

# Nonequilibrium dynamics in quantum field theory at high density: The “tsunami”

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The dynamics of a dense relativistic quantum fluid out of thermodynamic equilibrium is studied in the framework of the  $\Phi^4$  scalar field theory in the large  $N$  limit. The time evolution of a particle distribution in momentum space (the tsunami) is computed. The effective mass felt by the particles in such a high density medium equals the tree level mass plus the expectation value of the squared field. The case of negative tree level squared mass is particularly interesting. In such a case dynamical symmetry restoration as well as dynamical symmetry breaking can happen. Furthermore, the symmetry may stay broken with a vanishing asymptotic squared mass showing the presence of out of equilibrium Goldstone bosons. We study these phenomena and identify the set of initial conditions that lead to each case. We compute the equation of state which turns out to depend on the initial state. Although the system does not thermalize, the equation of state for asymptotically broken symmetry is of radiation type. We compute the correlation functions at equal times. The two point correlator for late times is the sum of different terms. One stems from the initial particle distribution. Another term accounts for the out of equilibrium Goldstone bosons created by spinodal instabilities when the symmetry is asymptotically broken. Both terms are of the order of the inverse of the coupling for distances where causal signals can connect the two points. The contribution of the out of equilibrium Goldstones exhibits scaling behavior in a generalized sense.

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## I. INTRODUCTION

Physical phenomena in high energy density situations cannot be treated with the usual perturbation methods. Self-consistent nonperturbative methods are necessary in order to describe the out-of-equilibrium dynamics of relaxation of quantum fields in such situations. The large  $N$  limit is a particularly powerful tool for scalar models. The need for a self-consistent method stems from the fact that the particle propagation in such situations depends on the detailed state of the system.

Relevant physical situations at high energy densities arise in the hadron (quark-gluon) plasma as in ultrarelativistic heavy ion collisions, in the interior of dense stars, and in the early universe [1,2].

We consider a dense relativistic quantum gas out of thermodynamic equilibrium. We investigate the nonperturbative dense regime in which the energy density is of the order  $m_R^4/\lambda$  where  $\lambda$  is the scalar self-coupling and  $m_R$  the physical mass in vacuum. In these conditions, even for very weakly coupled theories non-linear effects are important and must be treated nonperturbatively. We thus use the large  $N$  limit that provides a nonperturbative scheme which respects the internal as well as the space-time symmetries, is renormalizable and permits explicit calculations. More precisely, we consider the  $O(N)$  vector model with quartic self-interaction and the scalar field in the vector representation of  $O(N)$ .

We study the physics of the in-medium effects at high energy density. In such out of equilibrium regime the effective mass felt by the particles is time-dependent and different from the tree level mass  $m_R^2$ . In the large- $N$  approximation, it takes the form

$$\mathcal{M}^2(t) = m_R^2 + \frac{\lambda}{2N} \langle \vec{\Phi}^2 \rangle(t), \quad (1.1)$$

where  $\vec{\Phi}(x)$  stands for a  $N$ -component scalar field with self-coupling  $\lambda$ .

We see that the quantum fluctuations of the scalar field directly contribute to the effective mass. We shall consider for simplicity translationally invariant situations. That is, invariant under spatial translations. Therefore, the effective mass and all one-point expectation values depend on time but not on space coordinates.

Obviously, these are not realistic descriptions for the vast and complex problems in ultrarelativistic heavy-ion collision. However, the physics of such a simplified problem is rich enough to get understanding of more involved situations.

We work in the small coupling regime  $\lambda \ll 1$ . The reason being that the different dynamical time scales are well separated in the  $\lambda \ll 1$  regime. For larger couplings the dynamical time scales become of the same order and different physical phenomena get mixed up.

We choose as initial conditions a Gaussian wave functional and  $\langle \vec{\Phi} \rangle(0) = \langle \dot{\vec{\Phi}} \rangle(0) = 0$ . The calculation of correlation functions is done in the general case  $\langle \vec{\Phi} \rangle(0) \neq 0$ .

As initial distribution of particles we will choose a “tsunami” [3,4]. That is, a distribution of particles in momentum space that we choose for simplicity spherical. Typically, such distribution is a shell peaked in a wave number  $k_0$  with a large density of particles  $\sim m_R^3/\lambda$ .

We have seen that these conditions do not determine completely the initial state, and that the remaining degrees of freedom can be interpreted as the quantum coherence between different  $k$ -modes. We choose two highly coherent initial states that we will call case I and case II.

As we see in Eq. (1.1) the in-medium effects give a positive contribution to the effective mass. In general, this con-

tribution to the effective mass initially oscillates with time. These oscillations can produce parametric resonance which is shut off here after a few oscillations by the damping of the oscillations due to decoherence. This resonant effect is negligible for small  $\lambda$ . (Parametric instabilities are shut off by the back reaction for no particle initial conditions [9–11]).

We present analytic solutions for initial particle distributions narrow in momentum space. Such self-consistent solutions express in close form in terms of elliptic functions, and reproduce the numerical solutions with very good accuracy till the damping of the oscillations becomes important (due to decoherence phenomena). The narrower is the initial distribution in momentum, the longer in time this effective solution holds.

We investigate in the present paper the case where  $m_R^2 > 0$  as well as the case where  $m_R^2 < 0$ .

The more interesting case corresponds to  $m_R^2 < 0$ . We have in such case spontaneously broken symmetry for low density and low energy states. For large initial energy density, the positive definite term  $(\lambda/2)\langle\vec{\Phi}^2\rangle(t)$  may overcome the negative squared mass  $m_R^2 < 0$  in Eq. (1.1) and the symmetry may be restored:  $\mathcal{M}^2(t) > 0$  (see Table II).

This happens at  $t=0$  for the case we call II provided the energy density  $E$  satisfies

$$E > \frac{|m_R|^2}{\lambda_R} k_0^2.$$

For late times the symmetry is restored in both cases, I and II, provided

$$E > \frac{|m_R|^4}{\lambda_R} + 2 \frac{|m_R|^2}{\lambda_R} k_0^2.$$

Moreover, we find in case II an interval of energies where the symmetry is initially unbroken [ $\mathcal{M}^2(0) > 0$ ] and where it is broken for asymptotic times:

$$\frac{|m_R|^2}{\lambda_R} k_0^2 < E < \frac{|m_R|^4}{\lambda_R} + 2 \frac{|m_R|^2}{\lambda_R} k_0^2.$$

That is, we have a dynamical symmetry breaking for these situations (see Table II).

We compute the asymptotic equation of state for this out of equilibrium system. That is, we derive for late times explicit formulas for the pressure as a function of the total energy. The equation of state we obtain explicitly depends on the initial state,  $\lambda$  and  $m_R$  (see Table III). Notice that even for asymptotic times the system does not thermalize. In particular, the distribution functions reach nonthermal limits for  $t \rightarrow \infty$ .

When the symmetry is asymptotically broken, the equation of state takes the radiation form

$$P = \frac{E}{3}$$

in spite of the fact that the system is out of equilibrium. Moreover, the Goldstone theorem is valid here in this out of

equilibrium situation. Namely, the effective squared mass  $\mathcal{M}^2(t)$  asymptotically vanishes when the symmetry is asymptotically broken.

The equation of state for asymptotically unbroken symmetry [Eqs. (7.7)] has the cold matter and the radiation equation of state as limiting cases. We explicitly compute the two point correlation functions.

For late times, the two point correlator at equal times  $C(|\vec{x}|, t)$  expresses as a sum of two or three terms:

$$C(|\vec{x}|, t) = C_{\text{origin}}(|\vec{x}|) + C_p(|\vec{x}|, t) + C_s(|\vec{x}|, t). \quad (1.2)$$

There is the time-independent piece  $C_{\text{origin}}(|\vec{x}|)$  concentrated around the origin. The pulse term,  $C_p(|\vec{x}|, t)$ , is due to the particles in the initial distribution that effectively propagate as free particles with mass  $\mathcal{M}_\infty$ ,

$$C_p(|\vec{x}|, t) = \frac{1}{\lambda|\vec{x}|} F(|\vec{x}| - 2v_g t - c),$$

where  $v_g$  is the group velocity [ $v_g < 1$  for unbroken symmetry and  $v_g = 1$  for dynamically broken symmetry where  $\mathcal{M}_\infty = 0$ ].  $F(u)$  is nonzero only around  $u = 0$ .

The last term in Eq. (1.2) is only present for dynamically broken symmetry,

$$C_s(|\vec{x}|, t) = \frac{K}{\lambda|\vec{x}|} Q\left(\frac{|\vec{x}|}{2t}\right),$$

where  $K$  is a constant. The function  $Q(u)$  is of the order  $O(1)$  only for  $0 < u < 1$  due to causality. When the order parameter  $\langle\vec{\Phi}\rangle(t)$  is identically zero we have  $Q(u) = \theta(1 - u)$ .

When the order parameter is nonzero,  $Q(u)$  oscillates with  $u$ . At a given time  $t$ , the number of oscillations of  $Q(u)$  in the interval  $0 < u < 1$  equals the number of oscillations performed by the order parameter  $\langle\vec{\Phi}\rangle(t)$  from time  $t=0$  till time  $t$ . That is, scaling exists for  $\langle\vec{\Phi}\rangle(0) \neq 0$  in a generalized sense since the function  $Q(u)$  changes each time  $\langle\vec{\Phi}\rangle(t)$  performs an oscillation. This is due to the appearance of an extra length scale, the initial value of the order parameter.

## II. THE MODEL

We consider the  $O(N)$ -invariant scalar field model with quartic self-interaction [3–5] with the scalar field in the vector representation of  $O(N)$ .

The action and Lagrangian density are given by

$$S = \int d^4x \mathcal{L},$$

$$\mathcal{L} = \frac{1}{2} [\partial_\mu \vec{\Phi}(x)]^2 - \frac{1}{2} m^2 \vec{\Phi}^2 - \frac{\lambda}{8N} (\vec{\Phi}^2)^2. \quad (2.1)$$

The canonical momentum conjugate to  $\vec{\Phi}(x)$  is

$$\vec{\Pi}(x) = \dot{\vec{\Phi}}(x), \quad (2.2)$$

and the Hamiltonian is given by

$$H = \int d^3x \left\{ \frac{1}{2} \vec{\Pi}^2(x) + \frac{1}{2} [\nabla \vec{\Phi}(x)]^2 + \frac{1}{2} m^2 \vec{\Phi}^2 + \frac{\lambda}{8N} (\vec{\Phi}^2)^2 \right\}. \quad (2.3)$$

In the present case we will restrict ourselves to a translationally invariant situation, i.e., eigenstates of the total momentum operator. In this case the order parameter  $\langle \vec{\Phi}(\vec{x}, t) \rangle$  is independent of the spatial coordinates  $\vec{x}$  and only depends on time.

The Heisenberg equations of motion for the field operator take the form

$$\left[ \partial^2 + m^2 + \frac{\lambda}{2N} \vec{\Phi}^2(x) \right] \vec{\Phi}(x) = 0. \quad (2.4)$$

The coupling  $\lambda$  is chosen to remain fixed in the large  $N$  limit. It is convenient to write the field in the Schrödinger picture as

$$\vec{\Phi}(x) = (\sigma(x), \vec{\eta}(x)) = (\sqrt{N} \phi(t) + \chi(x), \vec{\eta}(x)), \quad (2.5)$$

with  $\langle \vec{\eta}(\vec{x}, t) \rangle = 0$  where  $\vec{\eta}$  represents the  $N-1$  ‘‘pions,’’ and  $\phi(t) = \langle \sigma(x) \rangle$ . Thus,  $\langle \chi(x) \rangle = 0$ .

#### A. The wave functional (Schrödinger picture)

We shall consider Gaussian wave functionals of the type [3,5–7,9]

$$\Psi[\vec{\Phi}(\cdot, t)] = \mathcal{N}^{1/2}(t) \Pi_{\vec{k}} \exp \left[ -\frac{A_{\vec{k}}(t)}{2} \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \right], \quad (2.6)$$

where

$$\vec{\eta}_{\vec{k}}(t) = \int d^3x \vec{\eta}(\vec{x}, t) e^{i\vec{k} \cdot \vec{x}}. \quad (2.7)$$

[Hence, we can assume  $A_{-\vec{k}}(t) = A_{\vec{k}}(t)$  without loss of generality.]

As shown below, such Gaussian wave functionals are solutions of the Schrödinger equation in the  $N = \infty$  limit. The Hamiltonian (2.3) in the large  $N$  limit is essentially a harmonic oscillator Hamiltonian with self-consistent, time-dependent frequencies:

$$H(t) = NVh_{cl}(t) - \frac{\lambda}{8N} \left( \sum_{\vec{k}} \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle \right)^2 + \sum_{\vec{k}} \left[ -\frac{1}{2} \frac{\delta^2}{\delta \vec{\eta}_{\vec{k}} \cdot \delta \vec{\eta}_{-\vec{k}}} + \frac{1}{2} \omega_{\vec{k}}^2(t) \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \right],$$

where  $h_{cl}(t)$  stands for the classical Hamiltonian

$$h_{cl}(t) = \frac{1}{2} \left[ \dot{\phi}^2(t) + m^2 \phi^2(t) + \frac{\lambda}{4} \phi^4(t) \right],$$

$$\omega_{\vec{k}}^2(t) = k^2 + m^2 + \frac{\lambda}{2} \left[ \phi^2(t) + \frac{1}{N} \langle \vec{\eta}^2(x) \rangle \right]. \quad (2.8)$$

The functional Schrödinger equation is then given by ( $\hbar = 1$ )

$$i \frac{\partial \Psi}{\partial t} = H \Psi. \quad (2.9)$$

More explicitly,

$$i \frac{\partial \Psi}{\partial t} = \left\{ NVh_{cl}(t) - \frac{\lambda}{8N} \left( \sum_{\vec{k}} \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle \right)^2 + \sum_{\vec{k}} \left[ -\frac{1}{2} \frac{\delta^2}{\delta \vec{\eta}_{\vec{k}} \cdot \delta \vec{\eta}_{-\vec{k}}} + \frac{1}{2} \omega_{\vec{k}}^2(t) \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \right] \right\} \Psi[\vec{\Phi}], \quad (2.10)$$

which then leads to a set of differential equations for  $A_{\vec{k}}(t)$ .

