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Research Article

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Coupled Versus Uncoupled Blow-Up Rates in Cooperative *n*-Species Logistic Systems

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Abstract: This paper ascertains the exact boundary blow-up rates of the large positive solutions of a class of cooperative logistic systems involving n species in a general domain of $\mathbb{R}^N$ of class $\mathcal{C}^{2+\nu}$, $0 < \nu < 1$. The problem models a population divided in groups whose individuals compete with those of the same group, while simultaneously they cooperate with the members of the remaining groups. Our main result provides with the exact blow-up rates along the edges of Ω from the values of the blow-up rates of the underlying uncoupled system. Rather astonishingly, these blow-up rates are independent of the strength of the cooperative effects, which play a secondary role in the analysis carried out in this paper. No previous result of this nature is available in the specialized literature for more than $n = 2$ species.

Keywords: Cooperative Systems, Large Solutions, Blow-Up Rates on the Boundary, Vanishing Weights at Variable Rates, Uniqueness of Large Solutions

MSC 2010: 35J57, 35K57, 35B44, 35Q92

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1 Introduction

In this paper we ascertain the blow-up rate of the classical positive solution of the singular boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \sum\limits_{j=1, j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega,\ 1 \le i \le n, \end{cases} \tag{1.1}$$

where $a_{ij} > 0$ and $p_i > 1$ are constants, Ω is a bounded subdomain of $\mathbb{R}^N$, $N \in \mathbb{N}$, of class $\mathcal{C}^{2+\nu}$ for some $\nu \in (0, 1)$, $\lambda_i \in \mathcal{C}^\nu(\bar{\Omega})$, and $\mathfrak{a}_i \in \mathcal{C}^\nu(\bar{\Omega})$ satisfy $\mathfrak{a}_i(x) > 0$ for all $x \in \Omega$ and $1 \le i, j \le n$. By a solution, u, of (1.1) it is understood any function $u \in \mathcal{C}^{2+\nu}(\Omega)$ satisfying the system and such that

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u_i(x) = +\infty, \quad 1 \le i \le n.$$

In this paper, we suppose that $\mathfrak{a}_i$ are power-like near $\mathfrak{a}_i^{-1}(0) \subset \partial\Omega$, in the sense that there exist $\mathfrak{b}_i, \gamma_i \in \mathcal{C}(\partial\Omega)$, with $\mathfrak{b}_i(z) > 0$ for all $z \in \partial\Omega$ and $\gamma_i \ge 0$ on $\partial\Omega$, such that

$$\lim_{\substack{x\to z \\ x\in\Omega,\ z\in\partial\Omega}} \frac{\mathfrak{a}_i(x)}{\mathfrak{b}_i(z)[\operatorname{dist}(x, \partial\Omega)]^{\gamma_i(z)}} = 1, \quad 1 \le i \le n. \tag{1.2}$$

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Problem (1.1) is the most natural way of coupling the fully uncoupled problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = +\infty & \text{on } \partial\Omega,\ 1 \le i \le n, \end{cases} \tag{1.3}$$

in a cooperative way, i.e., with $a_{ij} > 0$ for all $1 \le i, j \le n$, $i \ne j$, so that the strong maximum principle holds, [30]. A simpler prototype with $n = 2$ of the system in (1.1) goes back to [34–36]. When $a_{ij} < 0$ for some $i \ne j$, the cooperative structure of (1.1) is lost and all the available comparison techniques in the context of cooperative systems might fail. Consequently, the results of this paper might not be true.

Since (1.3) consists of n-uncoupled singular boundary value problems of logistic type, and the logistic equation is the most standard one in population dynamics and mathematical biology, [27, 37, 38], the problem of analyzing the singular problem (1.1) should deserve a significative attention in spatial ecology. Indeed, the solutions of (1.1) provide us with the asymptotic profiles of the positive solutions of wide classes of cooperative parabolic systems in the presence of spatial heterogeneities, [2, 3, 27]. A more realistic model would be to have different diffusion rates for each species, measured by $d_i > 0$, $1 \le i \le n$, but dividing the i-th equation by d_i, one is naturally driven to deal with (1.1).

Although there is a huge amount of literature devoted to the existence and uniqueness of large positive solutions for the single generalized logistic equation, as it becomes apparent by simply looking at [4–10, 12–14, 19, 20, 23–25, 32, 33, 40–43], and the rather complete list of references in [27], and even there are some fairly astonishing multiplicity results for large positive solutions, [31], the literature on systems is very short. Moreover, the few existing references deal with either two species, $n = 2$, as [2, 3, 16] and the very recent works [15] and [29], or with the general case of $n \ge 2$ species but in the radially symmetric case, where the maximum principle gives the uniqueness of the positive solution of (1.1) based on a very tricky use of the strong maximum principle, as it has been recently established by the authors, [28]. This is the first paper available in the literature where the exact blow-up rates of each of the components of a n-species system have been ascertained.

Throughout this paper, for every $z \in \partial\Omega$, we set

$$\mu_i(z) := \frac{\gamma_i(z) + 2}{p_i - 1}, \quad 1 \le i \le n. \tag{1.4}$$

According to [9, 14, 22], $\mu_i(z)$, $1 \le i \le n$, provide us with the blow-up rates on $\partial\Omega$ of the positive solutions of the uncoupled problem (1.3), in the sense that for every solution of (1.3), $\ell = (\ell_1, \dots, \ell_n)$, one has that

$$\lim_{\substack{x \to z \\ x \in \Omega, z \in \partial\Omega}} \frac{\ell_i(x)}{[\operatorname{dist}(x, \partial\Omega)]^{-\mu_i(z)}} = \Big(\frac{\mu_i(z)(\mu_i(z) + 1)}{\mathfrak{b}_i(z)}\Big)^{\frac{1}{p_i - 1}}, \quad 1 \le i \le n.$$

The main result of this paper provides us with the blow-up rates of all positive solutions of (1.1) in terms of $\mu_i(z)$, $1 \le i \le n$, defined in (1.4). Precisely, for any given $z \in \partial\Omega$, suppose the n equations of (1.1) have been re-ordered so that

$$0 < \mu_n(z) \le \mu_{n-1}(z) \le \cdots \le \mu_1(z). \tag{1.5}$$

Then, the next result holds.

Theorem 1.1. *Let $z \in \partial\Omega$ be such that* (1.5) *is satisfied, and consider the next partition of the subscripts set*

$$\begin{cases} I_+ := \{i \in \{1, \dots, n\} : \mu_i(z) + 2 - \mu_1(z) > 0\}, \\ I_0 := \{i \in \{1, \dots, n\} : \mu_i(z) + 2 - \mu_1(z) = 0\}, \\ I_- := \{i \in \{1, \dots, n\} : \mu_i(z) + 2 - \mu_1(z) < 0\}. \end{cases} \tag{1.6}$$

Let $k \in \{1, \dots, n\}$ be such that

$$I_M := \{i \in \{1, \dots, n\} : \mu_i(z) = \mu_1(z)\} = \{1, \dots, k\}.$$

Then, any positive solution of (1.1), $u = (u_1, \dots, u_n)$, *satisfies*

$$\lim_{\substack{x \to z \\ x \in \Omega}} \frac{u_i(x)}{[\operatorname{dist}(x, \partial\Omega)]^{-\alpha_i(z)}} = A_i(z), \quad 1 \le i \le n, \tag{1.7}$$

where

$$\alpha_i(z) := \begin{cases} \mu_i(z) & \text{if } i \in I_+ \cup I_0, \\ \dfrac{\mu_1(z) + \gamma_i(z)}{p_i} & \text{if } i \in I_-, \end{cases} \tag{1.8}$$

and

$$A_i(z) := \begin{cases} \Big(\dfrac{\mu_i(z)(\mu_i(z)+1)}{\mathfrak{b}_i(z)}\Big)^{\frac{1}{p_i-1}} & \text{if } i \in I_+, \\ \Big[\dfrac{1}{\mathfrak{b}_i(z)} \displaystyle\sum_{j=1}^{k} a_{ij} \Big(\dfrac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)}\Big)^{\frac{1}{p_j-1}}\Big]^{\frac{1}{p_i}} & \text{if } i \in I_-, \\ A_{0,i} & \text{if } i \in I_0, \end{cases} \tag{1.9}$$

where $A_{0,i}$ stands for the unique positive solution of the equation

$$\mathfrak{b}_i(z)x^{p_i} - \mu_i(z)(\mu_i(z)+1)x = \sum_{j=1}^{k} a_{ij} \Big(\frac{\mu_1(z)(\mu_1(z)+1)}{\mathfrak{b}_j(z)}\Big)^{\frac{1}{p_j-1}}.$$

Essentially, Theorem 1.1 establishes that the values of the blow-up rates at z of any solution of (1.1) only depend on the precise way the blow-up rates at z of the solution of (1.3) are interrelated, rather than on the size of the coupling coefficients of the problem, a_{ij}, $i \neq j$, as one might have expected from the every beginning. Indeed, by Theorem 1.1, the coupling coefficients only alter the values of $A_i(z)$. In particular, (1.8) provides us with the components of (1.1) whose blow-up rates equal the corresponding blow-up rates of (1.3), as well as with the blow-up rates of the remaining components. Note that, although I_0 and I_- might be empty, owing to (1.5), $1 \in I_+$. Hence, by Theorem 1.1, the first component of (1.1) blows up with exactly the same rate as the first component of the uncoupled problem (1.3), i.e., $\alpha_1(z) = \mu_1(z)$. Moreover, $\mu_1(z)$ plays a significant role in the blow-up rates of $\alpha_i(z)$ for all $i \in I_-$.

