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# Darboux transformations for super-symmetric KP hierarchies

Q.P. Liu <sup>a,b</sup>, Manuel Mañas <sup>c</sup><sup>a</sup> *Beijing Graduate School, China University of Mining and Technology, Beijing 100083, China*<sup>b</sup> *The Abdus Salam International Centre for Theoretical Physics, 34100 Trieste, Italy*<sup>c</sup> *Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain*

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## Abstract

We construct Darboux transformations for the super-symmetric KP hierarchies of Manin–Radul and Jacobian types. We also consider the binary Darboux transformation for the hierarchies. The iterations of both type of Darboux transformations are briefly discussed. © 2000 Elsevier Science B.V. All rights reserved.

## 1. Introduction

The first super-symmetric KP (SKP) hierarchy was introduced by Manin and Radul [5] more than a decade ago. Another relevant super-symmetrization is due to Mulase and Rabin [8]. These systems have been subject of extensive studies from both mathematical and physical viewpoints. In particular we stress the possible applications in two dimensional super-symmetric quantum gravity [1]. There are now many results for SKP hierarchies such as the description of their solution space in the framework of the universal super-Grassmann manifold [12], the construction of their additional symmetries [6] and tau functions [7].

In our previous papers [3,4], we constructed the Darboux transformations for the Manin–Radul's super-symmetric KdV (SKdV) hierarchy and its reductions. We found that, as in ordinary case, Darboux transformations constitute a very efficient tool for

the generation of solutions. The SKdV hierarchies are reductions of SKP hierarchies, so a natural task is to construct Darboux transformations for SKP hierarchies itself. It should be remarked that such generalization is not so straightforward as one may suppose. The reason is that the SKP hierarchies incorporates both even time and odd time flows while the SKdV hierarchy only has the even time flows. Very recently, Araytn et al. have pointed out that the natural candidate for the Darboux transformation does not preserve the odd flows [2].

The purpose of the paper is to present proper Darboux transformations for the SKP hierarchies. For that aim we must recall, following Ueno and Yamada [12], that there two SKP hierarchies, say  $SKP_k$ ,  $k = 0, 1$ . In fact, by reversing the signs of the odd times we go from  $SKP_0$  to  $SKP_1$ . We will show that the natural candidate for elementary Darboux transformation is a map  $SKP_k \rightarrow SKP_{k+1, \text{mod} 2}$ ; thus, when composed with a reversion of odd times gives new solution of the  $SKP_k$ . Observe also that when this elementary Darboux transformation is composed

E-mail address: manuel@darboux.fis.ucm.es (M. Mañas).

twice we get a transformation  $\text{SKP}_k \rightarrow \text{SKP}_k$ . We also consider the binary type of Darboux transformation for these hierarchies.

The paper is organized as follows. In the next section, we recall all the relevant facts and formulae for the Manin–Radul and Jacobian SKP hierarchies. In Section 3 the reader may find our main results. We construct both the so called elementary and binary Darboux transformations. We show that while the binary one has a rather straightforward generalization, the elementary one needs to be modified. We discuss the iterations of both type of Darboux transformations briefly.

## 2. Super-symmetric KP hierarchies

In this section we remind the reader the basic aspects of the two main SKP hierarchies: the Manin–Radul and Jacobian hierarchies.

The proper setting for the SKP hierarchies is the Sato's formalism which we now recall [7,8,11,12]. We consider the algebra  $\mathcal{S}$  of super pseudo-differential operators in the following form

$$\mathcal{S} = \sum_{n \leq N} a_n(x, \theta, \mathbf{t}, \boldsymbol{\tau}) D^n, \quad N \in \mathbb{N}$$

here the coefficients  $a_n$  are taken from a super-commutative algebra

$$\mathcal{S} = \mathbb{C}[[x, \mathbf{t}]] \otimes \Lambda(\theta, \boldsymbol{\tau}) \otimes \mathcal{A}$$

where  $x$  is an even variable,  $\theta$  an odd variable,  $\mathbf{t} = \{t_n\}_{n=1}^\infty$ ,  $\boldsymbol{\tau} = \{\tau_n\}_{n=1}^\infty$ , and  $\mathcal{A}$  is a finite or infinite dimensional Grassmann algebra over  $\mathbb{C}$ . The super-differential operator  $D$  is given by

$$D = \theta \frac{\partial}{\partial x} + \frac{\partial}{\partial \theta}$$

which satisfies

$$D^2 = \partial := \frac{\partial}{\partial x}, \quad D^{-1} = \theta + \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial x} \right)^{-1}.$$

