\{0,0,0,0,0,1,0,0,0,-\frac{3332}{143},-\frac{1904}{11},-952,-5984,-85085,-2167218\}$

5.4.2 Subcase 2: $C_{2,n-k-1}^{n-1} \neq 0$ and $C_{2,n-k-2}^{n-2} \neq 0$.

Solutions subjected to these additional constraints appear for $p \geq 3$, i.e., $k \geq 12$. The diagonal Δ has its $3+s$ first entries equal to zero as in (5.35)-(5.36), but the remaining nonvanishing entries have no easily recognizable pattern, and no zeroes have to appear. The structure constants of the nilradical are characterized by the factor sequence

$$\frac{C_{2,i+s}^{k+i+s}}{C_{2,i+s+1}^{k+i+s+1}} = \frac{i-7-2s}{i+2p-2s-j}, \quad 2p+2 \leq i \leq 4p+6, \tag{5.38}$$

and where j is constrained by p and s depending on the parity of n and the value of k. More precisely:

1. For odd values of n and $k = 4p$, there exists an $n + 1 = (10p + 2s + 10)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq p - 4$ and $8 \leq j \leq 2p - 2s + 1$.

2. For even values of n and $k = 4p$, there do not exist cohomologically rigid solutions.

3. For odd values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+12)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq p-4$ and $8 \leq j \leq 2p-2s+1$.

4. For even values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+13)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq p-3$ and $7 \leq j \leq 2p-2s+1$.

5. For odd values of n and $k = 4p+2$, there do not exist cohomologically rigid solutions.

6. For even values of n and $k = 4p+2$, there exists an $n+1 = (10p+2s+15)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $0 \leq s \leq p-3$ and $7 \leq j \leq 2p-2s+1$.

7. For $k = 4p+3$, there do not exist cohomologically rigid solutions, regardless of the parity on n.

The following table presents the lowest-dimensional cohomologically rigid solutions corresponding to this subcase:

Table 5.3: Lowest dimensional solutions for Subcase 2.

$\dim \mathfrak{n}$	$\boldsymbol{k}$	Δ
40	13	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, -\frac{1998557}{194446}, -\frac{2985595}{97223}, -\frac{6456225}{97223}, -\frac{13282053}{97223}, -\frac{45764301}{194446}, -\frac{43441288}{97223}, -\frac{335042213}{194446}\right\}$
40	13	$\left\{0^2, 1, -1, \frac{1}{5}, 0, -\frac{9317}{2470}, -\frac{27161}{2470}, -\frac{25917}{1235}, -\frac{31782}{1235}, -\frac{58629}{2470}, -\frac{16027}{2470}, \frac{471163}{2470}\right\}$
40	13	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, -\frac{2537}{102}, -\frac{11032}{153}, -\frac{24085}{153}, -\frac{5863}{17}, -\frac{32923}{34}, -\frac{401005}{153}, -\frac{4202891}{306}\right\}$
42	13	$\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{715}{34}, -\frac{1378}{17}, -\frac{26853}{119}, -\frac{73947}{119}, -\frac{520113}{238}, -\frac{2895530}{357}, \frac{-14343043}{238}\}$
42	14	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, -\frac{20119}{969}, -\frac{153853}{2907}, -\frac{282385}{2907}, -\frac{52987}{323}, -\frac{93731}{323}, -\frac{9053}{19}, -\frac{2622697}{2907}\right\}$
42	14	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, -\frac{4745}{578}, -\frac{6136}{289}, -\frac{76121}{2023}, -\frac{115959}{2023}, -\frac{283371}{4046}, -\frac{158285}{2023}, -\frac{353951}{4046}\right\}$
42	14	$\left\{0^2, 1, -1, \frac{1}{5}, 0, -\frac{1001}{380}, -\frac{275977}{45220}, -\frac{90023}{11305}, -\frac{71571}{22610}, \frac{76443}{9044}, \frac{1425457}{45220}, \frac{33407}{340}\right\}$
44	14	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{87035}{5168}, -\frac{73463}{1292}, -\frac{1196673}{9044}, -\frac{2524737}{9044}, -\frac{22311201}{36176}, -\frac{24418735}{18088}, -\frac{43441288}{97223}, -\frac{335042213}{194446}\right\}$
47	16	$\left\{0^2, 1, -5, 7, -4, 1, \frac{21917}{323}, \frac{816326}{3553}, \frac{1923088}{3553}, \frac{377858}{323}, \frac{885456}{323}, \frac{2778490}{323}, \frac{217347689}{7106}, \frac{1588345421}{7106}\right\}$
47	16	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, \frac{27981499}{1069776}, \frac{24341191}{267444}, \frac{57588575}{267444}, \frac{122804099}{267444}, \frac{1093212263}{1069776}, \frac{133386825}{59432}, \frac{2101240531}{356592}, \frac{4052486529}{118864}\right\}$
47	16	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, 0, \frac{748839}{81719}, \frac{2620562}{81719}, \frac{6026233}{81719}, \frac{1076861}{7429}, \frac{1819180}{7429}, \frac{232695697}{572033}, \frac{65841958}{81719}, \frac{267268274}{81719}\right\}$
47	16	$\left\{0^2, 1, -1, \frac{1}{5}, 0^2, \frac{1183}{437}, \frac{15413}{1748}, \frac{74447}{4370}, \frac{36155}{1748}, \frac{24915}{1748}, -\frac{6655}{437}, -\frac{62465}{437}, -\frac{6514053}{4370}\right\}$

5.4.3 Subcase 3: $C^{n-1}_{2,n-k-1} \neq 0$, $C^{n-2}_{2,n-k-2} \neq 0$ and $C^{n-3}_{2,n-k-3} \neq 0$.

In this subcase, the solutions appear for $p \geq 3$, thus $k \geq 13$. The diagonal Δ has its $3+s$ first entries are zero as before, the remaining entries being nonzero. The relevant factor sequence that determines the structure constants of the nilradical is given by

$$\frac{C^{k+i+s}_{2,i+s}}{C^{k+i+s+1}_{2,i+s+1}} = \frac{i-8-2s}{i+2p-2s-j}, \quad 2p+2 \leq i \leq 4p+6, \tag{5.39}$$

with j constrained by s and p. The analysis provides the following cases:

1. For odd values of n and $k = 4p$, there exists an $n+1 = (10p+2s+10)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $-1 \leq s \leq p-4$ and $10 \leq j \leq 2p-2s+1$.
2. For even values of n and $k = 4p$, there do not exist cohomologically rigid solutions.
3. For odd values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+12)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $-1 \leq s \leq p-4$ and $10 \leq j \leq 2p-2s+1$.
4. For even values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+13)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $-1 \leq s \leq p-3$ and $9 \leq j \leq 2p-2s+1$.
5. For odd values of n and $k = 4p+2$, there do not exist cohomologically rigid solutions.
6. For even values of n and $k = 4p+2$, there exists an $n+1 = (10p+2s+15)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $-1 \leq s \leq p-3$ and $9 \leq j \leq 2p-2s+1$.
7. For $k = 4p+3$, there do not exist cohomologically rigid solutions, regardless of the parity on n.

Table 5.4: Lowest dimensional solutions with $n \leq 50$ for Subcase 3.

$\dim \mathfrak{n}$	k	Δ
40	13	$\left\{0, 0, 1, -\frac{9}{2}, 3, -\frac{1}{2}, -\frac{891}{34}, -\frac{5313}{68}, -\frac{2933}{17}, -\frac{12933}{34}, -\frac{36459}{34}, -\frac{245773}{68}, -\frac{879219}{34}\right\}$
42	14	$\left\{0^2, 1, -\frac{9}{2}, 3, -\frac{1}{2}, -\frac{4433}{204}, -\frac{35035}{612}, -\frac{962}{9}, -\frac{18713}{102}, -\frac{22703}{68}, -\frac{44737}{68}, -\frac{1012121}{612}\right\}$
47	16	$\left\{0^2, 1, -8, 10, -4, \frac{1}{2}, \frac{55055}{646}, \frac{95277}{323}, \frac{226686}{323}, \frac{985611}{646}, \frac{2318591}{646}, \frac{7295717}{646}, \frac{30283877}{646}, \frac{140233457}{323}\right\}$
47	16	$\left\{0^2, 1, -\frac{9}{2}, 3, -\frac{1}{2}, 0, \frac{188097}{7429}, \frac{671671}{7429}, \frac{3228417}{14858}, \frac{3492414}{7429}, \frac{7953275}{7429}, \frac{39831759}{14858}, \frac{128959077}{14858}, \frac{487959836}{7429}\right\}$
49	17	$\left\{0^2, 1, -8, 10, -4, \frac{1}{2}, \frac{801944}{11305}, \frac{358917}{1615}, \frac{39561}{85}, \frac{3845109}{4522}, \frac{17478197}{11305}, \frac{70572931}{22610}, \frac{164397571}{22610}, \frac{510384407}{22610}\right\}$
49	17	$\left\{0^2, 1, -\frac{9}{2}, 3, -\frac{1}{2}, 0, \frac{4719}{230}, \frac{144508}{2185}, \frac{304281}{2185}, \frac{109971}{437}, \frac{964304}{2185}, \frac{1738341}{2185}, \frac{28170363}{17480}, \frac{9591842}{2185}\right\} \frac{241548541}{874}, \frac{1393530272}{437}\right\}$
50	17	$\left\{0^2, 1, -\frac{25}{2}, 25, -\frac{35}{2}, 5, \frac{3212359}{14858}, \frac{4851210}{7429}, \frac{10021665}{7429}, \frac{18255630}{7429}, \frac{33456995}{7429}, \frac{69751682}{7429}, \frac{200439395}{7429}, \frac{794583244}{7429}, \frac{7199857252}{7429}\right\}$
52	17	$\left\{0^2, 1, -\frac{9}{2}, 3, -\frac{1}{2}, 0, \frac{4719}{230}, \frac{144508}{2185}, \frac{304281}{2185}, \frac{109971}{437}, \frac{964304}{2185}, \frac{1738341}{2185}, \frac{28170363}{17480}, \frac{9591842}{2185}, \frac{50647275}{1748}\right\}$

5.4.4 Subcase 4: $C^{n}_{2,n-k} \neq 0$ and $C^{n-1}_{2,n-k-1} = 0$

Structurally, this subcase differs from the previous ones. Isolated solutions to the equations (5.20) exist for $p \geq 2$ (hence $k = 8$), implying that $n = 28$ is the lowest dimension for which Lie algebras of this type appear. As before, the first $3 + s$ entries of the diagonal Δ are zero, and again, after the first normalized entry there appear additional zeroes. The structure constants of the nilradical satisfy the relations

$$C^{2k+2s+8}_{2,k+2s+8} = C^{2k+2s+i+6}_{i,k+2s+8}, \quad 3 \leq i \leq n - 2k - 2s - 7. \tag{5.40}$$

Depending on the rest class of k, the following possibilities are given:

1. For $k = 4p$, there exists an $n + 1 = (10p + 2s + 9)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $2p - 6 \leq s \leq 2p - 4$.

2. For $k = 4p + 1$, there exists an $n + 1 = (10p + 2s + 10)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for $2p - 4 \leq s \leq 2p - 2$.

3. For $k = 4p + 2$, there exists an $n + 1 = (10p + 2s + 13)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $2p - 3 \leq s \leq 2p - 1$.

