On compactness results of Lions-Peetre type for bilinear operators

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Abstract

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples, let E be a Banach space and let T be a bilinear operator such that $\|T(a,b)\|_E \leq M_j \|a\|_{A_j} \|b\|_{B_j}$ for $a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1$. If $T : A_j^\circ \times B_j^\circ \longrightarrow E$ compactly for $j = 0$ or 1, we show that T may be uniquely extended to a compact bilinear operator from the complex interpolation spaces generated by $\bar{A}$ and $\bar{B}$ to E. Furthermore, the corresponding result for the real method is given and we also study the case when E is replaced by a couple (E_0, E_1) of Banach function spaces on the same measure space.

Keywords: Compact bilinear operators; complex interpolation; real interpolation; interpolation of compact bilinear operators among L_p spaces.
2010 MSC: Primary 46B70, 47B07. Secondary 46B50, 47B38.

1. Introduction

The problem of interpolation of compact bilinear (multilinear) operators was considered for the first time by Calderón [6] in his seminal paper on the complex method and it is receiving attention by several authors in recent years. See the papers by Fernandez and Silva [12], Fernández-Cabrera and Martínez [13, 14] and Cobos, Fernández-Cabrera and Martínez [7]. Moreover, quantitative results in terms of the measure of non-compactness have been established by Mastyło and Silva [20] and Besoy and Cobos [5]. A motivation for all these investigations

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[1] The authors have been supported in part by MTM2017-84058-P (AEI/FEDER, UE).

is the fact that compact bilinear operators arise rather naturally in harmonic analysis. See, for example, the papers by Bényi and Torres [3], Bényi and Oh [2] and Hu [15]. In particular, commutators of Calderón-Zygmund bilinear operators acting on L_p-spaces are compact (see [3] and [7]).

For the case of linear operators, the famous results established by Lions and Peetre [19] play an important role in the proofs of all compactness theorems (see, for example, [9, 10, 8]). Lions-Peetre results refer to the degenerate situations when one of the Banach couples reduces to a single Banach space. Similarly, for bilinear operators, Lions-Peetre type results are also important tools in the research on interpolation properties of compact bilinear operators.

Working with Banach couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ and bilinear operators T, often the starting assumption is that

$$T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E_0 + E_1 \quad \text{boundedly with the restrictions}$$

$$T : A_j \times B_j \longrightarrow E_j \quad \text{being also bounded for} \quad j = 0, 1. \tag{1.1}$$

However, sometimes in applications T does not satisfy (1.1) but only that there are constants $M_j > 0$ such that

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1. \tag{1.2}$$

Under the assumption (1.1), Lions-Peetre type theorems have been established in [13, Theorems 5.1 and 5.2]. Note that if $A_0 = A_1$ and $B_0 = B_1$, then (1.1) and (1.2) are the same. However, if $E_0 = E_1$ but $A_0 \neq A_1$ and $B_0 \neq B_1$, then (1.2) is weaker than (1.1). Fernández-Cabrera and Martínez have shown by means of examples (see [13, Example 4.2] and [14, Counterexamples 4.2 and 4.3]) that under the assumption (1.2), compactness of the interpolated operator can fail in some cases where assuming (1.1) the interpolated operator is compact. In fact, under assumption (1.2), it is not known a bilinear Lions-Peetre compactness result for $E_0 = E_1$. Accordingly, we prove in this paper such a theorem.

We start by recalling in Section 2 the most familiar interpolation methods. Then we establish the bilinear Lions-Peetre compactness result. We proceed with the help of duality, using the results of Ramanujan and Schock [21].

Finally, in Section 3, we study the non-degenerated case $E_0 \neq E_1$ but assuming that (E_0, E_1) is a couple of Banach function spaces on the same measure space, with E_0 having absolutely continuous norm, and we prove a compactness theorem under assumption (1.2). In particular, the results applies to couples of L_p spaces with $(E_0, E_1) = (L_{r_0}(\Omega), L_{r_1}(\Omega))$ and $1 \leq r_0 < \infty$. We conclude the paper with a result for the case $r_0 = \infty$ when the measure space (Ω, μ) is finite.

