Radial quantum number of Laguerre-Gauss modes

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We introduce an operator linked with the radial index in the Laguerre-Gauss modes of a two-dimensional harmonic oscillator in cylindrical coordinates. We discuss ladder operators for this variable, and confirm that they obey the commutation relations of the su(1,1) algebra. Using this fact, we examine how basic quantum optical concepts can be recast in terms of radial modes.

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I. INTRODUCTION

An optical vortex is a light field exhibiting a pure screw phase dislocation along the propagation axis; i.e., an azimuthal phase dependence $\exp(i\ell\varphi)$. The integer ℓ plays the role of a topological charge: the phase changes its value by ℓ cycles of 2π in any closed circuit about the axis, while the amplitude is zero there [1].

One of the most intriguing properties of vortices is that they carry orbital angular momentum (OAM). This was first realized by Allen and co-workers [2] for the important instance of Laguerre-Gauss (LG) laser modes. Furthermore, they demonstrated that these modes carry an OAM of $\ell\hbar$ per photon along the propagation direction.

A useful feature of optical OAM is that it can be easily manipulated and transferred; this has opened new horizons in various fields, ranging from mechanical micromanipulation [3] to imaging sciences [4,5], as well as potential astronomical [6,7] and communication applications [8]. Beyond optical wavelengths, OAM now plays a major role in electron [9–12], x-ray [13–15], and radio-frequency engineering [16–18]. The core observation that individual photons also carry OAM brings the most exciting possibilities for employing this variable in the quantum regime, and a number of uses has already been demonstrated [19–24].

Despite this intense activity, very little attention has been paid thus far to the radial index p of the LG modes. Usually, it is stated that for $p > 0$, the modes are multiringed with $p + 1$ radial nodes. Beyond this short mention, no physical meaning is attached to this quantity. Two recent papers, however, have presented challenging and interesting insights into this issue [25,26]. Our purpose here is to present a simple comprehensive analysis of this variable.

The two aforementioned papers considered optical modes, governed by the paraxial wave equation. These modes are ultimately suitably rescaled wave functions of the stationary states of a two-dimensional quantum oscillator, under the Schrödinger equation [27]. Since this latter system can properly model other interesting vortices arising in different media (as in plasmas [28], superfluids [29], and Bose-Einstein condensates [29]), the oscillator will serve as our thread, bearing in mind that the results can be immediately translated to the optical case.

II. STATIONARY STATES OF A TWO-DIMENSIONAL OSCILLATOR

A. Cartesian coordinates

To be as self-contained as possible, we briefly review the example of an isotropic two-dimensional quantum harmonic oscillator of mass m and natural frequency ω, with coordinates in two orthogonal axes, say x and y [30,31]. The Hamiltonian can be compactly written as $\hat{H} = \hbar\omega(\hat{n} + \hat{\mathbb{1}})$, where the total number operator $\hat{n}$ is

$$\hat{n} = \hat{n}_x + \hat{n}_y = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y \, , \tag{2.1}$$

and the annihilation and creation operators fulfill the canonical commutation relations $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbb{1}}$, with $j,k \in \{x,y\}$. Since the spectrum of $\hat{n}_j$ is composed of all non-negative integers n_j, the energies are given by $E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1)$ and these eigenvalues are $(n_x + n_y + 1)$-fold degenerate.

Elements of the Fock basis are the common eigenvectors of $\hat{n}_x$ and $\hat{n}_y$:

$$|n_x, n_y\rangle = \frac{1}{\sqrt{n_x! n_y!}} (\hat{a}_x^\dagger)^{n_x} (\hat{a}_y^\dagger)^{n_y} |0,0\rangle \, , \tag{2.2}$$

where $|0,0\rangle$ is the ground state. The stationary states of the oscillator are the product of Hermite-Gauss modes, as the oscillations in each axes are kinematically independent:

$$\Psi_{n_x n_y}(x,y) = \sqrt{\frac{\alpha^2}{\pi \; 2^{n_x+n_y} n_x! n_y!}} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \times \exp[-\alpha^2(x^2 + y^2)/2] \, , \tag{2.3}$$

with $\alpha = \sqrt{m\omega/\hbar}$. To recover the equivalent beam solutions, one needs to take $\alpha = \sqrt{2}$, since the paraxial wave equation (in adimensional coordinates) coincides with the Schrödinger equation for the oscillator when $m = 2\hbar$ and $\omega = 1$.