4. For $k = 4p + 3$, there exists an $n + 1 = (10p + 2s + 16)$-dimensional cohomologically rigid Lie algebra $\mathfrak{r}$ for any $2p - 5 \leq s \leq 2p - 3$.

Table 5.5: Lowest dimensional solutions for Subcase 4.

$\dim \mathfrak{n}$	k	Δ
28	8	$\left\{0^2, 1, \frac{24}{11}, 3, \frac{40}{11}, \frac{51}{11}, \frac{84}{11}, \frac{322}{11}\right\}$
29	9	$\left\{0^2, 1, -2, 1, \frac{85}{7}, 37, 112, 714\right\}$
31	9	$\left\{0^3, 1, -1, 0, \frac{475}{42}, 50, 206, 1764\right\}$
33	9	$\left\{0^4, 1, 0, 0, \frac{65}{6}, 65, 351, 3900\right\}$
34	10	$\{0^3, 1, -2, 1, 17, 55, 148, 474, 3462\}$
36	10	$\left\{0^4, 1, -1, 0, \frac{518}{33}, \frac{209}{3}, 243, 993, 9240\right\}$
38	10	$\left\{0^5, 1, 0, 0, \frac{490}{33}, \frac{945}{11}, 378, 1925, 22275\right\}$
35	11	$\left\{0^2, 1, \frac{52}{25}, -\frac{2}{25}, -4, -\frac{17}{2}, -14, -\frac{588}{25}, -\frac{1272}{25}, -\frac{519}{2}\right\}$
37	11	$\left\{0^3, 1, \frac{49}{15}, 3, -1, -\frac{161}{18}, -23, -\frac{267}{5}, -154, -\frac{2079}{2}\right\}$
38	12	$\left\{0^2, 1, \frac{13}{17}, -\frac{26}{17}, -\frac{72}{17}, -\frac{13}{2}, -\frac{287}{34}, -\frac{184}{17}, -\frac{264}{17}, -\frac{1023}{34}, -\frac{4807}{34}\right\}$
40	12	$\left\{0^3, 1, \frac{410}{221}, \frac{32}{221}, -4, -\frac{4333}{442}, -\frac{3850}{221}, -\frac{6450}{221}, -\frac{11886}{221}, -\frac{58971}{442}, -\frac{178673}{221}\right\}$
42	12	$\left\{0^4, 1, \frac{7400}{2499}, 3, -1, -\frac{526}{51}, -\frac{468}{17}, -\frac{7305}{119}, -\frac{51733}{357}, -\frac{378576}{833}, -\frac{414128}{119}\right\}$

5.5 Cohomological rigidity of the families

The proof of the rigidity of the previous solutions follows by application of the Hochschild–Serre factorization. Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ be the rank-one solvable Lie algebra with nilradical $\mathfrak{n}$ possessing the diagonal Δ. For an arbitrary 1-cochain $f \in C^1(\mathfrak{r}, \mathfrak{r})$

$$f(X_i) = \sum_{i=1}^{n} a_i^j X_j + b_i T, \quad 1 \leq i \leq n, \tag{5.41}$$

that is $\mathfrak{t}$-invariant we necessarily have $b_i = 0$. The image by the coboundary operator is given by

$$df(X_1, X_i) = \left(a_1^1 + a_i^j - a_{i+1}^{i+1}\right) X_{i+1}, \quad 2 \leq i \leq n-1,$$
$$df(X_i, X_j) = \left(a_i^i + a_j^j - a_{i+j+k-2}^{i+j+k-2}\right) C_{i,j}^{i+j+k-2} X_{i+j+k-2}, \; 2 \leq i \leq j + n + k - 2, \; i + j \leq n - k + 2.$$

From this we extract a basis of $Z^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}$:

$$f(X_1)=X_1,\ f(X_j)=(j+k-2)\,X_j,\ 2\le j\le n \tag{5.42}$$

corresponding to the linear operator $ad(T)$, hence $Z^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}=B^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}$ and $H^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}=0$. It follows that

$$\dim B^2(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}=n-1. \tag{5.43}$$

Now let $\varphi\in C^2(\mathfrak{n},\mathfrak{n})$ be a generic 2-cochain defined by

$$\varphi(X_i,X_j)=\sum_1^n\alpha_{i,j}^rC_r+\beta_{i,j}T, \tag{5.44}$$

Applying the coboundary operator leads to the identities (k and s are fixed):

$$\begin{aligned}\mathrm{d}\varphi(X_1,X_i,X_j)&=\Big(\alpha_{i,j}^{i+j+k-2}-\alpha_{1,j}^{j+k-1}C_{i,j+1}^{i+j+k-1}-\alpha_{1,i}^{i+1}C_{i+1,j}^{i+j+k-1}\\&\quad-\alpha_{i+1,j}^{i+j+k-1}-\alpha_{i,j+1}^{i+j+k+1}-\alpha_{1,i+j+k-2}^{i+j+k-1}C_{i,j}^{i+j+k-2}\Big)X_{i+j+k-1},\\\mathrm{d}\varphi(X_i,X_j,X_l)&=\Big(\alpha_{j,l}^{j+l+k-2}C_{i,j+l+k-2}^{i+j+2k+l-4}-\alpha_{i,l}^{i+l+k-2}C_{j,i+l+k-2}^{i+j+2k+l-4}\\&\quad+\alpha_{i,j}^{i+j+k-2}C_{l,i+j+k-2}^{i+j+2k+l-4}-\alpha_{l,i+j+k-2}^{i+j+l+2k-4}C_{i,j}^{i+j+k-2}\\&\quad-\alpha_{j,i+l+k-2}^{i+j+l+2k-4}C_{i,l}^{i+l+k-2}+\alpha_{i,j+l+k-2}^{i+j+l+2k-4}C_{j,l}^{j+l+k-2}\Big)X_{i+j+l+2k-4}\end{aligned} \tag{5.45}$$

We distinguish the choice of fundamental parameters of the preceding system according to the various constraints used in the analysis of the different cases:

1. For the nilradicals belonging to the subcase $C_{2,n-k}^n=0$, the generic form of a cocycle is given by

$$\begin{aligned}\varphi(X_i,X_j)=&\Bigg((-1)^s\alpha_{2,7+2s}^{7+k+2s}+\sum_{t=0}^{i-3}\left(-\alpha_{1,2+t}^{3+t}+\alpha_{1,9-i+2s+t}^{10-i+2s+t}\right)\\&+\sum_{t=0}^{i+j-2s-10}\left(\alpha_{1,7+k+2s+t}^{8+k+t+2s}-\alpha_{1,9-i+2s+t}^{10-i+2s+t}\right)\Bigg)C_{i,j}^{i+j+k-2}X_{i,j}^{i+j+k-2},\end{aligned} \tag{5.46}$$

   so that a basis of 2-cocycles is given by the cocycle classes of

$$\alpha_{2,7+2s}^{7+k+2s};\quad\alpha_{1,j}^{j+1},\quad 2\le j\le n-1, \tag{5.47}$$

2. For the remaining cases with $C_{2,n-k}^n\neq 0$, the cocycles adopt the form

$$\begin{aligned}\varphi(X_i,X_j)=&\Bigg((-1)^{s+1}\alpha_{2,9+2s}^{9+k+2s}+\sum_{t=0}^{i-3}\left(-\alpha_{1,2+t}^{3+t}+\alpha_{1,11-i+2s+t}^{12-i+2s+t}\right)\\&+\sum_{t=0}^{i+j-2s-12}\left(\alpha_{1,9+k+2s+t}^{10+k+t+2s}-\alpha_{1,11-i+2s+t}^{12-i+2s+t}\right)\Bigg)C_{i,j}^{i+j+k-2}X_{i,j}^{i+j+k-2}\end{aligned} \tag{5.48}$$

   with basis

$$\alpha_{2,9+2s}^{9+k+2s};\quad\alpha_{1,j}^{j+1},\quad 2\le j\le n-1, \tag{5.49}$$

In both cases $\dim Z^2(\mathfrak{n},\mathfrak{r})^t=n-1$, thus from (5.43) it follows that

$$\dim H^2(\mathfrak{n},\mathfrak{r})^t=0, \tag{5.50}$$

showing the cohomological rigidity of these algebras.

5.5.1 Other isolated cohomologically rigid solutions

Besides the three main types described above, for some values low of k and n, there appear cohomological rigid Lie algebras whose nilradical does not satisfy any of the factor sequences (5.37), (5.38), (5.39), (5.40). This indicates that, from a certain dimension onwards, for a fixed value of k, new factor sequences have to be introduced, in order to obtain new types of solutions. Due to this fact, a full classification is probably not possible, albeit for fixed dimensions, a complete table of the various possible factor sequences can be obtained, eventually allowing an extrapolation to higher dimensions. Another conceivable strategy, that has been partially used in this work for high values of k, is to set a certain number of the parameters α_j in the diagonal (5.18) equal to zero, i.e., imposing additional constraints on the dimension of $C^1(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{n}]$ and inspect the minimal dimension for which a solution to the equations (5.20) depending on only one parameter exists. The cohomology of such isolated solutions has then be computed, allowing to derive cohomologically rigid algebras whose nilradical does not belong to the main types described above. The following table enumerates the solutions for dimensions $n \leq 56$ found by this procedure.

Table 5.6: Other isolated cohomologically rigid solutions

$\dim \mathfrak{n}$	k	Δ
39	12	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{55}{2}, -\frac{878}{7}, -\frac{3117}{7}, -\frac{12891}{7}, -\frac{241989}{14}, -\frac{1439933}{7}\right\}$
40	13	$\left\{0^2, 1, -5, 7, -4, -\frac{20681}{374}, -\frac{5215}{34}, -\frac{5585}{17}, -\frac{12102}{17}, -\frac{67467}{34}, -\frac{446677}{34}, -\frac{39588077}{374}\right\}$
41	13	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{715}{34}, -\frac{1378}{17}, -\frac{26853}{119}, -\frac{73947}{119}, -\frac{520113}{238}, -\frac{2895530}{357}\right\}$
42	13	$\left\{0^3, 1, -\frac{10}{3}, 3, -1, -\frac{4627}{102}, -\frac{8554}{51}, -\frac{7785}{17}, -\frac{63569}{51}, -\frac{147873}{34}, -\frac{615329}{17}, -\frac{42938467}{102}\right\}$
42	14	$\left\{0^2, 1, -5, 7, -4, -\frac{30649}{646}, -\frac{829535}{7106}, -\frac{68915}{323}, -\frac{118338}{323}, -\frac{26169}{38}, -\frac{1121263}{646}, -\frac{32265271}{7106}\right\}$
44	14	$\left\{0^3, 1, -\frac{10}{3}, 3, -1, -\frac{72667}{1938}, -\frac{39494}{323}, -\frac{91287}{323}, -\frac{584903}{969}, -\frac{906257}{646}, -\frac{1413269}{323}, -\frac{30078347}{1938}\right\}$
49	16	$\left\{0^3, 1, -5, 7, -4, 1, \frac{2408}{19}, \frac{110670}{209}, \frac{316738}{209}, \frac{75114}{19}, \frac{212069}{19}, \frac{803374}{19}, \frac{7495085}{19}, \frac{1274187707}{209}\right\}$
51	16	$\left\{0^4, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, 104, 540, \frac{16852}{9}, \frac{17567}{3}, 19734, \frac{802945}{9}, \frac{2989675}{3}, 19951542\right\}$
52	17	$\left\{0^3, 1, -7, 14, -12, 5, \frac{313796}{1311}, \frac{4962456}{5681}, \frac{945708}{437}, \frac{6169163}{1311}, \frac{4490915}{437}, \frac{11168586}{437}, \frac{115303838}{1311}, \frac{332435922}{437}, \frac{5046314533}{437}\right\}$
53	17	$\left\{0^4, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, \frac{5746}{69}, \frac{26860}{69}, \frac{81532}{69}, \frac{640123}{207}, \frac{1668238}{207}, \frac{4934215}{207}, \frac{20201675}{207}, \frac{220088839}{414}\right\}$
54	17	$\left\{0^4, 1, -5, 7, -4, 1, \frac{4269}{23}, \frac{19380}{23}, \frac{57936}{23}, \frac{150163}{23}, \frac{388713}{23}, \frac{1143987}{23}, \frac{4665700}{23}, \frac{48039654}{23}, \frac{942321832}{23}, -\frac{3885024}{23}, -\frac{43575246}{23}, -\frac{105666980678}{2783}\right\}$
54	18	$\left\{0^3, 1, -7, 14, -12, 5, \frac{453389}{2185}, \frac{1523136}{2185}, \frac{385968}{247}, \frac{16988433}{5681}, \frac{2402785}{437}, \frac{23268102}{2185}, \frac{52825188}{2185}, \frac{33833332}{437}, \frac{13043524332}{39767}\right\}$
56	18	$\left\{0^4, 1, -5, 7, -4, 1, \frac{17991}{115}, \frac{14994}{23}, \frac{40392}{23}, \frac{91683}{23}, \frac{198653}{23}, \frac{2260843}{115}, \frac{1206608}{23}, \frac{4553926}{23}, \frac{47612189}{46}\right\}$