The evolution equations for  $A_{\vec{k}}(t)$  and  $\mathcal{N}(t)$  are obtained by taking the functional derivatives and comparing powers of  $\vec{\eta}_{\vec{k}}$  on both sides. We obtain the following evolution equations:

$$i \dot{A}_{\vec{k}}(t) = A_{\vec{k}}^2(t) - \omega_{\vec{k}}^2(t), \quad (2.11)$$

$$\mathcal{N}(t) = \mathcal{N}(0) \exp \left\{ -i \int_0^t dt' \left[ 2NVh_{cl}(t') - \frac{\lambda}{4N} \left( \sum_{\vec{k}} \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle(t') \right)^2 + N \sum_{\vec{k}} A_{\vec{k}}(t') \right] \right\} \quad (2.12)$$

with  $A_{\vec{k}}(t) = A_{R\vec{k}}(t) + iA_{I\vec{k}}(t)$ .

The time dependence of the normalization factor  $\mathcal{N}(t)$  is completely determined by that of the  $A_{\vec{k}}(t)$  as a consequence of unitary time evolution. Notice that the first two terms in the right-hand side (RHS) of Eq. (2.10) were missing in Ref. [3] as well as the corresponding first two terms inside the phase in Eq. (2.12).

Using the expression for the wave functional (2.6) we have

$$\langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle = \frac{\int \int \Pi_{\vec{q}} \mathcal{D} \vec{\eta}_{\vec{q}} e^{-A_{R\vec{q}}(t) \vec{\eta}_{\vec{q}} \cdot \vec{\eta}_{-\vec{q}} - A_{I\vec{q}}(t) \vec{\eta}_{\vec{q}} \cdot \vec{\eta}_{-\vec{q}}} \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle}{\int \int \Pi_{\vec{q}} \mathcal{D} \vec{\eta}_{\vec{q}} e^{-A_{R\vec{q}}(t) \vec{\eta}_{\vec{q}} \cdot \vec{\eta}_{-\vec{q}} - A_{I\vec{q}}(t) \vec{\eta}_{\vec{q}} \cdot \vec{\eta}_{-\vec{q}}}} = \frac{N}{2A_{R\vec{k}}(t)}, \quad (2.13)$$

and

$$\langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle = \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle \delta_{\vec{k}, -\vec{k}'}, \quad (2.14)$$

leading to the self-consistency condition in the infinite volume limit,

$$\frac{\langle \vec{\eta}^2(x) \rangle}{N} = \frac{1}{N} \int \frac{d^3k}{(2\pi)^3} \langle \vec{\eta}_{\vec{k}}(t) \cdot \vec{\eta}_{-\vec{k}}(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2A_{R\vec{k}}(t)}. \quad (2.15)$$

The expectation value of the Heisenberg equation of motion (2.4) yields the equation of motion for the order parameter  $\phi(t)$  [3]:

$$\dot{\phi}(t) + m^2\phi(t) + \frac{\lambda}{2} \left[ \phi^2(t) + \frac{\langle \vec{\eta}^2(x) \rangle}{N} \right] \phi(t) = 0, \quad (2.16)$$

where we used that  $\langle \vec{\eta} \rangle = 0$  and  $\langle \vec{\eta}^2 \vec{\eta} \rangle = 0$ .

Equations (2.11)–(2.12) together with Eq. (2.16) define the time evolution of this quantum state in the infinite  $N$  limit.

### B. The field modes (Heisenberg picture)

The Riccati equation (2.11) can be linearized by writing  $A_{\vec{k}}(t)$  in terms of the functions  $\varphi_{\vec{k}}(t)$  as

$$A_{\vec{k}}(t) = -i \frac{\dot{\varphi}_{\vec{k}}^*(t)}{\varphi_{\vec{k}}^*(t)}, \quad (2.17)$$

leading to the mode equations [3]

$$\ddot{\varphi}_{\vec{k}}^* + \omega_{\vec{k}}^2(t) \varphi_{\vec{k}}^* = 0. \quad (2.18)$$

The relation (2.17) defines the mode functions  $\varphi_{\vec{k}}$  up to an arbitrary multiplicative constant that we choose such that the Wronskian takes the value

$$\mathcal{W}_{\vec{k}} \equiv \varphi_{\vec{k}} \dot{\varphi}_{\vec{k}}^* - \dot{\varphi}_{\vec{k}} \varphi_{\vec{k}}^* = 2i. \quad (2.19)$$

The functions  $\varphi_{\vec{k}}$  have a very simple interpretation: they obey the Heisenberg equations of motion for the pion fields obtained from the Hamiltonian (2.3). Therefore, we can write the Heisenberg field operators as

$$\vec{\eta}(\vec{x}, t) = \int \frac{d^3\vec{k}}{\sqrt{2}(2\pi)^3} [\vec{a}_{\vec{k}} \vec{\varphi}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} + \vec{a}_{\vec{k}}^\dagger \varphi_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}}], \quad (2.20)$$

where  $\vec{a}_{\vec{k}}, \vec{a}_{\vec{k}}^\dagger$  are the time independent annihilation and creation operators with the usual canonical commutation relations. Thus, the  $\varphi_{\vec{k}}(t)$  are the mode functions of the field.

From Eq. (2.17) we obtain the following useful relations:

$$A_{R\vec{k}}(t) = \frac{1}{|\varphi_{\vec{k}}(t)|^2}, \quad A_{I\vec{k}}(t) = -\frac{1}{2} \frac{d}{dt} \ln |\varphi_{\vec{k}}(t)|^2. \quad (2.21)$$

Then, using these relations and Eq. (2.15) we express the fluctuation in terms of the mode functions:

$$\langle \eta^2 \rangle(t) \equiv \frac{\langle \vec{\eta}^2(x) \rangle}{N} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} |\varphi_{\vec{k}}(t)|^2. \quad (2.22)$$

### C. Definition of the particle number

Since in a time dependent situation the definition of the particle number is not unique, we choose to *define* the particle number with respect to the eigenstates of the instantaneous Hamiltonian (2.8) at the *initial time*, i.e.,

$$\hat{n}_{\vec{k}} = \frac{1}{N\omega_{\vec{k}}} \left[ -\frac{1}{2} \frac{\delta^2}{\delta \vec{\eta}_{\vec{k}} \cdot \delta \vec{\eta}_{-\vec{k}}} + \frac{1}{2} \omega_{\vec{k}}^2 \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \right] - \frac{1}{2}. \quad (2.23)$$

Here,  $\omega_{\vec{k}}$  is the frequency (2.8) evaluated at  $t=0$ .

The expectation value of the number operator in the time evolved state is then

$$\begin{aligned} n_{\vec{k}}(t) &= \langle \Psi | \hat{n}_{\vec{k}} | \Psi \rangle \\ &= \frac{[A_{R\vec{k}}(t) - \omega_{\vec{k}}]^2 + A_{I\vec{k}}^2(t)}{4\omega_{\vec{k}} A_{R\vec{k}}(t)} = \frac{\Delta_{\vec{k}}^2(t) + \delta_{\vec{k}}^2(t)}{4[1 + \Delta_{\vec{k}}(t)]}, \end{aligned} \quad (2.24)$$

where  $\Delta_{\vec{k}}(t)$  and  $\delta_{\vec{k}}(t)$  are defined through the relations

$$A_{R\vec{k}}(t) = \omega_{\vec{k}} [1 + \Delta_{\vec{k}}(t)], \quad A_{I\vec{k}}(t) = \omega_{\vec{k}} \delta_{\vec{k}}(t). \quad (2.25)$$

In terms of the mode functions  $\varphi_{\vec{k}}(t)$  and  $\dot{\varphi}_{\vec{k}}(t)$ , the expectation value of the number operator is given by

$$n_{\vec{k}}(t) = \frac{1}{4\omega_{\vec{k}}} [|\dot{\varphi}_{\vec{k}}(t)|^2 + \omega_{\vec{k}}^2 |\varphi_{\vec{k}}(t)|^2] - \frac{1}{2}. \quad (2.26)$$

For initially broken symmetry the frequencies in Eq. (2.23) are modified for low  $k$  according to Eq. (3.17).

### D. Initial conditions

We will take initial conditions for the field expectation value of the form

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = 0, \quad (2.27)$$

since we can make  $\dot{\phi}(0)$  to vanish by a shift in time.

The initial quantum state defined by Eq. (2.6) is determined giving  $A_{R\vec{k}}(0)$  and  $A_{I\vec{k}}(0)$  for all  $\vec{k}$  or equivalently, using the relations (2.25), giving  $\Delta_{\vec{k}} \equiv \Delta_{\vec{k}}(0)$  and  $\delta_{\vec{k}} \equiv \delta_{\vec{k}}(0)$ :

$$A_{R\vec{k}}(0) = \omega_{\vec{k}} [1 + \Delta_{\vec{k}}], \quad A_{I\vec{k}}(0) = \omega_{\vec{k}} \delta_{\vec{k}}. \quad (2.28)$$

We can invert Eq. (2.24) expressing  $\Delta_{\vec{k}}(t)$  in terms of  $\delta_{\vec{k}}(t)$  and the particle number  $n_{\vec{k}}(t)$ ,

$$\Delta_{\vec{k}}(t) = 2 \left[ n_{\vec{k}}(t) \pm \sqrt{n_{\vec{k}}^2(t) + n_{\vec{k}}(t) - \frac{\delta_{\vec{k}}^2(t)}{4}} \right]. \quad (2.29)$$

(This formula was derived in Ref. [3] for the case where  $\delta_{\vec{k}}(t) \equiv 0$ .)

The  $\pm$  sign in Eq. (2.29) is *not* the sign of  $m_R^2$ . The  $\pm$  possibility is due to the fact that the values of  $n_k^- \neq 0$  and  $\delta_k^-$  do not fix completely the initial state for the modes. We shall call case I to the upper sign in Eq. (2.29) and case II to the lower sign.

Thus, we can express the initial conditions in terms of the initial distribution of particles  $n_k^-(0)$ , the phase of the wave functional  $\delta_k^-$  and the choice of sign in Eq. (2.29). For simplicity, we will consider spherically symmetric particle distributions and with Gaussian form,

$$n_k^-(0) = \frac{\hat{N}_0}{I} \exp\left[-\frac{(k-k_0)^2}{\hat{\sigma}^2}\right], \quad (2.30)$$

where  $\hat{N}_0$  is the total number of particles per unit volume and  $I$  is a normalization factor. We shall always consider  $\hat{N}_0 \gg m^3$ . As we shall see below (Sec. IV A) it will be convenient to consider  $\hat{N}_0 \sim m^3/\lambda$ .

That is, our initial distribution of particles is a spherical shell in momentum space peaked around a momentum  $k_0$ . However, as seen in Ref. [3] and below features of the dynamics contain essential ingredients even for this simplified problem. [Nonspherically symmetrical distributions can be also treated in this scheme. Just the number of equations is doubled or tripled and the momentum integrals in Eq. (2.22) become double or triple integrals to be computed numerically at each time step.]

As the initial conditions are spherically symmetric and the evolution equations are invariant under rotations the solutions will be spherically symmetric. So, the dependence on  $\vec{k}$  is only through the modulus  $k$ .

The initial conditions on the mode functions follow from the relation (2.17) and the initial conditions (2.28) on  $A_k(t)$  plus the Wronskian constraint (2.19). Thus, the initial conditions on the mode functions are

$$\varphi_k(0) = \frac{1}{\sqrt{\Omega_k}}, \quad \dot{\varphi}_k(0) = -\frac{i\Omega_k + \omega_k \delta_k}{\sqrt{\Delta_k}}, \quad (2.31)$$

with  $\Omega_k$  defined by

$$\Omega_k \equiv A_{Rk}(0) = \omega_k [1 + \Delta_k]. \quad (2.32)$$

We consider initial conditions with

$$\delta_k(0) = 0. \quad (2.33)$$

The effect of taking  $\delta_k \neq 0$  is discussed below.

We then see from Eq. (2.29) that for modes with large occupation number ( $n_k \gg 1$ ), case I corresponds to initial conditions with  $|\varphi_k(0)| \ll 1$  and  $|\dot{\varphi}_k(0)| \gg 1$ ; while case II corresponds to  $|\varphi_k(0)| \gg 1$  and  $|\dot{\varphi}_k(0)| \ll 1$ . See for more details Secs. IV A and IV B.

That is, the physical meaning of the  $\pm$  signs in Eq. (2.29) for  $n_k \gg 1$  and  $\delta_k = 0$  is as follows. The upper sign (+) corresponds to an initial  $k$ -mode with small initial amplitude and large initial velocity and vice versa for the lower (-) sign.

If we neglect the interaction during early times we can write

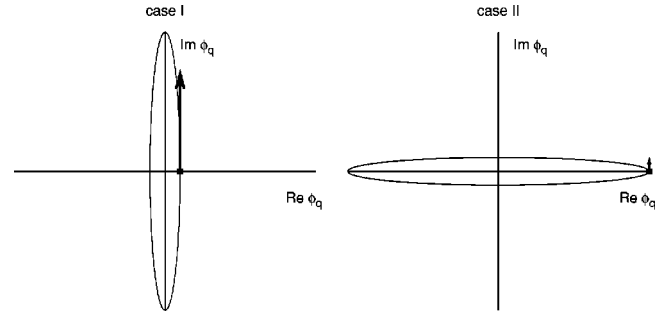


FIG. 1. The mode functions in the complex  $\varphi$  plane for short times and initial conditions  $n_k^- \sim 1/g \gg 1$ ,  $\delta_k^- = 0$ . Case I is on the left and case II on the right. The dot and the arrow represents the initial values of  $\varphi_k$  and its derivative, respectively.

$$\varphi_k(t) = \frac{1}{\sqrt{\Omega_k}} \cos(\omega_k t) - i \frac{\sqrt{\Omega_k}}{\omega_k} \sin(\omega_k t),$$

which describes an ellipse in the complex  $\varphi$  plane (see Fig. 1). The ratio of the half-axis of the ellipse goes as

$$\frac{\omega_k}{\Omega_k} = \frac{1}{1 + \Delta_k}.$$

This ratio is  $\ll 1$  ( $\gg 1$ ) in case I (II) when  $n_k \gg 1$  as follows from Eq. (2.29) and as can be seen in Fig. 1.

Taking  $\delta_k \neq 0$  keeps the elliptic trajectory with the additional effect of rotating its axis clockwise for increasing  $\delta_k$ . For an initial state with no particles ( $\Delta_k = 0$ ,  $\delta_k = 0$ ), the initial conditions become:  $\varphi_k(0) = 1/\sqrt{\omega_k}$  and  $\dot{\varphi}_k(0) = -i\sqrt{\omega_k}$ . These were the initial conditions used in Refs. [9–11].