For our purposes, the solution at $t = 0$ is enough. The wave function at any other time can be obtained in a simple way by using the explicit form of the propagator. For beams, where the role of time is played by the coordinate z along the symmetry axis, this propagation brings about additional interesting points, such as the Gouy phase.

B. Cylindrical coordinates

The axes x and y do not enjoy a privileged role in the problem. Since the energy is invariant under rotations in the xy plane, we could as well have chosen any other rotated reference frame. To take a better advantage of this symmetry, we consider the z component of the angular momentum, $\hat{L}_z = \hbar\hat{\ell}$, with $\hat{\ell} = i(\hat{a}_y^\dagger \hat{a}_x - \hat{a}_x^\dagger \hat{a}_y)$, and use the rotated bosonic operators [32]

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}(\hat{a}_x \mp i\hat{a}_y)\,, \quad \hat{a}_\pm^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_x^\dagger \pm i\hat{a}_y^\dagger)\,, \tag{2.4}$$

where $[\hat{a}_j,\hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbb{1}}$, with $j,k \in \{+,-\}$. We can then check that

$$\hat{n} = \hat{n}_+ + \hat{n}_-\,, \quad \hat{\ell} = \hat{n}_+ - \hat{n}_-\,, \tag{2.5}$$

whose interpretation is direct: the system can be envisioned now as consisting of "quanta" with positive (counterclockwise rotation around z) and negative (clockwise rotation around z) orbital angular momentum.

The Fock basis $\{|n_+,n_-\rangle\}$ of the common eigenvectors of $\hat{n}_+$ and $\hat{n}_-$ can be constructed much in the same way as in Eq. (2.2). However, it will prove useful to consider instead the continuous set

$$|\eta\rangle = \exp\left(-\tfrac{1}{2}|\eta|^2 + \eta\hat{a}_+^\dagger - \eta^*\hat{a}_-^\dagger + \hat{a}_+^\dagger\hat{a}_-^\dagger\right)|0,0\rangle\,, \tag{2.6}$$

parametrized by the complex number $\eta = r\exp(-i\varphi)$. The states $|\eta\rangle$ constitute an orthonormal basis, whose properties have been reviewed in depth in Ref. [33]. In the representation they generate $[\psi(\eta) = \langle\eta|\psi\rangle]$, the action of the basic operators is

$$\begin{aligned} \hat{a}_+\psi(\eta) &= \left(\frac{\eta}{2} + \frac{\partial}{\partial\eta^*}\right)\psi(\eta)\,, \\ \hat{a}_-\psi(\eta) &= -\left(\frac{\eta^*}{2} + \frac{\partial}{\partial\eta}\right)\psi(\eta)\,, \end{aligned} \tag{2.7}$$

while for the adjoints we have

$$\hat{a}_+^\dagger = \hat{a}_- - \eta^*\,, \quad \hat{a}_-^\dagger = \hat{a}_+ - \eta\,. \tag{2.8}$$

Since the exponential acting on the vacuum in Eq. (2.6) is not unitary, the creation and destruction operators are not conjugates one of the other under the usual boson conjugation.

Given the above, $\hat{n}$ and $\hat{\ell}$ act in this space as

$$\begin{aligned} \hat{n} &\mapsto \frac{r^2}{2} - \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) - 1\,, \\ \hat{\ell} &\mapsto -i\frac{\partial}{\partial\varphi}\,. \end{aligned} \tag{2.9}$$