5.6 Conclusions

Using symbolic computer packages, a computational analysis of rank one solvable Lie algebras with a filiform nilradical and an eigenvalue spectrum for the torus $\mathfrak{t}$ given by $\text{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$, subjected to the additional constraint $C_{2,3}^{k+3} = 0$ has been carried out, according to the algorithmic procedure proposed in [35], and extending the results of that work to arbitrary values of k. It turns out that this analysis is far more complicated than that corresponding to the case $C_{2,3}^{k+3} \neq 0$, leading to a continuous series of cohomologically rigid Lie algebras, as well as to an increasing number of algebras that only appear for certain values of k and the dimension n of the nilradical. A relevant albeit expected fact is that for the constraints $C_{2,3}^{k+3} = C_{2,n-k}^{n} = 0$, only one rigid Lie algebra appears from a certain dimension onwards, the remaining solutions to the quadratic equations being parameterized families that can be deformed nontrivially. In particular, for dimensions $n > m(k)$ no geometrically rigid subjected to the conditions $C_{2,3}^{k+3} = C_{2,n-k}^{n} = 0$ exist. This case can thus be considered as completely classified. On the contrary, for $C_{2,n-k}^{n} \neq 0$ we only obtain discrete solutions, i.e, rigid algebras that appear merely for certain values of k and n and only within a certain range of dimensions. For a given value of k and n, several structurally different Lie algebras appear, which makes the search of generating functions a hopeless task, at least for the general description. A full classification of this case is probably impossible, as for increasing values of k new types of special solutions that don't follow any of the patterns determined by the factor sequences of the type (5.37), (5.38), (5.39) and (5.40) appear, implying that new factor sequences have to be defined. Albeit the number of such sequences is surely infinite, the analysis could be refined, analyzing for which values of the constants α_j, β_j the rational functions

$$\frac{C_{2,i+s}^{i+k+s}}{C_{2,i+s+1}^{i+k+s+1}} = \frac{\alpha_1 i + \alpha_2 s + \alpha_3}{\beta_1 i + \beta_2 s + \beta_3},\ s \geq 0 \tag{5.51}$$

provide isolated solutions to the quadratic Jacobi conditions (5.20). It is still unclear whether all admissible factor sequences of this type can be described globally in terms of some kind of generating function.

In the preceding discussion, we have restricted to the generic shape of cohomologically rigid Lie algebras. In particular, we have not considered the non-rational (and even purely complex) solutions to the Jacobi equations, which are also known to exist and provide rigid Lie algebras [20]. On the other hand, as expected, there are also various (families of) rigid algebras with varying cohomology, the dimension of which depends on the parameters k and s. As follows from the computational analysis, all these geometrically rigid solutions are subjected to the following two constraints in common

$$C_{2,3}^{k+3} = 0, \quad C_{2,n-k}^{n} \neq 0. \tag{5.52}$$

However, for the algebras having nonzero cohomology the quadratic Rim map (see e.g. [8, 10] and references therein) is not conclusive, so that the rigidity has to be established by other means. While this is cumbersome but realizable for low values of k, the increasing number of isolated solutions with non-vanishing cohomology makes a direct approach quite impractical for values $k > 5$. For these cases, the best method is to analyze the nilpotence conditions in the scheme defined by the Jacobi conditions [5]. However, even this ansatz is not devoid from technical difficulties, as the analysis of nilpotent elements is computationally far from being an easy task. The authors are currently developing an algorithmic method in the language MATHEMATICA® in order to systematize the analysis of geometrically rigid Lie algebras corresponding to the eigenvalue spectrum (5.13), and decide whether the deformation parameters are subjected to some nilpotence condition. We hope to report on some advances in this context in future work.

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Capítulo 6

Construction of rank one solvable rigid Lie algebras with nilradicals of decreasing nilpotence index

6.1 Introduction

In this chapter develops the results obtained in the article *Construction of rank one solvable rigid Lie algebras with nilradicals of decreasing nilpotence index* published in Axioms in 2023, in which the rigidity of one rank solvable Lie algebras with a minimum characteristic sequence is studied, these algebras present a spectrum substantially different from that of previous works, since a new parameter, q, is introduced into it. Using the Maurer-Cartan equations, the structural propietes of nilpotente Lie algebras of dimension N, with anipontency index $n-k$ are determinated.
In spite of the fact that the work of R. Carles on rigid Lie algebras presents a clear picture concerning their generic structural properties [1, 2, 3, 4], unifying previous approaches [5, 6] and establishing a subdivision of rigid algebras into six principal types [2], the problem of classifying and characterizing rigid Lie algebras is far from being solved in satisfactory manner. Although the cohomological tools have been shown to be an effective alternative [7], the existence of a purely geometrical notion of rigidity shows that other procedures, such as the Jacobi schemes [8], must be further developed and refined in order to obtain reliable classifications, even in comparatively low dimensions. Solvable Lie algebras are of special relevance among the rigid ones, as they correspond to a class of algebras that cannot be fully classified beyond low dimensions. In this context, the study of the weight systems of maximal tori of derivations [9, 10, 11] is a powerful technique to analyze the rigidity independently of cohomological tools, and several algorithmic procedures to determine rigid Lie algebras and construct them systematically from the eigenvalue spectra of maximal tori have been developed [12, 13], eventually leading to a classification of low dimensional solvable rigid algebras [14, 15, 16, 17] as well as the discovery of various rigid hierarchies in arbitrary dimension, in both the cohomological and geometrically rigid cases [18, 19, 20]. With the application of symbolic computer packages, further generalizations of some of the previous results have been made possible, as well as the determination of new series of geometrically rigid Lie algebras or the explicit computation of the integrability obstructions that appear in the cohomological approach [21, 23]. In this context, recently various works have been devoted to the systematic analysis and classification of solvable rigid Lie algebras of rank one associated to various types of eigenvalue spectra ([24, 25, 26] and references therein), showing the possibility of a unified description of ample classes of spectra in dependence of one or more parameters by means of generating functions.

In this work we proceed with the study of eigenvalue spectra of one-dimensional tori, but focusing on the construction of rank one solvable cohomologically rigid Lie algebras such that the nilradical $\mathfrak{n}$ has a nilpotence index $\dim\mathfrak{n} - k$ for $k \geq 2$, hence enlarging to lower nilpotent indices some of the constructions and results already known for the filiform case. In particular, we show that for arbitrary integers $k \geq 2$, $q \geq 2k$ and $N \geq k+q+2$ there exists a real solvable Lie algebra of rank one with a maximal torus of derivations $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1\ldots, N+q-k-1)$ such that the nilradical has the nilpotence index $N-k$ and the characteristic sequence $(N-k, 1^k)$. Some possible generalizations of this spectrum analysis are outlined, as well as some comments on the possibility of obtaining geometrically rigid Lie algebras based on them.

Unless otherwise stated, any Lie algebra in this work is finite-dimensional and defined over the field of real numbers $\mathbb{R}$.

6.1.1 General properties of nilpotent Lie algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra. For any $X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]$ we consider the decreasing sequence of dimensions of the Jordan blocks of the adjoint operator $\mathrm{ad}(X)$

$$c(X) = (c_1(X), c_2(X), \cdots, c_k(X), 1), \quad c_i(X) \geq c_{i+1}(X) \geq 1. \tag{6.1}$$

As $c(X)$ constitutes a similarity invariant, it determines an invariant $c(\mathfrak{n})$ defined as

$$c(\mathfrak{n}) = \sup\{c(X) \mid X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]\}. \tag{6.2}$$

and called the characteristic sequence of $\mathfrak{n}$ (see e.g. [18] and references therein). A vector X such that $c(X) = c(\mathfrak{n})$ will be called a characteristic vector of $\mathfrak{n}$. Another invariant is given by the dimensions of the central descending sequence, given recursively by

$$C^0(\mathfrak{n}) = \mathfrak{n},\ C^k(\mathfrak{n}) = \left[\mathfrak{n}, C^{k-1}(\mathfrak{n})\right],\ k \geq 1. \tag{6.3}$$

This sequence further determines the so-called associated graded Lie algebra $\mathfrak{gr}(\mathfrak{n}) = \mathfrak{g}_1(\mathfrak{n}) \oplus \cdots \oplus \mathfrak{g}_r(\mathfrak{n})$ with

$$\mathfrak{g}_k(\mathfrak{n}) = C^{k-1}(\mathfrak{n})/C^k(\mathfrak{n}),\ k \geq 1. \tag{6.4}$$

The Lie algebra $\mathfrak{n}$ is called naturally graded if the isomorphism of Lie algebras $\mathfrak{n} \simeq \mathfrak{gr}(\mathfrak{n})$ holds. We denote by $Der(\mathfrak{n})$ the Lie algebra of derivations of $\mathfrak{n}$, i.e., the space of linear maps $D : \mathfrak{n} \to \mathfrak{n}$ satisfying the condition

$$D([X, Y]) = [D(X), Y] + [X, D(Y)], \quad X, Y \in \mathfrak{n}. \tag{6.5}$$

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus of derivations is any Abelian subalgebra of $Der(\mathfrak{g})$ the generators of which are semisimple.*

Elements in a (maximal) torus are simultaneously diagonalizable in the complex extension of the base field, i.e., $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})$ admit a diagonal matrix over $\mathbb{C}$ for some basis. As shown in [27], maximal tori of the complexified Lie algebra $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}$ are conjugate by an inner automorphism, which implies that their dimension is a scalar invariant of the Lie algebra, commonly referred to as the rank of $\mathfrak{n}$, and denoted by $\mathrm{r}(\mathfrak{n})$.