### III. EVOLUTION EQUATIONS

The evolution equations for the expectation value of the field  $\phi(t)$  and for the mode functions  $\varphi_k(t)$ , Eq. (2.16) and Eq. (2.18), respectively, are:

$$\frac{d^2 \phi(t)}{dt^2} + \left\{ m_B^2 + \frac{\lambda}{2} [\phi^2(t) + \langle \eta^2 \rangle_B(t)] \right\} \phi(t) = 0, \quad (3.1)$$

$$\frac{d^2 \varphi_k(t)}{dt^2} + \left\{ k^2 + m_B^2 + \frac{\lambda}{2} [\phi^2(t) + \langle \eta^2 \rangle_B(t)] \right\} \varphi_k(t) = 0, \quad (3.2)$$

with the self-consistent condition

$$\langle \eta^2 \rangle_B(t) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} |\varphi_k(t)|^2. \quad (3.3)$$

The above evolution equations must be renormalized. This is achieved by demanding that the equations of motion be finite. The divergent pieces are absorbed into a redefinition of the mass and coupling constant:

$$\begin{aligned}\mathcal{M}_d^2(t) &\equiv m_B^2 + \frac{\lambda}{2} [\langle \eta^2 \rangle_B(t) + \phi^2(t)] \\ &= m_R^2 + \frac{\lambda_R}{2} [\langle \eta^2 \rangle_R(t) + \phi^2(t)].\end{aligned}\quad (3.4)$$

A detailed derivation of the renormalization prescriptions requires a WKB analysis of the mode functions  $\varphi_k(t)$  that reveals their ultraviolet properties. Such an analysis has been performed in Refs. [3,5,9]. In summary, quadratic and logarithmic divergences are absorbed in the mass term while the coupling constant absorbs a logarithmically divergent piece [5,8]. The renormalized quantum fluctuations take the form

$$\langle \eta^2 \rangle_R(t) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[ |\varphi_k(t)|^2 - \frac{1}{k} + \frac{\theta(k-\kappa)}{2k^3} \mathcal{M}_d^2(t) \right] \quad (3.5)$$

with  $\kappa$  an arbitrary renormalization scale.

We define now dimensionless quantities choosing the physical mass in vacuum  $|m_R|$  as unit of mass:

$$\begin{aligned}q &\equiv \frac{k}{|m_R|}; \quad \tau \equiv |m_R|t; \quad \zeta^2(\tau) \equiv \frac{\lambda_R \phi^2(t)}{2|m_R|^2}; \\ \varphi_q(\tau) &\equiv \sqrt{|m_R|} \varphi_k(t); \\ g &\equiv \frac{\lambda_R}{8\pi^2}; \quad \omega_q(\tau) \equiv \frac{\omega_k(t)}{|m_R|}; \quad \Omega_q \equiv \frac{\Omega_k}{|m_R|}; \\ g\Sigma(\tau) &\equiv \frac{\lambda_R}{2|m_R|^2} \langle \eta^2 \rangle_R(t).\end{aligned}\quad (3.6)$$

In terms of these dimensionless quantities the equations of motion (3.1),(3.2) become

$$\left[ \frac{d^2}{d\tau^2} \pm 1 + \zeta^2(\tau) + g\Sigma(\tau) \right] \zeta(\tau) = 0, \quad (3.7)$$

$$\left[ \frac{d^2}{d\tau^2} + q^2 \pm 1 + \zeta^2(\tau) + g\Sigma(\tau) \right] \varphi_q(\tau) = 0; \quad (3.8)$$

$$g\Sigma(\tau) = g \int_0^\infty q^2 dq \left[ |\varphi_q(\tau)|^2 - \frac{1}{q} + \frac{\theta(q-1)}{2q^3} \frac{\mathcal{M}_d^2(\tau)}{|m_R|^2} \right], \quad (3.9)$$

where we have chosen the renormalization scale  $\kappa = |m_R|$  for simplicity and the sign  $\pm$  corresponds here to the sign of  $m_R^2$ .

In Eqs. (3.7),(3.8),

$$\mathcal{M}^2(\tau) \equiv \frac{\mathcal{M}_d^2(\tau)}{|m_R|^2} = \alpha + \zeta^2(\tau) + g\Sigma(\tau) \quad (3.10)$$

plays the role of a time dependent effective squared mass.  $\alpha \equiv \text{sgn}(m_R^2) = \pm 1$ .

Depending on whether  $\mathcal{M}^2(0)$  is positive or negative the symmetry will be initially unbroken or broken.

In Ref. [3] a different definition for the mass unit and for  $\Sigma(\tau)$  were used. The advantage to use  $|m_R|$  here stems from

the fact that it is independent of the initial number of particles. The present choice yields much simpler expressions for the final results.

The initial conditions become in dimensionless variables for  $\mathcal{M}^2(0) > 0$ :

$$\zeta(0) = \zeta_0, \quad \dot{\zeta}(0) = 0, \quad (3.11)$$

$$\varphi_q(0) = \frac{1}{\sqrt{\Omega_q}}, \quad \dot{\varphi}_q(0) = -\frac{i\Omega_q + \omega_q \delta_q}{\sqrt{\Omega_q}}, \quad (3.12)$$

with

$$\begin{aligned}\Omega_q &\equiv \omega_q [1 + \Delta_q], \quad \omega_q = \sqrt{q^2 + \mathcal{M}^2(0)}, \\ \Delta_q &= 2 \left[ n_q(0) \pm \sqrt{n_q^2(0) + n_q(0) - \frac{\delta_q^2}{4}} \right].\end{aligned}\quad (3.13)$$

From Eqs. (2.33) and (2.30) we have

$$\delta_q = 0, \quad n_q(0) = \frac{N_0}{I} \exp \left[ -\frac{(q-q_0)^2}{\sigma^2} \right], \quad (3.14)$$

with  $N_0 = \hat{N}_0 / |m_R|^3$  and  $\sigma = \hat{\sigma} / |m_R|$ .

Therefore,

$$g\Sigma(\tau) = g \int_0^\infty q^2 dq \left[ |\varphi_q(\tau)|^2 - \frac{1}{q} + \frac{\theta(q-1)}{2q^3} \mathcal{M}^2(\tau) \right]. \quad (3.15)$$

To perform the numerical evolution, we have introduced a momentum cutoff  $\Lambda$ . The quantum fluctuations become for finite cutoff,

$$\begin{aligned}g\Sigma(\tau) &= \frac{1}{1 - \frac{g}{2} \log \Lambda} \left\{ \int_0^\Lambda q^2 dq |\varphi_q(\tau)|^2 - \frac{g}{2} \Lambda^2 \right. \\ &\quad \left. + \frac{g}{2} \log \Lambda [\alpha + \zeta^2(\tau)] \right\} + O\left(\frac{g}{\Lambda^2}\right).\end{aligned}\quad (3.16)$$

This is a positive quantity [up to  $O(g)$ ].  $\alpha = \text{sgn}(m_R^2) = \pm 1$ .

In the case where the symmetry is initially broken ( $\mathcal{M}^2(0) < 0$ ) the only change in Eqs. (3.11)–(3.16) is that the initial frequencies  $\omega_q$  are modified for low  $q$  as follows [10]:

$$\omega_q = \begin{cases} \sqrt{q^2 + |\mathcal{M}^2(0)|} & \text{for } q^2 < 1 + |\mathcal{M}^2(0)|, \\ \sqrt{q^2 + \mathcal{M}^2(0)} & \text{for } q^2 > 1 + |\mathcal{M}^2(0)|. \end{cases} \quad (3.17)$$

#### IV. EARLY TIME DYNAMICS

We study now the time evolution of a narrow particle distribution  $n_q(0)$  peaked at  $q = q_0$ . We show below that we can approximate the dynamics for early times using a single mode  $\varphi_{\text{eff}}(\tau)$ . We solve the time evolution of  $\varphi_{\text{eff}}(\tau)$  in close form in terms of elliptic functions. Moreover, we compare

these analytic results with the full numerical solution of Eqs. (3.7)–(3.9).

For  $m_R^2 < 0$ , we only consider initial particle peaks with  $q_0^2 > 2$ . Thus, they are well outside possible spinodally resonant bands. For simplicity, we consider  $\zeta_0 = 0$  in this section and in Secs. V, VI, and VII.

### A. Case I

In this case, we have  $\Delta_q = 2[n_q(0) + \sqrt{n_q^2(0) + n_q(0)}]$ . We see from Eq. (3.13) that the modes with  $n_q(0) \ll 1$  have  $\Omega_q \approx \omega_q$ . Therefore,  $\varphi_q(0)$  and  $\dot{\varphi}_q(0)$  are of order  $g^0$  [see Eq. (3.12)]. Thus, their contribution to  $g\Sigma(\tau)$  will be of order  $g$  for early times.

On the other hand, we have for modes with  $n_q(0) = O(1/g)$  [as  $g \ll 1$  this implies  $n_q(0) \gg 1$ ]

$$\Delta_q \approx 4n_q(0) \gg 1, \quad \Omega_q \approx 4\omega_q n_q(0) \gg 1. \quad (4.1)$$

Therefore, Eqs. (3.12) imply that these modes have  $|\varphi_q(0)| \ll 1$ .

Thus,

$$g\Sigma_I(0) = O(g), \quad (4.2)$$

$$\mathcal{M}^2(0) = \alpha + O(g) = \pm 1 + O(g), \quad (4.3)$$

where  $\alpha = \text{sgn}(m_R^2) = \pm 1$ . For these modes with  $n_q(0) = O(1/g)$ ,  $|\dot{\varphi}_q(0)| \gg 1$ . Therefore, these modes will then grow and their contribution will dominate  $g\Sigma(\tau)$  for early times.

We always consider that  $n_q(0) = O(1/g)$  for some interval in  $q$ . Hence, its contribution to  $g\Sigma(\tau)$  will be of order one and will have an important effect on the dynamics. In such conditions, the total number of particles  $N_0$  per unit volume is also of the order  $O(1/g)$ .

For the initial conditions considered, Eq. (3.14), these dominant modes are in a small interval centered at  $q = q_0$ , and they are in phase at least for short times. More precisely, the  $q$ -modes will stay in phase for  $(q - q_0)\tau \ll 2\pi$ . The modes which are in phase contribute coherently to  $g\Sigma(\tau)$  (each one with a contribution proportional to its occupation number).

Hence, for small  $\tau$  a good approximation is to consider that all the particles are in a single mode with  $q = q_0$ . This approximation will apply as long as the modes with large occupation number stay in phase.

In these conditions Eq. (3.16) yields

$$g\Sigma(\tau) = gq_0^2 |\varphi_{\text{eff}}(\tau)|^2 \Delta q + O(g) + O(g\sigma), \quad (4.4)$$

where  $\Delta q \approx \sigma$  [see Eq. (3.14)]. Thus

$$\frac{d^2 \varphi_{\text{eff}}}{d\tau^2} + \omega_{q_0}^2 \varphi_{\text{eff}}(\tau) + g\Delta q q_0^2 |\varphi_{\text{eff}}(\tau)|^2 \varphi_{\text{eff}}(\tau) = 0, \quad (4.5)$$

with  $\omega_{q_0} = \sqrt{q_0^2 + \alpha}$  [recall  $\alpha = \text{sgn}(m_R^2) = \pm 1$  and we choose  $q_0^2 > 2$ ].

We also have

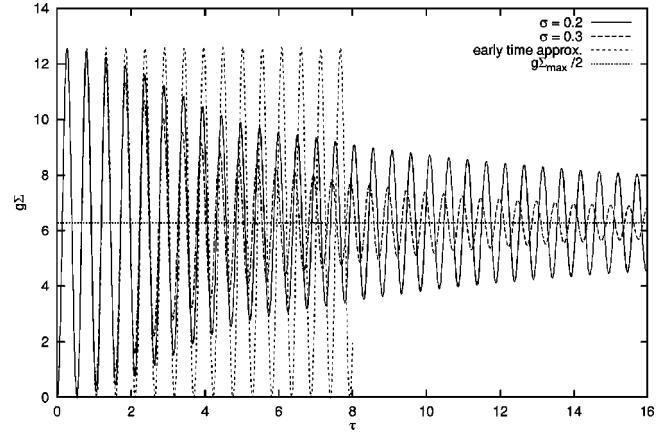


FIG. 2. Case I.  $m_R^2 > 0$ . Unbroken symmetry;  $gN_0 = 250$ ,  $q_0 = 5$ ,  $\zeta_0 = 0$ , and  $g = 10^{-7}$ . Comparison between numerical solutions and the early time approximation (4.7).

$$N_0 = 4\pi q_0^2 \Delta q n_{\text{eff}} \Rightarrow \Omega_{\text{eff}} = \frac{N_0 \omega_{q_0}}{\pi q_0^2 \Delta q}. \quad (4.6)$$

Solving the nonlinear differential equation (4.5) with the initial conditions (3.12) and (4.6), we obtain  $\varphi_{\text{eff}}(\tau)$ . We find, from Eq. (4.4),

$$g\Sigma_I(\tau) = (q_0^2 + \alpha) \left( 1 + \frac{1}{1 - 2k^2} \right) \times \left[ \frac{1}{1 - k^2 \text{sn}^2 \left( \sqrt{\frac{q_0^2 + \alpha}{1 - 2k^2}} \tau, k \right)} - 1 \right], \quad (4.7)$$

with  $\text{sn}(z, k)$  the Jacobi sine function and

$$k = \sqrt{\frac{1}{2} \left( 1 - \frac{1}{1 + \frac{g\Sigma_{I \text{max}}}{q_0^2 + \alpha}} \right)}. \quad (4.8)$$

The function (4.7) is non-negative and oscillates between zero and

$$g\Sigma_{I \text{max}} = (q_0^2 + \alpha) \left[ \sqrt{1 + \frac{2gN_0}{\pi(q_0^2 + \alpha)^{3/2}}} - 1 \right], \quad (4.9)$$

with period

$$T = 2 \sqrt{\frac{1 - 2k^2}{q_0^2 + \alpha}} K(k), \quad (4.10)$$

where  $K(k)$  stands for the complete elliptic integral of the first kind. [Notice that Eq. (4.8) implies  $1 - 2k^2 > 0$ ].