As $[\hat{\ell},\hat{n}] = 0$, the basis $\{|n_+,n_-\rangle\}$ can be reinterpreted as common eigenvectors of $\hat{n}$ and $\hat{\ell}$, with eigenvalues $n = n_+ + n_-$ and $\ell = n_+ - n_-$, respectively. The stationary states in this basis can be readily obtained using Eqs. (2.9); the final result is

$$\Psi_{n\ell}(r,\varphi) = A_{n\ell}(r)\,e^{i\ell\varphi}\,, \tag{2.10}$$

where the normalized amplitude is

$$A_{n\ell}(r) = \frac{\sqrt{2\alpha^2 p!}}{\sqrt{(p+|\ell|)!}}\,e^{-\alpha^2 r^2/2}(\alpha r)^{|\ell|}L_p^{|\ell|}(\alpha^2 r^2)\,, \tag{2.11}$$

$L_p^\ell(x)$ are the generalized Laguerre polynomials and we have written $p = (n - |\ell|)/2$. The probability distribution $|\Psi_{n\ell}(r,\varphi)|^2$ shows p dark concentric rings.

III. QUANTUM OPTICS WITH RADIAL MODES

A. The radial number operator

Since the number of dark rings is $p = (n - |\ell|)/2$, the operator

$$\hat{p} = \frac{1}{2}(\hat{n} - |\hat{\ell}|) = \begin{cases} \hat{n}_- & \text{for} \quad \ell > 0\,, \\ \hat{n}_+ & \text{for} \quad \ell < 0\,, \end{cases} \tag{3.1}$$

seems to be a sensible definition for the radial-number operator of the Laguerre-Gauss modes. According to Eq. (2.9), in differential form it reads

$$\hat{p} \mapsto -\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{i}{2}\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{r^2}{2} - \frac{1}{2}\,. \tag{3.2}$$

Incidentally, it coincides with the operator found in Ref. [26] by setting $r \to \alpha r$.

To simplify the following discussion, let us, for the time being, relabel the stationary states $|n,\ell\rangle$ as $|p,\ell\rangle$, where p indicates the radial mode eigenvalue; i.e.,

$$\hat{p}|p,\ell\rangle = p|p,\ell\rangle\,. \tag{3.3}$$

As heralded in the Introduction, we are interested in exploring the Hilbert space associated with the radial number p, while keeping the OAM ℓ fixed. At first sight, one might look for the canonical conjugate variable to $\hat{p}$. Since, according to Eq. (3.1), $\hat{p} = \hat{n}_-$ or $\hat{n}_+$ (depending on the sign of ℓ), such a variable would be a phase $\hat{\phi}_-$ or $\hat{\phi}_+$. This means that if we denote by $\hat{e} = \exp(i\hat{\phi}_\pm)$ the exponential of such a putative phase, the corresponding commutation relation will read [34]

$$[\hat{e},\hat{p}] = \hat{e}\,. \tag{3.4}$$

This immediately implies

$$\hat{e} = \sum_{p=0}^{\infty} |p,\ell\rangle\langle p+1,\ell|\,, \tag{3.5}$$

so that

$$\hat{e}|p,\ell\rangle = |p-1,\ell\rangle\,, \quad \hat{e}^\dagger|p,\ell\rangle = |p+1,\ell\rangle\,. \tag{3.6}$$

Whereas the spectrum of $\hat{\ell}$ is unbounded, including all the integer numbers, the spectrum of $\hat{p}$ is semibounded, as it comprises only non-negative integers. This indicates that the action of $\hat{e}$ as a ladder operator fails at $p = 0$, and consequently it cannot be unitary:

$$\hat{e}\hat{e}^\dagger = \hat{\mathbb{1}}\,, \quad \hat{e}^\dagger\hat{e} = \hat{\mathbb{1}} - \hat{\mathscr{P}}_0, \tag{3.7}$$

where $\hat{\mathscr{P}}_0 = |0,\ell\rangle\langle 0,\ell|$ is the projector on the "vacuum."