2. Bilinear compactness results of Lions-Peetre type

In what follows the scalar field is $\mathbb{C}$, the set of complex numbers.

Let A, B, E be complex Banach spaces. We put $U_A = \{a \in A \,:\, \|a\|_A \leq 1\}$ for the closed unit ball of A and define U_B similarly. Let $T : A \times B \longrightarrow E$ be a bilinear operator. We say that the operator T is *bounded* if

$$\|T\|_{A \times B, E} = \sup \left\{ \|T(a, b)\|_E \,:\, a \in U_A,\ b \in U_B \right\} < \infty.$$

If T is bounded, we write $T \in \mathcal{L}(A \times B, E)$. The operator T is said to be *compact* if

$$T(U_A \times U_B) = \big\{T(a, b) \ : \ a \in U_A,\, b \in U_B\big\}$$

is relatively compact in E or, equivalently, if for any bounded sequence $(z_n) \subseteq A \times B$, the sequence (Tz_n) has a convergent subsequence in E (see [3, Proposition 1]). Examples of compact bilinear operators can be found in [21, 3, 2, 15, 14, 7]. See also [16] for same examples of non-compact bilinear operators.

As it is show in [3, Proposition 3], the set of all compact bilinear operators from $A \times B$ into E is a closed subspace of $\mathcal{L}(A \times B, E)$.

Let E^* be the dual space of E. If $T \in \mathcal{L}(A \times B, E)$, following Ramanujan and Schock [21], we define the *adjoint operator* $T^\times$ of T as the linear map

$$T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{C})$$

given by

$$(T^\times f)(a, b) = f(T(a, b)).$$

It turns out that $\|T\|_{A\times B,E} = \|T^\times\|_{E^*,\mathcal{L}(A\times B,\mathbb{C})}$. Furthermore,

$$T \text{ is compact if and only if } T^\times \text{ is compact} \tag{2.1}$$

(see [21, Theorem 2.6]).

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, two Banach spaces A_j which are continuously embedded in the same Hausdorff topological vector space. We write A_j° for the closure of $A_0 \cap A_1$ in the norm of A_j. The Banach couple $\bar{A}$ is said to be *regular* if $A_j^\circ = A_j$ for $j = 0, 1$. If this is the case, the dual couple $\bar{A}^* = (A_0^*, A_1^*)$ is a Banach couple because $A_j^* \hookrightarrow (A_0 \cap A_1)^*$ for $j = 0, 1$. Here $\hookrightarrow$ means continuous embedding.

Consider the closed strip $D = \{z \in \mathbb{C} : 0 \leq \mathrm{Re} z \leq 1\}$ and define $\mathcal{F}(\bar{A})$ to be the space of all functions g from D into $A_0 + A_1$ such that g is bounded and continuous on D, analytic on the interior of D, with $g(j + it) \in A_j$ for all $t \in \mathbb{R}$, $j = 0, 1$, and the functions $t \to g(j + it)$ are continuous from $\mathbb{R}$ into A_j and tend to zero as $|t| \to \infty$. The space $\mathcal{F}(\bar{A})$ becomes a Banach space with the norm

$$\|g\|_{\mathcal{F}(\bar{A})} = \max_{j=0,1}\{\sup_{t\in\mathbb{R}} \|g(j + it)\|_{A_j}\}.$$

For $0 < \theta < 1$, the *complex interpolation space* $[A_0, A_1]_\theta$ consists of all $a \in A_0 + A_1$ such that $a = g(\theta)$ for some $g \in \mathcal{F}(\bar{A})$. We endow $[A_0, A_1]_\theta$ with the norm

$$\|a\|_{[A_0,A_1]_\theta} = \inf\{\|g\|_{\mathcal{F}(\bar{A})} \ : \ g(\theta) = a,\, g \in \mathcal{F}(\bar{A})\}.$$

See [6, 4, 22, 18].

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the *real interpolation space* $(A_0, A_1)_{\theta,q}$ is formed of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t}\right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1 \ , \ a_j \in A_j\}$$

is the Peetre's K-functional. See [19, 4, 22, 1].