The elliptic solution (4.7) correctly predicts the amplitude of the first oscillation and the oscillation period (if the initial distribution of particles is not too wide around  $q = q_0$ ). See Fig. 2.

The amplitude of the numerical solution is well reproduced by the elliptic solution (4.7) till damping becomes

important. The oscillation period keeps well reproduced by Eq. (4.7) for longer times than the amplitude. (See Fig. 2.) We see from the numerical solution of the exact equations (3.7)–(3.9) that  $g\Sigma(\tau)$  exhibit significantly damped oscillations after a few periods.

The integral over  $q$  for  $g\Sigma(\tau)$  [Eq. (3.15)] gets damped with time due to the loss of coherence between the different  $q$  modes in the distribution peak. We can apply here the adiabatic approximation (see Appendix A). In the late time limit,  $g\Sigma(\tau)$  therefore tends to the value  $g\Sigma_{I \max}/2$ . In addition, the narrower is the peak, the slower the oscillations in  $g\Sigma(\tau)$  are damped. See Fig. 2.

The time scale where the numerical and the early times solution deviate in amplitude, essentially depends on the width  $\sigma$  of the initial distribution. The smaller is  $\sigma$ , the latest the early time solution (4.7) holds. Notice that Eq. (4.7) gives  $g\Sigma(\tau)$  to zero order in  $\sigma$ , whereas the damping of the oscillations is given by higher orders in  $\sigma$ . In addition, we have seen from the numerical resolution that the early time evolution depends on  $g$  only through  $gN_0$  as predicted by Eqs. (4.8),(4.9).

When there is particle production through parametric resonance, particles in the initial peak distribution are annihilated (in order to conserve the total energy). This reduces the contribution of the initial peak to  $g\Sigma(\tau)$  whereas the oscillations due to the created particles give a contribution to  $g\Sigma(\tau)$ . These oscillations are due to the coherence between the created particles at different  $q$ .

Such changes in the distribution of particles influence the asymptotic value of  $g\Sigma(\tau)$  for late times. Contrary to the vacuum initial conditions, parametric resonance shuts off here by the damping of the oscillations and not due to back-reaction as in Ref. [9]. Therefore, for small  $g$  the influence of parametric resonances is highly suppressed. When parametric resonance is appreciable, it makes the dynamics to depend on  $g$  and not only through the combination  $gN_0$ .

In case I, as  $\mathcal{M}^2(0) = \alpha = \text{sgn}(m_R^2) = \pm 1$ . The symmetry is initially spontaneously broken or not depending on whether  $m_R^2$  (the squared tree level mass) is negative or positive.

## B. Case II

In this case,  $\Delta_q = 2[n_q(0) - \sqrt{n_q^2(0) + n_q(0)}]$ . As in case I,  $g\Sigma(\tau)$  is dominated for short times by the modes with large occupation numbers which are in phase at small  $\tau$  due to the initial conditions (3.14). Thus, we can do the same approximation as in case I, considering that all particles are in a single mode with  $q = q_0$ .

We have for the modes with  $q \cong q_0$ ,  $n_q(0) = O(1/g) \gg 1$ ,

$$\Delta_q \cong -1 + \frac{1}{4n_q(0)}, \quad \Omega_q \cong \frac{\omega_q}{4n_q(0)} \ll 1. \quad (4.11)$$

Thus, for  $\tau=0$ ,

$$g\Sigma_{II}(0) = \frac{gN_0}{\pi\omega_{q_0}} + O(g) + O(g\sigma), \quad (4.12)$$

$$\mathcal{M}^2(0) = \alpha + g\Sigma_{II}(0), \quad (4.13)$$

where  $\alpha = \text{sgn}(m_R^2) = \pm 1$ .

Equation (4.12) is a third degree equation in  $g\Sigma_{II}(0)$  since  $\omega_{q_0} = \sqrt{q_0^2 + \alpha + g\Sigma_{II}(0)}$ . The explicit solution is given in Appendix D. We find, in the limiting cases,

$$g\Sigma_{II}(0) \begin{cases} gN_0 \gg (q_0^2 + \alpha)^{3/2} \left( \frac{gN_0}{\pi} \right)^{2/3} - \frac{1}{3}(q_0^2 + \alpha) + O\left( \frac{q_0^2 + \alpha}{gN_0} \right)^{2/3}, \\ gN_0 \ll (q_0^2 + \alpha)^{3/2} \frac{gN_0}{\pi\sqrt{q_0^2 + \alpha}} + O\left( \frac{gN_0}{q_0^2 + \alpha} \right)^2. \end{cases} \quad (4.14)$$

For early times Eq. (3.16) becomes

$$g\Sigma(\tau) = gq_0^2 \Delta q |\varphi_{\text{eff}}(\tau)|^2 + O(g) + O(g\sigma). \quad (4.15)$$

Therefore,

$$\frac{d^2 \varphi_{\text{eff}}}{d\tau^2} + (q_0^2 + \alpha) \varphi_{\text{eff}}(\tau) + gq_0^2 + \Delta q |\varphi_{\text{eff}}(\tau)|^2 \varphi_{\text{eff}}(\tau) = 0. \quad (4.16)$$

Using the initial conditions given by Eq. (3.12) and  $\delta_{\text{eff}}(0) = 0$  we get from Eq. (4.11),

$$\Omega_{\text{eff}} = \frac{\pi q_0^2 \Delta q \omega_{q_0}}{N_0}, \quad (4.17)$$

where we have used that  $n_{\text{eff}} = N_0/4\pi q_0^2 \Delta q$ .

Thus, the solution of Eq. (4.16) with the specified initial conditions can be written as

$$g\Sigma_{II}(\tau) = g\Sigma_I(\tau + T/2) = (q_0^2 + \alpha) \left( 1 + \frac{1}{1 - 2k^2} \right) \left[ \frac{1}{1 - k^2 s n^2 \left( \sqrt{\frac{q_0^2 + \alpha}{1 - 2k^2}}(\tau + T/2), k \right)} - 1 \right] \quad (4.18)$$

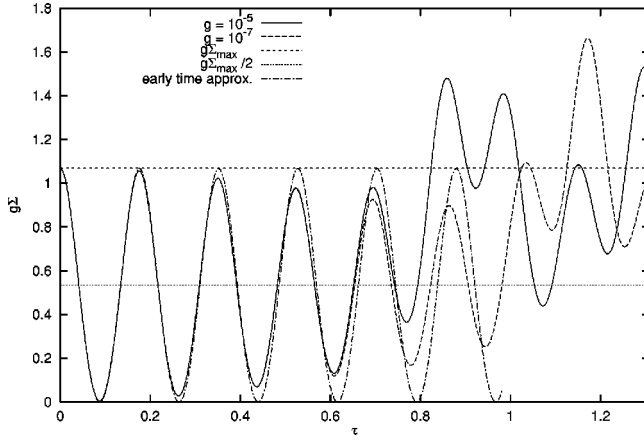


FIG. 3. Case II.  $m_R^2 < 0$ .  $g\Sigma(\tau)$  as a function of  $\tau$ . Dynamically broken symmetry;  $gN_0 = 4.478$ ,  $q_0 = 1.3083$ ,  $\sigma = 0.05233$ , and  $\zeta_0 = 0$ . Comparison between numerical solutions and the early time approximation (4.18). For late times  $g\Sigma(\tau)$  tends to 1, thus  $\mathcal{M}_\infty^2 = 0$  (see Fig. 6).

where  $sn(z, k)$  is the Jacobi sine function,  $T$  is the real period

$$T = 2 \sqrt{\frac{1 - 2k^2}{q_0^2 + \alpha}} K(k),$$

$K(k)$  stands for the complete elliptic integral of the first kind and

$$k = \sqrt{\frac{1}{2} \left( 1 - \frac{1}{1 + \frac{g\Sigma_{II \max}}{q_0^2 + \alpha}} \right)}. \quad (4.19)$$

The expression for  $g\Sigma_{II \max}$  is

$$g\Sigma_{II \max} = g\Sigma_{II}(0) = \frac{gN_0}{\pi \sqrt{q_0^2 + \alpha + g\Sigma_{II}(0)}}. \quad (4.20)$$

Notice that the relation  $g\Sigma_{II}(\tau) = g\Sigma_I(\tau + T/2)$  is true for given values of  $q_0$ ,  $\alpha$ , and  $k$ .

Here  $g\Sigma(\tau)$  oscillates between zero and  $g\Sigma_{II \max}$ , while the effective squared mass  $\mathcal{M}^2(\tau)$  oscillates between its initial value  $\mathcal{M}^2(0) = \pm 1 + g\Sigma_{II \max}$  and its tree level value  $\pm 1$  at the minima of  $g\Sigma(\tau)$ . We see that this approximation gives us correctly the amplitude of the first oscillation and the oscillation period; but not the damping of the oscillation that is due to the dephasing of the initial particle distribution. The broader is the initial particle distribution, the more effective the dephasing works and the faster the damping occurs.

As in case I, the smaller is  $\sigma$  (width of the initial particle distribution), the later the early time solution (4.18) holds (see Figs. 2 and 3). Recall that the damping of the oscillations is given by higher orders in  $\sigma$  while Eq. (4.18) gives  $g\Sigma(\tau)$  to zero order in  $\sigma$ . These higher orders in  $\sigma$  will also break the relation:  $g\Sigma_{II}(\tau) = g\Sigma_I(\tau + T/2)$ ; that we have found at zeroth order.

In case II,  $\mathcal{M}^2(0) = \alpha + g\Sigma_{II \max}$ . [Recall  $\alpha = \text{sgn}(m_R^2) = \pm 1$ .] As  $g\Sigma_{II \max} \geq 0$ , for  $m_R^2 > 0$  the symmetry is initially unbroken. Instead, for  $m_R^2 < 0$ , the symmetry is initially unbroken for  $g\Sigma_{II \max} > 1$ . For  $g\Sigma_{II \max} < 1$  the symmetry is initially spontaneously broken.

## V. INTERMEDIATE AND LATE TIME DYNAMICS

We discuss here the intermediate and late time behavior of the quantum fluctuations and the effective squared mass. We mean by late time, times later than the spinodal time and than the damping time. We consider  $\zeta_0 = 0$ .

We observe from the numerical results the following common features in the late time behavior for a wide range of initial conditions. The asymptotic constant values of the magnitudes (mass, pressure, number of particles) depend on  $g$ , for small  $g$ , only through the combination  $gN_0$  (except when parametric resonance is important). Energy is conserved to one part in  $10^7$  confirming the precision of our numerical calculations.

We have also seen that the mass tends to its limiting value oscillating with an amplitude that decays at least as  $\sim 1/\tau$ . A similar asymptotic decay has been found in this model for zero particle initial conditions and  $\zeta(0) \neq 0$  [10]. In Sec. VII we derive the asymptotic equation of state.

### A. $m_R^2 > 0$ ( $\alpha = +1$ )

We have for cases I and II that

$$\mathcal{M}^2(\tau) = 1 + g\Sigma(\tau), \quad (5.1)$$

although  $g\Sigma(\tau)$  has a different expression in each case. As  $g\Sigma(\tau) \geq 0$  the symmetry is unbroken for all  $\tau$ .

The adiabatic approximation holds (see Appendix A) since parametric resonance is negligible in the weak coupling regime considered here. Hence, the asymptotic value of  $g\Sigma(\tau)$  is  $g\Sigma_{\max}/2$  and the asymptotic squared mass goes to (see Fig. 2)

$$\mathcal{M}_\infty^2 = 1 + \frac{g\Sigma_{\max}}{2} > 0. \quad (5.2)$$

This result is in good agreement with the numerical calculations (see Table I).

We find that the mass tends to this value oscillating with an amplitude that decays at least as  $\sim 1/\tau$ . The initial peak of particles becomes lower and wider, and the total number of particles slightly decreases compared to its initial value.

### B. $m_R^2 < 0$ ( $\alpha = -1$ )

We have for both case I and case II that

$$\mathcal{M}^2(\tau) = -1 + g\Sigma(\tau). \quad (5.3)$$

Recall that in case I the symmetry is initially broken since  $\Sigma_I(0) = 0$ . In case II,  $\Sigma_{II}(0) = \Sigma_{II \max}$  and the symmetry is initially broken (unbroken) for  $g\Sigma_{II \max} < 1$  ( $g\Sigma_{II \max} > 1$ ).

We have two different asymptotic regimes (see Table II).

TABLE I. Initial and late effective mass and symmetry for  $m_R^2 > 0$ .

$m_R^2 > 0$	Case I	Case II
$\mathcal{M}^2(0)$	+1	$+1 + g\Sigma_{II \max}$
Initial symmetry	unbroken	unbroken
$\mathcal{M}_\infty^2$	$+1 + \frac{g\Sigma_{I \max}}{2}$	$+1 + \frac{g\Sigma_{II \max}}{2}$
Late time symmetry	unbroken	unbroken

(a)  $g\Sigma_{\max}/2 > 1$ . Asymptotically unbroken symmetry. In this regime the results are similar to those in the previous subsection V A, with

$$\mathcal{M}_\infty^2 = -1 + \frac{g\Sigma_{\max}}{2} > 0. \quad (5.4)$$

This is indeed the value we obtain numerically (see Fig. 4). We find that for small  $g$  there is no appreciable spinodal resonance here. This is so because  $\mathcal{M}^2(\tau)$  oscillates around the positive value  $\mathcal{M}_\infty^2$ , although for some intervals of time  $\mathcal{M}^2(\tau) < 0$ . In this regime the symmetry is restored for late times due to the presence of a high density of particles. In particular, we have already noticed that for case I,  $\mathcal{M}^2(0) = -1$ . This does not change appreciably the dynamics and the symmetry gets restored provided  $g\Sigma_{\max}/2 > 1$ .

