

QUASI COMPLETELY CONTINUOUS MULTILINEAR OPERATORS

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ABSTRACT. We introduce the concept of quasi completely continuous multilinear operators and use this concept to characterize, for a wide class of Banach spaces X_1, \dots, X_k , the multilinear operators $T : X_1 \times \dots \times X_k \rightarrow X$ with an X -valued Aron-Berner extension.

1. INTRODUCTION AND BACKGROUND

Every bounded linear operator $T : X \rightarrow Y$ between two Banach spaces induces a dual map $T^* : Y^* \rightarrow X^*$ which in turn induces $T^{**} : X^{**} \rightarrow Y^{**}$. When we identify X and Y with their canonical images in, respectively, X^{**} and Y^{**} , then T^{**} is an extension of T . We recall that T is weakly compact if, and only if, T^{**} maps X^{**} into Y .

A natural question is to consider a bounded multilinear map $T : X_1 \times X_2 \times \dots \times X_n \rightarrow Y$ (where X_1, X_2, \dots, X_n, Y are Banach spaces) and ask what property or properties of T might be the multilinear equivalent of weak compactness in the sense that they characterize the existence of a (unique) canonical extension T^{**} mapping $X_1^{**} \times X_2^{**} \times \dots \times X_n^{**}$ into Y .

Two observations are relevant when studying this question. The first is the existence of $n!$ possibly different “natural” extensions of T to multilinear operators mapping $X_1^{**} \times X_2^{**} \times \dots \times X_n^{**}$ into Y^{**} . These maps are separately weak* to weak* continuous if and only if they all coincide (see the discussion on the Aron-Berner later in this section).

The second observation is as follows. For some Banach spaces X_1, X_2, \dots, X_n (e.g. C^* -algebras) T has a (unique) canonical Aron-Berner extension T^{**} . In that situation, the weak compactness of T does imply that T^{**} takes its values in Y (see [2] and [20]). But, for $n \geq 2$, the converse may fail (compare [20]).

However, Bombal and Villanueva showed that when the X_j are unital abelian C^* -algebras, the necessary and sufficient condition for T^{**} to take values in Y is not the weak compactness of T but the property that T be *completely continuous* [5]. Roughly speaking this means that T is jointly sequentially continuous, when each X_j is given its weak topology. Wright

First author partially supported by I+D MCYT project no. MTM 2008-02186, and Junta de Andalucía grants FQM 0199, 3737 and second author partially supported by I+D MCYT project no. MTM2004-01308.

and Ylinen investigated the situation where the C^* -algebras are allowed to be non-commutative. Then the "obvious" generalisation of [5] does not work. Complete continuity is no longer the correct condition. What is required is a new condition, *quasi-complete continuity* [23]. This turns out to be equivalent to joint sequential continuity of T , when each X_j is equipped with the *algebra strong* topology* in the sense introduced by Sakai [19, p. 20], and Y has its norm topology.

An appropriate extension of the algebra strong* topology from the setting of C^* -algebras to general Banach spaces has been recently introduced in [16]. The development of the strong* topology in the setting of general Banach spaces makes natural the question of whether the above characterization obtained by Wright and Ylinen remains valid for Banach spaces.

Let X_1, \dots, X_k be Banach spaces satisfying property (V) of Pełczyński and let X be any Banach space. J. Gutiérrez and the second author of the present note stated in [12], that a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ has an X -valued Aron-Berner extension if, and only if, T is unconditionally converging (see definition in §2). However, in [24, 25], Ylinen pointed out that a lemma used in [12] needed an extra hypothesis for one of the implications. We show in §2 how to obtain the results claimed in [12] by applying a refinement of the original arguments. The results of [22] show that the original characterization of [5, 12] remains valid whenever X^* is weakly sequentially complete; this covers the important case where X is a C^* -algebra.

In §3 we investigate these questions in the setting of general Banach spaces and give the following answers: whenever X_1, \dots, X_k are Banach spaces satisfying Pełczyński's property (V) (see [8] for the definition) and such that, for each $1 \leq i \leq k$, the strong* topology and the w-right topology coincide on bounded subsets of X_i , and $T : X_1 \times \dots \times X_k \rightarrow X$ is a multilinear operator, then the (unique) Aron-Berner extension of T is X -valued if and only if T is quasi completely continuous.

Notation and Background

Throughout this paper, given a Banach space X , B_X will stand for the closed unit ball of X . Let X_1, \dots, X_k be Banach spaces, the symbol $L^k(X_1, \dots, X_k; X)$ will denote the space of all k -linear operators from $X_1 \times \dots \times X_k$ into X ; $L^0(X)$ will be identified with the scalar field, and if X is the scalar field, we write $L^k(X_1, \dots, X_k)$ in place of $L^k(X_1, \dots, X_k; X)$ (In this paper, linear and multilinear operators are always required to be bounded, though this property is sometimes re-stated for clarity).