Is turns out that

$$(A_0, A_1)_{\theta,1} \hookrightarrow [A_0, A_1]_\theta \hookrightarrow (A_0, A_1)_{\theta,\infty}.$$

The space $A_0 \cap A_1$ is dense in $[A_0, A_1]_\theta$, and also in $(A_0, A_1)_{\theta,q}$ if $q < \infty$ (see [4, 22]).

Let $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be other Banach couples. By $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ we mean that T is a bounded bilinear operator $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E_0 + E_1$ whose restriction to $A_j \times B_j$ defines a bounded bilinear operator from $A_j \times B_j$ into E_j for $j = 0, 1$. We write $\|T\|_j$ for the norm of $T : A_j \times B_j \longrightarrow E_j$.

The following compactness results of Lions-Peetre type are consequence of [13, Theorems 5.1 and 5.3].

Theorem 2.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let E be a Banach space. Assume that $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E$ is a bounded bilinear operator such that the restriction $T : A_j \times B_j \longrightarrow E$ is compact for $j = 0$ or 1. Let $0 < \theta, \eta < 1$ and $1 \le p, q \le \infty$. Then the following holds.*

(i) $T : [A_0, A_1]_\theta \times [B_0, B_1]_\eta \longrightarrow E$ is compact.

(ii) $T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\eta,q} \longrightarrow E$ is compact.

Theorem 2.2. *Let A, B be Banach spaces and let $\bar{E} = (E_0, E_1)$ be a Banach couple. Assume that $T : A \times B \longrightarrow E_0 \cap E_1$ is a bounded bilinear operator such that any of the restrictions $T : A \times B \longrightarrow E_j$ is compact for $j = 0$ or 1. Let $0 < \theta < 1$ and $1 \le q \le \infty$. Then the following holds.*

(i) $T : A \times B \longrightarrow [E_0, E_1]_\theta$ is compact.

(ii) $T : A \times B \longrightarrow (E_0, E_1)_{\theta,q}$ is compact.

Sometimes in applications we do not have that $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ but only that the bilinear operator T is defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ and that there are constants $M_j > 0$ such that

$$\|T(a, b)\|_{E_j} \le M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1. \tag{2.2}$$

We denote by $\mathcal{B}(\bar{A} \times \bar{B}, \bar{E}) = \mathcal{B}(\bar{A} \times \bar{B}, (E_0, E_1))$ those operators which satisfy (2.2).

Assumption (2.2) was the one used by Calderón [6, 10.1] for establishing the bilinear (and multilinear) interpolation theorem for the complex method (see also [4, 4.4]).

If $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$, it is not difficult to check that T may be uniquely extended to a bilinear operator $T_j : A_j^\circ \times B_j^\circ \longrightarrow E_j$, $j = 0, 1$. We write $\|T\|_j = \|T_j\|_{A_j^\circ \times B_j^\circ, E_j} = M_j$, $j = 0, 1$. We say that $T : A_j^\circ \times B_j^\circ \longrightarrow E_j$ is compact if T_j does it.

Note that in the case of Theorem 2.2 where $A_0 = A_1 = A$ and $B_0 = B_1 = B$, the fact that $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ coincides with $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$. Hence Theorem 2.2 does not change working with the weaker assumption. However, Theorem 2.1 fails if we replace

$$T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E \quad \text{boundedly}$$

by

$$T \in \mathcal{B}(\bar{A} \times \bar{B}, (E, E)).$$

Indeed, the example given in [14, Counterexample 4.2 and Remark 4.4] with the couples $\bar{A} = (\ell_p, \ell_p(2^{-m}))$, $\bar{B} = (\ell_{p'}(2^m), \ell_{p'})$, $1 < p < \infty$, $1/p + 1/p' = 1$, $0 < \eta$, $\theta < 1$, the space $E = \mathbb{C}$ and the operator

$$T(\xi, \eta) = \sum_{m=-\infty}^{\infty} \xi_m 2^{-m} \eta_{-m} \;, \quad \xi = (\xi_m) \;, \; \eta = (\eta_m) \tag{2.3}$$

shows that $T \in \mathcal{B}(\bar{A} \times \bar{B}, (\mathbb{C}, \mathbb{C}))$ but Theorem 2.1/(i) and (ii) fails if $\eta \neq \theta$. On the other hand, as it is pointed out in [13, Example 4.2] working with the same operator T as in (2.3), even though $\eta = \theta$, if $T \in \mathcal{B}(\bar{A} \times \bar{B}, (E, E))$ then Theorem 2.1/(ii) may fail if $1/p + 1/q < 1$.