(b)  $g\Sigma_{\max}/2 < 1$ . Asymptotically broken symmetry. In this regime for intermediate times (times earlier than the spinodal time  $\tau_s$ )  $\mathcal{M}^2(\tau)$  oscillates around the negative value,

$$-\mu^2 = -1 + \frac{g\Sigma_{\max}}{2} < 0, \quad (5.5)$$

giving rise to spinodal resonances. (See Figs. 3, 5, and 6.) The dynamics turns to be similar as for a constant squared mass  $-\mu^2$ . That is, the spinodally resonant band is in the

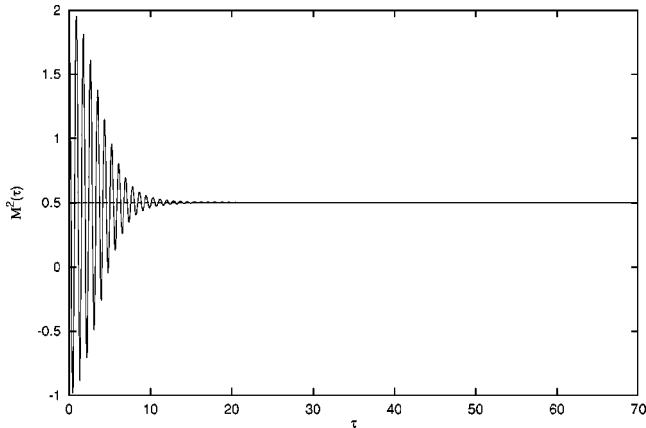


FIG. 4. Case II.  $m_R^2 < 0$ .  $\mathcal{M}^2(\tau)$  as a function of  $\tau$ . The symmetry is unbroken.  $gN_0 = 67.96$ ,  $q_0 = 7.071$ ,  $\sigma = 0.4243$ , and  $\zeta_0 = 0$ . Thus,  $g\Sigma_{II \max} = 3.000$  [see Eq. (4.20)] and  $\mathcal{M}_\infty^2 = -1 + g\Sigma_{II \max}/2 = 0.5000$  [See Eq. (5.4)].

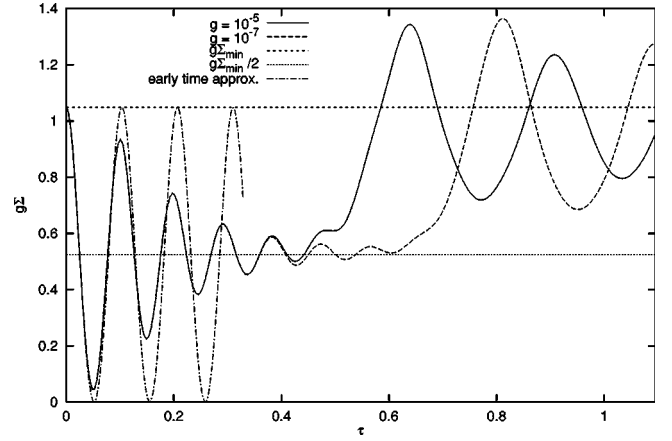


FIG. 5. Case II.  $m_R^2 < 0$ .  $g\Sigma(\tau)$  as a function of  $\tau$ . Dynamically broken symmetry;  $gN_0 = 5.101$ ,  $q_0 = 1.5336$ ,  $\sigma = 0.2191$ , and  $\zeta_0 = 0$ . Comparison between numerical solutions and the early time approximation (4.18). For late times (not shown in the figure)  $g\Sigma(\tau)$  tends to 1.

$q$ -interval from  $q=0$  to  $q=\mu$  and the spinodal time  $\tau_s$  is the same as for a constant squared mass of  $-\mu^2$  (see Appendix C).

We can further distinguish (1) when  $g\Sigma_{\max} < 1$ ,  $\mathcal{M}^2(\tau)$  is always negative for times  $\tau < \tau_s$ , and (2) when  $g\Sigma_{\max} > 1$ ,  $\mathcal{M}^2(\tau)$  can be temporarily positive due to oscillations. This happens in case I where  $\mathcal{M}^2(0) = -1$ .

In case II,  $\mathcal{M}^2(0) = -1 + g\Sigma_{\max} > 0$ , but after a time of order  $T/2$ ,  $\mathcal{M}^2(\tau) < 0$ , and  $\mathcal{M}^2(\tau)$  continues to oscillate around the negative value given by Eq. (5.5). Thus, in case II we have dynamical symmetry breaking (see Table II).

At time  $\tau_s$ , the spinodal resonance has created enough particles [of the order  $O(1/g)$ ] to give an important contribution to  $g\Sigma(\tau)$ . This finally makes  $g\Sigma(\tau)$  oscillate around 1. Thus,  $\mathcal{M}^2(\tau)$  oscillates around zero and the spinodal resonance stops. The particles created by the spinodal resonance are coherent. This gives new oscillations to  $\mathcal{M}^2(\tau)$ . These oscillations get damped and the squared mass goes for late times to its asymptotic value,

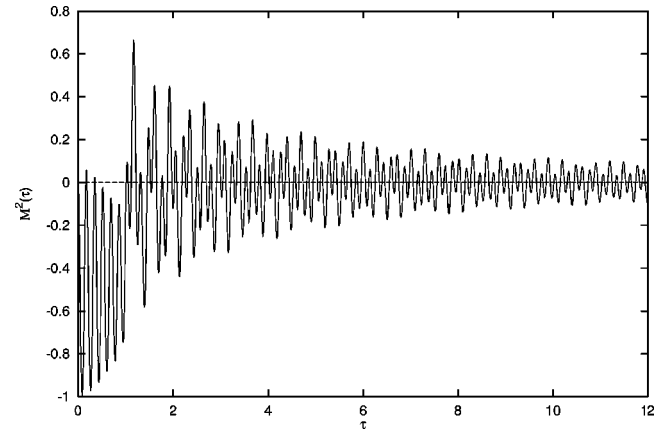


FIG. 6. Case II.  $m_R^2 < 0$ .  $\mathcal{M}^2(\tau)$  as a function of  $\tau$ . The symmetry is dynamically broken.  $gN_0 = 4.478$ ,  $q_0 = 1.3083$ ,  $\sigma = 0.05233$ , and  $\zeta_0 = 0$ . (The same initial conditions and  $g$  as in Fig. 3.)

TABLE II. Initial and late effective mass and symmetry for  $m_R^2 < 0$ .

$m_R^2 < 0$	Case I		Case II		
$\mathcal{M}^2(0)$	-1		-1 + $g\Sigma_{II \max}$		
Initial symmetry	broken		if $g\Sigma_{II \max} < 1$ broken	if $g\Sigma_{II \max} > 1$ unbroken	
	if $\frac{g\Sigma_I \max}{2} < 1$	if $\frac{g\Sigma_I \max}{2} > 1$	if $\frac{g\Sigma_{II \max}}{2} < 1$	if $\frac{g\Sigma_{II \max}}{2} > 1$	
$\mathcal{M}_\infty^2$	0	$-1 + \frac{g\Sigma_I \max}{2}$	0	$-1 + \frac{g\Sigma_{II \max}}{2}$	
Late time symmetry	broken	unbroken	broken	unbroken	
Change of symmetry	no change	dynamical symmetry restoration	no change	dynamical symmetry breaking	no change

$$\mathcal{M}_\infty^2 = 0. \quad (5.6)$$

The vanishing of the effective mass is accompanied by the presence of Goldstone bosons out of equilibrium as in [3,9]. Particles created by spinodal resonances remain in the  $q$ -interval from 0 to  $\mu$ . This creation of particles depletes the initial peak of particles keeping the total energy conserved. For  $\tau < \tau_s$  the dynamic depends on  $g$ , for small  $g$ , only through the combination  $gN_0$ . However,  $\tau_s$  depends explicitly on  $g$ .

## VI. ENERGY

After renormalization, taking dimensionless variables and introducing a momentum cut-off, the renormalized energy can be written as [9]

$$E_{\text{ren}} = \frac{2|m_R|^4}{\lambda_R} \epsilon, \quad (6.1)$$

$$\begin{aligned} \epsilon = & \frac{1}{2} \zeta^2(\tau) + \frac{\alpha}{2} \zeta^2(\tau) + \frac{1}{4} \zeta^4(\tau) + \frac{1}{4} \frac{1-\alpha}{2} \\ & + \frac{g}{2} \int_0^\Lambda q^2 dq [|\dot{\phi}_q(\tau)|^2 + \omega_q^2(\tau) |\varphi_q(\tau)|^2] \\ & - \frac{1}{4} [g\Sigma(\tau)]^2 + O(g), \end{aligned} \quad (6.2)$$

where  $\alpha = \text{sgn}(m_R^2)$ .

One can easily check that the energy is conserved using the renormalized equations of motion (3.7)–(3.9).

Using the initial conditions (3.11), (3.12), (3.14), and  $\zeta_0 = 0$  we obtain for the energy at  $\tau = 0$

$$\epsilon = \frac{1}{4} \frac{1-\alpha}{2} + \frac{g}{2} \int_0^\Lambda q^2 dq \left( \omega_q + \frac{\omega_q^2}{\Omega_q} \right) - \frac{1}{4} [g\Sigma(0)]^2 + O(g). \quad (6.3)$$

This equation gives the energy in terms of the initial data.  $\epsilon - 1/4(1-\alpha)/2$  is always positive because we consider  $\zeta_0 = 0$  and  $\dot{\zeta}_0 = 0$ .

### A. Case I

Let us consider an initial narrow distribution of particles which is peaked at  $q = q_0$ . Thus, we can use Eqs. (4.1) and (6.3) and consider all particles in a single mode with  $q = q_0$  as in previous sections. (Recall that in this case  $\omega_{q_0}^2 = q_0^2 + \alpha$ .) The energy is then given by

$$\epsilon = \frac{gN_0}{2\pi} (q_0^2 + \alpha) + \frac{1}{4} \frac{1-\alpha}{2} + O(g) + O(g\sigma). \quad (6.4)$$

### B. Case II

Under the same approximations [now using Eq. (4.11)], the energy is given by

$$\begin{aligned} \epsilon = & (q_0^2 + \alpha) \frac{g\Sigma_{II}(0)}{2} + \left[ \frac{g\Sigma_{II}(0)}{2} \right]^2 + \frac{1}{4} \frac{1-\alpha}{2} \\ & + O(g) + O(g\sigma), \end{aligned} \quad (6.5)$$

where  $\omega_{q_0}^2 = q_0^2 + \alpha + g\Sigma(0)$  and  $g\Sigma_{II}(0)$  is given by the relation (4.12), i.e.,  $g\Sigma_{II}(0) = gN_0 / \pi \sqrt{q_0^2 + \alpha + g\Sigma_{II}(0)}$ .

### C. $g\Sigma_{\max}$ in terms of the energy

The energy and  $g\Sigma_{\max}$  have different expressions in cases I and II [see Eq. (4.9) vs Eq. (4.20)]. However, it is important to notice that  $g\Sigma_{\max}$  has the same expression in terms of the energy both for cases I and II,

$$g\Sigma_{\max} = \sqrt{(q_0^2 + \alpha)^2 + 4 \left( \epsilon - \frac{1}{4} \frac{1-\alpha}{2} \right)} - (q_0^2 + \alpha). \quad (6.6)$$

We have verified that Eqs. (6.4)–(6.6) are valid numerically for initial particle distributions with width  $\sigma < 1$ .

## VII. EQUATION OF STATE

We derive here the equation of state (i.e., the pressure as a function of the energy) for asymptotic times. As shown below, the asymptotic equation of state depends on the initial state. We consider  $\zeta_0=0$  and  $\delta_q=0$ . Notice that  $\zeta_0=0$  and  $\dot{\zeta}_0=0$  implies  $\zeta(\tau)=0$  for all  $\tau$ .

### A. Sum rule

A further expression for the energy follows by evaluating the right-hand side of Eq. (6.1) in the  $\tau \rightarrow \infty$  limit. Using Eqs. (B10) and (B13) yields

$$\begin{aligned} \epsilon = & \frac{1}{4} \frac{1-\alpha}{2} + \int_0^\Lambda q^4 dq M_q^2(\infty) + \mathcal{M}_\infty^2 g \Sigma_\infty \\ & - \frac{1}{4} (g \Sigma_\infty)^2 + O(g). \end{aligned} \quad (7.1)$$

[The cosine terms in Eqs. (B10) and (B13) for late time are fastly oscillant in  $q$  and thus they do not contribute to the  $q$ -integral in the infinite time limit.]

Equation (7.1) allows us to express the integral over  $M_q^2(\infty)$  in terms of known quantities: the initial data and  $\mathcal{M}_\infty^2$ ,

$$\begin{aligned} \int_0^\Lambda q^4 dq M_q^2(\infty) = & \epsilon - \frac{1}{4} \frac{1-\alpha}{2} - \mathcal{M}_\infty^2 g \Sigma_\infty \\ & + \frac{1}{4} (g \Sigma_\infty)^2 + O(g). \end{aligned} \quad (7.2)$$

This sum rule holds for  $\zeta_0=0$  and  $\delta_q=0$ .

### B. Pressure and equation of state

The renormalized pressure can be written as

$$P_{\text{ren}}(\tau) = \frac{2|m_R|^4}{\lambda_R} p(\tau), \quad (7.3)$$

$$\begin{aligned} p(\tau) = & -\epsilon + \dot{\zeta}^2 + g \int_0^\Lambda q^2 dq \left[ |\dot{\varphi}_q(\tau)|^2 + \frac{q^2}{3} |\varphi_q(\tau)|^2 \right] \\ & + O(g). \end{aligned} \quad (7.4)$$

We analogously evaluate the pressure in the infinite time limit with the result

$$p_\infty = -\epsilon + \frac{4}{3} \int_0^\Lambda q^4 dq M_q^2(\infty) + \mathcal{M}_\infty^2 g \Sigma_\infty. \quad (7.5)$$

Using now the sum rule (7.2) and  $\mathcal{M}_\infty^2 = \alpha + g \Sigma_\infty$  yields

$$p_\infty = \frac{1}{3} \epsilon - \frac{\alpha}{3} g \Sigma_\infty - \frac{1}{3} \frac{1-\alpha}{2}. \quad (7.6)$$

[Recall  $\alpha = \text{sgn}(m_R^2)$ .]

We consider narrow distributions of particles centered at  $q=q_0$ . To obtain the equation of state, we make the approximation of considering all particles in a single mode with  $q=q_0$ , and we express the pressure as a function of the energy.

The equation of state is the same for case I and case II.

We have to distinguish between unbroken and broken symmetry for  $\tau=\infty$ .

(1)  $\mathcal{M}_\infty^2 > 0$

In this regime  $g \Sigma_\infty = g \Sigma_{\text{max}}/2$  with  $g \Sigma_{\text{max}}$  given by Eq. (6.6). Thus, the equation of state (7.6) becomes

$$\begin{aligned} p_\infty = & \frac{1}{3} \epsilon - \frac{\alpha}{6} \left[ \sqrt{(q_0^2 + \alpha)^2 + 4 \left( \epsilon - \frac{1}{4} \frac{1-\alpha}{2} \right)} - (q_0^2 + \alpha) \right] \\ & - \frac{1}{3} \frac{1-\alpha}{2}. \end{aligned} \quad (7.7)$$

This equation gives  $p_\infty$  as a function of  $\epsilon$  for an initial momentum distribution centered at  $q_0$ . Therefore, the equation of state explicitly depends on the initial conditions.

