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Laminaciones por superficies de Riemann en superficies Kähler

Laminations by Riemann surfaces in Kähler surfaces

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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LAMINACIONES POR SUPERFICIES DE RIEMANN EN SUPERFICIES KÄHLER

LAMINATIONS BY RIEMANN SURFACES IN KÄHLER SURFACES

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Memoria presentada para optar al grado de
Doctor en Ciencias Matemáticas por

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“Wildness, Ed. We’re running out of it, even up here in Alaska.
People need to be reminded that the world is unsafe and unpredictable,
and at a moment’s notice, they could lose everything, like that.
I do it to remind them that chaos is always out there,
lurking beyond the horizon. That, plus, sometimes,
Ed, sometimes you have to do something bad,
just to know you’re alive.”

Northern Exposure

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En algún sitio escuché que cada uno de nosotros somos el promedio de la gente que nos rodea (¡somos armónicos!) y, por fortuna para mí, me rodea gente espectacular.

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Y ahora, si se me permite, voy a pasar a hablar directamente con el interlocutor. Vosotros, sí, vosotros que habéis buscado con ahínco este párrafo: Luis, Laura, Fonsi, Silvia, Simone, Alba, Giovanni, Marta, Nacho, Johnny, Blanca, Javi, Alvarito, Diego, Espe, Ali, Quesada, Andrea, Roger y demás alumnos de doctorado de la facultad. Había pensado escribir aquí un compendio de algunas situaciones cómico-festivas-laborales que pasamos juntos, pero esos recuerdos los tenemos todos y son cosas que ya sabéis, así que os voy a decir algo que, quizá no sepáis: sois grandes, sois muy grandes, enormes. Como amigos y como matemáticos. Y yo, de mayor, quiero ser como vosotros.

No puedo, ni tampoco quiero, olvidarme de dar las gracias a mi familia: a mis padres, Javier y Loli, por todos los sacrificios que tuvieron que hacer para criar seis hijos; a mis hermanos mayores, Javi, María, Raquel e Irene, por cuidarnos a los dos últimos cuando éramos enanos y a Marta, mi hermana pequeña, por ser mi asidua compañera de castigos.

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0.1 Resumen

0.1.1 Introducción

Uno de los objetos de estudio más importantes de las matemáticas modernas son las ecuaciones diferenciales. Este tipo de ecuaciones pueden modelizar desde el crecimiento de la población de una determinada especie a los movimientos de los planetas. Son, de hecho, una de las piedras angulares de las ciencias y de las matemáticas.

Aunque hubo numerososos matemáticos anteriormente, podríamos decir que fue alrededor de 1900 cuando el estudio de las ecuaciones diferenciales alcanzó la importancia que disfruta a día de hoy. Fue debido al nuevo enfoque desarrollado por Poincaré quién introdujo técnicas y argumentos topológicos en el estudio de las ecuaciones, dejando a un lado la búsqueda de soluciones exactas y centrándose en los aspectos cualitativos. De hecho, quizá el ejemplo más famoso de este tipo de enfoque es el Teorema de Poincaré-Bendixson que clasifica los posibles límites de órbitas acotadas en ecuaciones diferenciales autónomas en $\mathbb{R}^2$. Se acumulan hacia singularidades, órbitas heteroclínicas o ciclos límites.

Aunque las ecuaciones diferenciales han sido ampliamente estudiadas, aún hay varias preguntas muy naturales sin respuesta. Quizá, la más interesante sea el problema número 16 de Hilbert, que se pregunta sobre la acotación del número de ciclos límite que puede tener un campo vectorial polinomial en $\mathbb{R}^2$. Écalle e Ilyashenko probaron que este número es finito, pero la cuestión de la existencia de una cota uniforme sobre el número de ciclos límites para campos vectoriales polinomiales de un grado fijado, sigue abierta.

Estos campos vectoriales polinomiales en $\mathbb{R}^2$ pueden verse como campos polinomiales en $\mathbb{C}^2$, y éstos, a su vez, como restricciones a una vista afín de un campo vectorial en $\mathbb{P}^2$. Ahora, las órbitas han dejado de ser curvas reales, son curvas complejas, es decir superficies de Riemann. Diremos que tenemos una foliación por superficies de Riemann de $\mathbb{P}^2$.

Así que, en este contexto, un análogo a tener un ciclo límite sería tener una curva cerrada invariante sin singularidades. Desafortunadamente, el Teorema del Índice de Camacho-Sad [CS82] implica que debemos tener, al menos, un punto singular en esta curva. Por tanto, debemos relajar nuestras exigencias y, en lugar de buscar una curva invariante, simplemente pediremos un conjunto cerrado invariante. Este conjunto tendría estructura de laminación por superficies de Riemann. Hasta el momento, no se sabe si existen este tipo de conjuntos en $\mathbb{P}^2$. Es lo que se conoce como el problema del minimal excepcional. La primera vez que fue estudiado en su forma moderna fue en el artículo de Camacho, Lins-Neto y Sad [CLNS92]. El problema análogo para foliaciones de codimensión uno en $\mathbb{P}^n$ con $n \geq 3$ fue resuelto por Lins-Neto en [LN99] donde probó que no existen estos conjuntos.

Éstos pueden ser unos buenos motivos para estudiar las laminaciones por superficies de Riemann, pero no son los únicos. El lector puede consultar el survey de Ghys [Ghy99] para ver diferentes ejemplos de laminaciones construidas desde otros contextos que muestran la importancia que juegan las laminaciones en ciertos sistemas dinámicos.

Del mismo modo que hay diferentes contextos donde aparecen las laminaciones por superficies de Riemann, hay muchas formas diferentes de estudiarlas, y multitud de aspectos que comprender.

0.1.2 Objetivos

El problema de intentar encontrar un embedding de una laminación en algún espacio ha sido muy estudiado. En este sentido, podemos destacar el trabajo de Deroin en [Dem], donde el autor es capaz de embeber una laminación por superficies de Riemann sin ciclos evanescentes (ver el artículo de Sullivan [Sul76]) en un espacio proyectivo de dimensión N, con N suficientemente grande. En el mismo sentido, Fornæss, Sibony y Wold prueban en [FSW11] que un limite proyectivo de variedades complejas de dimensión n puede ser embebido en $\mathbb{P}^{2n+1}$. De hecho, constuir laminaciones por límites proyectivos resulta ser especialmente importante, ya que, Alcalde-Cuesta, Lozano-Rojo y Macho-Stadler, en [ACLRMS11], prueban que bajo unas condiciones bastante generales, las laminaciones con un Cantor en la transversal siempre se pueden construir como límites proyectivos.

En los dos casos mencionados, partimos de una laminación dada, y queremos embeberla en estos espacios. Sin embargo, podríamos considerar el razonamiento contrario. Es decir, dada una variedad, queremos saber cómo son las laminaciones embebidas en ella. El primer paso es estudiar las foliaciones de estas variedades. Hemos indicado anteriormente que las laminaciones en $\mathbb{P}^2$ tienen singularidades, pero eso no es cierto al estudiar otras variedades. Por ejemplo, Ghys clasifica en [Ghy96] las foliaciones de codimensión uno sin singularidades en variedades homogéneas.

Otro problema interesante es averiguar si se pueden asociar medidas a la laminación y qué tipos de medidas serían. El primer intento que uno puede hacer en este sentido es intentar definir una medida transversal invariante, sin embargo, las laminaciones que las admiten son bastante escasas. Pero, afortunadamente, si relajamos nuestras expectativas, siempre podemos encontrar una medida armónica. Este resultado fue probado por Garnett en [Gar83] para foliaciones sin singularidades, y por Berndtsson y Sibony en [BS02] cuando el conjunto de singularidades de una laminación tiene dimensión de Hausdorff menor o igual que 2.

Sin embargo, una vez que la existencia está asegurada, es importante averiguar la unicidad. Ésta no es trivial y depende mucho de la foliación que estemos considerando. Por ejemplo, Lozano-Rojo, en [LR11], hay laminaciones minimales que admiten dos medidas transversalmente invariantes mutuamente singulares. Del mismo modo, una laminación con infinitas medidas transversales invariantes, puede encontrarse en [FSW11], donde los autores utilizan un ejemplo debido a Furstenberg para construir tal laminación. Deroin, en [Der09], usa también el ejemplo de Furstenberg para construir una foliación sin medidas transversas invariantes, pero con infinitas medidas armónicas.Merece la pena mencionar que en este artículo, además, se dejan abiertas cuatro cuestiones y esta tesis indaga sobre la tercera de ellas.

Así que, necesitamos estudiar cada laminación por separado. Consideremos, por ejemplo, una foliación de Riccati. Estas foliaciones son uno de los ejemplos más sencillos de comportamiento caótico en una foliación. Son transversas a una fibración de fibra $\mathbb{P}^1$ salvo en una cantidad finita de puntos de la base donde la fibra es invariante. En este caso, Bonatti y Gómez-Mont probaron la unicidad de la medida en [BGM01] mediante el uso del flujo geodésico.

0.1.3 Resultados

Otra situación donde se obtuvo la unicidad fue para laminacion embebidas en $\mathbb{P}^2$, probada por Fornæss y Sibony en [FS05]. Este artículo es el punto de partida de esta tesis. Nosotros hemos podido generalizar ese resultado para laminaciones embebidas en superficies Kähler. Además, el cuidadoso estudio de un entorno de una singularidad hiperbólica llevado a cabo en [FS10], nos permite, tras una pequeña modificacion de los argumentos, probar un teorema similar cuando permitimos este tipo de singularidades.

La razón por la cual la unicidad es importante en este tipo de medidas es porque puede ser vista como un atractor global para la dinámica de la laminacion. Del mismo modo, puede ser entendido como un análogo a un teorema de independencia del parámetro inicial en un sistema dinámico. Trataremos de explicar esta afirmación más detenidamente en el Capítulo 3.

Por tanto, el Teorema Principal obtenido en esta tesis es el siguiente.

Teorema 0.1. *Sea (M, ω) una superficie Kähler homogénea compacta con una laminación por superficies de Riemann $\mathcal{L}$ que es minimal y transversalmente Lipschitz embebida en la superficie. Si $\mathcal{L}$ no admite ninguna corriente cerrada invariante dirigida por la laminación, entonces existe una única corriente armónica de masa uno dirigida por la laminación.*

La demostración hace uso de la teoría de intersección desarrollada por Fornæss y Sibony en [FS05]. Por tanto, según la clasificación de superficies homogéneas compactas de Tits [Tit63], hay sólo cuatro tipos diferentes de superficies que estudiar. Estas superficies son las siguientes: toros de dimensión compleja dos, el producto de una curva elíptica por una recta proyectiva, $\mathbb{P}^1 \times \mathbb{P}^1$ y $\mathbb{P}^2$.

Esencialmente, en [FS05], donde el teorema está probado para $\mathbb{P}^2$, los autores reducen el problema de probar la unicidad a un problema de calcular puntos de intersección cuando la laminación está perturbada por una familia de automorfismos cercana a la identidad. En este caso, la familia de automorfismos tiene una recta de puntos fijos. Mediante el control del comportamiento de la laminación y de la familia de automorfismos cerca de esta esta recta, y mediante argumentos de continuación de la distancia transversal entre placas, son capaces de encontrar una cota superior para el número de estos puntos de intersección.

En nuestros casos, podríamos no tener esta recta invariante. Por tanto, aunque podemos usar la teoría de intersección de [FS05], la prueba para el resto de superfices será diferente a la de $\mathbb{P}^2$. El factor común al resto de las superficies es la estructura producto en el fibrado tangente, que nos permitirá trabajar con nociones naturales de verticalidad y horizontalidad.

Esta tesis es, esencialmente, la combinación de dos artículos [PG13a] y [PG13b]. En el primero, resolvemos el caso no singular y en el segundo, el caso con singularidades hiperbólicas. El caso sin singularidades está motivado por el problema de dilucidar la existencia o no de laminaciones embebidas en tales superficies mediante el estudio de las propiedades que estas laminaciones debieran tener. Sin embargo, hasta el momento no se ha conseguido dar ningún ejemplo explícito. De este modo, usando argumentos similares, podemos extender este resultado para el caso de laminaciones con singularidades hipérbolicas, donde las hipótesis del teorema se cumplen de forma genérica. La demostración principal del teorema se desarrolla en $\mathbb{T}^2$ para el caso no singular y en $\mathbb{P}^1 \times \mathbb{P}^1$ para el caso con singularidades.

La organización de este texto es la siguiente. En el Capítulo 1, incluiremos las nociones necesarias para la mejor comprensión del texto, desde las primeras definiciones en la teoría de corrientes y laminaciones hasta la teoría de la intersección desarrollada en [FS05], que nos permitirá reducir la prueba del teorema a contar puntos de intersección. En el Capítulo 2, probaremos el Teorema Principal de esta tesis, primero para laminaciones sin singularidades, y después con ellas. Por último, el Capítulo 3 consiste en una discusión sobre dónde y cómo este Teorema se puede aplicar.

0.1.4 Conclusiones

Si bien es cierto que anteriormente mencionamos la necesidad de estudiar cada laminación por separado, a partir de los resultados obtenidos en esta tesis, parece que el comportamiento de las laminaciones es similar en cualquier superficie Kähler homogénea compacta. De esta forma, se puede generalizar el problema de minimal excepcional a este contexto:

¿Existe alguna laminación no singular embebida en alguna superficie Kähler homogénea compacta que no admita corrientes dirigidas?

Si bien es cierto que se pueden dar ejemplos de laminaciones no singulares no triviales en algunos toros complejos, todas ellas admiten corrientes cerradas. Así pues, esta pregunta sigue abierta para investigaciones futuras.

En lo concerniente a laminaciones con singularidades, es sabido que las hipótesis bajos las cuales hemos obtenido nuestros teoremas son bastante generales. Además, particularizando para el caso de $\mathbb{P}^1 \times \mathbb{P}^1$, podemos dar una prueba relativamente sencilla de que esto es así. Asimismo hemos probado que toda laminación transversalmente Lipschitz no singular en $\mathbb{P}^1 \times \mathbb{P}^1$ sin curvas compactas no admite corrientes cerradas dirigidas. Del mismo modo, se puede probar que una foliación de $\mathbb{P}^1 \times \mathbb{P}^1$ sin curvas compactas invariantes y con, a lo sumo, singularidades hiperbólicas, soporta una única corriente armónica. De este modo, queda eliminada la hipótesis de la minimalidad.

0.2 Summary

One of the most important parts of modern Mathematics is the study of differential equations. These equations can modelize from the growth of a population to the motion of the planets. Actually, they are one the cornerstones of Mathematics and Science.

Although there were several earlier mathematicians who studied differential equations, it was around 1900 when the study of these equations gained importance, mainly because of the work of Poincaré who introduced topological techniques to their qualitative study. In fact, maybe the most famous example of a topological result in differential equations is the Poincaré-Bendixson theorem, which classifies the possible limit behaviour of a bounded orbit in an autonomous differential equation on $\mathbb{R}^2$. They can accumulate either to a singularity, to an heteroclinic trajectory or to a limit cycle.

Even though differential equations have been widely studied, there are some natural questions which are still unsolved. Perhaps, the most interesting one is Hilbert's 16th theorem which enquires about the boundedness on the number of limit cycles of a polynomial vector field in $\mathbb{R}^2$. It was proven by Écalle and Ilyashenko that this number is finite, but it remains unsolved if there is any uniform bound on the number of finite cycles for polynomial vector fields of a fixed degree.

These polynomial vector fields on $\mathbb{R}^2$ can be seen as polynomial vector fields in $\mathbb{C}^2$ and these ones as restrictions to an affine view of a vector field in $\mathbb{P}^2$. Now, the orbits are no longer real curves, but are complex

curves, namely Riemann surfaces. We say that we have a foliation by Riemann surfaces of $\mathbb{P}^2$.

In this way, having an invariant closed curve without singularities is the analogous to the existence of a limit cycle. Unluckily, this cannot happen in $\mathbb{P}^2$ because every invariant curve for these foliations must contain a singularity by the Theorem of the Index of Camacho-Sad [CS82]. Hence, if we relax our demands and, instead of searching for an invariant curve, we search for any closed invariant set without singularities, this set would have the structure of a lamination by Riemann surfaces. The existence of such sets is unknown in $\mathbb{P}^2$ so far. This problem is known as the minimal exceptional set problem and was firstly studied on its modern statement in the article of Camacho, Lins-Neto and Sad, [CLNS92]. If we consider the same problem for foliations of codimension one in $\mathbb{P}^n$ with $n \geq 3$, Lins-Neto showed in [LN99] that there cannot be any exceptional minimal set.

This is one motivation for studying laminations by Riemann surfaces, but it is not the only one. The reader can check the survey by Ghys [Ghy99] for examples of laminations constructed from other different sources showing the important role laminations play on certain dynamical systems.

As beforementioned, there are many different contexts where laminations can appear and likewise, there are many different approaches to their study and many different characteristics to understand.

For instance, the problem of the embeddability of abstract laminations has been widely studied. For an example of this approach we can refer to [Der08], where Deroin finds embeddings of any Riemann surface lamination without vanishing cycles (see Sullivan's paper [Sul76]) in $\mathbb{P}^N$ for certain N big enough. In the same direction, Fornæss, Sibony and Wold proved in [FSW11] that a lamination arising from projective limits of complex manifolds of dimension n can be embedded in $\mathbb{P}^{2n+1}$. This technique for constructing laminations became important because of the results obtained by Alcalde-Cuesta, Lozano-Rojo and Macho-Stadler in [ACLRMS11], where they show that under certain hypothesis concerning the transversal behavior, any lamination transversely Cantor is a projective limit.

In both cases mentioned above, we are given a lamination and we embed it in these spaces. However, we can make the converse reasoning. Namely, we are given a manifold and we want to know how are the laminations embedded in it. The first step is studying the foliations of these manifolds. It was mentioned before that every foliation of $\mathbb{P}^n$ has singularities, but this is no longer true if we study other manifolds. For instance, in [Ghy96], Ghys classifies the foliations without singularites on homogeneous manifolds.

Another interesting problem is to find out wether we can associate measures to a lamination and which kind of measures are these. The first attemp that one can try in order to find this association would be trying to define an invariant transversal measure, nonetheless a lamination admitting such a measure is very uncommon. But luckily, if we relax our expectations, we can always find a harmonic measure. This result was proven in [Gar83] for foliations without singularities and, in [BS02], when the set of singularities of the lamination has Hausdorff dimension lower or equal to 2.

However, once the existence is ensured, it is important to check the unicity. This is not trivial and depends strongly on the foliation. For instance, Lozano-Rojo has proven in [LR11] that there are minimal laminations with two transversely invariant measures mutually singular. Likewise, an example of a lamination with an infinite amount of transversely invariant measures can be found in [FSW11], where Fornæss, Sibony and Fornæss-Wold use an example given by Furstenberg to construct a lamination with this property. Deroin, in [Dem], constructs a foliation by Riemann surfaces of a manifold from the Furstenberg example without transversely invariant measures, but with several harmonic measures. Moreover, in the end of this article, the author leaves four open questions and this thesis is devoted to study the third one of them.

Thus, we need to study each lamination almost separately. Consider, for instance, a Riccati foliation. These foliations are the very first example of chaotic behavior of a lamination. They are transverse to a fibration with fibers $\mathbb{P}^1$ everywhere except in a discrete set of points of the base. In this situation, the unicity of the harmonic measure was proven by Bonatti and Gómez-Mont in [BGM01] by using the geodesic flow.

Another situation where this unicity is obtained is in a lamination embedded in $\mathbb{P}^2$, proven in [FS05]. This article is the starting point of this thesis. We could generalize this result for every compact homogeneous Kähler surface. Furthermore, the careful study of the behavior in a neighbourhood of a hyperbolic singularity carried out in [FS08] allows us to, with a small modification of the original argument, to prove the same theorem allowing hyperbolic singularities in the lamination.

The reason it is so important to have the unicity on these measures is because it can be seen as a global attractor for the dynamics of the lamination. It can also be understood like an analogous to a result of independence of the initial parameters on a dynamical system. We will try to explain these assertion more carefully in Chapter 3.

Hence, the Main Theorem in this thesis is the following

Theorem 0.1 (Main theorem)**.** *Let (M, ω) be a homogeneous compact Kähler surface with a minimal transversely Lipschitz lamination by Riemann surfaces with only hyperbolic singularities $\mathcal{L}$ embedded on it. Suppose that $\mathcal{L}$ does not admit any directed invariant closed current. Then there exists a unique harmonic current of mass one directed by $\mathcal{L}$.*

The proof is based on the intersection theory that Fornæss and Sibony developed in [FS05]. Then, according to Tits' classification [Tit63], there are only four different kinds of surfaces to consider. These surfaces are the following: two dimensional complex tori, the product of a projective line and a elliptic curve, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

Basically, in [FS05], where this theorem was proven for $\mathbb{P}^2$, the authors reduce the problem of proving uniqueness to a problem of computing intersection points when the lamination is perturbed by a family of automorphisms close to the identity. In that case, the family of automorphisms has a line of fixed points. By controlling the behavior near this line, and by arguments of continuation of the transversal distance between plaques, the authors were able to find a bound for the amount of these intersection points.

In our case, we might not have this invariant line. For instance, for two dimensional complex tori, the automorphisms close to the identity are translations. Then, for automorphisms which are close to the identity, there are no fixed points. Hence, although we can use the intersection theory of [FS05], the proof for the rest of the desired surfaces will be different to the one of $\mathbb{P}^2$. The common feature for the rest of the surfaces is that they have a product structure in the tangent bundle, which allows us to work with natural notions of verticality and horizontality.

This thesis is essentially the combination of two papers: [PG13a] and [PG13b]. In the first one, we solve the non singular case, and in the second one the case allowing only hyperbolic singularities. The non singular case is motivated by the problem of elucidate the existence or not of laminations embedded in the surfaces under consideration, studying the properties such lamination should have. However, there are no explicit examples of these laminations so far. In this way, similar arguments allow us to extend this result for laminations with hyperbolic singularities. In this setting, the hypotheses of our theorem hold generically. The main proof of the theorem is made in $\mathbb{T}^2$ in the non singular case, whereas in the case with singularities, it is done in $\mathbb{P}^1 \times \mathbb{P}^1$.

The organization of this dissertation goes as follows. In Chapter 1, we include some necessary preliminar knowledge, from the very basics about currents and laminations to intersection theory of directed currents, which allows us to reduce the proof of the theorem to a problem of computing intersection points. In Chapter 2, we prove the Main Theorem of this thesis firstly for laminations without singularities and later allowing them. Finally, Chapter 3 consists of a discussion about where and how this theorem can be applied.

Chapter 1

Preliminaries

1.1 Laminations and Foliations

1.1.1 Definitions and examples

Definition 1.1. We say $(X, \mathcal{L}, E)$ is a *lamination by Riemann surfaces* with singular set $E \subset X$, if X is a compact topological space such that for every $p \notin E$ we can find local charts $\phi_i : \Delta \times \mathscr{T}_i \to X$ where Δ is the unit disk and $\mathscr{T}_i$ is a topological space. These charts satisfy that the change of coordinates is $\phi_i^{-1} \circ \phi_j(z,t) = (\phi_{ij}^1(z,t), \phi_{ij}^2(t))$ with ϕ_{ij}^2 continuous and ϕ_{ij}^1 holomorphic in the first variable and continuous in the second one. These local charts are called *flow boxes*, and the sets $\phi_i(\Delta \times \{\alpha\})$ are the *plaques* of the flow box.