All these problems thus place this interpretation on shaky grounds. For this reason, we prefer to follow an alternative route. To this end, we observe that to increase (decrease) the radial number by one unit, with ℓ unchanged, we need to create (annihilate) one positive quantum and one negative quantum, namely,

$$\begin{aligned} \hat{k}_+|p,\ell\rangle &= \hat{a}_-^\dagger\hat{a}_+^\dagger|p,\ell\rangle \propto |p+1,\ell\rangle\,, \\ \hat{k}_-|p,\ell\rangle &= \hat{a}_-\hat{a}_+|p,\ell\rangle \propto |p-1,\ell\rangle\,. \end{aligned} \tag{3.8}$$

One can check that

$$[\hat{k}_+,\hat{k}_-] = -2\hat{k}_z\,, \quad [\hat{k}_z,\hat{k}_+] = \hat{k}_+\,, \quad [\hat{k}_z,\hat{k}_-] = -\hat{k}_-\,, \tag{3.9}$$

with $\hat{k}_z = (\hat{n} + \hat{\mathbb{1}})/2$. This means that if we define $\hat{k}_\pm = \hat{k}_x \pm i\hat{k}_y$, we have

$$[\hat{k}_x,\hat{k}_y] = -i\hat{k}_z\,, \quad [\hat{k}_y,\hat{k}_z] = i\hat{k}_x\,, \quad [\hat{k}_z,\hat{k}_x] = i\hat{k}_y\,, \tag{3.10}$$

that is, they are the generators of the su(1, 1) algebra, as first noticed in Ref. [25].

B. Radial coherent states

To explore the issue in more detail, it is convenient to give some basic background on some well-known irreducible representations (irreps) of SU(1,1), which are excellently reviewed in Ref. [35] and whose role in quantum optics is difficult to underestimate [36–45]. The Casimir operator for this group is $\hat{K}^2 = \hat{k}_z^2 - \hat{k}_x^2 - \hat{k}_y^2$, which can be expressed as $\hat{K}^2 = k(k-1)\hat{\mathbb{1}}$, where the Bargmann index k labels the different irreps (this index plays the role of spin for rotations). In our case, a simple calculation shows that $k = (|\ell| + 1)/2$, so that $k = 1/2, 1, 3/2, \ldots$, which corresponds to the so-called positive discrete series, for which $\hat{k}_z$ is diagonal and has a discrete spectrum. In the Fock basis $\{|n_+,n_-\rangle\}$, the basis states of the irrep k are $\{|k,k+p\rangle\}$, with $p = 0,1,\ldots$, and hence

$$\hat{k}_z|k,k+p\rangle = (k+p)|k,k+p\rangle\,, \tag{3.11}$$

while the ladder operators act as

$$\begin{aligned} \hat{k}_+|k,k+p\rangle &= \sqrt{(2k+p)(p+1)}\,|k,k+p+1\rangle\,,\\ \hat{k}_-|k,k+p\rangle &= \sqrt{p(2k+p-1)}\,|k,k+p-1\rangle\,. \end{aligned} \tag{3.12}$$

Note that we can make the identification $|p,\ell\rangle \leftrightarrow |k,k+p\rangle$, provided $k = (|\ell| + 1)/2$.

Since $\hat{k}_-|k,k\rangle = 0$, this state can be taken as the vacuum. Indeed, $\hat{D}(\xi) = \exp(\xi\hat{k}_+ - \xi^*\hat{k}_-)$ are truly displacement operators, so according to the Perelomov prescription [46], the set

$$|\zeta\rangle = \hat{D}(\xi)|k,k\rangle \tag{3.13}$$

constitutes a family of *bona fide* coherent states parametrized by the pseudo-Euclidean unit vector $\mathbf{n} = (\sinh\omega\cos\varphi, \sinh\omega\sin\varphi, \cosh\omega)$, with $\xi = (\omega/2)\exp(i\varphi)$ and $\zeta = \tanh(\omega/2)\exp(-i\varphi)$.

By expanding the exponential and employing the disentangling theorem, we get the decomposition

$$|\zeta\rangle = (1-|\zeta|^2)^k \sum_{p=0}^{\infty} \sqrt{\frac{\Gamma(2k+p)}{p!\,\Gamma(2k)}}\,\zeta^p |k,k+p\rangle\,, \tag{3.14}$$

and by projecting over the complete basis $|\eta\rangle$ we get the corresponding wave function $\Psi_\zeta(r,\varphi)$ in the transverse parameters. The expression can be simplified into an exponential form using the identity

$$\exp\left(\frac{\gamma x}{\gamma - 1}\right) = (1-\gamma)^{1+|\ell|} \sum_{p=0}^{\infty} \gamma^p L_p^{|\ell|}(x)\,, \tag{3.15}$$