Remark 1.2. The assignations $\partial\Omega \to [0, +\infty)$, $z \mapsto \alpha_i(z)$, are uniformly continuous for all $1 \le i \le n$.

From Theorem 1.1 the next result follows readily.

Theorem 1.3. *Problem* (1.1) *admits a unique positive solution.*

In the special case $n = 3$, one can easily sketch how the blow-up rates given by (1.8) depend on the blow-up rates given by (1.4). By regarding

$$\mu := (\mu_1, \mu_2, \mu_3) \in [0, +\infty)^3$$

as an independent variable, we can divide $[0, +\infty)^3$ into several portions, taking into account the relationships between the components of the variable μ. Figure 1 shows a partition of the set of values of the parameters, $\mu \in [0, +\infty)^3$, into thirteen complementary zones according to the nature of the values of the blow-up rates of the solutions of (1.1),

$$\alpha_i := \alpha_i(\mu), \quad i = 1, 2, 3,$$

depending on μ.

The central portion of Figure 1 stands for the closed hexagonal prism

$$\mu_i + 2 - \mu_j \ge 0 \quad \text{for all } i, j \in \{1, 2, 3\},\ i \neq j,$$

where, according to Theorem 1.1,

$$\alpha := (\alpha_1, \alpha_2, \alpha_3) = (\mu_1, \mu_2, \mu_3).$$

Hence, it consists of the set of values of μ where all the blow-up rates of the coupled cooperative problem (1.1) equal the corresponding blow-up rates of the uncoupled one (1.3). Set

$$\gamma_i := \gamma_i(\mu) = \mu_i(p_i - 1) - 2, \quad 1 \le i \le n.$$

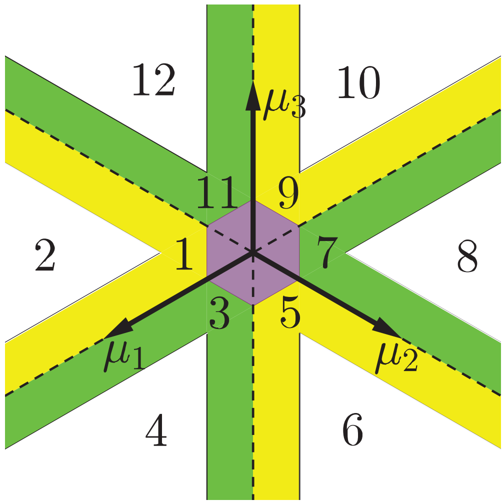

Figure 1. The thirteen zones where α changes its nature, depending on μ.

Order	Zone	α_1	α_2	α_3
$\mu_2 \le \mu_3 \le \mu_1$	1	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	μ_3
	2	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	$\frac{\mu_1+\gamma_3}{p_3}$
$\mu_3 \le \mu_2 \le \mu_1$	3	μ_1	μ_2	$\frac{\mu_1+\gamma_3}{p_3}$
	4	μ_1	$\frac{\mu_1+\gamma_2}{p_2}$	$\frac{\mu_1+\gamma_3}{p_3}$
$\mu_3 \le \mu_1 \le \mu_2$	5	μ_1	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
	6	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
$\mu_1 \le \mu_3 \le \mu_2$	7	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	μ_3
	8	$\frac{\mu_2+\gamma_1}{p_1}$	μ_2	$\frac{\mu_2+\gamma_3}{p_3}$
$\mu_1 \le \mu_2 \le \mu_3$	9	$\frac{\mu_3+\gamma_1}{p_1}$	μ_2	μ_3
	10	$\frac{\mu_3+\gamma_1}{p_1}$	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3
$\mu_2 \le \mu_1 \le \mu_3$	11	μ_1	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3
	12	$\frac{\mu_3+\gamma_1}{p_1}$	$\frac{\mu_3+\gamma_2}{p_2}$	μ_3

Table 1. Blow-up rates α versus blow-up rates μ.

Table 1 collects all the values of the blow-up rates α in each of the remaining zones according to Theorem 1.1. Surrounding the central prisma and labeled by the first six odd integers, we have represented the set of values of μ for which two of the blow-up rates α_i equal the corresponding blow-up rates μ_i. Labeled by the first six even integers, we find the set of values of μ for which exactly one of the α_i equals the corresponding μ_i. As it has been already commented after the statement of Theorem 1.1, there is not any value of μ for which the three blow-up rates α_i can differ from the corresponding blow-up rates μ_i.

The results of this paper can be easily generalized to cover the case where the coefficients a_{ij} are replaced by positive Hölder continuous functions, $a_{ij} \in \mathcal{C}^\nu(\Omega)$, $1 \le i \le n$. Also in this case, the blow-up rates of the solution of (1.1) at $z \in \partial\Omega$ are given by (1.8), while the coefficients $A_i(z)$ are given by (1.9), by replacing a_{ij} by $a_{ij}(z)$ for all $1 \le i, j \le n$. Moreover, under the appropriate circumstances, the $\gamma_i(z)$, introduced in (1.2), might satisfy $\gamma_i(z) > -2$ without affecting substantially most of the results. It should be noted that power-like nonlinearities, as those considered in this paper, always satisfy the classical conditions of Keller [17] and Osserman [39].

The structure of this paper is the following. Section 2 collects the necessary results to obtain existence of solutions for (1.1) as well as the necessary comparison results for the proofs of Theorems 1.1 and 1.3. Section 3 provides us with the construction of a supersolution for the singular problem in a ball and of a subsolution for the problem in an annulus. Finally, in Section 4 we complete the proofs of Theorems 1.1 and 1.3, and the proof of Remark 1.2.

2 Existence of Large Solutions

The main goal of this section is to sketch the proof of the existence of solution for (1.1). Some previous results were already found in [2] and [28]. Given a smooth subdomain $D \subset \mathbb{R}^N$, we consider the operator $\mathfrak{L}\colon [\mathcal{C}^{2+\nu}(D)]^n \to [\mathcal{C}^\nu(D)]^n$ defined by

$$(\mathfrak{L}u)_i = -\Delta u_i - \lambda_i(\cdot)u_i - \sum_{j=1, j\ne i}^{n} a_{ij}u_j, \qquad 1 \le i \le n.$$

By the main result of [30], there exists a unique $\sigma \in \mathbb{R}$, the *principal eigenvalue* of $\mathfrak{L}$ under Dirichlet homogeneous conditions, such that the linear eigenvalue problem

$$\mathfrak{L}\varphi = \sigma\varphi \quad \text{in } D, \qquad \varphi = 0 \quad \text{on } \partial D,$$

admits a positive eigenfunction $\varphi \in [\mathcal{C}^{2+\nu}(D)]^n$, $\varphi_i \geq 0$, $1 \leq i \leq n$, $\varphi \neq 0$. Such a value of σ will be denoted by $\sigma[\mathfrak{L}, D]$.

Throughout the rest of this paper, for any given $u \in [\mathcal{C}(D)]^n$, it is said that $u > 0$ in D if $u_i \geq 0$ for all $1 \leq i \leq n$ but $u \neq 0$. Similarly, given $u \in [\mathcal{C}^1(\bar{D})]^n$, it is said that u is *strongly positive* in D, $u \gg 0$, if for any $1 \leq i \leq n$,

$$u_i(x) > 0 \quad \text{for all } x \in D \qquad \text{and} \qquad \frac{\partial u}{\partial n_x}(x) < 0 \quad \text{for all } x \in u^{-1}(0) \cap \partial D,$$

where n_x is the outward unit normal vector to D at $x \in \partial D$.

The following characterization of the strong maximum principle, going back to [30], holds.