Let us consider some limiting cases.

In the limit,  $\epsilon - 1/4(1-\alpha)/2 \ll (q_0^2 + \alpha)^2$ , we have the equation of state

$$p_\infty = \frac{1}{3} \left( \epsilon - \frac{1}{4} \frac{1-\alpha}{2} \right) \left( 1 - \frac{\alpha}{q_0^2 + \alpha} \right) - \frac{1}{4} \frac{1-\alpha}{2}. \quad (7.8)$$

For  $\alpha = +1$  ( $m_R^2 > 0$ ) this equation reduces to

$$p_\infty = \frac{\epsilon}{3} \left( 1 - \frac{1}{q_0^2 + 1} \right), \quad (7.9)$$

and interpolates between a cold matter (for  $q_0 \ll 1$ ), and a radiation (for  $q_0 \gg 1$ ) equation of state.

In the opposite limit  $\epsilon - 1/4(1-\alpha)/2 \gg (q_0^2 + \alpha)^2 > 1$  we have a radiation type equation of state  $p_\infty = 1/3 \epsilon$ .

(2)  $\mathcal{M}_\infty^2 = 0$

This can only happen if  $\alpha = -1$  ( $m_R^2 < 0$ ).  $\mathcal{M}_\infty^2 = -1 + g \Sigma_\infty = 0$  implies  $g \Sigma_\infty = 1$  and we have a radiation type equation of state,

$$p_\infty = \frac{\epsilon}{3}. \quad (7.10)$$

The pressure is continuous at the boundary between broken and unbroken symmetry, but its derivative with respect to the energy have a discontinuity  $1/3 1/(q_0^2 + 1)$ . [See Eq. (7.10) vs Eq. (7.7).]

We see that the equation of state is determined by the energy, the momentum  $q_0$  around which the initial peak of particles is centered and the sign of the physical mass in vacuum  $m_R^2$ . But it does not depend whether we are in case I or II. (See Table III.)

## VIII. THE INFINITE TIME LIMIT

The spectral distribution does not take here the Bose-Einstein form in the infinite time limit. This was already the

TABLE III. Asymptotic equation of state in the different situations.

	$\mathcal{M}_\infty^2=0$	$m_R^2<0$	$\mathcal{M}_\infty^2>0$	$m_r^2>0$
Equation of state	$p_\infty=\frac{\epsilon}{3}$	$p_\infty=\frac{\epsilon}{3}+\frac{q_0^2-1}{6}\left[\sqrt{1+\frac{4\left(\epsilon-\frac{1}{4}\right)}{(q_0^2-1)^2}}-1\right]-\frac{1}{3}$		$p_\infty=\frac{\epsilon}{3}+\frac{q_0^2+1}{6}\left[\sqrt{1+\frac{4\epsilon}{(q_0^2+1)^2}}-1\right]$

case in Refs. [3,9–11] in the infinite  $N$  limit. It is not obvious that subdominant  $1/N$  corrections can render the spectral distribution of the Bose-Einstein type since the distribution functions in the  $N=\infty$  limit differ from Bose-Einstein distributions on an amount of order one.

On the other hand, the quantum modes do interact between themselves for all finite times in the high energy density situations considered here and in Refs. [3], [9–11]. We are in a different situation than for quantum modes around the vacuum where the modes are free in the  $N=\infty$  limit. That is, in the vacuum case equilibration cannot be achieved unless  $1/N$  contributions are included.

One may think that the lack of asymptotic thermalization is linked to the translationally invariant character of the states considered since an infinite number of conservation laws holds in such situation [12]. Then, the equilibration time scale can be infinity in the  $N=\infty$  limit. That is, it can grow with  $N$ , and being infinity for  $N=\infty$ , it cannot be recovered by  $1/N$  corrections.

However, classical calculations in a two-fields scalar field theory averaging over space-dependent plane-wave-type solutions yielded also nonthermal distributions for late times [13]. These distributions resemble qualitatively to the asymptotic distributions in the large  $N$  limit [9–11]. In the other hand, quantum calculations with inhomogeneous states for fermions in 1+1 dimensions yielded approximately Fermi-Dirac distributions [15]. However, as noticed in a different context, Boltzmann-like distributions do not necessarily imply thermal equilibrium [16].

Last but not least, a model may reach for late times a stationary situation which is nonthermal but turbulent as explicitly shown in Ref. [14]. The issue of late time thermalization or nonthermalization is clearly very important and we hope to deal with these deep problems in the future.

## IX. CORRELATION FUNCTIONS AND BOSE CONDENSATE

The equal time correlation function is given for an arbitrary time  $\tau$  by

$$\begin{aligned} & \langle \eta^a(\vec{r}, \tau) \eta^b(\vec{0}, \tau) \rangle - \langle \eta^a(\vec{r}, \tau) \rangle \langle \eta^b(\vec{0}, \tau) \rangle \\ &= \delta^{a,b} C(\vec{r}, \tau) \\ &= \delta^{a,b} \int \frac{d^3q}{2(2\pi)^3} |\varphi_q(\tau)|^2 e^{iq \cdot \vec{r}}. \end{aligned} \quad (9.1)$$

The initial conditions are specified in Eqs. (3.11)–(3.14); and as they are rotationally invariant,

$$C(r, \tau) = \frac{1}{4\pi r} \int_0^\infty q dq |\varphi_q(\tau)|^2 \sin(qr). \quad (9.2)$$

In this section we consider  $\zeta_0=0$  and  $\zeta_0 \neq 0$  with  $\zeta_0 \ll 1$ .

### A. Early time

We have the same results for  $\zeta_0=0$  and for  $\zeta_0 \ll 1$ .

#### 1. Case I

At  $\tau=0$  the correlation function is of order one, because  $|\varphi_q(0)|^2 \leq 1$ . [ $|\varphi_q(0)|^2 \sim 1$  for the nonoccupied modes and  $|\varphi_q(0)|^2 \ll 1$  for the highly occupied modes, see Eq. (4.1).] For the highly occupied modes we have  $|\dot{\varphi}_q(0)|^2 = O(1/g)$ . Thus, for early times  $\tau = O(1)$  these modes will have  $|\varphi_q(\tau)|^2 = O(1/g)$ . This makes the correlation function be of order  $1/g$  near the origin for early times, as we see in Fig. 7.

#### 2. Case II

In this case at  $\tau=0$ , the modes have  $|\varphi_q(0)|^2 = O(1/g)$ . This makes the correlation function be of order  $1/g$  near the origin for  $\tau=0$ , as we see in Figs. 10 and 12.

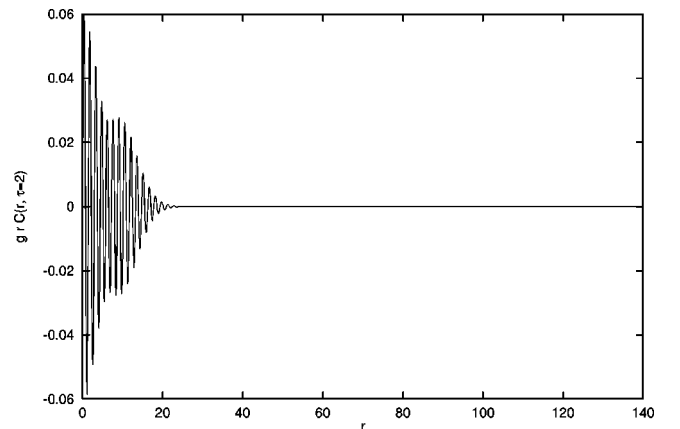


FIG. 7. Case I.  $m_R^2 > 0$ . Unbroken symmetry.  $grC(r, \tau=2)$  for  $g=10^{-7}$  and initial conditions:  $\zeta_0=0$ ,  $gN_0=250$ ,  $q_0=4.0$ , and  $\sigma=0.3$ .

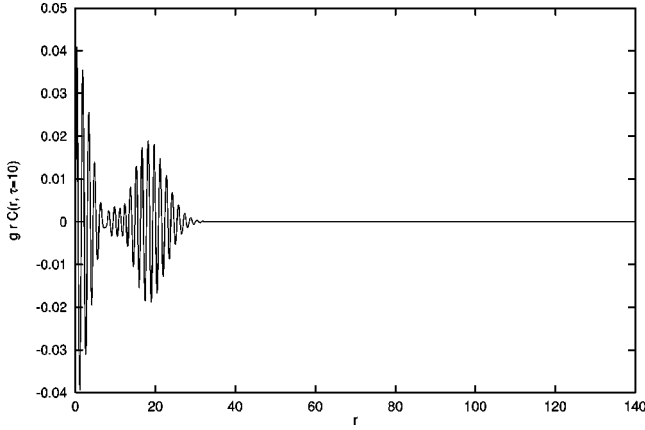


FIG. 8. Case I.  $m_R^2 > 0$ . Unbroken symmetry.  $grC(r, \tau=10)$  with the same  $g$  and initial conditions as in Fig. 7.

### B. Late time

The late time behavior of the correlation function depends whether the symmetry is dynamically broken or not. However, its behavior is the same for both cases, I and II.

We have to distinguish the two regimes.

#### 1. $\mathcal{M}_\infty^2 > 0$

In this regime we have the same results for  $\zeta_0 = 0$  and for  $\zeta_0 \ll 1$ .

We observe that at intermediate times a spherical pulse develops. This spherical pulse propagates with a constant radial speed given by the radial group velocity for  $q = q_0$  and an amplitude that decreases as  $1/r$ . The radial width of the pulse,  $L$ , remains approximately constant (see Figs. 8 and 9).

The group velocity is asymptotically given by

$$v_g = \left. \frac{d\omega_{q^\infty}}{dq} \right|_{q=q_0} = \left. \frac{d}{dq} \sqrt{q^2 + \mathcal{M}_\infty^2} \right|_{q=q_0} = \frac{q_0}{\sqrt{q_0^2 + \mathcal{M}_\infty^2}}. \quad (9.3)$$

We recall that here  $\mathcal{M}_\infty^2 = \alpha + g \Sigma_{\max}/2 > 1$  [see Eqs. (5.2) and (5.4)]. [ $\alpha = \text{sgn}(m_R^2) = \pm 1$ ].

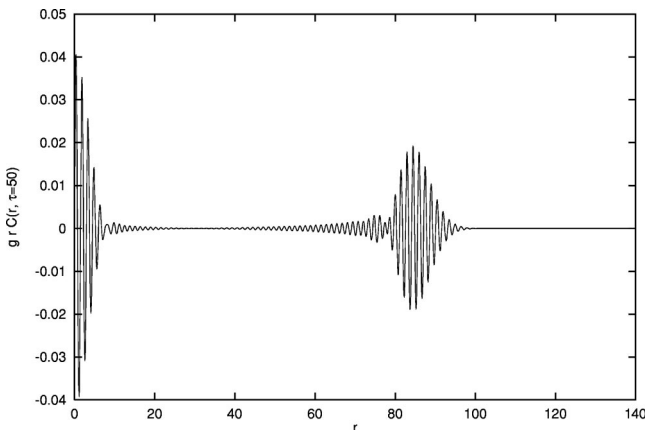


FIG. 9. Case I.  $m_R^2 > 0$ . Unbroken symmetry.  $grC(r, \tau=50)$  with the same  $g$  and initial conditions as in Fig. 7.

Asymptotically, the correlation function becomes the sum of two terms

$$C(r, \tau) = C_{\text{origin}}(r) + C_p(r, \tau), \quad (9.4)$$

where  $C_{\text{origin}}(r)$  is the correlation function near the origin. This term is asymptotically time-independent.

The pulse contribution to the correlation function,  $C_p(r, \tau)$ , has approximately the asymptotic form

$$C_p(r, \tau) = \frac{1}{gr} P(r - 2v_g \tau - c). \quad (9.5)$$

Here  $c$  is a constant of order one, and  $P(u)$  is of the order  $\mathcal{O}(1)$  only for  $-L/2 < u < L/2$  where  $L$  is the width of the pulse (i.e., the pulse is localized around  $r \approx 2v_g \tau + c$ ).

The pulse term is due to the particles in the initial distribution that effectively propagate as free particles. This is so since the effective mass in the mode equations (3.2) becomes asymptotically constant and hence the modes effectively decouple from each other. The pulse term is absent when there are no particles in the initial state [11].

In summary, the correlator is of order  $\mathcal{O}(1/g) \gg 1$  for  $2v_g \tau + c - L/2 < r < 2v_g \tau + c + L/2$  whereas causality makes it to fall to  $\mathcal{O}(1)$  values for  $r > 2v_g \tau + c + L/2$ .

#### 2. $\mathcal{M}_\infty^2 = 0$

In this regime the symmetry is broken and there were spinodal resonances for earlier times.

Since the effective mass vanishes asymptotically the mode with  $q = 0$  behaves as

$$\varphi_0(\tau) = \tau^{-\infty} L + K \tau. \quad (9.6)$$

The Wronskian guarantees that neither of the complex coefficients  $L, K$  can vanish [11]. This linear growth with time can be interpreted as an out of equilibrium novel form of Bose Einstein condensation. We analyze below the contribution of this condensate  $C_s(r, \tau)$  to the correlation function.

We have studied both  $\zeta_0 = 0$  and  $\zeta_0 \neq 0$  (with  $\zeta_0 \ll 1$ ).

(a)  $\zeta_0 = 0$ . For late time ( $\tau \gg \tau_s$ ), the particles created due to the spinodal resonance contribute to the correlation function with a term  $C_s(r, \tau)$  of order  $1/g$  when  $r$  is in the interval  $(0, 2\tau)$ . This term decays as  $1/r$ .

There is in addition a pulse term,  $C_p(r, \tau)$ , in the correlator that moves away from the origin with unit velocity [remember Eq. (9.3) and that  $\mathcal{M}_\infty^2 = 0$ ]. See Figs. 10 and 11.

Thus, the correlation function is asymptotically given by

$$C(r, \tau) = C_{\text{origin}}(r) + C_s(r, \tau) + C_p(r, \tau). \quad (9.7)$$

Here,  $C_{\text{origin}}(r)$ , the correlation near the origin is asymptotically time-independent.

