RANDOM CONSTRUCTIONS IN BELL INEQUALITIES: A SURVEY

CARLOS PALAZUELOS

Abstract. Initially motivated by their relevance in foundations of quantum mechanics and more recently by their applications in different contexts of quantum information science, violations of Bell inequalities have been extensively studied during the last years. In particular, an important effort has been made in order to quantify such Bell violations. Probabilistic techniques have been heavily used in this context with two different purposes. First, to quantify how common the phenomenon of Bell violations is; and second, to find large Bell violations in order to better understand the possibilities and limitations of this phenomenon. However, the strong mathematical content of these results has discouraged some of the potentially interested readers. The aim of the present work is to review some of the recent results in this direction by focusing on the main ideas and removing most of the technical details, to make the previous study more accessible to a wide audience.

Introduction

Bell inequalities have attracted much attention in the last years. Their original interest as a key tool in the study of foundations of quantum mechanics has been nowadays surpassed by the relevance of these inequalities in different contexts such as quantum cryptography, communication complexity protocols and generation of trusted random numbers. In addition, Bell inequalities have been shown to be intimately related with some problems in computer science and the theory of operator algebras, capturing an increasing interest from the corresponding communities. Given the great importance of probabilistic techniques in those fields as well as in different areas of quantum information, it is not surprising that they are also very useful in the context of Bell inequalities. In fact, random constructions have been a key tool to solve some questions which had remained open for a long time in the field. However, despite their potential usefulness, these techniques and results are still far from being considered natural by many people working on quantum nonlocality. The aim of the present work is to review some of the most important results in the context of Bell inequalities for which probabilistic techniques have played a crucial role. Here, we will focus on the main ideas without paying attention to the technical details with the hope that this makes the previous works more appealing for non-experts. Hence, this work should not be understood as a general survey on Bell inequalities, for which the reader can find some other references [15], [16], [38], [49], [52]. In particular, we will deliberately skip some standard topics such as connections with other areas, physical applications of the results and so on, with the upside of going directly to the important points of our discussion.

Let us start by introducing the basic objects of study. Bell inequalities were first considered by Bell in [8] as a way to clarify an apparently metaphysical dispute on the completeness of quantum mechanics as a model of Nature arising from the work [23]. Given two spatially separated quantum systems, controlled by Alice and Bob respectively, and described by a bipartite quantum state ρ, Bell showed that certain probability distributions obtained from an experiment in which Alice and Bob perform some measurements x and y respectively from a set of N measurements, each of which has K possible outputs a and b, respectively, cannot be explained by a classical model[1]. More precisely, if $P = (P(a,b|x,y))_{x,y;a,b=1}^{N,K}$ denotes the probability distribution[2] obtained in such an experiment, we say that P is a *classical (or local) probability distribution* if it can be written as

$$P(a,b|x,y) = \int_\Omega P_\omega(a|x)Q_\omega(b|y)d\mathbb{P}(\omega)$$

for every x, y, a, b, where $(\Omega, \mathbb{P})$ is a probability space and for every $\omega \in \Omega$ we have $P_\omega(a|x) \geq 0$ and $\sum_a P_\omega(a|x) = 1$ for all a, x (resp. $Q_\omega(b|y) \geq 0$ and $\sum_b Q_\omega(b|y) = 1$ for all b, y). We denote the set of classical probability distributions by $\mathcal{L}$. It is easy to see that $\mathcal{L}$ is a polytope (that is, it is a compact convex set with a finite number of extreme points) and the inequalities describing its facets are called *Bell inequalities*. On the other hand, we say that P is a *quantum probability distribution* if

$$P(a,b|x,y) = tr(E_x^a \otimes F_y^b \rho)$$

for every x, y, a, b, where ρ is a density operator acting on the tensor product of two Hilbert spaces $H_1 \otimes H_2$ and $(E_x^a)_{x,a}$, $(F_y^b)_{y,b}$ are two sets of operators representing POVMs acting on H_1 and H_2 respectively. That is, $E_x^a \geq 0$ and $\sum_a E_x^a = \mathbb{1}_{H_1}$ for all a, x (resp. $F_y^b \geq 0$ and $\sum_b F_y^b = \mathbb{1}_{H_2}$ for all b, y). We denote the set of quantum probability distributions by $\mathcal{Q}$. It is easy to see that this set is convex and it verifies $\mathcal{L} \subset \mathcal{Q}$. As was shown by Bell, the converse inclusion fails; equivalently, there exist quantum probability distributions violating some Bell inequalities.

In the beginning of this theory a slightly simpler scenario was considered. In the particular case where Alice's and Bob's measurements are binary (that is, $a_x, b_y = \pm 1$ for every x, y) one can consider the joint correlation:

$$\gamma_{x,y} = \mathbb{E}[a_x \cdot b_y] = P(1,1|x,y) + P(-1,-1|x,y) - P(-1,1|x,y) - P(1,-1|x,y)$$

for every $x, y = 1, \dots, N$. By plugging the definition of classical (resp. quantum) probability distribution in the previous expression one justifies the definition of *classical (or local) correlation matrices* as those matrices $(\gamma_{x,y})_{x,y=1}^N$ which can be expressed in the form

$$\gamma_{x,y} = \int_\Omega A_x(\omega)B_y(\omega)d\mathbb{P}(\omega)$$

[1]Formally, Bell talked about a local hidden variable model (LHVM).

[2]Note that P is not a probability distribution itself. For every fixed x, y we have that $(P(a,b|x,y))_{a,b=1}^K$ is a probability distribution. However, it is standard to use this terminology. Sometimes they are also called *behaviors*, because of the terminology used in [49].

for every x, y, where $(\Omega, \mathbb{P})$ is a probability space and for every $\omega \in \Omega$ we have $A_x(\omega) \in \{-1, 1\}$ for every x (resp. $B_y(\omega) \in \{-1, 1\}$ for every y); and *quantum correlation matrices* as those matrices $(\gamma_{x,y})_{x,y=1}^N$ of the form

$$\gamma_{x,y} = tr(A_x \otimes B_y \rho),$$

for every x, y, where ρ is a density operator acting on the tensor product of two Hilbert spaces $H_1 \otimes H_2$ and $(A_x)_x$, $(B_y)_y$ are families of self-adjoint operators acting on H_1 and H_2 respectively satisfying $\max_{x,y}\{\|A_x\|, \|B_y\|\} \leq 1$. Since the different contexts will be clear along the work, we will also denote by $\mathcal{L}$ and $\mathcal{Q}$ the set of classical and quantum correlation matrices respectively. Note that in order to consider this simpler context we made two simplifications: We restrict to binary measurements and we only considered the joint correlations of Alice's and Bob's measurements (and not the marginals). In particular, $\mathcal{L}$ is again a polytope defined by its facets, now called *correlation Bell inequalities*, contained in the convex set $\mathcal{Q}$. Bell's work actually showed that this inclusion is strict even for the simplest case $x, y = 1, 2$ (which implies the result for probability distributions).

In this work it will be useful to understand Bell inequalities in a slightly more general sense as *dual objects* of probability distributions, and talk about functionals instead of inequalities. More precisely, *any* real tensor $M = (M_{x,y}^{a,b})_{x,y;a,b=1}^{N,K}$ (resp. real matrix $M = (M_{x,y})_{x,y=1}^N$) defines a Bell functional (resp. correlation Bell functional) by considering the dual action on probability distributions (resp. correlation matrices):

$$\langle M, P\rangle = \sum_{x,y;a,b} M_{x,y}^{a,b} P(a,b|x,y) \quad (\text{resp. } \langle M, \gamma\rangle = \sum_{x,y} M_{x,y}\gamma_{x,y}).$$

Given M, we will denote its *classical value* and its *quantum value* respectively by

$$\omega(M) = \sup\big\{|\langle M, P\rangle| : P \in \mathcal{L}\big\} \quad \text{and} \quad \omega^*(M) = \sup\big\{|\langle M, P\rangle| : P \in \mathcal{Q}\big\}.$$

This picture allows us to not only describe Bell inequality violations[3]:

$$M \ \text{ such that } \quad LV(M) := \frac{\omega^*(M)}{\omega(M)} > 1,$$

but also to quantify these violations by means of the above quantity LV. From a general point of view, the quantity LV can be understood as a quantification of how much better one can perform certain tasks if quantum resources are used instead of classical ones (see [28], [30], [38], [39] for more information). Bell's theorem can be re-stated as: $LV(M) > 1$ for some correlation Bell functionals M (see also [19] for the famous CHSH inequality).

Although the previous description only considers the bipartite case, it is straightforward to extend all the definitions to the multipartite setting. The multipartite scenario will be very important in our work for two reasons. First of all, in the tripartite case we will observe new phenomena which cannot be found in the scenario of two parties. Secondly, considering n parties introduces a new parameter in the problem and, as we will see, the answer to different questions can strongly depend on it.

Once we know that violations of Bell inequalities exist, two natural questions arise:

[3]Note that any Bell functional M defines an inequality by writing $\langle M, P\rangle \leq \omega(M)$, $P \in \mathcal{L}$.

1. *How common is the phenomenon of quantum nonlocality?*
2. *How large can Bell inequality violations (that is, the quantity $LV(M)$) be?*

Regarding the first question, there are two key objects in our picture, Bell functionals (resp. correlation Bell functionals) and probability distributions (resp. correlation matrices). On the one hand, one can study the probability of having $LV(M) > 1$ if we pick a Bell functional (resp. correlation Bell functional) M at random or study the expected value of $LV(M)$ if we follow this procedure. On the other hand, one can pick a quantum probability distribution (resp. quantum correlation matrix) $P \in \mathcal{Q}$ at random and ask how likely the event $P \notin \mathcal{L}$ is. However, two main obstacles appear when studying these questions.

The first problem is that it is not so clear what "picking these objects at random" means. For that, one has to consider a probability measure on the corresponding sets and that is not always easy nor natural. In fact, the sampling procedure is particularly tricky for quantum probability distributions (resp. quantum correlation matrices) since here one has to impose a certain structure which is not required when one samples Bell functionals. Regarding the definition of quantum probability distributions and quantum correlation matrices above one could think about sampling states and measurements independently. However, this procedure presents several problems such as deciding the dimension of the corresponding Hilbert spaces, the lack of a natural way to sample general measurements and so on. Although we will see below that some of these problems can be circumvented in the case of correlation matrices, they are there when sampling quantum probability distributions. Some works have also considered the problem of fixing one of these elements (either the state or the measurements) and sampling the other one at random. In fact, considering random measurement seems to be an interesting problem from the experimental point of view since it is directly connected to the absence of a shared reference frame in Bell experiments.

A second problem is that the quantities $\omega(M)$ and $\omega^*(M)$ are in general very difficult to compute. The study [49] performed by Tsirelson clarified the context of bipartite correlation Bell functionals. Indeed, Tsirelson gave an alternative description of the quantum value $\omega^*(M)$ for correlation Bell functionals which allows us, in particular, to understand the quantity $\mathcal{D}_2(N) = \sup\big\{LV(M) : M = (M_{x,y})_{x,y=1}^N\big\}$[4] via the so called *Grothendieck constants*. However, the situation is more intricate in the case of general bipartite Bell functionals as well as in the tripartite correlation case. If we define $\mathcal{D}_2(N,K) = \sup\big\{LV(M) : M = (M_{x,y}^{a,b})_{x,y;a,b=1}^{N,K}\big\}$, it is not clear how large this quantity can be as a function of N and K and the same happens for the quantity $\mathcal{D}_3(N)$ in the tripartite correlation scenario. This leads us to study question 2. above.

Nevertheless, while the use of techniques of probability theory is somehow unavoidable to analyze question 1. above (because of the nature of the question itself!) it is not clear a priori why we should link the quantification of Bell inequality violations, that is, question 2., with the use of random techniques. It turns out that, as in many other contexts in quantum information theory (and other disciplines) random objects are very suitable to show extreme behaviors. In particular, the use of random techniques has been proven

[4] The subscript 2 in $\mathcal{D}_2(N)$ refers to the bipartite case.

extremely useful to find examples of (correlation) Bell functionals M showing a large gap between their quantum value and their classical value; that is, large lower bounds for $LV(M)$. In this survey, we will review some of the recent results about how large the quantities $\mathcal{D}_3(N)$ and $\mathcal{D}_2(N,K)$ can be and what kind of objects we need to use in order to make them large.