Theorem 2.1. *The following assertions are equivalent:*
(a) $\sigma[\mathfrak{L}, D] > 0$.
(b) *There exists* $\bar{u} \in [\mathcal{C}^2(D)]^n \cap [\mathcal{C}(\bar{D})]^n$ *such that* $\bar{u} > 0$ *in* D *and*

$$\mathfrak{L}\bar{u} \geq 0 \quad \text{in } D,$$

and, for some $1 \leq i_0 \leq n$, *either* $\bar{u}_{i_0} > 0$ *on* ∂D, *or else*

$$(\mathfrak{L}\bar{u})_{i_0} > 0 \quad \text{in } D.$$

Should this be the case, $\bar{u}$ *is said to be a positive strict supersolution of* $\mathfrak{L}$ *in* D.
(c) *The operator* $\mathfrak{L}$ *satisfies the strong maximum principle in* D, *in the sense that, for every* $h \in [\mathcal{C}^\nu(\bar{D})]^n$, $u \in [\mathcal{C}^{2+\nu}(\bar{D})]^n$ *and* $w \in [\mathcal{C}^{2+\nu}(\partial D)]^n$ *satisfying*

$$\mathfrak{L}u = h \geq 0 \quad \text{in } D, \qquad u = w \geq 0 \quad \text{on } \partial D,$$

with some of these inequalities strict, one has that $u \gg 0$ *in* D.

Using Theorem 2.1, one can easily show the monotonicity of the principal eigenvalue with respect to the potentials $\lambda_i(\cdot)$ and the coefficients a_{ij}, $1 \leq i, j \leq n$. Actually, this result was established by [30, Theorem 3.2]. As a result, if we assume that

$$\underline{\lambda}_i(x) \leq \bar{\lambda}_i(x) \quad \text{and} \quad \underline{a}_{ij} \leq \bar{a}_{ij} \quad \text{for all } x \in \bar{\Omega} \text{ and } 1 \leq i, j \leq n,$$

with some of these inequalities strict, then, setting

$$(\underline{\mathfrak{L}}u)_i := -\Delta u_i - \underline{\lambda}_i(\cdot)u_i - \sum_{j=1, j\neq i}^{n} \underline{a}_{ij}u_j, \quad (\bar{\mathfrak{L}}u)_i := -\Delta u_i - \bar{\lambda}_i(\cdot)u_i - \sum_{j=1, j\neq i}^{n} \bar{a}_{ij}u_j,$$

we find that

$$\sigma[\underline{\mathfrak{L}}, D] > \sigma[\bar{\mathfrak{L}}, D]. \tag{2.1}$$

Next, for every $w \in [\mathcal{C}^{2+\nu}(\partial\Omega)]^n$, $w > 0$, we consider the non-homogeneous Dirichlet boundary value problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \sum\limits_{j=1, j\neq i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega, \\ u_i = w_i & \text{on } \partial\Omega, \ 1 \leq i \leq n. \end{cases} \tag{2.2}$$

Using Theorem 2.1, the results of [34–36] can be easily adapted to obtain the next one.

Theorem 2.2. *Suppose* (2.2) *admits a subsolution* $\underline{u} \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *and a supersolution* $\bar{u} \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *satisfying* $\underline{u} \leq \bar{u}$. *Then* (2.2) *possesses a solution* $u \in [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n$ *such that* $\underline{u} \leq u \leq \bar{u}$. *Actually,* (2.2) *possesses a minimal and a maximal solution in the interval* $[\underline{u}, \bar{u}]$.

Using Theorems 2.1 and 2.2, the abstract results of [2, Section 3] can be easily adapted, almost *mutatis mutandis*, to get the next one.

Theorem 2.3. *Problem* (2.2) *has a unique positive solution,* $\theta_{[\Omega,w]}$. *Moreover, for every positive subsolution* $\underline{u}$ *(resp. supersolution* $\bar{u}$*) of* (2.2),

$$\underline{u} \leq \theta_{[\Omega,w]} \quad (\text{resp. } \bar{u} \geq \theta_{[\Omega,w]}).$$

Proof. Adapting [2, Theorem 3.7], it suffices to construct a supersolution $\bar{u}$ such that $\bar{u}_i > 0$ for all $1 \le i \le n$. In the special case $\mathfrak{a}_i(x) > 0$ for all $x \in \partial\Omega$ and $1 \le i \le n$, one can take $\bar{u} = (M, \dots, M)$ for some $M > 0$ sufficiently large. In general, we may proceed as follows.

Since $\partial\Omega$ is smooth, $\partial\Omega$ possesses finitely many components, Γ_k, $1 \le k \le m$. For each $\varepsilon > 0$ and $1 \le k \le m$, denote

$$\Omega_k^\varepsilon := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_k) < \varepsilon\}.$$

Let

$$\lambda := \max_{1 \le i \le n} \|\lambda_i\|_\infty + 1, \quad a := \max_{1 \le i,j \le n} a_{ij},$$

and let $\bar{\mathfrak{L}}$ be the operator

$$(\bar{\mathfrak{L}}u)_i := -\Delta u_i - \lambda u_i - \sum_{j=1, j \ne i}^{n} a u_j.$$

Thanks to (2.1),

$$\sigma[\mathfrak{L}, \Omega_k^\varepsilon] > \sigma[\bar{\mathfrak{L}}, \Omega_k^\varepsilon] \quad \text{for all } \varepsilon > 0 \text{ and } 1 \le k \le m.$$

On the other hand, by the uniqueness of the principal eigenvalue,

$$\sigma[\bar{\mathfrak{L}}, \Omega_k^\varepsilon] = \sigma[-\Delta - \lambda - (n-1)a, \Omega_k^\varepsilon] = \sigma[-\Delta, \Omega_k^\varepsilon] - \lambda - (n-1)a, \quad 1 \le k \le m.$$

Thus, since

$$\lim_{\varepsilon \downarrow 0} |\Omega_k^\varepsilon| = 0, \quad 1 \le k \le m,$$

where $|\Omega_k^\varepsilon|$ stands for the Lebesgue measure of Ω_k^ε, the Faber–Krahn inequality, going back to [11] and [18], yields

$$\lim_{\varepsilon \downarrow 0} \sigma[\mathfrak{L}, \Omega_k^\varepsilon] = +\infty, \quad 1 \le k \le m$$

(see [21, Theorem 5.1]). Therefore, ε can be shortened, if necessary, so that $\min_{1 \le k \le m} \sigma[\mathfrak{L}, \Omega_k^\varepsilon] > 0$. Fix $\varepsilon > 0$ satisfying the last inequality and, for each $1 \le k \le m$, let

$$\varphi_k := (\varphi_{k,1}, \dots, \varphi_{k,n})$$

be a principal eigenfunction associated to $\sigma[\mathfrak{L}, \Omega_k^\varepsilon]$. As $\varphi_k \gg 0$, it is apparent that

$$\min\Big\{\min_{\Gamma_k} \varphi_{k,i} : 1 \le i \le n\Big\} > 0, \quad \min\Big\{\min_{\Omega \cap \partial\Omega_k^{\varepsilon/2}} \varphi_{k,i} : 1 \le i \le n\Big\} > 0 \quad \text{for all } 1 \le k \le m. \tag{2.3}$$

Subsequently, we consider the auxiliary function Φ defined through

$$\Phi := \begin{cases} \varphi_k & \text{in } \bar{\Omega}_m^{\varepsilon/2}, \ 1 \le k \le m, \\ g & \text{in } \Omega_{\text{int}} := \Omega \setminus \Big(\bigcup_{k=1}^{m} \bar{\Omega}_k^{\varepsilon/2}\Big), \end{cases}$$

where g is any $C^{2+\nu}$-extension of the function $\varphi_1 \otimes \cdots \otimes \varphi_m$ to the open set Ω_{int} with the special requirement that $\inf_{\Omega_{\text{int}}} g > 0$. Such a function exists because of (2.3). Then $\tau\Phi$ is a supersolution of (2.2) for sufficiently large $\tau > 1$. Indeed, by (2.3), there exists $\tau_0 \ge 1$ such that

$$\tau\Phi > w \quad \text{on } \partial\Omega \text{ for all } \tau > \tau_0.$$

Moreover, for every $1 \le k \le m$ and $1 \le i \le n$, we find that, in $\Omega_k^{\varepsilon/2} \cap \Omega$,

$$-\Delta(\tau\Phi_i) - \lambda_i(\cdot)\tau\Phi_i - \sum_{j=1, j \ne i}^{n} a_{ij}\tau\Phi_j = (\mathfrak{L}(\tau\Phi))_i = \tau(\mathfrak{L}(\varphi_k))_i = \tau\sigma[\mathfrak{L}, \Omega_k^\varepsilon]\varphi_{k,i} > 0 \ge -\mathfrak{a}_i(\cdot)(\tau\Phi_i)^{p_i}.$$