A central role is played by certain extensions of bounded multilinear maps. Let X_1, \dots, X_k , and X be Banach spaces. If T is an element in $L^k(X_1, \dots, X_k; X)$, then there are $k!$ possibly different multilinear operators in $L^k(X_1^{**}, \dots, X_k^{**}; X^{**})$ called the Aron-Berner extensions of T [1].

There are several equivalent ways to define these extensions. We will adopt the following.

Let X_1, \dots, X_k , and X be Banach spaces and $T : X_1 \times \dots \times X_k \rightarrow X$ a k -linear operator. Let $\pi : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ (denoted $i \mapsto \pi_i$) be a permutation. We define the Aron-Berner extension of T associated to π

$$AB(T)_\pi : X_1^{**} \times \dots \times X_k^{**} \rightarrow X^{**}$$

by

$$AB(T)_\pi(z_1, \dots, z_k) = \text{weak}^* - \lim_{\alpha_{\pi_1}} \dots \text{weak}^* - \lim_{\alpha_{\pi_k}} T(x_1^{\alpha_1}, \dots, x_k^{\alpha_k}),$$

where $(z_1, \dots, z_k) \in X_1^{**} \times \dots \times X_k^{**}$ and, for $1 \leq i \leq k$, $(x_i^{\alpha_i})_{\alpha_i} \subset X_i$ is a net weak* convergent to z_i . $AB(T)_\pi$ is bounded and has the same norm as T .

The extensions $AB(T)_\pi$ are the same as the extensions constructed by Aron and Berner for polynomials in [1]. If we consider $\pi = \text{id}$ (identity permutation), then the extension $AB(T)_\pi$ is called the *Davie-Gamelin extension* (compare [7]). We also recall that there is a permutation π for which $AB(T)_\pi$ is separately weak*-weak* continuous if and only if the extensions $AB(T)_\sigma$ are the same for all permutations σ . For the definition and references on Aron-Berner extensions see, for instance, [9].

In [5, Theorem 1] F. Bombal and the second author of this note proved that, whenever we assume that, for each $i \neq j$, every bounded linear operator from X_i into X_j^* is weakly compact (this condition is satisfied whenever X_1, \dots, X_k have property (V) of Pelczynski, in particular when every X_i is a C^* -algebra), then every multilinear operator T in $L^k(X_1, \dots, X_k; X)$ has a (unique) separately weak*-weak* continuous extension $AB(T)$ in $L^k(X_1^{**}, \dots, X_k^{**}; X^{**})$.

The algebra strong*-topology of a C^* -algebra A is the topology generated by the seminorms $x \mapsto \phi(x^*x + xx^*)^{\frac{1}{2}}$ where ϕ ranges over the positive linear functionals on A . As a consequence of the noncommutative generalisation of the little Grothendieck inequality (see [13], [17]), the strong* topology of A coincides with the local convex topology generated by all seminorms of the form $x \mapsto \|Sx\|$ where it is required that S is a bounded linear map from A into a Hilbert space.

Let X be a Banach space. Following [16], the *strong* topology* of X (denoted by $s^*(X, X^*)$) is defined as the locally convex topology of X generated by the seminorms $x \mapsto \|x\|_S := \|Sx\|$ where S is a bounded linear map from X to a Hilbert space H . Analogously, we get the *w-right topology* of X if in this description the words "a Hilbert space" are replaced by "a reflexive Banach space" (compare [15] and [16]).

We shall also need the following definition taken from [14]. A JB^* -triple is a complex Banach space E with a continuous triple product $(a, b, c) \mapsto \{a, b, c\}$ that is conjugate linear in b and symmetric bilinear in a and c for which each operator $L(a, a) : E \rightarrow E$, of the form $x \mapsto \{a, a, x\}$, is

hermitian with non-negative spectrum satisfying $\|L(a, a)\| = \|a\|^2$ and $L(a, a)(\{x, y, z\}) = \{L(a, a)(x), y, z\} - \{x, L(a, a)(y), z\} + \{x, y, L(a, a)(z)\}$. Every C^* -algebra is a JB^* -triple with respect to

$$\{x, y, z\} := 2^{-1}(xy^*z + zy^*x),$$

every JB^* -algebra is a JB^* -triple with triple product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*,$$

and the Banach space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB^* -triple with respect to $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$.

2. UNCONDITIONALLY CONVERGING MULTILINEAR OPERATORS

As we said in the introduction, the problem of characterising those multilinear operators T whose Aron-Berner extensions take their values in the codomain space of T , was partially treated by Gutiérrez and Villanueva in [12]. There are, however, subtle difficulties in the proofs of some results in the just quoted paper. In this section we refine the original arguments to avoid the gaps in those proofs.

We begin by recalling the following definitions.

Definition 2.1. *Given Banach spaces X_1, \dots, X_k, X , a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ is unconditionally converging (UC) if for every weakly unconditionally Cauchy (w.u.C.) series $\sum_n x_i^n \subset X_i$ ($1 \leq i \leq k$) the sequence*

$$T(s_1^m, \dots, s_k^m)$$

converges in norm, where

$$s_i^m := \sum_{n=1}^m x_i^n.$$

A different notion of unconditionally converging multilinear operators and polynomials, giving rise to a strictly wider class, was used in [11, 10]. However, the definition given above, used in [4, 12], seems to be more appropriate for most applications.