Finally, let us remark that, despite the restriction imposed in the title of this survey, the number of works on Bell inequalities using probabilistic tools is very large. Here, we will focus on those results studying asymptotic behaviors. That is, those results describing the probabilistic nature of a problem when (some of) the parameters go to infinity. With this in mind, let us remind the reader of the standard asymptotic notation, which will be constantly used in this work. Given two nonnegative functions $f(n)$ and $g(n)$ on the natural numbers, we will write $f(n) = O\big(g(n)\big)$ (resp. $f(n) = \Omega\big(g(n)\big)$) if there exist a constant C and a natural number n_0 such that $f(n) \leq Cg(n)$ (resp. $f(n) \geq Cg(n)$) for every $n \geq n_0$. On the other hand, we will write $f(n) = o\big(g(n)\big)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

The survey is organized as follows. Section 1 is devoted to explaining some results about the probabilistic nature of quantum nonlocality in the bipartite correlation case. As we will see, the deep understanding of this situation thanks to Tsirelson's work allows us to study the problem in a very natural way. In Section 2 some results about lower and upper bounds for tripartite correlation Bell inequality violations will be reviewed. This will lead us to the first examples of *unbounded violations of tripartite correlation Bell inequalities*. Section 3 will deal with some results about the probabilistic nature of multipartite Bell inequalities in some particular cases as a function of the number of parties n. Finally, in Section 4 some recent results about general bipartite Bell inequalities and the quantity $\mathcal{D}_2(N,K)$ will be reviewed. In particular, we will explain some random constructions in this setting and compare them with some new results from computer science.

1. Bipartite correlation Bell inequalities

As we mentioned in the Introduction Bell's work showed the strict inclusion $\mathcal{L} \subsetneq \mathcal{Q}$ (see also [19]). This picture was completed by Tsirelson in the case of correlation matrices, by showing that the set $\mathcal{Q}$ is not much bigger than the set $\mathcal{L}$. More precisely, one has $\mathcal{L} \subsetneq \mathcal{Q} \subsetneq K_G^{\mathbb{R}}\mathcal{L}$, where $1.67696... \leq K_G^{\mathbb{R}} \leq 1.78221...$ is the real Grothendieck constant[5]. The following result is the standard statement of the corresponding theorem.

Theorem 1.1 (Tsirelson)**.**

$$\mathcal{D}_2(N) := \sup\left\{\frac{\omega^*(M)}{\omega(M)} : M = (M_{x,y})_{x,y=1}^N\right\} \leq K_G^{\mathbb{R}} \quad \textit{for every } N. \tag{1.1}$$

In fact, the Grothendieck constant can be defined by $K_G^{\mathbb{R}} := \sup_N \mathcal{D}_2(N)$. Theorem 1.1 is a consequence of Grothendieck's inequality and a result proved by Tsirelson [49]

[5]The exact value of the Grothendieck constant is still unknown in both the real and the complex case (see [12] for the most recent progress).

which states that $\gamma = (\gamma_{x,y})_{x,y=1}^N$ is a quantum correlation matrix if and only if there exist a real Hilbert space $H_{\mathbb{R}}$ and unit vectors $u_1, \dots, u_N, v_1, \dots, v_N$ in $H_{\mathbb{R}}$ such that

$$\gamma_{x,y} = \langle u_x, v_y \rangle \text{ for every } x, y = 1, \dots, N. \tag{1.2}$$

In [49] the author posed the open question of whether a similar result to Theorem 1.1 holds in the tripartite case. This is related to the lack of Grothendieck's inequality for trilinear forms and we will analyze this question in the following section.

In [1] the authors tackled the question of how likely it is for a random bipartite correlation Bell functional[6] M to verify that the quotient of $\omega^*(M)$ by $\omega(M)$ is strictly larger than one. In order to study this problem, one first needs to define a way of sampling these inequalities. In [1], the authors considered random $N \times N$ matrices M sampled from $\{-1,1\}^{N^2}$ with respect to the uniform measure. In fact, although the authors focused on sign matrices, the same techniques they use can be applied to study more general random matrices, such as real gaussian matrices. The main result in [1] states the following.

Theorem 1.2. *If $M = \big(\epsilon_{x,y}\big)_{x,y=1}^N$ is a random matrix sampled from $\{-1,1\}^{N^2}$ with respect to the uniform measure, then*

$$\lim_{N\to\infty} \mathbb{P}\Big\{M : 1.5638... > \frac{\omega^*(M)}{\omega(M)} > 1.2011...\Big\} = 1.$$

Theorem 1.2 implies that for almost any correlation Bell functional its quantum value is strictly larger than its classical one. On the other hand, according to Theorem 1.1 and the comments below, the above equation also tells us that with probability tending to one these inequalities are *not so close* to the optimal value $K_G^{\mathbb{R}} \in [1.67, 1.79]$ for the quotient between $\omega^*(M)$ and $\omega(M)$. Let us also mention that, although very natural, the way of sampling bipartite correlation Bell inequalities considered in [1] is very particular. Indeed, this procedure does not consider all bipartite correlation Bell inequalities since, for instance, all the entries of the matrices have the same absolute value 1. One can think about some other ways of sampling them by considering for instance a bipartite correlation Bell functional $M \in \mathbb{R}^{N^2}$ as an element in the unit sphere with respect to the norm $\|M\| = \sum_{x,y=1}^N |M_{x,y}|$. Then, one can naturally define a probability distribution on this sphere.

Since the most important part of Theorem 1.2 is the lower bound for the quotient, we will just briefly explain how to prove that estimate. To this end, the authors first showed that $\lim_{N\to\infty} \mathbb{P}\big\{M : \omega(M) \leq \big(1.6651... + o(1)\big)N^{\frac{3}{2}}\big\} = 1$. This is obtained as an application of the Chernoff bound to the random variable $\sum_{x,y=1}^N \epsilon_{x,y} t_x s_y$ for a fixed choice of signs $t_x, s_y = \pm 1$, $x, y = 1, \cdots, N$; and a counting argument to consider the 2^{2N} possible choices of signs. In order to prove the estimate $\lim_{N\to\infty} \mathbb{P}\big\{M : \omega^*(M) \geq \big(2 - o(1)\big)N^{\frac{3}{2}}\big\} = 1$, from where one obtains the lower bound in Theorem 1.2, the authors used a clever construction based on the Marcenko-Pastur law, which describes the behavior of the singular values of the random matrix M. If we call L and R the

[6]The work [1] deals with bipartite XOR games, but the problem is completely equivalent.

$n \times m$ matrices whose columns are respectively the left and right singular vectors of the matrix M associated to its m largest singular values, basic linear algebra shows that

$$\sum_{x,y=1}^{N} M_{x,y}\langle u_x, v_y\rangle = \sum_{i=1}^{m} \lambda_i, \tag{1.3}$$

where the m-dimensional vectors u_x and v_y are the rows of L and R respectively and $(\lambda_i)_{i=1}^m$ are the corresponding singular values. At the same time, the Marcenko-Pastur law [34] tells us that if one defines the function $f(s) = \frac{1}{2\pi}\int_{s^2}^4 \sqrt{\frac{4}{x} - 1}\, dx$ on $[0, 2]$, for every $\epsilon > 0$ the number m of singular values satisfying $\lambda_i > (2 - \epsilon)\sqrt{N}$ belongs to the interval $\big[\big(f(2-\epsilon) - o(1)\big)N, \big(f(2-\epsilon) + o(1)\big)N\big]$ with probability tending to 1 as N goes to infinity. Hence, the quantity (1.3) is lower bounded by $(2-\epsilon)\big(f(2-\epsilon) - o(1)\big)N^{\frac{3}{2}}$ with probability tending to one. The technical part of the proof in [1] consists in adapting the Marcenko-Pastur law to show that one can assume that for a given $\delta > 0$, all the vectors u_x and v_y have norm at most $\sqrt{f(2-\epsilon) + \delta}$ with probability tending to one[7]. Therefore, by considering the normalized version of the previous vectors $\tilde{u}_x$, $\tilde{v}_y$, one obtains that the quantum correlation $\gamma = \big(\langle \tilde{u}_x, \tilde{v}_y\rangle\big)_{x,y=1}^N$ verifies

$$\omega^*(M) \geq \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} = \frac{1}{f(2-\epsilon) + \delta}\sum_{i=1}^{m} \lambda_i \geq \frac{(2-\epsilon)\big(f(2-\epsilon) - o(1)\big)}{f(2-\epsilon) + \delta} N^{\frac{3}{2}}.$$

Since ϵ and δ can be made arbitrarily small one obtains the desired estimate.

Interestingly, the authors show in [1] that $(2 + o(1))N^{\frac{3}{2}}$ is also an upper bound for the quantum value $\omega^*(M)$, so it is optimal. However, the best lower bound for the classical value $\omega(M)$ obtained in [1] is far from the upper bound explained above (see [1, Section 4] for details). Obtaining the exact asymptotic value of $\omega(M)$ appears to be a challenging open problem.

In [24] the same problem was considered from the correlation matrices point of view. That is, if one picks a quantum correlation matrix at random, what is the probability that it is nonlocal? The equivalent reformulation (1.2) of a quantum correlation gives a natural sampling procedure which avoids most of the problems that we mentioned in the Introduction: one can pick the vectors $u_1, \cdots, u_N, v_1, \cdots, v_N$ independently and uniformly distributed on the unit sphere of $\mathbb{R}^m$. It is well known that this is exactly the same as sampling independent normalized m-dimensional real gaussian vectors. It is easy to see that if one fixes any finite m, the probability that a quantum correlation matrix sampled according to the previous procedure is nonlocal tends to one as N tends to infinity (see [24, Section 2] for details). However, this kind of sampling does not say too much about the structure of quantum correlation matrices. Indeed, this is because the set of these matrices of (large) order N which can be written as in (1.2), when the vectors u_x and v_y have a fixed (small) dimension m, is very small compare to the set of all quantum correlation matrices. The interesting case is that where m and N are of the

---

[7] In fact, the modified Marcenko-Pastur law proved in [1, Theorem 3] gives the probability for each of these vectors to have norm larger than $\sqrt{f(2-\epsilon) + \delta}$. Then, the authors proved that Eq. (1.3) is not affected if one rules out those vectors with large norm.

same order. The main result in [24] gives indeed an answer to the considered problem as a function of $\alpha = m/N$:

Theorem 1.3. *Let $u_1, \cdots, u_N, v_1, \cdots, v_N$ be $2N$ vectors sampled independently according to the uniform measure on the unit sphere of $\mathbb{R}^m$ and denote by $\gamma = (\langle u_i, v_j\rangle)_{i,j=1}^N$ the corresponding quantum correlation matrix. Let us also denote $\alpha = m/N$. If $\alpha \leq \alpha_0 \approx 0.004$, then γ is nonlocal with probability tending to one as N tends to infinity. On the other hand, if $\alpha > 2$, then γ is local with probability tending to one as N tends to infinity.*

There is a considerable gap between α_0 and 2. In fact, one should not expect this result to be optimal. This is because the proof of the first part of the previous theorem is based on an approximation of gaussian matrices by orthogonal ones proved in [26] and the main result in [1], where N seems artificially larger than m. However, the important point of the previous statement is that it shows a nontrivial phase transition for the nonlocal properties of γ as a function of the parameter $\alpha = m/N$.

The study in [24] was continued and improved in [25]. In this last work, the authors quantified the *classicality* and the *quantumness* of certain random matrices γ, when they are seen as correlation matrices. Note, that the crucial quantities here are

$$\pi(\gamma) := \sup\{|\langle\gamma, M\rangle| : \omega(M) \leq 1\} \quad \text{and} \quad \gamma_2(\gamma) := \sup\{|\langle\gamma, M\rangle| : \omega^*(M) \leq 1\}. \tag{1.4}$$

Notations π and γ_2 are due to the fact that these quantities coincide with the π and the γ_2- tensor norm of γ, when the matrix γ is seen as an element of $\ell_\infty^n \otimes \ell_\infty^n$. Here, we will ignore the tensor product point of view and will just explain the key ideas to understand the main results in [25]. To this end, first note that in the new picture (1.4) $\pi(\gamma) \geq \gamma_2(\gamma)$ for every matrix γ, and the goal in [25] is to prove that, when properly sampled, for most of the γ this is a strict inequality (in fact, the point is to show that there is a gap).