Note that, transversely to the plaques we are only asking for continuity. But, in this thesis, we will often deal with transversely Lipschitz laminations. In this case, the topological spaces $\mathscr{T}_i$ are metrizable, the function ϕ_{ij}^2 is Lipschitz and so it is ϕ_{ij}^1 in the second variable. If $E = \emptyset$ we say that it is a non singular lamination. In fact, for the sake of simplicity, the set of the singularities E of lamination $(X, \mathcal{L}, E)$ will be the smallest set such that we can find neighbourhoods as described in Definition 1.1 for every point p in $X \setminus E$.

As mentioned in the introduction, laminations are related to foliations. Actually, in this thesis we will sometimes deal with holomorphic foliations.

Definition 1.2. Let $\mathscr{M}$ be a complex manifold of dimension m. A *holomorphic foliation* $\mathcal{F}$ of $\mathscr{M}$ by Riemann surfaces with singular set E is given by an atlas $\mathcal{U} = \{U_i, \phi_i\}$ of $\mathscr{M} \setminus E$, with $\phi_i : \Delta \times \Delta^{m-1} \to \mathscr{M}$,

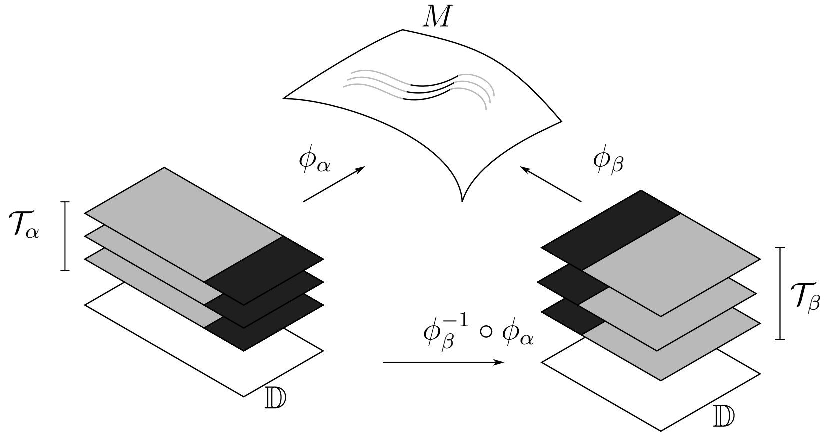

Figure 1.1: Change of coordinates between flow boxes

where Δ is the unit disk and Δ^{m-1} is the $m-1$-dimensional polydisk. The change of coordinates satisfies that can be written like $\phi_i^{-1} \circ \phi_j(z,t) = (\phi_{ij}^1(z,t), \phi_{ij}^2(t))$, where ϕ_{ij}^1 and ϕ_{ij}^2 are holomorphic in each variable.

Obviously, any holomorphic foliation is a lamination, and from certain holomorphic foliations, we can extract laminations which are not foliations. We can see this situation in the following well known example.

Example 1 (Suspension). Consider the surface $S = \mathbb{D} \setminus \{1/2, -1/2\}$. Its fundamental group is the free group generated by two elements. Let $\Phi : \tilde{S} \to S$ be a conformal universal covering of S and $\Gamma \in Aut\{\tilde{S}\}$ is the group of Deck transformations of Φ which is isomorphic to the fundamental group of S. Consider a isomorphism π from Γ to a Schottky group $G \subset Aut(\mathbb{P}^1)$ of two generators f_1, f_2.

Let us recall the definition of a Schottky group. Let $U \subset \mathbb{P}^1$ an open set bounded by $2l$ Jordan curves $\tau_1, \tau_l', \ldots, \tau_l, \tau_l'$, if there exists $f_1, \ldots, f_l \in Aut(\mathbb{P}^1)$ such that $f_j(\tau_j) = \tau_j'$ and $f_j(U) \cap U = \emptyset$, we say that the subgroup generated by $f_1, \ldots, f_j$ is a *Schottky group* and Schottky groups are free groups. More information about Kleinian and Schottky groups can be found in [Mas88].

Now, over the product manifold $\tilde{S} \times \mathbb{P}^1$, which carries the trivial horizontal holomorphic foliation, we can consider the action of Γ as follows. For every $\alpha \in \Gamma$, a point $(\tilde{s}, p) \in \tilde{S} \times \mathbb{P}^1$ is sent to $(\alpha\tilde{s}, \pi(\alpha)p)$. The group Γ acts properly and freely over $\tilde{S} \times \mathbb{P}^1$ and by considering the quotient, we get a complex manifold M_G which is a fibration over S with fiber $\mathbb{P}^1$. This manifold is endowed with a holomorphic foliation, coming from the horizontal one we mentioned above, transversal everywhere to the fibration. The transversal dynamics of the foliation is given by the group G.

Since G is a special case of Kleinian group, it has a limit set $\Lambda(G)$, which is the smallest closed set invariant by G. In this case, as G is a Schottky group, its limit set is a perfect nowhere dense set. So, we can extract a lamination from M_G that can be understood as a fibration with fiber $\Lambda(G)$ over S, and this lamination is not a holomorphic foliation because its ambient space is not a manifold.

Example 2 (Projective limits)**.** Suppose we have a family of Riemann surfaces $\{S_i\}_{i\in\mathbb{N}}$ together with a family of holomorphic maps $\{f\}_{i\in\mathbb{N}}$, $f_i : S_{i+1} \to S_i$ of degree $d_i \geq 2$. The projective limit is the subset

$$X = \{(x_i)_{i\in\mathbb{N}} |\, f_i(x_{i+1}) = x_i\}$$

of the product space $\prod S_i$. It has a structure of lamination by Riemann surfaces with a Cantor set in the transversal. Examples of laminations constructed like this are widely studied in [FSW11].

This construction is specially important because of a theorem stated in [ACLRMS11]. By allowing more flexibility on the maps, the authors proved that any non singular transversely Cantor lamination with a simple enough transversal dynamic arises from a suitable projective limit.

Example 3. (Holomorphic motions) This example will be useful later. We need to recall the definition of a really important concept in one dimensional complex dynamics. This is the concept of holomorphic motion which was introduced by Mañé, Sad and Sullivan in [MSS83] in order to study perturbations of Julia sets.

Definition 1.3. Let $\mathscr{T}$ be a subset of $\mathbb{P}^1$. A *holomorphic motion* of $\mathscr{T}$ is a map $f : \Delta \times \mathscr{T} \to \mathbb{P}^1$ such that:

- for any fixed $t \in \mathscr{T}$, the map $f_{(.,t)}(z) := f(z,t)$ is holomorphic in Δ

- for any fixed z the map $f_{(z,.)}(t) := f(t,z)$ is an injection and

- the mapping $f_{(0,.)}$ is the identity on A.

It is easy to realize that laminations embedded in complex surfaces can be seen as local holomorphic motions close to regular points: if we fix t and move z we obtain a parametrization of a plaque.

Notice that, in the definition above our conditions seem very flexible: we require holomorphicity in one variable and only injectivity in the other one. Nevertheless, in [MSS83], the authors obtain the so-called λ-Lemma

Theorem 1.4. *If $f : \Delta \times \mathscr{T} \to \mathbb{P}^1$ is a holomorphic motion, then f has an extension to $F : \Delta \times \overline{\mathscr{T}} \to \mathbb{P}^1$ such that*

- *F is a holomorphic motion of $\mathscr{T}$*
- *each $F_{(z,.)}(t) := F(z,t)$ is quasiconformal*
- *F is jointly continuous in both variables.*

As a direct consequence of this theorem we obtain that the transversal regularity of every lamination embedded in a surface will be, at least, quasiconformal. This theorem has been refined several times and one of these refinements given by Bers and Royden [BR86] include some estimates that will be worthful for us.

Example 4. (Levi-flats) Let $\mathscr{M}$ be a complex manifold of dimension n and X is a C^1 real submanifold of codimension 1. For every point $p \in X$, the tangent space contains a unique complex subspace of complex dimension $n-1$, say C_p. In this way, we obtain a distribution $\mathfrak{C}$. In the case $\mathfrak{C}$ is integrable, X carries a foliation with $n-1$-dimensional complex leaves. We say that X is a *Levi-flat*. Therefore, if $\mathscr{M}$ is a surface, X carries a structure of lamination by Riemann surfaces.

Lins-Neto proved in [LN99] that there are no real analytic Levi-flats in $\mathbb{P}^n$ if $n \geq 3$, however their existence in $\mathbb{P}^2$ is still unknown. Nonetheless, this is not true for every compact surface, since Ohsawa gave in [Ohs06] a complete classification of real analytic Levi-flats in complex tori of dimension 2.

The laminations we will deal with in this dissertation are embedded in surfaces. More explicitely, a laminated set $(X, \mathcal{L})$ by Riemann surfaces is said to be embedded in a manifold if there exists an injection $\Phi : X \to M$ such that the complex structure of the leaves as well as the transversal regularity of $(X, \mathcal{L})$ come from Φ.

1.1.2 Holonomy and Monodromy

In some examples, like suspensions, we can find a global transversal space where the whole transversal dynamics can be seen. Nonetheless, this situation is very uncommon, so we need some extra flexibility to code the transversal behavior of the laminations. This is the holonomy pseudogroup. This concept is fundamental in foliation theory and a wider explanation can be found in [CC00]. This book also includes several interesting explicit examples of holonomy groups and pseudogroups. For a more concise definition in the case of holomorphic foliations, see [Zak01].

Consider $\alpha : [0,1] \to L$ a loop in the leaf L with basepoint p. This loop can be covered by a finite number of flow boxes, $U_1, U_2, \dots U_l$, such that there is a partition $t_1 = 0, t_2, \dots, t_l = 1$ of $[0,1]$ with $\alpha([t_i, t_{i+1}]) \in U_i$ for every $i = 1, 2, \dots, l$ and $U_1 = U_l$. Let $\varphi_i : \mathbb{D} \times \mathscr{T}_i$ be the coordinates charts of the flow boxes, then if $\varphi_1^{-1}(p) = (z_1, \tau_1)$ the change of coordinates from U_1 to U_2 gives a homeomorphism from a neighborhood of $\tau_1 \in \mathscr{T}_1$ to an open set of $\mathscr{T}_2$ and τ_1 is sent to τ_2. Repeating this process, after $l-1$ iterations we finally come back to U_1, so, considering the composition of all these homeomorphism from $\mathscr{T}_i$ to $\mathscr{T}_{i+1}$, we can associate a homeomorphism Hol_α from V_α to V'_α neighborhoods of $\tau_1 \in \mathscr{T}_1$. The regularity of these homeomorphisms is the transversal regularity of the lamination. This function Hol_α is called the *holonomy function associated to the loop α*. Since this function is not defined in all $\mathscr{T}_1$, we need to consider the holonomy pair (Hol_α, V_α).

Remark 1.1.1. The function Hol_α does not depend on the choice of the intermediate transversals $\mathscr{T}_i, i = 2, \dots, n$.

Remark 1.1.2. The germ of the map Hol_α only depends on the homotopy class of α.

Remark 1.1.3. If we consider another flow box B containing p, with coordinates $\phi_B : B \to \mathbb{D} \times \mathscr{T}_B$, there exists a homeomorphism $h : \mathscr{T}_1 \to \mathscr{T}_B$ having the same regularity than the transversal one of the lamination such that the germ of the holonomy $Hol'_\alpha : \mathscr{T}_B \to \mathscr{T}_B$ satisfies that $Hol'_\alpha = h^{-1} \circ Hol_\alpha \circ h$.

Definition 1.5. Let L be a leaf of a lamination by Riemann surfaces and $p \in L$ a point contained in a flow box B mapped into $\mathbb{D} \times \mathscr{T}$. Then for every $[\alpha] \in \pi(L, p)$ we can associate a germ of a map $Hol_\alpha : \mathscr{T} \to \mathscr{T}$. This is the so-called *monodromy mapping.*

We say that the image of $\pi(L, p)$ by the monodromy mapping is the *monodromy pseudogroup.*

Theorem 1.6 (Hector [Hec72], Epstein, Millet, Tischler [EMT77])**.** *If $(X, \mathcal{L}, E)$ is a lamination with $E = \emptyset$, the leaves having trivial monodromy pseudogroup are generic.*

It could actually happen that there were no leaves with non trivial holonomy. This one would be the "simple" transversal behavior mentioned before appearing in the hypothesis of the main theorem in [ACLRMS11]. So in this situation, the lamination arises as a projective limit.

On the other hand, we have the following interesting result:

Theorem 1.7 (Bonatti, Langevin, Moussu [BLM92])**.** *If X is a minimal exceptional set for a foliation $\mathcal{F}$ on $\mathbb{P}^2$, there exists a leaf of $\mathcal{F}$ with contractive holonomy.*

1.1.3 Singular Laminations

Example 5 (Singular foliations)**.** So far, we have just given examples of laminations without singularities. The most natural way to introduce an example with singularities is throughout singular holomorphic foliations. Let us restrict to foliations in $\mathbb{P}^2$. A holomorphic vector field

$$\chi = \sum_{i=1}^{3} P_i \frac{\partial}{\partial z_i}$$

on $\mathbb{C}^3$ with P_i homogeneous polynomials of the same degree induces a holomorphic foliation in $\mathbb{P}^2$ and every complex vector field in $\mathbb{P}^2$ has singular points. In this setting, we will say that the foliation is saturated if the singular set is finite. By the Theorem of the Index of Camacho-Sad, [CS82], there is a germ of a leaf passing through any singularity. This analytic set is called separatrix of the singularity. Therefore if we consider the analytic continuation of a local separatrix L, and we look at its adherence $\overline{L}$, this will be an invariant set for the foliation. Then, if $\overline{L}$ is not a Riemann surface, it has structure of lamination by Riemann surfaces with singularities.

There are several kinds of singularities of a vector field on a surface according to their local behavior. Since we are assuming they are discrete, we can take a holomorphic coordinate chart (x, y) centered on one of them p, and the vector field can be expressed in this chart like

$$F(x, y)\frac{\partial}{\partial x} + G(x, y)\frac{\partial}{\partial y},$$

with F and G holomorphic functions satisfying that $F(0,0) = G(0,0) = 0$. Consider the the matrix

$$J = \begin{pmatrix} \frac{\partial F}{\partial x}(0,0) & \frac{\partial G}{\partial x}(0,0) \\ \\ \frac{\partial F}{\partial y}(0,0) & \frac{\partial G}{\partial y}(0,0) \end{pmatrix}.$$

This matrix depends on the chart we choose.

Definition 1.8. Let $(0,0)$ be a singularity of a complex vector field $F(x,y)\frac{\partial}{\partial x} + G(x,y)\frac{\partial}{\partial y}$ on $U \subset \mathbb{C}^2$. Let λ_1, λ_2 be the the eigenvalues of the matrix above. We will say that the singularity is *irreducible* if it satisfies one of the following conditions:

1. $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{C} \setminus (\mathbb{N} \cup 1/\mathbb{N})$
2. $\lambda_1\lambda_2 = 0$ and $\lambda_1 + \lambda_2 \neq 0$.

We say that λ is the *characteristic value* of the singularity, where $\lambda = \lambda_1/\lambda_2$ if we are in the first situation and $\lambda = 0$ in the second one.

For every chart, we will obtain the same characteristic value λ or its inverse $1/\lambda$. The name irreducible singularity is due to the following theorem:

Theorem 1.9 (Seidenberg)**.** *Let χ be a complex vector field on a compact complex surface $\mathscr{M}$ with a discrete set of singularities. There exist a complex surface $\tilde{\mathscr{M}}$ and $\Pi : \mathscr{M} \to \tilde{\mathscr{M}}$ a birational map, such that the vector field induced by χ on $\tilde{\mathscr{M}}$ has only irreducible singularities.*

We will say that an irreducible singularity of a complex vector field in a surface is *hyperbolic* if $\lambda = \lambda_1/\lambda_2 \notin \mathbb{R}$. Poincaré showed that there exists a linearizable neighborhood of a hyperbolic singularity. This means, an open set around the singularity and a change of coordinates (x,y) such that, in these coordinates, the vector field can be written $\lambda x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Note that if a foliation on a coordinate chart (x,y) of a surface, since it has codimension one, is given by the orbits of the vector field

$$\chi = F(x,y)\frac{\partial}{\partial x} + G(x,y)\frac{\partial}{\partial y},$$

it can be also seen like the invariant varieties of the holomorphic 1-form

$$\gamma = G(x,y)dx - F(x,y)dy.$$

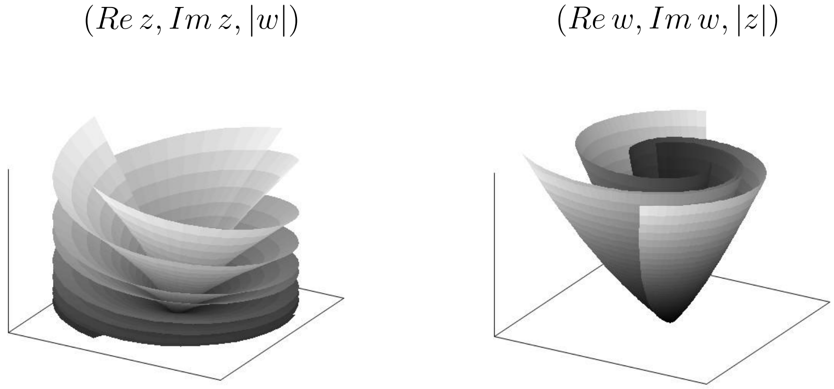

Figure 1.2: Representation of leaves of $zdw - (0.75 + 0.2\sqrt{-1})wdz$ close to $(0, 0)$

Hence, on a neighborhood of a hyperbolic singularity for the foliation given by a 1-form γ, we can find some new coordinates (x', y') where the foliation can be written as

$$\gamma' = ydx' - \lambda x'dy',$$

where $\lambda = \lambda_1/\lambda_2 \notin \mathbb{R}$.

Then, we can extend the definition of a hyperbolic singularity to laminations by Riemann surfaces.

Definition 1.10. Let $(X, \mathcal{L}, E)$ be a lamination by Riemann surfaces with singularities embedded on a compact complex surface $\mathscr{M}$, with E discrete. We say that $p \in E$ is a *hyperbolic singularity* if we can find $U \subset \mathscr{M}$ a neighborhood of p and holomorphic coordinates (z, w) centered at p such that the leaves of $(X, \mathcal{L}, E)$ are invariant varieties for the holomorphic 1-form $\omega = zdw - \lambda wdz$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Note that this definition needs an analytic structure around the singularity, thus it is not defined for an abstract lamination.

Unlike the non singular case, it is very easy to find a leaf with non trivial holonomy. If we consider a foliation by Riemann surfaces and we take the separatrix of a hyperbolic singularity, it has non trivial holonomy. We just need to consider a small loop around the singularity, and depending on the orientation given to the loop, it will be contracting or expanding.

1.2 Currents

Currents and pluripotential theory will be the main tools in the proof of our results. In this section, we will recall the necessary background on these topics in order to follow the discussion. More information about it can be found in Demailly's book [Dem] for a deep and rigorous treatment of currents, and in the survey of Dinh and Sibony [DS13] for an approach more oriented towards dynamics.

Let $\mathscr{M}$ be a homogeneous compact complex surface. Consider a differential l-form γ on $\mathscr{M}$. In local coordinates $z = (z_1, z_2)$, this form can be written as

$$\gamma(z) = \sum_{|I|+|J|=l} \gamma_{IJ} dz_I \wedge d\overline{z}_J,$$

where $I = (i_1, \dots, i_p) \in \{1,2\}^p$, $J = (j_1, \dots, j_q) \in \{1,2\}^q$. In this expression, we denote by $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ and γ_{IJ} is a function with complex values. We say that γ is a form of *bidegree* (p, q) if the decomposition above has non zero coeficients γ_{IJ} only if $|I| = p$ and $|J| = p$. We define the conjugate of γ like

$$\overline{\gamma}(z) = \sum \overline{\gamma}_{IJ} d\overline{z}_I \wedge dz_J,$$

and obviously the conjugate of a (p, q) form is a (q, p) form. We say that a form γ is *real* if $\gamma = \overline{\gamma}$.

If we apply the operator d to a (p, q)-form we will obtain a sum of a $(p+1, q)$-form and a $(p, q+1)$-form. So we could split the operator d as a sum $d = \partial + \overline{\partial}$. Since $dd = 0$ we can conclude that $\partial\partial = \overline{\partial\partial} = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

The operator d sends real forms to real forms, but ∂ and $\overline{\partial}$ do not. We can, however, define the operator $d^c = \frac{\sqrt{-1}}{2\pi}(\overline{\partial} - \partial)$ which is real and satisfies that $dd^c = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.

We will say that a $(1, 1)$ form ω is *Hermitian* if it can be written as

$$\omega(z) = \sqrt{-1} \sum_{i,j=1,2} \omega_{ij} dz_i \wedge d\overline{z}_j$$

and the matrix $(\omega_{ij}(z))$ is Hermitian and positive definite at every point. Hermitian forms define the so-called Hermitian metrics on manifolds. Moreover, a complex surface $\mathscr{M}$ endowed with a Hermitian closed $(1, 1)$-form ω is called a *Kähler manifold.*

We can define a topology in the space of (p, q)-forms C^k of a compact complex manifold as follows. Suppose α is a C^k (p, q)-form and U is a coordinate open set with holomorphic coordinates $z_1, z_2, \ldots, z_n$. In these coordinates, α can be expressed as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J$$

and for every compact subset $V \subset U$ and $l \leq k$ define the seminorms

$$p_L^s(\alpha) = \sup_{x \in V} \max_{|I|=p,|J|=q,|s|\leq l} |D^s \alpha_{IJ}|$$

where $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, $|s| = \sum_{i=1}^n s_i$ and $D^s = \frac{\partial^{|s|}}{\partial^{s_1},\ldots,\partial^{s^n}}$. Considering a finite atlas of the manifold $\mathcal{U} = \{U_j\}$, these collection of seminorms varying on l, V and U_j induces the C^k topology on (p, q)-forms.

A *current* S on a compact complex surface $\mathscr{M}$ of bidimension (p, q) (or bidegree $(2-p, 2-q)$) and order k is a continuous $\mathbb{C}$-linear functional on the space of the C^k (p, q)-forms with the C^k topology. Unless we metion otherwise, we consider order 0 currents. We will write $\langle S, \varphi\rangle$ or $S(\varphi)$ to indicate the value of S on φ. The differential operators on currents are defined by duality. For instance, for d, the current dT is defined to hold that $\langle dT, \varphi\rangle = \langle T, d\varphi\rangle$. The rest of operators are defined analogously.

A simple, but important, example of (p, p) current is the integration current on a subvariety $Y \subset \mathscr{M}$ of dimension p, denoted $[Y]$. Namely, for every test form ψ

$$[Y](\psi) = \int_Y \psi.$$

Moreover, any (p, q) form α induces a current T_α of bidimension $(n - p, n - q)$ in the following way:

$$T_\alpha(\phi) = \int_{\mathscr{M}} \alpha \wedge \phi$$

for every test $(n - p, n - q)$ form ϕ. These currents are often referred as smooth currents. Actually, any current can be approximated by smooth currents.

The wedge product of two currents is not always defined. However, it can be defined the wedge product between a smooth current and any current in the following way. If S is a current of bidimension (p_1, q_1) on $\mathscr{M}$ and α a (p_2, q_2) form with $p_1 \geq p_2$ and $q_1 \geq q_2$, we can define the current $S \wedge \alpha$ as

$$S \wedge \alpha(\phi) = \langle S, \alpha \wedge \phi\rangle$$

for every $(p_1 - p_2, q_1 - q_2)$ test form ϕ.