Lastly, in Ω_{int}, we have that, for every $1 \le i \le n$,

$$-\Delta(\tau\Phi_i) = -\tau\Delta g_i \ge \lambda_i(\cdot)g_i + \sum_{j=1, j \ne i}^{n} a_{ij}\tau g_j - \mathfrak{a}_i(\cdot)(\tau g_i)^{p_i}$$

for sufficiently large $\tau > 1$, because $\mathfrak{a}_i$ and g_i are bounded away from zero in Ω_{int} and $p_i > 1$ for all $1 \le i \le n$. This ends the proof. □

Theorem 2.3 entails that the mapping

$$(0,+\infty) \to [\mathcal{C}^{2+\nu}(\bar{\Omega})]^n, \quad m \mapsto \theta_{[\Omega,\vec{m}]},$$

where $\vec{m} := (m,\dots,m)$, is strongly increasing, in the sense that $m_1 < m_2$ implies $\theta_{[\lambda,\Omega,\vec{m}_1]} \ll \theta_{[\lambda,\Omega,\vec{m}_2]}$. Hence, the point-wise limit

$$\theta_{[\Omega,\infty]}(x) := \lim_{m\uparrow+\infty} \theta_{[\Omega,\vec{m}]}(x), \quad x \in \Omega, \tag{2.4}$$

is well defined. In fact, the next result holds.

Theorem 2.4. *There exists a minimal and a maximal positive solution of* (1.1), $L^{\min}$ *and* $L^{\max}$, *respectively, in the sense that any solution,* L, *of* (1.1) *satisfies*

$$L^{\min}(x) \le L(x) \le L^{\max}(x) \quad x \in \Omega.$$

Moreover, the point-wise limit (2.4) *provides us with the minimal solution*

$$L^{\min} = \theta_{[\Omega,\infty]},$$

while the maximal solution is given by

$$L^{\max} = \lim_{\delta\downarrow 0} \theta_{[\Omega_\delta,\infty]},$$

where we have denoted

$$\Omega_\delta := \{x \in \Omega : d(x,\partial\Omega) > \delta\}, \quad \delta > 0.$$

3 Two Pivotal Technical Results Under Radial Symmetry

In this section, for every $R > 0$, we consider the auxiliary problem

$$\begin{cases} -\psi_i'' - \dfrac{N-1}{r}\psi_i' = \lambda\psi_i + \displaystyle\sum_{j=1,j\ne i}^{n} a_{ij}\psi_j - b_i(R-r)^{\gamma_i}\psi_i^{p_i}, & 0 < r < R, \\ \psi_i'(0) = 0, \quad \psi_i(R) = +\infty, & 1 \le i \le n, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$, $\gamma_i \ge 0$, $p_i > 1$, $a_{ij} > 0$ and $b_i > 0$ for all $1 \le i, j \le n$. Without loss of generality, we can assume that the equations in (3.1) have been reordered so that

$$0 < \mu_n \le \mu_{n-1} \le \dots \le \mu_1,$$

where, as in (1.4), for each $1 \le i \le n$, μ_i is defined by

$$\mu_i := \frac{\gamma_i + 2}{p_i - 1}.$$

As in (1.6), we consider

$$\begin{cases} I_+ := \{i \in \{1,\dots,n\} : \mu_i + 2 - \mu_1 > 0\}, \\ I_0 := \{i \in \{1,\dots,n\} : \mu_i + 2 - \mu_1 = 0\}, \\ I_- := \{i \in \{1,\dots,n\} : \mu_i + 2 - \mu_1 < 0\}. \end{cases} \tag{3.2}$$

Let $k \ge 1$ be such that

$$I_M := \{i \in \{1,\dots,n\} : \mu_i = \mu_1\} = \{1,\dots,k\},$$

and set

$$\alpha_i := \begin{cases} \mu_i & \text{if } i \in I_+ \cup I_0, \\ \dfrac{\mu_1 + \gamma_i}{p_i} & \text{if } i \in I_-, \end{cases} \tag{3.3}$$

and

$$\bar{A}_i := \begin{cases} \left(\dfrac{\mu_i(\mu_i+1)}{b_i}\right)^{\frac{1}{p_i-1}} & \text{if } i \in I_+, \\ \left[\dfrac{1}{b_i}\sum\limits_{j=1}^{k} a_{ij}\left(\dfrac{\mu_1(\mu_1+1)}{b_j}\right)^{\frac{1}{p_j-1}}\right]^{\frac{1}{p_i}} & \text{if } i \in I_-, \\ \bar{A}_{0,i} & \text{if } i \in I_0, \end{cases} \tag{3.4}$$

where $\bar{A}_{0,i}$ stands for the unique positive solution of

$$b_i x^{p_i} - \mu_i(\mu_i+1)x = \sum_{j=1}^{k} a_{ij}\left(\frac{\mu_1(\mu_1+1)}{b_j}\right)^{\frac{1}{p_j-1}}.$$

The first result of this section reads as follows.

Lemma 3.1. *For each $\varepsilon > 0$, there exists a constant $C := C(\varepsilon, n)$ such that the function $\overline{\psi}_\varepsilon := (\overline{\psi}_{\varepsilon,1}, \dots, \overline{\psi}_{\varepsilon,n})$, defined by*

$$\overline{\psi}_{\varepsilon,i}(r) := (1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2 + C, \quad 0 \le r < R,\ 1 \le i \le n,$$

provides us with a supersolution of (3.1).

Proof. By definition, $\overline{\psi}_\varepsilon$ is smooth and satisfies the boundary conditions. Hence, $\overline{\psi}_\varepsilon$ is a supersolution of (3.1) if, and only if,

$$\begin{aligned} -\left(\frac{r}{R}\right)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1)(R-r)^{-\alpha_i-2} - \frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r)^{-\alpha_i-1} - \frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i} \\ \ge \lambda\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2 + C\right] + \sum_{j=1,j\ne i}^{n} a_{ij}\left[(1+\varepsilon)\bar{A}_j(R-r)^{-\alpha_j}\left(\frac{r}{R}\right)^2 + C\right] \\ - b_i(R-r)^{\gamma_i}\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2 + C\right]^{p_i} \end{aligned} \tag{3.5}$$

for every $0 < r < R$ and $0 \le i \le n$. Now, multiply (3.5) by $(R-r)^{\alpha_i+2}$ if $i \in I_+ \cup I_0$, and by $(R-r)^{-\gamma_i+\alpha_i p_i}$ if $i \in I_-$. Then, $\overline{\psi}_\varepsilon$ is a supersolution if, and only if, for every $0 < r < R$,

$$\begin{aligned} -\left(\frac{r}{R}\right)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1) - \frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r) - \frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^2 \\ \ge \lambda\left[(1+\varepsilon)\bar{A}_i(R-r)^2\left(\frac{r}{R}\right)^2 + C(R-r)^{\alpha_i+2}\right] + \sum_{j=1,j\ne i}^{n} a_{ij}\left[(1+\varepsilon)\bar{A}_j(R-r)^{\alpha_i+2-\alpha_j}\left(\frac{r}{R}\right)^2 + C(R-r)^{\alpha_i+2}\right] \\ - b_i(R-r)^{\alpha_i+2+\gamma_i}\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2 + C\right]^{p_i} \end{aligned} \tag{3.6}$$

if $i \in I_+ \cup I_0$, and

$$\begin{aligned} -\left(\frac{r}{R}\right)^2(1+\varepsilon)\bar{A}_i\alpha_i(\alpha_i+1)(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i-2} - \frac{(N+3)r}{R^2}(1+\varepsilon)\bar{A}_i\alpha_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i-1} \\ - \frac{2N}{R^2}(1+\varepsilon)\bar{A}_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i} \\ \ge \lambda\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_i}\left(\frac{r}{R}\right)^2 + C(R-r)^{-\gamma_i+\alpha_i p_i}\right] \\ + \sum_{j=1,j\ne i}^{n} a_{ij}\left[(1+\varepsilon)\bar{A}_j(R-r)^{-\gamma_i+\alpha_i p_i-\alpha_j}\left(\frac{r}{R}\right)^2 + C(R-r)^{-\gamma_i+\alpha_i p_i}\right] \\ - b_i(R-r)^{\alpha_i p_i}\left[(1+\varepsilon)\bar{A}_i(R-r)^{-\alpha_i}\left(\frac{r}{R}\right)^2 + C\right]^{p_i} \end{aligned} \tag{3.7}$$

if $i \in I_-$.