According to the general structure theory, a real or complex solvable Lie algebra $\mathfrak{r}$ admits the decomposition as semidirect sum

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}, \tag{6.6}$$

satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \tag{6.7}$$

where $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{r}$ (the nilradical) and $\overrightarrow{\oplus}$ denotes the action of $\mathfrak{t}$ on $\mathfrak{n}$ by linearly nil-independent outer derivations. The dimension of $\mathfrak{t}$ is further upper bounded by the following inequality

$$\dim \mathfrak{n} - \dim [\mathfrak{n}, \mathfrak{n}] \geq \dim \mathfrak{t}. \tag{6.8}$$

6.1.2 Solvable rigid Lie algebras

Let $\mathcal{L}^n$ denote the variety of n-dimensional Lie algebras $\mathfrak{g} = \left(\mathbb{K}^n, [\,,\,]_{\mathfrak{g}}\right)$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The general linear group $GL(n, \mathbb{K})$ acts naturally on $\mathcal{L}^n$ by changes of basis:

$$(f \star \mathfrak{g})(X, Y) = f^{-1}\left([f(X), f(Y)]_{\mathfrak{g}}\right), \; f \in GL(n, \mathbb{K}), \; X, Y \in \mathfrak{g}. \tag{6.9}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ is therefore identified with the Lie algebras that are isomorphic to $\mathfrak{g}$.

Definition 2 *A Lie algebra $\mathfrak{g}$ is rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

This definition of rigidity, although mainly topological, admits various equivalent reformulations in analytical or algebraic terms (see e.g. [1, 5, 16]). In this context, using the adjoint cohomology of Lie algebras [7, 28, 29], several criteria to ensure rigidity have been proposed [15, 30, 31]:

Proposition 1 *Let $\mathfrak{g}$ be a Lie algebra. If the condition $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$ holds, then $\mathfrak{g}$ is rigid.*

According to this result, we say that a Lie algebra $\mathfrak{g}$ is cohomologically rigid if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$. This criterion, albeit not necessary for rigidity, has been extremely useful in the analysis of large classes of rigid Lie algebras, and has further allowed a detailed comparison with rigid algebras whose cohomology is not zero. Using the quadratic Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \to H^3(\mathfrak{g}, \mathfrak{g})$ defined by

$$\mathrm{Sq}(\psi)(X_i, X_j, X_k) := \psi(\psi(X_i, X_j), X_k) + \psi(\psi(X_j, X_k), X_i) + \psi(\psi(X_k, X_i), X_j), \tag{6.10}$$

another sufficiency criterion for rigidity was proved in [32, 33]. This criterion states that if Sq is an injective map, then $\mathfrak{g}$ is a rigid Lie algebra.

We also recall briefly the Hochschild-Serre factorization theorem [28, 34], that provides a practical procedure for explicitly computing the cohomology spaces of semidirect sums of Lie algebras. Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ denote a solvable Lie algebra such that $\mathfrak{t}$ is Abelian and the operators $\mathrm{ad}_{\mathfrak{r}} T$ $(T \in \mathfrak{t})$ are diagonal. Then the adjoint cohomology $H^p(\mathfrak{r}, \mathfrak{r})$ satisfies the following isomorphism

$$H^p(\mathfrak{r}, \mathfrak{r}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}}, \tag{6.11}$$

where

$$H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = \left\{[\varphi] \in H^b(\mathfrak{n}, \mathfrak{r}) \mid (T.\varphi) = 0, \; T \in \mathfrak{t}\right\} \tag{6.12}$$

is the space of $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$. Invariance of cocycles is determined by the condition

$$(T.\varphi)(Z_1, \cdots, Z_b) = [T, \varphi(Z_1, \cdots, Z_b)] - \sum_{s=1}^{b} \varphi(Z_1, \cdots, [T, Z_s], \cdots, Z_b). \tag{6.13}$$

Observing that $H^p(\mathfrak{t}, \mathbb{R}) = \bigwedge^p \mathfrak{t}$, it can be easily justified that $H^p(\mathfrak{r}, \mathfrak{r}) = 0$ is equivalent to the identities $H^b(\mathfrak{n}, \mathfrak{r}) = 0$ for $0 \leq b \leq p$. If in addition $\mathfrak{r}$ is a complex solvable rigid Lie algebra, the decomposition theorem of Carles implies that the torus $\mathfrak{t}$ is indeed a maximal external torus of derivations of the nilradical $\mathfrak{n}$ [2].

6.2 Structural properties of the nilpotent Lie algebra $\mathfrak{n}_{N,k}^{0}$

For any $k \geq 1$ and $N \geq 2k+1$ let $\mathfrak{n}_{N,k}^{0}$ be the Lie algebra with nonvanishing commutators

$$\begin{aligned}
&[X_1, X_j] = X_{j+1}, \quad k+1 \leq j \leq N-1, \\
&[X_2, X_j] = X_{j+2}, \quad k+1 \leq j \leq N-2, \\
&\qquad \dots \\
&[X_k, X_j] = X_{j+k}, \quad k+1 \leq j \leq N-k,
\end{aligned} \tag{6.14}$$

over the basis $\mathcal{B} = \{X_1, \dots, X_N\}$. The central descending sequence is given by

$$C^s(\mathfrak{n}_{N,k}^{0}) = \langle X_{k+s}, \dots, X_N \rangle, \quad 2 \leq s \leq N-k; \quad C^{N+1-k}(\mathfrak{n}_{N,k}^{0}) = 0,$$

showing that $\mathfrak{n}_{N,k}^{0}$ is nilpotent with nilpotence index $N-k$. It is straightforward to verify that the characteristic sequence of the Lie algebra is given by $c(\mathfrak{n}_{N,k}^{0}) = (N-k, 1^k)$. We further observe that $\mathfrak{n}_{N,k}^{0}$ is naturally graded only for $k=1$, in which case $\mathfrak{n}_{N,k}^{0}$ is isomorphic to the model filiform Lie algebra L_N [14]. In a certain sense, the algebras defined by (6.14) constitute an extension of the models of Bratzlavsky type (see [10, 23]) to lower characteristic sequences.

For later use, it is convenient to consider the Maurer-Cartan equations of $\mathfrak{n}_{N,k}^{0}$. If $\left\{\omega^1, \dots, \omega^N\right\}$ denotes the dual basis of $\mathcal{B}$, these are given by

$$\begin{aligned}
\mathrm{d}\omega^p &= 0, \quad 1 \leq p \leq k+1, \\
\mathrm{d}\omega^r &= \sum_{a=1}^{r-k-1} \omega^a \wedge \omega^{r-a}, \quad k+2 \leq r \leq 2k \\
\mathrm{d}\omega^s &= \sum_{a=1}^{k} \omega^a \wedge \omega^{s-a}, \quad 2k+1 \leq s \leq N.
\end{aligned} \tag{6.15}$$

If now $\theta = \sum_{\ell=1}^{N} a_\ell \mathrm{d}\omega^\ell \in \mathcal{L}(\mathfrak{n}_{N,k}^{0}) = \mathbb{R}\left\{d\omega_i\right\}_{1\leq i\leq N}$ is a generic linear combination of the 2-forms in (6.15), it is straightforward to verify that

$$\bigwedge^{k} \theta \equiv 0, \quad \bigwedge^{k-1} \theta \neq 0.$$

The quantity

$$j_0\left(\mathfrak{n}_{N,k}^{0}\right) = \text{máx}\left\{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{n}_{N,k}^{0})\right\} = k \tag{6.16}$$

depends only on the structure of $\mathfrak{n}_{N,k}^{0}$, and constitutes a numerical invariant of the Lie algebra [35].[1]

Lemma 1 *For any $k \geq 1$ and $N \geq 2k+1$, the rank of $\mathfrak{n}_{N,k}^{0}$ is two.*

Let $f(X_\ell) = \sum_{s=1}^{N} f_\ell^s X_s$ be the expression of a derivation of $\mathfrak{n}_{N,k}^{0}$. As the centre is generated by X_N, it follows immediately that $f(X_N) = f_N^N X_N$. Evaluation of the derivation condition (6.5) for $X = X_1$, $Y = X_{N-1}$ shows in particular that

$$f(X_{N-1}) = \sum_{s=1}^{k} f_{N-1}^s X_s + f_{N-1}^{N-1} X_{N-1} + f_{N-1}^N X_N.$$

[1] This actually means that $\mathfrak{n}_{N,k}^{0}$ possesses $N-2k$ functionally independent invariants for the coadjoint representation.

Now, computation for the pair $X = X_{k+1}$, $Y = X_{N-1}$ implies that

$$f_{N-1}^{s} = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^{1} = 0.$$

Iterating the computation for the pair $X = X_1$, $Y = X_{N-p}$ (for $N - p \geq k+1$) first shows that $f(X_{N-p}) = \sum_{s=1}^{k} f_{N-p}^{s} X_s + \sum_{q=N-p}^{N} f_{N-p}^{q} X_q$, while evaluation of (6.5) for $X = X_{k+1}$, $Y = X_{N-p}$ successively leads to the conditions

$$f_{N-p}^{s} = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^{p} = 0.$$

From these identities we conclude that $f(X_q) = \sum_{s=q}^{N} f_q^s X_s$ for $q \geq k+1$. Considering now the pair $X = X_m$, $Y = X_{k+1}$ for $m \leq k$, we obtain

$$\sum_{s=k+1}^{N-m} f_{k+1}^{s} X_s - \sum_{s=1}^{m} f_m^s X_s = \sum_{s=m+k+1}^{N} f_{m+k+1}^{s} X_s,$$

from which it follows by iteration on the value of m that $f(X_m) = \sum_{s=m}^{N} f_m^s X_s$, showing that the matrix of f is triangular. In order to compute the semisimple derivations, it therefore suffices to consider a generic diagonal derivation $\Phi(X_i) = \lambda_i X_i$. From the commutators in (6.14) the following relations are easily obtained:

$$\lambda_i + \lambda_j = \lambda_{i+j}, \quad 1 \leq i \leq k, \ k+1 \leq j \leq N-i. \tag{6.17}$$

Considering $i = 1$, it follows for $s \geq 2$ that

$$\lambda_{k+s} = \lambda_1 + \lambda_{k+s-1} = 2\lambda_1 + \lambda_{k+s-2} \cdots = (s-1)\lambda_1 + \lambda_{k+1}.$$

On the other hand, for $1 < i \leq k$ the relation

$$\lambda_i + \lambda_{k+1} = \lambda_{i+k+1} = i\lambda_1 + \lambda_{k+1} \tag{6.18}$$

implies that $\lambda_i = i\lambda_1$. It follows that there exist two diagonalizable derivations F_1 and F_2 with eigenvalues

$$\begin{aligned} \operatorname{spec}(F_1) &= (1, 2, \ldots, k, 0, 1, 2, \ldots, (N-k-1)), \\ \operatorname{spec}(F_2) &= (0, 0, \ldots, 0, 1, 1, 1, \ldots, 1), \end{aligned} \tag{6.19}$$

from which we conclude that the rank of $\mathfrak{n}_{N,k}^{0}$ is two. We denote a maximal torus of $\mathfrak{n}_{N,k}^{0}$ by $\mathfrak{t}_0$.