Let us first assume that γ is sampled as a gaussian matrix and consider its Singular Value Decomposition $\gamma = UDV$. If we define now the Bell functional $M = UV$, it is trivial to check that

$$\pi(\gamma) \geq \frac{1}{\omega(M)}|\langle\gamma, M\rangle| = \frac{1}{\omega(M)}tr(D).$$

Now, since γ is an $N\times N$ gaussian matrix, the behavior of $tr(D)$ for large n is described by the Marcenko-Pastur law (as $\sim 8N^{3/2}/3\pi$) while, at the same time, it is shown that the functional M defined above behaves as a Haar distributed orthogonal matrix (see [24, Proposition 1.5.] for details). This last observation allows the authors in [25] to use an argument on the area of spherical caps (see [25, Proposition 3.1]) to show that with high probability when n tends to infinity, $\omega(M) \leq (\sqrt{15/16}+o(1))N$. One can therefore obtain a lower bound for $\pi(\gamma)$.

The analysis of $\gamma_2(\gamma)$ performed in [25] is much simpler than the one for $\pi(\gamma)$. Indeed, Tsirelson's description of quantum correlation matrices (1.2) provides a nice way of writing the quantity $\gamma_2(\gamma)$ in (1.4) and it is not difficult to show that with high probability as n tends to infinity $\gamma_2(\gamma) = (1 \pm o(1))tr(D)/N$. An immediate consequence of the previous explanation is:

Theorem 1.4. *If γ is an $N \times N$ gaussian matrix, then with high probability as N tends to infinity we have*

$$\pi(\gamma) \geq \big(\sqrt{16/15} + o(1)\big)\gamma_2(\gamma).$$

In particular, the matrix $(1/\gamma_2(\gamma))\gamma$ is a $N \times N$ quantum correlation matrix (by construction) and it is nonclassical with high probability as N tends to infinity.

We must note that Theorem 1.3 and Theorem 1.4 are not comparable a priori because the way of sampling correlation matrices in both results is completely different. However, the techniques used to prove Theorem 1.4 are actually more general than those used for the first theorem. In fact, the main result in [25] is stated for a quite general family of random matrices (see [25, Theorem 4.1]). In particular, the same techniques can be applied when correlation matrices are sampled as $\gamma = G_1G_2$, where G_i are independent and properly normalized rectangular gaussian matrices and, in this case, the situation is exactly the same as the one considered in Theorem 1.3. Using similar computations to the ones explained above for gaussian matrices one can show that the constant α_0 in Theorem 1.3 can be replaced by $\alpha_1 = 0.1269$ (see [25, Theorem 4.4]), which reduces the corresponding gap in a considerable way.

Some open questions are posed in [24] and [25]. Regarding the work [24], it would be interesting to understanding the situation when $\alpha = 1$, reducing the gap $[\alpha_1, 2]$ (where α_1 was introduced in the above paragraph) and also studying the existence of a threshold in the phase transition. On the other hand, finding better lower bounds for the value $\pi(\gamma)$, where γ is a random gaussian matrix, would imply a better understanding of the main result in [25].

We finish this section going back to some of our initial comments in the Introduction about the problems to sample quantum probability distributions and quantum correlation matrices because of their particular form. While the way of sampling in Theorem 1.3 is motivated by this point of view, Theorem 1.4 says, somehow, that this is not so important. Indeed, it says that we can sample "any" random matrix and then we can make it quantum by properly normalizing (via its γ_2-norm). We remark that this is so because of the very particular form of correlation matrices. Indeed, when we work in this setting, both $\mathcal{L}$ and $\mathcal{Q}$ are very nice sets and, in particular, for any matrix γ we can find a suitable (small enough) constant c such that the new matrix $c\gamma$ belongs to $\mathcal{Q}$. Interestingly, from this point of view Theorem 1.4 says that in most directions in $\mathbb{R}^{N^2}$ (according to a gaussian sampling), the borders of $\mathcal{Q}$ and $\mathcal{L}$ do not coincide (there is actually a gap). However, when we work with probability distributions the situation is completely different and we must be much more careful with the probability measure we consider. On the other hand, the way of sampling in Theorem 1.3 can be particularly interesting if one wants to have a good control of the observables used in the quantum correlation matrices, which are obtained from Tsirelson's theorem. This is related to the way to sample random measurements (see Section 3.3).

2. Tripartite correlation Bell inequalities: Unbounded violations

2.1. **Extensions of Tsirelson's result to the multipartite setting.** In order to study Bell violations in the multipartite case and regarding the importance of the maximally entangled state in the setting of two parties: $|\psi\rangle = 1/\sqrt{d}\sum_{i=1}^d |ii\rangle$ (which can be used to construct any quantum correlation matrix as in (1.2) by considering $d = 2^N$ and suitable observables), it seems very reasonable to consider the generalized n-partite d-dimensional GHZ state: $|\psi\rangle = 1/\sqrt{d}\sum_{i=1}^d |i\rangle^{\otimes n}$. On the other hand, any possible strategy looking for unbounded Bell violations should definitely exploit the absence of *the* Grothendieck inequality in the multilinear framework. However, one should make this statement more precise. Grothendieck's inequality can be stated in many equivalent ways (see [21, Page 172]) and it turns out that, in the multilinear case, the corresponding generalizations of those statements are not equivalent anymore. In fact, although several of the possible extensions of Grothendieck's inequality to the multilinear setting have been proved to be false, *a few of them* remain valid in the new context (see [9], [10], [18], [48]). The following generalization of Grothendieck's inequality was proved in [9], [48][8].

Theorem 2.1. *Let $n, N \geq 2$ and d be positive integers, let $T = (T_{i_1,\cdots,i_n})_{i_1,\cdots,i_n=1}^N$ be a real tensor and for every $k \in \{1,\dots,n\}$ and $i_k \in \{1,\dots,N\}$ let x_{i_k} be an element in the unit ball of a complex Hilbert space H of dimension d. Then,*

$$\Big|\sum_{i_1,\cdots,i_n=1}^N T_{i_1,\cdots,i_n}\langle x_{i_1},\cdots,x_{i_n}\rangle\Big| \leq 2^{\frac{3n-5}{2}} K_G^{\mathbb{C}}\cdot\omega(T),$$

where $\langle x_{i_1},\cdots,x_{i_n}\rangle = \sum_{k=1}^d x_{i_1}(k)\cdots x_{i_n}(k)$ and $K_G^{\mathbb{C}}$ is the complex Grothendieck constant. Here, $x_{i_j}(k)$ denotes the k-th component of the vector x_{i_j}.

In [39, Theorem 11] the authors used another version of Theorem 2.1 to show that the largest Bell violation achievable by three parties[9] sharing a d-dimensional GHZ state is upper bounded by $4\sqrt{2}K_G^{\mathbb{C}}$, independently of the number of inputs N and the dimension d. This was the first result proving that a nontrivial family of states cannot give large Bell violations and it can be seen as a generalization of Theorem 1.1. In [39] the authors posed the problem of whether an analogous result could be proved for Schmidt states: $|\psi_\alpha\rangle = \sum_{i=1}^d \alpha_i|i\rangle^{\otimes n}$, where $\alpha = (\alpha_i)_{i=1}^d$ verifies $\sum_{i=1}^n |\alpha_i| = 1$; and provided in addition a direct connection between such a problem and an open question in the context of operator algebras. Two years latter this question was answered in [13] by using a surprisingly easy argument. The authors in [13] realized that the upper bound for the GHZ state can be obtained in a straightforward manner from Theorem 2.1 and extended the result to Schmidt states by using a nice expansion of each of these states in terms of non-normalized GHZ states (see [13, Theorem 1]).

Theorem 2.2. *Let T be an n-partite correlation Bell functional. Then, for every n-partite quantum correlation γ constructed with the state $|\psi_\alpha\rangle = \sum_{i=1}^d \alpha_i|i\rangle^{\otimes n}$ we have*

[8] In fact, we state here a slightly modified version for T real, which involves a modification in the constant (see [13, Theorem 9] for details).

[9] The result can be generalized to n parties straightforwardly.

that $\langle T, \gamma \rangle \leq 2^{\frac{3n-5}{2}} K_G^{\mathbb{C}} \cdot \omega(T)$, independently of the number of inputs N and the local dimension d.

In [13] an exhaustive study of Carne's extension of the Grothendieck inequality to the multilinear case [18] was performed to conclude that an analogous result to Theorem 2.2 can also be stated when the parties share a *clique-wise entangled state* (see [13, Theorem 2]). This implies in particular that for tripartite correlation Bell functional, the amount of Bell violation achievable by an arbitrary stabilizer states is uniformly bounded.

2.2. **Unbounded violations of tripartite Bell inequalities.** The previous results rule out most of the candidates one would first study in order to find large Bell violations in the multipartite setting. Then, as in many other contexts, it becomes natural to study the behavior of *random states*. A standard way of sampling random pure quantum states $|\psi\rangle \in \mathbb{C}^d$ is by using the uniform measure on the unit sphere of the Hilbert space $\mathbb{C}^d$ or, equivalently, using gaussian random variables $|\psi\rangle = \sum_{i=1}^d g_i |i\rangle \in \mathbb{C}^d$.[10] When we are dealing with more than one system this last notation is usually extended to $|\psi\rangle = \sum_{i_1,\cdots,i_n=1}^d g_{i_1,\cdots,i_n} |i_1 \cdots i_n\rangle \in (\mathbb{C}^d)^{\otimes n}$. In the case of two and three systems it is also very common to use random unitaries sampled according to the Haar measure in the unitary group $\mathbb{U}_d$. We write $|\psi\rangle = \sum_{i,j=1}^d U(i,j)|ij\rangle \in (\mathbb{C}^d)^{\otimes 2}$ or $|\psi\rangle = \sum_{i,j,k=1}^d U_i(j,k)|ijk\rangle \in (\mathbb{C}^d)^{\otimes 3}$, where in the last expression the unitaries $(U_i)_{i=1}^d$ are sampled in the cartesian product $\prod_d \mathbb{U}_d$. The unitary approach has been extensively used in the theory of quantum channels and entanglement theory.

In [39] it was proved for the first time that, in contrast to the bipartite case, there are tripartite correlation Bell inequalities which lead to unbounded violations.

Theorem 2.3. *For every dimension $d \in \mathbb{N}$, there exist $D \in \mathbb{N}$, a pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^D \otimes \mathbb{C}^D$ and a Bell functional $T = (T_{i,j,k})_{i,j,k=1}^{2^{d^2}, 2^{D^2}, 2^{D^2}}$ such that the violation of such a functional by $|\psi\rangle$ is $\Omega(\sqrt{d})$.*

This theorem implies that there does not exist a uniform constant C such that $\mathcal{D}_3(N) \leq C$ independently of the number of inputs N; giving in this way a negative answer to the question posed by Tsirelson in [49]. It is advisable to extend the previous definition to $\mathcal{D}_3(N,d) = \sup \{\omega_d^*(T)/\omega(T) : T = (T_{x,y,z})_{x,y,z=1}^N\}$, where here $\omega_d^*(T)$ denotes the quantum value of T when we only consider tripartite quantum correlations constructed with quantum state of local dimension d. Then, it is important to point out that, in order to obtain unbounded violations, one must allow to increase both the number of inputs N and the dimension of the Hilbert space of the system d. Indeed, if we fix one of these parameters, then the amount of violation is upper bounded by a constant depending on it.

Theorem 2.4. *The following upper bound holds:*

$$\mathcal{D}_3(N,d) = O(\sqrt{k}), \quad \text{where} \quad k = \min\{N, d\}.$$

[10] One needs to normalize the gaussian state to have the same distribution.