Let us show that

$$\alpha_i + 2 - \alpha_j > 0 \quad \text{for all } i \in I_+ \text{ and } 1 \le j \le n,\ j \ne i. \tag{3.8}$$

Indeed, if $i \in I_+$ and $j \in I_+ \cup I_0$, $j \neq i$, we have that

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \mu_j \geq \mu_i + 2 - \mu_1 > 0,$$

by the definition of I_+ (see (3.2)). When $j \in I_-$, we may proceed as follows. According to (3.2),

$$\frac{\gamma_j + 2}{p_j - 1} < \mu_1 - 2, \quad j \in I_-.$$

So, multiplying by $p_j - 1$ we deduce that $\gamma_j < p_j\mu_1 - 2p_j - \mu_1$, $j \in I_-$, and dividing by p_j yields

$$\frac{\gamma_j}{p_j} + \frac{\mu_1}{p_j} < \mu_1 - 2, \quad j \in I_-.$$

Equivalently,

$$\alpha_j < \mu_1 - 2, \quad j \in I_-, \tag{3.9}$$

and therefore

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \alpha_j > \mu_i + 2 - \mu_1 + 2 > 2 > 0$$

for every $i \in I_+$ and $j \in I_-$.

Analogously, the following estimates hold:

$$\begin{cases} \alpha_i + 2 - \alpha_j = 0 & \text{for every } i \in I_0,\ 1 \leq j \leq k, \\ \alpha_i + 2 - \alpha_j > 0 & \text{for every } i \in I_0,\ k+1 \leq j \leq n,\ j \neq i. \end{cases} \tag{3.10}$$

Indeed, by (3.3), $\alpha_i = \mu_i$ for all $i \in I_0$. Similarly, since $\{1, \dots, k\} \subset I_+$, we have $\alpha_j = \mu_j$ for all $1 \leq j \leq k$. Moreover, $\mu_j = \mu_1$ for all $1 \leq j \leq k$. Thus,

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \mu_j = \mu_i + 2 - \mu_1 = 0$$

for all $i \in I_0$ and $1 \leq j \leq k$, which shows the validity of the identities of (3.10). In order to check the inequalities of (3.10), we can argue as follows. Pick $i \in I_0$ and $j \in \{k+1, \dots, n\}$, $j \neq i$. Suppose $j \in I_+ \cup I_0$. Then, by (3.3) and taking into account that, by construction, $\mu_j < \mu_1$ for all $k+1 \leq j \leq n$, we find that

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \mu_j > \mu_i + 2 - \mu_1 = 0,$$

because $i \in I_0$. Now, suppose that $j \in I_-$. Then, thanks to (3.3) and (3.9),

$$\alpha_i + 2 - \alpha_j = \mu_i + 2 - \alpha_j > \mu_i + 2 - \mu_1 + 2 = 2 > 0.$$

Therefore, (3.10) holds.

Next, we will see that

$$-\gamma_i + \alpha_i p_i - \alpha_i - 2 > 0 \quad \text{for all } i \in I_-. \tag{3.11}$$

Indeed, by the definition of μ_i and since $i \in I_-$,

$$\frac{\gamma_i + 2}{p_i - 1} + 2 < \mu_1, \quad i \in I_-.$$

Thus, adding γ_i at both sides of this inequality and taking common factor $\gamma_i + 2$ yields

$$(\gamma_i + 2)\frac{p_i}{p_i - 1} < \mu_1 + \gamma_i, \quad i \in I_-.$$

Hence, by (3.3),

$$\frac{\gamma_i + 2}{p_i - 1} < \frac{\mu_1 + \gamma_i}{p_i} = \alpha_i, \quad i \in I_-,$$

whence (3.11).

Lastly, we will establish that

$$\begin{cases} -\gamma_i + \alpha_i p_i - \alpha_j = 0 & \text{for all } i \in I_-,\ 1 \le j \le k, \\ -\gamma_i + \alpha_i p_i - \alpha_j > 0 & \text{for all } i \in I_-,\ k+1 \le j \le n,\ j \ne i. \end{cases} \tag{3.12}$$

By (3.3), $-\gamma_i + \alpha_i p_i = \mu_1$ for all $i \in I_-$. Thus,

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j = \mu_1 - \mu_j = 0$$

for all $i \in I_-$ and $1 \le j \le k$, whence the identities of (3.12). Now, pick $i \in I_-$ and $k+1 \le j \le n$, $i \ne j$. Suppose $j \in I_+ \cup I_0$. Then, due to (3.3),

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j = \mu_1 - \mu_j > 0,$$

and hence (3.12) holds in this case. Now, suppose $j \in I_-$. Then, by (3.9),

$$-\gamma_i + \alpha_i p_i - \alpha_j = \mu_1 - \alpha_j > 2 > 0,$$

and therefore (3.12) is satisfied.

By (3.3) and the definition of μ_i,

$$\alpha_i + 2 + \gamma_i - \alpha_i p_i = \mu_i + 2 + \gamma_i - \mu_i p_i = 0 \quad \text{for all } i \in I_+ \cup I_0.$$

Thus,

$$\lim_{r \uparrow R} \Big(b_i (R-r)^{\alpha_i + 2 + \gamma_i} \Big[(1+\varepsilon)\bar{A}_i (R-r)^{-\alpha_i} \Big(\frac{r}{R}\Big)^2 + C \Big]^{p_i} \Big) = b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i} \tag{3.13}$$

for all $C \ge 0$ and $i \in I_+ \cup I_0$. Hence, thanks to (3.8), (3.10) and (3.13), we can extend (3.6) to $r = R$ by letting $r \uparrow R$. Similarly, (3.7) can be extended to $r = R$ by (3.11) and (3.12). More precisely, at $r = R$, (3.6) provides us with

$$\begin{cases} -(1+\varepsilon)\bar{A}_i \alpha_i(\alpha_i + 1) \ge -b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i}, & i \in I_+, \\ -(1+\varepsilon)\bar{A}_i \alpha_i(\alpha_i + 1) \ge \sum_{j=1}^{k} a_{ij}(1+\varepsilon)\bar{A}_j - b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i}, & i \in I_0, \end{cases} \tag{3.14}$$

while (3.7) at $r = R$ becomes

$$0 \ge \sum_{j=1}^{k} a_{ij}(1+\varepsilon)\bar{A}_j - b_i (1+\varepsilon)^{p_i} \bar{A}_i^{p_i}, \quad i \in I_-. \tag{3.15}$$

Due to (3.4) and using that $(1+\varepsilon) < (1+\varepsilon)^{p_i}$ (since $p_i > 1$ for all $1 \le i \le n$), it is easily seen that (3.14) and (3.15) are true. In the derivation one should note that $\alpha_j = \mu_j = \mu_1$ for all $1 \le j \le k$, by construction. Actually, all inequalities in (3.14) and (3.15) are strict, because $(1+\varepsilon) < (1+\varepsilon)^{p_i}$. By continuity, this entails the existence of $\delta := \delta(\varepsilon, n) > 0$ such that (3.6) and (3.7) are satisfied for all $r \in [R-\delta, R)$. Therefore, choosing a sufficiently large $C > 0$, we can assume that (3.5) holds in $(0, R)$, because, for each $1 \le i \le n$, the function $b_i(R-r)^{\gamma_i}$ is positive and bounded away from zero in $[0, R-\delta]$. The proof is complete. □

The next result provides us with a universal subsolution on an annulus.