1.2.1 Positivity

Usually, in the literature, there are three notions of positivity for differential forms. However, in the case of surfaces, these three definitions are equivalent, so we will give just one of them.

Definition 1.11. A (p,p) form γ in a complex surface $\mathscr{M}$ is *positive* if it can be written like

$$\gamma = \sum_{i=1}^{l} \gamma_i(\sqrt{-1})\alpha_i \wedge \overline{\alpha}_i$$

for certain $(p,0)$ forms α_i and coefficients $\gamma_i > 0$.

Criterion. A (p,p)-form γ in $\mathscr{M}$ is positive if and only if for every p-dimensional complex submanifold S endowed with its canonical orientation, $\gamma_{|S}$ is a volume form on S.

By duality, we can define the concept of positivity for currents

Definition 1.12. Let S be a current of bidimension (p,p) such that for every $(2-p,2-p)$ positive form γ the measure $S \wedge \gamma$ is positive. Then we say that S is a *positive current.*

For a positive (p,p) current S on a Kähler surface $(\mathscr{M},\omega)$, we will define the mass of S on a compact set K as

$$\|S\| = \int_K S \wedge \omega^{2-p}.$$

1.2.2 Positive Directed Currents

In this paragraph, we will relate currents to laminations. On abstract laminations, we only have differential structure along the leaves. However, our laminations will be embedded on complex surfaces, so we will have a complex differential structure on the ambient space which, along the leaves, is coherent with the one of the lamination.

Let $(X,\mathcal{L},E)$ be a lamination embedded in a surface $\mathscr{M}$. Then, for every $p \in X$, we can find a small flow box U and a continuous map $\phi : U \to \Delta^2$, holomorphic along the plaques such that in Δ^2 the image of the plaques satisfies the Pfaffian equation $\{dw = 0\}$. In this way, if we define the $(1,0)$ form $\gamma = \phi^*(dw)$, then the plaques D_t of the lamination satisfy that $[D_t] \wedge \gamma_j = 0$.

Definition 1.13. A $(1,1)$ current T on $\mathscr{M}$ is *weakly directed* by the lamination $\mathcal{L}$ if $T \wedge \gamma_j = 0$.

Then, by definition, the first example of weakly directed laminations is the current of integration on a plaque $[D]$. Nevertheless, if we consider a function f with support contained on $[D]$ the current $f[D]$ is also a directed current. Given that we were searching for a unicity property we need to impose an extra condition to the currents which is that they be harmonic. In this case, $f[D]$ is a harmonic directed current if and only if f is harmonic with support contained on D. By the maximum modulus principle f is constant, so the flexibility has been significantly reduced. These are the kind of currents we will deal with.

Definition 1.14. Let $(X, \mathcal{L}, E)$ be a lamination embedded on $\mathscr{M}$. We will say that a current T is a *positive harmonic directed current* if it can be decomposed in flow boxes like

$$T = \int h_\alpha [\Gamma_\alpha] d\mu(\alpha)$$

with $[\Gamma_\alpha]$ the integration current on the plaque Γ_α, μ a transversal measure and h_α a positive harmonic function on the plaque Γ_α.

It was proven in [FWW09] that T is a $\partial\overline{\partial}$-closed positive current in a surface $\mathscr{M}$ directed by a lamination $(X, \mathcal{L}, E)$ if and only if it can be written locally as the definition above. This fact is no longer true for laminations embedded in higher dimensional manifolds.

The existence of directed harmonic positive currents for foliations without singularites was proven by Garnett in [Gar83]. Later, Berndtsson and Sibony [BS02] proved its existence for foliations with a pluripolar set E of singularities. This proof was generalized in [FS05] for a C^1 lamination embedded in a manifold $\mathscr{M}$. In case $\mathscr{M}$ is a surface the proof goes as follows. See A for the basics on Functional Analysis and Chapter I of [Dem] for a explicit definition of the topologies of currents.

Theorem 1.15 ([FS05])**.** *Let $(X, \mathcal{L}, E)$ be a lamination with a pluripolar set of singularities embedded in a suface $\mathscr{M}$. There exists a positive directed harmonic current of mass one.*

Proof. Let $\{\gamma_i\}$ be a family of continuous $(1,0)$ forms, such that $\gamma_i \wedge [\Gamma_\alpha] = 0$ for every Γ_α plaque in a flow box B.

Consider the set $\mathcal{C}$ of all the directed positive currents of mass one with support in the laminated set. Therefore, in a flow box, $T \in \mathcal{C}$ can be seen as $T = i\|T\|\gamma \wedge \overline{\gamma}$, for $\|T\|$ a positive measure.

Take a transversal $\mathscr{T} = \{z = 0\}$ and let π be the projection of the plaques on $\mathscr{T}$. The measure T can be decomposed along π in a measure ν_α on each plaque. In this way, for any ϕ a C^∞ $(1,1)$-form with support on the flow box

$$\langle T, \phi \rangle = \int \langle \nu_\alpha i\gamma \wedge \overline{\gamma}, \phi \rangle d\mu(\alpha) = \int \langle \tilde{\nu}_\alpha [\Gamma_\alpha], \phi \rangle d\mu(\alpha)$$

where $\tilde{\nu}_\alpha$ are measures.

Now, let $\mathcal{V}_N$ be the space of continuous functions on X with support on N flow boxes and C^2 on the leaves. This space is endowed with the supnorm on X and the C^2 topology on the leaves. If we consider a plaque of a flow box, written as $(z, f(z))$, we can define

$$\sqrt{-1}\partial_b\overline{\partial}_b\psi = \sqrt{-1}\frac{\partial^2\psi(z, f(z))}{\partial z \partial \overline{z}} dz \wedge d\overline{z}.$$

We can extend the action of T to $\partial_b\overline{\partial}_b\psi$ for $\psi \in \mathcal{V}_N$. Consider ξ_j a partition of unity associated to $\{B_j\}_{j=1}^N$ where $\operatorname{Supp}\phi = \cup_{i=1}^N B_j$. We define

$$\langle T, \partial_b\overline{\partial}_b\psi \rangle = \sum_j \int \langle [\Gamma_\alpha]\tilde{\nu}_\alpha, \Delta_\alpha(\xi_j\psi) \rangle d\mu(\alpha),$$

where Δ_α is the Laplacian on the plaque Γ_α. So T is continuous on $\mathcal{V}_N$.

If T_n converges towards T in the weak topology, then the sequence will also converge weakly in the dual of $\mathcal{V}_N$.

We define $\mathcal{W}_N = \mathscr{X} + \mathcal{V}_N$, with $\mathscr{X}$ noting the space of the $C^{1,1}$ forms on $\mathscr{M}$ endowed with the topology of the supremum on X. Consider $\tilde{\mathcal{W}}_N$ the Banach completion of this space. Since T acts on $\mathcal{W}_N$, it can be extended to a continuous linear functional on $\tilde{\mathcal{W}}_N$.

Therefore, there is a natural map $\Lambda : \mathcal{C} \to \tilde{\mathcal{W}}'_N$ such that, since a subsequence T_n in C has a subsequence that converges weakly to a current T, then $\Lambda(\mathcal{C})$ is also a compact convex set.

If we denote by $\mathcal{D}$ the space of exact forms $\sqrt{-1}\partial\overline{\partial}\phi$ with ϕ a C^∞ function on $\mathscr{M}$, we can define $\mathcal{B}_N := \mathcal{D} + \mathcal{V}_N \subset \mathcal{W}_N$.

Suppose that $\Lambda(\mathcal{C}) \cap \mathcal{B}_N^T = \emptyset$, where $\mathcal{B}_N^T$ is the set of the elements of $\mathcal{W}'_N$ vanishing on every element of $\mathcal{B}_N$. Then, by Hahn-Banach Theorem there exists an element $\sqrt{-1}\partial\overline{\partial}\phi_N + \sqrt{-1}\partial_b\overline{\partial}_b\psi_N$ of $\mathcal{B}_N$ such that

$$\langle T, \sqrt{-1}\partial\overline{\partial}\phi_N + i\partial_b\overline{\partial}_b\psi_N \rangle \geq \delta > 0$$

for every $T \in C$. In particular, $u_N := \phi_N + \psi_N$ is subharmonic on the leaves of the lamination. Hence, u_N attains its maximum on a point $z_0 \in E$. Since E is pluripolar, we can find a ball centered on z_0, $B(z_0, r)$ and a plurisubharmonic function v on a neighborhood of $B(z_0, r)$, such that $E \cap B(z_0, r) \subset \{v = -\infty\}$. But considering the subharmonic function $u_N - \frac{\delta}{2}|z - z_0|^2 + \epsilon v$, it attains its maximum in a point z_1 close to z_0 if ϵ small enough, and $z_1 \notin E$. In this way, we obtain a contradiction.

Therefore, there exists $T_N \in \mathcal{C}$ vanishing in $\mathcal{B}_N$. So let T be a weak limit of this sequence. The current T is positive, directed and for every continuous function ψ which is C^2 on leaves,

$$T(\psi) = \int \langle [\Gamma_\alpha]\tilde{\nu}_\alpha, \Delta_\alpha \psi \rangle d\mu(\alpha) = 0$$

hence $\langle \tilde{\nu}_\alpha, \Delta_\alpha \psi \rangle = 0$ for μ almost every point and, in consequence, $\tilde{\nu}_\alpha$ is a positive harmonic function on μ almost every plaque. □

In order to understand the role of the harmonic functions appearing in the decomposition a little more, we need to consider the following key

Remark 1.2.1 ([Mat12]). If we consider two decompositions of a directed harmonic current T in a flow box B as stated in Definition 1.14,

$$T_{|B} = \int h_\alpha [\Gamma_\alpha] d\mu(\alpha) = \int h'_\alpha [\Gamma_\alpha] d\mu'(\alpha)$$

then

$$h_\alpha d\mu(\alpha) = h'_\alpha d\mu'(\alpha) \tag{1.1}$$

for μ almost every point in the transversal. Hence, if we take a loop γ with basepoint p on a plaque $\Gamma_0 \subset B_0$ and we cover it by flow boxes $B_0, B_1, \ldots, B_l$ there is a unique way of extending the harmonic function h_p appearing in the decomposition of T in B associated to the plaque Γ_0 along the loop. In this way, when we return to p, the value in Γ_0 of the extended function $\tilde{h}_0$ might have changed, but it satisfies the equality (1.1) if μ' is the pushforward of the original μ by the holonomy map Hol_γ. Therefore these harmonic functions are not well defined on the leaves but they are on their universal covering.

This observation allows us to prove the following

Proposition 1.16. *[Sul76] Let $(X, \mathcal{L}, E)$ be a lamination in a complex surface $\mathscr{M}$ and let T be a directed closed current. Then, in flow boxes, T can be written as*

$$T = \int [\Gamma_\alpha] d\mu(\alpha),$$

for μ a holonomy invariant transversal measure.

Conversely, if μ is a holonomy invariant transversal measure, we can construct a directed closed positive current associated to μ.

Proof. Since $T(d\varphi) = 0$ for every form φ supported on a flow box $U = \Delta \times \Delta'$ and such that $d\varphi$ is a $(1,1)$-form, then if we integrate by parts, $dh_\alpha = 0$ for μ almost every point in the transversal so h_α is a constant and the local expression of T in the flow box can be normalized such that this constant is one.

This holds in every flow box, so consider a loop γ with basepoint p and passing through the flow boxes $U_1, \dots, U_n$ and define the pushforward of the measure $\mu' = Hol\gamma_* \mu$. It gives us that $T_{|U} = \int [\Gamma_\alpha] d\mu(\alpha) = \int [\Gamma_\alpha] d\mu'(\alpha)$. Hence $d\mu(\alpha) = d\mu'(\alpha)$ and μ is holonomy invariant.

For the second part of the proposition, we are given μ, a transversely invariant measure, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be a covering of the lamination by flow boxes. Consider $\{\psi_\alpha\}_{\alpha\in\Lambda}$ a partition of unity associated to $\mathcal{U}$. We will show that the current defined as $T := \sum_\alpha \int \psi_\alpha [\Gamma_t] d\mu(t)$ is closed.

Indeed, let φ be a form such that $d\varphi$ has bidegree $(1,1)$. Then, by integrating by parts and using the fact that ψ_α is 0 on the boundary of the plaques

$$\begin{aligned} T(d\varphi) &= \sum_\alpha \int \int_{\Gamma_t} d\varphi \psi_\alpha d\mu(t) \\ &= -\sum_\alpha \int \int_{\Gamma_t} \varphi d\psi_\alpha d\mu(t), \end{aligned}$$

but, for every flow box U, the invariance of μ allows us to exchange the sum and the integral, so we get

$$\begin{aligned} T_{|U}(d\varphi) &= -\int \int_{\Gamma_t} \sum_\alpha \varphi d\psi_\alpha d\mu(t) \\ &= -\int \int_{\Gamma_t} \varphi \sum_\alpha d\psi_\alpha d\mu(t) \\ &= -\int \int_{\Gamma_t} \varphi d\left(\sum_\alpha \psi_\alpha\right) d\mu(t) = 0. \end{aligned}$$

Therefore $T(d\varphi) = 0$. □

Proposition 1.17. *Let $(X, \mathcal{L}, E)$ be a minimal lamination in a compact Kähler surface $(\mathscr{M}, \omega)$ and let T be a directed closed current of mass one. Suppose that we are in one of the following situations,*

1. *$E = \emptyset$ and there exists a loop with contractive holonomy,*

2. *all the singularities in the lamination are hyperbolic,*

then $\operatorname{Supp} T$ *is a compact Riemann surface.*

Proof. In both hypotheses we have a loop γ with contractive holonomy. The first one by assumption and in the second one as it was stated at the end of subsection 1.1.3. Fix $p \in \gamma$ a basepoint. The point p is regular, so we can consider a flow box B_p centered at p with $\psi : B_p \to \Delta \times T \subset \Delta^2$. The current T in B_p can be written as

$$T = \int [\Gamma_t] d\mu(t)$$

where μ is holonomy invariant.

Since Hol_γ is contractive, we can find two subsets $V_p \subset U_p$ of $\mathscr{T}$ such that $Hol_\gamma(U_p) = V_p$. Iterating the holonomy, due to its contractiveness, $Hol_\gamma^n(U_p) \to \{0\}$. On the other hand, μ is holonomy invariant, hence $\mu(\{p\}) = \mu(U_p) > 0$, which means that μ has an atomic mass at p.

Let L be the leaf passing through p and notice that $\operatorname{Supp} T = \overline{L}$. Let us suppose that $\overline{L}$ has no singular points. Then, we can cover $\overline{L}$ with a finite number of flow boxes, having a finite number of plaques belonging to L. Otherwise, if there is a flow box $B_0 \approx \Delta \times \mathscr{T}_0$ with Γ_{t_n} plaques of L, then, by the holonomy invariance of μ,

$$\mu(\mathscr{T}_0) \geq \sum_{n \in \mathbb{N}} \mu(t_n) = \infty.$$

Hence the mass of T would not be one.

In the second assumption, since s_0 is a hyperbolic singular point in the support of T then both separatrices are contained in the support of T. Let S_0 be one of them. Consider a singular neighborhood and a small loop γ contained on S_0 surrounding the singularity, it has contractive holonomy, and reasoning as before, it can have mass only on the separatrices. This situation occurs around every singularity. For regular points, we can repeat the argument above. Thus $\operatorname{Supp} T = \overline{L}$ with L the analytic continuation of a local separatrix, and $\overline{L}$ must be a compact Riemann surface. □

As a corollary, due to the Theorem 1.7, we can ensure that, if a minimal set for a holomorphic foliation in $\mathbb{P}^2$ carries a closed current, it must be a compact Riemann surface. In this sense, Rebelo has obtained interesting results in [Reb13] for singular holomorphic foliations in algebraic surfaces, relating closed currents, compact leaves and infinite trajectories of the 1-dimensional real flow introduced in [BLM92].

1.2.3 Construction of Positive Directed Closed and Harmonic Currents

The proof of the existence of directed harmonic currents given in Theorem 1.15 is not constructive. The most common way of obtaining closed and harmonic directed currents is by an averaging process à la Ahlfors. This method was introduced by Goodman and Plante ([GP79], [Pla75]) to construct holonomy invariant measures in foliations, which would correspond to closed currents. Afterwards, it was modified by Fornæss and Sibony [FS05] to produce harmonic directed currents. We include here an overview of this averaging process. We will restrict the staments to our setting.

Theorem 1.18 (Goodman, Plante[GP79])**.** *Let $(X, \mathcal{L}, E)$ be a lamination with a finite set of singularities E in a Kähler surface $(\mathscr{M}, \omega)$. Let $\phi : \mathbb{C} \to L$ the universal covering of a parabolic leaf and define the currents $\tau_r := \frac{[\phi(\Delta_r)]}{A(r)}$, where Δ_r is the disk of radius r and $A(r)$ the area of $\phi(\Delta_r)$. Then, every limit current T of τ_r in the weak topology is a directed closed current of mass one.*

Actually, as we said above, this theorem is more general. In its usual statement, the averaging sequence of increasing subsets is more flexible. It just needs to satisfy a condition about the growth of the area.

If we do not have any closed current, we will not have any image of $\mathbb{C}$ directed by the lamination. We need to modify this procedure in order to have a constructive way to obtain harmonic directed currents. In this situation, every leaf L of the lamination is hyperbolic, and if $\phi : \mathbb{D} \to L$ is the universal covering then

$$\int_{\mathbb{D}} (1 - |\xi|)|\phi'(\xi)|^2 d\lambda(\xi) = \infty.$$

This estimate above suggests that the area of the image increases very fast and is crucial to prove

Theorem 1.19 ([FS05])**.** *Let $(X, \mathcal{L}, E)$ a lamination on a Kähler surface $(\mathscr{M}, \omega)$ and $\phi : \mathbb{D} \to L$, the universal cover of a leaf L. Define $T_r =$*

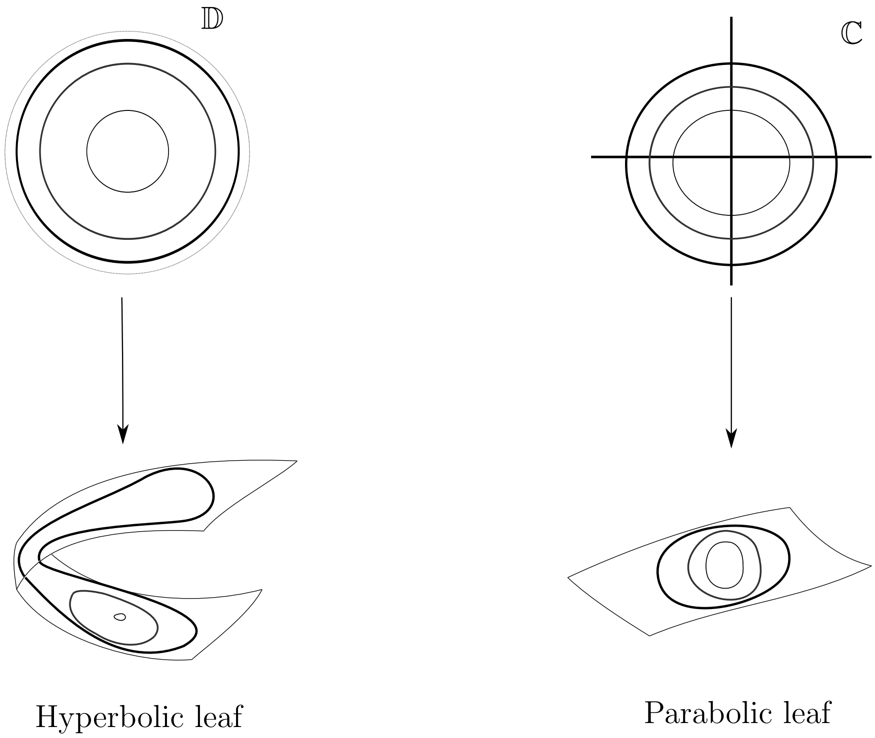

Figure 1.3: Coverings of different leaves

$\phi_*(\log^+ \frac{r}{|\xi|}[\Delta_r])$ *and average by its mass* $\tau_r := \frac{T_r}{\|T_r\|}$. *Then every limit current of* τ_r *is a harmonic directed current of the lamination of mass one.*

At this point, we can explain better the claim given in the introduction, where we say that unicity is important because it can be seen as a global atractor for the dynamics of the lamination. We have just seen how to construct a harmonic current as a very natural process consisting on averaging the integration current of images of a increasing sequence of concentric disks in the universal cover of the leaf by their area. Hence, *a priori*, different leaves of the laminations could generate different positive harmonic currents. Even, since this limit is taken in the weak topology, two different currents can be in the accumulation set of an averaging process starting from only one leave. However, if we have the unicity of directed harmonic currents, these phenomena cannot occur: at the end of this process we will always obtain the same positive directed harmonic current.

1.3 Intersection Theory

First of all, we will revisit [FS05] to clarify our exposition, further details can be found in that reference. Let (M,ω) be a homogeneous Kähler surface and T a real harmonic current of bidegree $(1,1)$ and order 0 in $\mathscr{M}$. If we denote the operator by $\square = (\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial})$, we say that T is $\square$ harmonic if $\square(T) = 0$. Then T can be decomposed as $T = \Omega + \partial S + \overline{\partial S}$, for a unique $\square$-harmonic form Ω of bidegree $(1,1)$ and a current S of bidegree $(0,1)$. The current S is not uniquely determined, but $\partial\overline{S}$ is. Moreover, T is closed if and only if $\partial\overline{S} = 0$.

Since $T = \Omega + \partial S + \overline{\partial S}$ with Ω and $\partial\overline{S}$ uniquely determined, the energy of T can be defined as

$$E(T) = \int \overline{\partial} S \wedge \partial \overline{S}$$

when $\overline{\partial}S$ is in L^2. Then $0 \leq E(T) < \infty$ and the energy depends only on T but not on the choice of S. Considering a scalar product $\langle\,,\rangle$ on the space of $\square$-harmonic forms, a real inner product and a seminorm are defined on $\mathcal{H}_e = \{T, \text{ with } E(T) < \infty\}$ as

$$\langle T_1, T_2\rangle_e = \langle \Omega_1, \Omega_2\rangle + \frac{1}{2}\left(\int \overline{\partial} S_1 \wedge \partial \overline{S}_2 + \overline{\partial} S_2 \wedge \partial \overline{S}_1\right)$$

$$\|T\|_e^2 = \langle \Omega, \Omega\rangle + \int \overline{\partial} S \wedge \partial \overline{S}.$$

With this seminorm we can define a Hilbert space H_e of classes $[T]$ as follows: T_1, T_2 are in the same class if and only if $T_1 = T_2 + i\partial\overline{\partial}u$ with $u \in L^1$ and u real.

Now, for T_1, T_2 currents, an intersection form Q is defined by

$$Q(T_1,T_2) = \int \Omega_1 \wedge \Omega_2 - \int (\overline{\partial} S_1 \wedge \partial \overline{S_2} + \overline{\partial} S_2 \wedge \partial \overline{S_1}).$$

Then $Q(T,T) = \int \Omega \wedge \Omega - 2E(T)$. This is a continuous bilinear form on H_e and $Q(T,T)$ is upper semicontinuous for the weak topology on H_e. If T is a harmonic positive current then $Q(T,T) \geq 0$. A class $[T]$ is positive if there is a positive harmonic current in the class $[T]$. Defining the hyperplane $\mathcal{H} = \{[T], [T] \in H_e, \int T \wedge \omega = 0\}$, it can be proven that Q is strictly negative definite on $\mathcal{H}$.