The upper bound as a function of d was first proved in [39] (see [14, Theorem 3] for an alternative proof). In fact, one can state a stronger result since it suffices that only one party has local dimension d. This tells us that Theorem 2.3 is optimal in the local dimension d. The (easier) upper bound as a function of N can be found in [14, Theorem 2] and it also admits an extension requiring that only one party has N inputs. In [14] the authors generalized the previous theorem to n parties by providing the upper bound $O(d^{\frac{n-2}{2}})$ whenever the shared state of the parties is restricted to have local dimension d on at least $n-2$ players and to $O(N^{\frac{n-2}{2}})$ whenever the number of inputs in the correlation Bell functional is at most N for at least $n-2$ players.

The key point to prove Theorem 2.3 is the use of random states, showing once more that these states exhibit unexpected extremal properties. The proof of the theorem relies on hard techniques from operator spaces and it is highly nonconstructive. In particular, it does not provide an explicit (nor even probabilistic) form of the Bell functional T attaining such a violation nor any control on the dimension D appearing in the statement of the theorem. Since we will explain below a more recent result improving Theorem 2.3, we will not say too much about the proof of the previous result. Instead, we will present some basic ideas in order to be able to discuss the analogies and differences with the later proofs.

The first idea in the proof of Theorem 2.3 is to reduce the problem to work with Hilbert spaces. Indeed, the initial problem consists in comparing two quantities $\omega(T)$ and $\omega^*(T)$, which are difficult to handle, for a given tensor $T=(T_{i,j,k})_{i,j,k=1}^N$. Let us consider another tensor $S=(S_{i,j,k})_{i,j,k=1}^m$ as an element in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ and define two new quantities (norms):

$$\|S\|_{\ell_2^m\otimes_\epsilon\ell_2^m\otimes_\epsilon\ell_2^m}=\sup\Big\{\Big|\sum_{i,j,k=1}^m S_{i,j,k}a_ib_jc_k\Big| : \|(a_i)_{i=1}^m\|_2, \|(b_j)_{j=1}^m\|_2, \|(c_k)_{j=1}^m\|_2\leq 1\Big\}$$

and

$$\|S\|_*=\sup\Big\{\Big\|\sum_{i,j,k=1}^m S_{i,j,k}A_i\otimes B_j\otimes C_k\Big\|_{M_{d^3}} : \|(A_i)_{i=1}^m\|_{RC}, \|(B_j)_{j=1}^m\|_{RC}, \|(C_k)_{k=1}^m\|_{RC}\leq 1\Big\},$$

where for a sequence of complex numbers $(z_j)_{j=1}^m$, $\|(z_j)_{j=1}^m\|_2$ is the euclidean norm and for a sequence of matrices $(Z_j)_{j=1}^m\subset M_d$ we define

$$\|(Z_j)_{j=1}^m\|_{RC}=\max\Big\{\Big\|\sum_{j=1}^m Z_jZ_j^\dagger\Big\|^{\frac{1}{2}}, \Big\|\sum_{j=1}^m Z_j^\dagger Z_j\Big\|^{\frac{1}{2}}\Big\}. \tag{2.1}$$

On the other hand, for every tensor S one can construct another tensor $T=(T_{i,j,k})_{i,j,k=1}^N$ with $N=2^m$ for which, up to a universal (known) constant,

$$\frac{\omega^*(T)}{\omega(T)}\simeq\frac{\|S\|_*}{\|S\|_{\ell_2^m\otimes_\epsilon\ell_2^m\otimes_\epsilon\ell_2^m}}.$$

Here, we should point out that one could explicitly construct both the Bell functional T and the observables to be used to compute $\omega^*(T)$ from the elements S, $(A_i)_{i=1}^m$,

$(B_j)_{j=1}^m$, $(C_k)_{k=1}^m$ used to compute $\|S\|_*$. In order to find a tensor S for which the quotient between $\|S\|_*$ and $\|S\|_{\ell_2^m\otimes_\epsilon\ell_2^m\otimes_\epsilon\ell_2^m}$ is large, one considers the random state $|\psi\rangle = \frac{1}{m}\sum_{i,j,k=1}^m U_i(j,k)|ijk\rangle \in (\mathbb{C}^m)^{\otimes 3}$. It follows from well known results in random matrix theory that $\langle\psi|\psi\rangle \simeq 1$. Then, by doubling indices one can naturally consider the element S in $\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}$ defined by $S_{i,i';j,j';k,k'} = \langle k|U_i^t|j\rangle\langle k'|U_{i'}^\dagger|j'\rangle$ and the matrices $A_{i,i'} = 1/\sqrt{m}|i\rangle\langle i'|$, $B_{j,j'} = 1/\sqrt{m}|j\rangle\langle j'|$, $C_{k,k'} = 1/\sqrt{m}|k\rangle\langle k'|$ for every $i,i',j,j',k,k' = 1,\cdots,m$. It is not difficult to check that for these matrices the quantity (2.1) is at most one, which implies

$$\|S\|_* \geq \frac{1}{m^{\frac{3}{2}}}\Big|\Big\langle\psi\Big|\sum_{i,i',j,j',k,k'=1}^m S_{i,i',j,j',k,k'}|i\rangle\langle i'|\otimes|j\rangle\langle j'|\otimes|k\rangle\langle k'|\Big|\psi\Big\rangle\Big| \tag{2.2}$$

$$= \frac{1}{m^{\frac{7}{2}}}\sum_{i,i'=1}^m tr\Big(\big(\bar{U}_i\otimes U_{i'}^\dagger\big)\big(U_i^t\otimes U_{i'}\big)\Big) = \frac{1}{m^{\frac{3}{2}}}tr\big(\mathbb{1}_{M_{m^2}}\big) = \sqrt{m}.$$

According to the previous comments, we would finish the proof if we could show that $\|S\|_{\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}} \lesssim 1$. Unfortunately, it turns out that this estimate is not true. The key result [39, Proposition 20] shows that for every $\epsilon > 0$ and for every $m\in\mathbb{N}$, there exist $N(\epsilon,m)\in\mathbb{N}$, some unitary matrices $U_i\in M_N$ and some matrices $H_{i,i'}\in M_{N^2}$ for every $i,i' = 1,\cdots,m$ such that if we define $F_{i,i'} = U_i^t\otimes U_{i'}^\dagger + H_{i,i'}$ and $S_{i,i';j,j';k,k'} = \langle kk'|F_{i,i'}|jj'\rangle$, we have that

$$\|S\|_{\ell_2^{m^2}\otimes_\epsilon\ell_2^{N^2}\otimes_\epsilon\ell_2^{N^2}} \lesssim 1 \quad\text{and}\quad \Big|\frac{1}{N^2}tr\big(H_{i,i'}(\bar{U}_i\otimes U_{i'})\big)\Big| < \epsilon \quad\text{for every } i,i' = 1,\cdots,m. \tag{2.3}$$

This means that, while the value of the norm $\|\cdot\|_{\ell_2^{m^2}\otimes_\epsilon\ell_2^{N^2}\otimes_\epsilon\ell_2^{N^2}}$ is smaller for the new functional S, the modification with respect to the previous functional does not affect essentially to the estimate (2.2). Hence, the result follows. As the reader can guess, it is precisely in the proof of the existence of these highly nontrivial matrices $H_{i,i'}$ that the explicitness of the result is lost.

In [40], [41] the author refined the preceding proof by using random gaussian matrices instead of random unitaries. The problem is again reduced to separating the norms $\|S\|_{\ell_2^m\otimes_\epsilon\ell_2^m\otimes_\epsilon\ell_2^m}$ and $\|S\|_*$ but the technical proof to find the previous unitaries is replaced by the use of a previous known result ([40, Theorem 16.6]), which seems to be due to Steen Thorbjørnsen. This result guarantees the existence of a family of $N\times N$ gaussian matrices $(G_i)_{i=1}^m$ for a certain N, for which the construction explained above can be done directly. The main advantage of this approach is that one can use directly these gaussian matrices and the matrices $H_{i,i'}$ are not needed anymore. Actually, one can follow the estimates behind these results and replace the previous transformation from the tensor S to T by a more sophisticated one (allowing $N\approx n^2$) to obtain a Bell functional $T = (T_{i,j,k})_{i,j,k=1}^{N^4,N^8,N^8}$ which can give violations of order $\sqrt{N}$ by using a quantum state in $\mathbb{C}^N\otimes\mathbb{C}^{N^2}\otimes\mathbb{C}^{N^2}$ (see [41, Remark 3.3] for details).

2.3. **Briet and Vidick's construction.** In the work [14] the authors gave another proof of Theorem 2.3 which considerably improved both the estimates on the parameters and the construction.

Theorem 2.5. *Let us assume that $N = 2^j$ for some natural number j. There exist a quantum pure state $|\psi\rangle$ in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ and a Bell functional $T = (T_{i,j,k})_{i,j,k=1}^{N^2,N^2,N^2}$ such that the violation of the functional by $|\psi\rangle$ is $\Omega\big(\sqrt{N}\log^{-5/2} N\big)$. Moreover, the observables used by each party are tensor products of Pauli matrices.*

More generally, for every N that is a power of 2 there exists an n-party Bell functional T with N^2 inputs for each party and a state $|\psi\rangle \in (\mathbb{C}^N)^{\otimes n}$ such that they can lead to a violation $\Omega\big((N\log^{-5} N)^{\frac{n-2}{2}}\big)$, where the observables used by each party are tensor products of Pauli matrices.

According to Theorem 2.4 and the comments below it, the previous result is optimal, up to a logarithmic factor[11], in the dimension of the Hilbert spaces and it is only quadratically off from the best upper bound as a function of the number of inputs N. The proof of Theorem 2.5 is probabilistic, being based again on the construction of a random Bell functional which interacts properly with a random pure state via some suitable observables. In fact, since the elements involved in Theorem 2.5 are very simple, we will first explain them and a brief explanation at a more mathematical level will be discussed later.

Let us consider the random state

$$|\varphi\rangle = \frac{1}{\|\bar{g}\|}\sum_{i,j,k=1}^{N} g_{i,j,k}|i,j,k\rangle, \tag{2.4}$$

where $\bar{g}$ is the corresponding non-normalized state. A key point in [14] is the use of Pauli matrices[12] to show that if one considers a six-index tensor $(S_{i,i';j,j';k,k'})_{i,i',j,j',k,k'=1}^{N}$, one can easily define a Bell functional $T = (T_{P,Q,R})_{P,Q,R}$ with the following properties: On the one hand, the inputs P, Q and R are indexed in the tensor product of j Pauli matrices. On the other hand, if the three parties use respectively the observables P, Q and R associated to the corresponding inputs, then it turns out that

$$\sum_{P,Q,R} T_{P,Q,R}\langle\varphi|P \otimes Q \otimes R|\varphi\rangle = N^3\langle\varphi|S|\varphi\rangle. \tag{2.5}$$

Here $S = \sum_{i,i';j,j';k,k'=1}^{N} S_{i,i';j,j';k,k'}|ijk\rangle\langle i'j'k'|$ is regarded as an element in M_{N^3}. Indeed, (2.5) follows easily from the fact that the set $\mathcal{P}_j = \{\text{Pauli matrices}\}^{\otimes j}$ forms an orthogonal basis of M_N with respect to the inner product defined as $\langle A, B\rangle = tr(AB^\dagger)$. Moreover, it is trivial to check that $\langle P, Q\rangle = N\delta_{P,Q}$ for every $P, Q \in \mathcal{P}_j$. Then, by considering the set $\mathcal{P}_j \otimes \mathcal{P}_j \otimes \mathcal{P}_j$ one obtains an orthogonal basis of M_{N^3}, and for every

[11] Pisier has shown in [41] that such a factor can be reduced to $\log^{-\frac{3}{2}} N$.

[12] The only property used by the authors is that the tensor products of j Pauli matrices form an orthogonal basis of M_N formed by observables. Any other such a system would equally work in the proof.

element S in this space one can write (its Fourier expansion)

$$S = \frac{1}{N^3} \sum_{P,Q,R \in \mathcal{P}_j} \langle S, P \otimes Q \otimes R \rangle P \otimes Q \otimes R.$$

Hence, if one defines $T = (T_{P,Q,R})_{P,Q,R \in \mathcal{P}_j}$ such that

$$T_{P,Q,R} = \langle S, P \otimes Q \otimes R \rangle = \sum_{i,i';j,j';k,k'=1}^{N} S_{i,i';j,j';k,k'} P_{i,i'} Q_{j,j'} R_{k,k'},$$

one has the desired property (2.5). Note that in order to obtain this property one needs to double indices as was done in the proof of Theorem 2.3.