Lemma 3.2. *Let $R_2 > R_1 > 0$ and consider the problem*

$$\begin{cases} -\psi_i'' - \dfrac{N-1}{r}\psi_i' = \lambda\psi_i + \sum_{j=1, j\ne i}^{n} a_{ij}\psi_j - \beta_i(r)(r-R_1)^{\gamma_i}\psi_i^{p_i}, & R_1 < r < R_2, \\ \psi_i(R_1) = +\infty, \quad \psi_i'(R_2) = 0, & 1 \le i \le n, \end{cases} \tag{3.16}$$

where all the coefficients satisfy the same requirements as in (3.1) and, for every $1 \le i \le n$, the function $\beta_i \in \mathcal{C}^\nu[R_1, R_2]$ satisfies $\beta_i(r) > 0$ for all $r \in [R_1, R_2]$. Then, for every $\varepsilon \in (0,1)$, there exists a negative constant $D := D(\varepsilon, n) < 0$ such that the function

$$\underline{\psi}_\varepsilon := (\underline{\psi}_{\varepsilon,1}, \dots, \underline{\psi}_{\varepsilon,n}),$$

defined by

$$\underline{\psi}_{\varepsilon,i}(r) := \max\{0, (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D\}, \quad R_1 < r \le R_2,\ 1 \le i \le n,$$

provides us with a weak subsolution of (3.1), *as discussed in* [1], *where the constants* α_i, $1 \le i \le n$, *are given by* (3.3) *and*

$$\underline{A}_i := \begin{cases} \Big(\dfrac{\mu_i(\mu_i+1)}{\beta_i(R_1)}\Big)^{\frac{1}{p_i-1}} & \text{if } i \in I_+, \\ \Big[\dfrac{1}{\beta_i(R_1)}\displaystyle\sum_{j=1}^{k} a_{ij}\Big(\dfrac{\mu_1(\mu_1+1)}{\beta_j(R_1)}\Big)^{\frac{1}{p_j-1}}\Big]^{\frac{1}{p_i}} & \text{if } i \in I_-, \\ \underline{A}_{0,i} & \text{if } i \in I_0, \end{cases} \tag{3.17}$$

where $\underline{A}_{0,i}$ *stands for the unique positive solution of*

$$\beta_i(R_1)x^{p_i} - \mu_i(\mu_i+1)x = \sum_{j=1}^{k} a_{ij}\Big(\frac{\mu_1(\mu_1+1)}{\beta_j(R_1)}\Big)^{\frac{1}{p_j-1}}.$$

Proof. As the maps $r \mapsto (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i}$ are strictly decreasing, taking any D satisfying

$$D < -(1-\varepsilon)\underline{A}_i(R_2-R_1)^{-\alpha_i} = -\min\{(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} : R_1 < r \le R_2\}$$

for all $1 \le i \le n$, there exist $\varrho_i(D) \in (R_1, R_2)$, $1 \le i \le n$, such that

$$\underline{\psi}_{\varepsilon,i} = \begin{cases} (1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D & \text{if } R_1 < r \le \varrho_i(D), \\ 0 & \text{if } \varrho_i(D) < r \le R_2,\ 1 \le i \le n. \end{cases}$$

Moreover, the mappings $D \mapsto \varrho_i(D)$, $1 \le i \le n$, can be chosen continuous, and

$$\lim_{D \downarrow -\infty} \varrho_i(D) = R_1, \quad 1 \le i \le n. \tag{3.18}$$

Thus, $\underline{\psi}_\varepsilon$ is a subsolution of (3.16) if, and only if, for every $1 \le i \le n$ and $R_1 < r \le \varrho_i(D)$,

$$\begin{aligned} &-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1)(r-R_1)^{-\alpha_i-2} + \frac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1)^{-\alpha_i-1} \\ &\quad \le \lambda[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D] + \sum_{j=1,j\ne i}^{n} a_{ij}[(1-\varepsilon)\underline{A}_j(r-R_1)^{-\alpha_j} + D] \\ &\qquad - \beta_i(r)(r-R_1)^{\gamma_i}[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D]^{p_i}. \end{aligned} \tag{3.19}$$

Next, we will adapt the proof of Lemma 3.1. Multiplying (3.19) by $(r-R_1)^{\alpha_i+2}$ when $i \in I_+ \cup I_0$ and by $(r-R_1)^{-\gamma_i+\alpha_i p_i}$ when $i \in I_-$, it becomes apparent that $\underline{\psi}_\varepsilon$ is a subsolution of (3.16) if, and only if,

$$\begin{aligned} &-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) + \frac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1) \\ &\quad \le \lambda[(1-\varepsilon)\underline{A}_i(r-R_1)^2 + D(r-R_1)^{\alpha_i+2}] \sum_{j=1,j\ne i}^{n} a_{ij}[(1-\varepsilon)\underline{A}_j(r-R_1)^{\alpha_i+2-\alpha_j} + D(r-R_1)^{\alpha_i+2}] \\ &\qquad - \beta_i(r)(r-R_1)^{\alpha_i+2+\gamma_i}[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D]^{p_i} \end{aligned} \tag{3.20}$$

for all $i \in I_+ \cup I_0$ and $R_1 < r \le \varrho_i(D)$, and

$$\begin{aligned} &-(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1)(r-R_1)^{-\alpha_i-2-\gamma_i+\alpha_i p_i} + \frac{N-1}{r}(1-\varepsilon)\underline{A}_i\alpha_i(r-R_1)^{-\alpha_i-1-\gamma_i+\alpha_i p_i} \\ &\quad \le \lambda[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i-\gamma_i+\alpha_i p_i} + D(r-R_1)^{-\gamma_i+\alpha_i p_i}] \\ &\qquad \sum_{j=1,j\ne i}^{n} a_{ij}[(1-\varepsilon)\underline{A}_j(r-R_1)^{-\gamma_i+\alpha_i p_i-\alpha_j} + D(r-R_1)^{-\gamma_i+\alpha_i p_i}] \\ &\qquad - \beta_i(r)(r-R_1)^{\alpha_i p_i}[(1-\varepsilon)\underline{A}_i(r-R_1)^{-\alpha_i} + D]^{p_i} \end{aligned} \tag{3.21}$$

for all $i \in I_-$ and $R_1 < r \le \varrho_i(D)$. Thanks to (3.8), (3.10), (3.11) and (3.12), letting $r \downarrow R_1$ in (3.20) and (3.21) yields

$$\begin{cases} -(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) \le -\beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, & i \in I_+, \\ -(1-\varepsilon)\underline{A}_i\alpha_i(\alpha_i+1) \le \sum_{j=1}^k a_{ij}(1-\varepsilon)\underline{A}_j - \beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, & i \in I_0, \\ 0 \le \sum_{j=1}^k a_{ij}(1-\varepsilon)\underline{A}_j - \beta_i(R_1)(1-\varepsilon)^{p_i}\underline{A}_i^{p_i}, & i \in I_-. \end{cases} \tag{3.22}$$

As $(1-\varepsilon) > (1-\varepsilon)^{p_i}$ for all $1 \le i \le n$, all inequalities in (3.22) are strict. Hence, by continuity, there exists $\delta = \delta(\varepsilon, n) > 0$ such that all inequalities in (3.20) and (3.21) hold in the interval $(R_1, R_1+\delta]$. Finally, by (3.18), $D < 0$ can be taken sufficiently negative so that

$$\max_{1\le i\le n} \varrho_i(D) \le R_1 + \delta.$$

This ends the proof. □

4 Proofs of the Main Results

As $\partial\Omega$ is smooth, the outward unit normal vector field to $\partial\Omega$ is well defined at every point of $\partial\Omega$. We will denote it by

$$n\colon \partial\Omega \to \mathbb{R}^N, \quad z \mapsto n_z.$$

Since $\partial\Omega \in \mathcal{C}^2$, Ω satisfies the *uniform interior sphere in the strong sense on* $\partial\Omega$ (see [26, Theorem 1.9]). So, there exists $R_0 > 0$ such that for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < R_0$, there is a point $\pi(x) \in \partial\Omega$ such that

$$t(x) := \operatorname{dist}(x, \partial\Omega) = |x - \pi(x)|, \quad B_{R_0}\Big(\pi(x) + R_0\frac{x-\pi(x)}{t(x)}\Big) \subset \Omega. \tag{4.1}$$

Moreover, R_0 can be shortened so that for every $z \in \partial\Omega$,

$$\bar{B}_{R_0}(z - R_0 n_z) \cap \partial\Omega = \{z\} \quad \text{and} \quad \bar{B}_{R_0}(z + R_0 n_z) \cap \bar{\Omega} = \{z\}. \tag{4.2}$$

4.1 Proof of Theorem 1.1

Fix $z \in \partial\Omega$ and $0 < \eta < 1$. By (1.2), there exist $\delta > 0$ such that for every $1 \le i \le n$,

$$(1-\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega)]^{\gamma_i(z)} < \mathfrak{a}_i(x) < (1+\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega)]^{\gamma_i(z)} \tag{4.3}$$

for all $x \in B_\delta(z) \cap \Omega$. Choose R_0 sufficiently small so that (4.1) and (4.2) hold, and $0 < R_0 < \delta$. Set

$$\Gamma := \bar{B}_{R_0/2}(z) \cap \partial\Omega.$$

It is rather clear that there exist $R > 0$ and $\varrho_0 > 0$ such that

$$B_R(y - (R+\varrho)n_y) \subset B_\delta(z) \cap \Omega$$

for all $y \in \Gamma$ and $0 \le \varrho \le \varrho_0$. Figure 2 sketches this construction scheme.