The following lemma is the correct version of [12, Lemma 3.3]. It appeared with a slight omission in [12], and an erratum with the present corrected version was published in [25] complementing [24]. For a Banach space X , $c(X)$ denotes the Banach space of convergent sequences in X .

Lemma 2.2. *Let Y, X be Banach spaces. Let $S : Y \longrightarrow c(X)$ be an operator. For every $n \in \mathbb{N}$ let $S_n : Y \longrightarrow X$ be the operator defined by $S_n(y) = S(y)_n$ and let $S_\infty : Y \longrightarrow X$ be the operator defined by $S_\infty(y) = \lim_n S_n(y)$. Then S is weakly compact if and only if the following three conditions hold:*

- (1) *For every $n \in \mathbb{N}$, S_n is weakly compact.*
- (2) *For every $z \in Y^{**}$, the limit $\lim_n S_n^{**}(z)$ exists.*

(3) S_∞ is weakly compact and, for every $z \in Y^{**}$,

$$S_\infty^{**}(z) = \lim_n S_n^{**}(z).$$

□

In [12], a defective version of the above Lemma was used. This created a gap in the proof of [12, Proposition 3.6] and, as a consequence, the proof of [12, Proposition 4.5] was also in doubt. We use now Lemma 2.2 to give a correct proof of [12, Proposition 3.6], which fixes the situation.

We recall the following definition from [12].

Definition 2.3. Let X be a Banach space. A subset $A \subseteq X$ is said to be a Dunford-Pettis set (DP set) if, for each weakly-null sequence (ϕ_n) in X^* , we have

$$\limsup_n \sup_{x \in A} |\phi_n(x)| = 0.$$

A sequence (x_n) in X is said to be a Dunford-Pettis sequence (DP sequence) if the set $\{x_n : n \in \mathbb{N}\}$ is a Dunford-Pettis set. A weak Cauchy Dunford-Pettis sequence is a sequence which is both weak Cauchy and Dunford-Pettis.

Let X_1, \dots, X_k be Banach spaces. A multilinear operator

$$T : X_1 \times \dots \times X_k \longrightarrow X$$

is called weakly continuous on DP sets if whenever we choose weakly Cauchy DP sequences $(x_i^n)_n \subset X_i$ ($1 \leq i \leq k$), it follows that the sequence $(T(x_1^n, \dots, x_k^n))_n$ is norm convergent.

We need the following result:

Proposition 2.4. Let X_1, \dots, X_k and X be Banach spaces. A multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ is weakly continuous on DP sets if and only if whenever we choose DP sequences $(x_i^n)_n \subset X_i$ such that (x_i^n) converges weakly to $x_i \in X_i$ ($1 \leq i \leq k$), it follows that the sequence $(T(x_1^n, \dots, x_k^n))_n$ converges in norm to $T(x_1, \dots, x_k)$.

Proof. Suppose first that T is weakly continuous on DP sets, and let $(x_i^n)_n$ be a DP sequence in X_i which converges weakly to x_i ($1 \leq i \leq k$). For every $1 \leq i \leq k$ consider now the sequence $(y_i^n)_n \subset X_i$ defined by

$$(y_i^n)_n = (x_i^1, x_i, x_i^2, x_i, x_i^3, x_i, \dots).$$

Clearly $(y_i^n)_n$ is a DP weakly Cauchy sequence. Therefore, our hypothesis tells us that the sequence $(T(y_1^n, \dots, y_k^n))_n$ is norm convergent. Clearly it can only converge to $T(x_1, \dots, x_k)$ which implies that $T(x_1^n, \dots, x_k^n)$ also converges to $T(x_1, \dots, x_k)$ which finishes this half of the proof.

The converse can be proved by following, as closely as possible, the reasonings in [3, Lemma 2.4 and Theorem 2.3]. □

Now we can give a correct proof of [12, Proposition 3.6].

Proposition 2.5. *Let X_1, \dots, X_k , and X be Banach spaces and $T : X_1 \times \dots \times X_k \longrightarrow X$ a multilinear operator such that one of its AB -extensions is X -valued. Then T is weakly continuous on DP sets.*

Proof. By Proposition 2.4, it suffices to prove that whenever we choose a DP sequence $(x_i^n) \subset X_i$ which converges weakly to x_i ($1 \leq i \leq k$) it follows that $\|T(x_1^n, \dots, x_k^n) - T(x_1, \dots, x_k)\| \rightarrow 0$. We express this condition by saying that T is DP weakly sequentially continuous. We proceed by induction on k . If $k = 1$ the result is known (see [12]). We suppose the result to be true for $k - 1$, and for each $1 \leq i \leq k$, we fix a DP sequence $(x_i^n) \subset X_i$ weakly convergent to $x_i^0 \in X_i$. We suppose without loss of generality that the Davie-Gamelin extension of T , $AB(T)$, is X -valued.