Next we establish the corresponding version of Theorem 2.1 for $T \in \mathcal{B}(\bar{A} \times \bar{B}, (E, E))$ in the remaining range of parameters.

Theorem 2.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let E be a Banach space. Assume that $T \in \mathcal{B}(\bar{A} \times \bar{B}, (E, E))$ and $T : A_j^\circ \times B_j^\circ \longrightarrow E$ compactly for $j = 0$ or 1. Let $0 < \theta < 1$ and $1 \le p, q < \infty$ with $1/p + 1/q \ge 1$. Then the following holds.*

(i) T may be uniquely extended to a compact bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to E.

(ii) T may be uniquely extended to a compact bilinear operator from $(A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}$ to E.

Proof. According to [4, Theorem 4.2.2], we have that $[A_0, A_1]_\theta = [A_0^\circ, A_1^\circ]_\theta$. Hence, in order to establish (i), without lost of generality we may assume that $\bar{A}$ is a regular couple, and also that $\bar{B}$ is regular.

By the bilinear interpolation theorem for the complex method [4, Theorem 4.4.1], T may be uniquely extended to a bounded bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to E. Hence, having in mind (2.1), to conclude that $T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow E$ compactly, it suffices to show that the linear operator

$$T^\times : E^* \longrightarrow \mathcal{L}([A_0, A_1]_\theta \times [B_0, B_1]_\theta, \mathbb{C}) \quad \text{is compact} \tag{2.4}$$

Put $X_j = \mathcal{L}(A_j \times B_j, \mathbb{C})$, $j = 0, 1$. Since $X_j \hookrightarrow \mathcal{L}((A_0 \cap A_1) \times (B_0 \cap B_1), \mathbb{C})$, we have that $\bar{X} = (X_0, X_1)$ is a Banach couple. Using the diagram of bounded bilinear operators

$$\begin{tikzcd}
A_0 \times B_0 \arrow[dr, "T"] & \\
& E \\
A_1 \times B_1 \arrow[ur, "T"] &
\end{tikzcd}$$

and going to adjoint operators, we get the diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{L}(A_0 \times B_0, \mathbb{C}) = X_0 \\ E^* \arrow[ur, "T^\times"] \arrow[dr, "T^\times"'] & \\ & \mathcal{L}(A_1 \times B_1, \mathbb{C}) = X_1. \end{tikzcd}$$

In addition, $T^\times : E^* \longrightarrow X_j$ is compact provided $T : A_j \times B_j \longrightarrow E$ is so. Therefore, we can apply the Lions-Peetre compactness theorem for bounded linear operators [4, Theorem 3.8.1/(i)], obtaining that

$$T^\times : E^* \longrightarrow [X_0, X_1]_\theta \quad \text{compactly.} \tag{2.5}$$

We claim that

$$[X_0, X_1]_\theta \hookrightarrow \mathcal{L}([A_0, A_1]_\theta \times [B_0, B_1]_\theta, \mathbb{C}). \tag{2.6}$$