According to (4.3), for every $y \in \Gamma$ and $0 \le \varrho \le \varrho_0$,

$$\mathfrak{a}_i(x) > (1-\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega]^{\gamma_i(z)} \ge (1-\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x, \partial B_R(y-(R+\varrho)n_y))]^{\gamma_i(z)}, \quad 1 \le i \le n, \tag{4.4}$$

for all $x \in B_R(y - (R+\varrho)n_y)$. Set

$$\begin{aligned} \bar{\lambda} &:= \max\{\|\lambda_i\|_\infty : 1 \le i \le n\}, \\ y_\varrho &:= y - (R+\varrho)n_y, \quad y \in \Gamma,\ \varrho \in [0, \varrho_0], \\ \bar{b}_i(z) &:= (1-\eta)\mathfrak{b}_i(z), \quad 1 \le i \le n, \end{aligned}$$

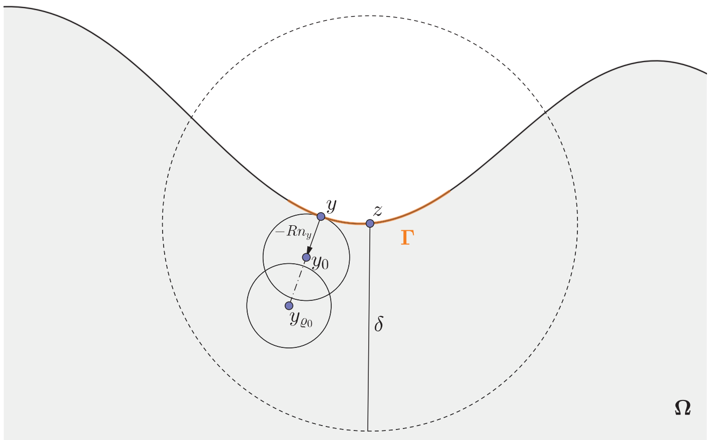

Figure 2. Scheme of the construction.

and consider, for each $y \in \Gamma$ and $0 < \varrho \le \varrho_0$, the problem

$$\begin{cases} -\Delta u_i = \bar{\lambda} u_i + \displaystyle\sum_{j=1, j\neq i}^{n} a_{ij} u_j - \bar{b}_i(z)(R - |x - y_\varrho|)^{\gamma_i(z)} u_i^{p_i} & \text{in } B_R(y_\varrho), \\ u_i = +\infty & \text{on } \partial B_R(y_\varrho),\ 1 \le i \le n. \end{cases} \tag{4.5}$$

By (4.4), any positive solution of (1.1),

$$L = (L_1, \dots, L_n) := (u_1, \dots, u_n),$$

is a *bounded* positive subsolution of (4.5) for every $y \in \Gamma$ and $0 < \varrho \le \varrho_0$.

Let $\varepsilon > 0$. Applying Lemma 3.1 to problem (3.1) with $\lambda = \bar{\lambda}$, $\gamma_i = \gamma_i(z)$, $b_i = \bar{b}_i(z)$, $\le i \le n$, we get the functions

$$\bar{\psi}_{\varepsilon,i}(r) = (1+\varepsilon)\bar{A}_i(z)(R-r)^{-\alpha_i(z)}\Big(\frac{r}{R}\Big)^2 + C, \quad 0 \le r < R,\ 1 \le i \le n,$$

where $\alpha_i(z)$, $\bar{A}_i(z)$, $1 \le i \le n$, are defined through (3.3) and (3.4). By the radial symmetry of (4.5), for every $y \in \Gamma$ and $0 < \varrho \le \varrho_0$, the functions

$$\bar{L}^{y_\varrho}_{\varepsilon,i}(x) := \bar{\psi}_{\varepsilon,i}(r), \quad x \in B_R(y_\varrho),\ r := |x - y_\varrho|,\ 1 \le i \le n,$$

provide us with a supersolution of (4.5). Hence, by Theorem 2.3,

$$L_i(x) \le \bar{L}^{y_\varrho}_{\varepsilon,i}(x), \quad x \in B_R(y_\varrho),\ 1 \le i \le n,$$

for every $y \in \Gamma$ and $0 < \varrho \le \varrho_0$. Consequently, we may infer

$$L_i(x) \le (1+\varepsilon)\bar{A}_i(z)[\operatorname{dist}(x, \partial B_R(y_0))]^{-\alpha_i(z)}\Big(\frac{|x - y_0|}{R}\Big)^2 + C \tag{4.6}$$

for all $y \in \Gamma$, $x \in B_R(y_0)$ and $1 \le i \le n$.

On the other hand, for every x sufficiently close to Γ, we have that $\operatorname{dist}(x, \Gamma) = \operatorname{dist}(x, \partial B_R(y_0))$, with $y_0 = \pi(x) - Rn_{\pi(x)}$. Thus, (4.6) implies

$$\limsup_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x, \Gamma)]^{-\alpha_i(z)}} \le (1+\varepsilon)\bar{A}_i(z), \quad 1 \le i \le n.$$

Therefore, letting $\varepsilon \downarrow 0$ yields

$$\limsup_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x, \Gamma)]^{-\alpha_i(z)}} \le \bar{A}_i(z), \quad 1 \le i \le n. \tag{4.7}$$

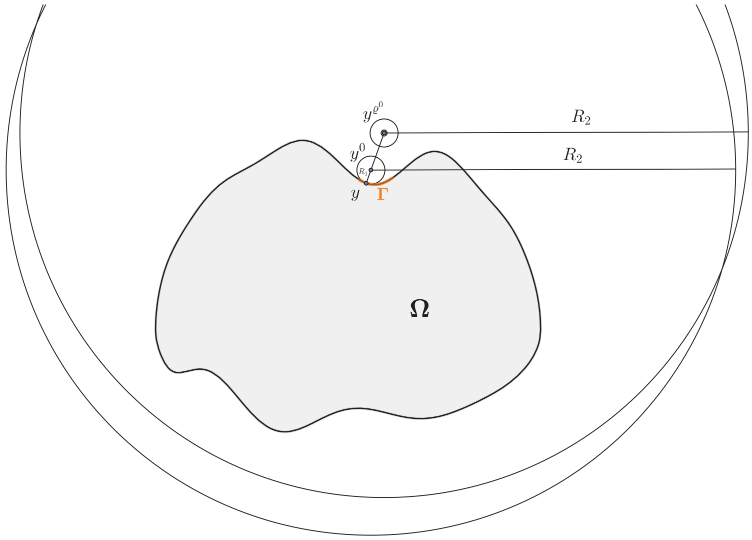

Figure 3. Scheme of the construction.

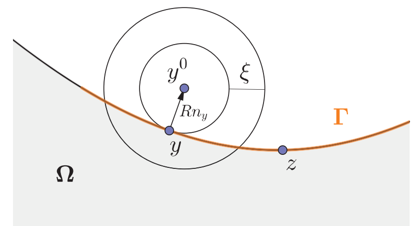

Figure 4. Magnification around y^0.

Now, we will construct a subsolution of (1.1) with the appropriate growth on Γ. Due to (4.2) and taking into account that Ω is bounded, there exist $R_2 > R_1 > 0$ and $\varrho^0 > 0$ such that

$$\Omega \subset \bigcap_{0\le\varrho\le\varrho^0} A_{R_1,R_2}(y + (R_1+\varrho)n_y) \quad \text{and} \quad \overline{\Omega} \cap \partial A_{R_1,R_2}(y + R_1 n_y) = \{y\}$$

for all $y \in \Gamma$ and $\varrho \in [0, \varrho^0]$, where, for every $z \in \mathbb{R}^N$ and $0 < r_1 < r_2$,

$$A_{r_1,r_2}(z) := \{x \in \mathbb{R}^N : r_1 < |x - z| < r_2\}.$$

As before, we will denote

$$y^\varrho := y + (R_1 + \varrho)n_y, \quad y \in \Gamma,\ 0 \le \varrho \le \varrho^0.$$

Figure 3 sketches this construction.