For any $z_k \in X_k^{**}$, the operator

$$T_{z_k} : X_1 \times \dots \times X_{k-1} \longrightarrow X$$

defined by

$$T_{z_k}(x_1, \dots, x_{k-1}) = AB(T)(x_1, \dots, x_{k-1}, z_k)$$

verifies that its Davie-Gamelin extension is X -valued. Therefore T_{z_k} is DP weakly sequentially continuous.

For every $n \in \mathbb{N}$ we define the operator $S_n : X_k \longrightarrow X$ by

$$S_n(x_k) = T(x_1^n, \dots, x_{k-1}^n, x_k)$$

and we also define $S_\infty : X_k \longrightarrow X$, by

$$S_\infty(x_k) = \lim_n S_n(x_k) = T(x_1^0, \dots, x_{k-1}^0, x_k).$$

Since T_{x_k} is DP weakly sequentially continuous, we deduce that S_∞ is well defined.

For the same reason we can define the operator

$$S : X_k \longrightarrow c(X)$$

by

$$S(x_k) = (S_n(x_k))_n.$$

We now observe:

- (i) For every $n \in \mathbb{N}$, S_n is weakly compact, because, for every $z_k \in X_k^{**}$,

$$S_n^{**}(z_k) = AB(T)(x_1^n, \dots, x_{k-1}^n, z_k) \in X.$$

- (ii) For every $z_k \in X_k^{**}$, the limit $\lim_n S_n^{**}(z_k)$ exists, because

$$\begin{aligned} \lim_n S_n^{**}(z_k) &= \lim_n AB(T)(x_1^n, \dots, x_{k-1}^n, z_k) = \\ &= \lim_n T_{z_k}(x_1^n, \dots, x_{k-1}^n) = T_{z_k}(x_1^0, \dots, x_{k-1}^0), \end{aligned}$$

since T_{z_k} is DP weakly sequentially continuous.

- (iii) S_∞ is weakly compact and, for every $z_k \in X_k^{**}$,

$$S_\infty^{**}(z_k) = \lim_n S_n^{**}(z_k).$$

To see the first statement in (iii), it suffices to observe that, for every z_k in X_k^{**} ,

$$S_\infty^{**}(z_k) = AB(T)(x_1^0, \dots, x_{k-1}^0, z_k) \in X.$$

For the second, let us notice that

$$\begin{aligned} S_\infty^{**}(z_k) &= AB(T)(x_1^0, \dots, x_{k-1}^0, z_k) = T_{z_k}(x_1^0, \dots, x_{k-1}^0) = \\ &= \lim_n T_{z_k}(x_1^n, \dots, x_{k-1}^n) = \lim_n S_n^{**}(z_k). \end{aligned}$$

Therefore we can use Lemma 2.2 to obtain that S is weakly compact. Again, this implies that S is DP weakly sequentially continuous. Hence, the sequence $(S(x_k^n))_n \subset c(X)$ converges in norm to $S(x_k^0)$, that is,

$$\lim_n \sup_m \|T(x_1^m, \dots, x_{k-1}^m, x_k^n) - T(x_1^m, \dots, x_{k-1}^m, x_k^0)\| = 0.$$

In particular,

$$(1) \quad \lim_n \|T(x_1^n, \dots, x_{k-1}^n, x_k^n) - T(x_1^n, \dots, x_{k-1}^n, x_k^0)\| = 0.$$

To finish the proof, we notice that

$$\begin{aligned} &\|T(x_1^n, \dots, x_{k-1}^n, x_k^n) - T(x_1^0, \dots, x_{k-1}^0, x_k^0)\| \leq \\ &\|T(x_1^n, \dots, x_{k-1}^n, x_k^n) - T(x_1^n, \dots, x_{k-1}^n, x_k^0)\| + \\ &+ \|T(x_1^0, \dots, x_{k-1}^0, x_k^0) - T(x_1^0, \dots, x_{k-1}^0, x_k^0)\|. \end{aligned}$$

The first of these terms can be made arbitrarily small by (1). The second can be made arbitrarily small because $T_{x_k^0}$ is DP weakly sequentially continuous. \square

Now the gaps in the proof of [12, Proposition 4.5] can be filled thanks to the preceding proposition. Therefore, the next result now follows from the previous results and [12, Proposition 4.5, Theorem 4.2].

Theorem 2.6. *Let X_1, \dots, X_k and X be non-zero Banach spaces. If a bounded multilinear operator $T : X_1 \times \dots \times X_k \rightarrow X$ has an X -valued Aron-Berner extension then T is UC. Moreover, X_1, \dots, X_k have property (V) if, and only if, for each Banach space X every UC operator $T : X_1 \times \dots \times X_k \rightarrow X$ has an X -valued Aron-Berner extension.* \square

3. QUASI COMPLETELY CONTINUOUS MULTILINEAR OPERATORS

We shall introduce the central notion of this note: *quasi completely continuous multilinear operators*. We shall make use of the strong* topology of a general Banach space (introduced in [16]) to generalise the concept developed by the last two authors of this note in the setting of C*-algebras [23]. In a vast class of Banach spaces (including C*-algebras, JB*-triples, spaces of operators between Hilbert spaces, etc.) quasi complete continuity will characterise those multilinear operators T whose Aron-Berner extensions take their values in the codomain space of T .