For any operator  $L = \sum_i a_i D^i \in \mathcal{S}$  we denote the truncations to various orders by

$$L_+ = \sum_{i \geq 0} a_i D^i, \quad L_- = \sum_{i < 0} a_i D^i, \quad L_0 = a_0$$

and its super-residue is

$$\text{res}(L) = a_{-1}.$$

We also follow the convention:

- Parentheses is intended to indicate that an operator has acted on an argument.
- Juxtaposition of operators to indicate an operator product.

The parity of a super quantity  $s$  is denoted by  $|s|$ ; i.e., if the quantity is even,  $|s| = 0$ , otherwise  $|s| = 1$ .

To work out the so called binary Darboux transformation, we need the adjoint operation  $*$ , which is defined as

$$D^* = -D, \quad (D^n)^* = (-1)^{\frac{n(n-1)}{2}} D^n,$$

$$(fg)^* = (-1)^{|f||g|} g^* f^*$$

for any  $f, g \in \mathcal{S}$ .

The super-vector fields on  $\mathcal{S}$  generating the SKP flows are given by

$$\partial_n := \frac{\partial}{\partial t_n},$$

$$D_n := \begin{cases} \frac{\partial}{\partial \tau_n} - \sum_{m \geq 1} \tau_m \frac{\partial}{\partial t_{n+m-1}} & \text{for the Manin–Radul SKP,} \\ \frac{\partial}{\partial \tau_n} & \text{for the Jacobian SKP.} \end{cases}$$

The action of  $\partial_n$  on an object  $f$  is sometimes written as  $f_{t_n}$ . They generate a Lie super-algebra with relations

$$[\partial_n, \partial_m] = 0, \quad [\partial_n, D_m] = 0,$$

$$\{D_n, D_m\} = \begin{cases} -2\partial_{n+m-1} & \text{for the Manin–Radul SKP,} \\ 0 & \text{for the Jacobian SKP,} \end{cases} \quad (1)$$

where  $[\cdot, \cdot]$  denotes the usual commutator and  $\{\cdot, \cdot\}$  the anti-commutator.

Now we introduce Sato's operator

$$W = 1 + \sum_{n=1}^{\infty} w_n(x, \theta, \mathbf{t}, \boldsymbol{\tau}) D^{-n}, \quad (2)$$

where the parity of the field  $w_n$  is given by  $(1 - (-1)^n)/2$ ; i.e.,  $w_{2k}$  are even and  $w_{2k-1}$  are odd, so that  $W$  is a homogeneous even operator.

Sato's equation for the  $SKP_k$  hierarchies of type  $k = 0, 1$  are given by

$$\partial_n W = -(W \partial^n W^{-1})_- W,$$

$$D_n W = -(-1)^k (W A_n W^{-1})_- W,$$

$$\text{with } A_n := \begin{cases} D^{2n-1}, & n \geq 1 \quad \text{for the Manin–Radul SKP,} \\ \frac{\partial}{\partial \theta} D^{2n-2}, & n \geq 1 \quad \text{for the Jacobian SKP.} \end{cases}$$

Two observations are in order now: Notice first that given a Sato operator  $W(x, \theta, t, \tau)$  for  $SKP_k$  the operator  $W(x, \theta, t, -\tau)$  is a Sato operator of  $SKP_{k+1} \pmod{2}$ ; i.e. both are essentially the same hierarchy. Secondly, for the Jacobian hierarchy we have  $A_n = D^{2n-1} - \theta D^{2n}$ .