Next, this approach is used to study laminar currents. Let $(X, \mathcal{L}, E)$ be a laminated set with singularities in (M,ω), a Kähler surface. There exists a unique equivalence class $[T]$ of harmonic currents of mass one directed by the lamination and maximizing $Q(T,T)$ given that Q is strictly concave on $\mathcal{H}$. However, this uniqueness is for equivalence classes, not for currents. It is necessary to assume some extra hypotheses:

Theorem 1.20. *Let $(X, \mathcal{L}, E)$ be a laminated set with singularities in a Kähler surface (M, ω). Suppose E is a locally complete pluripolar set with 2-dimensional Hausdorff measure $\Lambda_2(E) = 0$. If there is no non-zero positive directed closed current, then there is a unique positive harmonic laminated current T of mass one maximizing $Q(T,T)$.*

This implies that under the same hypotheses, when $Q(T,T) = 0$ for every T positive laminated harmonic current, there exists a unique positive laminated harmonic current of mass one.

Finally, the case of a minimal lamination on $\mathbb{P}^2$ is considered, and it is proven that $Q(T,T) = 0$ for every T positive harmonic laminated current when the lamination is transversely Lipschitz or when the current has finite transversal energy. Fornaess and Sibony prove that a lamination in $M = \mathbb{P}^2$ verifies the following condition:

Condition 1. There exist:

- A family of automorphisms Φ_ϵ of $\mathscr{M}$ such that $\Phi_\epsilon \to id$ when $\epsilon \to 0$,
- a covering by flow boxes $\mathcal{U}$,
- a natural number $N_0 > 0$
- and a positive number $\epsilon_0 > 0$

such that for every ϵ with $|\epsilon| < \epsilon_0$ and for every pair of plaques Γ_α and Γ_β in a flow box of $\mathcal{U}$, the number of intersection points between Γ_α and $\Gamma_\beta^\epsilon = \Phi_\epsilon(\Gamma_\beta)$ is bounded from above by N_0.

Theorem 1.21. *[FS05] Let $(X, \mathcal{L})$ be a transversely Lipschitz lamination in a Kähler homogeneous compact surface $(\mathscr{M}, \omega)$ with no closed leaves satisfying Condition 1. For every harmonic directed current T of mass one $Q(T,T) = 0$.*

Proof. We know that if T is a $(1,1)$ positive directed harmonic current it can be written as

$$T = \int_A [\Gamma_\alpha] h_\alpha d\mu(\alpha)$$

in a flow box $\Delta \times A$, where h_α is a positive harmonic function in the plaque Γ_α. Hence, the pushforward of the current $T_\epsilon = (\Phi_\epsilon)_*(T)$ in a flow box can be written as

$$T_\epsilon = \int_A h'_\beta [\Gamma^\epsilon_\beta] d\mu'(\beta).$$

And the geometric self-intersection is defined in the flow box evaluated on a function ϕ as follows

$$T \wedge_g T_\epsilon(\phi) = \int \sum_{p \in J^\epsilon_{\alpha,\beta}} h_\alpha(p) {h'_\beta}^\epsilon(p) d\mu(\alpha) d\mu'(\beta)$$

where $J^\epsilon_{\alpha,\beta}$ are the intersection points between Γ_α and Γ^ϵ_β.

Since the lamination verifies Condition 1, the number of intersection points is bounded by N_0 which independent of ϵ. Therefore,

$$|(T \wedge_g T_\epsilon)(\phi)| \leq K \|\phi\|_\infty \int_{d_{min}(\Gamma_\alpha, \Gamma_\beta) \leq C\epsilon} N_0 d\mu(\alpha) d\mu(\beta) \to 0$$

because μ has no mass on single points.

Now, we need to prove that $Q(T, T) = \int T \wedge T = 0$. Since we are working on homogeneous Kähler surfaces, it is enough to prove this for smoothings $T^\delta, T^{\delta'}_\epsilon$, $Q(T^\delta, T^{\delta'}_\epsilon) \to 0$ when δ, δ' are small enough compared to ϵ, and δ, δ' and ϵ go to 0.

The estimate on the geometric wedge product is stable under small translations T_ϵ of T, so we can think of smoothing a current as an average of small translations.

Let ϕ be a test function supported in some local flow box. By definition, the value of the geometric wedge product on ϕ is

$$\langle T \wedge T_\epsilon \rangle_g(\phi) = \int \sum_{p \in J^\epsilon_{\alpha,\beta}} h_\alpha(p) {h'_\beta}^\epsilon(p) d\mu(\alpha) d\mu'(\beta).$$

But if we fix a plaque Γ^ϵ_β we can look for points in it which are also points of a plaque Γ_α and we write the intersection product as

$$\langle T \wedge T_\epsilon \rangle_g(\phi) = \int \left(\int_{\Gamma^\epsilon_\beta} [\phi h_\alpha {h'_\beta}^\epsilon](p) i\partial\overline{\partial} \log |w - f_\alpha(z)| d\mu(\alpha) \right) d\mu'(\beta).$$

These expressions are small when ϵ is small. The same applies when we do this for translations within small neighborhoods $U(\epsilon)$ of the identity in $Aut_0(M)$ and their smooth averages T^δ. So, if we consider ϕT^δ as a smooth test form we get

$$\langle T_\epsilon, \phi T^\delta \rangle = \int \left(\int_{\Gamma^\epsilon_\beta} [\phi h^\epsilon_\beta](p) T^\delta \right) d\mu(\beta).$$

Repeating the process, considering the averaging over small translations of T_ϵ, we get that $T_\epsilon^{\delta'} \wedge T^\delta(\phi) \to 0$ when $\delta, \delta' << \epsilon$ and $\epsilon \to 0$. Since this argument is made over flow boxes, we need to consider a partition of unity associated to the covering by flow boxes of the function ϕ such that we can obtain that this limit is zero in every flow box. Hence, $T_\epsilon^{\delta'} \wedge T^\delta = Q(T_\epsilon^{\delta'}, T^\delta) \to 0$. Therefore $Q(T,T) = 0$. □

Hence, we can state

Theorem 1.22 ([FS05]). *Let $(X, \mathcal{L})$ be a C^1 laminated compact set in $\mathbb{P}^2$, without compact curves, then X has a unique positive directed closed harmonic current T of mass 1.*

The hypothesis mentions neither minimality nor closed currents because by Hurder-Mitsumatsu [HM91], absence of compact curves implies no directed positive closed currents, and in $\mathbb{P}^2$ there is only one $X' \subset X$ minimal set (see [Zak01], for instance).

Note, that the proof of the theorem above is done on regular flow boxes. If we want to extend the result to laminations with singularities, we need to control the behavior close to the singularities, which is the only place where the intersection could be not 0.

We will show that Condition 1 holds for transversely Lipschitz laminations embedded in homogeneous compact Kähler surfaces. However, this is not always the case for every lamination. For laminations which are just transversely continuous, we can obtain weaker results.

Definition 1.23. A harmonic directed current T which can be written in flow boxes as $T = \int h_\alpha[\Gamma_\alpha] d\mu(\alpha)$, has *finite transverse energy* if in some local flow box

$$\int \log|\alpha - \beta| d\mu(\alpha) d\mu(\beta) > -\infty.$$

Condition 2. There exist:

- A family of automorphisms Φ_ϵ of $\mathscr{M}$ such that $\Phi_\epsilon \to id$ when $\epsilon \to 0$,
- a covering by flow boxes $\mathcal{U}$,
- a big positive number $A > 0$,
- and a small positive number $\epsilon_0 > 0$

such that for every ϵ with $|\epsilon| < \epsilon_0$ and for every pair of plaques Γ_α and Γ_β in a flow box of $\mathcal{U}$, the number of intersection points between Γ_α and $\Gamma_\beta^\epsilon = \Phi_\epsilon(V_\beta)$ is bounded from above by $A \log \frac{1}{|\epsilon|}$.

Therefore, we can state a theorem analogous to Theorem 1.21 for general laminations.

Theorem 1.24. *[FS05] Let $(X, \mathcal{L})$ be a lamination in $(\mathscr{M}, \omega)$, a Kähler homogeneous compact surface with no closed leaves satisfying Condition 2. For every harmonic directed current T of finite transverse energy and mass one, the self-intersection is $Q(T,T) = 0$.*

The proof is mostly the same as the one in the transversely Lipschitz case, but they differ on the estimates of the geometric self-intersection. See [FS05].

In [FS08], the case of holomorphic foliations with only hyperbolic singularities is considered. If the family of automorphisms satisfies certain general conditions that we will mention later, then the authors prove that the self-intersection in a neighbourhoood of the singularities is zero. Therefore, once we have proved that non singular transversely Lipschitz laminations embedded in these surfaces satisfy Condition 1, in order to prove the case with hyperbolic singularities, we just need to verify that these laminations also satisfy this Condition outside the singular neighborhoods. But the family of automorphisms must verify these general conditions mentioned above.

Chapter 2

Main Theorem

2.1 Statement and overview of the Theorem

The main theorem of this dissertation is the following

Theorem 2.1. *Let $(\mathcal{M}, \omega)$ be a homogeneous compact Kähler surface containing a minimal transversely Lipschitz lamination $\mathcal{L}$ by Riemann surfaces with hyperbolic singularities. If there are no closed currents directed by $\mathcal{L}$, then there is a unique directed harmonic current of mass one.*

It was explained in the preliminaries that our aim is to prove that every lamination transversely Lipschitz satisfies Condition 1 outside the singular neighborhoods, so we can apply Theorem 1.21 and get that the self-intersection of every directed harmonic current ought to be zero. Once we prove this, by the intersection theory explained in the preliminaries, we obtain the theorem above.

This theorem will be proven separately for each one of the surfaces under consideration, namely $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{T}^2$ and $\mathbb{P}^1 \times \mathbb{T}^1$, which together with the proof for $\mathbb{P}^2$ carried out in [FS05] and [FS08], complete the theorem for every Kähler homogeneous compact surface.

The common feature in the surfaces under consideration is their natural product structure. In the case without singularities, we will just need to consider a family of automorphisms that moves only horizontally or vertically, whereas the case with singularities will require a more complicated family of automorphisms. However, in both cases there will be a big open coordinate chart whose closure is the total surface and, in this big coordinate chart, the family of automorphisms will be seen as a family of translations. Hence, our main study will be focused ion the behavior of these laminations under translations. This is completely different than the situation in the case of $\mathbb{P}^2$ where the key was obtaining good expresions of the lamination close to a line which was fixed by the family of automorphisms under consideration.

Although the case with singularities includes the non singular case, we deal with both cases separately. The arguments differ in the choice of the family of automorphims. The families we choose in the non singular case are much easier than the ones in the case with singularities, so dealing with both cases independently allows us to understand the arguments better.

2.2 Lemmas and remarks

Let $(X, \mathcal{L}, E)$ be a lamination embedded in a surface M, and $p \in X \setminus E$ a regular point. If (z, w) are local coordinates around $p = (z_0, w_0)$ and

$$a\frac{\partial}{\partial z} + b\frac{\partial}{\partial w}$$

is a tangent vector to the lamination at p with $b \neq 0$, then we can take a polydisk $\Delta_{\delta,\delta'}$ centered at (z_0, w_0) such that $\{z = z_0\}$ is a local transversal and the plaques are parametrized as

$$\Gamma_w = \{(z, f_w(z)), z \in \Delta_{\delta,\delta'}(z_0)\}.$$

These are the sort of flow boxes we will consider in our arguments. Locally, the lamination can be seen as a holomorphic motion (see Example 3).

Proposition 2.2 (Bers-Royden [BR86]). *If we have a lamination in the unit polydisk of $\mathbb{C}^2$ where the leaves are*

$$\Gamma_w = \{(z, f_w(z)), z \in \Delta\}$$

satisfying $f_w(0) = w$ and $f_0(z) \equiv 0$ then the function $F(z, w) = f_w(z)$ is a holomorphic motion and we get the estimate

$$\frac{|w_0 - w_1|^{\frac{1+|z|}{1-|z|}}}{K} \leq |f_{w_0}(z) - f_{w_1}(z)| \leq K|w_0 - w_1|^{\frac{1-|z|}{1+|z|}}.$$

Consequently, if we just consider the polydisk $\Delta_{\delta/2,\delta'}$ we get the following remark.

Remark 2.2.1. We can always find a flow box around every regular point small enough to satisfy that

$$\frac{|w_0 - w_1|^2}{C} \leq |f_{w_0}(z) - f_{w_1}(z)| \leq C|w_0 - w_1|^{\frac{1}{2}}.$$

Note that the constant C is not exactly the constant K in the proposition, because we have to normalize the domain.

Remark 2.2.2. In the case of a transversely Lipschitz lamination the estimate is stronger

$$\frac{|w_0 - w_1|}{C} \leq |f_{w_0}(z) - f_{w_1}(z)| \leq C|w_0 - w_1|.$$

These considerations show the importance local estimates will have in the proofs. We still need to recall two lemmas from [FS05] which will be essential for our argument.

Lemma 2.3. *There is a number $1 > c_0 > 0$ such that, for every holomorphic function g defined on the unit disk $\mathbb{D}$, with $|g| < 1$ and having N zeros on $\mathbb{D}_{1/2}$, then $|g| < c_0^N$ on $\mathbb{D}_{1/2}$.*

Proof. Define $M_\alpha(z) = \frac{z+\alpha}{1+z\overline{\alpha}}$, the Möbius biholomorphism of the disk which sends 0 to α. Then by defining

$$c_0 = \sup_{|\alpha|\leq 1/2, |z|\leq 1/2} |M_{-\alpha}(z)| < 1,$$

we will see that we obtain the desired estimated.

Indeed, suppose that g has a zero at a with $|a| < 1/2$. If we set $f(z) = g(M_a(z))$, f is a holomorphic function that goes from the unit disk to the unit disk with $f(0) = 0$, hence by Schwarz's lemma, $|z| > |f(z)| = |g(M_a(z))|$ on the disk. Therefore, $|M_{-a}(z)| > |g(M_a(M_{-a}(z)))| = |g(z)|$ for $|z| < 1$.

Now, suppose that g has another zero at b. By applying Schwarz's lemma to the function $\frac{g}{M_{-a}} \circ M_b$, we get that $|\frac{g(M_b(z))}{M_{-a}(M_b(z))}| < |z|$. Now, by undoing the substitution, we get $|\frac{g(z)}{M_{-a}(z)}| < |M_{-b}(z)|$. Hence $|g(z)| < |M_{-a}(z)M_{-b}(z)|$.

Thus, if $a_1, a_2 \ldots, a_N$ are zeros of g on $\mathbb{D}_{1/2}$, by repeating this process, we obtain that $|g(z)| < |\prod_{i=1}^N M_{-a_i}(z)|$ on $|z| < 1$. Then, $|g(z)| < c_0^N$ for $|z| < 1/2$ if we take c_0 as above. ∎

Lemma 2.4. *Let g be a holomorphic function on the disk $\mathbb{D}$ with $|g| < 1$. If $|g| < \eta < 1$ on $\mathbb{D}_{1/4}$ then $|g| < \sqrt{\eta}$ on $\mathbb{D}_{1/2}$.*

Proof. Note that $\log|g| \leq \log\eta$ when $|z| \leq 1/4$. Since $\log|g(z)| - \log\eta\frac{\log|z|}{\log 1/4}$ is subharmonic in the annulus $1/4 < |z| < 1$, it reaches its maximum on its boundary. Then, $\log|g(z)| - \log\eta\frac{\log|z|}{\log 1/4} < 0$ in the annulus, so $\log|g| < \max\left\{\log\eta\frac{\log|z|}{\log 1/4}, \log\eta\right\}$. This implies that if $|z| < 1/2$, then $\log|g| < \log\eta/2$. $\square$

The first lemma will allow us to relate transversal distances with the number of zeros, whereas the second one will be very important in controlling the estimates when moving among flow boxes.

2.3 Nonsingular Case

2.3.1 Complex Tori

We want to study minimal laminations by Riemann surfaces embedded holomorphically in two dimensional tori. Then $\mathbb{T}^2 = \frac{\mathbb{C}^2}{\Lambda}$, and we have a locally injective projection $\pi : \mathbb{C}^2 \to \mathbb{T}^2$ which induces the complex structure on $\mathbb{T}^2$. Foliations on complex tori has been widely studied and classified. The classification for non singular foliations is done in the article of Ghys [Ghy96] and singular holomorphic foliations were classified by Brunella in [Bru10]. Regarding the case of 2-dimensional tori, one can see that only algebraic tori carry holomorphic foliations of codimension 1 with singularites.

Since the embedding is holomorphic, the flow boxes are open sets U on $\mathbb{C}^2$ where π is injective and we can write every plaque as a graph of a holomorphic function of z (horizontal flow box) or w (vertical flow box). Explicitly:

Definition 2.5. We say that a polydisk $U = \Delta_\delta(p_1) \times \Delta_{\delta'}(p_2) \subset \mathbb{C}^2$ is a *horizontal flow box* for a lamination $(X, \mathcal{L}) \subset \mathbb{T}^2$ centered at $p = (p_1, p_2)$ if $\pi_{|U}$ is injective and the plaques of $\mathcal{L}$ in $\pi(U)$ are

$$\Gamma_w = \{\pi(p_1 + z, w + f_w(z)),\, z \in \Delta_\delta\}$$

for every $w \in \pi(\{p_1\} \times \Delta_{\delta'}(p_2)) \cap X$ with f_w holomorphic and satisfying $f_w(0) = 0$. We can define analogously the notion of *vertical flow boxes*.

Definition 2.6. We say a point p of the lamination is *horizontal* if $(1,0)$ is a tangent vector to the lamination in p. If $(0,1)$ is a tangent vector to the lamination we say that p is a *vertical point*.

Definition 2.7. Let $(X,\mathcal{L})$ be a laminated compact set in $\mathbb{T}^2$. We say that the lamination has *invariant complex line segments* if there is an affine line Y in $\mathbb{C}^2$ and $U \subset \mathbb{C}^2$ open set, such that $Y \cap U \subset \pi^{-1}(X) \cap U$.

A typical situation with invariant complex line segments is a lamination where every leaf lifts to an affine complex line in the covering $\mathbb{C}^2$. We will refer to this situation as *holomorphically flat* laminations. In this sense, there is a paper of Ohsawa [Ohs06] where he proves that every C^∞ Levi-flat in $\mathbb{T}^2$ contains a complex segment. Hence if the foliation induced is minimal, it can only be holomorphically flat. Note that all the leaves of holomorphically flat laminations are parabolic. Hence, in these cases there is always a directed closed current.

Moreover, if we recall the discussion about foliations on complex tori in the beginning of this subsection and we look at their classification [Ghy96], it is easy to see that nonsingular holomorphic foliations in tori have always holomorphically flat leaves. Hence, these foliations will not satisfy our hypotheses. However, in [Bru10], the author proves that every leaf in a codimension 1 non singular holomorphic foliation in $\mathbb{T}^n$ with $n \geq 3$ accumulates towards the singular set, leaving unsolved the case of $\mathbb{T}^n$ when $n = 2$. If there is a leaf that does not accumulate towards the singular set, it induces a structure of non singular lamination in a set of $\mathbb{T}^2$ which is not a holomorphic foliation of the whole torus, and this lamination still might satisfy our hypotheses.

As we know, $\mathbb{T}^2$ is a complex connected Lie group, so the connected component of the identity of the group of automorphisms of the surface $\mathbb{T}^2$ is $Aut_0(\mathbb{T}^2) = \mathbb{T}^2$, and we will denote by

$$\tau_{(\epsilon_1,\epsilon_2)}(x_1,x_2) = (x_1+\epsilon_1, x_2+\epsilon_2)$$

a translation on $\mathbb{C}^2$ where $x_1, x_2, \epsilon_1, \epsilon_2 \in \mathbb{C}$. These translations induce the automorphisms on $\mathbb{T}^2$.

Proposition 2.8. *Let $(X,\mathcal{L})$ be a minimal lamination by Riemann surfaces embedded on a torus $\mathbb{T}^2 = \mathbb{C}^2/\Lambda$. If there exists $\epsilon_n \to 0$ such that $\tau_{(0,\epsilon_n)}(\mathcal{L}) = \mathcal{L}$, then either every point is vertical or there are no vertical points.*

Proof. Suppose there is a $p = (p_1, p_2)$ with vertical tangent. We can find a vertical flow box, $\psi_p : \mathscr{T}_p \times \Delta_\delta \to \mathbb{C}^2$ where $\psi_p(z, w) \to (z + f_z(w), p_2 + w)$, with π injective on the image of ψ_p.

Consider the plaque Γ_p passing through p. There are two options. The first one is that, for n big enough, the moved plaque $\Gamma_p^{\epsilon_n}$ is another plaque on the flow box. In this situation, the local transversal distance between Γ_p and $\Gamma_p^{\epsilon_n}$ is

$$d_z(\Gamma_p, \Gamma_p^{\epsilon_n}) = |p_1 + f_{p_1}(w) - p_1 - f_{p_1}(w - \epsilon_n)| > 0$$

for every n and for every z. But this means that $\frac{f_{p_1}(z) - f_{p_1}(z - \epsilon_n)}{\epsilon_n}$ has no zeros for any n big enough. On the other hand, this sequence converges uniformly to $f'_{p_1}(w)$, which has a zero at $w = 0$. Therefore, by Hurwitz's theorem, $f'_{p_1}(w) = 0$ for every z in the flow box. By analytical continuation, every point of the leaf is vertical and, because of the minimality, every point in the lamination is vertical.

The second option would be that for n big enough, the translation induces an automorphism on each leaf. In that case, the leaves are all vertical as well. □

Hence, if there were no vertical points, the lamination could be covered by horizontal flow boxes only, and for every point p we get a holomorphic function by analytic continuation, f_{L_p} such that $\pi(z, f_{L_p}(z)), z \in \mathbb{C}$ parametrizes L_p. On the other hand, if every point is vertical, then the lamination is holomorphically flat.

Proposition 2.9. *Let L be a leaf of a lamination $(X, \mathcal{L})$ embedded in a torus $\mathbb{T}^2$ and suppose that there is a holomorphic function $f_L : \mathbb{C} \to \mathbb{C}$ that parametrizes L by $\pi(z, f_L(z))$. Then f_L is linear. Then $\overline{L}$ contains a holomorphically flat laminated set.*

Proof. Applying Hurwitz's Theorem, we can ensure that either $\overline{L}$ contains a vertical leaf (namely a leaf whose points are vertical) or every leaf in $\overline{L}$ is a horizontal graph. If we are in the first situation, we have already obtained the desired statement. Hence, let us suppose that we are in the second one. If every leaf in $\overline{L}$ is a horizontal graph, then it means that there are no vertical points in it. Thus, assuming f'_L is not constant, there is a sequence z_n with $|f'_L(z_n)| \to \infty$. But $\pi(z_n, f_L(z_n))$ has a convergent subsequence in $\mathcal{L}$, $\pi(z_{n_k}, f_L(z_{n_k})) \to (z_0, w_0) \in \mathbb{T}^2$ and the unitary tangent in each point $\pi(z_{n_k}, f_L(z_{n_k}))$ is

$$\frac{(1, f'(z_{n_k}))}{\sqrt{1 + \|f'(z_{n_k})\|}}.$$

This sequence converges to the vector $(0,1)$ which is the unitary tangent vector to the lamination at (z_0, w_0). Therefore it is a vertical point, which lead us to a contradiction with the theorem assumptions.