Definition 3.1. Given Banach spaces X_1, \dots, X_k, X , a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ is quasi completely continuous (QCC) if whenever we choose $s^*(X_i, X_i^*)$ Cauchy sequences $(x_i^n)_n \subset X_i$ ($1 \leq i \leq k$), it follows that the sequence $(T(x_1^n, \dots, x_k^n))_n$ is norm convergent.

Definition 3.2. Given Banach spaces X_1, \dots, X_k, X , a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ is right quasi completely continuous (RQCC) if whenever we choose w-right Cauchy sequences $(x_i^n)_n \subset X_i$ ($1 \leq i \leq k$), it follows that the sequence $(T(x_1^n, \dots, x_k^n))_n$ is norm convergent.

Using the same standard reasoning as in Proposition 2.4 it can be proved that a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$ is quasi completely continuous if and only if for every sequence $(x_i^n)_n \subset X_i$ which is $s^*(X_i, X_i^*)$ -convergent to $x_i \in X_i$ ($1 \leq i \leq k$) we have

$$\lim_n \|T(x_1^n, \dots, x_k^n) - T(x_1, \dots, x_k)\| = 0.$$

The corresponding statement for right quasi completely continuous multilinear operators is also true.

As in the linear case, since w-right convergent sequences are strong* convergent to the same limit, we get that every quasi completely continuous operator is right quasi completely continuous.

As mentioned in [12], multilinear operators with an X -valued Aron-Berner extension seem to be the natural multilinear replacement of weakly compact linear operators. Since every weakly compact linear operator is pseudo weakly compact [15] (the linear version of RQCC), the following result is not unexpected.

Proposition 3.3. Let $T : X_1 \times \dots \times X_k \longrightarrow X$ be a multilinear operator. If any of the Aron-Berner extensions of T is X -valued, then T is right quasi completely continuous.

Proof. The proof follows along the lines of Proposition 2.5. \square

We shall prove next that right quasi completely continuous multilinear operators are unconditionally converging. First we need a lemma.

Lemma 3.4. Let $\sum_n x_n$ be a w.u.C. series in X . For every $m \in \mathbb{N}$ let $z_m := \sum_{n=1}^m x_n$. Then the sequence $(z_m) \subset X$ is w-right Cauchy.

Proof. Let $\sum_n x_n$ be as in the hypothesis. Let $U : c_0 \longrightarrow X$ be the unique operator mapping each e_n to x_n , and let $\sigma_m = \sum_{n=1}^m e_n$ be the summing basis of c_0 . Then (σ_m) is an algebra strong* Cauchy sequence in (the commutative C^* -algebra) c_0 . Since the algebra strong* and the w-right topologies coincide on the closed unit ball of c_0 , it follows that (σ_m) is w-right Cauchy in c_0 (compare [16, Theorem 5.6]). Lemma 12 in [15] tells us that U is w-right to w-right continuous, and it follows that (z_m) is w-right Cauchy. \square

The next proposition follows immediately from the previous lemma and the definitions.

Proposition 3.5. *Given Banach spaces X_1, \dots, X_k, X and a multilinear operator $T : X_1 \times \dots \times X_k \longrightarrow X$, if T is right completely continuous, then it is also unconditionally converging.* \square

For the next theorem we need two technical lemmas. We state them for RQCC operators, but the corresponding results, with entirely similar proofs, are also true for QCC operators. The proof of the first one, standard by now, can be easily done following the lines of [3, Theorem 2.3 and Lemma 2.4].

Lemma 3.6. *Let $T : X_1 \times \dots \times X_k \longrightarrow X$ be a multilinear operator. Then T is RQCC if and only if for all w-right Cauchy sequences $(x_i^n)_n \subset X_i$ ($1 \leq i \leq k$) at least one of which is w-right-null, we have*

$$\lim_{n \rightarrow \infty} \|T(x_1^n, \dots, x_k^n)\| = 0.$$

\square

Lemma 3.7. *Let X_1, \dots, X_k, X be Banach spaces and let $T : X_1 \times \dots \times X_k \longrightarrow X$ be a multilinear operator. Then T is right quasi completely continuous if and only if for all w-right Cauchy sequences $(x_i^n)_n \subset X_i$ ($1 \leq i \leq k$) at least one of which, say $(x_1^n)_n$, is w-right-null, we have*

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \|T(x_1^m, x_2^n, x_3^n, \dots, x_k^n)\| = 0.$$