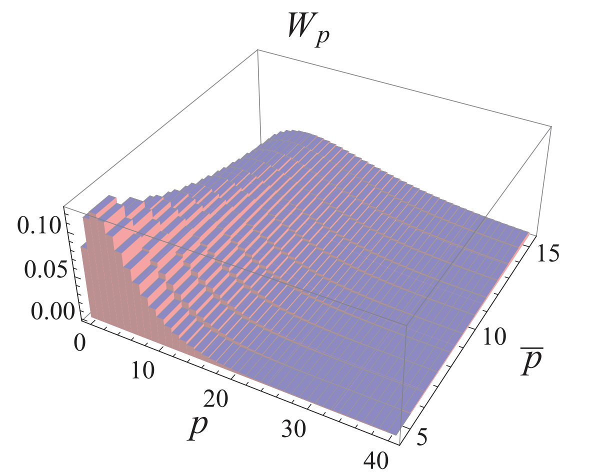

FIG. 1. (Color online) Probability distribution W_p for a coherent state $|\zeta\rangle$ with $\ell = 1$ as as a function of p and the ring average number $\bar{p}$.

so the final result is

$$\Psi_\zeta(r,\varphi) = \sqrt{\frac{\alpha^2}{\pi|\ell|!}}\left[\frac{1-|\zeta|^2}{(1-\zeta)^2}\right]^{(|\ell|+1)/2} \times e^{(\zeta+1)/(\zeta-1)\,\alpha^2 r^2/2}(\alpha r)^{|\ell|} e^{i\ell\varphi}\,. \tag{3.16}$$

We see that the Perelomov coherent states are polynomial-Gauss modes at $t = 0$; a subfamily of hypergeometric-Gauss modes upon evolution, as discussed in Ref. [47]. They are also eigenstates of the OAM and shape invariant in the time evolution.

The average number of sharp rings in the state $|\zeta\rangle$ is

$$\bar{p} = \frac{|\zeta|^2}{|\zeta|^2 - 1}(|\ell| + 1)\,, \tag{3.17}$$

and the statistical distribution of rings $W_p = |\langle p,\ell|\zeta\rangle|^2$ is

$$W_p = \frac{(|\ell|+1)^{|\ell|+1}(p+|\ell|)!}{p!\,|\ell|!}\,\frac{\bar{p}^p}{(\bar{p}+|\ell|+1)^{p+|\ell|+1}}\,. \tag{3.18}$$

In Fig. 1 we have plotted this distribution for a coherent state with $\ell = 1$ and different values of $\bar{p}$. In Fig. 2, we have plotted

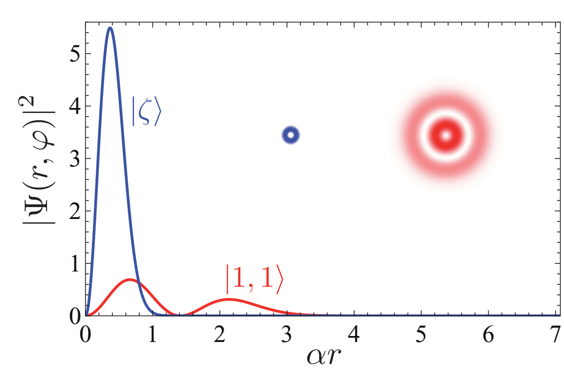

FIG. 2. (Color online) Intensity profiles $|\Psi(r,\phi)|^2$ for a radial coherent state $|\zeta\rangle$ written as in (3.16), with $\langle p\rangle = 1$ and for an eigenstate $|p,\ell\rangle$ with $p = 1$ and $\ell = 1$. In the inset, we show the corresponding density plots, in the same order.