Embedding (2.6) is a consequence of the abstract result [20, Theorem 2.1]. For completness, we include the arguments. Consider the bilinear mapping Φ assigning to any $a \in A_0 \cap A_1$ and $R \in X_0 \cap X_1$ the functional $\Phi(a, R)(b) = R(a, b)$ where $b \in B_0 \cap B_1$. Since

$$\|\Phi(a, R)\|_{B_j^*} = \sup \left\{ |R(a, b)| \, : \, \|b\|_{B_j} \leq 1 \, , \, b \in B_0 \cap B_1 \right\} \leq \|R\|_{X_j} \|a\|_{A_j},$$

we have that $\Phi \in \mathcal{B}(\bar{A} \times \bar{X}, \bar{B}^*)$. By [4, Theorem 4.4.2], we get that Φ may be uniquely extended to a bilinear mapping from $[A_0, A_1]_\theta \times [X_0, X_1]_\theta$ to $[B_0^*, B_1^*]^\theta$ with norm at most 1. The space $[B_0^*, B_1^*]^\theta$ is the so-called upper complex space (see [4, p. 89]). The duality theorem [4, Theorem 4.5.1] gives that $[B_0^*, B_1^*]^\theta = [B_0, B_1]_\theta^*$. Therefore, for any $a \in A_0 \cap A_1$, $R \in X_0 \cap X_1$ and $b \in B_0 \cap B_1$, we obtain

$$|R(a, b)| = |\Phi(a, R)(b)| \leq \|R\|_{[X_0, X_1]_\theta} \|a\|_{[A_0, A_1]_\theta} \|b\|_{[B_0, B_1]_\theta}.$$

This shows that $R \in \mathcal{L}([A_0, A_1]_\theta \times [B_0, B_1]_\theta, \mathbb{C})$. Since $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$, embedding (2.6) follows.

Combining (2.5) and (2.6) we derive that $T^\times$ in (2.4) is compact and, therefore, $T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow E$ compactly. This establishes (i).

Since $(A_0, A_1)_{\theta,1} \hookrightarrow [A_0, A_1]_\theta$ and $(B_0, B_1)_{\theta,1} \hookrightarrow [B_0, B_1]_\theta$, when $p = q = 1$, statement (ii) is a consequence of (i). For the case $1 \leq 1/p + 1/q < 2$, the proof of (ii) follows the same steps as for (i) but using now that $(A_0^\circ, A_1^\circ)_{\theta,p} = (A_0, A_1)_{\theta,p}$ [4, Theorem 3.4.2/(d)], the bilinear interpolation theorem for the real method [14, Theorem 4.1] with $1/p + 1/r' = 1 + 1/q'$ and $1 < r \leq \infty$, $1/r + 1/r' = 1$ and duality formula $(B_0^*, B_1^*)_{\theta,q'} = (B_0, B_1)_{\theta,q}^*$ [4, Theorem 3.7.1]. This time the required embedding between operator spaces read

$$(X_0, X_1)_{\theta,r'} \hookrightarrow \mathcal{L}((A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}, \mathbb{C}) \, , \quad 1/r + 1/r' = 1.$$

This completes the proof. □

Remark 2.4. Note that in Theorem 2.1 (as in Lions-Peetre compactness results for linear operators) the case of the complex method is a consequence of the case of the real method because we have the the following factorization for the operator

$$[A_0, A_1]_\theta \times [B_0, B_1]_\eta \hookrightarrow (A_0, A_1)_{\theta,\infty} \times (B_0, B_1)_{\eta,\infty} \xrightarrow{\quad T \quad} E.$$

However, in Theorem 2.3, the complex and real cases are independent because $1/p + 1/q \geq 1$.

3. Bilinear operators with target in Banach function spaces

Let (Ω, μ) be a σ-finite measure space. We denote by $\mathcal{M}$ the collection of all (equivalence classes of) scalar-valued μ-measurable functions on Ω which are finite μ-almost everywhere. The space $\mathcal{M}$ becomes a complete metric space with the topology of convergence in measure on sets of finite measure.

Following [1, 11], we say that a Banach space E of functions in $\mathcal{M}$ is a *Banach function space* if the following four properties hold:

(a) Whenever $g \in \mathcal{M}$, $f \in E$ and $|g(x)| \leq |f(x)|$ μ-a.e., then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

(b) If $f_n \to f$ μ-a.e., and if $\liminf_{n\to\infty} \|f_n\|_E < \infty$, then $f \in E$ and $\|f\|_E \leq \liminf_{n\to\infty} \|f_n\|_E$.