Proof. Let T be RQCC. Suppose the result is not true. In that case there exist w-right Cauchy sequences $(x_i^n)_n \subset X_i$ ($2 \leq i \leq k$), a w-right null sequence $(x_1^m)_m \subset X_1$ and $\epsilon > 0$ such that, for all $m \in \mathbb{N}$,

$$(2) \quad \sup_{n \in \mathbb{N}} \|T(x_1^m, x_2^n, x_3^n, \dots, x_k^n)\| > \epsilon.$$

Let $m(1) = 1$ and let $n(1) \in \mathbb{N}$ be such that

$$\|T(x_1^{m(1)}, x_2^{n(1)}, x_3^{n(1)}, \dots, x_k^{n(1)})\| > \epsilon.$$

For any fixed $1 \leq n \leq n(1)$, the linear operator

$$T(\cdot, x_2^n, \dots, x_k^n) : X_1 \longrightarrow X$$

is RQCC. Therefore, there exists $m(2) > m(1)$ such that, for every $m \geq m(2)$, for every $1 \leq n \leq n(1)$,

$$\|T(x_1^{m(2)}, x_2^n, x_3^n, \dots, x_k^n)\| < \epsilon.$$

Applying (2), we get the existence of $n(2) > n(1)$ such that

$$\|T(x_1^{m(2)}, x_2^{n(2)}, x_3^{n(2)}, \dots, x_k^{n(2)})\| > \epsilon.$$

Proceeding inductively we get the existence of a w-right null sequence $(x_1^{m(p)})_p \subset X_1$ and w-right Cauchy sequences $(x_2^{n(p)})_p \subset X_2, \dots, (x_k^{n(p)})_p \subset X_k$ such that, for every $p \in \mathbb{N}$,

$$\|T(x_1^{m(p)}, x_2^{n(p)}, x_3^{n(p)}, \dots, x_k^{n(p)})\| > \epsilon,$$

in contradiction with Lemma 3.6. The converse is clear. \square

Let us recall that a Banach space X is said to be *sequentially right* if every pseudo weakly compact operator with domain X is weakly compact (see [15]). Following the ideas of [12], we can prove.

Theorem 3.8. *Let X_1, \dots, X_k be non zero Banach spaces. The following are equivalent:*

- (a) *For every $1 \leq i \leq k$, X_i is sequentially Right.*
- (b) *For every Banach space X and every right quasi completely continuous operator $T : X_1 \times \dots \times X_k \longrightarrow X$, all of the Aron-Berner extensions of T are X -valued.*
- (c) *For every Banach space X and every right quasi completely continuous operator $T : X_1 \times \dots \times X_k \longrightarrow X$, T has an X -valued Aron-Berner extension.*

Proof. We prove first that (a) implies (b). We consider T as in the hypothesis of (b). Without loss of generality, we choose the Davie-Gamelin extension of T among all of its Aron-Berner extensions, and we call it $AB(T)$. We reason by induction on k . When $k = 1$, $AB(T)$ is nothing but T^{**} , and the definition of sequentially Right space tells us that T is weakly compact, hence T^{**} is X -valued.

Now suppose that the result is true for every $1 \leq j \leq k - 1$. Fix $(x_1, \dots, x_{k-1}) \in X_1 \times \dots \times X_{k-1}$ and define

$$T_{x_1, \dots, x_{k-1}} : X_k \longrightarrow X$$

by

$$T_{x_1, \dots, x_{k-1}}(x) = T(x_1, \dots, x_{k-1}, x).$$

Clearly, $T_{x_1, \dots, x_{k-1}}$ is also RQCC; therefore, using the fact that X_k is sequentially Right we get that $T_{x_1, \dots, x_{k-1}}$ is weakly compact. Therefore, for every $z_k \in X_k^{**}$ we have that

$$AB(T)(x_1, \dots, x_{k-1}, z_k) = (T_{x_1, \dots, x_{k-1}})^{**}(z_k) \in X$$

Fix now $z_k \in X_k^{**}$ and define

$$T_{z_k} : X_1 \times \dots \times X_{k-1} \longrightarrow X$$

by

$$T_{z_k}(x_1, \dots, x_{k-1}) = AB(T)(x_1, \dots, x_{k-1}, z_k).$$

The previous comments guarantee that T_{z_k} is indeed X -valued.

Let us see that T_{z_k} is RQCC. Fix w-right Cauchy sequences $(x_j^n)_n \subset X_j$ ($1 \leq j \leq k-1$). For every $n \in \mathbb{N}$, define $S_n : X_k \longrightarrow X$ by $S_n(x) = T(x_1^n, \dots, x_{k-1}^n, x)$.

Previous reasonings tell us that S_n is weakly compact for all $n \in \mathbb{N}$.

We define the operator

$$S : X_k \longrightarrow c(X)$$

by

$$S(x_k) = (S_n(x_k)) = T(x_1^n, \dots, x_{k-1}^n, x_k).$$

It is easy to check that S is well defined and continuous. Indeed Lemma 3.7 tell us that S is RQCC. Hence, S is weakly compact. Now, Lemma 2.2 shows that, for every $z_k \in X_k^{**}$, the sequence

$$(S_n^{**}(z_k)) = AB(T)(x_1^n, \dots, x_{k-1}^n, z_k) = T_{z_k}(x_1^n, \dots, x_{k-1}^n)$$

is convergent in X . That is, T_{z_k} is completely continuous. By the induction hypothesis, all of its Aron-Berner extensions are X -valued. Therefore $AB(T)$ is X -valued.