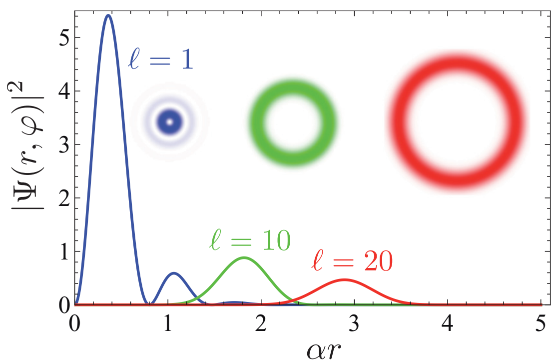

FIG. 3. (Color online) Intensity profiles $|\Psi(r,\varphi)|^2$ for intelligent states with $M = 10$ and $\tau = 1/2$ for several values of ℓ: $\ell = 1$, $\ell = 10$, and $\ell = 20$. The inset shows the corresponding density plots, in the same order.

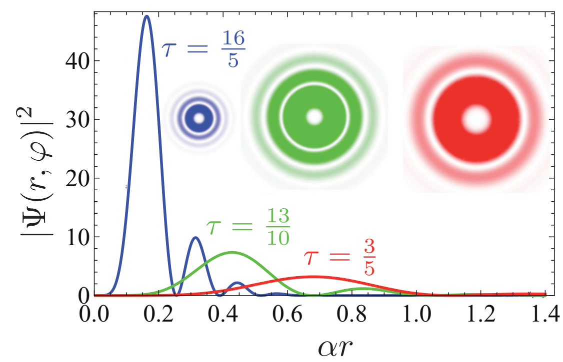

FIG. 4. (Color online) Intensity profiles $|\Psi(r,\varphi)|^2$ for intelligent states with $M = 11$ and $\ell = 3$ for several values of τ: $\tau = 16/5$, $\tau = 13/10$, and $\tau = 3/5$. The inset shows the corresponding density plots, in the same order.

$|\Psi_\zeta(r,\varphi)|^2$ for a coherent state with $\ell = 1$ and $\langle \hat{p} \rangle = 1$, and the corresponding distribution for the Laguerre-Gauss eigenstate with $p = 1$. The striking differences can be appreciated at a glance.

It is worth mentioning that there is an alternative definition of coherent states, due to Barut and Girardello [48]:

$$\hat{k}_- |\zeta\rangle_{\mathrm{BG}} = \zeta |\zeta\rangle_{\mathrm{BG}} \,, \tag{3.19}$$

which appears as a reasonable generalization of the standard coherent states as eigenstates of the annihilation operator. This equation can be solved in the $|p,\ell\rangle$ basis, yielding

$$|\zeta\rangle_{\mathrm{BG}} = \frac{|\zeta|^{\ell/2}}{\sqrt{I_\ell(2|\zeta|)}} \sum_{p=0}^{\infty} \frac{\zeta^p}{\sqrt{p!(p+|\ell|)!}} |p,\ell\rangle \,, \tag{3.20}$$

where $I_\ell(x)$ is the modified Bessel function. Projecting again in the transverse coordinates we get, after some calculations,

$$\Psi_{\zeta,\mathrm{BG}}(r,\varphi) = \sqrt{\frac{\alpha^2}{\pi I_{|\ell|}(2|\zeta|^2)}} e^{(\zeta^2 - \alpha^2 r^2/2)} J_{|\ell|}(2\zeta r) e^{i\ell\varphi} \,. \tag{3.21}$$

This set of coherent states are thus realized as Bessel-Gauss modes. However, these solutions are not shape invariant, which runs against the notion of coherence.

C. Radial intelligent and squeezed states

The commutation relations (3.9) imply that these operators cannot be measured simultaneously, which is reflected by the uncertainty relation

$$\Delta \hat{k}_x \, \Delta \hat{k}_y \geqslant \tfrac{1}{2} |\langle \hat{k}_z \rangle| \,, \tag{3.22}$$

where $\Delta \hat{A} = [\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2]^{1/2}$ stands for the variance. According to the standard definition, squeezing occurs whenever [49]

$$(\Delta \hat{k}_x)^2 \leqslant \tfrac{1}{2} |\langle \hat{k}_z \rangle| \quad \text{or} \quad (\Delta \hat{k}_y)^2 \leqslant \tfrac{1}{2} |\langle \hat{k}_z \rangle| \,. \tag{3.23}$$

Intelligent states are those for which (3.22) holds as an equality. The coherent states (3.16) and (3.21) are intelligent but not squeezed.