On the other hand, let us assume that $\chi, \nu, \zeta : \mathcal{P}_j \to \{-1, 1\}$ achieve the lower bound $\omega(T)$. By defining the hermitian matrices $X = \sum_{P \in \mathcal{P}_j} \chi(P) P$, $Y = \sum_{Q \in \mathcal{P}_j} \nu(Q) Q$, $Z = \sum_{R \in \mathcal{P}_j} \zeta(R) R$ in M_N one can write

$$\omega(T) = \sum_{P,Q,R \in \mathcal{P}_j} T_{P,Q,R} \chi(P) \nu(Q) \zeta(R) = \langle S, X \otimes Y \otimes Z \rangle.$$

Then, the fact $\|X\|_2 = \|Y\|_2 = \|Z\|_2 = N^{3/2}$ implies that

$$\omega(T) \leq N^{9/2} \sup_{X,Y,Z \in B(\mathrm{Herm}(N))} \langle S, X \otimes Y \otimes Z \rangle, \tag{2.6}$$

where $B(\mathrm{Herm}(N))$ denotes the unit ball of the hermitian $N \times N$ matrices with respect to the Frobenius norm.

By looking at (2.5), a potential good choice for the tensor S is $S_{i,i';j,j';k,k'} = g_{i,j,k} g_{i',j',k'}$ for every i, i', j, j', k, k'. Indeed, if one considers this element it is straightforward to check that

$$\omega^*(T) \geq N^3 \langle \varphi | S | \varphi \rangle = N^3 \|\bar{g}\|_2^2 \simeq N^6 \quad \text{with high probability over } g, \tag{2.7}$$

where the last estimate follows from well known results on gaussian variables. Therefore, Theorem 2.5 would follow if one proved that the quantity $\sup_{X,Y,Z \in B(\mathrm{Herm}(N))} \langle T, X \otimes Y \otimes Z \rangle$ is upper bounded by N up to, maybe, some logarithmic terms. Unfortunately, it can be deduced from known results that this quantity is $\Omega(N\sqrt{N})$. The key point in the construction by Briet and Vidick is to remove some of the indices of S. More precisely, they consider the tensor S defined by $S_{i,i';j,j';k,k'} = g_{i,j,k} g_{i',j',k'}$ whenever $i \neq i'$, $j \neq j'$, $k \neq k'$ and $S_{i,i';j,j';k,k'} = 0$ otherwise. It is very easy to see that the corresponding values $\omega^*(T)$ has the same order N^6, since the number of removed terms is negligible in the estimate (2.7). On the other hand, the main result in [14] shows that the classical value of the new functional does decrease by the previous modification:

$$\sup_{X,Y,Z \in B(\mathrm{Herm}(N))} \langle S, X \otimes Y \otimes Z \rangle \lesssim N \log^{5/2} N \quad \text{with high probability over } g. \tag{2.8}$$

This estimate allows us to obtain Theorem 2.5. Note that here, as in the proof of Theorem 2.3, the definition of the Bell functional T can lead to complex coefficients $T_{P,Q,R} = \sum_{i,i';j,j';k,k'=1}^{N} T_{i,i';j,j';k,k'} P_{i,i'} Q_{j,j'} R_{k,k'}$. However, it is trivial that either the real part or the imaginary part must lead to a similar violation up to a constant 2.

The proof of Eq. (2.8) is based on a technical ϵ-net construction using a decomposition of elements in $B(\mathrm{Herm}(N))$ as linear combinations of normalized projections. In their proof the authors show that the corresponding estimate holds for such projections with a sufficiently good concentration so that, by applying a counting argument on the elements of the net, they obtain the result.

A more careful look at the previous argument allows us to see some analogies with the proof of Theorem 2.3 (and the subsequent improvement in [41]). Indeed, one can see that the proof by Briet and Vidick is also reduced to separating two norms in $\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}$, the first one being again the norm $\|S\|_{\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}}$. However, in this case the second norm is:

$$\|S\|_{3,3}=\Big\|\sum_{i,i';j,j';k,k'}^{m} S_{i,i';j,j';k,k'}|i,j,k\rangle\langle i',j',k'|\Big\|_{M_{N^3}}.$$

The reader should note that, while in the proof of Theorem 2.3 the key point was to study the norm of the identity map on $\otimes_{i=1}^{3}\mathbb{C}^{m^2}$ when this space is endowed with the norms $\|\cdot\|_{\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}\otimes_\epsilon\ell_2^{m^2}}$ and $\|\cdot\|_*$ respectively, in the proof of Theorem 2.5 one must compute the norm of the rearrangement map $R:\otimes_{i=1}^{3}\mathbb{C}^{m^2}\to\otimes_{i=1}^{2}\mathbb{C}^{m^3}$ defined by $R(|ii'\rangle|jj'\rangle|kk'\rangle)=|ijk\rangle|i'j'k'\rangle$. Note that doubling indices is an essential point to define this map.

Two points are key to understanding the improvement in Theorem 2.5 with respect to the previous results. The first one is that the transformation from the tensor S in $\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}$ to the Bell functional $T=T(S)$ does not imply an increase in the number of inputs. That is, one has $m=N$. The second point is that the authors gave an essentially optimal separation from the norms they considered in $\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}\otimes\mathbb{C}^{m^2}$ (see [41, Section 2] for details). Finally, we should mention that Theorem 2.5 is not just an existence result, but the proof shows that such an estimate happens with high probability in the choice of the gaussian variables g.

Theorem 2.5 can be understood as a result proving that tripartite correlation Bell functionals give large violations with high probability when they are properly sampled. However, this sampling is rather artificial since, in particular, it implies doubling indices in the coefficients. Regarding the results we explained in Section 1, a plausible next step in the problem is to study the behavior of $T=(\epsilon_{i,j,k})_{i,j,k=1}^{N}$, where T is sampled from $\{-1,1\}^{N^3}$ according to the uniform measure. In fact, this is essentially equivalent to consider $T=(g_{i,j,k})_{i,j,k=1}^{N}$, a family of independent real gaussian variables. It is not difficult to see that $\mathbb{E}\big[\omega(T)\big]\lesssim N^2$ in this case. Any nontrivial estimate on $\mathbb{E}\big[\omega^*(T)\big]$ would definitely be an interesting result. On the other hand, despite the sharpness of Theorem 2.5 there is still a gap with respect to the best known upper bound for the multipartite Bell violation as a function of the number of inputs. It would be interesting to know if one can indeed attain violation of order $\sqrt{N}$ by using a tripartite correlation Bell functional with N inputs per party. Finally, any explicit (non-probabilistic) construction of a tripartite correlation Bell functional leading to unbounded violations would be also very interesting.

3. Correlation Bell inequalities for a large number of parties

3.1. **Symmetric XOR games.** In the works [2], [3] and [4] Ambainis and coauthors studied the scenario of binary inputs correlation Bell inequalities with *many* parties $T = (T_{x_1,\cdots,x_n})_{x_1,\cdots,x_n\in\{0,1\}}$. In this case, it is well known that the quotient between $\omega^*(T)$ and $\omega(T)$ can be equal to $2^{\frac{n}{2}}$ for some particular functionals T ([5], [35]). Moreover, it is also known that such a violation is optimal in this setting [52], [53]. The reader should immediately note that this problem is completely different from the one considered in the previous section, where unbounded Bell violations were proved by considering only a fixed number of parties n (say three) and increasing the number of inputs N. In order to clarify the connection between the works [2], [3], [4] and the explanation in this survey, let us remind the reader that any correlation Bell functional $T = (T_{x_1,\cdots,x_n})_{x_1,\cdots,x_n}$ satisfying $\sum_{x_1,\cdots,x_n}|T_{x_1,\cdots,x_n}| = 1$ can be trivially written as $T = \big(\pi(x_1,\cdots,x_n)f(x_1,\cdots,x_n)\big)_{x_1,\cdots,x_n}$, where π is a probability distribution over the set of inputs and $f : X_1\times\cdots\times X_N \to \{-1,1\}$ is a function. This gives a correspondence between the classical (resp. quantum) value of a correlation Bell functional and the classical (resp. quantum) bias (probability of winning minus probability of losing) of an XOR game. The corresponding game is defined by the distribution π over the set of questions (inputs) and the predicate function, given by $V(x_1,\cdots,x_n,a_1,\cdots,a_n) = 1$ if and only if $a_1\oplus\cdots\oplus a_n = \frac{1}{2}(f(x_1,\cdots,x_n)+1)$. Then, a *symmetric XOR game* is a correlation Bell functional where π is the uniform probability distribution over the set of inputs and the function f is invariant under permutations of the parties[13]. Note that for the case of binary inputs $x_i = 0,1$, a symmetric XOR game T can be described by means of a sequence of $n+1$ bits $(M_0,M_1,\cdots,M_n)$, where $M_j = f(x_1,\cdots,x_n)$ whenever $\sum_{i=1}^n x_i = j$. Then, one can define a probability distribution on the set of these Bell functionals by picking the $(n+1)$-bit string $(M_0,M_1,\cdots,M_n)$ uniformly at random. The main results in [2] and [3] imply that for every $\epsilon > 0$ there exist positive constants $C_1(\epsilon)$, $C_2(\epsilon)$ such that for a high enough n we have

$$\mathbb{P}\Big\{T : \frac{\omega^*(T)}{\omega(T)} \in \Big[C_1(\epsilon)\sqrt{\log n}, C_2(\epsilon)\sqrt{\log n}\Big]\Big\} \geq 1-\epsilon. \tag{3.1}$$

Therefore, large Bell violations occur with probability very close to one on the set of all symmetric XOR games. On the other hand, since the violation $2^{\frac{n}{2}}$ mentioned before is attained on symmetric XOR games, one sees that the violation $\sqrt{\log n}$ is far from being optimal.

The key point to prove Eq. (3.1) is a simplification of both quantities $\omega(T)$ and $\omega^*(T)$, when T is a symmetric XOR games [4]. On the one hand, in [3] the authors proved that in order to study $\omega(T)$ in this case, it suffices to look at $n+1$ deterministic correlations $\gamma_0,\cdots,\gamma_n\in\mathcal{L}$. These correlations are very easy to describe in terms of the strategy followed by the players to play the XOR game corresponding to T. Indeed, γ_k is the correlation obtained if the players follow the deterministic strategy denoted by $(00)^k(01)^{n-k}$, $k = 0,\cdots,n$. Here, for a fixed k the previous notation means that the first

[13]We will not talk about symmetric correlation Bell functionals since these should be defined as those functionals which are invariant under permutations of the parties without any extra restriction.

k players answer always the output 0 while the next $n-k$ players answer the output 0 if they are asked question 0 and they output 1 if they are asked question 1. Interestingly, one can show in addition that only two of these strategies ($k=0$ and $k=n$) are relevant on average. Indeed, [3, Theorem 4] and [3, Theorem 5] show, respectively, that

$$\mathbb{E}\Big\{ \max\big\{\big|\langle T,\gamma_0\rangle\big|, \big|\langle T,\gamma_n\rangle\big|\big\}\Big\} = \frac{0.8475...+o(1)}{n^{\frac{1}{4}}}, \tag{3.2}$$

and

$$\mathbb{P}\Big\{ \max_{1\leq k\leq n-1} \big|\langle T,\gamma_k\rangle\big| \geq \frac{c}{n^{\frac{1}{4}}}\Big\} = O\Big(\frac{1}{n}\Big) \quad \text{for any constant } c>0. \tag{3.3}$$

A simplification in the analysis of the classical value of a symmetric XOR game allows the authors in [3] to prove Eq. (3.2) and Eq. (3.3) by using classical probabilistic techniques. One can deduce from these equations that for every $\epsilon>0$ there exist positive constants $C_1(\epsilon)$, $C_2(\epsilon)$ such that the probability of the classical value $\omega(T)$ being in $[C_1(\epsilon)n^{-1/4}, C_2(\epsilon)n^{-1/4}]$ is at least $1-\epsilon$. This is done by using that the random variable $\max\big\{\big|\langle T,\gamma_0\rangle\big|, \big|\langle T,\gamma_n\rangle\big|\big\}$ converges (as n tends to infinity) to the sum of two Gaussian random variables with known mean and variance (see [2, Section 5.2] for details). However, note that this does not allow us to state the previous estimate for fixed constants C_1 and C_2 and probability $1-o(1)$. This lack in the concentration of measure seems to be due to the nature of the value $\omega(T)$ rather than to the analysis performed by the authors.