An essential ingredient in our forthcoming considerations are the wave functions, which are solutions  $\varphi \in \mathcal{Y} \rightarrow \mathcal{A}$  of the following equations

$$\partial_n \varphi = P_n \varphi, \quad (-1)^k D_n \varphi = Q_n \varphi, \quad (3)$$

where  $P_n$  and  $Q_n$  are defined in terms of the dressing operator  $W$  as follows

$$P_n := (W \partial^n W^{-1})_+, \quad Q_n := (W A_n W^{-1})_+,$$

and accordingly we also have their adjoint counterparts

$$\partial_n \varphi^* := -P_n^* \varphi^*, \quad (-1)^k D_n \varphi^* := -Q_n^* \varphi^*. \quad (4)$$

A second equivalent formulation of the SKP hierarchies is the Zakharov–Shabat formalism [12], which can be regarded as the compatibility condition of the linear systems Eq. (3)

$$\partial_n P_m - \partial_m P_n + [P_m, P_n] = 0$$

$$\partial_n Q_m - (-1)^k D_m P_n + [Q_m, P_n] = 0$$

$$\begin{aligned} & (-1)^k D_m Q_n + (-1)^k D_n Q_m - \{Q_n, Q_m\} \\ &= \begin{cases} -2P_{n+m-1} & \text{for the Manin–Radul SKP,} \\ 0 & \text{for the Jacobian SKP.} \end{cases} \end{aligned}$$

For the Manin–Radul SKP hierarchy there is a third formulation: the Lax form. It is obtained by dressing the operator  $D$  with  $W$ , getting thus the Lax operator

$$L = W D W^{-1} = D + \sum_{n=1}^{\infty} u_n D^{-n+1}$$

and the corresponding Lax representation

$$\partial_n L = -[(L^{2n})_-, L] = [(L^{2n})_+, L],$$

$$\begin{aligned} (-1)^k D_n L &= -\{(L^{2n-1})_-, L\} \\ &= \{(L^{2n-1})_+, L\} - 2L^{2n}. \end{aligned}$$

### 3. Darboux transformations

In this section, we discuss the Darboux transformations for the SKP hierarchy. We first consider so called elementary Darboux transformation. We will see that the naive candidate does not qualify as a proper one, but by examining the failure we show how to remedy it and obtain a meaningful Darboux transformation. The iteration of such transformation leads us to super ‘Wronski’ determinant representation for the solution of the susy KP hierarchies. This is the generalization of the one found by Ueno et al. [12] in the framework of the universal super-Grassmann manifold.

Next we discuss the binary Darboux transformation. We show that unlike the elementary one, the binary Darboux transformation does enjoy a straightforward generalization. As in the SKdV case, the iteration of such transformation provides us the solution of the SKP hierarchy in the form of super-Grassmann determinants.

#### 3.1. Elementary Darboux transformations

We first introduce odd gauge operator

$$T := \varphi D \varphi^{-1} = D - \varphi^{-1} (D \varphi) \quad (5)$$

where  $\varphi$  is an even invertible wave function of the linear system (3). Indeed, as shown in previous papers [3,4], this very operator supplies us the Darboux transformation for the SKdV equation. We first observe

**Lemma 1.** *The gauge operator  $T$  satisfies*

$$(\partial_n T) T^{-1} = -(T P_n T^{-1})_-,$$

$$(-1)^k (D_n T) T^{-1} = (T Q_n T^{-1})_-.$$

**Proof.** First observe

$$\begin{aligned}\partial_n T &= -\partial_n(\varphi^{-1}(D\varphi)) = -D(\varphi^{-1}\partial_n \varphi) \\ &= -D(\varphi^{-1}P_n \varphi)\end{aligned}$$

and

$$\begin{aligned}(TP_n T^{-1})_- &= (\varphi D\varphi^{-1}P_n \varphi D^{-1}\varphi^{-1})_- \\ &= \varphi(D\varphi^{-1}P_n \varphi D^{-1})_- \varphi^{-1} \\ &= \varphi(D\varphi^{-1}P_n \varphi)D^{-1}\varphi^{-1}.\end{aligned}$$

Second,

$$D_n T = -D_n(\varphi^{-1}(D\varphi)) = D(\varphi^{-1}(D_n \varphi))$$

and

$$\begin{aligned}(TQ_n T^{-1})_- &= (\varphi D\varphi^{-1}Q_n \varphi D^{-1}\varphi^{-1})_- \\ &= \varphi(D\varphi^{-1}Q_n \varphi D^{-1})_- \varphi^{-1} \\ &= \varphi(D\varphi^{-1}Q_n \varphi)D^{-1}\varphi^{-1}.\end{aligned}$$