Indeed, these intelligent states are solutions of the eigenvalue problem [50]

$$(\hat{k}_x - i\lambda \hat{k}_y) |\Psi_\lambda\rangle = \Lambda |\Psi_\lambda\rangle \,, \quad \lambda \in \mathbb{R} \,. \tag{3.24}$$

Although they have been investigated from various perspectives [51–53], we follow here the comprehensive approach of Ref. [54], which starts by noting that the state $\exp(i\tau\hat{k}_y)|k,k\rangle$ is intelligent provided $\lambda = \cosh\tau$ [with eigenvalue $\Lambda = -(k + M)\sinh\tau$, and $M = 0,1,\ldots$ an integer number]. Then, the most general intelligent state can be written as

$$|\Psi_{\ell,M}(\tau)\rangle = \exp(i\tau\hat{k}_y) \big|\kappa_M^k(\tau)\big\rangle \,, \tag{3.25}$$

where τ is the squeezing parameter and the seed state $|\kappa_M^k\rangle$ can be expressed as

$$\big|\kappa_M^k(\tau)\big\rangle = \sum_{p=0}^{M} c_p^k(\tau) |k,k+p\rangle \,. \tag{3.26}$$

The coefficients c_p^k can be obtained as a recursion relation; the final result reads

$$c_p^k = \binom{M}{p} \frac{\tanh^p \tau}{\binom{2k+p-1}{p}^{1/2}} \, c_0^k \,, \tag{3.27}$$

and c_0^k is fixed by the normalization of the state. The infinite family (3.25) of states parametrized by k and M is actually squeezed.

Next we need to project $\exp(i\tau\hat{k}_y)|\kappa_M^k(\tau)\rangle$ on the basis $|k,k+p\rangle$. To this end, we recall that the action of $\exp(i\tau\hat{k}_y)$ on a basis state $|k,k+p\rangle$ is given in terms of SU(1,1) Wigner d functions [55]

$$\exp(i\tau\hat{k}_y)|k,k+p\rangle = \sum_{p'=0}^{\infty} d_{k+p,k+p'}^k(-\tau) \, |k,k+p'\rangle \,. \tag{3.28}$$

In this way, we get, expressed in transverse coordinates,

$$\Psi_{\ell M}(r,\varphi,\tau) = \sum_{p'=0}^{\infty} \sum_{p=0}^{M} d_{k+p',k+p}^k(-\tau) \, c_p^k(\tau) \, A_{p\;2k-1}(r) e^{i\ell\varphi} \,, \tag{3.29}$$

where $A_{p\ell}$ has been defined in Eq. (2.11). The effect of increasing ℓ, for fixed p and τ, is illustrated by plotting the intensity profile $|\Psi_{\ell,M}(r,\varphi,\tau)|^2$ as a function of r in Fig. 3: this intensity tends to a Gaussian-like shape. The effect of increasing τ for fixed ℓ and M is illustrated in Fig. 4, and leads to the appearance of rings as we increase τ.

For large values of ℓ (more concretely, for k and p large, but $p/k \ll 1$), one has at hand a compact asymptotic approximation to the d functions, namely [56],

$$d^{k}_{k+p,k}(\tau) \simeq \frac{1}{[(k+p)^2 - k^2]^{1/4}} \, e^{-k(\tau-\tau_p)^2/2}\,, \tag{3.30}$$

with $\cosh \tau_p = (k+p)/k$, and whose Gaussian nature is evident.

IV. CONCLUDING REMARKS

In summary, we have provided a handy toolbox to deal with the radial index of Laguerre-Gauss modes and shown how it can be used to construct a consistent quantum theory of this variable. We stress that this is more than an academic curiosity, since recent experiments in our laboratory [57] have confirmed that the radial degree of freedom of single photons can be manipulated individually in a quantum regime. It is our hope that the results presented here will inspire novel quantum protocols and algorithms using such a "forgotten quantum number."

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