Let $\mathfrak{r}_0 = \mathfrak{t}_0 \overrightarrow{\oplus} \mathfrak{n}$ be a solvable Lie algebra such that the torus $\mathfrak{t}_0$ is generated by two diagonalizable derivations T_1, T_2 with eigenvalues as given in (6.19). Although it is not essential for the following, using the properties of the root system associated to solvable Lie algebras [13], the rigidity of $\mathfrak{r}_0$ can be shown directly without applying cohomological methods. Thus $\mathfrak{r}_0$ defines a series of rank two solvable rigid Lie algebras for any $k \geq 2$ and $q \geq 2k$. In particular, for $k = 1$ we recover the rigid Lie algebra associated to the model filiform Lie algebra L_n [36]. For higher values of k, the algebra can be seen as the counterpart of the model algebra for characteristic sequences $c(\mathfrak{n}_{N,k}^{0}) = (N-k, 1^k)$.[2]

[2]Incidentally, the algebras $\mathfrak{r}_0$ are actually cohomologically rigid.

6.2.1 Generation of rank one solvable Lie algebras

In this section, we analyze how to derive nilpotent Lie algebras that have rank one using the Lie algebra $\mathfrak{n}_{N,k}^{0}$, and such that the eigenvalues of a maximal torus are given in terms of (6.19). Considering a linear combination $F_1 + qF_2$, we get a diagonal derivation with eigenvalues

$$\mathrm{spec}(F_1 + qF_2) = (1, 2, \ldots, k, q, q+1, q+2, \ldots, (N+q-k-1)), \quad q \neq 0. \tag{6.20}$$

In this context, it can be asked whether, starting from the nilpotent Lie algebra $\mathfrak{n}_{N,k}^{0}$, we can get another nilpotent Lie algebra that is isomorphic to a nontrivial deformation of $\mathfrak{n}_{N,k}^{0}$ and such that it has rank one, with a torus $\mathfrak{t}$ whose eigenvalues are given by (6.20). A first example in this direction was already given in [18] for fixed dimension, where the cohomological rigidity of the solvable Lie algebra $\mathfrak{r}_{q+4,q}$ with commutators

$$\begin{aligned} [T, X_j] &= \mu_j X_j, \\ [X_1, X_j] &= X_{j+1}, \quad 3 \leq j \leq q+3, \\ [X_2, X_j] &= X_{j+2}, \quad 3 \leq j \leq q+2, \\ [X_3, X_4] &= X_{q+4}, \end{aligned} \tag{6.21}$$

with $q \geq 4$, $\mu_1 = 1, \mu_2 = 2$ and $\mu_s = q + s - 3$ for $3 \leq s \leq q+4$, was proved. In this case, the spectrum of the torus generated by T is given by

$$\mathrm{spec}(T) = (1, 2, q, q+1, q+2, \ldots, 2q+1), \quad q \geq 4, \tag{6.22}$$

thus belongs to type (6.20) with $k = 2$ and $N = q+4$. We further observe that the nilradical is isomorphic to the deformation $\mathfrak{n}_{q+4,2}^{0} + \varphi$, where $\varphi(X_3, X_4) = X_{q+4}$ defines a nontrivial cocycle. The addition of this cocycle implies in particular that F_2 cannot be a derivation of the deformed algebra, from which the rank reduction follows.

The Lie algebra (6.21) can actually be seen as the first element in a series of solvable Lie algebras of rank one with vanishing cohomology. To this extent, consider $N \geq q+4$ and the skew-symmetric 2-form φ on $\mathfrak{n}_{N,2}^{0}$ defined by

$$\varphi(X_3, X_j) = X_{q+j}, \quad 4 \leq j \leq N-q. \tag{6.23}$$

It is immediate to verify that φ is a 2-cocycle of $\mathfrak{n}_{N,2}^{0}$. In order to prove that the cohomology class of φ is nonzero, we consider the following 2-form on the (linearly) deformed Lie algebra $\mathfrak{n}_{N,2}^{0} + \varepsilon\varphi$:

$$\theta = \mathrm{d}\omega^N = \omega^1 \wedge \omega^{N-1} + \omega^2 \wedge \omega^{N-2} + \varepsilon\, \omega^3 \wedge \omega^{N-3}.$$

For any $\varepsilon \neq 0$ we have $\theta \wedge \theta \wedge \theta \neq 0$, while for $\varepsilon = 0$ the index of a generic 2-form over $\mathfrak{n}_{N,2}^{0}$ is 2 (see equation (6.16)), showing that both algebras are not isomorphic, hence implying that $[\varphi] \neq 0$.

Let $\mathfrak{n}_{2,q,N} = \mathfrak{n}_{N,2}^{0} + \varphi$. Repeating the argumentation of Lemma 1, it follows at once that any derivation f of $\mathfrak{n}_{2,q,N}$ is triangular. Assuming that f is a diagonal derivation, it satisfies in particular the conditions (6.17) and (6.18) for $k = 2$, so that

$$\begin{aligned} f(X_1) &= \lambda_1 X_1,\ f(X_2) = 2\lambda_1 X_2,\ f(X_3) = \lambda_3 X_3, \\ f(X_j) &= ((j-3)\lambda_1 + \lambda_3)\, X_j, \quad 4 \leq j \leq N. \end{aligned}$$

In addition to these constraints, the condition $f([X_3, X_j]) = [f(X_3), X_j] + [X_3, f(X_j)]$ must be fulfilled, leading to the eigenvalue identities

$$\lambda_3 + (j-3)\lambda_1 + \lambda_3 = (q+j-3)\lambda_1 + \lambda_3, \quad j \geq 4, \tag{6.24}$$

from which $\lambda_3 = q$ follows at once. We conclude that $\mathfrak{n}_{2,q,N}$ has rank one with a maximal torus $\mathfrak{t}$ having eigenvalues as given in (6.20) for $k = 2$.

Proposition 2 *For any $q \geq 4$ and $N \geq q+4$ the solvable Lie algebra $\mathfrak{r}_{2,q,N} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_{2,q,N}$ is rigid with vanishing cohomology $H^2\left(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)$.*

The proof follows by application of the Hochschild-Serre factorization theorem [28]. It is straightforward to verify that any invariant 1-cochain $\varphi \in C^1\left(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)$ has the form

$$\varphi(X_i) = a_i^i X_i, \quad 1 \leq i \leq N.$$

For the coboundary operator we have the nonvanishing entries

$$\begin{aligned}
\mathrm{d}\varphi(X_1, X_j) &= (a_1^1 + a_j^j - a_{1+j}^{j+1})X_{1+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_2, X_j) &= (a_2^2 + a_j^j - a_{2+j}^{j+1})X_{2+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_3, X_j) &= (a_3^3 + a_j^j - a_{q+j}^{q+j})X_{q+j}, \quad j \geq 4,
\end{aligned}$$

from which it follows at once that $\mathrm{d}\varphi = 0$ only if

$$a_2 = 2a_1, \quad a_j = (q + j - 3)a_1, \quad j \geq 3,$$

further showing that $\dim B^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}} = N - 1$. On the other hand, a $\mathfrak{t}$-invariant 2-form has the shape

$$\begin{aligned}
\mathrm{d}\varphi(X_1, X_j) =& b_{1,j}^{j+1} X_{1+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_2, X_j) =& b_{2,j}^{2+j} X_{2+j}, \quad j \geq 3,\\
\mathrm{d}\varphi(X_i, X_j) =& b_{i,j}^{i+j+q-3} X_{i+j+q-3}, \quad j \geq 4,
\end{aligned}$$

Imposing the condition $\mathrm{d}\varphi = 0$ leads to the system of coefficients

$$\begin{aligned}
b_{2,j}^{j+2} - b_{1,j}^{1+j} - b_{2,j+1}^{j+3} + b_{1,j+2}^{j+3} = 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{2,j}^{j+2} - b_{3,j+2}^{j+2+q} + b_{2,j+q}^{j+q+2} = 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{1,j}^{j+1} = 0, \quad j \geq 5.
\end{aligned}$$

The system can be solved recursively. A routine computation shows that, as a basis of independent coefficients, we can choose

$$b_{3,4}^{4+q}, \; b_{2,3}^{5}, \; b_{1,j}^{j+1} \quad \text{for} \quad 3 \leq j \leq N-1,$$

implying that $\dim Z^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}} = N-1$. It follows at once from this identity that $\dim H^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}}$ 0, showing that the algebra is cohomologically rigid.

6.3 The solvable Lie algebras $\mathfrak{r}_{k,q,N}$

As the preceding proof does not essentially depend on the value of k, it is naturally suggested that the result can be easily generalized to nilradicals with characteristic sequence $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$ for arbitrary values $k \geq 3$, by considering the 2-cocycle class of $\mathfrak{n}_{N,k}^0$ defined by

$$\varphi\left(X_{k+1}, X_j\right) = X_{k+q}, \quad k+1 \leq j \leq N-q. \tag{6.25}$$

Consider the Maurer-Cartan equations of $\mathfrak{n}_{N,k}^0 + \varepsilon\varphi$. It is immediate to verify that the 2-form

$$\theta = \mathrm{d}\omega^N = \sum_{p=1}^{k} \omega^p \wedge \omega^{N-p} + \varepsilon\, \omega^{k+1} \wedge \omega^{N-k-1}$$

satisfies the identity $\bigwedge^{k+1}\theta \neq 0$ for any $\varepsilon \neq 0$, showing that the deformation $\mathfrak{n}_{N,k}^{0} + \varepsilon\varphi$ is not isomorphic to $\mathfrak{n}_{N,k}^{0}$. Further, using Lemma 1, the same reasoning as that used in equation (6.24) shows that $\mathfrak{n}_{k,q,N} = \mathfrak{n}_{N,k}^{0} + \varphi$ has rank one with a maximal torus $\mathfrak{t}$ possessing the eigenvalues

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1, q+2, \ldots, q+N-k-1)\,, \quad q \geq 2k. \tag{6.26}$$

In analogy with the previous case, we define the solvable real Lie algebra of rank one $\mathfrak{r}_{k,q,N} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_{k,q,N}$. Over a basis $\{T, X_1, \ldots, X_N\}$ with $N \geq 2q+2-k$, the precise brackets are given by

$$\begin{aligned} [T, X_i] &= i\, X_i, \quad 1 \leq i \leq k \\ [T, X_j] &= (q+j-k-1)X_j, \quad k+1 \leq j \leq N \\ [X_a, X_j] &= X_{j+a}, \quad 1 \leq a \leq k,\ k+1 \leq j \leq N-a, \\ [X_{k+1}, X_j] &= X_{q+j}, \quad k+2 \leq j \leq N-q. \end{aligned} \tag{6.27}$$

Proposition 3 *For any $k \geq 2$, $q \geq 2k$ and $N \geq 2q+2-k$ the solvable Lie algebra $\mathfrak{r}_{k,q,N}$ is cohomologically rigid.*

The proof is completely analogous to that of Proposition 1, for which reason we omit the detailed computations. The results above show that for any $k \geq 2$ and any dimension $N \geq 3k+2$ such Lie algebras exist, with $N = 8$ (for $k = 2$) being the lowest dimension for which an eigenvalue spectrum as given in (6.26) appears. The series $\mathfrak{r}_{k,q,N}$ hence gives a partial answer to a question formulated in [19], namely finding conditions for the existence of rank one rigid Lie algebras such that the nilradical has a given characteristic sequence.