This contradiction arises from the fact that f'_L was supposed unbounded. So it is bounded and by Liouville's theorem it is constant. Thus, the lamination induced on $\overline{L}$ is holomorphically flat. $\square$

We can conclude that a lamination on a torus has no invariant complex segments if and only if every leaf has horizontal and vertical points. Equivalently, a minimal lamination $\mathcal{L}$ has no complex segments if and only if there exists a neighborhood of the identity $U \subset Aut_0(\mathbb{T}^2)$ such that the minimal lamination $\mathcal{L}$ is not invariant for any automorphism $\Phi \in U$.

Theorem 2.10. *If $(X, \mathcal{L})$ is a transversely Lipschitz lamination by Riemann surfaces in $\mathbb{T}^2$ without invariant complex line segments, then it satisfies Condition 1 for Φ_ϵ horizontal or vertical translations.*

Proof. Since $\mathcal{L}$ has no complex segments, every leaf has vertical and horizontal points. For every vertical point p_v, we can take a relatively compact vertical flow box $\mathscr{T}_{p_v} \times \Delta_\delta$ such that f'_α has a finite number K_{p_v} of zeros on $\Delta_{\delta/2}$ for every $\alpha \in \mathscr{T}_{p_v}$ due to the absence of complex invariant line segments. These flow boxes will be called special vertical flow boxes. Since the set of the vertical points is closed, it is also compact, so it admits a finite covering by special flow boxes. We make an analogous argument for horizontal points, and, in this way, collecting all the flow boxes we obtain a relatively compact open set of the lamination, and the complement can be covered by polydisks which can be seen as horizontal or vertical flow boxes for our convenience.

Lemma 2.11. *Considering the family of horizontal translations, for every $p = (p_0, p_1)$ horizontal point, there exist:*

- *a horizontal flow box U_p, where the plaques are expressed like $\Gamma_{t_1} = (p_0 + z, t + f_t(z))$,*
- *a natural number N_p,*
- *a real number $\epsilon_p > 0$*

such that Γ_{t_1} and $\Gamma^{\epsilon}_{t_2}$ intersect each other at most in N_p points, and Γ_{t_1} and $\Gamma^{\epsilon}_{t_1}$ always intersect each other, for every t_1, t_2 in the transversal if $|\epsilon| < \epsilon_p$.

Proof. We start by considering a horizontal flow box $\Delta'_\delta(p_0) \times \mathscr{T}'(p_1)$. The plaque passing through $p = (p_0, p_1)$, $\Gamma_{p_1} = (p_0 + z, p_1 + f_{p_1}(z))$ satisfies that $f'_{p_1}(0) = 0$, and f'_{p_1} has $N_p > 0$ zeros in $\Delta_{\delta'/2}(p_0)$.

Since $\frac{f_{p_1}(z) - f_{p_1}(z-\epsilon)}{\epsilon} \to f'_{p_1}(z)$ when $\epsilon \to 0$, by Hurwitz's theorem, there exists ϵ'_0 such that $\frac{f_{p_1}(z) - f_{p_1}(z-\epsilon)}{\epsilon}$ has N_p zeros in $\Delta_{\delta'/2}(p_0)$ for ϵ with modulus smaller than ϵ'_0. This condition holds for every close enough plaques.

Next, we shrink the flow box to $\Delta_\delta(p_0) \times \mathscr{T}(p_1)$ in order to verify that there are a $\xi > 0$ and a $\epsilon_0 < \epsilon'_0$ such that

$$\left| \frac{f_t(z) - f_t(z-\epsilon)}{\epsilon} \right| > \xi$$

for every t in the transversal, $|\epsilon| < \epsilon_0$ and $z \in \partial\Delta_{\delta/2}(p_0)$ and still satisfying that $\frac{f_t(z) - f_t(z-\epsilon)}{\epsilon}$ has N_p zeros inside $\Delta_{\delta/2}(p_0)$.

Then, if $\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}$ intersect in N points, by Lemma 2.3, $d_z(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) < c^N|\epsilon|$, if $z \in \Delta_{\delta/2}(p_0)$. Then

$$c^N|\epsilon| > d_z(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) > d_z(\Gamma_{t_2}, \Gamma^{\epsilon}_{t_2}) - d_z(\Gamma_{t_2}, \Gamma_{t_1}) > \xi|\epsilon| - d_z(\Gamma_{t_1}, \Gamma_{t_2}).$$

Then, we get that $d_z(\Gamma_{t_1}, \Gamma_{t_2}) > (\xi - c^N)|\epsilon|$ in $\partial\Delta_{\delta/2}(p_0)$. By Lipchitzness, $d_z(\Gamma_{t_1}, \Gamma_{t_2}) > \frac{(\xi - c^N)|\epsilon|}{C^2}$ for every $z \in \Delta_{\delta/2}(p_0)$.

On the other hand,

$$d_z(\Gamma_{t_2}, \Gamma^{\epsilon}_{t_2}) > d_z(\Gamma_{t_1}, \Gamma_{t_2}) - d_z(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) > \left(\frac{\xi - c^N}{C^2} - c^N \right) |\epsilon|.$$

Therefore, if N is big enough, this last number is positive, which would imply that Γ_{t_2} and $\Gamma^{\epsilon}_{t_2}$ would not intersect, so we would get a contradiction. □

Since the set of horizontal points is compact, we can find a finite covering by flow boxes $U_1, \ldots, U_{k_h}$ centered in $p_1, \ldots, p_{k_h}$ respectively as we did in the previous lemma. Let us call $N_h = \max_i N_{p_i}$, and $\epsilon_h = \min_i \epsilon_{p_i}$.

We can reason analogously for vertical points and vertical translations, and we get a covering by flow boxes $V_1, \ldots, V_l$, N_v and ϵ_v in the same way. Finally, take $\epsilon_0 = \min(\epsilon_h, \epsilon_v)$ and $N_1 = \max\{N_h, N_v\}$.

For $\epsilon < \epsilon_0$, $\tau_{(0,\epsilon)}(z,w) = (z, w+\epsilon)$ is a vertical translation, and we suppose that we have a horizontal flow box where we have N' intersection points between two plaques, $\Gamma_\alpha, \Gamma_\beta$ when we move one of them by the translation $\Gamma_\beta^\epsilon = \tau_{(0,\epsilon)}(\Gamma_\beta)$. In this case, the transversal distance defined on every $z \in \Delta_\delta$ is $d_z(\Gamma_\beta, \Gamma_\alpha) = |\alpha + f_\alpha(z) - \beta - f_\beta(z)|$, and as $\mathcal{L}$ is a transversely Lipschitz lamination, we have

$$\frac{|\alpha - \beta|}{C} < d_z(\Gamma_\alpha, \Gamma_\beta) < C|\alpha - \beta|$$

for certain global constant C independent of the flow box. Since Γ_α and Γ_β^ϵ intersect, there is z_0 with $d_{z_0}(\Gamma_\alpha, \Gamma_\beta) = \epsilon$. Hence

$$\frac{\epsilon}{C^2} < d_z(\Gamma_\alpha, \Gamma_\beta) < C^2\epsilon.$$

There is also a constant $b > 1$ such that the following holds: if Γ_1 and Γ_2 are two plaques in a flow box with $d_z(\Gamma_1, \Gamma_2)$, the transversal distance on it, and Γ_1', Γ_2' are their continuations in an adjacent flow box with the transversal distance $d_z'(\Gamma_1, \Gamma_2)$ then

$$\frac{\min d_z'(\Gamma_1', \Gamma_2')}{b} \leq \min d_z(\Gamma_1, \Gamma_2) \leq \max d_z(\Gamma_1, \Gamma_2) \leq b \max d_z'(\Gamma_1', \Gamma_2').$$

This b depends on neither the flow box nor the plaques.

As we have a finite covering, and every leaf has vertical points, we can reach a special vertical flow box following a path with at most M changes of flow boxes where M is a global bound. Hence, we get

$$\frac{|\epsilon|}{C^2 b^M} < d_z(\Gamma_{\alpha_0}, \Gamma_{\beta_0}) < C^2 b^M |\epsilon|$$

where α_0 and β_0 are the analytic continuation of the plaques.

Due to the transversal Lipschitzness of the lamination, we can find a global constant K' such that, for every flow box continuing Γ_α and Γ_β, say $\Gamma_{\alpha'}, \Gamma_{\beta'}$ we have

$$\frac{d_z(\Gamma_{\alpha'}, \Gamma_{\beta'})}{K'|\epsilon|} < \frac{1}{b^2}.$$

By Lemma 2.3, there is $c < 1$ such that

$$\frac{d_z(\Gamma_\alpha, \Gamma_\beta^\epsilon)}{K'|\epsilon|} < c^{N'} < \frac{1}{b^2},$$

then we can see this transversal distance in the next plaques, and considering the distortion, it satisfies that

$$\frac{d_z'(\Gamma_{\alpha'}, \Gamma_{\beta'}^\epsilon)}{K'|\epsilon|} < bc^{N'} < 1.$$

Hence, in a bigger disk, by Lemma 2.4, they would differ at most by $(bc^{N'})^{1/2}$. Repeating the argument until we arrive to the vertical special flow box, we get that $d'_z(\Gamma_{\alpha_0}, \Gamma^{\epsilon}_{\beta_0}) < K'|\epsilon| b^2 c^{N'/2^M}$.

So, we should have that, by triangular inequality,

$$d_z(\Gamma_{\beta_0}, \Gamma^{\epsilon}_{\beta_0}) \geq d_z(\Gamma_{\alpha_0}, \Gamma_{\beta_0}) - d_z(\Gamma_{\alpha_0}, \Gamma^{\epsilon}_{\beta_0}) \geq \left(\frac{1}{C^2 b^M} - K' b^2 c^{N'/2^M} \right) |\epsilon|$$

but, if N' is big enough to make $\frac{1}{C^2 b^M} > K'|\epsilon| b^2 c^{N'/2^M}$, this would mean that $\Gamma_{\beta'}, \Gamma^{\epsilon}_{\beta'}$ does not intersect each other, but they do. Therefore, making $N_0 = \max\{N', N_1\}$, we obtain the N_0 appearing in Condition 1 for vertical translations.

This argument can be made analogously for horizontal translations.

□

Theorem 2.12. *Let $(X, \mathcal{L})$ be a lamination by Riemann surfaces in $\mathbb{T}^2$ without invariant complex line segments. Then the lamination satisfies Condition 2*

Proof. The proof is similar to the previous one, but the estimates are slightly different. We will try to be consistent with the notation of the Theorem 2.10. Here, since the lamination is a holomorphic motion, we can take horizontal and vertical flow boxes as we said before, such that

$$\frac{|\alpha - \beta|^2}{C} \leq |\alpha + f_\alpha(z) - \beta - f_\beta(z)| \leq C|\alpha - \beta|^{1/2}.$$

We can consider a covering by flow boxes as before, where these inequalities hold for transversal distances, and taking ϵ_0 small enough to assure that a plaque on a special horizontal flow box and the same plaque moved by a horizontal translation have to intersect each other.

We need to understand the behavior of the lamination under the action of $\tau_{(\epsilon,0)}$. Assume that we have N crossing points on a vertical flow box. In this case, the following inequality holds

$$\frac{\epsilon^4}{K} \leq d_z(\Gamma_\alpha, \Gamma_\beta) \leq K|\epsilon|^{1/4}$$

for certain $K > 2$ non depending on ϵ. Then, we can reach a special horizontal flow box by a path in at most M changes of flow boxes and α' and β' are the corresponding plaques in this flow box. Hence

$$\frac{|\epsilon|^{4^M}}{K^M b^M} < d_z(\Gamma'_\alpha, \Gamma'_\beta) < b^M K^M |\epsilon|^{1/4^M}.$$

By similar arguments, we can find a constant c verifiying the estimate

$$\frac{d_z(\Gamma_\alpha, \Gamma_\beta^\epsilon)}{K|\epsilon|^{1/4^M}} < c^N < \frac{1}{b^2}.$$

Hence, as in the Lipschitz case,

$$d_z(\Gamma'_\alpha, \Gamma'^\epsilon_\beta) < b^2 c^{N/2^M} K|\epsilon|^{1/4^M}.$$

But, by triangular inequality again,

$$d_z(\Gamma'_\beta, \Gamma'^\epsilon_\beta) > d_z(\Gamma'_\alpha, \Gamma'_\beta) - d_z(\Gamma'_\alpha, \Gamma'^\epsilon_\beta) > \frac{|\epsilon|^{4^M}}{K^M b^M} - b^2 c^{N/2^M} K|\epsilon|^{1/4^M}$$

and if

$$N > \frac{(4^M - (1/4)^M)\log|\epsilon|}{1/2^M \log c} - \frac{\log(2b^{M+2}K^{M+1})}{1/2^M \log c} = A \log\frac{1}{|\epsilon|} + B$$

then $d_z(\Gamma'_\beta, \Gamma'^\epsilon_\beta) > \frac{|\epsilon|^{4^M}}{2K^M b^M} > 0$, hence $\Gamma'_\beta, \Gamma'^\epsilon_\beta$ would not intersect each other. The contradiction arises if N is too big compared to $-\log|\epsilon|$. □

Corollary 2.13. *Let $(X, \mathcal{L})$ be a transversely Lipschitz lamination in $\mathbb{T}^2$ with no directed positive closed currents. Then there is a unique harmonic current T of mass one directed by the lamination. In particular, there is only one minimal set.*

2.3.2 Products of curves

In this section we will deal with the case of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{T}^1 \times \mathbb{P}^1$. We have a slightly different definition of verticality and horizontality here, but it is still natural based on their standard parametrizations. We define $\phi_1 : \mathbb{C} \to \mathbb{P}^1$ as $\phi_1(w) = [1 : w]$, and $\phi_2 : \mathbb{C} \to \mathbb{P}^1$ as $\phi_2(z) = [z : 1]$. For $\mathbb{T}^1$, since $\pi : \mathbb{C} \to \mathbb{T}^1$ is locally injective, there exists $\delta > 0$ such that $\pi_{|\Delta_\delta(z)}$ is injective for every $z \in \mathbb{C}$. So, every p in $X = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{T}^1 \times \mathbb{P}^1$ admits a parametrization $\varphi = (\varphi_1, \varphi_2)$ where φ_i are injective restrictions to disks of those functions.

Definition 2.14. An open subset $U \subset X$ is a *horizontal flow box centered* on $p = (p_1, p_2)$ if there is a parametrization as above with $\varphi(z_0, w_0) = (p_1, p_2)$, a disk D_1 centered at 0, a subset A contained on a disk D_2 centered at 0, such that the plaques of $\mathcal{L}_{|U}$ are parametrized by $\varphi(z_0 + z, w_0 + \alpha + f_\alpha(z))$ for every $\alpha \in A$.

Definition 2.15. We will say that a point p of the lamination is *horizontal* if $\pi_2(T_p\mathcal{L}) = 0$.

We define analogously *vertical flow boxes* and *vertical points*, and we can cover our lamination by horizontal or vertical flow boxes. Note that if p is a horizontal point we can take a horizontal flow box on a neighborhood of p, and if $\varphi(z_0, w_0) = p$, then $f_0'(0) = 0$.

Proposition 2.16. *Every minimal lamination $(X, \mathcal{L})$ in $\mathbb{T}^1 \times \mathbb{P}^1$ either has horizontal points, or is $\mathbb{T}^1 \times \{p\}$. Furthermore, if $(X, \mathcal{L})$ is embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ and there is a leaf L without horizontal points, then $L = (f(p), p)$ is a closed leaf for $f : \mathbb{P}^1 \to \mathbb{P}^1$ holomorphic.*

Proof. The proof is analogous to Proposition 2.8. We can consider a covering only with vertical flow boxes and, beginning with a vertical plaque Γ_α with a parametrization $\varphi(f_\alpha(z), z)$, we can extend f_α to obtain a holomorphic function from $\mathbb{P}^1$ to the first factor of the surface. If the first factor is $\mathbb{P}^1$, this function is rational, but if the first factor is $\mathbb{T}^1$ there are no nonconstant holomorphic functions from $\mathbb{P}^1$ to $\mathbb{T}^1$.

□

Clearly, the same is true for vertical points in $\mathbb{P}^1 \times \mathbb{P}^1$. So every lamination $(X, \mathcal{L})$ embedded in it without compact curves has vertical and horizontal points.

Theorem 2.17. *Let $(X, \mathcal{L})$ be a lamination without compact leaves having only one minimal set in $M = \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that the point $p = ([1 : 0], [1 : 0])$ is neither vertical nor horizontal and belongs to the minimal set. Let Φ_ϵ be the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ defined as $\Phi_\epsilon([z_1 : z_2], [w_1 : w_2]) = ([z_1 + \epsilon z_2 : z_2], [w_1 : w_2])$. Then, the lamination verifies Condition 1 if it is transversely Lipschitz or Condition 2 otherwise, for this family of automorphisms.*

Proof. We will explain the Lipschitz case. The only difference with non Lipschitz case is that the last one has slightly more complicated inequalities as we could see in Theorem 2.12.

First of all, we notice that $[1 : 0] \times \mathbb{P}^1$ is invariant for every Φ_ϵ and we consider a flow box B_0 centered at p.

$$\begin{aligned} \varphi : \Delta_\delta \times A &\to B_0 \subset \mathbb{P}^1 \times \mathbb{P}^1 \\ (z, w) &\mapsto ([1 : z], [1 : f_w(z) + w]) \end{aligned}$$

small enough to hold that $0 < |\frac{f_0'(0)}{2}| < |f_w'(z)| < 2|f_0'(0)|$. We cover $\mathbb{P}^1 \times \mathbb{P}^1 \setminus B_0$ by horizontal or vertical flow boxes, and we obtain a covering $\mathcal{B} = \{B_i\}$.

The automorphism Φ_ϵ sends (z, w) to $(\frac{z}{1+\epsilon z}, w)$, so the transversal distance between a plaque Γ_β of $\mathcal{L}$ and Γ_β^ϵ, the same one moved by Φ_ϵ, is

$$\begin{aligned} d_z(\Gamma_\beta, \Gamma_\beta^\epsilon) &= \left|\beta + f_\beta(z) - \beta - f_\beta\left(\frac{z}{1-\epsilon z}\right)\right| \\ &= \left|f_\beta(z) - f_\beta\left(\frac{z}{1-\epsilon z}\right)\right| \\ &\geq k\left|z - \frac{z}{1-\epsilon z}\right| \\ &= k\left|\epsilon\frac{z^2}{1-\epsilon z}\right| \end{aligned}$$

for $k = |f_0'(0)|/4$ if ϵ small enough.

In this situation,

$$\max_{|z|\leq\delta} d_z(\Gamma_\beta, \Gamma_\beta^\epsilon) = \max_{|z|=\delta} d_z(\Gamma_\beta, \Gamma_\beta^\epsilon) \geq k|\epsilon||\delta|^2/2.$$

Now, we repeat the argument. Consider two plaques Γ_α and Γ_β^ϵ which intersect each other in N points. Following a path, we reach B_0 in at most M changes of flow boxes which is independent of the plaques. Let α' and β' be the analytic continuation of the original plaques, and by same reasoning of Theorem 2.10, we obtain that, $d_z(\Gamma_{\beta'}^\epsilon, \Gamma_{\alpha'}) \leq K'|\epsilon|b^2c^{N'/2^M}$ if $|z| \leq \delta$, in fact for $z = 0$, $d_0(\Gamma_{\beta'}^\epsilon, \Gamma_{\alpha'}) = |\alpha' - \beta'| \leq K'|\epsilon|b^2c^{N'/2^M}$, then

$$d_z(\Gamma_{\alpha'}, \Gamma_{\beta'}) \leq C|\alpha' - \beta'| \leq CK'|\epsilon|b^2c^{N'/2^M}.$$

Finally,

$$\begin{aligned} \left|\frac{k\epsilon\delta^2}{2}\right| &\leq \max_{|z|\leq\delta} d_z(\Gamma_{\beta'}^\epsilon, \Gamma_{\beta'}) \\ &\leq \max_{|z|\leq\delta} d_z(\Gamma_{\beta'}^\epsilon, \Gamma_{\alpha'}) + \max_{|z|\leq\delta} d_z(\Gamma_{\beta'}, \Gamma_{\alpha'}) \\ &\leq K'|\epsilon|b^2c^{N'/2^M} + CK'|\epsilon|b^2c^{N'/2^M} \end{aligned}$$

then if N is big enough to hold $k|\epsilon||\delta|^2/2 > (C+1)K'|\epsilon|b^2c^{N'/2^M}$, a contradiction arises. So the number of intersection points is bounded by certain N_0. $\square$

Theorem 2.18. *Let $(X, \mathcal{L})$ be a lamination without compact leaves and having only one minimal set embedded in $M = \mathbb{T}^1 \times \mathbb{P}^1$. Let $\Phi_\epsilon([z_1], [w_1 : w_2]) = ([z_1 + \epsilon], [w_1 : w_2])$ be a family of automorphisms. Then, the lamination verifies Condition 1 if it is Lipschitz or Condition 2 otherwise, for this family of automorphisms.*

Proof. The proof of this theorem is similar to theorems 2.10 and 2.12. Since $\mathcal{L}$ has no compact leaves, there are non horizontal points. Hence, we just need to take a finite covering of the horizontal points by special horizontal flow boxes, find ϵ_0 small enough to hold that every plaque in these flow boxes intersects itself when we move it by Φ_ϵ if $|\epsilon| < \epsilon_0$, and get the same contradiction we obtain in theorems 2.10 and 2.12. □

2.3.3 End of the argument

In the past two subsections, we prove that, under different hypotheses, laminations embedded on the surfaces under study satisfy Conditions 1 or 2, depending on whether the lamination is transversely Lipschitz or not. As stated at the beginning of this section, and proven in the previous one, this is sufficient to ensure the unicity of positive harmonic currents directed by the lamination. Let us state explicitely the theorem that we have obtained with the wider generality we have so far.

Theorem 2.19. *Let $\mathcal{L}$ be a transversely Lipschitz lamination. If we are in one of the following situations*

- *it has a unique minimal set, it is embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ without invariant closed curves,*
- *it has a unique minimal set, it is embedded in $\mathbb{P}^1 \times \mathbb{T}^1$ without invariant closed curves,*
- *or it is embedded in $\mathbb{T}^2$ without invariant complex segments, then*

every harmonic current of mass one T directed by the lamination satisfies $Q(T, T) = 0$.

If the lamination is not transversely Lipschitz, then every harmonic current of mass one T directed by the lamination with finite transverse energy satisfies that $Q(T, T)$.

Corollary 2.20. *Let $\mathcal{L}$ be a transversely Lipschitz lamination by Riemann surfaces without directed closed currents. If we are in one of the following situations*

- *it has a unique minimal set and is embedded in* $\mathbb{P}^1 \times \mathbb{P}^1$,

- *it has a unique minimal set and is embedded in* $\mathbb{P}^1 \times \mathbb{T}^1$,

- *or it is embedded in* $\mathbb{T}^2$*, then*

the lamination has only one directed harmonic current of mass one.

Note that, in the case of $\mathbb{T}^2$, this Corollary implies the unicity of the minimal set. In the last section, we will sharpen these results and obtain some interesting corollaries.