(c) For every $\Gamma \subseteq \Omega$ with $\mu(\Gamma) < \infty$, we have that $\chi_\Gamma \in E$.

(d) For every $\Gamma \subseteq \Omega$ with $\mu(\Gamma) < \infty$ there is a constant $c_\Gamma > 0$ such that $\int_\Gamma |f|\, d\mu \leq c_\Gamma \|f\|_E$ for every $f \in E$.

Examples of Banach function spaces are the Lebesgue spaces L_p, Lorentz spaces $L_{p,q}$ and Orlicz spaces L^Φ (see, [23, 18, 1, 11]).

Let (Γ_n) be a sequence of μ-measurable sets of Ω. We put $\Gamma_n \to \emptyset$ $\mu-$a.e. if the characteristic functions χ_{Γ_n} converge to 0 pointwise μ-a.e.

We say that a function $f \in E$ has *absolutely continuous norm* if $\|f\chi_{\Gamma_n}\|_E \to 0$ for every sequence (Γ_n) satisfying that $\Gamma_n \to \emptyset$ $\mu-$a.e. The space E is said to have absolutely continuous norm if every function of E has absolutely continuous norm.

The following criterion for compactness is useful (see [1, p. 31] and [17, Lemma I.1.1]).

Lemma 3.1. *Let E be a Banach function space and let $K \subseteq E$ a subset formed by functions with absolutely continuous norm. Then K is relatively compact in E if and only if K is relatively compact in $\mathcal{M}$ and for any $\Gamma_n \to \emptyset$ $\mu-$a.e. and any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|f\chi_{\Gamma_n}\|_E \leq \varepsilon$ for any $f \in K$ and $n \geq N$.*

If E is a Banach function space then $E \hookrightarrow \mathcal{M}$ (see [1, Theorem I.1.4]). Hence, if E_0 and E_1 are Banach function space on Ω, we have that (E_0, E_1) is a Banach couple.

Let $0 < \theta < 1$. If E_0 or E_1 has absolutely continuous norm, then

$$[E_0, E_1]_\theta = \left\{ f \in \mathcal{M} \,:\, |f(x)| = |f_0(x)|^{1-\theta} |f_1(x)|^\theta,\, f_j \in E_j,\, j = 0, 1 \right\}$$

and

$$\|f\|_{[E_0,E_1]_\theta} = \inf\{\max(\|f_0\|_{E_0}, \|f_1\|_{E_1}) : |f| = |f_0|^{1-\theta} |f_1|^\theta\}$$

(see [18, Theorem 4.1.14]). In particular $[E_0, E_1]_\theta$ is a Banach function space.

The next theorem complements the results of [14, Section 4].

Theorem 3.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples. Assume that (Ω, μ) is a σ-finite measure space, let $\bar{E} = (E_0, E_1)$ be a couple of Banach function spaces on Ω, let $0 < \theta < 1$ and $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$. If $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ compactly and E_0 has absolutely continuous norm, then T may be uniquely extended to a compact bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to $[E_0, E_1]_\theta$.*

Proof. By the bilinear interpolation theorem [4, Theorem 4.4.1], the operator T may be uniquely extended to a bounded bilinear operator $T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow [E_0, E_1]_\theta$. To check that the extension is compact we rely on Theorem 2.3 and Lemma 3.1. Since $T \in \mathcal{B}(\bar{A} \times \bar{B}, (E_0 + E_1, E_0 + E_1))$ and $T : A_0^\circ \times B_0^\circ \longrightarrow E_0 + E_1$ is compact, it follows from Theorem 2.3/(i) that $T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow E_0 + E_1$ is compact. Let $W = T(U_{[A_0,A_1]_\theta} \times U_{[B_0,B_1]_\theta})$. Then W is relatively compact in $E_0 + E_1$ and so in $\mathcal{M}$. Furthermore, $[E_0, E_1]_\theta = E_0^{1-\theta} E_1^\theta$ has absolutely continuous norm because E_0 does (see [18, Remark in p. 245]). Whence, the subset $W \subseteq [E_0, E_1]_\theta$ is formed by functions with absolutely continuous norm. Consider any sequence $(\Gamma_n) \subseteq \Omega$ with $\Gamma_n \to \emptyset$ $\mu-$a.e. and any $\varepsilon > 0$. Let R_n be the bilinear operator $R_n(f, g) = T(f, g)\chi_{\Gamma_n}$. Clearly $R_n \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$. Since $T : A_0^\circ \times B_0^\circ \longrightarrow E_0$ compactly, if follows from Lemma 3.1 that there is $N \in \mathbb{N}$ such that $\|R_n\|_0 \leq (\frac{\varepsilon}{\|T\|_1^\theta})^{1-\theta}$ for any $n \geq N$. Moreover, $\|R_n\|_1 \leq \|T\|_1$. Hence, the bilinear interpolation theorem yields that if $n \geq N$ then