The results above can be slightly refined. Still considering the eigenvalue spectrum (6.20), the requirement $q \geq 2k$ can be relaxed under certain circumstances. Indeed, for the values $k+2 < q < 2k$ and dimension $N = k+q+2$, the nilpotent Lie algebra given by the commutators

$$\begin{aligned} [X_i, X_j] &= X_{i+j}, \quad 1 \leq i \leq k, \quad k+1 \leq j \leq N-i, \\ [X_{k+1}, X_{k+2}] &= X_N \end{aligned} \tag{6.28}$$

with $k \geq 3$ also leads to a rank-one solvable Lie algebras with vanishing cohomology, hence to a rigid Lie algebra. It is straightforward to verify that this case also arises as a deformation of the nilpotent algebras $\mathfrak{n}_{k,q,k+q+2}$. Observe that for $k = 2$, it coincides with the case $q = 2k$, and hence do not provides any additional solution for this value.

In Table 1, we enumerate the spectra of type (6.20) and with $k \geq 3$ and the range of values $k+2 \leq q \leq 2k$ that give rise to a rank-one solvable rigid Lie algebra for dimensions $10 \leq N \leq 20$. A comparison of the dimensions shows that for any dimension $N \geq 10$ there exists at least one solvable rigid Lie algebra satisfying the above constraints. Adding those solutions for which $q \geq 2k$ holds, we conclude that for any fixed dimension $N \geq 10$, there are several nonisomorphic solvable rigid Lie algebras of rank one and eigenvalue spectrum of the type (6.20). In this context, a question that is still unanswered is whether any cohomologically rigid solvable Lie algebra of rank one with the eigenvalue spectrum (6.20) is characterized by the fact that its nilradical is isomorphic to a deformation of the nilpotent Lie algebra $\mathfrak{n}_{N,k}^{0}$.

Table 6.1: Cohomologically rigid Lie algebras with spectrum (6.26) and $k+2 \leq q \leq 2k$.

spec($\mathfrak{t}$)	k	q	N
$(1,2,3,5,6,7,8,9,10,11)$	3	5	10
$(1,2,3,6,7,8,9,10,11,12,13)$	3	6	11
$(1,2,3,4,6,7,8,9,10,11,12,13)$	4	6	12
$(1,2,3,4,7,8,9,10,11,12,13,14,15)$	4	7	13
$(1,2,3,4,8,9,10,11,12,13,14,15,16,17)$	4	8	14
$(1,2,3,4,5,7,8,9,10,11,12,13,14,15)$	5	7	14
$(1,2,3,4,5,8,9,10,11,12,13,14,15,16,17)$	5	8	15
$(1,2,3,4,5,9,10,11,12,13,14,15,16,17,18,19)$	5	9	16
$(1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17)$	6	8	16
$(1,2,3,4,5,10,11,12,13,14,15,16,17,18,19,20,21)$	5	10	17
$(1,2,3,4,5,6,9,10,11,12,13,14,15,16,17,18,19)$	6	9	17
$(1,2,3,4,5,6,10,11,12,13,14,15,16,17,18,19,20,21)$	6	10	18
$(1,2,3,4,5,6,7,9,10,11,12,13,14,15,16,17,18,19)$	7	9	18
$(1,2,3,4,5,6,11,12,13,14,15,16,17,18,19,20,21,22,23)$	6	11	19
$(1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,18,19,20,21)$	7	10	19
$(1,2,3,4,5,6,12,13,14,15,16,17,18,19,20,21,22,23,24,25)$	6	12	20
$(1,2,3,4,5,6,7,11,12,13,14,15,16,17,18,19,20,21,22,23)$	7	11	20
$(1,2,3,4,5,6,7,8,10,11,12,13,14,15,16,17,18,19,20,21)$	8	10	20

6.4 Conclusions

In this work certain results of [10] and [18] concerning rank one solvable rigid Lie algebras have been extended to the case of nilradicals having characteristic sequence $c(\mathfrak{n}) = (n-k, 1^k)$ for arbitrary $k \geq 2$, a one dimensional torus of derivations with eigenvalues (6.26) and dimensions $N \geq k+q+2$. It also solves a subsidiary question formulated in [19], providing minimal dimensions for which rank one rigid Lie algebra with a certain characteristic sequence can appear. The guiding principle has been to consider certain deformations of the nilpotent Lie algebra $\mathfrak{n}^0_{N,k}$ that imply the existence of a unique diagonal derivation, hence guaranteeing that the rank is one. However, this approach merely constitutes one of the multiple possibilities that are conceivable. Rigid algebras structurally analogous but not related to $\mathfrak{n}^0_{N,k}$ can also be constructed along similar lines. We start considering the eigenvalue sequence $\Lambda = (1,2,4,\ldots,9,18,19,\ldots 37)$. A routine computation shows that the 28-dimensional nilpotent algebra defined by

$$\begin{aligned} [X_a, X_j] &= X_{j+a}, \;\; 1 \leq a \leq 2; \quad 9 \leq j \leq 28-a, \\ [X_a, X_j] &= X_{j+a+1}, \;\; 3 \leq a \leq 8; \quad 9 \leq j \leq 27-a, \\ [X_9, X_{10}] &= X_{28} \end{aligned}$$

has rank one, with a maximal torus having the eigenvalues Λ. The corresponding extension of the nilradical by the torus determines a rank one solvable rigid Lie algebra with vanishing cohomology. In contrast to the series derived from $\mathfrak{n}_{N,k}^{0}$, the eigenvalues of Λ are not obtainable as a linear combination of the elements in (6.19), as there are two jumps (i.e. discontinuities) in the eigenvalue sequence.

The preceding example can be generalized to arbitrary dimensions considering the eigenvalue sequence

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1, \ldots, 2q+1, p, p+1, \ldots, 2p+1) \tag{6.29}$$

with $N = k+q+p+4$, $k \geq 2$, $q \geq 2k$ and $p \geq 2q$. The nilpotent Lie algebra defined by the brackets

$$\begin{aligned}
&[X_i, X_j] = X_{i+j}, && 1 \leq i \leq k, && q+k+3 \leq j \leq N-i,\\
&[X_i, X_j] = X_{i+j+q-k-1}, && k+1 \leq i \leq q+k+2, && q+k+3 \leq j \leq N-i-j+q-k,\\
&[X_{q+k+3}, X_{q+k+4}] = X_N
\end{aligned}$$

can be easily verified to be of rank one and to admit the eigenvalues (6.29). The corresponding solvable extension again leads to solvable rigid Lie algebras.

A larger number of jumps in the eigenvalue sequence can be introduced along the same lines. As an example with three jumps, the lowest possible spectrum is given by

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, 4, 5, \ldots, 9, 11, 12, \ldots, 23, 25, 26, \ldots, 51)\,. \tag{6.30}$$

It corresponds to the eigenvalues of a maximal torus of derivations of the 48-dimensional nilpotent Lie algebra

$$\begin{aligned}
[X_i, X_j] =& X_{i+j}, \quad 1 \leq i \leq 2,\ 22 \leq j \leq 48-i,\\
[X_i, X_j] =& X_{i+j+1}, \quad 3 \leq i \leq 8,\ 22 \leq j \leq 47-i\\
[X_i, X_j] =& X_{i+j+1}, \quad 9 \leq i \leq 21,\ 22 \leq j \leq 46-i\\
[X_{22}, X_{23}] =& X_{48},
\end{aligned}$$

and the solvable extension is again cohomologically rigid. Clearly, this case can also be generalized to arbitrary dimension, leading to another family of rank one solvable rigid Lie algebras.

A direct extrapolation to a sequence with $s \geq 2$ jumps would lead to eigenvalue spectra of the type

$$\mathrm{spec}(\mathfrak{t}) = (1, \ldots, k_1, k_2, \ldots, 2k_2+1, k_3, \ldots, 2k_3+1, \ldots, k_{s+1}, \ldots, 2k_{s+1}+1) \tag{6.31}$$

with $k_{m+1} \geq 2k_m$ for $m \geq 1$ and $k_1 \geq 2$. A nilpotent algebra admitting the preceding spectrum would have dimension $N = \sum_{i=1}^{s+1} k_i + 2s$, albeit it is not entirely obvious that the rank is still one, and that the corresponding extension has indeed vanishing cohomology. The problem, which is certainly worthy to be inspected in detail, would require to find a generic nilpotent Lie algebra $\mathfrak{N}$ of rank s that plays the analogue role of $\mathfrak{n}_{N,k}^{0}$, such that the spectrum (6.31) could be obtained as a linear combination (in analogy to equation (6.20)) of the corresponding eigenvalues of the torus generators and nilpotent algebras admitting these eigenvalues as a deformation of $\mathfrak{N}$. Appropriate algorithmic methods are currently being developed to tackle the problem computationally.

On the other hand, from the Jacobi scheme associated to the eigenvalue spectrum (6.26), it follows that a decreasing nilpotence index allows the existence of different characteristic sequences, with the rigidity type (cohomological or geometrical) being deeply related to the particular structure of the characteristic sequence.[3] In other words, the eigenvalue spectrum (6.26) does not uniquely determine the nilradical. Consider for instance the spectrum

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, 4, 5, 6, 7, 8, 9, 10, 11)$$

in dimension $N = 10$. From the Jacobi equations equations we deduce that there exist two nilpotent Lie algebras admitting these eigenvalues. One leads to the rigid solvable algebra $\mathfrak{r}_{2,4,11}$, while the second is given by

$$\begin{aligned}
&[X_1, X_i] = X_{i+1}, && 4 \leq i \leq 9,\\
&[X_3, X_i] = C^8_{3,4} X_{i+4}, && 4 \leq i \leq 5,\\
&[X_3, X_6] = C^{10}_{3,6} X_{10}, &&\\
&[X_4, X_5] = (C^8_{3,4} - C^{10}_{3,6}) X_{10}, &&
\end{aligned} \tag{6.32}$$

where $C^8_{3,4}$ and $C^{10}_{3,6}$ are free. Moreover, for any values of the parameters, the latter nilpotent Lie algebra admits a second diagonal derivation, implying that the rank one solvable extension cannot be rigid.

This shows that, besides considering deformations of a given nilpotent algebra of appropriate rank, there is another more systematic approach, namely studying all nilpotent algebras that admit a certain one-dimensional torus. This is essentially the same as studying the Jacobi scheme, and a systematic analysis of the Jacobi equations would lead to a classification of all algebras admitting a diagonal derivation of a specific type. The drawback of this ansatz is that for each of the obtained solutions, it must be analyzed separately whether the rank is one or higher, and in the former case, the potential rigidity (either cohomological or geometrical) must also be considered case by case. The problem is of interest, but demands the implementation of adequate algorithms to appropriately separate the solutions.