□

This gauge operator allows us to define, in the realm of the standard Darboux transformation, the transformed Sato's operator

$$\hat{W} := TWD^{-1}. \quad (6)$$

The susy KP flows on this transformed operator are

**Lemma 2.** *The operator  $\hat{W}$  satisfies*

$$\begin{aligned}\partial_n \hat{W} &= -(\hat{W}\partial^n \hat{W}^{-1})_- \hat{W}, \\ (-1)^{k+1} D_n \hat{W} &= -(\hat{W}A_n \hat{W}^{-1})_- \hat{W}.\end{aligned}$$

**Proof.** For the even evolution we have

$$\begin{aligned}\partial_n \hat{W} &= -(TP_n T^{-1})_- TWD^{-1} - T(W\partial^n W^{-1})_- WD^{-1} \\ &= (TP_n T^{-1})_+ TWD^{-1} - TW\partial^n D^{-1} \\ &= (T(W\partial^n W^{-1})_+ T^{-1})_+ TWD^{-1} - TW\partial^n D^{-1} \\ &= (TW\partial^n W^{-1} T^{-1})_+ TWD^{-1} - TW\partial^n D^{-1} \\ &= -(\hat{W}\partial^n \hat{W}^{-1})_- \hat{W},\end{aligned}$$

while for the odd time evolution

$$\begin{aligned}(-1)^k D_n \hat{W} &= (TQ_n T^{-1})_- TWD^{-1} \\ &\quad + T(WA_n W^{-1})_- WD^{-1} \\ &= -(TQ_n T^{-1})_+ TWD^{-1} + TWA_n D^{-1} \\ &= -(T(WA_n W^{-1})_+ T^{-1})_+ TWD^{-1} \\ &\quad + TWA_n D^{-1} \\ &= -(TWA_n W^{-1} T^{-1})_+ TWD^{-1} \\ &\quad + TWA_n D^{-1} \\ &= (\hat{W}A_n \hat{W}^{-1})_- \hat{W}.\end{aligned}$$

□

We see then that while the even evolutions are preserved by the suggested transformation the odd time evolutions are not.

At the level of wave functions for any given solution  $\psi$  of Eq. (3) a new function  $\hat{\psi}$  as

$$\begin{aligned}\hat{\psi} &:= T\psi, \quad \hat{P}_n := (TP_n T^{-1})_+, \\ \hat{Q}_n &= (TQ_n T^{-1})_+\end{aligned} \quad (7)$$

then

**Proposition 1.** *The new function  $\hat{\psi}$  satisfies*

$$\begin{aligned}\partial_n \hat{\psi} &= \hat{P}_n \hat{\psi}, \\ (-1)^{k+1} D_n \hat{\psi} &= \hat{Q}_n \hat{\psi}.\end{aligned}$$

**Proof.** We first compute

$$\begin{aligned}\partial_n \hat{\psi} &= ((\partial_n T)T^{-1} + TP_n T^{-1})\hat{\psi} = (TP_n T^{-1})_+ \hat{\psi} \\ \text{and then we calculate} \\ (-1)^k D_n \hat{\psi} &= ((-1)^k (D_n T)T^{-1} - TQ_n T^{-1})\hat{\psi} \\ &= -(TQ_n T^{-1})_+ \hat{\psi}.\end{aligned}$$

□

Hence, we observe that is true that the  $SKP_k$  odd flows will not be preserved because there is a change  $\tau_n \rightarrow -\tau_n$  involved within the transformation. In fact, this Darboux transformation generates a map

$SKP_k \rightarrow SKP_{k+1}$  from solutions of the  $SKP_k$  to solutions of  $SKP_{k+1}$ ,  $k \bmod 2$ . That is:

**Theorem 1.** *Given a Sato operator  $W$  of the  $SKP_k$  hierarchy then  $\hat{W}$  is a Sato operator of the  $SKP_{k+1}$  ( $k \bmod 2$ ).*

Moreover, for

$$\tilde{W}(x, \theta, \mathbf{t}, \boldsymbol{\tau}) := \hat{W}(x, \theta, \mathbf{t}, -\boldsymbol{\tau})$$

we have

**Corollary 1.** *Given a Sato operator  $W$  of the  $SKP_k$  hierarchy then  $\tilde{W}$  is again a Sato operator of the  $SKP_k$  hierarchy.*

Let us remark that, on the level of Lax formulation of the Manin–Radul  $SKP_0$  hierarchy, this point is treated in Section 3 of [2]. Their observation is that the Darboux transformation does preserve only the bosonic flows. But, as we have seen this can be remedied by reversing the sign of the fermionic flows. However, there is even another solution to this problem: the idea is that the single step of the transformation preserves the even flows but changes the odd flows in the way  $\tau_n \rightarrow -\tau_n$ . Therefore, we do successively two steps of such transformation which do preserve the odd flows as well.