Clearly (b) implies (c). To see that (c) implies (a), suppose that one the spaces, say X_1 , is not sequentially Right. Then, there exists a Banach space X and an operator $S : X_1 \longrightarrow X$ which is RQCC and not weakly compact. For every ($2 \leq i \leq k$) we consider a nonzero form $x_i^* \in X_i^*$. It is obvious that the multilinear operator

$$T : X_1 \times \dots \times X_k \longrightarrow X$$

defined by

$$T(x_1, \dots, x_k) = x_2^*(x_2) \cdots x_k^*(x_k) S(x_1)$$

is RQCC. At the same time, the (unique) Aron-Berner extension of T is given by

$$AB(T)(z_1, \dots, z_k) = z_2(x_2^*) \cdots z_k(x_k^*) S^{**}(z_1)$$

which clearly is not X -valued. \square

Therefore, putting together our results, we have the following relations between classes of multilinear operators $T : X_1 \times \dots \times X_n \longrightarrow X$.

$$CC \subset QCC \subset RQCC \subset UC$$

and

$$ABX \subset RCC,$$

where CC are the completely continuous operators and ABX are the operators with an X -valued Aron-Berner extension.

The inverse containments are in general false. The containments between CC and ABX are conditional on the Dunford-Pettis and Reciprocal Dunford-Pettis Property of the spaces [12]. The containment of RQCC in ABX is conditional on the spaces being *sequentially Right*.

Let X_1, \dots, X_k , and X be Banach spaces. Suppose that for each $1 \leq i \leq k$, X_i has property (V). From [5, Theorem 1] it follows that for each continuous multilinear operator $T : X_1 \times \dots \times X_k \rightarrow X$ there exists a unique separately weak*-weak* continuous multilinear operator

$$T^{**} : X_1^{**} \times \dots \times X_k^{**} \rightarrow X^{**},$$

which extends T and satisfies $\|T^{**}\| = \|T\|$.

We can now get [23, Corollary 3.6] as a consequence of our results. We recall first that every JB*-triple has property (V) (compare [6]).

Theorem 3.9. *Let A_1, \dots, A_k be JB*-triples (or C*-algebras), X a Banach space, and $T : A_1 \times \dots \times A_k \rightarrow X$ a multilinear operator. The following assertions are equivalent:*

- (a) *T is QCC.*
- (b) *T is RQCC.*
- (c) *T is UC.*
- (d) *The unique Aron-Berner extension of T is X -valued.*

Proof. Statement (a) implies (b), and (b) implies (c) in general, not only for JB*-triples. That (c) implies (d) follows from [12, Theorem 4.2] and (d) implies (b) in general. In the case of JB*-triples, Remark 5.3 in [16] implies that RQCC operators coincide with QCC operators, and that finishes the proof. \square

Let X_1, \dots, X_k and X be Banach spaces. It is not hard to see that a multilinear operator $T : X_1 \times \dots \times X_k \rightarrow X$ is QCC (respectively, RQCC) if, and only if, it is jointly sequentially strong*-to-norm (respectively, w-right-to-norm) continuous. When X_1, \dots, X_k are JB*-triples (or C*-algebras), this continuity was enough to show that the unique Aron-Berner extension of T is X -valued (compare Theorem 3.9). We might also consider the hypothesis of T being separately strong*-to-norm (respectively, w-right-to-norm) continuous, that is, whenever we fix $k-1$ variables we get a strong*-to-norm (respectively, w-right-to-norm) continuous linear operator. It seems natural to ask whether under this weaker hypothesis on T , it follows that T has an Aron-Berner extension which is X -valued. The following example shows that the answer is, in general, negative.

Example 3.10. Let $T : c_0 \times c_0 \rightarrow c_0$ be the operator defined by $T(x, y) := xy$. Let x be an element in c_0 . It is clear that the mappings

$$T(x, \cdot), T(\cdot, x) : c_0 \rightarrow c_0$$

$$y \mapsto T(x, y) = T(y, x) = xy$$

are weakly compact. From [15, Corollary 5], it now follows that T is separately w-right-to-norm continuous and strong*-to-norm continuous by Corollary 5.9 in [16]. However the unique Aron-Berner extension of T is the natural product of ℓ_∞ , which is not c_0 -valued.

The above example shows that separate w-right-to-norm continuity is not enough to guarantee that a multilinear operator $T : X_1 \times \cdots \times X_k \rightarrow X$ has an Aron-Berner extension which is X -valued. To finish this section, we shall give an example of a multilinear operator which is not jointly strong*-to-norm (respectively, w-right-to-norm) continuous and admits an Aron-Berner extension which is X -valued; this example shows that the stronger hypothesis of T being jointly strong*-to-norm (respectively, w-right-to-norm) continuous is too restrictive.