2.4 Singular Case

We need to prove Condition 1 outside the singular neighborhoods and apply the results of [FS05] to them. For this reason, we need to control the situation in the singular neighborhoods and the family of automorphisms given in the previous section could not give this control. Therefore, we need to study a wider class of automorphisms in order to get the desired result.

2.4.1 Case of $\mathbb{P}^1 \times \mathbb{P}^1$

We consider $\mathbb{P}^1 \times \mathbb{P}^1$ with the Fubini-Study metric in each factor. Since it is a product space then $T(\mathbb{P}^1 \times \mathbb{P}^1) = T\mathbb{P}^1 \times T\mathbb{P}^1$. Hence, we have a notion of verticality and horizontality in the tangent bundle defined in the natural way.

Assume that the lines $[1 : 0] \times \mathbb{P}^1$ and $\mathbb{P}^1 \times [1 : 0]$ do not contain any singularity, $p = ([1 : 0], [1 : 0]) \in \mathcal{L}$ and $T_p\mathcal{L}$ is neither vertical nor horizontal.

Therefore, we have four different charts covering $\mathbb{P}^1 \times \mathbb{P}^1$, $\psi_i : \mathbb{C}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3, 4$ defined as follows:

a) $\psi_1(z, w) = ([z : 1], [w : 1])$,

b) $\psi_2(z, w) = ([1 : z], [w : 1])$,

c) $\psi_3(z, w) = ([z : 1], [1 : w])$,

d) $\psi_4(z, w) = ([1 : z], [1 : w])$.

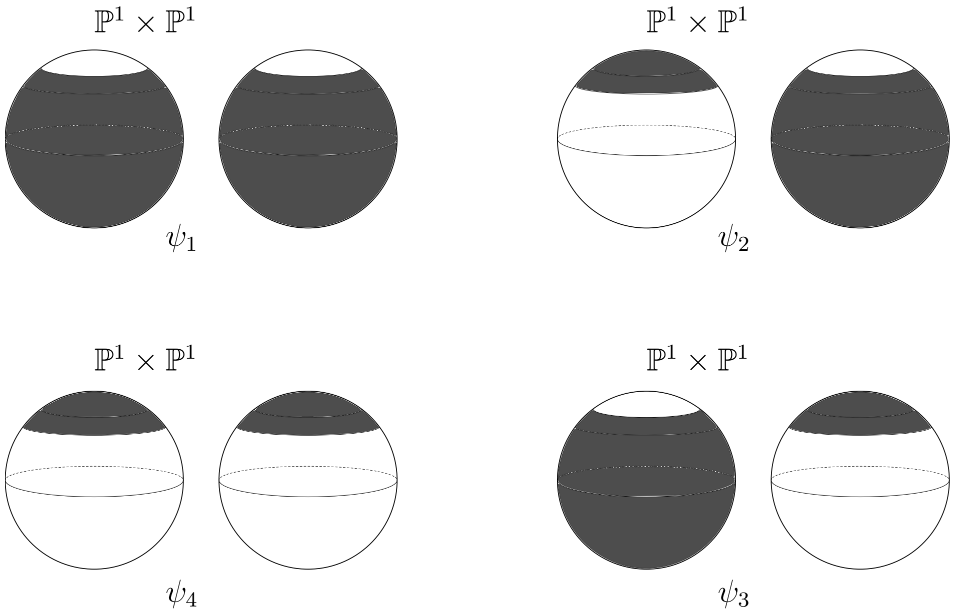

Figure 2.1: Sketch of the covering of $\mathbb{P}^1 \times \mathbb{P}^1$

Clearly every singularity is contained in the image of ψ_1.

The family of automorphisms we are searching for is

$$\Phi_\epsilon([z_1 : z_2], [w_1 : w_2]) = ([z_1 + \epsilon v_1 z_2 : z_2], [w_1 + \epsilon v_2 w_2 : w_2])$$

for a suitable vector (v_1, v_2). However, we have to choose it carefully according to the behavior of the lamination in a neighborhood of a singularity.

Let $s_1, s_2, \ldots, s_n$ be the singularities. Since they are hyperbolic, there exist A_{i_A} a linearizable neighborhood around $\psi_1^{-1}(s_{i_A})$ and a change of coordinates $\phi_{i_A} : A_{i_A} \to \Delta^2_{\delta,\delta'}$ with $\phi_{i_A}(\psi_{i_A}^{-1}(s_{i_A})) = (0,0)$ such that in the new coordinates (z', w'), the leaves of the lamination are integral varieties of the 1-form $w'dz' - \lambda_{i_A} z'dw'$, with this λ_{i_A} veryfing that $\lambda_{i_A} \notin \mathbb{R}$. Hence, the separatrices are $\{w' = 0\}$ and $\{z' = 0\}$. Φ_ϵ would act as a translation by $(\epsilon v_1, \epsilon v_2)$ in $\psi_1^{-1}(A_{i_A}) = \Delta^2_{\delta,\delta'}$, namely $\Phi_\epsilon(z, w) = (z + \epsilon v_1, w + \epsilon v_2)$.

Next we define $\Phi_\epsilon^{i_A} = \phi_{i_A}^{-1} \Phi_\epsilon \phi_{i_A}$, and $\Phi_\epsilon^{i_A}$ has to hold the conditions of [FS10]: it can be written as $(\alpha(\epsilon), \beta(\epsilon)) + (z', w') + \epsilon O(z', w')$ with $\alpha'(0), \beta'(0) \neq 0$ and $\frac{\beta'(0)}{\alpha'(0)} \neq \lambda_{i_A}$. Notice that $(\alpha'(0), \beta'(0)) = (D\phi_{i_A}^{-1})_{\Phi_\epsilon(0,0)}(v_1, v_2) =: (v_1^{i_A}, v_2^{i_A})$. The third element of the sum appears if and only if ϕ_{i_A} is not linear. In fact, it is not linear because in that case the lamination would have a directed closed current, the integration current on the separatrix, which would be a projective line. These conditions must hold around every singularity. Therefore we have to choose a vector (v_1, v_2) such that:

(i) $v_1^{i_A}, v_2^{i_A} \neq 0$ and $\frac{v_2^{i_A}}{v_1^{i_A}} \neq \lambda_{i_A}$,

(ii) (v_1, v_2) is unitary,

(iii) $v_1, v_2 \neq 0$ and (v_2, v_1) is not a tangent vector to the lamination at p,

(iv) (v_1, v_2) is tangent to the lamination at certain point $p' \in \mathbb{C}^2 \setminus (\bigcup A_{i_A})$.

So, we have fixed (v_1, v_2) and we have the family of automorphisms Φ_ϵ. The next step is choosing a good covering of the lamination $\mathcal{L}$ as follows:

(1) We already have linearizable neighborhoods of the singularities where [FS10] can be applied, we will denote them by A_{i_A}. We will call them *singular neighborhoods*.

(2) We need a neighborhood U_0 of p, because it is a fixed point for every element of the family of automorphisms. We will find it by using ψ_4.

(3) Afterwards, we cover $\mathbb{P}^1 \times [1:0] \setminus U_0$ via ψ_3 with two types of flow boxes, horizontal $W^a_{j_W}$ and and vertical $W^t_{i_W}$. The superindices come from "along" and "transversal", referring to the behavior of the laminations with respect to the automorphisms.

(4) Same for $[1:0] \times \mathbb{P}^1 \setminus U_0$ with ψ_2. We obtain $V^t_{i_V}$ and $V^a_{j_V}$.

(5) And finally, by using ψ_1, we consider flow boxes $B^a_{j_B}$ and $B^t_{i_B}$ covering the rest of the points of $\mathbb{P}^1 \times \mathbb{P}^1$ depending on whether every plaque is transversal to the motion or not, respectively.

Lemma 2.21. *There is a flow box U_0 centered at $p = ([1:0],[1:0])$ biholomorphic to $\Delta_\delta \times \mathscr{T}$ and an $\epsilon'_0 > 0$ such that, if Γ_w and $\Gamma^\epsilon_{w'}$ intersect each other in N'_0 points, then the vertical distance in $|z| = \delta$ verifies*

$$d_z(\Gamma_w, \Gamma_{w'}) > c_0|\epsilon|$$

with certain $c_0 > 0$ for every ϵ with $|\epsilon| < \epsilon'_0$.

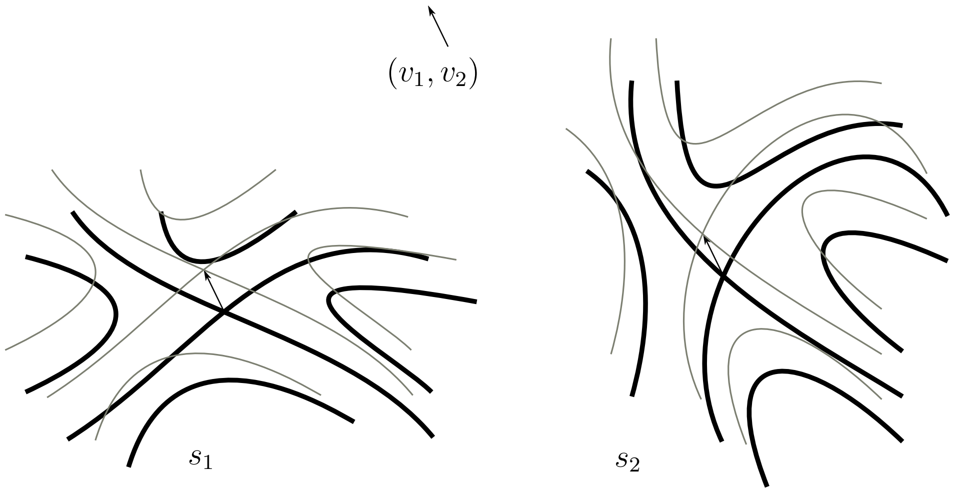

Figure 2.2: Election of the vector

Proof. We will use ψ_4. Consider a horizontal flow box $U_0' = \Delta_\delta \times \mathscr{T}$ centered at p; Δ_δ is a disk centered at 0, and $\mathscr{T}$ is a topological space containing 0. The points in the flow box can be written as $(z, w + f_w(z))$, where f_w are holomorphic functions satisfying $f_w(0) = 0$ for every $w \in \mathscr{T}$.

Since $f_0'(0) \neq 0$ and (v_2, v_1) is not a scalar multiple of $(1, f_0'(0))$, we can choose U_0 verifying that $m < |f_w'(z)| < M$, $|f_w'(z) - \frac{v_1}{v_2}| > m_0 > 0$ for every $(z, w) \in \Delta_\delta \times \mathscr{T}$, and as $f_w(z) = g_w(z)z$ for certain holomorphic function g_w varying continuously with w. Furthermore, we can also require $m < |g_w(z)| < M$ and $|g_w(z) - \frac{v_1}{v_2}| > m_0 > 0$ for every $(z, w) \in \Delta_\delta \times \mathscr{T}$.

Now, we want to find δ_0 small enough to get that if Γ_w and $\Gamma_{w'}^\epsilon$ intersect each other in N_0 points, then the vertical distance in z satisfies

$$d_z(\Gamma_w, \Gamma_{w'}) > d_z(\Gamma_{w'}, \Gamma_{w'}^\epsilon) - d_z(\Gamma_w, \Gamma_{w'}^\epsilon) > c_0|\epsilon|$$

with certain $c_0 > 0$ for every z with $|z| = \delta_0$. The idea is to find a lower bound for d_z. Since $\mathcal{L}$ is transversely Lipschitz, we can find the bound for Γ_0 and later shrink the transversal to ensure that every plaque holds the inequality.

In the domain of ψ_4,

$$\Phi_\epsilon(z, w) = \left(\frac{z}{1 + \epsilon v_1 z}, \frac{w}{1 + \epsilon v_2 w}\right),$$

then

$$\Gamma_0^\epsilon = \left\{\left(\frac{z}{1 + \epsilon v_1 z}, \frac{f_0(z)}{1 + \epsilon v_2 f_0(z)}\right), z \in \Delta_\delta\right\}.$$

Hence, if we fix $z \in \Delta_\delta$ such that $z' = \frac{z}{1+\epsilon v_1 z} \in \Delta_\delta$, then $z = \frac{z'}{1-\epsilon v_1 z'}$. Thus, the transversal distance at a point z is

$$d_z(\Gamma_0, \Gamma_0^\epsilon) = \left| f_0(z) - \frac{f_0(\frac{z}{1-\epsilon v_1 z})}{1 + \epsilon v_2 f_0(\frac{z}{1-\epsilon v_1 z})} \right|.$$

We can write it as follows

$$\begin{aligned}
d_z(\Gamma_0, \Gamma_0^\epsilon) &= \left| z g_0(z) - \frac{(\frac{z}{1-\epsilon v_1 z}) g_0(\frac{z}{1-\epsilon v_1 z})}{1 + \frac{z\epsilon v_2}{1-\epsilon v_1 z} g_0(\frac{z}{1-\epsilon v_1 z})} \right| \\
&= \left| z g_0(z) - \frac{z g_0(\frac{z}{1-\epsilon v_1 z})}{1 + z\epsilon \left(-v_1 + v_2 g_0(\frac{z}{1-\epsilon v_1 z})\right)} \right| \\
&= \left| \frac{z \left[g_0(z) - \epsilon z g_0(z) \left(v_2 g_0\left(\frac{z}{1-\epsilon v_1 z}\right) - v_1 \right) - g_0\left(\frac{z}{1-\epsilon v_1 z}\right) \right]}{1 + \epsilon z \left(-v_1 + v_2 g_0 \left(\frac{z}{1-\epsilon v_1 z}\right)\right)} \right| \\
&\geq \left| \frac{z}{1 + z\epsilon \left(-v_1 + v_2 g_0\left(\frac{z}{1-\epsilon v_1 z}\right)\right)} \right| (F - G),
\end{aligned}$$

where

$$\begin{aligned}
F &:= \left| \epsilon z g_0(z) \left(v_2 g_0 \left(\frac{z}{1-\epsilon v_1 z} \right) - v_1 \right) \right|, \\
G &:= \left| g_0(z) - g\left(\frac{z}{1-\epsilon v_1 z} \right) \right|.
\end{aligned}$$

We are searching for a lower bound of this last expression. F is obviously greater than $|\epsilon||z|mm_0|v_2|$ so we have to find an upper bound for G. We observe that $\frac{z}{1-\epsilon v_1 z} = z + \frac{\epsilon v_1 z^2}{1-\epsilon v_1 z}$, and considering Taylor expansion of g_0 at 0, we obtain that

$$\begin{aligned}
\left| g_0(z) - g\left(\frac{z}{1-\epsilon v_1 z}\right) \right| &= \left| \sum_{n=p}^{\infty} a_n z^n - \sum_{n=p}^{\infty} a_n \left(z + \frac{\epsilon v_1 z^2}{1-\epsilon v_1 z} \right)^n \right| \\
&= |\epsilon v_1 z^{p+1} h_\epsilon(z)|,
\end{aligned}$$

with $|h_\epsilon(z)|$ bounded by a number $M_0 > 0$ for every z in the disk and every ϵ small enough.

Thus, by replacing these bounds in the previous expression,

$$d_z(\Gamma_0, \Gamma_0^\epsilon) \geq \frac{|z|\,[|\epsilon||z|m|v_2|m_0 - |\epsilon v_1 z^{p+1} h(z)|]}{1 + z\epsilon\left(-v_1 + v_2 g_0\left(\frac{z}{1-\epsilon v_1 z}\right)\right)}$$
$$\geq \frac{|\epsilon z^2|}{1 + z\epsilon\left(-v_1 + v_2 g_0\left(\frac{z}{1-\epsilon v_1 z}\right)\right)}(mm_0|v_2| - v_1|z|^p M_0).$$

Now, we choose ϵ_0' such that if $|\epsilon| < \epsilon_0'$ then

$$\frac{1}{1 + z\epsilon\left(-v_1 + v_2 g_0\left(\frac{z}{1-\epsilon v_1 z}\right)\right)} > \frac{1}{2},$$

for every $z \in \Delta_\delta$, and if we set δ to satisfy that $mm_0|v_0| > 2|v_1|\delta^p M_0$, then

$$\min_{|z|=\delta} d_z(\Gamma_0, \Gamma_0^\epsilon) > \frac{\delta^2|\epsilon|mM_0|v_2|}{4}.$$

Therefore

$$\min_{|z|=\delta} d_z(\Gamma_w, \Gamma_{w'}) \geq$$
$$\geq \min_{|z|=\delta} d_z(\Gamma_{w'}, \Gamma_{w'}^\epsilon) - \max_{|z|=\delta} d_z(\Gamma_w, \Gamma_{w'}^\epsilon)$$

then, by applying Lemma 2.3,

$$\min_{|z|=\delta} d_z(\Gamma_w, \Gamma_{w'}) \geq \frac{\delta^2|\epsilon|mM_0|v_2|}{4} - c_0^N K|\epsilon|.$$

Hence if N_0' is big enough and $N > N_0'$,

$$\min_{|z|=\delta} d_z(\Gamma_w, \Gamma_{w'}) \geq \frac{\delta^2|\epsilon|mM_0|v_2|}{8} > 0.$$

Consequently, the number c_0 we were searching for is

$$c_0 = \frac{\delta^2 mM_0|v_2|}{8}.$$

$\square$

Lemma 2.22. *There is a covering of $\mathbb{P}^1 \times [1:0] \setminus U_0$ by flow boxes of two different types, W_{jW}^a and W_{iW}^t and an $\epsilon_1 > 0$, verifying that for every ϵ such that $|\epsilon| < \epsilon_1$,*

- *if Γ_w is a plaque in W_{jW}^a then $\Gamma_w^\epsilon \cap \Gamma_w \neq \emptyset$;*

- *if* Γ_z *and* $\Gamma_{z'}$ *are plaques in* $W^t_{i_W}$ *satisfying that* $\max d_w(\Gamma_z, \Gamma^\epsilon_{z'}) < \frac{|v_1||\epsilon|}{2}$ *then* $\min d_w(\Gamma_z, \Gamma_{z'}) > \frac{|v_1||\epsilon|}{2}$.

Proof. In order to prove this lemma we use ψ_3. In this chart, an automorphism behaves as $\Phi_\epsilon(z, w) = (z + \epsilon v_1, \frac{w}{1+\epsilon v_1 w})$ which is a horizontal translation in $w = 0$. We want to cover the points of $w = 0$ which are not in U_0. It is a compact set, so we will find a finite covering.

If q is a point with horizontal tangent, we take a horizontal flow box centered at q where $f_0'(z) = 0$ if and only if $z = 0$. We will proof that for ϵ small enough, Γ_0 and Γ_0^ϵ intersect each other and by Hurwitz's theorem (see A) we can find a flow box centered at q verifying this for every plaque in it.

We can write $\Gamma_0 = \{(z, f_0(z)),\ z \in \Delta_{\delta'}\}$ with $f_0(0) = 0$ and $f'(0) = 0$ and $\Gamma_0^\epsilon = \{(z + \epsilon v_1, \frac{f_0(z)}{1+\epsilon v_2 f_0(z)}),\ z \in \Delta_{\delta'}\}$, so we want to compute if the function

$$f_0(z) - \frac{f_0(z - \epsilon v_1)}{1 + \epsilon v_2 f_0(z - \epsilon v_1)}$$

has any zero. The number of zeros of that function is the same as the number of zeros of

$$\begin{aligned} g_0^\epsilon(z) =& \frac{1}{\epsilon}\left(f_0(z) - \frac{f_0(z - \epsilon v_1)}{1 + \epsilon v_2 f_0(z - \epsilon v_1)}\right) \\ =& \frac{1}{\epsilon}\left(f_0(z) - f_0(z - \epsilon v_1) - \frac{f_0^2(z - \epsilon v_1)\epsilon v_2}{1 + \epsilon v_2 f_0(z - \epsilon v_1)}\right). \end{aligned}$$

Then, $\lim_{\epsilon \to 0} g_0^\epsilon(z) = f_0'(z)v_1 - f_0^2(z)v_2$ which has a finite number of zeroes in Δ_δ. By Hurwitz's theorem again, there is ϵ_1 such that if $|\epsilon| < \epsilon_1$, $g_0^\epsilon(z)$ has the same number of zeros than the limit. Then Γ_0^ϵ and Γ_0 intersect each other, as do nearby enough plaques. We cover these points by flow boxes $W^a_{j_W}$.

Now, if q is a non horizontal point in $w = 0$, we can take a vertical flow box around it $(z + f_z(w), w)$ and $\Gamma_z^\epsilon = (z + \epsilon v_1 + f_z(w), \frac{w}{1+\epsilon v_2 w})$. If $\max d_w(\Gamma_z, \Gamma^\epsilon_{z'}) < |v_1 \epsilon|/2$, then

$$\min d_w(\Gamma_z, \Gamma_{z'}) \geq \min d_w(\Gamma_z, \Gamma^\epsilon_{z'}) - \max d_w(\Gamma_{z'}, \Gamma^\epsilon_{z'}) = |\epsilon v_1| - |v_1 \epsilon|/2 > |\epsilon v_1|/2.$$

In this way we obtain the flow boxes $W^t_{i_W}$.

So, finally, we can cover $\{w = 0\} \setminus U_0$ by a finite number of flow boxes. □

We can cover $[1 : 0] \times \mathbb{P}^1$ analogously and obtain the same result for open sets $V^t_{i_V}$ and $V^a_{j_V}$.

Lemma 2.23. *There is a covering of* $[1:0] \times \mathbb{P}^1 \setminus U_0$ *by flow boxes of two different types,* $V^a_{j_V}$ *and* $V^t_{i_V}$ *and an* $\epsilon_2 > 0$*, verifying that for every* ϵ *such that* $|\epsilon| < \epsilon_2$*,*

- *if* Γ_z *is a plaque in* $V^a_{j_V}$ *then* $\Gamma^\epsilon_z \cap \Gamma_z \neq \emptyset$*;*
- *if* Γ_w *and* $\Gamma_{w'}$ *are plaques in* $V^t_{i_V}$ *satisfying that* $\max d_z(\Gamma_w, \Gamma^\epsilon_{w'}) < \frac{|v_2||\epsilon|}{2}$ *then* $\min d_z(\Gamma_w, \Gamma_{w'}) > \frac{|v_2||\epsilon|}{2}$.

Define $W := \bigcup(W^a_{j_W}) \cup \bigcup(W^t_{i_W})$ and $V := \bigcup(V^a_{j_V}) \cup \bigcup(V^t_{i_V})$.

Lemma 2.24. *There is a covering of* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (U_0 \cup V \cup W \cup A)$ *by flow boxes of two different types,* $B^a_{j_B}$ *and* $B^t_{i_B}$*, and an* $\epsilon_3 > 0$ *such that if* $|\epsilon| < \epsilon_3$*,*

- *if* Γ_w *is a plaque in* $B^a_{j_B}$ *then* $\Gamma^\epsilon_w \cap \Gamma_w \neq \emptyset$*;*
- *if* Γ_z *and* $\Gamma_{z'}$ *are plaques in* $B^t_{i_B}$ *satisfying* $\max d_w(\Gamma_z, \Gamma^\epsilon_{z'}) < \frac{|\epsilon|}{2}$ *then* $\min d_w(\Gamma_z, \Gamma_{z'}) > \frac{|\epsilon|}{2}$

Proof. We use ψ_1 because every point of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (U_0 \cup W \cup V \cup A)$ is on its domain. In this chart, Φ_ϵ works as a translation by the vector $(\epsilon v_1, \epsilon v_2)$, and there is a point p' on this open set whose tangent space contains (v_1, v_2).