Given two distinct solutions of the system Eq. (3)  $\theta_0$  (even) and  $\theta_1$  (odd), we construct an operator

$$T_e = \partial + \alpha D + a \quad (8)$$

where the coefficients  $\alpha$  and  $a$  are given in terms of  $\theta_0$  and  $\theta_1$

$$\alpha = D \ln \text{sdet } \mathcal{F}, \quad a = -\frac{\text{sdet } \hat{\mathcal{F}}}{\text{sdet } \mathcal{F}} \quad (9)$$

with the super-matrices  $\mathcal{F}$  and  $\hat{\mathcal{F}}$  defined as

$$\mathcal{F} := \begin{pmatrix} \theta_0 & \theta_1 \\ D\theta_0 & D\theta_1 \end{pmatrix}, \quad \hat{\mathcal{F}} := \begin{pmatrix} \partial\theta_0 & \partial\theta_1 \\ D\theta_0 & D\theta_1 \end{pmatrix}.$$

Here  $\text{sdet}$  means the super-determinant or Berezian of a super-matrix.

We define as well

$$\hat{W} := T_e W \partial^{-1} \quad (10)$$

so that

$$\begin{aligned} \hat{w}_1 &:= w_1 + \alpha, & \hat{w}_2 &:= w_2 - \alpha w_1 + a, \\ \hat{w}_{n+2} &:= w_{n+2} + (-1)^{|w_{n+1}|} \alpha w_{n+1} + a w_n \\ &\quad + \alpha(Dw_n) + w_{n,x}, \quad (n \geq 1). \end{aligned}$$

Then

**Theorem 2.** *The operator  $\hat{W}$  defined by Eq. (10) satisfies the  $SKP_k$  hierarchy*

$$\begin{aligned} \partial_n \hat{W} &= -(\hat{W} \partial^n \hat{W}^{-1})_- \hat{W}, \\ D_n \hat{W} &= -(\hat{W} A_n \hat{W}^{-1})_- \hat{W}, \quad (n \geq 1). \end{aligned}$$

So in the  $SKP$  case, the proper elementary Darboux transformation is generated by the compound operator  $T_e$ .

This Darboux transformation can be iterated and the solutions can be formed in terms of super ‘Wronski’ determinants as in super KdV case.

**Proposition 2.** *Given  $2n$  distinct solutions  $\theta_i$  ( $i = 0, \dots, 2n-1$ ) of the linear system (3) with parities  $|\theta_i| = (-1)^i$ , then the new Sato’s operator is given by*

$$\hat{W} = T_e[n] W \partial^{-n}, \quad T_e[n] = \partial^n + \sum_{i=0}^{2n-1} a_i D^i \quad (11)$$

where the coefficients  $a_i$  of the operator  $T_e[n]$  are given by solving the linear equations

$$T_e[n](\theta_i) = 0, \quad i = 0, \dots, 2n-1. \quad (12)$$

To obtain the explicit transformations between fields, one has to solve the linear algebraic system Eq. (12) first, then compare the coefficients of the different powers  $D^i$  of Eq. (11). The system Eq. (12) can be solved easily as we did in [3,4]. To this end, we introduce new variables

$$\mathbf{a}^{(0)} := (a_0, a_2, \dots, a_{2n-2}),$$

$$\mathbf{a}^{(1)} := (a_1, a_3, \dots, a_{2n-1}),$$

$$\boldsymbol{\theta}^{(0)} := (\theta_0, \theta_2, \dots, \theta_{2n-2}),$$

$$\boldsymbol{\theta}^{(1)} := (\theta_1, \theta_3, \dots, \theta_{2n-1}),$$

$$\mathbf{b}^{(i)} := \partial^n \boldsymbol{\theta}^{(i)}, \quad \mathcal{W}^{(i)} := \begin{pmatrix} \boldsymbol{\theta}^{(i)} \\ \partial \boldsymbol{\theta}^{(i)} \\ \vdots \\ \partial^{n-1} \boldsymbol{\theta}^{(i)} \end{pmatrix}, \quad i = 0, 1$$

so that the linear algebraic system Eq. (12) can be reformulated as

$$(\mathbf{a}^{(0)}, \mathbf{a}^{(1)})\mathcal{W} = -(\mathbf{b}^{(0)}, \mathbf{b}^{(1)}) \quad (13)$$

where

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}^{(0)} & \mathcal{W}^{(1)} \\ D\mathcal{W}^{(0)} & D\mathcal{W}^{(1)} \end{pmatrix}$$