At the same time, it was proved in [2, Theorem 1, Theorem 2] that

$$\lim_{n\to\infty} \mathbb{P}\Big[C_1\frac{\sqrt{\ln n}}{n^{\frac{1}{4}}} \leq \omega^*(T) \leq C_2\frac{\sqrt{\ln n}}{n^{\frac{1}{4}}}\Big] = 1 \quad \text{for certain constants } C_1, C_2>0. \tag{3.4}$$

We see that this result is, in terms of concentration, stronger than the previous one for the classical value of the game, since here we can fix the constants C_1, C_2 and get probability $1-o(1)$. To show Eq. (3.4) the authors first used a previous result in [52], [53] stating that the quantum value of a symmetric XOR game is given by $\omega^*(M) = \max_{|z|=1}\Big|\frac{1}{2^n}\sum_{j=0}^n (-1)^{M_j}\binom{n}{j} z^j\Big|$. Then, computing $\omega^*(M)$ reduces to the maximization of the absolute value of a polynomial in one complex variables. On the other hand, the previous expression is trivially upper and lower bounded by functions depending on the real and imaginary part of the corresponding polynomial,

$$\max_{\alpha\in[0,2\pi]}\Big|\frac{1}{2^n}\sum_{j=0}^n (-1)^{M_j}\binom{n}{j}\cos(j\alpha)\Big|, \quad \max_{\alpha\in[0,2\pi]}\Big|\frac{1}{2^n}\sum_{j=0}^n (-1)^{M_j}\binom{n}{j}\sin(j\alpha)\Big|.$$

If $(M_0, M_1, \cdots, M_n)$ are chosen at random, these expressions reduce to random trigonometric polynomials studied in [45] and the problem is actually reduced to studying the quantities $M_n(t) = \max_{\alpha\in[0,2\pi]}\big|P_n(x,t)\big|$ for $P_n(x,t) = \sum_{m=0}^n r_m\psi_m(t)\cos mx$, where $(\psi_m)_m$ is the Rademacher system. Although this is the key object of study in [45], the results in this work do not fit directly into the problem considered in [2] and obtaining the estimate (3.4) requires a careful study of the particular case.

3.2. **Sampling quantum n-partite states.** A different way of studying Bell violations consists in sampling quantum states according to some random procedure and looking at the violations they can produce. Note that this is not the same as sampling quantum correlations at random as it was considered in Section 1. In the current situation one should regard both the POVMs and the Bell functionals as free parameters in the problem. One can then define a natural measure of how nonlocal a quantum state is (see [37]). However, the great freedom in the problem makes it difficult to handle. An intimately related problem is to study the probability of a quantum state being nonlocal with respect to a particular family of Bell functionals. This is the context considered in [22], where the authors restricted themselves to the setting of n-partite binary inputs correlation Bell functionals $(T_{x_1,\cdots,x_n})_{x_1,\cdots,x_n\in\{0,1\}}$. It is well known that in this case the set of correlation Bell functionals[14] can be replaced by the single nonlinear functional [53], [54]

$$\sum_{1\leq i\leq n}\sum_{x_i\in\{0,1\}}\Big|\prod_{j=1}^{n}\frac{A_0^j+(-1)^{x_j}A_1^j}{2}\Big|\leq 1,$$

where $A_0^j, A_1^j=\pm 1$, $1\leq j\leq n$ represent the deterministic measurement results for the pair of measurements in site j. Hence, one can define the function $LV:\mathbb{S}^{2d^n-1}\to\mathbb{R}^+$ on the unit sphere of $(\mathbb{C}^d)^{\otimes n}$; that is, the set of pure quantum state $|\psi\rangle$ of n d-dimensional systems, by

$$LV(|\psi\rangle):=\sup\sum_{1\leq i\leq n}\sum_{x_i\in\{0,1\}}\Big\langle\psi\Big|\bigotimes_{j=1}^{n}\frac{A_0^j+(-1)^{x_j}A_1^j}{2}\Big|\psi\Big\rangle.$$

Here, the supremum runs over all possible choices of pair of observables A_0^j, A_1^j per site $j=1,\cdots,n$. Note also that we use a similar notation to the one used for Bell functionals $LV(M)$ on purpose since both quantities measure the largest Bell violation of different "objects" (states and Bell functionals respectively).

The main result in [22] is the following, which describes the probabilistic behavior of LV with respect to the uniform measure on $\mathbb{S}^{2d^n-1}$.

Theorem 3.1. *Let n and d be two natural numbers larger than 1, let $|\psi\rangle$ be a unit vector distributed according to the uniform measure in the unit sphere $\mathbb{S}^{2d^n-1}$ of $(\mathbb{C}^d)^{\otimes n}$ and denote $\mathcal{A}_v=\{|\psi\rangle : LV(|\psi\rangle)>v\}$. Then, the following inequality holds:*

$$\mathbb{P}(\mathcal{A}_v)\leq 2\Big(\frac{n2^{n+1}d^2}{\delta}+2\Big)^{2d^2n}e^{-\frac{(v-\delta-c_{d,n})^2(\frac{d}{2})^n}{9\pi^3}}. \tag{3.5}$$

Here, δ is any positive number, $v>c_{d,n}+\delta$ and $c_{d,n}=(\frac{2}{d})^{\frac{n}{2}}+\frac{d-2}{2}$.

As we mentioned in the previous section, for binary inputs correlation Bell functionals the optimal quotient between the quantum and the classical value is $2^{\frac{n}{2}}$. Note that Eq. (3.5) implies that for large n most pure states do not get even close to this violation. For the particular case $d=2$ (qubits) one has that $c_{d,n}=1$ for every n, so as long as $v\geq cn$ for a suitable constant c, $\mathbb{P}(\mathcal{A}_v)\to 0$ as n goes to infinity. It was conjectured

[14] Here, we really mean the equations defining the facets of the set $\mathcal{L}$.

in [42] that this is also true if v is of order $\sqrt{n \log n}$ but this problem seems to remain open (see [22, Section I] for details). On the other hand, if $d \geq 3$ Eq. (3.5) implies that most of the states do not violate any of these inequalities for n large enough. Indeed, note that $\lim_n c_{d,n} = \frac{d-2}{2} < 1$, so we can choose $\delta > 0$ with $\delta + c_{d,n} < v < 1$ such that $\lim_n \mathbb{P}(\mathcal{A}_v) = 0$ super exponentially. It could be interesting to study this problem when one samples not-necessarily-pure states. Although the way to sample in that case is not so obvious, there are very interesting works showing different behaviors in that situation (see [7], [47], [55] and the references therein).

Theorem 3.1 is proved via a concentration-type argument. First, the authors consider an ϵ-net for the set $\mathcal{B}$ of possible pairs of observables A_0^j, A_1^j per site $j = 1, \cdots, n$. This allows them to fix a finite set $\mathcal{B}_\epsilon$ of these elements to work with. On the other hand, for a fixed $B \in \mathcal{B}_\epsilon$, one considers the corresponding function $LV^B(|\psi\rangle) : \mathbb{S}^{2d^n-1} \to \mathbb{R}^+$. It turns out that this function is quite regular in terms of its Lipschitz constant. Then, one uses this regularity in two different ways. First of all, it allows us to pass from taking the supremum on $\mathcal{B}$ to considering $\mathcal{B}_\epsilon$ by means of the inequality

$$\mathbb{P}(\mathcal{A}_v) := \mathbb{P}\big\{|\psi\rangle : \sup_{B \in \mathcal{B}} LV^B(|\psi\rangle) > v\big\} \leq \mathbb{P}\big\{|\psi\rangle : \sup_{B \in \mathcal{B}_\epsilon} LV^B(|\psi\rangle) > v - \delta\big\},$$

where here $\epsilon = \delta / d^2 N 2^{N+1}$. Secondly, for a fixed $B \in \mathcal{B}_\epsilon$ one can compute the expectation $\mathbb{E}[LV^B]$ and use Levy's Lemma to bound the probability of the points for which the function is far from its expectation $\mathbb{P}\big(|f - E[f]| > \epsilon\big) \leq 2 \exp\big(-\frac{(n+1)\epsilon^2}{9\pi^3 \lambda}\big)$, where here $f : \mathbb{S}^n \to \mathbb{R}$ is a real function and λ is its Lipschitz constant with respect to the euclidean distance. Putting these ingredients together allows the authors to obtain Eq. (3.5) by using a counting argument.

3.3. **Sampling random measurements.** In [32], [46], [51], the authors considered quantum correlations sampled in such a way that the quantum state is fixed and the observables are chosen at random. This is a very interesting problem from an experimental point of view since it is motivated by the requirement of well calibrated devices and a common reference frame between the parties in the standard Bell experiments. Showing that random measurements produce Bell violations implies that the previous assumptions can be removed. In [32], [51] the authors considered the n-partite 2-dimensional GHZ state: $|\psi\rangle = 2^{-1/2} \sum_{i=1}^2 |i\rangle^{\otimes n}$ and studied the correlations when each party j measures with two possible observables $A_{1,j}$, $A_{2,j}$ chosen at random according to the law: $A_{i,j} = n_{i,j} \cdot \sigma$, where $n_{i,j}$ are uniform and independent vectors on the unit sphere of $\mathbb{C}^3$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is defined via the Pauli matrices. Numerical computations suggest that the corresponding correlations are nonlocal with probability tending to one as n goes to infinity. This behavior is also studied when the two measurement per party are chosen so that n_1^j and n_2^j are orthogonal but still random to show that nonlocality is even more likely in this last case. Again, numerical evidences in [46] suggest that the correlations produced by each party performing N randomly chosen measurements on $|\psi\rangle = 2^{-1/2} \sum_{i=1}^2 |i\rangle^{\otimes 2}$ lead to a similar asymptotic behavior. Since the techniques in these papers are a bit different from those considered in the current survey we will not explain them in detail. In fact, it would be very interesting to understand some of the results presented in these papers from a more analytical point of view by (maybe) using some of the techniques explained in other sections of this survey. However, we refer the reader to the original works, where many other results can be found as well as some very nice explanations about the experimental interpretation of the different ways of sampling.

4. Unbounded violations for bipartite Bell inequalities

The setting of general Bell functionals is much richer than the corresponding one for correlations. The new parameter given by the number of outputs K leads to study the quantity $\mathcal{D}_n(N,K,d)$ analogous to $\mathcal{D}_n(N,d)$ introduced in Section 2.2, where n is the number of parties. The application of the parallel repetition theorem [43] to the magic square game (or any pseudo-telepathy game [11]) and also the work [31] show that $\mathcal{D}_2(N,K,d)$ cannot be upper bounded by a uniform constant independent of the parameters N, K and d (see [28, Introduction] for details). That is, there exist *unbounded violations of bipartite Bell inequalities*. However, the unbounded Bell violations obtained from the previously mentioned results are far from the bets known upper bounds provided in [28, Theorem 14] (see also [38]):

Theorem 4.1. *The following upper bound holds:*

$$\mathcal{D}_2(N,K,d) = O\big(h\big), \quad \textit{where} \quad h = \min\big\{N,K,d\big\}.$$

In the paper [29] the authors used a random construction to show the lower bound $\mathcal{D}_2([2^{\log^2 N/2}]^N, N, N) = \Omega(\sqrt{N}/\log^2 N)$, where $[x]$ denotes the smallest natural number p such that $x \leq p$, improving the existing estimates so far. Since this result has been recently improved in two different ways we will omit the details in this survey. In the current section we will explain a more recent result which improves and simplifies the work in [29]. Based on a random construction, in [28] the authors proved that $\mathcal{D}_2(N,N,N) = \Omega(\sqrt{N}/\log N)$. According to Theorem 4.1 the previous estimate is only quadratically off from the best upper bounds in all the parameters of the problem at the same time. Actually, the most relevant point of this result with respect to the previous ones is the decrease in the number of inputs from exponential to polynomially many. We will also explain the existence of a bipartite Bell functionals providing the lower bound $\mathcal{D}_2(2^N/N, N, N) = \Omega(N/\log^2 N)$ in Section 4.2. This gives an essentially optimal lower bound for $\mathcal{D}_2(N,K,d)$ as a function of the number of outputs K and the dimension of the Hilbert space d. On the negative side, this last example requires an exponential number of inputs.