We already mentioned that the spectrum may lead to Lie algebras that, albeit being rigid, are not cohomologically rigid. As an example that illustrates how geometrically rigid Lie algebras arise in this context, consider $k = 3$, $q = 8$, $N = 13$ and the torus $\mathfrak{t}$ with eigenvalue spectrum $(1, 2, 3, 8, 9, \ldots, 17)$. The nilpotent Lie algebra $\mathfrak{m}$ given by

$$\begin{aligned}
&[X_1, X_2] = X_3, &&\\
&[X_1, X_j] = X_{j+1}, && 4 \leq j \leq 12,\\
&[X_2, X_j] = X_{j+2}, && 4 \leq j \leq 11,\\
&[X_4, X_5] = X_{13} &&
\end{aligned} \tag{6.33}$$

admits $\mathfrak{t}$ as a maximal torus of derivations. The corresponding solvable extension $\mathfrak{R} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{m}$ has a one-dimensional adjoint cohomology space, generated by the cocycle class ψ defined by

$$\begin{aligned}
&\psi(X_2, X_j) = (j-4) X_{j+2}, && 5 \leq j \leq 11,\\
&\psi(X_3, X_j) = -X_{j+3}, && 4 \leq j \leq 10.
\end{aligned} \tag{6.34}$$

Although this cocycle is not integrable, using the Rim map (6.10) it can be easily verified that

$$\mathrm{Sq}(\psi)(X_2, X_3, X_5) = 3X_{10} \neq 0,$$

from which we deduce that $\mathrm{Sq} : H^2(\mathfrak{R}, \mathfrak{R}) \to H^3(\mathfrak{R}, \mathfrak{R})$ is injective. Following the criterion in [33], $\mathfrak{R}$ is rigid with nonvanishing cohomology. It is worthy to be observed that, as happened for

[3]This phenomenon cannot occur for filiform algebras, as these correspond to the maximal possible nilpotence index.

the filiform case, a same eigenvalue spectrum can lead to either cohomologically or geometrically rigid Lie algebras depending on the dimension of the nilradical (see e.g. [4, 12, 23, 26]). The interesting fact that distinguishes this type of eigenvalue spectrum from those associated to filiform algebras is that $\mathfrak{m}$ has characteristic sequence $c(\mathfrak{m}) = (10, 2, 1)$, and the natural question that arises is whether it is the lowest dimensional hierarchy of a series that generalizes recent constructions of geometrically rigid algebras (as that e.g. proposed in [23]) to characteristic sequences of the type $c(\mathfrak{m}) = (c_1, c_2, 1^{c_1+c+2})$. In a more wide context, it can be asked what conditions must be satisfied by the elements of a sequence of integers $\{c_1, \ldots, c_s\}$ in order to imply the existence of a nilradical with characteristic sequence $(c_1, \ldots, c_s, 1^{s+1})$ associated to a rigid Lie algebra of rank one. A complete answer to this question will probably require the use of symbolic computer packages, due to the relatively high dimensions and the number of solutions of the Jacobi equations involved.

Summarizing, there are various potential ways to generalize the results of this work to wider classes of spectra, leading to solvable Lie algebras with nilradicals of varying characteristic sequence, either searching for appropriate nilpotent algebras that serve as "model.ªnd studying their derivations, or by a direct approach using the Jacobi equations, given an eigenvalue spectrum. Although computationally cumbersome, a complete classification of rank-one solvable Lie algebras of this type, up to a given dimension, is conceivable, and may lead to a further understanding of rigidity for rank one Lie algebras. Work in these various directions is currently in progress.

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Apéndice A

I Código para el programa Mathematica

En este apéndice se muestra el código para el programa Mathematica que se ha utilizado para cálculo de la dimensión del segundo grupo de cohomología para álgebras de rango 1 por medio de la factorización de Hochschild-Serre, a continuación se expone un ejemplo de la ejecución de dicho código para el álebra $\mathfrak{Br}_{4,8}$.
Antes de ejecutar el código que se describe a continuación, se debe definir el correspondiente álgebra de Lie por medio de sus corchetes, para ello es necesario definir dos matrices, una la relación del toro maximal con el nilradical y la otra la de los corchetes entre los elementos del nilradical. Para un espectro dado del álgebra, se comienza definiendo la matriz del toro, el parámetro **NN** es la dimensión del nilradical.

```
MatPeso = Flatten[Normal[SparseArray[{{1, j_}/;1 ≤ j ≤ NN → λ[j]}, {1, NN}]]]
```

Donde $\lambda[j]$ es el peso correspondiente al vector $X[j]$.
A continuación se establece el sistema de pesos:

```
Peso(i_):=MatPeso[[i]];
m(i_, j_, k_):=KroneckerDelta [Peso[[i]] + Peso[[j]], Peso[[k]]];
MatToro = Normal[SparseArray[{{1, j_} → MatPeso[[j]]X(j)}, {1, NN}]];
```

Matriz de los corchetes del nilradical:

```
MatN = Normal[SparseArray[{{j_, i_}/;1 ≤ i ≤ NN/;j > i/;2 ≤ j ≤ NN →
c[i, j, k]Sum[a[i, j, k]m[i, j, k]X[k], {k, 1, NN}]}, {NN, NN}]];
```

Donde $c[i,j,k]$ es la constante de estructura correspondiente al corchete $[X_i, X_j]$ en el vector X_k.
En el siguiente paso se completa la matriz del nilradical con la parte antisimétrica.

```
MatNS = −Transpose[MatN];
MatNilradical = MatN + MatNS;
```

A continuación se establecen las condiciones que han de cumplir los corchetes:

```
K[0, X[i_]]:=0;
K[X[i_], 0]:=0;
K[T[l_], 0]:=0;
K[X[i_], X[i_]]:=0;
K[X[i_], X[j_]]:= − K[X[j], X[i]];
K[a_X[i_], X[k_]]:=aK[X[i], X[k]];
K[X[i_], a_X[k_]]:=aK[X[i], X[k]];
K[a_X[i_] + b_X[j_], X[k_]]:=aK[X[i], X[k]] + bK[X[j], X[k]];
```

```
K[X[i_], a_X[j_] + b_X[k_]]:=aK[X[i], X[j]] + bK[X[i], X[k]];
K[X_ + Y_, Z_]:=K[X, Z] + K[Y, Z]
K[X_, Y_ + Z_]:=K[X, Y] + K[X, Z]
K[T_, a_X_]:=aK[T, X]
K[a_T_,  X_]:=aK[T, X]
```

I.1 Definición de los 1-cobordes invariantes y sus propiedades:

```
CB[X[i_]]:=If[i<=NN, z[i, i]X[i], 0]
COB = Flatten[Table[CB[X[i]], {i, 1, NN}]];
CAR = Table[z[i, i], {i, 1, NN}];
```

I.2 Operador coborde.

```
B3[i_, j_]:=K[X[i], CB[X[j]]] − K[X[j], CB[X[i]]] − CB[K[X[i], X[j]]]
CB[a_X_]:=aCB[X];
CB[0]:=0;
CB[K[X_ + Y_, Z_]]:=CB[K[X, Z]] + CB[K[Y, Z]];
K[CB[X_ + Y_], W_]:=K[CB[X], W] + K[CB[Y], W];
K[W_, CB[X_ + Y_]]:=K[W, CB[X]] + K[W, CB[Y]];
```

Propiedad de invarianza de las cocadenas.

```
CB[T[l_]] = 0;
C3T = Table[B3[i, j], {i, 1, NN}, {j, i + 1, NN}];
C3T1 = DeleteCases[Flatten[C3T], 0]/.X[i_] → 1;
SOO = Flatten[Solve[C3T == 0, CAR]];
JJJ = CAR/.SOO;
SO3 = Solve[JJJ == 0];
LOL = C3T/.SO3;
POL = Solve[LOL == 0, CAR];
UIP = Flatten[Keys[POL]];
```

La dimensión de B^2 viene dada por:

```
Length[UIP]
```

2-cocadenas invariantes.

Aquí se va a buscar los cobordes del álgebra. Para ello se define el operador coborde y sus propiedades, se buscan las 2-cocadenas que van a cero al aplicar el operador coborde.

Relaciones generales

```
Ψ[X_ + Y_, Z_]:=Ψ[X, Z] + Ψ[Y, Z]
Ψ[X_, Y_ + Z_]:=Ψ[X, Y] + Ψ[X, Z]
Ψ[K[X_ + Y_, Z_], W_]:=Ψ[K[X, Z], W] + Ψ[K[Y, Z], W]
Ψ[W_, K[X_ + Y_, Z_]]:=Ψ[W, K[X, Z]] + Ψ[W, K[Y, Z]]
K[Ψ[X_ + Y_, Z_], W_]:=K[Ψ[X, Z], W] + K[Ψ[Y, Z], W]
K[W_, Ψ[X_ + Y_, Z_]]:=K[W, Ψ[X, Z]] + K[W, Ψ[Y, Z]]
```

$\Psi[T[\text{l\_}], X[\text{i\_}]]$:=0
$\Psi[X[\text{i\_}], T[\text{l\_}]]$:=0
$\Psi[X[\text{i\_}], X[\text{i\_}]]$:=0
$\Psi[0, X[\text{i\_}]]$:=0
$\Psi[X[\text{i\_}], 0]$:=0
$\Psi[\text{a\_X\_}, \text{Z\_}]$:=$a * \Psi[X, Z]$
$\Psi[\text{Z\_}, \text{a\_X\_}]$:=$a * \Psi[Z, X]$
$c[\text{i\_}, \text{i\_}, \text{j\_}] = 0;$
Definición de 2-cocadenas invariantes por el toro
$c[\text{i\_}, \text{j\_}, \text{p\_}]$:=If[EW[[$i$]] + EW[[$j$]]===EW[[$p$]], Signature[$\{i, j\}$] $*$ q[Sort[$\{i, j\}$][[1]],
, Sort[$\{i, j\}$][[2]], p], 0] $\Psi[X[\text{i\_}], X[\text{j\_}]]$:=Sum[$c[i, j, p]X[p]$, $\{p, 1, \text{NN}\}$]
DERF[l_, i_, j_]:=$K[T[l], \Psi[X[i], X[j]]] - K[X[i], \Psi[T[l], X[j]]] + K[X[j], \Psi[T[l], X[i]]] -$
$-\Psi[K[T[l], X[i]], X[j]] + \Psi[K[T[l], X[j]], X[i]] - \Psi[K[X[i], X[j]], T[l]]$
Condición de las 2-cocadenas invariantes
Matriz de las 2-cocadenas t-invariantes
KOCI = Table[$\Psi[X[i], X[j]]$, $\{i, 1, \text{NN}\}$, $\{j, i+1, \text{NN}\}$];
NCOC = DeleteCases[Flatten[KOCI], 0]/.$X[\text{i\_}] \to 1$;
DERC[l_, i_, j_]:=$K[X[l], \Psi[X[i], X[j]]] - K[X[i], \Psi[X[l], X[j]]] + K[X[j], \Psi[X[l], X[i]]] -$
$\Psi[K[X[l], X[i]], X[j]] + \Psi[K[X[l], X[j]], X[i]] - \Psi[K[X[i], X[j]], X[l]]$
SOCO = Table[DERC[i, j, l], $\{i, 1, \text{NN}\}$, $\{j, i+1, \text{NN}\}$, $\{l, j+1, \text{NN}\}$];
SOTO = DeleteCases[Flatten[SOCO], 0]/.$X[\text{i\_}] \to 1$;
LR = Flatten[Solve[SOTO == 0, NCOC]];
SL = KOCI/.LR;
KOKO = KOCI/.LR;
PR = Flatten[Solve[SL == 0, NCOC]];
PRR = Keys[PR];
La dimensión de Z^2 es:
Length[PR]

La dimensión del segundo grupo de cohomología es:
DIH2 = Length[PR] − Length[UIP]

I.3 Ejemplo: Implementación del código.