$$\|R_n\|_{[A_0,A_1]_\theta \times [B_0,B_1]_\theta, [E_0,E_1]_\theta} \leq \frac{\varepsilon}{\|T\|_1^\theta} \|T\|_1^\theta = \varepsilon.$$

In other words, for every $h = T(f, g) \in W$ and $n \geq N$, we have that $\|T(f, g)\chi_{\Gamma_n}\|_{[E_0,E_1]_\theta} \leq \varepsilon$. Consequently, according to Lemma 3.1, we derive that

$$T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow [E_0, E_1]_\theta \quad \text{is compact.}$$

□

For $1 \leq p_0, p_1 \leq \infty$, $0 < \theta < 1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$, we know that $[L_{p_0}(\Omega), L_{p_1}(\Omega)]_\theta = L_p(\Omega)$ (see [4] or [22]). As a consequence of the preceding result we have:

Corollary 3.3. *Let (Ω_k, μ_k) be σ-finite measure spaces for $k = 0, 1, 2$. Suppose $0 < \theta < 1$, $1 \leq p_j, q_j, r_j \leq \infty$, $j = 0, 1$, and put $1/p = (1 - \theta)/p_0 + \theta/p_1$, $1/q = (1 - \theta)/q_0 + \theta/q_1$ and $1/r = (1 - \theta)/r_0 + \theta/r_1$. Suppose that*

$$T \in \mathcal{B}\big((L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)) \times (L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)),\ (L_{r_0}(\Omega_2), L_{r_1}(\Omega_2))\big).$$

If $T : L_{p_0}(\Omega_0)^\circ \times L_{q_0}(\Omega_1)^\circ \longrightarrow L_{r_0}(\Omega_2)$ is compact and $r_0 < \infty$, then T may be uniquely extended to a compact bilinear operator from $L_p(\Omega_0) \times L_q(\Omega_1)$ to $L_r(\Omega_2)$.

Corollary 3.3 complements the results of [7, Section 5]. We close the paper with a result for the case $r_0 = \infty$

Theorem 3.4. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples. Assume that (Ω, μ) is a finite measure space. Let $1 \leq r_1 \leq \infty$, $0 < \theta < 1$ and $1/r = \theta/r_1$. Suppose that $T \in \mathcal{B}\big(\bar{A} \times \bar{B}, (L_\infty(\Omega), L_{r_1}(\Omega))\big)$ with $T : A_0^\circ \times B_0^\circ \longrightarrow L_\infty(\Omega)$ compactly. Then T may be uniquely extended to a compact bilinear operator from $[A_0, A_1]_\theta \times [B_0, B_1]_\theta$ to $L_r(\Omega)$.*

Proof. Applying the bilinear interpolation theorem [4, Theorem 4.4.1], T may be uniquely extended to a bounded bilinear operator $T : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow L_r(\Omega)$. To show compactness of T we prove that T can be uniformly approximated by compact bilinear operators. Take any $\varepsilon > 0$. Since $T\big(U_{A_0^\circ} \times U_{B_0^\circ}\big)$ is relatively compact in $L_\infty(\Omega)$, we can find a finite set $\{h_1, \cdots, h_n\} \subseteq L_\infty(\Omega)$ such that

$$T\big(U_{A_0^\circ} \times U_{B_0^\circ}\big) \subseteq \bigcup_{j=1}^{n} B(h_j, \varepsilon)$$