4.1. Unbounded violation with polynomially many parameters.

In [28] the authors introduced the following construction: Consider a fixed number of ± 1 signs $\epsilon_{x,a}^k$ with $x,a,k = 1,\cdots,N$. For a constant K define the vectors $|\tilde{u}_x^a\rangle = \frac{1}{\sqrt{NK}}(1, \epsilon_{x,a}^1, \cdots, \epsilon_{x,a}^N)$ in $\mathbb{C}^{N+1}$ for every $x,a = 1,\cdots,N$, and

a) *Bell functional* $\big(\tilde{M}_{x,y}^{a,b}\big)_{x,y;a,b=1}^{N,N+1}$:

$$\tilde{M}_{x,y}^{a,b} = \begin{cases} \frac{1}{N^2}\sum_{k=1}^N \epsilon_{x,a}^k \epsilon_{y,b}^k & \text{if } \ x,y,a,b = 1,\cdots,N\,, \\ 0 & \text{otherwise.} \end{cases}$$

b) *POVMs* $\{\tilde{E}_x^a\}_{x,a=1}^{N,N+1}$ acting on $\mathbb{C}^{N+1}$:

$$\tilde{E}_x^a = \begin{cases} |\tilde{u}_x^a\rangle\langle\tilde{u}_x^a| & \text{for } a = 1, \cdots, N, \\ \mathbb{1} - \sum_{a=1}^n \tilde{E}_x^a & \text{for } a = N+1 \end{cases}$$

for $x = 1, \cdots, N$.

c) *State*: Let $|\varphi_\alpha\rangle = \sum_{i=1}^{N+1} \alpha_i |ii\rangle$, where $(\alpha_i)_{i=1}^{N+1}$ is a decreasing and non-negative sequence satisfying $\sum_{i=1}^{N+1} \alpha_i^2 = 1$.

The following result can be found in [28, Theorem 2].

Theorem 4.2. *There exist universal positive constants C and K such that for every natural number N there exists a choice of signs $\{\epsilon_{x,a}^k\}_{x,a,k=1}^N$ satisfying that $\{\tilde{E}_x^a\}_{x,a=1}^{N,N+1}$ define POVMs for every x, $\omega(\tilde{M}) \leq C \log N$ and*

$$\sum_{x,y;a,b=1}^{N,N+1} \tilde{M}_{x,y}^{a,b} \langle\varphi_\alpha|\tilde{E}_x^a \otimes \tilde{E}_y^b|\varphi_\alpha\rangle \geq \frac{1}{K^2}\alpha_1 \sum_{i=2}^{N+1} \alpha_i. \tag{4.1}$$

Moreover, the probability of the elements (choices of signs) satisfying Eq. (4.1) when they are chosen independently and uniformly at random tends to 1 exponentially fast as N tends to infinity.

According to the previous theorem, the estimate $LV(\tilde{M}) = \Omega(\sqrt{N}/\log N)$ follows by considering an asymmetric state of the form $\alpha_1 = 1/\sqrt{2}$ and $\alpha_i = 1/\sqrt{2N}$. As in the case of tripartite correlation Bell functionals (see Section 2), a probabilistic construction provides a good lower bound for $\mathcal{D}_2(N,N,N)$ which almost matches the known upper bounds.

The estimate $\omega(\tilde{M}) = O(\log N)$ in Theorem 4.2 can be obtained from well known probabilistic results. The proof in [28] was based on Chevet's inequality (see also [44] for a slight improvement based on standard estimates for gaussian variables).

The key to understanding Eq. (4.1) is to consider the following family of rank-one operators $E_x^a = 1/\sqrt{N}|u_x^a\rangle\langle 0|$ for every $x, a = 1, \cdots N$, where $|u_x^a\rangle = 1/\sqrt{NK}(\epsilon_{x,a}^1, \cdots, \epsilon_{x,a}^N)$. Then, it is not difficult to see that if one defines the rank-one operator $\eta = |00\rangle\langle\psi|$, where $|\psi\rangle = 1/\sqrt{N}\sum_{i=1}^N |ii\rangle$, one obtains

$$\sum_{x,y;a,b=1}^{N} \tilde{M}_{x,y}^{a,b} tr(E_x^a \otimes E_b^y \eta) = \Omega(\sqrt{N}). \tag{4.2}$$

The problem here is that, for a fixed x, the elements E_x^a, $a = 1, \cdots, N$, do not define a POVM (the operators are not even selfadjoint!) and that the operator η is not a state. Nevertheless, the previous family of operators should be understood as a family of non-positive POVMs. The interesting point here is that the nice properties of these operators assure that, by replacing the vectors u_x^a by $\tilde{u}_x^a$ as in the statement of Theorem 4.2, one obtains a new family of operators $\tilde{E}_x^a = |\tilde{u}_x^a\rangle\langle\tilde{u}_x^a|$ satisfying $\sum_{a=1}^N \tilde{E}_x^a \leq \mathbb{1}$. The new elements $\tilde{E}_x^a$ are defined by adding more entries to the matrices defining the operators E_x^a so that they are positive and verify the required property on their sum. In particular,

E_x^a "is encoded" in the first column of $\tilde{E}_x^a$ up to an extra element equal to 1. Then, the reader can guess and easily check that if one considers the state $|\varphi\rangle = 1/\sqrt{2}\big(|00\rangle + |\psi\rangle\big)$, the terms $|00\rangle\langle\psi|$ and $|\psi\rangle\langle 00|$ appearing in $\rho = |\varphi\rangle\langle\varphi|$ have the same effect as in Eq. (4.2). Indeed, the rest of the terms in ρ do not play any role. Actually, a similar argument can be used for a general state $|\varphi_\alpha\rangle = \sum_{i=1}^{N+1} \alpha_i |ii\rangle$ to obtain the estimate (4.1) (see [28, Section 3] for details).

Finally, the reason to pass from M to $\tilde{M}$ is that the previous argument gives an element $\big(tr(\tilde{E}_x^a \otimes \tilde{E}_b^y \rho)\big)_{x,y,a,b=1}^N$ for which $\sum_{a=1}^N \tilde{E}_x^a \leq \mathbb{1}$ for every x. Hence, by naively adding an extra output $a = N+1$ per measurement x, the previous modification on the element M allows one to complete the family of operators to obtain $\sum_{a=1}^{N+1} \tilde{E}_x^a = \mathbb{1}$.

Interestingly, if one plugs the $(N+1)$-dimensional maximally entangled state $|\psi\rangle$ in Eq. (4.1), no large violation is obtained. It is an open question whether one can get large Bell violations on $\tilde{M}$ by using the maximally entangled state, even if we do not restrict its dimension. In [28] the authors modified the functional $\tilde{M}$ to obtain such an example. More precisely, [28, Theorem 3] shows the existence of a Bell functional $\bar{M}$ with 2^{N^2} inputs and $N+1$ outputs per party, and POVMs $\{\bar{E}_x^a\}_{x,a}$ acting on $\mathbb{C}^{N+1}$ such that $\bar{M}$ and $\{\bar{E}_x^a\}_{x,a}$ verify the same properties as in the statement of Theorem 4.2 and, in addition, $\sup\{|\langle\bar{M}, P\rangle| : P \in \mathcal{Q}_{max}\} = O(\log N)$, where $\mathcal{Q}_{max}$ denotes the set of all quantum probability distributions P constructed with the maximally entangled state in any dimension. Previous results in this direction were obtained in [33], [50], where the authors proved that certain quantum probability distributions cannot be written by using the maximally entangled state. We will not explain [28, Theorem 3] in detail since its proof is quite technical and it is based on certain estimates on completely bounded norms. Furthermore, an improvement of the previous construction was made in [44, Section 3], where the author gave an explicit Bell functional with $2^N/N$ inputs and N outputs per party satisfying the same properties as $\bar{M}$. The proof of this result is based on techniques from quantum information theory.

We must emphasize at this point that the original aim of Theorem 4.2 was to show the existence of Bell functionals for which the quantum value is much larger than its classical value. Here, the use of random techniques is only a mathematical tool to construct such a functional, mainly due to our limitations to compute $\omega(M)$ and $\omega^*(M)$ for explicit functionals M. This random construction leads to the estimate $\mathcal{D}_2(N,N,N) = \Omega(\sqrt{N}/\log N)$. A natural open question is whether one can find another element M leading to the lower bound $\mathcal{D}_2(N,N,N) = \Omega(N)$ (up to maybe logarithmic factors), which would be optimal in all the parameters at the same time. The reader should compare these results to those explained at the beginning of the section.

On the other hand, as in the case of Theorem 2.5, one can understand the statement of Theorem 4.2 as a result providing large Bell violations with high probability when the Bell functionals are properly sampled. In fact, while in the case of tripartite correlations the sampling procedure was quite artificial, involving in particular a duplication of indices, in the current situation the sampling looks quite natural if one ignores the added extra zeros, which do not play an important role. In this sense, Theorem 4.2 can be understood as an analogous result to Theorem 1.2, now stating that random Bell functionals lead to unbounded violations with high probability. However, the reader could argue (and we would agree!) that some other natural samplings can be done in the current situation. One could actually consider completely random functionals M defined as $M_{x,y}^{a,b} = g_{x,y}^{a,b}$ for every x, y, a, b, where $g_{x,y}^{a,b}$ are independent real gaussian variables. At this moment in time, we do not have non-trivial estimates for $LV(M)$ in this case.

The defect in the sampling criterium becomes more extreme when one samples quantum probability distributions. This has motivated the works in this direction to focus on very particular situations. An example of this can be found in [6], where the authors study the violation of the CGLMP bipartite Bell inequality M_{CGMLP}^K with binary inputs and K possible outputs per measurement [20]. Actually, even though the inequality M_{CGMLP}^K is fixed, the authors in [6] need an assumption about the best POVMs for that inequality to be able to analyze its probabilistic behavior. Once the measurements are fixed, the problem of sampling quantum probability distributions is reduced to sampling quantum states. The main result in [6] gives the expected value of the violation for the Bell inequality M_{CGMLP}^K with respect to the uniform measure on K-dimensional bipartite pure states; that is, when the pure states are uniformly sampled from the unit sphere of $\mathbb{C}^{K^2}$. This is done analytically for $K = 2$ and in the range of large K, while for intermediate values it is done numerically. The key point of this study is that the previous restrictions allow the author to write the quantum value of the corresponding inequality as a function of the Schmidt coefficients of the states (see [6, Eq. (3)]). Interestingly, even for this restricted setting the authors need to use nontrivial techniques from random matrix theory to analyze the problem.

4.2. **A comment about some sharp explicit constructions.** We will finish this work by briefly mentioning two fully explicit constructions introduced in [17]. Since this survey is focused on random constructions, we will not explain these examples in detail and we refer the reader to the original paper. However, the relevance of these examples forces us to mention them. The first one, called *Hidden matching game*, gives a bipartite Bell functional with 2^N inputs and N outputs per party, which leads to a violation of order $\sqrt{N}/\log N$. Although this order is slightly worse than the one in Theorem 4.2 and some of the parameters are exponentially higher, this Bell functional presents a very particular form. Indeed, it is introduced via a two-prover one-round game G for which $\omega^*(G) = 1$. However, we must point out that in [17] the authors show that the quotient between the quantum and the classical bias of the game G is $\beta^*(G)/\beta(G) = \Omega(\sqrt{N}/\log N)$. From here one can obtain a Bell functional M (which is not a two-prover one-round game anymore and it has different quantum and classical values from G) such that $\omega^*(M)/\omega(M) = \Omega(\sqrt{N}/\log N)$ (see [17, Section 2], for a discussion about the relation between the bias of a game and the value of Bell functionals).