The contribution of the pulse,  $C_p(r, \tau)$ , has the form

$$C_p(r, \tau) = \frac{1}{gr} P(r - 2\tau - c). \quad (9.8)$$

The contribution from the particles created by spinodal resonance,  $C_s(r, \tau)$ , has the form

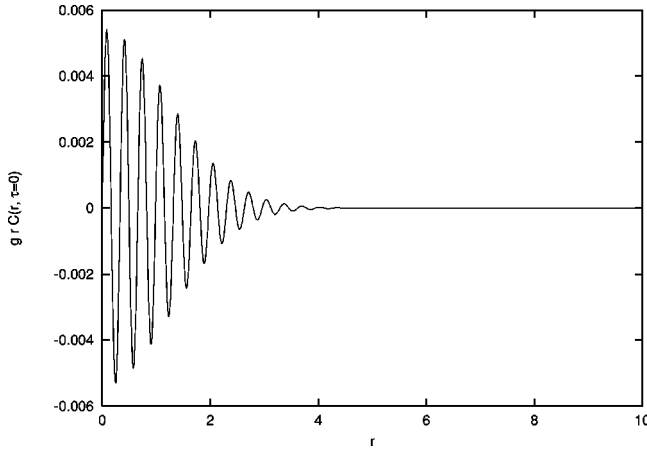


FIG. 10. Case II.  $m_R^2 < 0$ . Dynamically broken symmetry.  $grC(r, \tau=0)$  for  $g=10^{-7}$  and initial conditions:  $\zeta_0=0$ ,  $gN_0=4.478$ ,  $q_0=1.3083$ , and  $\sigma=0.07850$ . For  $r>5$ , we have  $grC(r, \tau=0) \approx 0$ .

$$C_s(r, \tau) = \frac{K}{gr} Q\left(\frac{r}{2\tau}\right), \quad (9.9)$$

where  $K$  is a constant and  $Q(u) = \theta(1-u)$ .

Introducing the dynamical correlation length  $\xi(\tau) \sim 2\tau$  and defining the variable  $u = r/\xi(\tau)$ ,  $C_s(r, \tau)$  can be written in the form

$$C_s(r, \tau) = \frac{K}{gu\xi(\tau)} \theta(1-u). \quad (9.10)$$

Using the customary notation for the scaling regime

$$C_s(r, \tau) = \frac{1}{[\xi(\tau)]^{2(1-z)}} I(u), \quad (9.11)$$

with the anomalous dynamical exponent  $z=1/2$  (the naive scaling length dimension of the field is 1). In this case the scaling function is given by

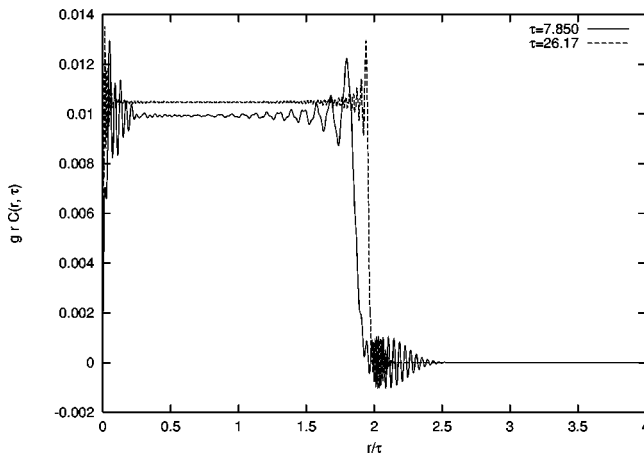


FIG. 11. Case II.  $m_R^2 < 0$ . Dynamically broken symmetry.  $grC(r, \tau)$  for  $\tau=7.850$  and  $\tau=26.17$  with the same  $g$  and initial conditions as in Fig. 10. We see that for late times  $grC_s(r, \tau) \approx K\theta(1-r/2\tau)$  [see Eq. (9.9)].

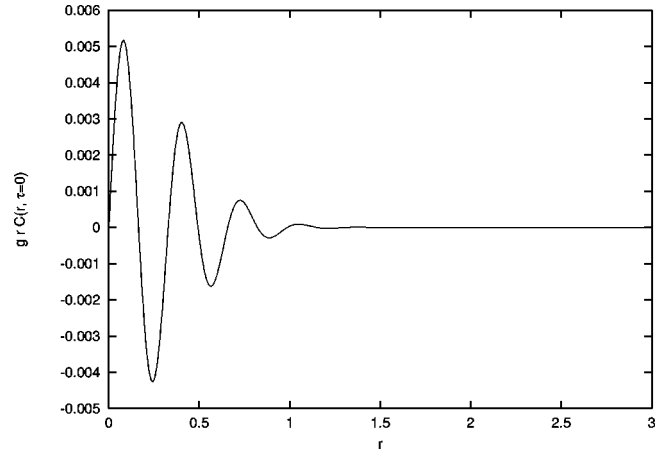


FIG. 12. Case II.  $m_R^2 < 0$ . Dynamically broken symmetry.  $grC(r, \tau=0)$  for  $g=10^{-7}$  and initial conditions:  $\zeta_0=0.2617 \times 10^{-5}$ ,  $gN_0=4.478$ ,  $q_0=1.3083$ , and  $\sigma=0.2617$ . For  $r>1.5$ , we have  $grC(r, \tau=0) \approx 0$ .

$$I(u) = \frac{K}{gu} \theta(1-u). \quad (9.12)$$

An analogous spinodal term  $C_s(r, \tau)$  in the correlator have been obtained for initially broken symmetry and no particles in the initial state in [11].

The spinodal term  $C_s(r, \tau)$  can be interpreted as follows. For times  $\tau$  later than  $\tau_s$  there is a zero momentum condensate formed by Goldstone bosons travelling at the speed of light and back-to-back. That is, massless particles emitted from the points  $(0, \tau)$  and  $(r, \tau)$  form propagating fronts which at time  $\tau$  are at a distance  $\tau - \tau_s$  from the origin and from  $r$ , respectively. These space-time points are causally connected for  $2(\tau - \tau_s) \geq r$ . Otherwise,  $C_s(r, \tau)$  is not of order  $1/g$  but of order one.

An alternative interpretation of the causality step function goes as follows. Signals are emitted at the speed of light from all points in the condensate. We have causal connection between the points 0 and  $r$  once signals starting from a given point arrive to both points. The earlier this happens is for the signals emitted from the half-away point at  $r/2$ . These signals need a time  $\tau = r/2$  to reach both points. Hence, the correlator is of the order  $1/g$  for  $2\tau > r$ .

Moreover, the analytical derivation of Eq. (9.12) from the low- $q$  behavior of the mode functions given in Ref. [11] also applies here. Notice that the scaling contribution of the Goldstone bosons to the correlator is different from a free massless propagator. We have here a  $1/r$  falloff whereas a free massless scalar field has a  $1/r^2$  falloff.

(b)  $\zeta_0 \neq 0$  (with  $\zeta_0 \ll 1$ ). For late time ( $\tau \gg \tau_s$ ), the particles created by spinodal resonance give a contribution  $C_s(r, \tau)$  of order  $1/g$  to the correlation function in the interval  $0 < r < 2\tau$ . See Figs. 12–14.

The correlation function is asymptotically given by

$$C(r, \tau) = C_{\text{origin}}(r) + C_s(r, \tau) + C_p(r, \tau), \quad (9.13)$$

where the time-independent piece  $C_{\text{origin}}(r)$  is the correlation near the origin.  $C_p(r, \tau)$  is the contribution of the pulse,

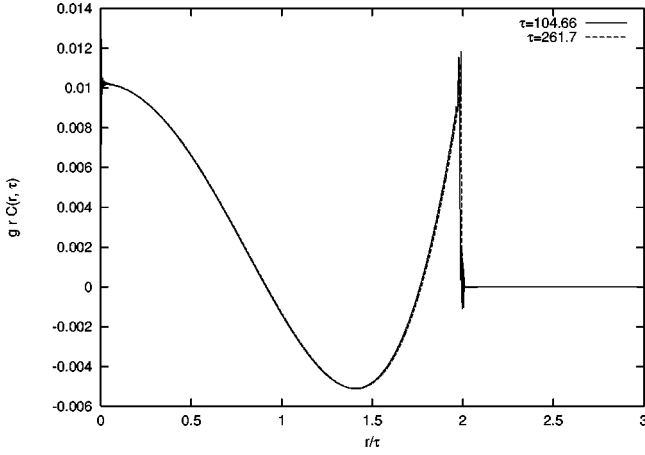


FIG. 13. Case II.  $m_R^2 < 0$ . Dynamically broken symmetry.  $grC(r, \tau)$  for  $\tau = 104.66$  and  $\tau = 261.7$  with the same  $g$  and initial conditions as in Fig. 12. For times  $\tau$  such that  $\zeta(\tau) \approx 0$  we see that  $grC_s(r, \tau) \approx KQ_\tau(r/2\tau)$  [see Eq. (9.15)].

$$C_p(r, \tau) = \frac{1}{gr} P(r - 2\tau - c), \quad (9.14)$$

and  $C_s(r, \tau)$  is the contribution from the particles created by spinodal resonance,

$$C_s(r, \tau) = \frac{1}{[\xi(\tau)]^{2(1-z)}} I_r(u), \quad (9.15)$$

with  $\xi(\tau) \sim 2\tau$  the correlation length,  $u = r/\xi(\tau)$ , and  $z = 1/2$ .

$I_r(u)$  is no longer given by Eq. (9.12) when  $\zeta_0 \neq 0$ .  $Q_\tau(u) = uI_r(u)$  now oscillates with  $u$ . At a given time  $\tau$ , the number of oscillations of  $Q_\tau(u)$  in the interval  $0 < u < 1$  equals the number of oscillations performed by the order parameter  $\zeta(\tau)$  from time  $\tau = 0$  till time  $\tau$ . That is, the scaling exists for  $\zeta_0 \neq 0$  in a generalized sense since the function

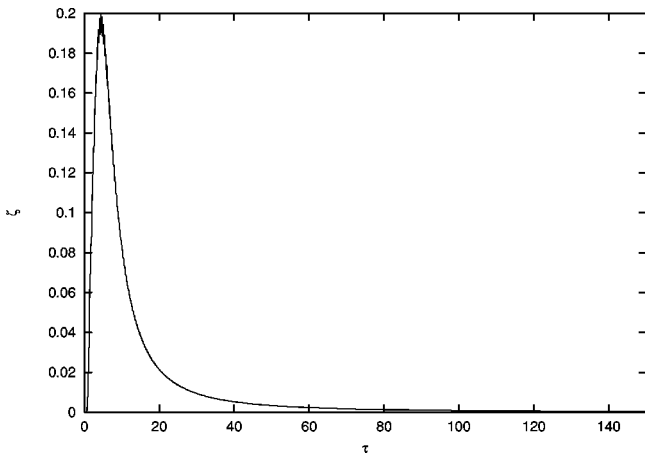


FIG. 14. Case II.  $m_R^2 < 0$ . Dynamically broken symmetry.  $\zeta(\tau)$  with the same  $g$  and initial conditions as in Fig. 12. For  $\tau \in [100, 262]$ ,  $\zeta(\tau) \approx 0$ .

$Q_\tau(u)$  changes each time  $\zeta(\tau)$  performs an oscillation. This is due to the appearance of the extra length scale  $\zeta_0$ . See Figs. 12–14.

As for unbroken symmetry, the pulse term is due to the particles in the initial distribution that effectively propagate as free particles. Both for  $\zeta_0 = 0$  and  $\zeta_0 \neq 0$  causality makes the correlator of order  $\mathcal{O}(1)$  for  $r > 2\tau + c$ . For  $r < 2\tau + c$  the correlations are of order  $\mathcal{O}(1/g) \gg 1$ .

## X. SUMMARY AND OVERVIEW

We have studied the time evolution of strongly out of equilibrium states in QFT of the tsunami type. That is, shell-type distributions of particles in momentum space. We work in the  $(\vec{\Phi}^2)^2$  theory in the large  $N$  limit.

We thoroughly explore the dynamics for all times as a function of the initial state and of the tree level mass in the vacuum. We concentrate on the dependence on initial number of particles for a thin shell momentum distribution. Besides solving numerically the time evolution we found novel analytic solutions for early times in the thin shell limit. These solutions express in closed form in terms of elliptic functions and they are valid till decoherence effects become important. We show in this way the presence of a whole variety of possible evolution patterns including symmetry breaking and/or unbreaking or restoring of symmetry as depicted in Tables I and II. The driving mechanism in the symmetry changes during time evolution is particle production through spinodal instabilities and/or decoherence of the quantum modes.

The rich manifold of physical evolutions uncovered here provide novel and nonperturbative mechanisms for particle production and relaxation that are out of thermodynamic equilibrium and cannot be described in the early stages by a hydrodynamic evolution. These new mechanisms could be of potential interest in the physics of ultrarelativistic heavy ion collisions.

Especially interesting in this context is the emergence of generalized scaling behaviors in the late times correlation functions as a landmark for the abundant production of Goldstone bosons out of equilibrium (due to the breaking of the symmetry and the vanishing of the effective mass for asymptotic times). These new scaling behaviors generalize the standard scaling found in previous works [11] and may constitute physical signals for the presence of massless Goldstone bosons produced through spinodal instabilities in these out of equilibrium situations.

Although our investigations with translationally invariant states clearly show a rich variety of physical phenomena, the necessity of more realistic states to describe collisions is fairly obvious. We defer for future research the very intensive numerical work needed to deal with such problems.