Shortening R_1 and ϱ^0 if necessary, it becomes apparent that there exists $\xi > 0$ such that

$$A_{R_1,R_1+\xi}(y^\varrho) \cap \Omega \subset (B_\delta(z) \cap \Omega) \quad \text{for all } y \in \Gamma \text{ and } \varrho \in [0, \varrho^0]. \tag{4.8}$$

Figure 4 shows a magnification of Figure 3 in a neighborhood of y^0. Using again (4.3) and thanks to (4.8), we obtain that

$$\mathfrak{a}_i(x) < (1+\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x, \partial\Omega)]^{\gamma_i(z)} \le (1+\eta)\mathfrak{b}_i(z)[\operatorname{dist}(x, \partial B_{R_1}(y^\varrho))]^{\gamma_i(z)} \tag{4.9}$$

for all $y \in \Gamma$, $\varrho \in [0, \varrho^0]$, $x \in A_{R_1,R_1+\xi}(y^0) \cap \Omega$, $1 \le i \le n$. For getting (4.9) in the entire Ω, we may proceed as follows. Set

$$K := \max_{1\le i\le n}\left\{\frac{\|\mathfrak{a}_i\|_\infty}{(R_1+\xi)^{\gamma_i(z)}} + (1+\eta)\mathfrak{b}_i(z)\right\},$$

and consider, for each $1 \le i \le n$, the piecewise linear function

$$\beta_i(r) = \begin{cases} (1+\eta)\mathfrak{b}_i(z), & R_1 \le r < R_1 + \dfrac{\xi}{2}, \\ K + \dfrac{K - (1+\eta)\mathfrak{b}_i(z)}{\xi/2}(r - (R_1+\xi)), & R_1 + \dfrac{\xi}{2} \le r < R_1 + \xi, \\ K, & R_1 + \xi \le r \le R_2. \end{cases}$$

Using the definition of β_i and (4.9), we have that

$$\mathfrak{a}_i(x) \le \beta_i(|x - y^\varrho|)[|x - y^\varrho| - R_1]^{\gamma_i(z)}, \quad 1 \le i \le n, \tag{4.10}$$

for all $y \in \Gamma$, $0 < \varrho \le \varrho^0$ and $x \in \Omega$.

Subsequently, we will consider the auxiliary problem

$$\begin{cases} -\Delta u_i = \underline{\lambda} u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \beta_i(r)(r-R_1)^{\gamma_i(z)} u_i^{p_i} & \text{in } A_{R_1,R_2}(y_\varrho), \\ u_i = +\infty & \text{on } \partial A_{R_1,R_2}(y_\varrho),\ 1 \le i \le n, \end{cases} \tag{4.11}$$

where

$$\underline{\lambda} := \min\{-\|\lambda_i\|_\infty : 1 \le i \le n\}, \quad r := |x - y^\varrho|.$$

Pick $\varepsilon \in (0,1)$. Then, due to Lemma 3.2, for every $y \in \Gamma$ and $0 < \varrho \le \varrho^0$, the function $L_\varepsilon^{y_\varrho} := (L_{\varepsilon,1}^{y_\varrho}, \dots, L_{\varepsilon,n}^{y_\varrho})$, defined for every $1 \le i \le n$ by

$$L_{\varepsilon,i}^{y_\varrho}(x) := \underline{\psi}_{\varepsilon,i}(r) = \max\{0, (1-\varepsilon)\underline{A}_i(z)(r-R_1)^{-\alpha_i(z)} + D\}, \quad x \in A_{R_1,R_2}(y^\varrho),$$

provide us with a positive subsolution for (4.11), where $\underline{A}_i(z)$ is defined through (3.17) for every $1 \le i \le n$.

By (4.10), the restriction $L_\varepsilon^{y_\varrho}|_\Omega$ provides us with a *bounded* positive subsolution of (1.1). Hence, by Theorem 2.3,

$$L_i(x) \ge \underline{L}_{\varepsilon,i}^{y^\varrho}(x), \quad x \in \Omega,\ 1 \le i \le n,$$

for all $y \in \Gamma$ and $0 < \varrho \le \varrho^0$. Thus, we can infer that

$$L_i \ge \underline{L}_{\varepsilon,i}^{y_0}, \quad 1 \le i \le n,$$

for all $y \in \Gamma$. The last inequality implies that

$$\liminf_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \ge (1-\varepsilon)\underline{A}_i(z), \quad 1 \le i \le n.$$

Hence, letting $\varepsilon \downarrow 0$ yields

$$\liminf_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \ge \underline{A}_i(z), \quad 1 \le i \le n. \tag{4.12}$$

Consequently, owing to (4.7) and (4.12), for each $z \in \partial\Omega$ and $0 < \eta < 1$, there exists a compact neighborhood of $z \in \partial\Omega$, Γ, such that

$$\underline{A}_i(z) \le \liminf_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \le \limsup_{\operatorname{dist}(x,\Gamma)\downarrow 0} \frac{L_i(x)}{[\operatorname{dist}(x,\Gamma)]^{-\alpha_i(z)}} \le \bar{A}_i(z)$$

for every $1 \le i \le n$. Therefore, letting $\eta \downarrow 0$ yields

$$\lim_{\substack{x\to z\\ x\in\Omega}} \frac{L_i(x)}{[\operatorname{dist}(x,\partial\Omega)]^{-\alpha_i(z)}} = A_i(z),$$

because

$$A_i(z) = \lim_{\eta\downarrow 0} \bar{A}_i(z) = \lim_{\eta\downarrow 0} \underline{A}_i(z).$$

This ends the proof of Theorem 1.1.

4.2 Proof of Theorem 1.3

Let $L := (L_1, \dots, L_n)$ and $M := (M_1, \dots, M_n)$ be two positive solutions of (1.1). Using (1.7) it is easily seen that the quotients

$$q_i(x) := \begin{cases} \dfrac{L_i(x)}{M_i(x)} & x \in \Omega, \\ 1 & x \in \partial\Omega,\ 1 \le i \le n, \end{cases}$$

are uniformly continuous in $\bar{\Omega}$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|q_i(x) - q_i(\pi(x))| = |q_i(x) - 1| < \varepsilon \quad \text{if } |x - \pi(x)| < \delta,\ 1 \le i \le n.$$

Thus, setting

$$Q_\xi := \{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \le \xi\}, \quad \xi > 0,$$

we find that

$$(1-\varepsilon)M_i \le L_i \le (1+\varepsilon)M_i \quad \text{in } Q_\delta,\ 1 \le i \le n.$$

Moreover, L is a solution of the problem

$$\begin{cases} -\Delta u_i = \lambda_i(x)u_i + \sum\limits_{j=1, j\ne i}^{n} a_{ij}u_j - \mathfrak{a}_i(x)u_i^{p_i} & \text{in } \Omega \setminus Q_\delta, \\ u_i = L_i & \text{on } \partial(\Omega \setminus Q_\delta),\ 1 \le i \le n, \end{cases} \tag{4.13}$$

$(1-\varepsilon)M$ is a subsolution of (4.13) and $(1+\varepsilon)M$ is a supersolution of (4.13). Therefore, by Theorem 2.3,

$$(1-\varepsilon)M_i \le L_i \le (1+\varepsilon)M_i \quad \text{in } \Omega \setminus Q_\delta,\ 1 \le i \le n.$$

Letting $\varepsilon \downarrow 0$ we find that $L = M$ in Ω. This ends the proof.

4.3 Proof of Remark 1.2

Setting

$$\mu_{\max}(z) := \max_{1\le j\le n} \mu_j(z), \quad z \in \partial\Omega,$$

by Theorem 1.1, for every $z \in \partial\Omega$ and $1 \le i \le n$, we have

$$\alpha_i(z) = \begin{cases} \mu_i(z) & \text{if } \mu_i(z) + 2 - \mu_{\max}(z) \ge 0, \\ \dfrac{\mu_{\max}(z) + \gamma_i(z)}{p_i} & \text{if } \mu_i(z) + 2 - \mu_{\max}(z) < 0. \end{cases}$$

Pick $z \in \partial\Omega$ and $1 \le i \le n$. Based on the continuity of $\mu_{\max}(z)$, it easy to check that α_i is continuous at z if

$$\mu_i(z) + 2 - \mu_{\max}(z) \ne 0.$$

Suppose

$$\mu_i(z) + 2 - \mu_{\max}(z) = 0 \tag{4.14}$$

for some $z \in \partial\Omega$, and let $(z_n)_{n\ge1} \subset \partial\Omega$ be a sequence such that $z_n \to z$ if $n \to +\infty$ and

$$\mu_i(z_n) + 2 - \mu_{\max}(z_n) < 0 \quad \text{for all } n \ge 1.$$

Then, invoking (4.14) and (1.4), shows that

$$\begin{aligned} \lim_{n\to\infty} \alpha_i(z_n) = \lim_{n\to\infty} \frac{\mu_{\max}(z_n) + \gamma_i(z_n)}{p_i} &= \frac{\mu_{\max}(z) + \gamma_i(z)}{p_i} \\ &= \frac{\mu_i(z) + 2 + \gamma_i(z)}{p_i} = \frac{\frac{\gamma_i(z)+2}{p_i-1} + 2 + \gamma_i(z)}{p_i} \\ &= \frac{(\gamma_i(z)+2)(\frac{1}{p_i-1} + 1)}{p_i} = \frac{\gamma_i(z) + 2}{p_i - 1} \\ &= \alpha_i(z). \end{aligned}$$

Therefore, α_i is also continuous at $z \in \partial\Omega$ if (4.14) holds. This ends the proof.

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