Proposition 3.11. *Let X_1, \dots, X_k and X be Banach spaces and let $T : X_1 \times \cdots \times X_k \rightarrow X$ be a multilinear operator. Then the following are equivalent:*

- a) *T is jointly w-right-to-norm continuous.*
- b) *There exists a positive constant M (depending only on T), reflexive Banach spaces R_1, \dots, R_k and bounded linear operators $T_i : X_i \rightarrow R_i$ satisfying, for each x_i in X_i ,*

$$\|T(x_1, \dots, x_k)\| \leq M \|x_1\|_{T_1} \cdots \|x_k\|_{T_k}.$$

Proof. The implication (b) \Rightarrow (a) is clear. We shall prove (a) \Rightarrow (b).

It follows from our hypothesis that the set

$$\mathcal{O} := \{(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k : \|T(x_1, \dots, x_k)\| \leq 1\}$$

is a neighbourhood of 0 in the w-right product topology of $X_1 \times \cdots \times X_k$. From the definition of the w-right product topology it follows that there is a positive constant δ such that for each $1 \leq i \leq k$ we can find a finite number of reflexive Banach spaces $R_1^1, \dots, R_{p_i}^i$ and bounded linear operators $T_{j_i}^i : X_i \rightarrow R_{j_i}^i$, ($1 \leq j_i \leq p_i$) such that $\mathcal{O} \supseteq \mathcal{O}'$, where

$$\mathcal{O}' := \left\{ (x_1, \dots, x_k) \in X_1 \times \cdots \times X_k : \|x_i\|_{T_{j_i}^i} \leq \delta, \forall 1 \leq i \leq k, 1 \leq j_i \leq p_i \right\}.$$

We denote

$$R_i := \bigoplus_{1 \leq j_i \leq p_i}^{\ell_\infty} R_{j_i}^i$$

and let $S_i : X_i \rightarrow R_i$ the bounded linear operator given by $S_i(x_i) := (T_{j_i}^i(x_i))_{j_i}$. Clearly R_i is reflexive for each $1 \leq i \leq k$.

Let $x_i \in X_i$, such that $\|x_i\|_{T_i} \neq 0$ for all i . Since the vector

$$\left(\frac{\delta}{\|x_1\|_{T_1}} x_1, \dots, \frac{\delta}{\|x_k\|_{T_k}} x_k \right)$$

is in $\mathcal{O}' \subseteq \mathcal{O}$, we have

$$\left\| T \left(\frac{\delta}{\|x_1\|_{T_1}} x_1, \dots, \frac{\delta}{\|x_k\|_{T_k}} x_k \right) \right\| \leq 1,$$

which implies

$$\|T(x_1, \dots, x_k)\| \leq \frac{1}{\delta^k} \|x_1\|_{T_1} \cdots \|x_k\|_{T_k}.$$

If, for some $1 \leq i_0 \leq k$, we have $\|x_{i_0}\|_{T_{i_0}} = 0$, then $\|tx_{i_0}\|_{T_{i_0}} = 0$, for every $t > 0$ and hence $\|tT(x_1, \dots, x_{i_0}, \dots, x_k)\| \leq 1$, for all $t > 0$ and $x_i \in X_i$ such that $\|x_i\|_{T_i} \leq \delta$, which gives $T(x_1, \dots, x_{i_0}, \dots, x_k) = 0$. \square

Example 3.12. Let $T : c_0 \times \ell_2 \rightarrow \ell_2$ be the bilinear operator defined by $T(a, b) := ab$. It is clear that the unique Aron-Berner extension of T is ℓ_2 -valued. We claim that T is not jointly w-right-to-norm continuous. Indeed, if T is jointly w-right-to-norm continuous, then by Proposition 3.11, there exist $M > 0$, reflexive Banach spaces R_1 and R_2 and bounded linear operators $T_1 : c_0 \rightarrow R_1$ and $T_2 : \ell_2 \rightarrow R_2$ satisfying

$$(3) \quad \|T(a, b)\| \leq M \|a\|_{T_1} \|b\|_{T_2},$$

for all $(a, b) \in c_0 \times \ell_2$. Let (e_n) and (h_n) be the canonical bases of c_0 and ℓ_2 , respectively. Since $\sum e_n$ is a w.u.C. series in c_0 , it follows, from [15, Lemma 13], that (e_n) is a w-right-null sequence in c_0 , and hence $\|e_n\|_{T_1} \rightarrow 0$. Therefore, the above inequality (3) gives

$$\begin{aligned} 1 &= \|h_n\| = \|T(e_n, h_n)\| \leq M \|e_n\|_{T_1} \|h_n\|_{T_2} \\ &\leq M \|T_2\| \|h_n\| \|e_n\|_{T_1} \leq M \|T_2\| \|e_n\|_{T_1} \rightarrow 0, \end{aligned}$$

which is impossible.

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