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### APPENDIX A: ADIABATIC APPROXIMATION FOR THE MODES

In this Appendix we use the adiabatic form of the modes to evaluate the quantum fluctuations  $\Sigma(\tau)$ . The modes  $\varphi_q(\tau)$  can be represented as [7]

$$\varphi_q(\tau) = \frac{1}{\sqrt{\mathcal{P}_q(\tau)}} [a_q e^{-i\int_0^\tau dx \mathcal{P}_q(x)} + b_q e^{i\int_0^\tau dx \mathcal{P}_q(x)}], \quad (\text{A1})$$

where  $a_q$  and  $b_q$  are constants and  $\mathcal{P}_q(\tau)$  depends on time. Inserting Eq. (A1) into Eq. (3.8) yields the following nonlinear differential equation for  $\mathcal{P}_q(\tau)$ :

$$\frac{\ddot{\mathcal{P}}_q(\tau)}{2\mathcal{P}_q(\tau)} - \frac{3}{4} \left( \frac{\dot{\mathcal{P}}_q(\tau)}{\mathcal{P}_q(\tau)} \right)^2 + \mathcal{P}_q^2(\tau) = q^2 + \mathcal{M}^2(\tau). \quad (\text{A2})$$

The initial conditions (3.12) combined with the Wronskian conservation yields

$$|a_q|^2 - |b_q|^2 = 1. \quad (\text{A3})$$

As long as  $\mathcal{P}_q(\tau)$  is a real function, we have for the squared modulus

$$|\varphi_q(\tau)|^2 = \frac{1}{\mathcal{P}_q(\tau)} \left[ |a_q|^2 + |b_q|^2 + 2|a_q b_q| \times \cos \left( 2 \int_0^\tau dx \mathcal{P}_q(x) + \alpha_q \right) \right], \quad (\text{A4})$$

where  $\alpha_q$  is a time independent phase.

When  $q$  belongs to a parametric resonant band  $\mathcal{P}_q(\tau)$  gets an imaginary part. We consider here the case where such instabilities have a negligible effect.

We are interested in the modes with large amplitudes  $|a_q| \gg 1, |b_q| \gg 1$ . That is, those in an interval of width  $\sigma$  around the peak momentum  $q_0$ . For such modes, thanks to Eq. (A3) we can approximate  $|a_q| \simeq |b_q|$ .

Inserting Eq. (A4) into the integral (3.15) for  $\Sigma(\tau)$ , and approximating the slowly varying factor  $1/\mathcal{P}_q(\tau)$  by its average on a period (that we will denote  $1/\hat{\mathcal{P}}_q$ ) yields

$$\Sigma(\tau) \simeq 2 \int \frac{q^2 dq}{\hat{\mathcal{P}}_q} |a_q|^2 \left[ 1 + \cos \left( 2 \int_0^\tau dx \mathcal{P}_q(x) + \alpha_q \right) \right]. \quad (\text{A5})$$

It follows from here the bounds

$$0 \leq \Sigma(\tau) \leq 4 \int \frac{q^2 dq}{\hat{\mathcal{P}}_q} |a_q|^2 \equiv \Sigma_{\max}.$$

For late times the integral of the oscillating cosinus in Eq. (A5) vanishes. Therefore,

$$\Sigma(\infty) = 2 \int \frac{q^2 dq}{\hat{\mathcal{P}}_q} |a_q|^2$$

and

$$\Sigma(\infty) = \frac{1}{2} \Sigma_{\max}. \quad (\text{A6})$$

### APPENDIX B: SLOWLY VARYING PARAMETERS IN THE LATE TIME MODE FUNCTIONS

Let us define the following slowly varying parameters [10]:

$$A_q(\tau) \equiv \frac{1}{2} e^{-i\omega_{q\infty}\tau} \left[ \varphi_q(\tau) - \frac{i}{\omega_{q\infty}} \dot{\varphi}_q(\tau) \right], \quad (\text{B1})$$

$$B_q(\tau) \equiv \frac{1}{2} e^{+i\omega_{q\infty}\tau} \left[ \varphi_q(\tau) + \frac{i}{\omega_{q\infty}} \dot{\varphi}_q(\tau) \right], \quad (\text{B2})$$

with

$$\omega_{q\infty} \equiv \sqrt{q^2 + \mathcal{M}_{\infty}^2}, \quad \mathcal{M}_{\infty}^2 \equiv \mathcal{M}^2(\infty). \quad (\text{B3})$$

These slowly varying parameters are asymptotically constant.

We can express the mode functions in terms of them as follows:

$$\varphi_q(\tau) = A_q(\tau) e^{i\omega_{q\infty}\tau} + B_q(\tau) e^{-i\omega_{q\infty}\tau}. \quad (\text{B4})$$

Thus the squared modulus of the modes is

$$|\varphi_q(\tau)|^2 = |A_q(\tau)|^2 + |B_q(\tau)|^2 + 2|A_q(\tau)B_q(\tau)| \times \cos[2\omega_{q\infty}\tau + \phi_q(\tau)], \quad (\text{B5})$$

where

$$A_q(\tau)B_q(\tau)^* = |A_q(\tau)B_q(\tau)| e^{i\phi_q(\tau)}. \quad (\text{B6})$$

The constancy of the Wronskian implies

$$|B_q(\tau)|^2 - |A_q(\tau)|^2 = \frac{1}{\omega_{q\infty}}, \quad (\text{B7})$$

plus terms with derivatives of the slowly varying parameters that vanish asymptotically.

We define a slowly varying modulus [10],

$$M_q(\tau) \equiv \sqrt{g} \sqrt{|A_q(\tau)|^2 + |B_q(\tau)|^2}. \quad (\text{B8})$$

The main contributions to the physical quantities comes from the modes with occupation numbers of order  $1/g$ , for these modes  $|A_q(\tau)|$  and  $|B_q(\tau)|$  are of order  $1/\sqrt{g}$ . Therefore,  $M_q(\tau)$  is of order 1 for these modes. Moreover, Eq. (B7) implies that

$$|B_q(\tau)|^2 = |A_q(\tau)|^2 [1 + O(g)], \quad (\text{B9})$$

and we can thus approximate the squared modulus as follows:

$$g|\varphi_q(\tau)|^2 = M_q(\tau)^2 \{1 + \cos[2\omega_{q\infty}\tau + \phi_q(\tau)]\} [1 + O(g)]. \quad (\text{B10})$$

We obtain from Eq. (B4) a similar formula for the squared modulus of the derivative of the mode function,

$$\begin{aligned} \dot{\phi}_q(\tau) &= i\omega_{q\infty}[A_q(\tau)e^{i\omega_{q\infty}\tau} - B_q(\tau)e^{-i\omega_{q\infty}\tau}] \\ &\quad + \dot{A}_q(\tau)e^{i\omega_{q\infty}\tau} + \dot{B}_q(\tau)e^{-i\omega_{q\infty}\tau}. \end{aligned} \quad (\text{B11})$$

Using  $\dot{A}_q(\infty) = \dot{B}_q(\infty) = 0$  and the same procedure we have used to derive the equation for the squared modulus, we obtain

$$\begin{aligned} g|\dot{\phi}_q(\tau)|^2 &= \omega_{q\infty}^2 M_q^2(\infty) \{1 - \cos[2\omega_{q\infty}\tau + \phi_q(\tau)]\} \\ &\quad \times [1 + O(g)][1 + O(\tau^{-1})]. \end{aligned} \quad (\text{B12})$$

Or equivalently,

$$\begin{aligned} g|\dot{\phi}_q(\tau)|^2 &= \omega_{q\infty}^2 \{g|\phi_q(\tau)|^2 - 2M_q^2(\tau)\cos[2\omega_{q\infty}\tau + \phi_q(\tau)]\} \\ &\quad \times [1 + O(g)][1 + O(\tau^{-1})]. \end{aligned} \quad (\text{B13})$$

### APPENDIX C: CALCULATION OF THE SPINODAL TIME IN THE PRESENCE OF THE TSUNAMI

Here, we obtain the early time solution for the modes in the spinodally resonant band, and we estimate the spinodal time,  $\tau_s$ , both for cases I and II. Recall that there is spinodal resonance provided  $m_R^2 < 0$  and  $g\Sigma_{\max} < 2$ .

Let us assume that the time scale for the development of spinodal instabilities,  $\tau_s$ , is longer than the time scale of damping of the oscillations in  $g\Sigma(\tau)$ ,  $\tau_d$ . This happens when the particle peak is wide enough.

Before entering on the calculation of  $\tau_s$ , let us remark that the numerical calculations show that the main effect of  $g\Sigma(\tau)$  for  $\tau < \tau_s$  (and small  $g$ ) is giving a constant positive term  $g\Sigma_{\max}/2$  to the effective squared mass. In fact, this turns to be true in general, not only when the oscillations of  $g\Sigma(\tau)$  are damped before  $\tau_s$ .

We have for  $\tau_d < \tau < \tau_s$

$$-\mu^2 \equiv M_{\text{eff}}^2(\tau) \approx 1 + \frac{g\Sigma_{\max}}{2} < 0. \quad (\text{C1})$$

Hence, an approximate equation for the modes reads

$$\left(\frac{d^2}{d\tau^2} + q^2 - \mu^2\right)\phi_q(\tau) = 0. \quad (\text{C2})$$

Thus, the modes with  $q$  in the interval between 0 and  $\mu$  are spinodally resonant.

We consider initial peaks of particles centered well outside the possible resonant bands. Thus, for a sufficiently narrow initial particle peak there are no particles in the spinodally resonant band. Therefore the initial condition for modes in the resonant band are

$$\begin{aligned} \phi_q(0) &= \frac{1}{\sqrt{\Omega_q}} = (q^2 + |\mathcal{M}^2(0)|)^{-1/4}, \\ \dot{\phi}_q(0) &= -i\sqrt{\Omega_q} = -i(q^2 + |\mathcal{M}^2(0)|)^{1/4}. \end{aligned} \quad (\text{C3})$$

The solution of Eq. (C1) for these modes is

$$\begin{aligned} \phi_q(\tau) &= \frac{1}{2\sqrt{\mu^2 - q^2}(q^2 + |\mathcal{M}^2(0)|)^{1/4}} \\ &\quad \times [(\sqrt{\mu^2 - q^2} - i\sqrt{q^2 + |\mathcal{M}^2(0)|})e^{\tau\sqrt{\mu^2 - q^2}} \\ &\quad + (\sqrt{\mu^2 - q^2} + i\sqrt{q^2 + |\mathcal{M}^2(0)|})e^{-\tau\sqrt{\mu^2 - q^2}}]. \end{aligned} \quad (\text{C4})$$

We thus obtain for the squared modulus neglecting the exponentially decreasing term

$$\begin{aligned} |\phi_q(\tau)|^2 &\approx \frac{1 + \mu^2}{4(\mu^2 - q^2)\sqrt{q^2 + |\mathcal{M}^2(0)|}} e^{2\tau\sqrt{\mu^2 - q^2}} \\ &= \frac{1 + \mu^2}{4\mu^2\left(1 - \frac{q^2}{\mu^2}\right)\sqrt{q^2 + |\mathcal{M}^2(0)|}} e^{2\tau\mu^2\sqrt{1 - q^2/\mu^2}}. \end{aligned} \quad (\text{C5})$$

The contribution of the spinodal band to  $g\Sigma(\tau)$  is given by

$$\Sigma_s(\tau) = \int_0^\mu q^2 dq |\phi_q(\tau)|^2. \quad (\text{C6})$$

Inserting Eq. (C5) into Eq. (C6) we obtain an estimation for the spinodal growth of the quantum fluctuations.

To approximately evaluate this integral, we can make further simplifications in Eq. (C5). As  $q/\mu < 1$  and the contribution of the modes with  $q \approx \mu$  is exponentially suppressed, we can expand in  $q/\mu$  to second order in the exponential and to zeroth order in the factor outside the exponential:

$$|\phi_q(\tau)|^2 \approx \frac{1 + \mu^2}{4\mu^2\sqrt{q^2 + |\mathcal{M}^2(0)|}} e^{2\tau\mu} e^{-\tau(q^2/\mu^2)}. \quad (\text{C7})$$

In addition the integrand has its maximum at  $q = O(0.1\mu)$  and  $0 < \mu < 1$ . Therefore we can do the approximations  $1 + \mu^2 \sim 1$  and  $\sqrt{q^2 + |\mathcal{M}^2(0)|} \sim 1$  (for both case I and case II). Thus,

$$|\phi_q(\tau)|^2 \approx \frac{1}{4\mu^2} e^{2\tau\mu} e^{-\tau(q^2/\mu^2)}. \quad (\text{C8})$$

Then the integral over  $q$  takes the value

$$\Sigma_s(\tau) \approx \frac{\sqrt{\pi}\mu}{16} \frac{e^{2\tau\mu}}{\tau^{3/2}}. \quad (\text{C9})$$

The spinodal time  $\tau_s$  is by definition, the time where the instabilities are shut off for all  $0 \leq q \leq \mu$ . This happens when the spinodal modes contribution  $\Sigma_s(\tau)$  compensates the initial (negative) value of  $M_{\text{eff}}^2(\tau)$  [see Eq. (C1)]:

$$g\Sigma_s(\tau_s) \approx \mu^2. \quad (\text{C10})$$

Therefore  $\tau_s$  is given by the following implicit equation:

$$\tau_s = \frac{1}{2\mu} \log \left[ \frac{16}{g\sqrt{\pi\mu}} \right] + \frac{3}{4\mu} \log(\mu\tau_s). \quad (\text{C11})$$

The spinodal times given by this equation are in good agreement with the numerical results.

#### APPENDIX D: THE INITIAL QUANTUM FLUCTUATIONS IN CASE II

The quantum fluctuations at initial time are related with the initial data through Eq. (4.12)

$$[g\Sigma_{II}(0)]^3 + (q_0^2 + \alpha)[g\Sigma_{II}(0)]^2 - \left(\frac{gN_0}{\pi}\right)^2 = 0.$$

For  $gN_0/2\pi > [(q_0^2 + \alpha)/3]^{3/2}$  the positive solution of this equation takes the form

$$g\Sigma_{II}(0) = -\frac{q_0^2 + \alpha}{3} + s_+ + s_-,$$

where

$$s_{\pm} = \frac{1}{2^{1/3}} \left(\frac{gN_0}{\pi}\right)^{2/3} \left[ 1 - \frac{1}{2} \left(\frac{2\pi}{gN_0}\right)^2 \left(\frac{q_0^2 + \alpha}{3}\right)^3 \pm \sqrt{1 - \left(\frac{2\pi}{gN_0}\right)^2 \left(\frac{q_0^2 + \alpha}{3}\right)^3} \right]^{1/3}.$$

For  $gN_0/2\pi < [(q_0^2 + \alpha)/3]^{3/2}$  the positive solution can be written as

$$g\Sigma_{II}(0) = \frac{q_0^2 + \alpha}{3} (2 \cos \beta - 1),$$

with

$$\cos 3\beta = 2 \left(\frac{3}{q_0^2 + \alpha}\right)^3 \left(\frac{gN_0}{2\pi}\right)^2 - 1.$$

In limiting cases we recover Eq. (4.14).

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