We change coordinates for simplicity. Let us consider the linear change of coordinates $R : \mathbb{C}^2 \to \mathbb{C}^2$ sending (v_1, v_2) to $(1, 0)$ and $(-\overline{v}_2, \overline{v}_1)$ to $(0, 1)$. We have obtained new coordinates (z', w') such that our family of automorphisms is a family of horizontal translations. Then, we can argue as we did in Theorem 2.10. We cover our new horizontal points on these new coordinates with flow boxes $B^a_{j_B}$. The rest of the points are transversal to the motions, hence they can be covered with flow boxes $B^t_{i_B}$.

The estimates appearing in the statement for $B^t_{i_B}$ follow from Remark 2.2.2 and the fact that $d_w(\Gamma_z, \Gamma^\epsilon_z) = \epsilon$. This finishes the proof of the lemma. □

Although we have several types of flow boxes covering the lamination in $\mathbb{P}^1 \times \mathbb{P}^1$, we can split them in three main types: flow boxes along the automorphisms which are $W^a_{j_W}, V^a_{j_V}, B^a_{j_B}$, transversal to the automorphisms $W^t_{i_W}, V^t_{i_V}$, $B^t_{i_B}$, U_0 and a singular flow box A_{i_A} for each singularity. We set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon'_0\}$ and $c_4 = \min\{c'_0, |v_1|/2, |v_2|/2, 1/2\}$. Now we are ready to prove that Condition 1 holds for $\mathcal{M} = \mathbb{P}^1 \times \mathbb{P}^1$ outside the singular neighborhoods for the chosen family of automorphisms.

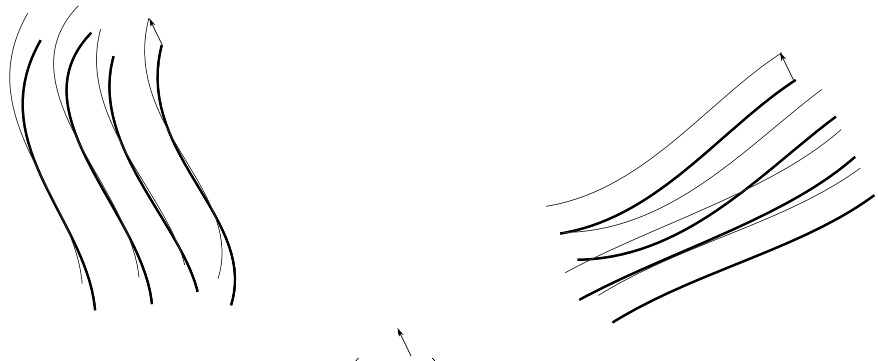

Figure 2.3: Behaviour of the lamination with respect to the automorphisms

Theorem 2.25. *Let $\mathcal{L}$ be a minimal transversely Lipschitz lamination with only hyperbolic singularities in $\mathbb{P}^1 \times \mathbb{P}^1$ and without directed closed currents. Then, it satisfies Condition 1 outside the singular neighborhoods.*

Proof. For the sake of simplicity, throughout the proof we will denote by $d_{max}(\Gamma_1, \Gamma_2)$ the maximum of the transversal distances in a flow box between the plaques Γ_1, Γ_2, and $d_{min}(\Gamma_1, \Gamma_2)$ the minimum.

By Lemma 2.3, if Γ_1 Γ_2^ϵ are plaques in the same regular flow box which intersect each other in N points, then the transversal distance satisfies that $d_{max}(\Gamma_1, \Gamma_2^\epsilon) < c^N|\epsilon|A$, for certain constants $c < 1$ and $A > 0$ not depending on the flow box. There exists $b > 0$ such that the distortion of the transversal distance in a change of flow boxes is bounded from above by b and by $1/b$ from below. This b arises from combining the constant in Remark 2.2.2 and the distortion of the distance when we change coordinates on the surface. Finally, there is also $M \in \mathbb{N}$ holding that, for every plaque in a flow box along the motion, we can find a path from this plaque to a plaque in a flow box transversal to the motion passing through at most M changes of flow boxes avoiding A_{i_A} and U_0 (unless we had started in U_0). This number M can also be chosen holding the same statement when starting from a flow box transversal to the motion and finishing in a tangential one.

Now, suppose two plaques, Γ_1 and Γ_2 in a flow box transversal to the motion satisfying that Γ_1 and Γ_2^ϵ have $N > N_0'$ intersection points for an ϵ with $|\epsilon| < \epsilon_0$. Hence $d_{max}(\Gamma_1, \Gamma_2^\epsilon) < c^N A|\epsilon|$. Consider a path as we said before joining this flow box transversal to the motion with another one along the motion, and let Γ_1' and Γ_2' be the corresponding continuation of the plaques. Then, by applying Lemma 2.4 when changing flow boxes, $d_{max}(\Gamma_1', \Gamma_2^{\epsilon'}) < b^M c^{N/2^M}|\epsilon|A$. Nevertheless, if $c^N A < c_4$ by the previous lemmas $d_{min}(\Gamma_1, \Gamma_2) > c_4|\epsilon|$. Following the path we can also conclude that $d_{min}(\Gamma_1, \Gamma_2) > \frac{|\epsilon|c_4}{b^M}$. Then,

$$d_{min}(\Gamma_1', \Gamma_1^{\epsilon'}) > d_{min}(\Gamma_1', \Gamma_2') - d_{max}(\Gamma_1', \Gamma_2^{\epsilon'}) \geq |\epsilon| \left(\frac{c_4}{b^M} - b^M c^{N/2^M} A \right)$$

There is $N_1 \in \mathbb{N}$ such that if $N > N_1$, the right side of the inequality above is bigger than zero, but if this happens, it would mean that Γ_1' and $\Gamma_1^{\epsilon'}$ do not have a common point. But they do if $|\epsilon| < \epsilon_0$. So N cannot be arbitrarily large.

Now, we argue when we start in a flow box along the motion. Consider Γ_1 and Γ_2 in it such that Γ_1 and Γ_2^ϵ intersect each other at N points. They also verify that $d_{max}(\Gamma_1, \Gamma_2^\epsilon) < c^N|\epsilon|A$. We construct a path to a transversal flow box, and we reach the continuation of the plaques Γ_1' and Γ_2'. They hold that $d_{max}(\Gamma_1', \Gamma_2^{\epsilon'}) < Ab^M c^{N/2^M}|\epsilon|$. Hence, there exists $N_2' \in \mathbb{N}$ such that, if $N > N_2'$, then $c^{N/2^M} Ab^M < c_4$. Therefore, by the previous lemmas, $d_{min}(\Gamma_1', \Gamma_2') > c_4|\epsilon|$. We follow the path back to the original flow box and we get that $d_{min}(\Gamma_2, \Gamma_2^\epsilon) > (c_4/b^M - c^N A)|\epsilon|$. So there is $N_2 > N_2'$ holding that $c_4/b^M - c^N A > 0$ for every $N > N_2$. But this would mean that there are no intersection points between Γ_2 and Γ_2^ϵ. The same contradiction arises.

In order to obtain the N_0 in Condition 1, take $N_0 = \max\{N_1, N_2\}$. $\square$

2.4.2 Case of $\mathbb{T}^1 \times \mathbb{P}^1$ and $\mathbb{T}^2$

These four different local behaviors we saw in the previous section describe also every behavior appearing in the two remaining surfaces to be studied. So we just need to put them in the right situation. Let us begin with $\mathbb{T}^1 \times \mathbb{P}^1$.

Let $\Pi_1 : \mathbb{T}^1 \times \mathbb{P}^1 \to \mathbb{T}^1$ and $\Pi_2 : \mathbb{T}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projections on each factor and $\pi : \mathbb{C} \to \mathbb{T}^1$ be the canonical projection in $\mathbb{T}^1$. Let $s_1, \ldots, s_n$ be the singularities of the lamination. We can find an automorphism of $\mathbb{T}^1 \times \mathbb{P}^1$ such that $\mathbb{T}^1 \times [1 : 0]$ does not contain any singularity, and an open simply connected relatively compact set U of $\mathbb{C}$, which is a neighborhood of a fundamental domain for the equivalence relation defining $\mathbb{T}^1$, containing only one preimage by π of the singularities.

In this case, we are going to search for a family of automorphisms as

$$\Phi_\epsilon([z],[w_1:w_2]) = ([z+v_1\epsilon],[w_1+\epsilon v_2 w_2 : w_2]).$$

So, in the chart $\psi_2(z,w) = ([z],[w:1])$ the automorphisms act as translations by a vector $(\epsilon v_1, \epsilon v_2)$. Thus, if we choose (v_1, v_2) satisfying the conditions i),ii) and iv) required in the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we can argue in a similar way: firstly, we need to cover $\mathbb{T}^1 \times [1:0]$ in a special way and then, the rest of the points are a compact set in the other chart where the automorphisms act as translations, so we can cover it as we did for $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 2.26. *There is a covering of* $\mathbb{T}^1 \times [1:0]$ *by flow boxes of two different types,* $V^a_{j_V}$ *and* $V^t_{i_V}$ *and an* $\epsilon_1 > 0$, *holding that if* $|\epsilon| < \epsilon_1$,

- *if* Γ_z *is a plaque in* $V^a_{j_V}$ *then* $\Gamma^\epsilon_z \cap \Gamma_z \neq \emptyset$;
- *if* Γ_w *and* $\Gamma_{w'}$ *are plaques in* $V^t_{i_V}$ *satisfying that* $\max d_w(\Gamma_z, \Gamma^\epsilon_{z'}) < \frac{|v_1||\epsilon|}{2}$ *then* $\min d_w(\Gamma_z, \Gamma_{z'}) > \frac{|v_1||\epsilon|}{2}$.

Proof. We work with ψ_1. In this chart $\Phi_\epsilon(z,w) = (z+\epsilon v_1, \frac{w}{1+\epsilon v_1 w})$. Hence, is a horizontal translation in $w = 0$. Notice that this is the same situation we dealed with in Lemma 2.23, therefore the proof is the same.

□

We set $V = \bigcup V^a_{j_V} \cup \bigcup V^t_{i_V}$.

Lemma 2.27. *There is a covering of* $\mathbb{T}^1 \times \mathbb{P}^1 \setminus V$ *by flow boxes of two different types,* $B^a_{j_B}$ *and* $B^t_{i_B}$, *and an* $\epsilon_2 > 0$ *such that if* $|\epsilon| < \epsilon_2$,

- *if* Γ_w *is a plaque in* $B^a_{j_B}$ *then* $\Gamma^\epsilon_w \cap \Gamma_w \neq \emptyset$;
- *if* Γ_z *and* $\Gamma_{z'}$ *are plaques in* $B^t_{i_B}$ *satisfying that* $\max d_w(\Gamma_z, \Gamma^\epsilon_{z'}) < \frac{|\epsilon|}{2}$ *then* $\min d_w(\Gamma_z, \Gamma_{z'}) > \frac{|\epsilon|}{2}$.

The behavior in the chart given by ψ_2 is a translation, so the proof is the same as in Lemma 2.24. Setting $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, both lemmas together let us prove the analogous to Theorem 2.25 for $\mathcal{M} = \mathbb{P}^1 \times \mathbb{T}^1$ by the same reasoning.

Finally, we deal with the case of $\mathbb{T}^2$. Let Λ be a lattice in $\mathbb{C}^2$, and let $\pi : \mathbb{C}^2 \to \mathbb{C}^2/\Lambda = \mathbb{T}^2$ be the canonical projection. If $\mathcal{L}$ is a minimal lamination with hyperbolic singularities embedded in $\mathbb{T}^2$, we can consider a simply connected relatively compact open neighborhood U of $(0,0)$ in $\mathbb{C}^2$ covering a fundamental domain of the equivalence relation defining $\mathbb{T}^2$, and containing only one preimage of the singularities inside it and none on its boundary. The family of automorphisms we will consider is $\Phi_\epsilon[(z,w)] = [(z+\epsilon v_1, w+\epsilon v_2)]$, with (v_1,v_2) chosen as before. Φ_ϵ lifts to a translation $\tilde{\Phi}_\epsilon : \mathbb{C}^2 \to \mathbb{C}^2$. We can argue as we did in Lemma 2.24 and we get the analogous to Theorem 2.25 when $\mathcal{M} = \mathbb{T}^2$ in the same way.

Chapter 3

Corollaries and Applications

3.1 Non singular case

One of the hypothesis of the statement of the Main Theorem in the non singular case of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{T}^1$ is the unicity of a minimal set for the lamination which seems to be a very strong condition. However, a modification of the proof chosing a different family of automorphisms leads us to a more interesting statement.

Theorem 3.1. *Every transversely Lipschitz lamination by Riemann surfaces without compact curves embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ satisfies Condition 1.*

Recall from Theorem 1.21 that this Theorem 3.1 would imply that every directed harmonic current of mass one verifies that its self-intersection is $Q(T,T) = 0$. Hence, if there are no closed currents there is only one harmonic positive current of mass one directed by the lamination. In particular, there is only one minimal set.

Whereas the proof included in Section 2 is similar to the case of $\mathbb{P}^2$, this new proof is more similar to the case of $\mathbb{T}^2$ where the statement above was already proven.

Proof. The key of this new proof is the fact that the adherence of every leaf has horizontal and vertical points, otherwise the lamination would contain a compact leaf.

Consider the family of automorphisms

$$\Phi_\epsilon = ([z_0 : z_1], [w_0 : w_1]) = ([z_0 + \epsilon z_1 : z_1], [w_0 : w_1])$$

The surface $\mathbb{P}^1 \times \mathbb{P}^1$ is parametrized with the charts of subsection 2.4.1. The automorphisms in φ_1 and φ_3 behave like horizontal translations, then we need to control the behavior on a neighborhood of the fixed line $[1:0]\times\mathbb{P}^1$ with the parametrizations ϕ_1 and ϕ_2. The automorphisms have the following expression in both of them

$$(z,w)\to\left(\frac{z}{1+\epsilon z},w\right).$$

We will begin the proof by covering the horizontal points of the fixed line.

The set of horizontal points on $[1:0]\times\mathbb{P}^1$ is compact, thus we just need to find a good neighborhood around every point and then to extract a finite subcovering.

Without loss of generality, we can work with φ_2. Suppose $p=([1:0],[1:p_1])$ is a horizontal point. We can take a flow box around p such that the plaques are

$$\Gamma_t=\{([1:z],[1:f_t(z)])\}$$

with $f_t(0)=t$ and $f'_{p_1}(0)=0$. Moved plaques have the expression

$$\Gamma_t^\epsilon=\left\{\left([1:z],\left[1:f_t\left(\frac{z}{1+\epsilon z}\right)\right]\right)\right\}.$$

Therefore, we need to estimate the number of zeroes of $f_{t_1}(z)-f_{t_2}\left(\frac{z}{1-\epsilon z}\right)$. Let us define

$$g_\epsilon(t_1,t_2,z)=\frac{f_{t_1}(z)-f_{t_2}\left(\frac{z}{1-\epsilon z}\right)}{\epsilon}.$$

Note that $\lim_{\epsilon\to 0}g_\epsilon(t,t,z)=-f'_t(z)z^2$ for every z in the flow box.

Then, we can consider $\delta_0>0$ such that $|f'_{p_1}(z)|>\xi$ for every z with $|z|=\frac{\delta_0}{2}$, and $f'_{p_1}(z)$ has N_0 zeros on $|z|<\delta_0/2$.

By Hurwitz's theorem, we can take ϵ_0 such that, for every ϵ with $|\epsilon|<\epsilon_0$ and $|z|=\delta_0/2$ then

$$M/2>|g_\epsilon(p_1,p_1,z)|>\frac{\xi\delta_0^2}{8},$$

and $g_\epsilon(p_1,p_1,z)$ has N_0+2 zeros in $|z|<\delta_0/2$ for every ϵ with $|\epsilon|<\epsilon_0$.

Now, take a transversal $\mathscr{T}_{p_1}$ where $g_\epsilon(t,t,z)$ has the same number of zeros on $|z|<\delta_0/2$ than $g_\epsilon(p_1,p_1,z)$ for every $t\in\mathscr{T}_{p_1}$ and every $|\epsilon|<\epsilon_0$. We can shrink it to a smaller transversal $\mathscr{T}'_{p_1}$ which is relatively compact on $\mathscr{T}_{p_1}$ and verifies that

$$M>|g_\epsilon(t,t,z)|>\frac{\xi\delta_0^2}{16},$$

in $|z| = \delta_0/2$ for every $t \in \mathscr{T}'_{p_1}$. Then, if Γ_{t_1} and $\Gamma^{\epsilon}_{t_2}$ intersect each other in $\Delta_{\delta_0/2}$, there exists a $z_0 \in \Delta_{\delta_0/2}$ such that

$$g_\epsilon(t_1, t_2, z_0) = f_{t_1}(z_0) - f_{t_2}(\frac{z}{1-\epsilon z}) = 0.$$

Hence $|f_{t_1}(z_0) - f_{t_2}(z_0)| = |f_{t_2}(\frac{z}{1-\epsilon z}) - f_{t_2}(z_0)| \leq M|\epsilon|$. Using the Lipchitzness, $d(\Gamma_{t_1}, \Gamma_{t_2}) \leq C^2 M|\epsilon|$.

Therefore $|f_{t_1}(z) - f_{t_2}\left(\frac{z}{1-\epsilon z}\right)| < (C^2+1)M|\epsilon|$, so if they intersect N times in $\Delta_{\delta_0/2}$ by Lemma 2.3 $d(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) < c^N_{p_1}|\epsilon|$ with $c_{p_1} < 1$ independent of t_1, t_2. Since $z = 0$ is fixed for all the automorphims $d(\Gamma_{t_1}, \Gamma_{t_2}) < C^2 c^N_{p_1}|\epsilon|$, we get

$$\begin{aligned} |\epsilon|\frac{\xi\delta_0^2}{16} &< \min_{|z|=\delta_0/2}\left|f_{t_2}(z) - f_{t_2}\left(\frac{z}{1-\epsilon z}\right)\right| && \leq \\ &\leq \max_{|z|<\delta_0/2}\left|f_{t_1}(z) - f_{t_2}\left(\frac{z}{1-\epsilon z}\right)\right| + \max_{|z|<\delta_0/2}|f_{t_1}(z) - f_{t_2}(z)| && \leq \\ &\leq (C^2+1)c^N|\epsilon| \end{aligned}$$

Therefore, if N is bigger than a suitable positive integer N_1, we would get a contradiction. Finally, we take $N_{p_1} = \max\{N_0+2, N_1\}$.

We can do this argument in order to obtain a neighborhood of every horizontal point in $[1:0] \times \mathbb{P}^1$. So we can obtain a finite subcovering by flow boxes $U_1 \dots U_l$, an $\epsilon_h > 0$ and a number N_h such that, two plaques Γ_{t_1} and $\Gamma^{\epsilon}_{t_2}$ have less than N_h intersection points, and if $t_1 = t_2$, they have at least one for every ϵ with $|\epsilon| < \epsilon_0$.

Once the horizontal points are covered, we need to cover the rest of the points of the fixed line. The set $[1:0] \times \mathbb{P}^1 \setminus \bigcup_{i=1}^{l} U_i$ is also compact, hence, by previous arguments, we will find a flow box around every point q in this set with the desired properties.

Around every of these points, we can take a vertical flow box, where the plaques are described like $\Gamma_t = (f_t(w), w)$. Then, when moved by the automorphisms $\Gamma^{\epsilon}_t = \left(\frac{f_t(w)}{1+\epsilon f_t(w)}, w\right)$, the transversal distance between both is

$$d_w(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) = \left|\frac{f_{t_1}(w) - f_{t_2}(w) + \epsilon f_{t_1}(w) f_{t_2}(w)}{1 + \epsilon f_{t_2}(w)}\right|.$$

We take a flow box $\mathscr{T}_q \times \Delta_{\delta_0}$ such that there is $\xi > 0$ with $1 > |f_t(w)| > \xi$ for every $t \in \mathscr{T}_q$, $|w| = \frac{\delta_0}{2}$ and $|\epsilon| < \epsilon_0$.

Let us suppose that Γ_{t_1} and $\Gamma^{\epsilon}_{t_2}$ intersect each other in N points in $\Delta_{\delta_0/2}$, then $d(\Gamma_{t_1}, \Gamma^{\epsilon}_{t_2}) < c^N|\epsilon|$ with $c < 1$.

On the other hand

$$\begin{aligned}
c^N|\epsilon| &\geq \max_{|w|<\delta_0/2}\left|\frac{f_{t_1}(w)-f_{t_2}(w)+\epsilon f_{t_1}(w)f_{t_2}(w)}{1+\epsilon f_{t_2}(w)}\right| = \\
&= \max_{|w|=\delta_0/2}\left|\frac{f_{t_1}(z)-f_{t_2}(w)+\epsilon f_{t_1}(w)f_{t_2}(w)}{1+\epsilon f_{t_2}(w)}\right| \geq \\
&\geq \max_{|w|=\delta_0/2}\frac{-|f_{t_1}(w)-f_{t_2}(w)|+|\epsilon||f_{t_1}(w)f_{t_2}(w)|}{|1+\epsilon f_{t_2}(w)|} \geq \\
&\geq \max_{|w|=\delta_0/2}\frac{-|f_{t_1}(w)-f_{t_2}(w)|+|\epsilon||f_{t_1}(w)f_{t_2}(w)|}{2} \geq \\
&\geq \frac{1}{2}\left[\min_{|w|=\delta_0/2}|\epsilon||f_{t_1}(w)f_{t_2}(w)| - \max_{|w|=\delta_0/2}|f_{t_1}(w)-f_{t_2}(w)|\right] \geq \\
&\geq \frac{|\epsilon|\xi^2}{2} - \max_{|z|=\delta_0/2}\frac{d_z(\Gamma_1,\Gamma_2)}{2} \geq \\
&\geq \frac{|\epsilon|\xi^2}{2} - \min_{|z|=\delta_0/2}\frac{d_z(\Gamma_1,\Gamma_2)}{2C^2},
\end{aligned}$$

where C is the Lipschitz constant of the lamination. Hence, we get that

$$\min_{|z|=\delta_0/2} d_z(\Gamma_1,\Gamma_2) \geq \frac{\xi^2-2c^N|\epsilon|}{C^2}.$$

Then, if N is big enough $\min_{|z|=\delta_0/2} d_z(\Gamma_1,\Gamma_2) \geq \frac{\xi^2|\epsilon|}{2C^2}$.

In this way, we can find a finite covering of these points $V_1,\dots,V_k$, a positive integer N_v and a positive number $\xi_v > 0$ such that, if Γ_{t_1} and $\Gamma^\epsilon_{t_2}$ intersect each other in more than N_v points, then

$$d_z(\Gamma_{t_1},\Gamma_{t_2}) > \frac{\xi_v^2|\epsilon|}{2C^2}$$

for every z in the boundary of the plaques. Let us simplify the constan $m = \frac{\xi_v^2}{2C^2}$

Once we have covered $[1:0]\times\mathbb{P}^1$ with $U_1,\dots,U_l,V_1,\dots,V_k$, we need to cover the rest of the points, namely $\mathbb{P}^1\times\mathbb{P}^1 \setminus \left(\bigcup_{i=1}^l U_i \cup \bigcup_{j=1}^k V_j\right)$. Around these points the local behavior is as a horizontal translation, thus we can cover them as we did in the case of the torus 2.10.