Then, we obtain

$$\begin{aligned} a_{2i+2} &= -\frac{\det(\mathcal{W}_i^{(0)} - \mathcal{W}_i^{(1)}(D\mathcal{W}^{(1)})^{-1}(D\mathcal{W}^{(0)}))}{\det(\mathcal{W}^{(0)} - \mathcal{W}^{(1)}(D\mathcal{W}^{(1)})^{-1}(D\mathcal{W}^{(0)}))} \\ &= -\frac{\text{sdet } \mathcal{W}_i}{\text{sdet } \mathcal{W}}, \quad i = 1, \dots, n-1, \end{aligned} \quad (14)$$

$$\begin{aligned} a_{2i-1} &= -\frac{\det((D\mathcal{W}^{(1)})_i - (D\mathcal{W}^{(0)})_i(\mathcal{W}^{(0)})^{-1}\mathcal{W}^{(1)})}{\det(D\mathcal{W}^{(1)} - (D\mathcal{W}^{(0)})(\mathcal{W}^{(0)})^{-1}\mathcal{W}^{(1)})}, \\ i &= 1, \dots, n. \end{aligned} \quad (15)$$

where the matrix  $\mathcal{W}_i^{(j)}$  is the  $\mathcal{W}^{(j)}$  with its  $i$ -th row replaced by  $\mathbf{b}^{(j)}$  and  $\mathcal{W} = \begin{pmatrix} \mathcal{W}_i^{(0)} & \mathcal{W}_i^{(1)} \\ D\mathcal{W}^{(0)} & D\mathcal{W}^{(1)} \end{pmatrix}$ , the matrix  $(D\mathcal{W}^{(1)})_i$  is the matrix  $D\mathcal{W}^{(1)}$  with its  $i$ -th row replaced by  $\mathbf{b}^{(1)}$  and the matrix  $(D\mathcal{W}^{(0)})_i$  is the matrix  $D\mathcal{W}^{(0)}$  with its  $i$ -th row replaced by  $\mathbf{b}^{(0)}$ .

By considering Eq. (11), we obtain the general formulae

$$\begin{aligned} \hat{w}_{2n-k} &= \sum_{j=k}^{2n} a_j \sum_{i=0}^{j-k} C_{i,j-k-i,j}, \quad k = 1, \dots, 2n, \\ \hat{w}_{2n+k} &= \sum_{j=0}^{2n} a_j \sum_{i=0}^j C_{i,j+k-i,j}, \quad k = 0, 1, \dots \end{aligned}$$

with  $w_0 = a_{2n} = 1$  and the  $a_i$  are given by Eq. (14)–Eq. (15) and

$$C_{i,j,k} = \begin{bmatrix} k \\ k-i \end{bmatrix} (-1)^{|w_j|(k-i)} (D^i w_j)$$

where  $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$  denotes the super-binomial coefficients.

### 3.2. Binary Darboux transformation

In this section we consider the extension of the well known binary Darboux transformation for the KP hierarchy to its super-symmetrizations.

For an eigenfunction  $\varphi$  and an adjoint eigenfunction  $\varphi^*$  satisfying the linear system Eq. (3) and Eq. (4) respectively, we may introduce a potential operator  $\Omega$  as follows

$$\begin{aligned} D\Omega(\varphi^*, \varphi) &= \varphi^* \varphi, \partial_n \Omega = \text{res}(D^{-1} \varphi^* P_n \varphi D^{-1}), \\ D_n \Omega &= \text{res}(D^{-1} \varphi^* Q_n \varphi D^{-1}) \end{aligned} \quad (16)$$

where we choose  $\varphi$  as an even quantity and  $\varphi^*$  as an odd one, so that our potential operator  $\Omega$  is even.

By lengthy calculation one can show that  $\Omega$  is well defined, i.e., Eq. (16) are consistent.