En esta sección se muestra un ejemplo de cómo se procede con el código de la sección anterior con un álgebra $\mathfrak{Br}_{4,8}$ definida y estudiada en el primer artículo.
Dicha álgebra está definida por los corc,hetes:

$$\begin{array}{ll} [X_1, X_j] = X_{j+1}, & 2 \leq j \leq n-1, \\ [X_2, X_j] = X_{j+4}, & 3 \leq j \leq 4, \\ [T, X_1] = X_1, \ [T, X_i] = (i+2)X_i, & 2 \leq i \leq 8. \end{array} \quad (1)$$

Antes de calcular la cohomología hay que definir el sistema de pesos y los corchetes de dicho álgebra.

Definición de los 1-cobordes invariantes y sus propiedades:

CB[$X[\text{i\_}]$]:=If[i<=NN, $z[i, i]X[i]$, 0]
COB = Flatten[Table[CB[$X[i]$], $\{i, 1, \text{NN}\}$]]
$\{X[1]z[1,1], X[2]z[2,2], X[3]z[3,3], X[4]z[4,4], X[5]z[5,5], X[6]z[6,6], X[7]z[7,7], X[8]z[8,8]\}$

CAR = Table[$z[i,i]$, $\{i, 1, \mathrm{NN}\}$]
{z[1,1],z[2,2],z[3,3],z[4,4],z[5,5],z[6,6],z[7,7],z[8,8]}
Relaciones de los cobordes:
B3[i_, j_]:=$K[X[i], \mathrm{CB}[X[j]]] - K[X[j], \mathrm{CB}[X[i]]] - \mathrm{CB}[K[X[i], X[j]]]$
CB[a_X_]:=aCB[X];
CB[0]:=0;
CB[K[X_ + Y_, Z_]]:=CB[$K[X, Z]$] + CB[$K[Y, Z]$];
K[CB[X_ + Y_], W_]:=K[CB[X], W] + K[CB[Y], W];
K[W_, CB[X_ + Y_]]:=K[W, CB[X]] + K[W, CB[Y]];
CB[T[l_]] = 0;
Tabla de los 2-cobordes t-invariantes:
C3T = Table[B3[i, j], $\{i, 1, \mathrm{NN}\}$, $\{j, i+1, \mathrm{NN}\}$]
{{X[3] z[1,1]+X[3] z[2,2]-X[3] z[3,3],X[4] z[1,1]+X[4] z[3,3]-X[4] z[4,4],X[5] z[1,1] +X[5] z[4,4]-X[5] z[5,5],X[6] z[1,1]+X[6] z[5,5]-X[6]z[6,6],X[7] z[1,1]+X[7] z[6,6]-X[7] z[7,7],X[8] z[1,1]+X[8] z[7,7]-X[8] z[8,8],0},{X[7] z[2,2]+X[7] z[3,3]-X[7] z[7,7],X[8] z[2,2]+X[8] z[4,4]-X[8] z[8,8],0,0,0,0},{0,0,0,0,0},{0,0,0,0},{0,0,0},{0,0},{0},{}}
C3T1 = DeleteCases[Flatten[C3T], 0]/.X[i_] → 1
{z[1,1]+z[2,2]-z[3,3],z[1,1]+z[3,3]-z[4,4],z[1,1]+z[4,4]-z[5,5],z[1,1]+z[5,5]-z[6,6], z[1,1]+z[6,6]-z[7,7],z[1,1]+z[7,7]-z[8,8],z[2,2]+z[3,3]-z[7,7],z[2,2]+z[4,4]-z[8,8]}
SOO = Flatten[Solve[C3T == 0, CAR]]
{z(2,2)→ 4 z(1,1),z(3,3)→ 5 z(1,1),z(4,4)→ 6 z(1,1),z(5,5)→ 7 z(1,1),z(6,6)→ 8 z(1,1),z(7,7)→ 9 z(1,1),z(8,8)→ 10 z(1,1)}
JJJ = CAR/. SOO
$\{z(1,1), 4z(1,1), 5z(1,1), 6z(1,1), 7z(1,1), 8z(1,1), 9z(1,1), 10z(1,1)\}$
SO3 = Solve[JJJ = 0]
$\{\{z(1,1) \to 0\}\}$
LOL=C3T/. SO3
$\{X(3)z(2,2) - X(3)z(3,3), X(4)z(3,3) - X(4)z(4,4), X(5)z(4,4) - X(5)z(5,5), X(6)z(5,5) - X(6)z(6,6), X(7)z(6,6) - X(7)z(7,7), X(8)z(7,7) - X(8)z(8,8), 0\}, \{X(7)z(2,2) + X(7)z(3,3) - X(7)z(7,7), X(8)z(2,2) + X(8)z(4,4) - X(8)z(8,8), 0, 0, 0, 0\}, \{0,0,0,0,0\}, \{0,0,0,0\}, \{0,0,0\}, \{0,0\}, \{0\}, \{\}$
POL=Solve[LOL=0,CAR]
$\{\{z(2,2) \to 0, z(3,3) \to 0, z(4,4) \to 0, z(5,5) \to 0, z(6,6) \to 0, z(7,7) \to 0, z(8,8) \to 0\}\}$
UIP=Flatten[Keys[POL]]
$\{z(2,2), z(3,3), z(4,4), z(5,5), z(6,6), z(7,7), z(8,8)\}$
La dimensión de B^2 es:
Length[UIP]
7

2-cocadenas invariantes.

Relaciones generales
Ψ[X_ + Y_, Z_]:=$\Psi[X, Z] + \Psi[Y, Z]$
Ψ[X_, Y_ + Z_]:=$\Psi[X, Y] + \Psi[X, Z]$
Ψ[K[X_ + Y_, Z_], W_]:=$\Psi[K[X, Z], W] + \Psi[K[Y, Z], W]$
Ψ[W_, K[X_ + Y_, Z_]]:=$\Psi[W, K[X, Z]] + \Psi[W, K[Y, Z]]$
K[Ψ[X_ + Y_, Z_], W_]:=$K[\Psi[X, Z], W] + K[\Psi[Y, Z], W]$
K[W_, Ψ[X_ + Y_, Z_]]:=$K[W, \Psi[X, Z]] + K[W, \Psi[Y, Z]]$
Ψ[T[l_], X[i_]]:=0

$\Psi[X[\text{i\_}], T[\text{l\_}]]$:=0
$\Psi[X[\text{i\_}], X[\text{i\_}]]$:=0
$\Psi[0, X[\text{i\_}]]$:=0
$\Psi[X[\text{i\_}], 0]$:=0
$\Psi[\text{a\_X\_}, \text{Z\_}]:=a*\Psi[X, Z]$
$\Psi[\text{Z\_}, \text{a\_X\_}]:=a*\Psi[Z, X]$
$c[\text{i\_}, \text{i\_}, \text{j\_}] = 0;$
Definición de 2-cocadenas invariantes por el toro
$c[\text{i\_}, \text{j\_}, \text{p\_}]:=$
If[EW[[i]] + EW[[j]]===EW[[p]], Signature[$\{i, j\}$] $*$ q[Sort[$\{i, j\}$][[1]], Sort[$\{i, j\}$][[2]], p], 0]
$\Psi[X[\text{i\_}], X[\text{j\_}]]$:=Sum[$c[i, j, p]X[p]$, $\{p, 1, \text{NN}\}$]
DERF[l_, i_, j_]:=$K[T[l], \Psi[X[i], X[j]]] - K[X[i], \Psi[T[l], X[j]]] + K[X[j], \Psi[T[l], X[i]]]-$
$\Psi[K[T[l], X[i]], X[j]] + \Psi[K[T[l], X[j]], X[i]] - \Psi[K[X[i], X[j]], T[l]]$
Condición de las 2-cocadenas invariantes
Matriz de las 2-cocadenas t-invariantes
KOCI = Table[$\Psi[X[i], X[j]]$, $\{i, 1, \text{NN}\}$, $\{j, i+1, \text{NN}\}$];
NCOC = DeleteCases[Flatten[KOCI], 0]/.X[i_] $\to$ 1;
DERC[l_, i_, j_]:=$K[X[l], \Psi[X[i], X[j]]] - K[X[i], \Psi[X[l], X[j]]] + K[X[j], \Psi[X[l], X[i]]]-$
$\Psi[K[X[l], X[i]], X[j]] + \Psi[K[X[l], X[j]], X[i]] - \Psi[K[X[i], X[j]], X[l]]$
SOCO = Table[DERC[i, j, l], $\{i, 1, \text{NN}\}$, $\{j, i+1, \text{NN}\}$, $\{l, j+1, \text{NN}\}$];

$\{\{\{X(8)(-q(1,3,4)) + X(8)q(1,7,8) + X(8)q(2,3,7) - X(8)q(2,4,8), 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0\}, \{0, 0\}, \{0\}, \{\}\}, \{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0\},$
$\{0, 0\}, \{0\}, \{\}\}, \{\{0, 0, 0, 0\}, \{0, 0, 0\}, \{0, 0\}, \{0\}, \{\}\}, \{\{0, 0, 0\}, \{0, 0\}, \{0\}, \{\}\},$
$\{\{0, 0\}, \{0\}, \{\}\}, \{\{0\}, \{\}\}, \{\{\}\}, \{\}\}$
SOTO = DeleteCases[Flatten[SOCO], 0]/.X[i_] $\to$ 1;
$\{-q(1,3,4) + q(1,7,8) + q(2,3,7) - q(2,4,8)\}$
LR = Flatten[Solve[SOTO == 0, NCOC]];
$\{q(2,4,8) \to -q(1,3,4) + q(1,7,8) + q(2,3,7)\}$
SL = KOCI/.LR;
$\{\{X(3)q(1,2,3), X(4)q(1,3,4), X(5)q(1,4,5), X(6)q(1,5,6), X(7)q(1,6,7), X(8)q(1,7,8), 0\},$
$\{X(7)q(2,3,7), X(8)(-q(1,3,4) + q(1,7,8) + q(2,3,7)), 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0\},$
$\{0, 0, 0\}, \{0, 0\}, \{0\}, \{\}\}$
KOKO = KOCI/.LR;
$\{\{X(3)q(1,2,3), X(4)q(1,3,4), X(5)q(1,4,5), X(6)q(1,5,6), X(7)q(1,6,7), X(8)q(1,7,8), 0\},$
$\{X(7)q(2,3,7), X(8)(-q(1,3,4) + q(1,7,8) + q(2,3,7)), 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0\}, \{0, 0, 0\}, \{0, 0\}, \{0\}, \{\}\}$
PR = Flatten[Solve[SL == 0, NCOC]];
$\{q(1,2,3) \to 0, q(1,3,4) \to 0, q(1,4,5) \to 0, q(1,5,6) \to 0,$
$q(1,6,7) \to 0, q(1,7,8) \to 0, q(2,3,7) \to 0\}$
PRR = Keys[PR];
$\{q(1,2,3), q(1,3,4), q(1,4,5), q(1,5,6), q(1,6,7), q(1,7,8), q(2,3,7)\}$
La dimensión de Z^2 es:
Length[PR]
7
La dimensión del segundo grupo de cohomología es:
DIH2 = Length[PR] − Length[UIP]
0