with $B(h_j, \varepsilon) = \big\{h \in L_\infty(\Omega) \ : \ \|h - h_j\|_{L_\infty(\Omega)} \leq \varepsilon\big\}$. By [1, Lemma IV.2.8], there is a partition of Ω into finitely many disjoint subsets $\Gamma_1 \cdots \Gamma_m$, each of positive measure, such that the linear operator

$$Ph = \sum_{k=1}^{m} \Big(\frac{1}{\mu(\Gamma_k)} \int_{\Gamma_k} h d\mu\Big) \chi_{\Gamma_k}$$

satisfies that

$$\|h_j - Ph_j\|_{L_\infty(\Omega)} \leq \varepsilon\,, \quad j = 1, \cdots, n.$$

It is clear that $\|P\|_{L_s(\Omega), L_s(\Omega)} \leq 1$ for $s = 1$ and $s = \infty$. Therefore, using the Riesz-Thorin theorem, we also have that $\|P\|_{L_s(\Omega), L_s(\Omega)} \leq 1$ for $1 < s < \infty$. Moreover, $P : L_r(\Omega) \longrightarrow L_r(\Omega)$ is compact because P has finite rank. Therefore, the bilinear operator

$$PT : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \longrightarrow L_r(\Omega) \quad \text{is compact.}$$

We estimate the norm of $T - PT$ by using the bilinear interpolation theorem. Since

$$\|PT\|_{A_1^\circ \times B_1^\circ, L_{r_1}(\Omega)} \leq \|P\|_{L_{r_1}(\Omega), L_{r_1}(\Omega)} \, \|T\|_1 \leq \|T\|_1,$$

we have $\|T - PT\|_1 \leq 2\|T\|_1$. As for the other restriction of $T - PT$, given any $a \in A_0 \cap A_1$, $b \in B_0 \cap B_1$ with $a \in U_{A_0}$ and $b \in U_{B_0}$, if we choose $1 \leq j \leq n$ such that $\|T(a, b) - h_j\|_{L_\infty(\Omega)} \leq \varepsilon$, then we have

$$\begin{aligned} &\|T(a, b) - PT(a, b)\|_{L_\infty(\Omega)} \\ &\quad \leq \|T(a, b) - h_j\|_{L_\infty(\Omega)} + \|h_j - Ph_j\|_{L_\infty(\Omega)} + \|Ph_j - PT(a, b)\|_{L_\infty(\Omega)} \\ &\qquad\qquad \leq 2\|T(a, b) - h_j\|_{L_\infty(\Omega)} + \|h_j - Ph_j\|_{L_\infty(\Omega)} \leq 3\varepsilon. \end{aligned}$$

Consequently,

$$\|T - PT\|_{[A_0, A_1]_\theta \times [B_0, B_1]_\theta, L_r(\Omega)} \leq \|T - PT\|_0^{1-\theta} \|T - PT\|_1^{\theta} \leq (3\varepsilon)^{1-\theta} (2\|T\|_1)^{\theta}.$$

This completes the proof. □

Corollary 3.5. *Let (Ω_j, μ_j) be σ-finite measure spaces, $j = 0, 1$, and let (Ω_2, μ_2) be a finite measure space. Suppose $1 \leq p_j, q_j, r_1 \leq \infty$, $j = 0, 1$. Let $0 < \theta < 1$, and put $1/p = (1 - \theta)/p_0 + \theta/p_1$, $1/q = (1 - \theta)/q_0 + \theta/q_1$ and $1/r = \theta/r_1$. Suppose that*

$$T \in \mathcal{B}\big((L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)) \times (L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)),\ (L_\infty(\Omega_2), L_{r_1}(\Omega_2))\big).$$

If $T : L_{p_0}(\Omega_0)^\circ \times L_{q_0}(\Omega_1)^\circ \longrightarrow L_\infty(\Omega_2)$ is compact, then T may be uniquely extended to a compact bilinear operator from $L_p(\Omega_0) \times L_q(\Omega_1)$ to $L_r(\Omega_2)$.

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