In terms of  $\Omega$ , we introduce the gauge operator

$$T := 1 - \varphi \Omega^{-1} D^{-1} \varphi^* \quad (17)$$

where we assume the invertibility of  $\Omega$ . The formal inverse of  $T$  is

$$T^{-1} = 1 + \varphi D^{-1} \Omega^{-1} \varphi^*. \quad (18)$$

It can be proven that

**Proposition 3.** The gauge operator  $T$  solves

$$\begin{aligned} (\partial_n T) T^{-1} &= -(TP_n T^{-1})_-, \\ (D_n T) T^{-1} &= -(TQ_n T^{-1})_-. \end{aligned} \quad (19)$$

This property qualifies  $T$  for generating a Darboux transformation [9]. Given a Sato operator  $W$  we introduce its transformed  $\hat{W}$  as follows

$$\hat{W} := TW. \quad (20)$$

**Proposition 4.** The operator  $\hat{W}$  is again a Sato operator.

**Proof.** We begun with

$$\begin{aligned} \partial_n \hat{W} &= -(TP_n T^{-1})_- TW - T(W\partial^n W^{-1})_- W \\ &= (TP_n T^{-1})_+ TW - TP_n W + T(W\partial^n W^{-1})_+ W \\ &\quad - TW\partial^n \\ &= (T(W\partial^n W^{-1})_+ T^{-1})_+ TW - TW\partial^n \\ &= (TW\partial^n W^{-1} T^{-1})_+ TW - TW\partial^n \\ &= -(\hat{W}\partial^n \hat{W}^{-1})_- \hat{W} \end{aligned}$$

and

$$\begin{aligned} D_n \hat{W} &= -(TQ_n T^{-1})_- TW - T(WA_n W^{-1})_- W \\ &= (TQ_n T^{-1})_+ TW - TQ_n W + T(WA_n W^{-1})_+ W \\ &\quad - TWA_n \\ &= (T(WA_n W^{-1})_+ W^{-1})_+ TW - TWA_n \\ &= -(\hat{W}A_n \hat{W}^{-1})_- \hat{W} \end{aligned}$$

Therefore, the transformed Sato's operator indeed is the solution of Sato's equation.

The explicit transformation is

$$\begin{aligned} \hat{w}_1 &= w_1 + \varphi \Omega^{-1} \varphi^*, \\ \hat{w}_2 &= w_2 - \varphi \Omega^{-1} ((D\varphi^*) + \varphi^* w_1), \\ \hat{w}_{2j+1} &= w_{2j+1} + (-1)^j \varphi \Omega^{-1} \left( \varphi^{*(j)} \right. \\ &\quad \left. + \sum_{k=0}^{j-1} (-1)^k ((D\varphi^* w_{2k+1})^{(j-k-1)} \right. \\ &\quad \left. - (\varphi^* w_{2k+2})^{(j-k-1)}) \right), \\ \hat{w}_{2j+2} &= w_{2j+2} + (-1)^{j+1} \varphi \Omega^{-1} \\ &\quad \times \left( (D\varphi^*)^{(j)} + (\varphi^* w_1)^{(j)} \right. \\ &\quad \left. + \sum_{k=1}^j (-1)^k ((D\varphi^* w_{2k})^{(j-k)} \right. \\ &\quad \left. + (\varphi^* w_{2k+1})^{(j-k)}) \right), \end{aligned}$$

for  $j = 1, 2, \dots$  and where  $f^{(i)} = \partial^i f / \partial x^i$ .

For any given solution  $\psi$  of Eq. (3) we define

$$\hat{\psi} := T\psi, \quad \hat{P}_n := (TP_n T^{-1})_+, \quad \hat{Q}_n := (TQ_n T^{-1})_+$$

and as before

**Proposition 5.** *The function  $\hat{\psi}$  satisfies*

$$\begin{aligned} \partial_n \hat{\psi} &= \hat{P}_n \hat{\psi}, \\ D_n \hat{\psi} &= \hat{Q}_n \hat{\psi}. \end{aligned}$$

For the Manin–Radul SKP hierarchy the transformation on the Lax level can be easily obtained as

$$\hat{L} = TLT^{-1}$$

and we found that  $\hat{L}$  satisfies

$$\begin{aligned} \hat{L}_{t_n} &= [(\hat{L}^{2n})_+, \hat{L}], \\ D_n \hat{L} &= \{(\hat{L}^{2n-1})_+, \hat{L}\} - 2\hat{L}^{2n}. \end{aligned}$$