Finally, suppose there are two plaques $\Gamma_1,\Gamma_2^\epsilon$ with more than N intersection points, with this $N > \max\{N_h, N_{p_1}, N_v\}$. This situation cannot occur in a horizontal flow box. Therefore, let us suppose that we are in a vertical flow box. Then, it implies that $d(\Gamma_1,\Gamma_2) > m|\epsilon|$ and $d(\Gamma_1,\Gamma_2^\epsilon) < c^N|\epsilon|$. By analytic continuation we will reach a flow box containing a horizontal point after a finite number of changes of flow boxes M and Γ'_1, Γ'_2 denote their analytic continuation. Using previous estimates $d(\Gamma'_1, \Gamma'_2) > \frac{m|\epsilon|}{b^M}$ and $d(\Gamma'_1, \Gamma'^{\epsilon}_2) < b^M c^N |\epsilon|$.

But in this case, this would mean that $d(\Gamma'_2, \Gamma'^{\epsilon}_2) \geq (\frac{m}{b^M} - b^M c^N)|\epsilon|$, so N cannot be arbitrarily large. $\square$

This new statement for $\mathbb{P}^1 \times \mathbb{P}^1$ is complemented with the following corollary.

Corollary 3.2. *Let $(X, \mathcal{L})$ be a transversely Lipschitz lamination by Riemann surfaces without compact leaves in $M = \mathbb{P}^1 \times \mathbb{P}^1$. Then there are no directed closed current of mass one.*

Proof. We know that if T is a closed current of mass one $T = \Omega + \partial S + \overline{\partial S}$ for a unique $\square-$harmonic form Ω and $\partial \overline{S} = 0$. As we proved previously, every directed harmonic current T satisfies $Q(T, T) = 0$. Then, $\int T \wedge T = \int \Omega \wedge \Omega = 0$.

The dimension of $H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$ is two. It is generated by $\omega_1 = \sqrt{-1} dz_1 \wedge d\overline{z}_1$ and $\omega_2 = \sqrt{-1} dz_2 \wedge d\overline{z}_2$ the Kähler forms on each factor satisfying that $\omega = \omega_1 + \omega_2$ is the Kähler form on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\int \omega \wedge \omega = 1$. In fact $2\omega_1$ and $2\omega_2$ are the only two $\square$-harmonic forms with self-intersection 0 and mass 1. Hence, Ω must be either $2\omega_1$ or $2\omega_2$.

Suppose, without loss of generality, that $\Omega = 2\omega_1$. We will establish that T is directed by dz_1 and, therefore, the lamination has a compact leaf like $\{p\} \times \mathbb{P}^1$.

As $T = 2\omega_1 + \partial S + \overline{\partial S}$, then $\int T \wedge (\sqrt{-1} dz_1 \wedge d\overline{z}_1) = 0$. Due to the positivity of T, we can assure that the positive measure $T \wedge \sqrt{-1} dz_1 \wedge d\overline{z}_1$ is 0.

Consider U a flow box in an affine chart (z_1, z_2). Inside this flow box, T is directed by a $(1,0)$ form $\gamma = a dz_1 + b dz_2$ for certain continuous complex valued functions a, b, namely the current $T \wedge \gamma$ of bidimension $(0, 1)$ is 0.

If $b = 0$, there is nothing to prove, so we suppose that $\operatorname{supp} b$ is not empty. By applying $g d\overline{z}_1$ to $T \wedge \gamma$ with $\operatorname{supp} g \subset \operatorname{supp} b$, we get

$$0 = T \wedge (a dz_1 + b dz_2)(g d\overline{z}_1) = T(gb\, dz_2 \wedge d\overline{z}_1) = T \wedge (dz_2 \wedge d\overline{z}_1)(gb),$$

for every g. Then $T \wedge (dz_2 \wedge d\overline{z}_1) = 0$. By conjugacy we get $T \wedge (dz_1 \wedge d\overline{z}_2) = 0$. This implies that $T \wedge dz_1 = 0$ on every flow box. Hence, T is directed by $\{dz_1 = 0\}$. $\square$

By joining both results we get the following corollary.

Corollary 3.3. *If $\mathcal{L}$ is a transversely Lipschitz lamination by Riemann surfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ without invariant compact leaves, there is only one directed positive harmonic current. In particular there is only one minimal set.*

3.2 Singular case

It is well known that foliations on $\mathbb{P}^2$ without algebraic leaves and having only hyperbolic singularities are generic in the space of foliations of $\mathbb{P}^2$ (see for instance [LN88]). However, only recently, Coutinho and Pereira in [CP06] extended this result for foliations by curves in arbitrarely projective varieties. This is the opposite situation to the non singular case, namely we do not know any examples of laminations embedded in the surfaces under consideration, but the singular case we considered happens to be the generic situation.

Proposition 3.4. *Let X be a minimal lamination containing a hyperbolic singularity. If X admits a directed closed current, X is a closed leaf.*

Furthermore, we can prove the following

Proposition 3.5. *Let $\mathcal{F}$ a holomorphic foliation with only hyperbolic singularities on $\mathbb{P}^1 \times \mathbb{P}^1$ without invariant closed curves. Then there is only one possibly singular minimal set. Therefore there exists a unique harmonic current directed by the foliation.*

Proof. Suppose there are two minimal sets X and X', and consider the lamination $\mathcal{L}$ given by the union of both of them. Since they come from a holomorphic foliation, $\mathcal{L}$ is transversely Lipschitz. Now, we can assume that $p = ([1:0],[1:0]) \in \mathcal{L}$ and $\mathbb{P}^1 \times \{p\} \cup \{p\} \times \mathbb{P}^1$ does not contain any singularity of $\mathcal{L}$.

In this setting, if we consider a vector (v_1, v_2) holding the three first conditions stated in Subsection 2.4.1, and making the substitution of the fourth one for

- (v_1, v_2) is tangent to the lamination in a point $p_1 \in X$ and in a point $p_2 \in X'$

we can repeat the same reasoning as before, and we obtain that $Q(T,T) = 0$ for every harmonic current directed by $\mathcal{L}$. Since $\mathcal{L}$ does not admit any directed closed current, there exists a unique positive harmonic current of measure one T and its support is a minimal set. Hence, there is only one minimal set. $\square$

Given that foliations with only hyperbolic singularities without algebraic leaves are generic in these surfaces, the main theorem can be applied generically. Although the genericity of this foliations is already proven in [CP06], the proof is quite complicated. Therefore, we would like to include here an easier proof for $\mathbb{P}^1 \times \mathbb{P}^1$, obtained essentially following the steps of the proof for $\mathbb{P}^2$ given in [Per07].

3.2.1 Genericity of Foliations in $\mathbb{P}^1 \times \mathbb{P}^1$

Let (d_1, d_2) be a pair of integers, and consider $X = (x_0 : x_1)$ and $Y = (y_0 : y_1)$, homogeneous coordinates of $\mathbb{P}^1$. We denote by Λ_{r_1,r_2} the bihomogeneous polinomials of bidegree (r_1, r_2) and a holomorphic foliation $\mathcal{F}$ of bidegree (d_1, d_2) on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by a vector field

$$\mathfrak{X} = A\frac{\partial}{\partial x_0} + B\frac{\partial}{\partial x_1} + C\frac{\partial}{\partial y_0} + D\frac{\partial}{\partial y_1}$$

where $A, B \in \Lambda_{d_1,d_2-1}$ and $C, D \in \Lambda_{d_1-1,d_2}$. We will denote $\mathfrak{X}_1 = A\frac{\partial}{\partial x_0} + B\frac{\partial}{\partial x_1}$ and $\mathfrak{X}_2 = C\frac{\partial}{\partial y_0} + D\frac{\partial}{\partial y_1}$.

Two different vector fields $\mathfrak{X}$ and $\mathfrak{X}'$ induce the same foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ if

$$\mathfrak{X} - \mathfrak{X}' = g_1\left(x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1}\right) + g_2\left(y_0\frac{\partial}{\partial y_0} + y_1\frac{\partial}{\partial y_1}\right)$$

with g_1, g_2 of bidegree $(d_1 - 1, d_2 - 1)$.

If the foliation has isolated singularities, we will say that $\mathcal{F}$ is saturated. Following [CS11], if $\mathcal{F}$ is a saturated foliation of bidegree (d_1, d_2) then it has $2d_1d_2 + 2$ singularities.

Let Σ_{d_1,d_2} be the vector space of vector fields inducing a foliation of bidegree (d_1, d_2). It is easy to check that $dim_{\mathbb{C}}\Sigma_{d_1,d_2} = 2d_1d_2 + 2d_1 + 2d_2$. Since $\mathfrak{X}, \mathfrak{X}' \in \Sigma_{d_1,d_2}$ induce the same foliation in $\mathbb{P}^1 \times \mathbb{P}^1$ if $\mathfrak{X} = \lambda\mathfrak{X}'$, the space of foliations of bidegree (d_1, d_2), $Fol(d_1, d_2)$ is a projective space of dimension $2d_1d_2 + 2d_2 + 2d_1 - 1$.

An algebraic curve C of bidegree (r_1, r_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the zeroes of a bihomogeneous polynomial $f \in \Lambda_{r_1,r_2}$.

If $\mathfrak{X}$ induces a foliation $\mathcal{F}$ of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^1$, the curve C is invariant for $\mathfrak{X} = \mathfrak{X}_1 + \mathfrak{X}_2$ if there are h_1, h_2 such that $\mathfrak{X}_i(f) = h_i f$ for $i = 1, 2$, bidegree of $h_1 = (r_1 - 1, r_2)$ and bidegree of $h_2 = (r_1, r_2 - 1)$.

Define the following sets

$$\begin{aligned}&\mathcal{C}_{r_1,r_2}(d_1, d_2) = \{\mathcal{F} \in Fol(d_1, d_2),\\ &\mathcal{F} \text{ has an invariant curve of bidegree } (r_1, r_2)\}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{D}_{r_1,r_2}(d_1, d_2) = \{(x, \mathcal{F}) \in \mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2),\\ &x \text{ belongs to an invariant curve of bidegree } (r_1, r_2)\}\end{aligned}$$

Proposition 3.6. *The sets $\mathcal{C}_{r_1,r_2}(d_1, d_2)$ and $\mathcal{D}_{r_1,r_2}(d_1, d_2)$ are closed algebraic sets.*

Proof. Define the set

$$\begin{aligned}&\mathcal{Z}_{r_1,r_2}(d_1, d_2) = \{(x, [(\mathfrak{X}, h_1, h_2)], [f]) \text{ such that}\\ &\mathfrak{X}_1(f) = h_1 f,\ \mathfrak{X}_2(f) = h_2 f \text{ and } f(x) = 0\}\end{aligned}$$

which is a closed algebraic subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}(\Sigma_{d_1,d_2} \times \Lambda_{r_1-1,r_2} \times \Lambda_{r_1,r_2-1}) \times \mathbb{P}(\Lambda_{r_1,r_2})$. For the sake of simplicity, we will denote by Σ_0 the set $\Sigma_{d_1,d_2} \times \Lambda_{r_1-1,r_2} \times \Lambda_{r_1,r_2-1}$. Consider the projection

$$\pi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}(\Sigma_0) \times \mathbb{P}(\Lambda_{r_1,r_2}) \to \mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2) \times \mathbb{P}(\Lambda_{r_1,r_2}).$$

The indeterminacy locus of π does not intersect $\mathcal{Z}_{r_1,r_2}(d_1, d_2)$; hence, π restricted to $\mathcal{Z}_{r_1,r_2}(d_1, d_2)$ is regular and holomorphic.

Given that $\mathcal{Z}_{r_1,r_2}(d_1, d_2)$ is a closed set then so it is $\pi(\mathcal{Z}_{r_1,r_2}(d_1, d_2))$. In this setting, $\mathcal{C}_{r_1,r_2}(d_1, d_2)$ is the image of $\pi_1 : \mathcal{Z}_{r_1,r_2}(d_1, d_2) \to Fol(d_1, d_2)$ and $\mathcal{D}_{r_1,r_2}(d_1, d_2)$ is the image of $\pi_2 : \mathcal{Z}_{r_1,r_2}(d_1, d_2) \to \mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2)$. □

Let $\mathcal{S}(d_1, d_2) = \{(x, \mathcal{F}) \in \mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2)$ such that $x \in \mathrm{Sing}\mathcal{F}\}$.

Proposition 3.7. *For every $d_1, d_2 \geq 1$, $\mathcal{S}(d_1, d_2)$ is an invariant irreducible variety of codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2)$.*

Proof. Consider the projection $\Pi : \mathcal{S}(d_1, d_2) \to \mathbb{P}^1 \times \mathbb{P}^1$.

$\Pi^{-1}(x)$ is a subvariety of $\{x\} \times Fol(d_1, d_2)$ contained in $\mathcal{S}(d_1, d_2)$ which is isomorphic to a projective space.

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is homogeneous, all the fibers are smooth, irreducible and biholomorphic. Therefore $\mathcal{S}(d_1, d_2)$ is irreducible (see [Sha94]).

Now, we will show that this set has codimension two. We just need to analize the fiber over $p = ([0:1],[0:1])$ and see that it has codimension two in $\{p\} \times Fol(d_1, d_2)$.

Let $H \in \Lambda_{d_1-1,d_2-1}$ be a bihomogeneous polynomial not vanishing at p and consider the vector fields

$$\mathfrak{X} = H\left(x_1\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_1}\right) + H\left(y_1\frac{\partial}{\partial y_0} + y_0\frac{\partial}{\partial y_1}\right)$$

$$\mathfrak{X}' = H\left(x_1\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_1}\right) - H\left(y_1\frac{\partial}{\partial y_0} + y_0\frac{\partial}{\partial y_1}\right)$$

Thus, $\mathfrak{X}(p)$ and $\mathfrak{X}'(p)$ generate the tangent space of the foliations not vanishing at p. Therefore, the space of foliations having a singularity at p have codimension 2. □

By the Theorem of the index of Camacho-Sad [CS82], if C is an invariant curve of bidegree (d_1, d_2) with $d_1, d_2 \neq 0$ then it contains a singularity of the foliation.

Proposition 3.8. *Suppose that there exists (r_1, r_2) for $r_1, r_2 \geq 1$ such that $\mathcal{C}_{r_1,r_2}(d_1, d_2) = Fol(d_1, d_2)$, with $d_1, d_2 \geq 1$. Then,*

$$\mathcal{S}(d_1, d_2) \subset \mathcal{D}_{r_1,r_2}(d_1, d_2).$$

Proof. Suppose $\mathcal{C}_{r_1,r_2}(d_1, d_2) = Fol(d_1, d_2)$ and consider the projection $\Pi : \mathbb{P}^1 \times \mathbb{P}^1 \times Fol(d_1, d_2) \to Fol(d_1, d_2)$.

Then,

$$\Pi(\mathcal{S}(d_1, d_2) \cap \mathcal{D}_{r_1,r_2}(d_1, d_2)) = \mathcal{C}_{r_1,r_2}(d_1, d_2) = Fol(d_1, d_2).$$

Since $\mathcal{S}(d_1, d_2)$ has codimension 2 and is irreducible, then $\mathcal{S}(d_1, d_2) \cap \mathcal{D}_{r_1,r_2}(d_1, d_2) = \mathcal{S}(d_1, d_2)$. In this case, there would be an invariant curve of bidegree (r_1, r_2) through every singularity . □

Let $\mathcal{F}$ be the foliation of $\mathbb{P}^2$ given by the holomorphic 1-form in $\mathbb{C}^3$

$$\Omega =$$

$$= x^{d_1-1}y^{d_2-1}z(x+y+z)\left(\lambda\frac{dx}{x} + \mu\frac{dy}{y} + \gamma\frac{dz}{z} - (\lambda+\mu+\gamma)\frac{dx+dy+dz}{x+y+z}\right).$$

It is shown in [Per07] that there is no algebraic leaf passing through the singular point $[\lambda : \mu : \gamma]$ if λ, μ, γ are $\mathbb{Z}$ linearly independent. Since the line $\{z = 0\}$ is invariant for the foliation and $[1 : 0 : 0], [0 : 1 : 0]$ are singular points in it, we can blow up the points and blow down the line to get a foliation $\mathcal{F}'$ of $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d_1, d_2) having a singular point that does not admit any invariant algebraic curve passing through it.

Therefore $\mathcal{S}(d_1, d_2) \not\subset \mathcal{D}_{r_1,r_2}(d_1, d_2)$ for $d_1, d_2 > 1$, and by Proposition 3.8 we get that $\mathcal{C}_{r_1,r_2} \neq Fol(d_1, d_2)$ for every r_1, r_2. Thus, its complementary is a Zariski open set of $Fol(d_1, d_2)$ for every $r_1, r_2 > 0$. Therefore, by Baire's theorem

$$\bigcap_{r_1,r_2=1} (Fol(d_1, d_2) \setminus \mathcal{C}_{r_1,r_2}(d_1, d_2))$$

is a dense set in $Fol(d_1, d_2)$.

Appendix A

Complex and Functional Analysis

We want to include here a small appendix containing some topics on Complex and Functional Analysis that appeared on this thesis. Functional Analysis have appeared in a very fleeting but important way in the preliminaries and Complex Analysis, in particular Hurwitz's Theorem, is crucial in the proofs of our theorems. This Appendix is far from being exhaustive, however it might become useful in the understanding of the previous discussion. For deeper details and information on Complex Analysis see [Con78] and [Rud91] on Functional Analysis, for instance.

A.1 Complex Analysis

Since the study of the laminations carried out in this thesis is mainly local, we recall some of the results of basic Complex Analysis we needed to achieve our aim.

We begin this section recalling the well-known

Theorem A.1 (Cauchy's Integral Formula). *Let $f : D \to \mathbb{C}$ be a holomorphic function with $D \subset \mathbb{C}$ a simply connected open set and γ a simple Jordan curve on D. For every point p in the interior of the curve γ*

$$f(p) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(\xi)}{(\xi - p)} d\xi.$$

In addition,

$$f^{(k)}(p) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(\xi)}{(\xi - p)^{k+1}} d\xi.$$

The first part of the theorem allows us to recover the value of a holomorphic function by mean of surrounding values, and the second one implies that the same occurs for the derivatives in a fixed point. Hence, if we have a sequence of holomorphic functions that converges uniformly on compact sets to another one, their derivatives coverge as well.

This property is an example of the rigidity of holomorphic functions, and the next Theorem is another example of this phenomenon.

Theorem A.2 (Liouville's Theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ a holomorphic function. If f is bounded then f is constant.*

The Fundamental Theorem of Algebra can be proven as a consequence of Liouville's Theorem.

This rigidity showed on the previous theorem involves only functions which are holomorphic in the entire complex plane. However, the Maximum Modulus Principle covers the rest of the cases.

Theorem A.3 (Maximum Modulus Principle). *If $f : D \to \mathbb{C}$ is holomorphic in a open set and $p \in D$ satisfies that $|f(p)| \geq |f(z)|$ for every $z \in D$. Then, f is constant.*

For instance, as a direct consequence of this Principle, we can assure that for every holomorphic function defined on a bounded open set, the maximum modulus is reached on the boundary.

The unit disk is the most special case of bounded open set, thus it deserves special atention. The following theorem studies this situation.

Theorem A.4 (Schwarz's Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function from the unit disk to itself with $f(0) = 0$. Then $|f(0)| \leq 1$ and $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$.*

Moreover, if $|f'(0)| = 1$ or $|f(p)| = |p|$ for some $p \neq 0$ then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for every $z \in \mathbb{D}$.

Along this thesis we faced several times with converging sequences of holomorphic functions. They were just plaques that accumulate towards each others or functions describing the distance among them in order to obtain the intersection points between plaques. Therefore, in order to bound the number of these zeros, we invoked time after time the following

Theorem A.5 (Hurwitz). *Let D be an open set and a subsequence $\{f_n\}_{n\in\mathbb{N}} \to f$ uniformly on compacts. Suppose $f \not\equiv 0$ and there is a* *closed disk centered on a of radius R, $\bar{B}(a,R)$ contained in D verifying that $f(z) \neq 0$ for $|z-a| = R$. Then, there exists a natural number N_0 such that for every $n \geq N_0$, f_n and f have the same number of zeros in $B(a,R)$.*

Moreover, as a inmediate consequence we obtain the following:

Corollary A.6. *Let $f_n : D \to \mathbb{C}$ and $f : D \to \mathbb{C}$ be holomorphic functions for $n \in \mathbb{N}$. If $f_n \to f$ and $f_n(z) \neq 0$ for every $z \in D$, then either $f \equiv 0$ or $f(z) \neq 0$ for every $z \in D$.*

A.2 Functional Analysis

Let X be a vector space over a field $\mathbb{K}$ endowed with a norm $\|\cdot\|$ which induces a topology on X. If this norm is complete we say that X is a Banach space.

Theorem A.7. *Let X be a Banach space. The unit ball $B_1 = \{x \in X, \|x\| \leq 1\}$ is compact if and only if $\dim X < \infty$.*

We denote by X' the dual of a normed space, namely

$$X' = \{T : X \to \mathbb{K}, \text{with } T \text{ linear and continuous}\}.$$

The elements T of the dual space X' are called functionals.

Proposition A.8. *A linear functional is continuous if and only if*

$$\sup_{x \in B_1} \|T(x)\| < \infty$$

with the defined norm.

Therefore, on X' we can define a norm, in the following way $|||T||| = \sup_{x \in B_1} \|T(x)\|$. Any dual space of a normed space is Banach with this norm.

Moreover, we can defined the bidual of a normed space X'', as the dual space of the dual $X'' = (X')'$ and its norm is defined likewise.

A normed space can be identified with a subspace of X''. For every $x \in X$, we define the linear functional on X'

$$L_x(T) = T(x), x \in X.$$

In this way, X is embedded in X'' which is a Banach space. Thus, we can define $\tilde{X}$ the Banach completion of a normed space, as the smallest Banach space containing X.

On a vector normed space X whose dual is X', we can define a new topology on X, the so-called weak topology. This is the coarsest topology on X such that $T \in X'$ is still continuous on X.

In the same spirit, we define the weak* topology on X' as the coarsest topology such that $L_x(T)$ is continous for every $x \in X$.

Theorem A.9 (Hahn-Banach)**.** *Let X be a topological vector space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and A, B be convex non-empty disjoints subsets of X.*

- *If A is open, then there exists $\lambda : X \to \mathbb{K}$ and $t \in \mathbb{R}$ such that $\mathrm{Re}(\lambda(a)) < t \leq \mathrm{Re}(\lambda(b))$ for every $a \in A$ and $b \in B$.*

- *If X is locally convex, A is compact and B is closed then there exists a continuous linear map $\lambda : V \to \mathbb{K}$ and $s, t \in \mathbb{R}$ such that $\mathrm{Re}(\lambda(a) < t < s < \mathrm{Re}(\lambda(b)$ for every $a \in A$ and $b \in B$.*

Although the unit ball of a normed space is not compact unless it is finite dimensional, we have the following:

Theorem A.10 (Banach-Alaoglu)**.** *Let X be a Banach space with a norm $\|\cdot\|$ and B_1 the unit ball. Then B_1 is compact in the weak* topology.*

This theorem above, allows us to extract a convergent subsequence for a sequence of linear functionals.

In the literature concerning currents (see [Dem]), authors use the term weak topology instead of weak*, as we defined in this appendix. We preserved the usual notation for currents in the discussion, and the usual notation in Functional Analysis in the Appendix.

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