The second example provided in [17] is the *KV game*, which was first introduced by Khot and Visnoi in [36] to show a large integrality gap for a SDP relaxation of certain complexity problems. In [17] the authors carried out a careful analysis of the game to show that it provides a Bell functional with $2^N/N$ inputs and N outputs per party, leading to a Bell violation of order $N/\log^2 N$. Since this order is attained on a quantum probability distribution constructed with the N-dimensional maximally entangled state, it follows that $\mathcal{D}_2(2^N/N, N, N) = \Omega(N/\log^2 N)$. According to Theorem 4.1, the previous result is essentially optimal in both the number of outputs and the dimension of the Hilbert space. Finally, this Bell functional is particularly interesting because it is a two-prover one-round game, which implies some additional structures (see [17] for details). As an interesting remark we should point out that the fact that the maximally entangled state is the key element to study the KV game is not a coincidence. Indeed, it was proved in [28, Theorem 10] that for any Bell functional with positive coefficients (in particular, any two-prover one-round game) the maximally entangled state always gives the largest violation up to a logarithmic factor in the dimension of the Hilbert space. In this sense, the KV game is very different from the Bell functional given in Theorem 4.2.

Finally, in the very recent paper [27] the authors introduced a method to reduce the number of inputs in (some) Bell functionals while preserving the quotient $\omega^*(M)/\omega(M)$. Then, the author applied that procedure to the KV game explained in the previous paragraph to obtain a new Bell functional KV^{red} (with possible signed coefficients) with $\simeq N^8$ inputs and N outputs per player an such that $\omega^*(\mathrm{KV}^{red})/\omega(\mathrm{KV}^{red}) = \Omega(N/\log^2 N)$. Moreover, since the corresponding quantum strategy can be obtained by using the N-dimensional maximally entangled state, this result shows $\mathcal{D}_2(N^8, N, N) = \Omega(N/\log^2 N)$. Although this estimate is still far from the best upper bound as a function of the number of inputs $O(N)$ in Theorem 4.1, it shows that the use of exponentially many inputs is not needed to obtain (almost) optimal estimates in the rest of the parameters. Interestingly, the reduction atom method proved in [27] is based on the introduction of some randomness in the coefficients of the Bell functionals. Indeed, roughly speaking, KV^{red} is obtained by randomly removing some (many) inputs from the KV game (see [27, Section A] for details).

Acknowledgments

The author would like to thank the referees for their many useful comments which helped to improve the readability of this paper.

Author's research was supported by the Spanish projects MTM2011-26912, Comunidad de Madrid QUITEMAD+ S2013/ICE-2801, and MINECO: ICMAT Severo Ochoa project SEV-2011-0087 and the "Ramón y Cajal" program (RYC-2012-10449).

References

[1] A. Ambainis, A. Backurs, K. Balodis, D. Kravcenko, R. Ozols, J. Smotrovs, M. Virza, *Quantum strategies are better than classical in almost any XOR game*, Automata, Languages, and Programming Lecture Notes in Computer Science Volume 7391, 25-37 (2012).

[2] A. Ambainis, J. Iraids, *Provable Advantage for Quantum Strategies in Random Symmetric XOR Games*. Available in arXiv:1302.2347.

[3] A. Ambainis, J. Iraids, D. Kravchenko, M. Virza, *Advantage of quantum strategies in random symmetric XOR games*, Mathematical and Engineering Methods in Computer Science, Lecture Notes in Computer Science, vol. 7721, 57-68 (2013).

[4] A. Ambainis, D. Kravchenko, N. Nahimovs, A. Rivosh, *Nonlocal quantum XOR games for large number of players*, Theory and Applications of Models of Computation, Lecture Notes in Computer Science, vol. 6108, 72-83 (2010).

[5] M. Ardehali, *Bell inequalities with a magnitude of violation that grows exponentially with the number of particles*, Phys. Rev. A, 46(9), 5375-5378 (1992).
[6] M. R. Atkin, S. Zohren, *Violations of Bell inequalities from random pure states.* Available in arXiv:1407.8233.
[7] G. Aubrun, S. Szarek, D. Ye, *Entanglement thresholds for random induced states*, Comm. Pure Appl. Math. 67, 129-171 (2014).
[8] J.S. Bell, *On the Einstein-Poldolsky-Rosen paradox*, Physics 1, 195 (1964).
[9] R. C. Blei, *Multidimensional extensions of the Grothendieck inequality and applications*, Arkiv fur Matematik, 17, 51-68 (1979).
[10] F. Bombal, D. Pérez-García, I. Villanueva, *Multilinear extensions of Grothendieck's theorem*, Q. J. Math. 55, 441-450 (2004) .
[11] G. Brassard, A. Broadbent, A. Tapp, *Quantum Pseudo-Telepathy*, Foundations of Physics, Volume 35, Issue 11, 1877-1907 (2005).
[12] M. Braverman, K. Makarychev, Y. Makarychev, A. Naor, *The Grothendieck constant is strictly smaller than Krivine's bound*, Forum of Mathematics, Pi, Volume 1 (2013).
[13] J. Briët, H. Buhrman, T. Lee, T. Vidick, *Multipartite entanglement in XOR games. Quantum Information Processing* 13, 334-360 (2013).
[14] J. Briet, T. Vidick, *Explicit lower and upper bounds on the entangled value of multiplayer XOR games*, Comm. Math. Phys. 321(1), (2013).
[15] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, S. Wehner, *Bell nonlocality*, Rev. Mod. Phys. 86, 419 (2014).
[16] H. Buhrman, R. Cleve, S. Massar, R. de Wolf, *Non-locality and Communication Complexity*, Rev. Mod. Phys. 82, 665 (2010).
[17] H. Buhrman, O. Regev, G. Scarpa, R. de Wolf, *Near-Optimal and Explicit Bell Inequality Violations*, IEEE Conference on Computational Complexity 2011: 157-166.
[18] T. K. Carne, *Banach Lattices and Extensions of Grothendieck's Inequality*, J. London Math. Soc., 21 (3), 496-516 (1980).
[19] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt, *Proposed Experiment to Test Local-Hidden-Variable Theories*, Phys. Rev. Lett. 23, 880 (1969).
[20] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, *Bell inequalities for arbitrarily high dimensional systems*, Phys. Rev. Lett. 88(4), 040404 (2002).
[21] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, (1993).
[22] R. C. Drumond, R. I. Oliveira, *Small violations of full correlations Bell inequalities for multipartite pure random states*, Physical Review A, 86 (1), 012117 (2012).
[23] A. Einstein, B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. 47, 777 (1935).
[24] C. E. González-Guillén, C. H. Jiménez, C. Palazuelos, I. Villanueva, *Sampling quantum nonlocal correlations with high probability.* Comm. Math. Phys. 344, 141-154 (2016).
[25] C. E. González-Guillén, C. Lancien, C. Palazuelos, I. Villanueva, *Random quantum correlations are generically non-classical.* Available in arXiv:1607.04203.
[26] C. González-Guillén, C. Palazuelos, I. Villanueva, *Distance between Haar unitary and random gaussian matrices.* J. Theor. Probab. DOI 10.1007/s10959-016-0712-6.
[27] M. Junge, T. Oikhberg, C. Palazuelos, *Reducing the number of inputs in nonlocal games*, J. Math. Phys. 57, 102203 (2016).
[28] M. Junge, C. Palazuelos, *Large violation of Bell inequalities with low entanglement*, Comm. Math. Phys. 306 (3), 695-746 (2011).
[29] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, M.M. Wolf, *Unbounded violations of bipartite Bell Inequalities via Operator Space theory.* Comm. Math. Phys. 300 (3), 715-739 (2010).
[30] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, M.M. Wolf, *Operator Space theory: a natural framework for Bell inequalities*, Phys. Rev. Lett. 104, 170405 (2010).
[31] J. Kempe, O. Regev, B. Toner, *The Unique Games Conjecture with Entangled Provers is False*, Proceedings of 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2008).

[32] Y.-C. Liang, N. Harrigan, S. D. Bartlett, T. Rudolph, *Nonclassical Correlations from Randomly Chosen Local Measurements*, Phys. Rev. Lett. 104, 050401 (2010).
[33] Y.-C. Liang, T. Vertesi, N. Brunner, *Semi-device-independent bounds on entanglement*, Phys. Rev. A 83, 022108 (2011).
[34] V. A. Marcenko, L. A. Pastur, *Distribution of eigenvalues for some sets of random matrices*, Math. USSR Sbornik, 1:457-483, (1967).
[35] D. Mermin, *Extreme quantum entanglement in a superposition of macroscopically distinct states*, Phys Rev Lett. 65 (15), 1838-1840 (1990).
[36] S. Khot, N. Vishnoi, *The unique games conjecture, integrality gap for cut problems and embeddability of negative type metrics into ℓ_1*, Proceedings of 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005).
[37] C. Palazuelos, *On the largest Bell violation attainable by a quantum state*, J. Funct. Anal., 267 (7), 1959-1985 (2014).
[38] C. Palazuelos, T. Vidick, *Survey on Nonlocal Games and Operator Space Theory*, J. Math. Phys. 57, 015220 (2016).
[39] D. Pérez-García, M.M. Wolf, C. Palazuelos, I. Villanueva, M. Junge, *Unbounded violation of tripartite Bell inequalities*, Comm. Math. Phys. 279 (2), 455-486 (2008).
[40] G. Pisier, *Grothendieck's Theorem, past and present*, Bull. Amer. Math. Soc. 49, 237-323 (2012). See also arXiv:1101.4195.
[41] G. Pisier, *Tripartite Bell inequality, random matrices and trilinear forms.* Available in arXiv:1203.2509.
[42] I. Pitowsky, *Macroscopic objects in quantum mechanics: A combinatorial approach*, Phys. Rev. A 70, 022103-1-6 (2004).
[43] R. Raz, *A Parallel Repetition Theorem*, SIAM Journal on Computing 27, 763-803 (1998).
[44] O. Regev, *Bell Violations through Independent Bases Games*, Quantum Inf. Comput., 12(1-2): 9-20 (2012).
[45] R. Salem, A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Mathematica 91(1), 245-301 (1954).
[46] P. Shadbolt, T. Vertesi, Y.-C. Liang, C. Branciard, N. Brunner, J. O'Brien, *Guaranteed violation of a Bell inequality without aligned reference frames or calibrated devices*, Scientific Reports 2, 470 (2012).
[47] S. Szarek, E. Werner, K. Zyczkowski, *How often is a random quantum state k-entangled?*, J. Phys. A: Math. Theor. 44, 045303 (2011).
[48] A. Tonge, *The Von Neumann inequality for polynomials in several Hilbert-Schmidt operators*, J. London Math. (2) 1, 519-526 (1978).
[49] B. S. Tsirelson, *Some results and problems on quantum Bell-type inequalities*, Hadronic J. Supp. 8(4), 329-345 (1993).
[50] T. Vidick, S. Wehner, *More nonlocality with less entanglement*, Phys. Rev. A 83, 052310 (2011).
[51] J. J. Wallman, Y.-C. Liang, S. D. Bartlett, *Generating nonclassical correlations without fully aligning measurements*, Phys. Rev. A 83, 022110 (2011).
[52] R. F. Werner, M. M. Wolf, *Bell inequalities and Entanglement*, Quant. Inf. Comp. 1(3), 1-25 (2001).
[53] R. F. Werner, M. M. Wolf, *All multipartite Bell correlation inequalities for two dichotomic observables per site*, Phys. Rev. A 64, 032112 (2001).
[54] M. Zukowski, C. Brukner, *Bell's theorem for general N-qubit states*, Phys. Rev. Lett. 88, 210401 (2002).
[55] K. Zyczkowski, K. A. Penson, I. Nechita, B. Collins, *Generating random density matrices*, J. Math. Phys. 52 (6), 062201 (2011).

Instituto de Ciencias Matemáticas (ICMAT), Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040, Madrid, Spain

E-mail address: `carlospalazuelos@mat.ucm.es`