Indeed, we have

$$\begin{aligned} \hat{L}_{t_n} &= [T_{t_n} T^{-1} + T(L^{2n})_+ T^{-1}, \hat{L}] \\ &= [- (TP_n T^{-1})_- + T(L^{2n})_+ T^{-1}, \hat{L}] \\ &= [(TP_n T^{-1})_+, \hat{L}] = [(TL^{2n} T^{-1})_+, \hat{L}] \\ &= [(\hat{L}^{2n})_+, \hat{L}] \end{aligned}$$

since  $(T(L^{2n})_- T^{-1})_+ = 0$

Likewise,

$$\begin{aligned} D_n \hat{L} &= \{(D_n T) T^{-1} + T(L^{2n-1})_+ T^{-1}, \hat{L}\} - 2\hat{L}^{2n} \\ &= \{- (TQ_n T^{-1})_- + T(L^{2n-1})_+ T^{-1}, \hat{L}\} - 2\hat{L}^{2n} \\ &= \{(T(L^{2n-1})_+ T^{-1})_+, \hat{L}\} - 2\hat{L}^{2n} \\ &= \{(\hat{L}^{2n-1})_+, \hat{L}\} - 2\hat{L}^{2n} \end{aligned}$$

because of  $(T(L^{2n-1})_- T^{-1})_+ = 0$ . The binary Darboux transformation can be iterated. To do this, we need to consider the effect of this transformation for adjoint wave functions  $\psi^*$ . It is indeed not difficult to find they are transformed as follows

$$\hat{\psi}^* = \psi^* - \varphi^* (\Omega(\varphi^*, \varphi))^{-1} \Omega(\psi^*, \varphi).$$

Another important fact is the following

$$\begin{aligned} \Omega(\psi^*, \hat{\psi}) &= \Omega(\psi^*, \psi) - \Omega(\psi^*, \varphi) \\ &\quad \times \Omega(\varphi^*, \varphi)^{-1} \Omega(\varphi^*, \psi) \end{aligned}$$

then as in the pure bosonic case [10], by iterations of the binary Darboux transformation leads to the following result

**Proposition 6.** *Let  $\varphi_i$  ( $i = 1, 2, \dots, n$ ) be  $n$  bosonic solutions of (3) and  $\varphi_i^*$  ( $i = 1, 2, \dots, n$ ) be  $n$  fermionic solutions of (4), let  $\mathcal{G} = (\Omega(\varphi_i^*, \varphi_j))$  be a Gramm type of matrix and  $\mathcal{G}_i$  be the matrix  $\mathcal{G}$  with its  $i$ -th row replaced by  $(\varphi_1, \dots, \varphi_n)$ . Define*

$$T[n] = 1 - \sum_{i=1}^n b_i D^{-1} \varphi_i^*, \quad b_i = \frac{\det \mathcal{G}_i}{\det \mathcal{G}}.$$

Then the new Sato's operator is given by

$$\hat{W} = T[n]W \quad (21)$$

From (21) we obtain the explicit transformation on the level of fields

$$\hat{w}_1 = w_1 + \sum_{j=1}^n b_j \varphi_j^*,$$

$$\hat{w}_2 = w_2 - \sum_{j=1}^n b_j ((D\varphi_j^*) + \varphi_j^* w_1),$$

$$\begin{aligned} \hat{w}_{2j+1} = & w_{2j+1} + (-1)^j \sum_{s=1}^n b_s \left( \varphi_s^{*(j)} \right. \\ & + \sum_{k=0}^{j-1} (-1)^k ((D\varphi_s^* w_{2k+1})^{(j-k-1)} \\ & \left. - (\varphi_s^* w_{2k+2})^{(j-k-1)}) \right) \end{aligned}$$

$$\begin{aligned} \hat{w}_{2j+2} = & w_{2j+2} + (-1)^{j+1} \\ & \times \sum_{s=1}^n b_s \left( (D\varphi_s^*)^{(j)} + (\varphi_s^* w_1)^{(j)} \right. \\ & + \sum_{k=1}^j (-1)^k ((D\varphi_s^* w_{2k})^{(j-k)} \\ & \left. + (\varphi_s^* w_{2k+1})^{(j-k)}) \right), \end{aligned}$$

for  $j = 1, 2, \dots$ .

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